Discontinuous Galerkin Methods for Conservation Laws

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Math 228B Numerical Solutions of Differential Equations

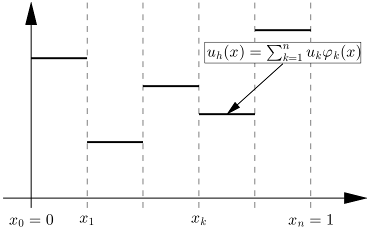
The Finite Volume Method = Galerkin FEM

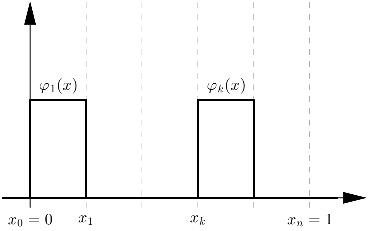
- Consider the 1-D conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$

- Look for solutions in space of piecewise constant functions V_h

$$u_h(x) = \sum_{k=1}^{n} u_k \varphi_k(x), \qquad \varphi_k(x) = \begin{cases} 1 & x_{k-1} < x < x_k \\ 0 & \text{otherwise} \end{cases}$$





The Finite Volume Method = Galerkin FEM

- Galerkin formulation: Find $u_h \in V_h$ such that

$$\int_0^1 \frac{\partial u_h}{\partial t} v\,dx + \int_0^1 \frac{\partial f(u_h)}{\partial x} v\,dx = 0, \quad \forall v \in V_h$$

- Set $v = \varphi_k = \begin{cases} 1 & x \in [x_{k-1}, x_k] \\ 0 & \text{otherwise} \end{cases}$

$$\int_{x_{k-1}}^{x_k} \frac{\partial u_h}{\partial t}\,dx + \int_{x_{k-1}}^{x_k} \frac{\partial f(u_h)}{\partial x}\,dx = 0 \Longleftrightarrow \int_{x_{k-1}}^{x_k} \frac{\partial u_h}{\partial t}\,dx + [f(u_h(x))]_{x_{k-1}}^{x_k} = 0$$

- Since u_h is discontinuous at x_k and x_{k-1}, use a numerical flux function $F(u_R, u_L)$ to obtain:

$$h\frac{\partial u_k}{\partial t} + F(u_{k+1}, u_k) - F(u_k, u_{k-1}) = 0$$

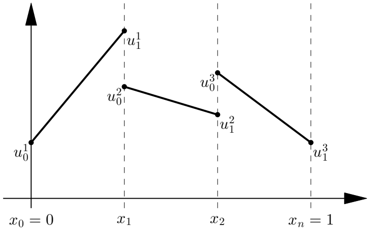
- This is a standard finite volume method on a uniform grid

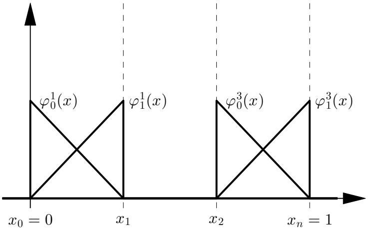
The Discontinuous Galerkin Method

- Generalize the Galerkin FEM approach to the space of piecewise polynomials of degree p
- Nodal representation with values u_i^k for local node i in element k:

$$u_h(x) = \sum_{k=1}^{n} \sum_{i=0}^{p} u_i^k \varphi_i^k(x)$$

- Example, piecewise linear functions ($p = 1$):





The Discontinuous Galerkin Method

- Galerkin formulation: Find $u_h \in V_h$ such that

$$\int_0^1 \frac{\partial u_h}{\partial t} v \, dx + \int_0^1 \frac{\partial f(u_h)}{\partial x} v \, dx = 0, \quad \forall v \in V_h$$

- Set $v = \varphi_i^k$ and integrate by parts

$$\int_{x_{k-1}}^{x_k} \frac{\partial u_h}{\partial t} \varphi_i^k \, dx + \Big[f(u_h(x)) \varphi_i^k(x) \Big]_{x_{k-1}}^{x_k} - \int_{x_{k-1}}^{x_k} f(u_h(x)) \frac{d\varphi_i^k}{dx} \, dx = 0$$

- Use a numerical flux function $F(u_R, u_L)$ at the discontinuities

$$\int_{x_{k-1}}^{x_k} \frac{\partial u_h}{\partial t} \varphi_i^k \, dx + F(u_0^{k+1}, u_p^k) \varphi_i^k(x_{k+1}) - F(u_0^k, u_p^{k-1}) \varphi_i^k(x_k)$$
$$- \int_{x_{k-1}}^{x_k} f(u_h(x)) \frac{d\varphi_i^k}{dx} \, dx = 0$$

