1 – DG Methods for Diffusion Problems

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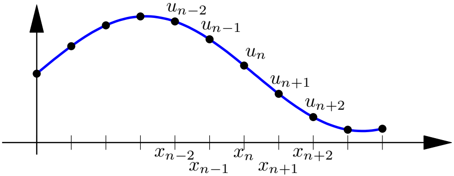
The Finite Difference Method (FDM)

- Consider linear convection: $u_t + u_x = 0$ for $x \in [0, 1]$, $u(0) = u(1)$
- Approximate u_x point-wise using difference formulas:

$$\frac{d}{dx}u(x_n) \approx \frac{u_{n+1} - u_{n-1}}{2\Delta x}$$



or high-order:

$$\frac{d}{dx}u(x_n) \approx \frac{u_{n+2} - 8u_{n+1} + 8u_{n-1} - u_{n-1}}{12\Delta x}$$

or one-sided (e.g. for stability, “upwinding”):

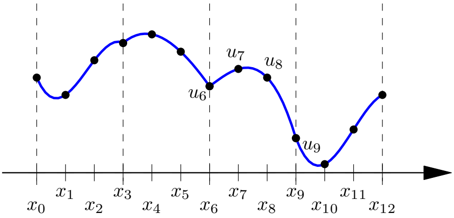
$$\frac{d}{dx}u(x_n) \approx \frac{3u_n - 16u_{n-1} + 36u_{n-2} - 48u_{n-3} + 25u_{n-4}}{25\Delta x}$$

- Simple, efficient, flexible
- Needs *structured neighborhood* of nodes – hard to generalize to unstructured grids in 2-D and 3-D

The Finite Element Method (FEM)

- Discretize domain into *elements* (intervals)
- Seek approximate solution in space of piecewise polynomials $\hat{X}$
- Impose equation weakly: Seek $\hat{u} \in \hat{X}$ such that for all $v \in \hat{X}$:

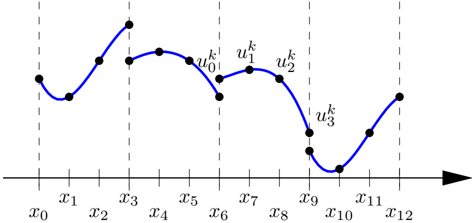
$$\begin{aligned} &\int_0^1 (\hat{u}_t + \hat{u}_x) v \, dx \\ &= \int_0^1 \hat{u}_t v \, dx + \int_0^1 \hat{u}_x v \, dx \\ &= \int_0^1 \hat{u}_t v \, dx - \int_0^1 \hat{u} v_x \, dx = 0 \end{aligned}$$

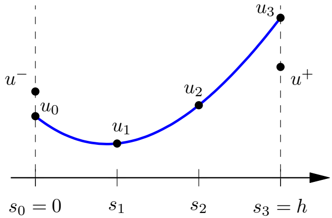


- Leads to semi-discrete system $M\boldsymbol{u}_t + K\boldsymbol{u} = 0$, with element-wise local M, K matrices
- M^{-1} dense $\Longrightarrow$ Explicit methods for $\boldsymbol{u}_t = -M^{-1}K\boldsymbol{u}$ not practical
- Also, unclear how to stabilize by upwinding (but other techniques exist, such as Streamline Upwind Petrov-Galerkin)

The Discontinuous Galerkin Method

- Do not enforce continuity – allow “jumps” between elements





- Galerkin formulation for single element $\kappa = [0, h]$: For all $v \in P^p(\kappa)$,

$$\int_0^h (\hat{u}_t + \hat{u}_x) v \, dx = \int_0^h \hat{u}_t v \, dx + \int_0^h \hat{u}_x v \, dx$$

$$= \int_0^h \hat{u}_t v \, dx - \int_0^h \hat{u} v_x \, dx + \mathcal{U}(u^+, u_p) v(h) - \mathcal{U}(u_0, u^-) v(0)$$

- *Numerical flux function* $\mathcal{U}(u_R, u_L)$ allows for stabilization by high-order upwinding, e.g. $\mathcal{U}(u_R, u_L) = u_L$

The Discontinuous Galerkin Method

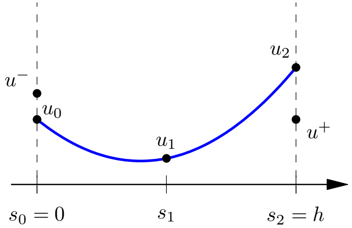
- The DG formulation leads to linear system of equations:

$$\boldsymbol{M}\boldsymbol{u}_t + \boldsymbol{K}\boldsymbol{u} + \begin{pmatrix} -u^- & 0 & \dots & 0 & u_p \end{pmatrix}^T = 0$$

