Abstract

We present a short note describing a condition number of the intersection of two Bézier curves.

Keywords: Bézier curve, Curve intersection, Condition number, Numerical analysis

Contents

1 Introduction

The problem of intersecting two planar Bézier curves is an important one in Computer Aided Geometric Design (CAGD). Many intersection algorithms have been described in the literature, both geometric ([SP86, SN90, KLS98]) and algebraic ([MD92]). Though the general convergence properties of these algorithms have been studied (e.g. [Sch09]), no condition number for the intersection problem has been described in the CAGD literature[1].

There are more generic condition numbers for rational polynomial systems ([HT15]) or nonlinear algebraic systems ([Hig02, Chapter 25, Section 25.4]). However, the condition numbers with an algebraic focus (rather than an analytic one) often require too much computation to be useful. The numerical analytic condition numbers are in some ways too general to be useful for planar Bézier curve intersection.

In this paper, we describe a simple relative root condition number for this intersection problem. Since tangent intersections are to transversal intersections as multiple roots are to simple roots of a function, this condition number is infinite for non-transversal intersections. We present a few examples verifying that the condition number increases as a family of intersections approach an ill-behaved intersection.

2 Preliminaries

Throughout, we'll refer to a *Bézier curve* as a parametric plane curve given by

$$b(s) = \sum_{j=0}^{n} \boldsymbol{b}_j B_{j,n}(s) \tag{2.1}$$

where $B_{j,n}(s) = \binom{n}{j}(1-s)^{n-j}s^j$ is a Bernstein polynomial. When the parameter $s \in [0,1]$, the coefficients $B_{j,n}(s) \in [0,1]$ as well and the evaluation is a convex combination of the set of *control points* $\boldsymbol{b}_j \in \mathbf{R}^2$.

A Bézier curve intersection is a root $\left[\begin{array}{c} \alpha \\ \beta \end{array}\right]$ of the function

$$F(s,t) = b_0(s) - b_1(t) \tag{2.2}$$

[1] As far as the author has been able to tell. In many CAGD textbooks, there is a long review of methods for intersecting two planar Bézier curves (e.g. [Far01, Sed16]) but no mention of conditioning.

where $b_0(s)$ and $b_1(t)$ are Bézier curves. Note that $F : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$.

Each component $x(s)$ and $y(s)$ of a Bézier curve is a polynomial in Bernstein form. For such a polynomial

$$p(s) = \sum_{j=0}^{n} p_j B_{j,n}(s) \tag{2.3}$$

the (absolute) condition number of evaluation is $\widetilde{p}(s) = \sum_{j=0}^{n} |p_j| \, B_{j,n}(s)$ ([FR87]) when the parameter s is in the unit interval.

3 Conditioning of Generic Root-finding

Consider a smooth function $F : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ with Jacobian $F_{\boldsymbol{x}} = J$. We want to consider a special class of functions of the form $F(\boldsymbol{x}) = \sum_j c_j \phi_j(\boldsymbol{x})$ where the basis functions ϕ_j are also smooth functions on $\mathbf{R}^n$ and each $c_j \in \mathbf{R}$. We want to consider the effects on a root $\boldsymbol{\alpha} \in \mathbf{R}^n$ of a perturbation in one of the coefficients c_j. We examine the perturbed function

$$G(x, \delta) = F(\boldsymbol{x}) + \delta \phi_j(\boldsymbol{x}). \tag{3.1}$$

Since $G(\boldsymbol{\alpha}, 0) = \mathbf{0}$, if J^{-1} exists at $\boldsymbol{x} = \boldsymbol{\alpha}$ then the implicit function theorem tells us that we can define $\boldsymbol{x}$ via

$$G(\boldsymbol{x}(\delta), \delta) = \mathbf{0}. \tag{3.2}$$

Taking the derivative with respect to δ we find that $\mathbf{0} = G_{\boldsymbol{x}} \boldsymbol{x}_\delta + G_\delta$. Plugging in $\delta = 0$ we find that $0 = J(\boldsymbol{\alpha}) \boldsymbol{x}_\delta + \phi_j(\boldsymbol{\alpha})$, hence we conclude that

$$\boldsymbol{x}(\delta) = \boldsymbol{\alpha} - J(\boldsymbol{\alpha})^{-1} \phi_j(\boldsymbol{\alpha}) \delta + \mathcal{O}\left(\delta^2\right). \tag{3.3}$$

