Note that for a vector valued $\mathbf{F}(\mathbf{x})$ and a scalar $v(\mathbf{x})$ we have a sort of product rule for divergence:

$$\nabla \cdot (v\mathbf{F}) = \nabla v \cdot \mathbf{F} + v\,(\nabla \cdot \mathbf{F}).$$

Thus a weak solution u of

$$u_t + \nabla \cdot \mathbf{F} = 0$$

must satisfy

$$\int_\Omega u_t v\, dV = -\int_\Omega v\,(\nabla \cdot \mathbf{F})\, dV = \int_\Omega \nabla v \cdot \mathbf{F} - \nabla \cdot (v\mathbf{F})\, dV = \int_\Omega \nabla v \cdot \mathbf{F}\, dV - \int_{\partial\Omega} (v\mathbf{F}) \cdot \mathbf{n}\, dS.$$

We are considering the "rotating flux" function

$$\mathbf{F} = u \begin{bmatrix} -y \\ x \end{bmatrix}$$

on the domain $(x, y) \in [-1, 1]^2$ with initial condition

$$u(x, y, 0) = \frac{1}{2\pi \cdot \frac{1}{8}^2} \exp\left(\frac{\left(x - \frac{1}{2}\right)^2 + y^2}{\frac{1}{8}^2} \right)$$

and boundary conditions

$$u(x, -1, t) = u(x, 1, t) = u(-1, y, t) = u(1, y, t) = 0.$$

Now our test functions must satisfy

$$\int_\Omega u_t v\, dV = \int_\Omega u\,(-y v_x + x v_y)\, dV - \int_{\partial\Omega} uv\,(-y n_x + x n_y)\, dS.$$

To use DG to solve this problem, our test functions and components of u will be degree p polynomials hence $u\,(-y v_x + x v_y)$ is degree $2p$ and $uv\,(-y n_x + x n_y)$ is degree $2p + 1$.

To make this concrete, we'll consider $p = 1$. The quadrature rule

$$\int_T f(\mathbf{x})\, d\mathbf{x} \approx \frac{|T|}{3}\left[f\left(\frac{C + v_0}{2}\right) + f\left(\frac{C + v_1}{2}\right) + f\left(\frac{C + v_2}{2}\right) \right]$$

is exact for quadratics (here $C = \frac{v_0 + v_1 + v_2}{3}$ is the centroid of T).

On each directed edge of ∂T, we have a parameterization $\gamma(s) = v_i + \frac{s+1}{2}(v_{i+1} - v_i)$. Writing $v_{i+1} - v_i = \begin{bmatrix} \Delta x_i \\ \Delta y_i \end{bmatrix}$ we have an outward[1] normal given by $\mathbf{n}_i = \frac{1}{|v_{i+1} - v_i|} \begin{bmatrix} \Delta y_i \\ -\Delta x_i \end{bmatrix}$ hence

$$\int_\gamma \begin{bmatrix} f \\ g \end{bmatrix} \cdot \mathbf{n}_i\, dS = \int_{-1}^{1} \frac{f(\gamma(s))\,\Delta y_i - g(\gamma(s))\,\Delta x_i}{|v_{i+1} - v_i|} \left|\gamma'(s)\right| ds = \int_{-1}^{1} \frac{f(\gamma(s))\,\Delta y_i - g(\gamma(s))\,\Delta x_i}{2}\, ds.$$

So if f, g are cubics, the Gaussian quadrature

$$\int_{-1}^{1} \frac{f(\gamma(s))\,\Delta y_i - g(\gamma(s))\,\Delta x_i}{2}\, ds \approx \frac{f\left(\gamma\left(-1/\sqrt{3}\right)\right)\Delta y_i - g\left(\gamma\left(-1/\sqrt{3}\right)\right)\Delta x_i}{2} + \frac{f\left(\gamma\left(1/\sqrt{3}\right)\right)\Delta y_i - g\left(\gamma\left(1/\sqrt{3}\right)\right)\Delta x_i}{2}$$

is exact. For our given function we have

$$2\int_\gamma vu \begin{bmatrix} -y \\ x \end{bmatrix} \cdot \mathbf{n}\, dS \approx -uv\left(\gamma\left(-\frac{1}{\sqrt{3}}\right) \cdot \Delta v_i\right) - uv\left(\gamma\left(\frac{1}{\sqrt{3}}\right) \cdot \Delta v_i\right).$$

[1] one can check that the cross product $\mathbf{n}_i \times (v_{i+1} - v_i) = \frac{\Delta x_i^2 + \Delta y_i^2}{|v_{i+1} - v_i|}$ points in the positive z-direction