- For example, with $p = 2$:

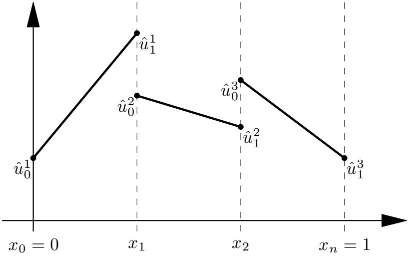
$$\begin{aligned} \boldsymbol{u}_t &= -\boldsymbol{M}^{-1}\boldsymbol{K}\boldsymbol{u} - \boldsymbol{M}^{-1}\begin{pmatrix} -u^- & 0 & u_2 \end{pmatrix}^T \\ &= \frac{1}{h}\begin{pmatrix} -6 & -4 & 1 \\ 2.5 & 0 & -1 \\ -4 & 4 & -3 \end{pmatrix}\begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix} + \frac{1}{h}\begin{pmatrix} 9 \\ -1.5 \\ 3 \end{pmatrix} u^- \end{aligned}$$

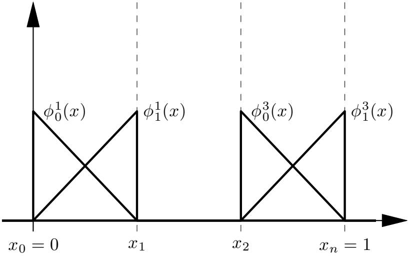
- Element-wise local FD-type stencil
- Stabilized, “upwinded” through u^-
- Extends naturally to other PDEs, N-D, unstructured meshes



The DG Scheme – Details of Discretization

- Consider the 1-D conservation law
$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$
- Seek a solution in space of piecewise polynomial functions X_h
- Nodal representation with values u_i^k for local node i in element k:
$$u_h(x) = \sum_{k=1}^{n} \sum_{i=0}^{p} u_i^k \phi_i^k(x)$$
- Example, piecewise linear functions ($p = 1$):





The DG Scheme – Details of Discretization

- Galerkin formulation: Find $u_h \in X_h$ such that

$$\int_0^1 \frac{\partial u_h}{\partial t} v\, dx + \int_0^1 \frac{\partial f(u_h)}{\partial x} v\, dx = 0$$

- Set $v = \phi_i^k$ and integrate by parts

$$\int_{x_{k-1}}^{x_k} \frac{\partial u_h}{\partial t} \phi_i^k\, dx + \left[f(u_h(x))\phi_i^k(x) \right]_{x_{k-1}}^{x_k} - \int_{x_{k-1}}^{x_k} f(u_h) \frac{d\phi_i^k}{dx}\, dx = 0$$

- Use a numerical flux function $F(u_R, u_L)$ at the discontinuities

$$\int_{x_{k-1}}^{x_k} \frac{\partial u_h}{\partial t} \phi_i^k\, dx + F(u_0^{k+1}, u_p^k)\phi_i^k(x_k) - F(u_0^k, u_p^{k-1})\phi_i^k(x_k)$$
$$- \int_{x_{k-1}}^{x_k} f(u_h) \frac{d\phi_i^k}{dx}\, dx = 0$$

The DG Scheme – Details of Discretization

- Example: $f(u) = u$, $F(u_R, u_L) = u_L$

$$\int_{x_{k-1}}^{x_k} \frac{\partial}{\partial t} \left(\sum_{k=1}^{n} \sum_{j=0}^{p} u_j^k \phi_j^k(x) \right) \phi_i^k \, dx - \int_{x_{k-1}}^{x_k} \left(\sum_{k=1}^{n} \sum_{j=0}^{p} u_j^k \phi_j^k(x) \right) \frac{d\phi_i^k}{dx} \, dx$$
$$+ u_p^k \phi_i^k(x_k) - u_p^{k-1} \phi_i^k(x_{k-1}) = 0$$

- Rearrange to obtain a linear system of equations

$$M^k \dot{u}^k - C^k u^k + \begin{bmatrix} -u_p^{k-1} & 0 & \cdots & 0 & u_0^k \end{bmatrix}^T = 0$$

