Abstract

We present a short note describing a condition number of the intersection of two Bézier curves. Since tangent intersections are to transversal intersections as multiple roots are to simple roots of a function, this condition number is infinite for non-transveral intersections.

Keywords: Bézier curve, Curve intersection, Condition number

Contents

1 Introduction

Placeholder

2 Problem conditioning

Consider a smooth function $F : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ with Jacobian $F_{\boldsymbol{x}} = J$. We want to consider a special class of functions of the form $F\left(\boldsymbol{x}\right) = \sum_j c_j \phi_j\left(\boldsymbol{x}\right)$ where the basis functions ϕ_j are also smooth functions on $\mathbf{R}^n$ and each $c_j \in \mathbf{R}$. We want to consider the effects on a root $\boldsymbol{\alpha} \in \mathbf{R}^n$ of a perturbation in one of the coefficients c_j. We examine the perturbed functions

$$G(x, \delta) = F\left(\boldsymbol{x}\right) + \delta \phi_j\left(\boldsymbol{x}\right). \tag{2.1}$$

Since $G\left(\boldsymbol{\alpha}, 0\right) = \mathbf{0}$, if J^{-1} exists at $\boldsymbol{x} = \boldsymbol{\alpha}$ then the implicit function theorem tells us that we can define $\boldsymbol{x}$ via

$$G\left(\boldsymbol{x}\left(\delta\right), \delta\right) = \mathbf{0}. \tag{2.2}$$

Taking the derivative with respect to δ we find that $\mathbf{0} = G_{\boldsymbol{x}} \boldsymbol{x}_{\delta} + G_{\delta}$. Plugging in $\delta = 0$ we find that $0 = J\left(\boldsymbol{\alpha}\right) \boldsymbol{x}_{\delta} + \phi_j\left(\boldsymbol{\alpha}\right)$, hence we conclude that

$$\boldsymbol{x}\left(\delta\right) = \boldsymbol{\alpha} - J\left(\boldsymbol{\alpha}\right)^{-1} \phi_j\left(\boldsymbol{\alpha}\right) \delta + \mathcal{O}\left(\delta^2\right). \tag{2.3}$$

This gives a relative condition number (for the root) of

$$\frac{\left\| J\left(\boldsymbol{\alpha}\right)^{-1} \phi_j\left(\boldsymbol{\alpha}\right) \right\|}{\left\| \boldsymbol{\alpha} \right\|}. \tag{2.4}$$

By considering perturbations in *all* of the coefficients: $\left|\delta_j\right| \leq \varepsilon \left|c_j\right|$, a similar analysis gives a root function

$$\boldsymbol{x}\left(\delta_0, \ldots, \delta_n\right) = \boldsymbol{\alpha} - J\left(\boldsymbol{\alpha}\right)^{-1} \sum_{j=0}^{n} \delta_j \phi_j\left(\boldsymbol{\alpha}\right) + \mathcal{O}\left(\varepsilon^2\right). \tag{2.5}$$

With this, we can define a root condition number

$$\kappa_{\boldsymbol{\alpha}} = \lim_{\varepsilon \to 0} \left(\sup \frac{\left\| \delta \boldsymbol{\alpha} \right\| / \varepsilon}{\left\| \boldsymbol{\alpha} \right\|} \right) = \lim_{\varepsilon \to 0} \left(\sup \frac{\left\| J\left(\boldsymbol{\alpha}\right)^{-1} \sum_j \delta_j \phi_j\left(\boldsymbol{\alpha}\right) \right\| / \varepsilon}{\left\| \boldsymbol{\alpha} \right\|} \right). \tag{2.6}$$

When $n = 1$, J^{-1} is simply $1/F'$ and we find

$$\kappa_\alpha = \frac{1}{|\alpha F'(\alpha)|} \sum_{j=0}^{n} |c_j \phi_j(\alpha)| . \tag{2.7}$$

This value is given by the triangle inequality applied to $\delta\alpha$ and equality can be attained since the sign of each $\delta_j = \pm c_j \varepsilon$ can be modified at will to make $\phi_j(\alpha)\delta_j = |\phi_j(\alpha)c_j| \varepsilon$.