The Discontinuous Galerkin Method

- Example: $f(u) = u$, $F(u_R, u_L) = u_L$

$$\int_{x_{k-1}}^{x_k} \frac{\partial}{\partial t}\left(\sum_{j=0}^{p} u_j^k \varphi_j^k(x)\right)\varphi_i^k\,dx - \int_{x_{k-1}}^{x_k}\left(\sum_{j=0}^{p} u_j^k \varphi_j^k(x)\right)\frac{d\varphi_i^k}{dx}\,dx$$
$$+\, u_p^k \varphi_i^k(x_k) - u_p^{k-1}\varphi_i^k(x_{k-1}) = 0$$

- Rearrange to obtain a linear system of equations

$$M^k \dot{\boldsymbol{u}}^k - C^k \boldsymbol{u}^k + \begin{pmatrix} -u_p^{k-1} \\ 0 \\ \vdots \\ 0 \\ u_p^k \end{pmatrix} = 0$$

for element k, with elementary matrices

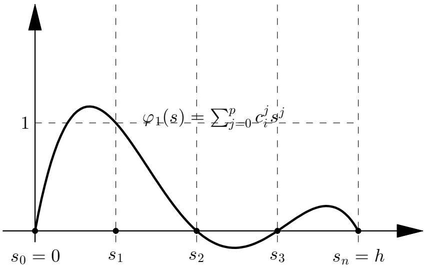
$$M_{ij}^k = \int_{x_{k-1}}^{x_k} \varphi_i^k \varphi_j^k\,dx \text{ and } C_{ij}^k = \int_{x_{k-1}}^{x_k} \frac{d\varphi_i^k}{dx}\varphi_j^k\,dx$$

Calculating Elementary Matrices

- Consider an element of degree p, width h, and a nodal basis at the points $s_i = h_i/p$, $i = 0, \ldots, p$
- Write basis functions in monomial form $\varphi_i(s) = \sum_{j=0}^{p} c_i^j s^j$
- Nodal basis functions are defined by

$$\varphi_i(s_k) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

- Produces a linear system of equations



Calculating Elementary Matrices

- The linear system of equations has the form

$$\begin{pmatrix} 1 & s_0 & s_0^2 & \cdots & s_0^p \\ 1 & s_1 & s_1^2 & \cdots & s_1^p \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & s_p & s_p^2 & \cdots & s_p^p \end{pmatrix} \begin{pmatrix} c_0^0 & c_1^0 & \cdots & c_p^0 \\ c_0^1 & c_1^1 & \cdots & c_p^1 \\ \vdots & \vdots & \ddots & \vdots \\ c_0^p & c_1^p & \cdots & c_p^p \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

  or $VC = I$, which gives the coefficient matrix $C = V^{-1}$
- Use Gaussian quadrature or explicit polynomial integration to compute the elementary matrices

$$M_{ij} = \int_0^h \varphi_i(s)\varphi_j(s)\,ds$$
$$C_{ij} = \int_0^h \varphi_i'(s)\varphi_j(s)\,ds$$

The DG method – General systems of conservation laws

- (Reed/Hill 1973, Lesaint/Raviart 1974, Cockburn/Shu 1989-)
- Consider a first-order system of conservation laws:

$$\boldsymbol{u}_t + \nabla \cdot \boldsymbol{F}(\boldsymbol{u}) = 0$$

- Triangulate domain Ω into elements $\kappa \in \mathcal{T}_h$
- Seek approximate solution $\boldsymbol{u}_h$ in space of element-wise polynomials:

$$\boldsymbol{V}_h^p = \{\boldsymbol{v} \in L^2(\Omega) : \boldsymbol{v}|_\kappa \in P^p(\kappa) \ \forall \kappa \in T_h\}$$

- Multiply by test function $\boldsymbol{v}_h \in \boldsymbol{V}_h^p$, integrate over element κ:

$$\int_\kappa \left[(\boldsymbol{u}_h)_t + \nabla \cdot \boldsymbol{F}(\boldsymbol{u}_h) \right] \boldsymbol{v}_h \, d\boldsymbol{x} = 0$$

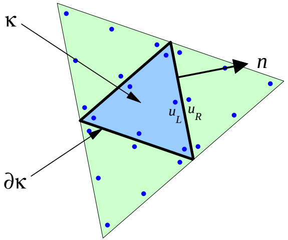
The DG method – General systems of conservation laws

- Integrate by parts:

$$\int_\kappa [(\boldsymbol{u}_h)_t]\, \boldsymbol{v}_h \, d\boldsymbol{x} - \int_\kappa \boldsymbol{F}(\boldsymbol{u}_h) \nabla \boldsymbol{v}_h \, d\boldsymbol{x} + \int_{\partial\kappa} \hat{\boldsymbol{F}}(\boldsymbol{u}_h^+, \boldsymbol{u}_h^-, \hat{\boldsymbol{n}}) \boldsymbol{v}_h^+ \, ds = 0$$

  with numerical flux function $\hat{\boldsymbol{F}}(\boldsymbol{u}_L, \boldsymbol{u}_R, \hat{\boldsymbol{n}})$ for left/right states $\boldsymbol{u}_L$,$\boldsymbol{u}_R$ in direction $\hat{\boldsymbol{n}}$ (Godunov, Roe, Osher, Van Leer, Lax-Friedrichs, etc)