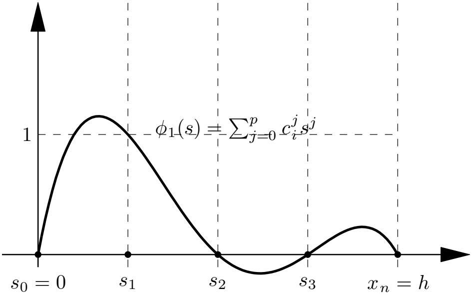
for element k, with elementary matrices

$$M_{ij}^k = \int_{x_{k-1}}^{x_k} \phi_i^k \phi_j^k \, dx \text{ and } C_{ij}^k = \int_{x_{k-1}}^{x_k} \frac{d\phi_i^k}{dx} \phi_j^k \, dx$$

Calculating Elementary Matrices

- Consider an element of degree p, width h, and a nodal basis at the points $s_i = h_i/p$, $i = 0, \ldots, p$
  - For p high (> 4), use Gauss-Lobatto points instead
- Write basis functions in monomial form $\phi_i(s) = \sum_{j=0}^{p} c_i^j s^j$
  - For p high (> 4), use an orthogonal basis instead
- Nodal basis functions are defined by

$$\phi_i(s_k) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$



- Produces a linear system of equations

Calculating Elementary Matrices

- The linear system of equations has the form

$$\begin{pmatrix} 1 & s_0 & s_0^2 & \cdots & s_0^p \\ 1 & s_1 & s_1^2 & \cdots & s_1^p \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & s_p & s_p^2 & \cdots & s_p^p \end{pmatrix} \begin{pmatrix} c_0^0 & c_1^0 & \cdots & c_p^0 \\ c_0^1 & c_1^1 & \cdots & c_p^1 \\ \vdots & \vdots & \ddots & \vdots \\ c_0^p & c_1^p & \cdots & c_p^p \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

  or $VC = I$, which gives the coefficient matrix $C = V^{-1}$
- Use Gaussian quadrature or explicit polynomial integration to compute the elementary matrices

$$M_{ij} = \int_0^h \phi_i(s)\phi_j(s)\,ds$$

$$C_{ij} = \int_0^h \phi_i'(s)\phi_j(s)\,ds$$

DG for Elliptic Problems – Historical Overview

- Enforcing Dirichlet conditions by penalties
  - Lions (1968), Babuška (1973) – Penalty term
  - Nitsche (1971) – additional terms in bilinear form for consistency
- Interior Penalty (IP) methods
  - Babuška and Zlámal (1973) – enforce C^1-continuity by penalties
  - Wheeler (1978), Arnold (1979) – Nitsche's method for spaces of discontinuous piecewise polynomials
- DG methods
  - Bassi and Rebay (1997) – apply RKDG to unknown and its gradient
  - Cockburn and Shu (1998) – generalized the ideas, the LDG method
- Unification
  - Arnold, Brezzi, Cockburn, Marini (2000,2002) – showed that most methods fit in a unified framework by choosing appropriate numerical fluxes

Second-order Equations

- Consider the 1-D Poisson equation

$$-\frac{d^2u}{dx^2} = f(x) \quad \text{in } [0,1]$$

with homogeneous Dirichlet conditions $u(0) = u(1) = 0$

- Standard Continuous Galerkin FEM would consider the space $X_{h,0}$ of continuous piecewise polynomials satisfying the Dirichlet conditions, and solve for $u_h \in X_{h,0}$ s.t.

$$\int_0^1 -\frac{d^2u_h}{dx^2} v\, dx = \int_0^1 \frac{du_h}{dx}\frac{dv}{dx}\, dx - \left[\frac{du_h}{dx} v\right]_0^1 = \int_0^1 \frac{du_h}{dx}\frac{dv}{dx}\, dx = \int_0^1 f\, v\, dx$$

for all $v \in X_{h,0}$

- With discontinuous functions, appropriate numerical fluxes must be chosen at all element boundaries

The 1-D Poisson Equation

- To define a DG discretization, first split into first order system:

$$-\sigma' = f(x), \quad u' = \sigma$$

- Multiply by test functions v, τ, integrate over an element, and integrate by parts to obtain the weak form

$$\int_{x_k}^{x_{k+1}} f(x)\, v\, dx = \int_{x_k}^{x_{k+1}} -\sigma' v\, dx = \int_{x_k}^{x_{k+1}} \sigma v'\, dx - [\hat{\sigma} v]_0^1$$

$$\int_{x_k}^{x_{k+1}} \sigma\, \tau\, dx = \int_{x_k}^{x_{k+1}} u'\, \tau\, dx = -\int_{x_k}^{x_{k+1}} u\, \tau'\, dx + [\hat{u}\tau]_0^1$$