On a given triangular element T with (ordered, local) vertices v_0, v_1, v_2 we have a map

$$R(x, y) = (1 - x - y)v_0 + xv_1 + yv_2$$

from the reference triangle T_0 to T, with this

$$\int_{R(T_0)} f \, dV = \int_{T_0} f(R(\mathbf{x})) \left|\det J\right| \, d\mathbf{x}.$$

Note that $|\det J| = 2\,|T|$. For identity functions φ_i such that $\varphi_i(v_j) = \delta_{ij}$, we must have $\varphi_i\left(R\left(v_j^{(0)}\right)\right) = \delta_{ij}$ (where $v_j^{(0)}$ are the nodes of the reference triangle) hence we must have $\varphi_0\left(R(x, y)\right) = 1-x-y$, $\varphi_1\left(R(x, y)\right) = x$ and $\varphi_2\left(R(x, y)\right) = y$ by uniqueness of these "hat" functions.

For $u_t = \dot{u}_0\varphi_0 + \dot{u}_1\varphi_1 + \dot{u}_2\varphi_2$ (again in local indices) we have

$$\int_T u_t \begin{bmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \end{bmatrix} dV = \frac{|T|}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} \dot{u}_0 \\ \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} \Longrightarrow M_T^{-1} = \frac{3}{|T|} \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}.$$

One can show that in the $p = 1$ case

$$\begin{bmatrix} \frac{\partial \varphi_i}{\partial x} & \frac{\partial \varphi_i}{\partial y} \end{bmatrix} = \frac{1}{2\,|T|} \begin{bmatrix} -\Delta y_1 & \Delta x_1 \\ -\Delta y_2 & \Delta x_2 \\ -\Delta y_0 & \Delta x_0 \end{bmatrix}$$

we can write the quadrature points as

$$\begin{bmatrix} q_0 & q_1 & q_2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \end{bmatrix} \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

hence computing

$$g_i = -y\frac{\partial \varphi_i}{\partial x} + x\frac{\partial \varphi_i}{\partial y}$$

at each of these three points can be accomplished via

$$\begin{aligned} G = \begin{bmatrix} g_i\left(q_j\right) \end{bmatrix} &= \frac{1}{12\,|T|} \begin{bmatrix} -\Delta y_1 & \Delta x_1 \\ -\Delta y_2 & \Delta x_2 \\ -\Delta y_0 & \Delta x_0 \end{bmatrix} \begin{bmatrix} -y_0 & -y_1 & -y_2 \\ x_0 & x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix} \\ &= \frac{1}{12\,|T|} \begin{bmatrix} \Delta v_1 & \Delta v_2 & \Delta v_0 \end{bmatrix}^T \begin{bmatrix} v_0 & v_1 & v_2 \end{bmatrix} \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix} \end{aligned}$$

We combine this with

$$Q = \begin{bmatrix} \varphi_i\left(q_j\right) \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

to compute

$$\int_T \varphi_j \left(-y\frac{\partial \varphi_i}{\partial x} + x\frac{\partial \varphi_i}{\partial y}\right) dV = \frac{|T|}{3} \sum_{k=0}^{2} \varphi_j(q_k) g_i(q_k).$$

This corresponds to taking the dot product of row j of Q with row i of G. But, due to the symmetry of Q these 9 values actually occur in

$$K = \frac{1}{6} \begin{bmatrix} \Delta v_1 & \Delta v_2 & \Delta v_0 \end{bmatrix}^T \begin{bmatrix} v_0 & v_1 & v_2 \end{bmatrix} Q^2$$

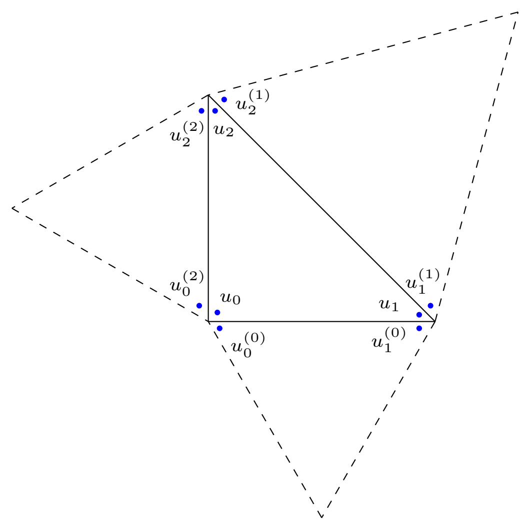
Thus far, we have

$$|T|\, M \begin{bmatrix} \dot{u}_0 \\ \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = K \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} - \int_{\partial T} (v\mathbf{F}) \cdot \mathbf{n}\, dS.$$

