UCB Math 228A, Fall 2013: Problem Set 3

Due October 10

1. Show that the Runge-Kutta methods below are fourth order accurate by verifying that the order conditions corresponding to all graphs of order ≤ 4 are satisfied.

a) RK4

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

b) Hammer-Hollingsworth

$$\begin{array}{c|cc} \frac{1}{2}-\frac{\sqrt{3}}{6} & \frac{1}{4} & \frac{1}{4}-\frac{\sqrt{3}}{6} \\ \frac{1}{2}+\frac{\sqrt{3}}{6} & \frac{1}{4}+\frac{\sqrt{3}}{6} & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

2. The method below for integrating $y' = f(y)$ one time step is often seen in practice, perhaps because it is short and simple, and requires a low amount of storage:

```
u0=u;
for j=s:-1:1
   u=u0+h/j*f(u);
end
```

Here, `h`, is the time step, `f` is the function $f(y)$, `u` is the previous solution, and `s` is a given positive integer.

a) Write the method as a Runge-Kutta method in terms of a Butcher array (include the $\boldsymbol{c}$ vector even if the code above assumes the system is autonomous).

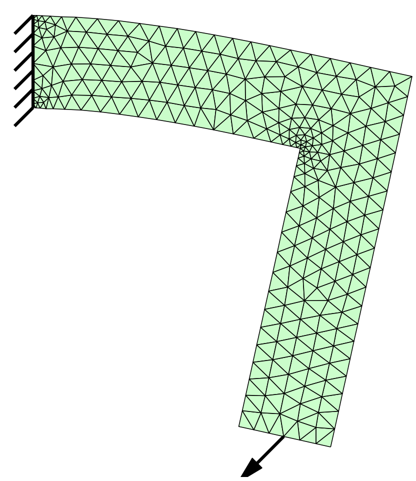
b) Find the order of the method.

c) Find the order of the method for the linear problem $y' = Ay + b$ (Hint: Consider the elementary differentials that the trees represent, most of them will be zero).

3. The file `struct_data.mat` on the course web page contains matrices and arrays that represent a transient linear elasticity problem of the form

$$M\ddot{u} + C\dot{u} + Ku = F, \tag{1}$$

with initial conditions $u(0) = u_0$. The file contains the matrices M, F, the arrays F, u_0, and two additional arrays p, t that only will be used for plotting. The damping matrix is given by $C = \alpha M + \beta K$ for some *Rayleigh damping coefficients* α, β. The function `struct_plot(p,t,u)` plots the solution as a deformed mesh, an example is shown below of the steady-state solution $Ku_{\text{ss}} = F$:



```
load struct_data
uss=K\F;
struct_plot(p,t,uss);
```

Turn page ⟶

3. **a)** Rewrite the system (1) as a first-order system with mass matrix,

$$\tilde{M}\dot{\tilde{u}} = \tilde{K}\tilde{u} + \tilde{F}, \tag{2}$$

where the matrices do not contain M^{-1} (that would result in computationally inefficient schemes).

b) Write down the Hammer-Hollingsworth scheme (see **1b**) for this problem, again making sure that the matrix M^{-1} is not used.

c) Implement a function

```
ytop = p3_3c(alpha,beta,T,h)
```

which integrates the system in time using the scheme in **b)**. The input parameters are the damping coefficients α, β, the final time T, and the time step h. The output `ytop` should be a vector of length $1 + T/h$ containing the y-displacements of the top-right corner for the times $0, h, 2h, \ldots, T$, which is the last component in the u-vector (note that this might be `u(end/2)` or `u(end)` in your $\tilde{u}$-vector, depending on how you ordered the unknowns). A simple test of the function is shown below:

```
ytop=[];
for damp=10.^(-2:0)
   ytop(:,end+1)=p3_3c(damp,damp,10,0.1);
end
plot(0:0.1:10,ytop);
```

d) Implement a function

```
slope = p3_3d(alpha,beta)
```

which computes y_h^{top} using **c)** with the time steps $h = 0.1 \cdot 2^{-i}$, $i = 0, \ldots, 4$, and the final time $T = 1.0$. Consider the solution for the smallest h the exact solution $y_{\text{exact}}^{\text{top}}$, and compute the errors

$$e_h = \max_{i=0,\ldots,T/h} |y_h^{\text{top}}(t_i) - y_{\text{exact}}^{\text{top}}(t_i)|. \tag{3}$$

The function should plot the errors versus the time steps in a log-log plot, and estimate and return the slope. Test the function with the command:

```
slope = p3_3d(1e-2,1e-2)
```

Code Submission: E-mail the MATLAB files `p3_3c.m`, `p3_3d.m`, and any supporting files to David at anderson@math.berkeley.edu as a zip-file named `lastname_firstname_3.zip`, for example `anderson_david_3.zip`.