- Galerkin formulation: Find $u_h, \sigma_h \in X_h$ s.t. for all elements k

$$\int_{x_k}^{x_{k+1}} \sigma_h v'\, dx = \int_{x_k}^{x_{k+1}} f(x)\, v\, dx + [\hat{\sigma}(u_h, \sigma_h) v]_0^1\,, \qquad \forall v \in X_h$$

$$\int_{x_k}^{x_{k+1}} \sigma_h\, \tau\, dx = -\int_{x_k}^{x_{k+1}} u_h\, \tau'\, dx + [\hat{u}(u_h)\tau]_0^1\,, \qquad \forall \tau \in X_h$$

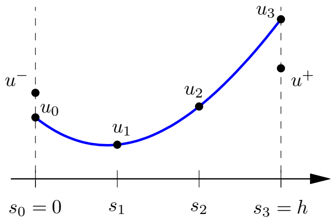
- Remains only to define the *numerical fluxes* $\hat{u}(u_h), \hat{\sigma}(u_h, \sigma_h)$

The BR1 Fluxes

- The BR1 fluxes:

$$\hat{u} = \{u_h\}, \qquad \hat{\sigma} = \{\sigma_h\}$$

  where $\{\cdot\}$ is the *averaging* operator



- For example, with notation according to the figure:

$$\hat{u}(0) = (u^- + u_0)/2 \text{ and } \hat{u}(h) = (u_3 + u^+)/2$$

$$\hat{\sigma}(0) = (\sigma^- + \sigma_0)/2 \text{ and } \hat{\sigma}(h) = (\sigma_3 + \sigma^+)/2$$

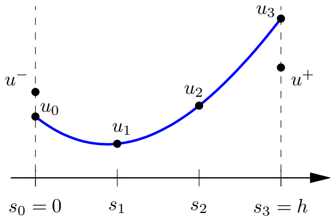
- Simple, intuitive (no preference to direction in equation)
- However, unstable and non-compact stencil

Interior Penalty (IP)

- In the *interior penalty* method, we set

$$\hat{u} = \{u_h\}$$
$$\hat{\sigma} = \{\nabla u_h\} + C_{11}[\![u_h]\!]$$

for some $C_{11} > 0$, where $\{\cdot\}$ is the *averaging* operator and $[\![\cdot]\!]$ is the *jump* operator



- For example, with notation according to the figure:

$$\hat{u}(0) = (u^- + u_0)/2 \text{ and } \hat{u}(h) = (u_3 + u^+)/2$$
$$\hat{\sigma}(0) = (u_h'|_{x=0^-} + u_h'|_{x=0^+})/2 + C_{11}(u^- - u_0)$$
$$\hat{\sigma}(h) = (u_h'|_{x=h^-} + u_h'|_{x=h^+})/2 + C_{11}(u_3 - u^+)$$

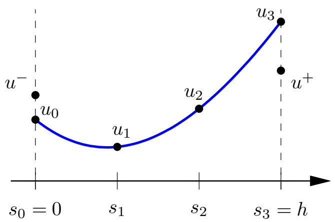
- Convergent with optimal order of accuracy
- However, C_{11} is problem dependent, introduces stiffness

The Local Discontinuous Galerkin (LDG) Method

- In the *LDG* method, we set

$$\hat{u} = \{u_h\} + C_{12}[\![u_h]\!]$$

$$\hat{\sigma} = \{\sigma_h\} + C_{11}[\![u_h]\!] - C_{12}[\![\sigma_h]\!]$$



- For the special cases $C_{11} = 0$ (minimal dissipation LDG) and $C_{12} = 1/2$ we get a simple upwind/downwind structure
- For example, with notation according to the figure:

$$\hat{u}(0) = u_0 \text{ and } \hat{u}(h) = u^+$$

$$\hat{\sigma}(0) = \sigma^- \text{ and } \hat{\sigma}(h) = \sigma_3$$

- Simple and general
- Convergent with optimal order of accuracy
- However, in general a non-compact stencil in higher dimensions

Higher Space Dimensions

- From Arnold, Brezzi, Cockburn, Marini (2002)
- Model problem:

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

- Rewrite as first-order system

$$\sigma = \nabla u, \quad -\nabla \cdot \sigma = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