This gives

$$\frac{\left\| J(\boldsymbol{\alpha})^{-1} \phi_j(\boldsymbol{\alpha}) \right\|}{\|\boldsymbol{\alpha}\|}. \tag{3.4}$$

as the relative condition number for a perturbation in c_j. By considering perturbations in *all* of the coefficients: $|\delta_j| \leq \varepsilon |c_j|$, a similar analysis gives a root function

$$\boldsymbol{x}(\delta_0, \ldots, \delta_n) = \boldsymbol{\alpha} - J(\boldsymbol{\alpha})^{-1} \sum_{j=0}^{n} \delta_j \phi_j(\boldsymbol{\alpha}) + \mathcal{O}\left(\varepsilon^2\right). \tag{3.5}$$

With this, we can define a root condition number

Definition 3.1. For a smooth function $F(\boldsymbol{x}) = \sum_j c_j \phi_j(\boldsymbol{x})$ parameterized by the coefficients $\{c_j\}$ with root $\boldsymbol{\alpha}$ and Jacobian $J(\boldsymbol{\alpha})$, we define a relative root condition number

$$\kappa_{\boldsymbol{\alpha}} = \lim_{\varepsilon \to 0} \sup \frac{\|\delta \boldsymbol{\alpha}\| / \varepsilon}{\|\boldsymbol{\alpha}\|} = \lim_{\varepsilon \to 0} \left(\sup_{|\delta_j| \leq \varepsilon |c_j|} \frac{\left\| J(\boldsymbol{\alpha})^{-1} \sum_j \delta_j \phi_j(\boldsymbol{\alpha}) \right\| / \varepsilon}{\|\boldsymbol{\alpha}\|} \right). \tag{3.6}$$

In [Hig02, Chapter 25, Section 25.4] a similar definition is given. Instead of bounding the perturbations component-wise, it bounds the entire perturbation vector

$$\lim_{\varepsilon \to 0} \left(\sup_{\|\boldsymbol{\delta}\| \leq \varepsilon \|\boldsymbol{c}\|} \frac{\left\| J(\boldsymbol{\alpha})^{-1} \sum_j \delta_j \phi_j(\boldsymbol{\alpha}) \right\| / \varepsilon}{\|\boldsymbol{\alpha}\|} \right) = \left\| J^{-1} F_{\boldsymbol{c}} \right\| \frac{\|\boldsymbol{c}\|}{\|\boldsymbol{\alpha}\|}. \tag{3.7}$$

While this has the benefit of having a closed form that is straightforward to compute, it may be less useful than the condition number given in Definition 3.1 since the larger coefficients can dominate the rest.

3.1 One-dimensional Case

When $n = 1$, due to the triangle inequality:

$$|\delta\alpha| = \left| J^{-1} \sum_{j=0}^{n} \delta_j \phi_j(\alpha) \right| \leq \frac{1}{|F'(\alpha)|} \sum_{j=0}^{n} |\delta_j \phi_j(\alpha)| . \tag{3.8}$$

The sign and magnitude of each δ_j can be chosen to make $\delta_j \phi_j(\alpha) = |c_j \phi_j(\alpha)| \varepsilon$, hence for these values equality holds in the triangle inequality:

$$\left| \sum_{j=0}^{n} \delta_j \phi_j(\alpha) \right| = \varepsilon \sum_{j=0}^{n} |c_j \phi_j(\alpha)| . \tag{3.9}$$

Thus we get a root condition number for a polynomial given in Bernstein form

$$\kappa_\alpha = \frac{1}{|\alpha F'(\alpha)|} \sum_{j=0}^{n} |c_j \phi_j(\alpha)| . \tag{3.10}$$

that agrees with the common definition ([FR87]) when the ϕ_j are specialized to Bernstein basis functions.

4 Conditioning of Bézier Curve Intersection

To define a condition number for the intersection of two planar Bézier curves, we write the difference as

$$F(s,t) = \begin{bmatrix} x_0(s) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y_0(s) \end{bmatrix} - \begin{bmatrix} x_1(t) \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ y_1(t) \end{bmatrix} . \tag{4.1}$$

We can show that there is a closed form for the condition number given by Definition 3.1, specialized to the 2-norm.