To handle the final integral, we need to utilize a sort of "upwind" condition. We consider

$$\operatorname{sign}\left(\beta(x,y)\cdot \mathbf{n}_i\right) = \operatorname{sign}\left(\begin{bmatrix} -y \\ x \end{bmatrix} \cdot \begin{bmatrix} \Delta y_i \\ -\Delta x_i \end{bmatrix} \right) = -\operatorname{sign}\left(x\Delta x_i + y\Delta y_i\right) = -\operatorname{sign}\left(\gamma(s)\cdot \Delta v_i\right).$$

If $\operatorname{sign}\left(\beta(x,y)\cdot \mathbf{n}_i\right) > 0$, we use u_i and u_{i+1} to parameterize our line. If not, then we use $u_i^{(i)}$ and $u_{i+1}^{(i)}$:



For example, against the (local) test function φ_i

$$2\int_{(\partial T)_0} \varphi_i u \left(\begin{bmatrix} -y \\ x \end{bmatrix} \cdot \mathbf{n}_0 \right) dS = \varphi_i u^{\pm}\big|_{s=-\frac{1}{\sqrt{3}}} \left(-\gamma_- \cdot \Delta v_0\right) + \varphi_i u^{\pm}\big|_{s=\frac{1}{\sqrt{3}}} \left(-\gamma_+ \cdot \Delta v_0\right).$$

Each of φ_i and u are lines, hence linear in s, this allows us to simplify

$$\varphi_0\big|_{(\partial T)_0} = \frac{1-s}{2}, \quad \varphi_1\big|_{(\partial T)_0} = \frac{1+s}{2}, \quad \varphi_2\big|_{(\partial T)_0} = 0, \quad u^{\pm}\big|_{(\partial T)_0} = \frac{1-s}{2} u_0^{\pm} + \frac{1+s}{2} u_1^{\pm}$$

so that

$$\begin{aligned}
2\int_{(\partial T)_i} \varphi_i u \left(\begin{bmatrix} -y \\ x \end{bmatrix} \cdot \mathbf{n}_i \right) dS &= \varphi_i u\big|_{s=-\frac{1}{\sqrt{3}}} \left(-\gamma_- \cdot \Delta v_i\right) + \varphi_i u\big|_{s=\frac{1}{\sqrt{3}}} \left(-\gamma_+ \cdot \Delta v_i\right) \\
&= \kappa_+ \left(\kappa_+ u_i + \kappa_- u_{i+1}\right)\left(-\gamma_- \cdot \Delta v_i\right) + \kappa_- \left(\kappa_- u_i + \kappa_+ u_{i+1}\right)\left(-\gamma_+ \cdot \Delta v_i\right) \\
2\int_{(\partial T)_i} \varphi_{i+1} u \left(\begin{bmatrix} -y \\ x \end{bmatrix} \cdot \mathbf{n}_i \right) dS &= \varphi_{i+1} u\big|_{s=-\frac{1}{\sqrt{3}}} \left(-\gamma_- \cdot \Delta v_i\right) + \varphi_{i+1} u\big|_{s=\frac{1}{\sqrt{3}}} \left(-\gamma_+ \cdot \Delta v_i\right) \\
&= \kappa_- \left(\kappa_+ u_i + \kappa_- u_{i+1}\right)\left(-\gamma_- \cdot \Delta v_i\right) + \kappa_+ \left(\kappa_- u_i + \kappa_+ u_{i+1}\right)\left(-\gamma_+ \cdot \Delta v_i\right) \\
2\int_{(\partial T)_i} \varphi_{i+2} u \left(\begin{bmatrix} -y \\ x \end{bmatrix} \cdot \mathbf{n}_i \right) dS &= 0
\end{aligned}$$

where $\kappa_{\pm} = \frac{1 \pm \frac{1}{\sqrt{3}}}{2}$.

1 Other Stuff

For a DG scheme $p = 1$, we consider the reference triangle T and

$$u|_T = u_0\varphi_0 + u_1\varphi_1 + u_2\varphi_2, \quad u_t|_T = \dot{u}_0\varphi_0 + \dot{u}_1\varphi_1 + \dot{u}_2\varphi_2$$

where $\varphi_0 = 1 - x - y$, $\varphi_1 = x$ and $\varphi_2 = y$.