- Multiply by test functions τ, v, integrate over element K, integrate by parts $\Rightarrow$ weak formulation:

$$\int_K \sigma \cdot \tau \, dx = -\int_K u \nabla \cdot \tau \, dx + \int_{\partial K} u \, n_K \cdot \tau \, ds$$

$$\int_K \sigma \cdot \nabla v \, dx = \int_K f \, v \, dx + \int_{\partial K} \sigma \cdot n_K \, v \, ds$$

Higher Space Dimensions

- Introduce finite element spaces for triangulation $\mathcal{T}_h = \{K\}$:

$$V_h := \{v \in L^2(\Omega) : v|_K \in \mathcal{P}_p(K) \quad \forall K \in \mathcal{T}_h\ \}$$

$$\Sigma_h := \{\tau \in [L^2(\Omega)]^2 : \tau|_K \in [\mathcal{P}_p(K)]^2 \quad \forall K \in \mathcal{T}_h\ \}$$

- The flux formulation: Find $u_h \in V_h$ and $\sigma_h \in \Sigma_h$ s.t.

$$\int_K \sigma_h \cdot \tau\, dx = -\int_K u_h \nabla \cdot \tau\, dx + \int_{\partial K} \hat{u}_K\, n_K \cdot \tau\, ds, \quad \forall \tau \in [\mathcal{P}_p(K)]^2$$

$$\int_K \sigma_h \cdot \nabla v\, dx = \int_K f\, v\, dx + \int_{\partial K} \hat{\sigma}_K \cdot n_K\, v\, ds, \qquad \forall v \in \mathcal{P}_p(K)$$

  for all elements $K \in \mathcal{T}_h$

- Need to define the *numerical fluxes* $\hat{u}_K$ and $\hat{\sigma}_K$

Higher Space Dimensions

- Denote the union of the element edges Γ, the interior edges $\Gamma^0 := \Gamma \backslash \partial\Omega$, and the trace space $T(\Gamma) := \prod_{K \in \mathcal{T}_h} L^2(\partial K)$
- For an interior edge e, with unit normal vectors n_1, n_2 define the *jump* and *average* of $q \in T(\Gamma)$ by

$$\{q\} = \frac{1}{2}(q_1 + q_2), \quad [\![q]\!] = q_1 n_1 + q_2 n_2$$

  and for $\sigma \in [T(\Gamma)]^2$ by

$$\{\sigma\} = \frac{1}{2}(q_1 + q_2), \quad [\![\sigma]\!] = \sigma_1 \cdot n_1 + \sigma_2 \cdot n_2$$

- For boundary edges, set

$$[\![q]\!] = qn, \quad \{\sigma\} = \sigma$$

- Note: The jump of a scalar is vector valued (in the normal direction), the jump of a vector is scalar

The Primal Formulation

- Summing over all K, the flux formulation can be written

$$\int_\Omega \sigma_h \cdot \tau \, dx = -\int_\Omega u_h \nabla_h \cdot \tau \, dx + \int_\Gamma [\![\hat{u}]\!] \cdot \{\tau\} \, ds + \int_{\Gamma^0} \{\hat{u}\}[\![\tau]\!] \, ds$$

$$\int_\Omega \sigma_h \cdot \nabla_h v \, dx - \int_\Gamma \{\hat{\sigma}\} \cdot [\![v]\!] \, ds - \int_{\Gamma^0} [\![\hat{\sigma}]\!]\{v\} = \int_\Omega f \, v \, dx$$

- With some manipulations, σ_h can be expressed as

$$\sigma_h = \sigma_h(u_h) := \nabla_h u_h - r([\![\hat{u}(u_h) - u_h]\!]) - l(\{\hat{u}(u_h) - u_h\})$$

where r, l are *lifting operators* defined by

$$\int_\Omega r(\phi) \cdot \tau \, dx = -\int_\Gamma \phi \cdot \{\tau\} \, ds, \quad \int_\Omega l(q) \cdot \tau \, dx = -\int_{\Gamma^0} q[\![\tau]\!] \, ds \quad \forall \tau \in \Sigma_h$$

The Primal Formulation

- This leads to the *primal formulation*

$$B_h(u_h, v) = \int_\Omega f\, v\, dx \qquad \forall v \in V_h$$

with the *primal form*

$$B_h(u_h, v) = \int_\Omega \nabla_h u_h \cdot \nabla_h v\, dx + \int_\Gamma ([\![\hat{u} - u_h]\!] \cdot \{\nabla_h v\} - \{\hat{\sigma}\} \cdot [\![v]\!])\, ds$$
$$+ \int_{\Gamma^0} (\{\hat{u} - u_h\}[\![\nabla_h v]\!] - [\![\hat{\sigma}]\!]\{v\})\, ds$$