Theorem 4.1. Let $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ be a transversal intersection of two planar Bézier curves $b_0(s)$ and $b_1(t)$. Define the vectors

$$\widetilde{F}_+ = \begin{bmatrix} \widetilde{x}_0(\alpha) + \widetilde{x}_1(\beta) \\ \widetilde{y}_0(\alpha) + \widetilde{y}_1(\beta) \end{bmatrix} \quad \text{and} \quad \widetilde{F}_- = \begin{bmatrix} \widetilde{x}_0(\alpha) + \widetilde{x}_1(\beta) \\ -\widetilde{y}_0(\alpha) - \widetilde{y}_1(\beta) \end{bmatrix} . \tag{4.2}$$

Then the root condition number of the intersection is the greater of two vector norms:

$$\kappa_{\alpha,\beta} = \frac{1}{\sqrt{\alpha^2 + \beta^2}} \max \left\{ \left\| J(\alpha,\beta)^{-1} \widetilde{F}_+ \right\|_2 , \left\| J(\alpha,\beta)^{-1} \widetilde{F}_- \right\|_2 \right\} . \tag{4.3}$$

Proof. Let the curve $b_0(s)$ be degree m and $b_1(t)$ be degree n. Then $F(s,t)$ can be written as $2(m+1)+2(n+1)$ terms:

$$F(s,t) = \sum_{i=0}^{m} c_i^{(1)} \begin{bmatrix} B_{i,m}(s) \\ 0 \end{bmatrix} + \sum_{i=0}^{m} c_i^{(2)} \begin{bmatrix} 0 \\ B_{i,m}(s) \end{bmatrix} + \sum_{j=0}^{n} c_j^{(3)} \begin{bmatrix} -B_{j,n}(t) \\ 0 \end{bmatrix} + \sum_{j=0}^{n} c_j^{(4)} \begin{bmatrix} 0 \\ -B_{j,n}(t) \end{bmatrix} . \tag{4.4}$$

Since $F(s,t) = b_0(s) - b_1(t)$ we have Jacobian $J(s,t) = \begin{bmatrix} b_0'(s) & -b_1'(t) \end{bmatrix}$. Since a transversal intersection, we have $\det J(\alpha,\beta) \neq 0$. In a perturbation, we replace each $c_j^{(k)}$ with a $\delta_j^{(k)}$ bounded by $\left|\delta_j^{(k)}\right| \leq \varepsilon \left|c_j^{(k)}\right|$. Writing $J^{-1} = \begin{bmatrix} \boldsymbol{v} & \boldsymbol{w} \end{bmatrix}$, we have

$$J(\boldsymbol{\alpha})^{-1} \sum_{j} \delta_j \phi_j(\boldsymbol{\alpha}) = \left[\sum_{i=0}^{m} \delta_i^{(1)} B_{i,m}(\alpha) + \sum_{j=0}^{n} \delta_j^{(3)} B_{j,n}(\beta) \right] \boldsymbol{v}$$

$$+ \left[\sum_{i=0}^{m} \delta_i^{(2)} B_{i,m}(\alpha) + \sum_{j=0}^{n} \delta_j^{(4)} B_{j,n}(\beta) \right] \boldsymbol{w} = \nu_1 \boldsymbol{v} + \nu_2 \boldsymbol{w} \quad (4.5)$$

where

$$|\nu_1| / \varepsilon \leq \sum_{i=0}^{m} \left| c_i^{(1)} \right| B_{i,m}(\alpha) + \sum_{j=0}^{n} \left| c_j^{(3)} \right| B_{j,n}(\beta) = \widetilde{x}_0(\alpha) + \widetilde{x}_1(\beta) = \mu_1 \quad (4.6)$$

$$|\nu_2| / \varepsilon \leq \sum_{i=0}^{m} \left| c_i^{(2)} \right| B_{i,m}(\alpha) + \sum_{j=0}^{n} \left| c_j^{(4)} \right| B_{j,n}(\beta) = \widetilde{y}_0(\alpha) + \widetilde{y}_1(\beta) = \mu_2. \quad (4.7)$$

As in (3.8), the bound can be attained by choosing the sign and magnitude of each perturbation so that $\delta_j^{(k)} B_{j,d} = \varepsilon \left| c_j^{(k)} \right| B_{j,d}$. The factor of ε can be cancelled to give condition number

$$\kappa_{\alpha,\beta} = \frac{1}{\sqrt{\alpha^2 + \beta^2}} \sup_{|\nu_k| \leq \mu_k} \left\| \nu_1 \boldsymbol{v} + \nu_2 \boldsymbol{w} \right\|_2 \quad (4.8)$$

$$= \sqrt{\frac{\sup_{|\nu_k| \leq \mu_k} \nu_1^2 (\boldsymbol{v} \cdot \boldsymbol{v}) + 2\nu_1 \nu_2 (\boldsymbol{v} \cdot \boldsymbol{w}) + \nu_2^2 (\boldsymbol{w} \cdot \boldsymbol{w})}{\alpha^2 + \beta^2}}. \quad (4.9)$$