$$\int_T u_t\varphi_i \, dV = \dot{u}_0 \int_T \varphi_i\varphi_0 \, dV + \dot{u}_1 \int_T \varphi_i\varphi_1 \, dV + \dot{u}_2 \int_T \varphi_i\varphi_2 \, dV$$

where these coefficients are given by the mass matrix

$$M = \frac{1}{24} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

We are considering the function

$$\mathbf{F} = u \begin{bmatrix} -y \\ x \end{bmatrix}$$

hence $\nabla v \cdot \mathbf{F} = v_x(-yu) + v_y(xu) = u(xv_y - yv_x)$ and we need to evaluate

$$\int_T \nabla\varphi_i \cdot \mathbf{F} \, dV = \sum_{j=0}^{2} u_j \int_T \varphi_j \left(x\frac{\partial \varphi_i}{\partial y} - y\frac{\partial \varphi_i}{\partial x} \right) dV$$

which gives the "stiffness" matrix

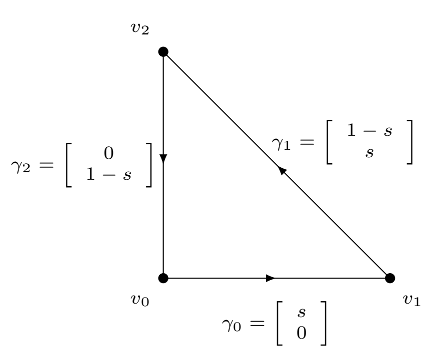
$$K = \frac{1}{24} \begin{bmatrix} 0 & -1 & 1 \\ -1 & -1 & -2 \\ 1 & 2 & 1 \end{bmatrix}.$$

As it turns out, each column of K is an eigenvector of M with the same eigenvalue, hence $M^{-1}K$ is easy to compute (this "matters" for future computations).

At this point we have

$$\frac{1}{24} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} \dot{u}_0 \\ \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \frac{1}{24} \begin{bmatrix} 0 & -1 & 1 \\ -1 & -1 & -2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} - \int_{\partial T} \left(\begin{bmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \end{bmatrix} \mathbf{F} \right) \cdot \mathbf{n} \, dS.$$

Along the reference triangle T, we have outward normals given by $\mathbf{n}_0 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $\mathbf{n}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{n}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.



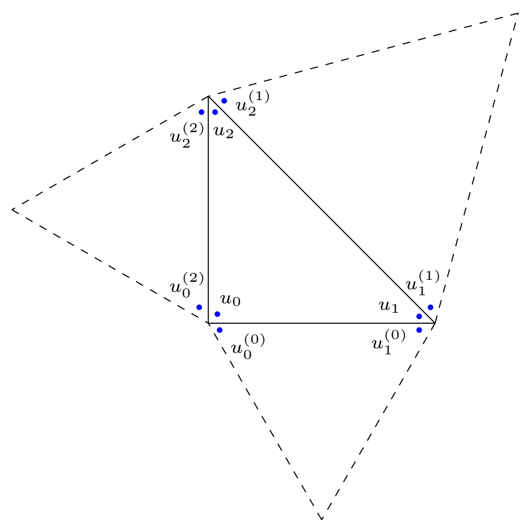
The final integral (noting that $\varphi_i \mathbf{F} = \varphi_i u \begin{bmatrix} -y \\ x \end{bmatrix}$) becomes

$$\int_{\partial T} (\varphi_i \mathbf{F}) \cdot \mathbf{n}\, dS = \int_0^1 (\varphi_i \mathbf{F}) \cdot \mathbf{n}_0 \, |\gamma_0'(s)| \; ds + \cdots$$
$$= \int_0^1 -\varphi_i x(s) u \, ds + \sqrt{2} \int_0^1 \varphi_i u \frac{x(s) - y(s)}{\sqrt{2}} \; ds + \int_0^1 \varphi_i y(s) u \, ds.$$

and computing this integrals we see

$$\int_{\partial T} \left(\begin{bmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \end{bmatrix} \mathbf{F} \right) \cdot \mathbf{n}\, dS = \left(\frac{1}{12} \begin{bmatrix} -1 & -1 & 0 \\ -1 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \frac{1}{12} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 3 \end{bmatrix} \right) \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix}$$
$$= (G_0 + G_1 + G_2) \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix}.$$

Rather than use the values u_0, u_1, u_2 in T, we instead "reach across" the edges of T as a sort of upwind condition:



All together, the update condition becomes

$$M \begin{bmatrix} \dot{u}_0 \\ \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = K \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} - G_0 \begin{bmatrix} u_0^{(0)} \\ u_1^{(0)} \\ 0 \end{bmatrix} - G_1 \begin{bmatrix} 0 \\ u_1^{(1)} \\ u_2^{(1)} \end{bmatrix} - G_2 \begin{bmatrix} u_0^{(2)} \\ 0 \\ u_2^{(2)} \end{bmatrix}.$$

In order to reduce the complexity of the computation, we produce here

$$M^{-1}K = 24K = \begin{bmatrix} 0 & -1 & 1 \\ -1 & -1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$
$$M^{-1}G_0 = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -4 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$
$$M^{-1}G_1 = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 3 & 1 \\ 0 & -1 & -3 \end{bmatrix}$$
$$M^{-1}G_2 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & -2 \\ 1 & 0 & 4 \end{bmatrix}.$$

For boundary edges, where there is no other triangle “across the edge”, we use the boundary condition.