- Standard FEM formulation without σ_h
- In implementations it is often easier to work directly with the flux formulation

Consistency and Conservation

- The numerical fluxes are *consistent* if for smooth functions v

$$\hat{u}(v) = v|_\Gamma, \qquad \hat{\sigma}(v, \nabla v) = \nabla v|_\Gamma$$

  $\Rightarrow$ consistency of the primal formulation and Galerkin orthogonality $B_h(u - u_h, v) = 0$, $\forall v \in V_h$

- The numerical fluxes are *conservative* if $\hat{u}(\cdot)$ and $\hat{\sigma}(\cdot, \cdot)$ are single-valued on Γ

  $\Rightarrow$ adjoint consistency of the primal form

Some DG Methods

- Some of the most important schemes are summarized below:

Method	$\hat{u}_K$	$\hat{\sigma}_K$	Stable
Bassi-Rebay (BR1)	$\{u_h\}$	$\{\sigma_h\}$	$\times$
Bassi-Rebay (BR2)	$\{u_h\}$	$\{\nabla_h u_h\} - \alpha_r([\![u_h]\!])$	$\inf_e \eta_e > 3$
Interior Penalty	$\{u_h\}$	$\{\nabla_h u_h\} + C_{11}[\![u_h]\!]$	$C_{11} > C_{11}^*$
LDG	$\{u_h\} + \boldsymbol{C}_{12} \cdot [\![u_h]\!]$	$\{\sigma_h\} + C_{11}[\![u_h]\!] - \boldsymbol{C}_{12}[\![\sigma_h]\!]$	$C_{11} > 0$

- $\alpha_r(\phi) = -\eta_e\{r_e(\phi)\}$ on an edge e, where r_e is defined by

$$\int_\Omega r_e(\varphi) \cdot \tau \, dx = -\int_e \varphi \cdot \{\tau\} \, ds, \qquad \forall \tau \in \Sigma_h, \varphi \in [L^1(e)]^2$$

- C_{11}^* is mesh dependent, explicit form derived by Shahbazi (2005)
- The methods BR2, IP, and LDG are all commonly used

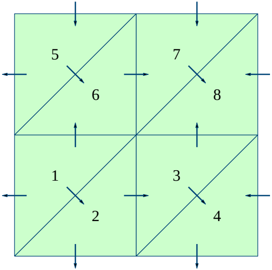
The LDG Method

- In the LDG method, we use the fluxes

$$\hat{\sigma}_K = \{\sigma_h\} + C_{11}[\![u_h]\!] - \boldsymbol{C}_{12}[\![\sigma_h]\!]$$

$$\hat{u}_K = \{u_h\} + \boldsymbol{C}_{12} \cdot [\![u_h]\!]$$

- Here, $C_{11} > 0$ (or zero for the *minimal dissipation LDG method*, Cockburn and Dong 2007)



- An important special case for $\boldsymbol{C}_{12}$ is the choice

$$\boldsymbol{C}_{12} = \frac{1}{2}(S_{K^+}^{K^-1} n^+ + S_{K^-}^{K^+} n^-)$$

where $S_{K^+}^{K^-} \in \{0, 1\}$ is a *switch* for the edge shared by K^- and K^+

- This leads to a simple upwind/downwind scheme:

$$\hat{\sigma}_K = C_{11}[\![u_h]\!] + \begin{cases} \sigma_h^+ & \text{if } S_{K^+}^{K^-} = 0 \\ \sigma_h^- & \text{if } S_{K^+}^{K^-} = 1 \end{cases}, \qquad \hat{u}_K = \begin{cases} u_h^- & \text{if } S_{K^+}^{K^-} = 0 \\ u_h^+ & \text{if } S_{K^+}^{K^-} = 1 \end{cases}$$

LDG Switch Functions

- *Natural switch*: Order the elements, let N_K be the index of element K, and set

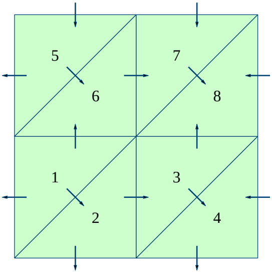
$$S_{K^+}^{K^-} = 1 \text{ if } N_{K^+} > N_{K^-}, \quad 0 \text{ otherwise.}$$