Now we seek to maximize the objective function $\theta(\nu_1, \nu_2) = \nu_1^2 (\boldsymbol{v} \cdot \boldsymbol{v}) + 2\nu_1 \nu_2 (\boldsymbol{v} \cdot \boldsymbol{w}) + \nu_2^2 (\boldsymbol{w} \cdot \boldsymbol{w})$ in the rectangle $[-\mu_1, \mu_1] \times [-\mu_2, \mu_2]$.

To find interior critical points, we solve the system $\theta_{\nu_1} = \theta_{\nu_2} = 0$:

$$\begin{bmatrix} 2\boldsymbol{v} \cdot \boldsymbol{v} & 2\boldsymbol{v} \cdot \boldsymbol{w} \\ 2\boldsymbol{v} \cdot \boldsymbol{w} & 2\boldsymbol{w} \cdot \boldsymbol{w} \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (4.10)$$

This system has the unique solution $\nu_1 = \nu_2 = 0$ unless $\|\boldsymbol{v}\|_2 \|\boldsymbol{w}\|_2 = |\boldsymbol{v} \cdot \boldsymbol{w}|$. By the Cauchy-Schwarz inequality, this can only occur if $\boldsymbol{v}$ and $\boldsymbol{w}$ are parallel; since J^{-1} is invertible, we know they are not. Hence $\theta(0,0) = 0$ is the only interior critical point and it is the global minimum.

Along the boundary of the rectangle, we fix one of ν_1 or ν_2 and the resulting univariate function is an up-opening parabola. For example, fixing $\nu_2 = c$ gives $\theta(\nu_1, c) = \nu_1^2 (\boldsymbol{v} \cdot \boldsymbol{v}) + \nu_1 [2c (\boldsymbol{v} \cdot \boldsymbol{w})] + c^2 (\boldsymbol{w} \cdot \boldsymbol{w})$ which has positive lead coefficient $\|\boldsymbol{v}\|_2^2$. The lead coefficient cannot be 0 since if $\boldsymbol{v}$ were the zero vector we'd have $\det J = 0$. Since θ is an up-opening parabola along the boundary, any critical point must be a local minimum.

Thus we know the maximum occurs at two of the four corners of the rectangle. Due to sign cancellation, this leads to one of two values $\theta = \mu_1^2 (\boldsymbol{v} \cdot \boldsymbol{v}) \pm 2\mu_1 \mu_2 (\boldsymbol{v} \cdot \boldsymbol{w}) + \mu_2^2 (\boldsymbol{w} \cdot \boldsymbol{w})$, the largest of which is $\mu_1^2 (\boldsymbol{v} \cdot \boldsymbol{v}) + 2\mu_1 \mu_2 |\boldsymbol{v} \cdot \boldsymbol{w}| + \mu_2^2 (\boldsymbol{w} \cdot \boldsymbol{w})$. Thus

$$\kappa_{\alpha,\beta} = \sqrt{\frac{\mu_1^2 (\boldsymbol{v} \cdot \boldsymbol{v}) + 2\mu_1 \mu_2 |\boldsymbol{v} \cdot \boldsymbol{w}| + \mu_2^2 (\boldsymbol{w} \cdot \boldsymbol{w})}{\alpha^2 + \beta^2}}. \quad (4.11)$$

Note that $\widetilde{F}_{\pm} = \begin{bmatrix} \mu_1 \\ \pm \mu_2 \end{bmatrix}$, so

$$\left\| J(\alpha, \beta)^{-1} \widetilde{F}_{\pm} \right\|_2 = \left\| \begin{bmatrix} \boldsymbol{v} & \boldsymbol{w} \end{bmatrix} \widetilde{F}_{\pm} \right\|_2 = \sqrt{\mu_1^2 (\boldsymbol{v} \cdot \boldsymbol{v}) \pm 2\mu_1 \mu_2 (\boldsymbol{v} \cdot \boldsymbol{w}) + \mu_2^2 (\boldsymbol{w} \cdot \boldsymbol{w})} \quad (4.12)$$

gives each of the two values that produce the maximum value. ■

4.1 Transversal Intersection

Consider the line $b_0(s) = \begin{bmatrix} 2s \\ 2s \end{bmatrix}$ and improperly parameterized line $b_1(t) = \begin{bmatrix} 4t^2 \\ 2 - 4t^2 \end{bmatrix}$ which intersect at $\alpha = \beta = 1/2$. At the intersection we have $J^{-1} = \frac{1}{8} \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$, so that $\boldsymbol{v} \cdot \boldsymbol{v} = \boldsymbol{w} \cdot \boldsymbol{w} = 5/64$ and $\boldsymbol{v} \cdot \boldsymbol{w} = 3/64$.