- Global problem: Find $\boldsymbol{u}_h \in \boldsymbol{V}_h^p$ such that this weighted residual is zero for all $\boldsymbol{v}_h \in \boldsymbol{V}_h^p$
- Error $= \mathcal{O}(h^{p+1})$ for smooth solutions



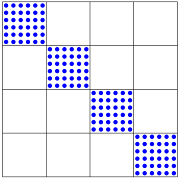
The DG Method – Observations

- Reduces to the finite volume method for $p = 0$:

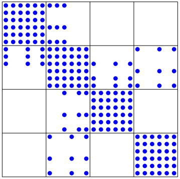
$$(\boldsymbol{u}_h)_t A_\kappa + \int_{\partial\kappa} \hat{\boldsymbol{F}}(\boldsymbol{u}_h^+, \boldsymbol{u}_h^-, \hat{\boldsymbol{n}})\, ds = 0$$

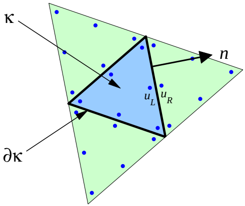
- Boundary conditions enforced naturally for any degree p
- Block-diagonal mass matrix (no overlap between basis functions)
- Block-wise compact stencil – neighboring elements connected

Mass Matrix



Jacobian





Convection-Diffusion, the LDG method

- Consider the convection-diffusion equation

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} - \mu \frac{\partial^2 u}{\partial x^2} = 0$$

- Split into system of first order equations:

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} - \mu \frac{\partial \sigma}{\partial x} &= 0 \\ \frac{\partial u}{\partial x} &= \sigma \end{aligned}$$

- Galerkin formulation: Find $u_h, \sigma_h \in V_h$ such that

$$\int_0^1 \frac{\partial u_h}{\partial t} v \, dx + \int_0^1 \left(\frac{\partial f(u_h)}{\partial x} - \mu \frac{\partial \sigma_h}{\partial x} \right) v \, dx = 0, \quad \forall v \in V_h$$

$$\int_0^1 \frac{\partial u_h}{\partial x} \tau \, dx = \int_0^1 \sigma_h \tau \, dx, \quad \forall \tau \in V_h$$

Convection-Diffusion, the LDG method

- Set $v, \tau = \varphi_i^k$ and integrate by parts

$$\int_{x_{k-1}}^{x_k} \frac{\partial u_h}{\partial t} \varphi_i^k \, dx + \Big[\left(f(u_h(x)) - \mu \sigma_h(x) \right) \varphi_i^k(x) \Big]_{x_{k-1}}^{x_k} - \int_{x_{k-1}}^{x_k} \left(f(u_h(x)) - \mu \sigma_h(x) \right) \frac{d\varphi_i^k}{dx} \, dx = 0, \qquad \forall i, k$$

$$\Big[u_h(x) \varphi_i^k(x) \Big]_{x_{k-1}}^{x_k} - \int_{x_{k-1}}^{x_k} u_h(x) \frac{d\varphi_i^k}{dx} \, dx = \int_{x_{k-1}}^{x_k} \sigma_h(x) \varphi_i^k \, dx, \qquad \forall i, k$$

- Use numerical flux functions $\hat{f}(u_R, u_L)$, $\hat{\sigma}(\sigma_R, \sigma_L)$, $\hat{u}(u_R, u_L)$ at the discontinuities
- Example: $f(u) = u$, $\hat{f}(u_R, u_L) = u_L$, $\hat{\sigma}(\sigma_R, \sigma_L) = \sigma_L$, $\hat{u}(u_R, u_L) = u_R$ (upwinding for the convection, LDG upwinding/downwinding for the diffusion)

Convection-Diffusion, the LDG method

- After discretization, this leads to the ODEs

$$M^k \dot{\boldsymbol{u}}^k - C^k \left(\boldsymbol{u}^k - \mu \boldsymbol{\sigma}^k \right) + \begin{pmatrix} -u_p^{k-1} + \mu \sigma_p^{k-1} \\ 0 \\ \vdots \\ 0 \\ u_p^k - \mu \sigma_p^k \end{pmatrix} = 0$$

$$M^k \boldsymbol{\sigma}^k = -C^k \boldsymbol{u}^k + \begin{pmatrix} -u_0^k \\ 0 \\ \vdots \\ 0 \\ u_0^{k+1} \end{pmatrix}$$

- For each element k, first solve for $\boldsymbol{\sigma}^k$, then insert into main equation as before