Simple, leads to beneficial matrix structure, but unstable if $C_{11} = 0$ in the original LDG method

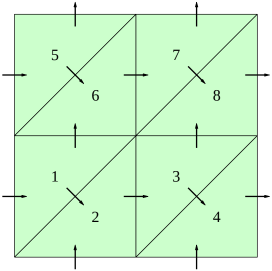
- *Consistent switch*: For example define any constant vector β and set

$$S_{K^+}^{K^-} = 1 \text{ if } n^+ \cdot \beta > 0, \quad 0 \text{ otherwise.}$$

- In general, any choice of switch leads to a *non-compact stencil*



Natural switch



Consistent switch

The Compact DG (CDG) Method

- To address the non-compactness of the LDG method and its sensitivity to the switch, Peraire and Persson developed the *Compact DG* method (2008)
- Recall the original LDG fluxes:

$$\hat{\sigma}_K = \{\sigma_h\} + C_{11}[\![u_h]\!] - \boldsymbol{C}_{12}[\![\sigma_h]\!]$$
$$\hat{u}_K = \{u_h\} + \boldsymbol{C}_{12} \cdot [\![u_h]\!]$$

- Now, introduce the *edge fluxes* σ_h^e on edge e by

$$\int_K \sigma_h^e \cdot \tau \, dx = -\int_K u_h \nabla \cdot \tau \, dx + \int_{\partial K} \hat{u}_K^e n_K \cdot \tau \, ds, \quad \forall \tau \in [\mathcal{P}_p(K)]^2$$

where

$$\hat{u}_K^e = \begin{cases} \hat{u}_K & \text{on edge } e\text{, as defined above} \\ u_h & \text{otherwise} \end{cases}$$

The Compact DG (CDG) Method

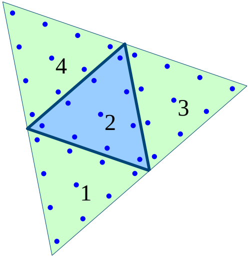
- The numerical fluxes for CDG are then simply given by

$$\hat{\sigma}_K^e = \{\sigma_h^e\} + C_{11}[\![u_h]\!] - \boldsymbol{C}_{12}[\![\sigma_h^e]\!]$$

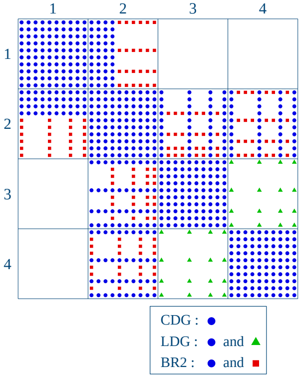
  on edge e

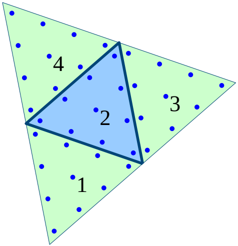
- The modification eliminates the non-compact terms in the primal form, while retaining all the good properties of the LDG scheme

- In addition, better stability properties are observed with in particular less sensitivity to the choice of switch function

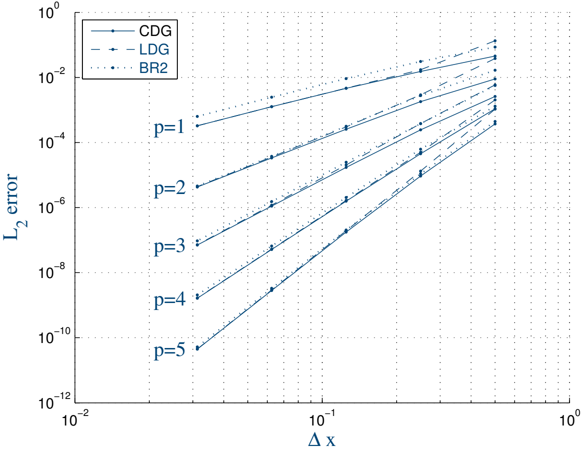


The CDG Method – Summary



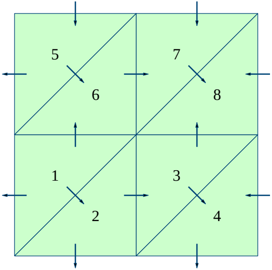


- Element-wise compact stencil
- Less connectivities than LDG/BR2/IP
- More accurate than LDG and BR2