Since the x-component of $F(s,t)$ can be written as $2s - 4t^2 = 2B_{1,1}(s) - 4B_{2,2}(t)$ and the y-component as $2s + 4t^2 - 2 = 2B_{1,1}(s) - 2B_{0,2}(t) - 2B_{1,2}(t) + 2B_{2,2}(t)$ we have

$$\mu_1 = 2B_{1,1}(\alpha) \qquad\qquad\qquad\qquad + 4B_{2,2}(\beta) = 2 \tag{4.13}$$
$$\mu_2 = 2B_{1,1}(\alpha) + 2B_{0,2}(\beta) + 2B_{1,2}(\beta) + 2B_{2,2}(\beta) = 3. \tag{4.14}$$

Following (4.11), this gives $\kappa_{\alpha,\beta} = \sqrt{202}/8 \approx 1.78$. This low condition number is somewhat unexpected since when using the resultant to eliminate each parameter, one of the two roots is a double root:

$$\operatorname{Res}_t\left(x_0(s) - x_1(t), y_0(s) - y_1(t)\right) = 64(2s-1)^2 \tag{4.15}$$
$$\operatorname{Res}_s\left(x_0(s) - x_1(t), y_0(s) - y_1(t)\right) = 4(2t-1)(2t+1). \tag{4.16}$$

4.2 Line-line Intersection with Poorly Behaved Coefficients

Consider the intersection of the lines $y = x$ and $y = 1 - x$ when $x \in [0,1]$. These correspond to the Bézier curves

$$b_0(s) = \left[\begin{array}{c} s \\ s \end{array}\right], \quad b_1(t) = \left[\begin{array}{c} t \\ 1-t \end{array}\right]. \tag{4.17}$$

By adding a scalar $D > 0$ to each component, we leave $F(s,t)$ and hence the solution unchanged. However, the coefficients of the curves change:

$$b_0(s) = \left[\begin{array}{c} D(1-s) + (1+D)s \\ D(1-s) + (1+D)s \end{array}\right], \quad b_1(t) = \left[\begin{array}{c} D(1-t) + (1+D)t \\ (1+D)(1-t) + Dt \end{array}\right]. \tag{4.18}$$

At the solution $\alpha = \beta = 1/2$, we have

$$\widetilde{F}_{\pm} = \left[\begin{array}{c} 2D+1 \\ \pm(2D+1) \end{array}\right] \tag{4.19}$$

and in either case $\left\| J^{-1}\widetilde{F}_{\pm} \right\|_2 = 2D + 1$.

So, we see the condition number $\kappa_{\alpha,\beta} = \sqrt{2}(2D+1)$ increases towards infinity as D does. This is what we expect as the coefficients grow so large that their ratio approaches 1.

4.3 Family of Lines Approaching Coincident

Consider a family of intersections in which one of the lines approaches the other:

$$b_0(s) = \left[\begin{array}{c} s \\ 1 \end{array}\right], \quad b_1(t) = \left[\begin{array}{c} t \\ (1+r)(1-t) + t \end{array}\right]. \tag{4.20}$$

These lines $y = 1$ and $rx + y = 1 + r$ intersect when $\alpha = \beta = 1$. However as $r \longrightarrow 0^+$, the lines become coincident: if $r = 0$ the single intersection becomes infinitely many.

At the solution, we have $\widetilde{F}_{\pm} = \left[\begin{array}{cc} 2 & \pm 2 \end{array}\right]^T$ and again have a condition number

$$\kappa_{\alpha,\beta} = \sqrt{\frac{4}{r^2} + \frac{4}{r} + 2} = \frac{2}{r} + 1 + \frac{r}{4} + \mathcal{O}\left(r^2\right). \tag{4.21}$$

that increases towards infinity as the parameter $r \longrightarrow 0^+$.

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