Switches and Null-space Dimensions

- Unlike the LDG scheme, the CDG scheme appears to be stable for $C_{11} = 0$ and an *inconsistent switch* such as highest element number
- Simple test [Sherwin et al 05]: Poisson problem, periodic boundary conditions, expected nullspace dimension = 1

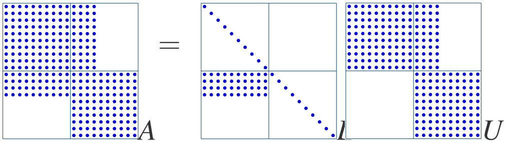


Nullspace dimension

Polynomial order p		1	2	3	4	5	6	7
Consistent switch	CDG	1	1	1	1	1	1	1
	LDG	1	1	1	1	1	1	1
Natural switch	CDG	1	1	1	1	1	1	1
	LDG	3	4	5	6	7	8	9

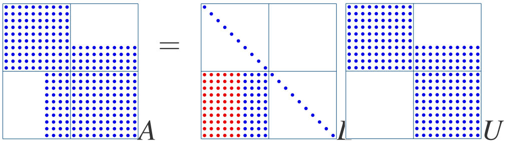
ILU and Switch Orientation

- Orientation of lower-triangular blocks important for ILU sparsity
- Take advantage of CDG's insensitivity to orientation



Switch 1:

Same LU storage



Switch 2:

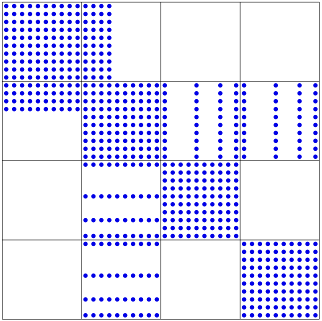
More LU storage

Switches and Null-space Dimensions

- No additional non-zeros in block-ILU(0) factorization using CDG
- Dense lower-triangular blocks using BR2 / IP

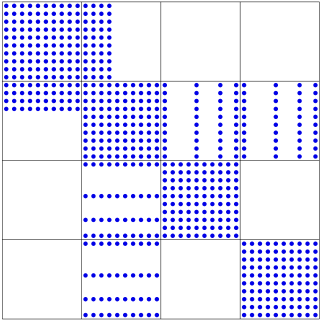
CDG

Stiffness Matrix



640 non-zeros

Block ILU(0)

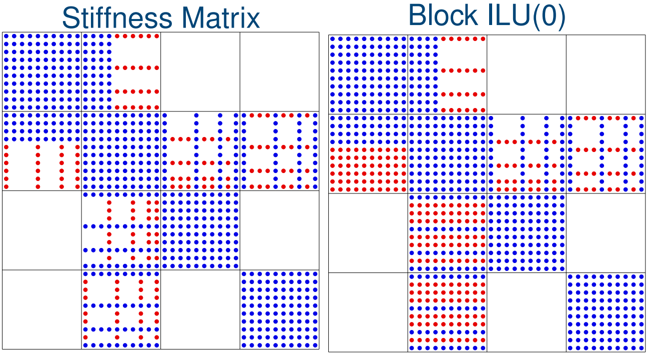


640 non-zeros

Switches and Null-space Dimensions

- No additional non-zeros in block-ILU(0) factorization using CDG
- Dense lower-triangular blocks using BR2 / IP

BR2 / IP

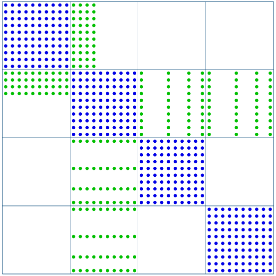


784 non-zeros

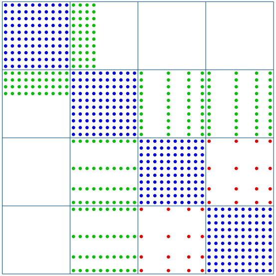
892 non-zeros

Matrix Representation

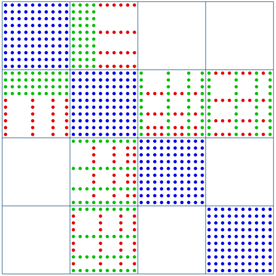
- Block matrix representation *fundamental for high performance*
  - Solver algorithms based on blocks
  - Up to 10 times higher performance with optimized BLAS
- Compact stencil $\Longrightarrow$ Matrix structure given by mesh connectivities
- Hard to store LDG/BR2/IP efficiently



CDG – 2 arrays



LDG – 3 arrays + struct



BR2 / IP – 3 arrays