When $n > 1$, the triangle inequality tells us that

$$\kappa_{\boldsymbol{\alpha}} = \lim_{\varepsilon \to 0} \left(\sup \frac{\|\delta\boldsymbol{\alpha}/\varepsilon\|}{\|\boldsymbol{\alpha}\|} \right) \leq \frac{1}{\|\boldsymbol{\alpha}\|} \sum_{j=0}^{n} |c_j| \left\| J\left(\boldsymbol{\alpha}\right)^{-1} \phi_j(\boldsymbol{\alpha}) \right\| . \tag{2.8}$$

However, this bound is only attainable if all $\phi_j(\boldsymbol{\alpha})$ are parallel. However, we'll seldom need to compute the exact condition number and are instead typically interested in the order of magnitude. In this case a lower bound

$$\frac{1}{\|\boldsymbol{\alpha}\|} \max_j |c_j| \left\| J\left(\boldsymbol{\alpha}\right)^{-1} \phi_j(\boldsymbol{\alpha}) \right\| \tag{2.9}$$

for $\kappa_{\boldsymbol{\alpha}}$ will suffice as an approximate condition number.

For an example, consider

$$\phi_0 = \left[\begin{array}{c} x_0 \\ 2 \\ 0 \end{array} \right], \phi_1 = \left[\begin{array}{c} 0 \\ x_1 \\ 3 \end{array} \right], \phi_2 = \left[\begin{array}{c} 2 \\ 0 \\ x_2 \end{array} \right], F = \phi_0 + 2\phi_1 + 3\phi_2, \boldsymbol{\alpha} = \left[\begin{array}{c} -6 \\ -1 \\ -2 \end{array} \right]. \tag{2.10}$$

For a given ε, the maximum root perturbation occurs when $\delta_0 = \varepsilon, \delta_1 = 2\varepsilon, \delta_2 = -3\varepsilon$ and gives $\left\| J\left(\boldsymbol{\alpha}\right)^{-1} \sum_j \delta_j \phi_j\left(\boldsymbol{\alpha}\right) \right\| = 4\sqrt{10}\varepsilon \approx 12.65\varepsilon$. The pessimistic triangle inequality bound gives $\sum_j |c_j| \left\| J\left(\boldsymbol{\alpha}\right)^{-1} \phi_j(\boldsymbol{\alpha}) \right\| \approx 14.64\varepsilon$ and the maximum individual perturbation is $2\sqrt{10}\varepsilon \approx 6.325\varepsilon$ (this occurs when $\delta_0 = \delta_1 = 0, \delta_2 = \pm 3\varepsilon$).

In this general framework, we can define a condition number both for a simple root of a polynomial in Bernstein form and for the intersection of two planar Bézier curves. For the first, $\phi_j(s) = \binom{n}{j}(1-s)^{n-j}s^j$ the Bernstein basis functions, a polynomial $p(s) = \sum_j b_j \phi_j(s)$ with a simple root $\alpha \in (0, 1]$ has root condition number

$$\kappa_\alpha = \frac{1}{\alpha\,|p'(\alpha)|} \sum_{j=0}^{n} |b_j \phi_j(\alpha)| = \frac{\widetilde{p}(\alpha)}{\alpha\,|p'(\alpha)|}. \tag{2.11}$$

For the intersection of a degree m curve $b_1(s)$ and a degree n curve $b_2(t)$, we have basis functions

$$\begin{aligned} \phi_{0,-1,1} = \left[\begin{array}{c} B_{0,m}(s) \\ 0 \end{array} \right], \phi_{0,-1,2} = \left[\begin{array}{c} 0 \\ B_{0,m}(s) \end{array} \right], \cdots, \\ \phi_{m,-1,1} = \left[\begin{array}{c} B_{m,m}(s) \\ 0 \end{array} \right], \phi_{m,-1,2} = \left[\begin{array}{c} 0 \\ B_{m,m}(s) \end{array} \right], \\ \phi_{-1,0,1} = \left[\begin{array}{c} -B_{0,n}(t) \\ 0 \end{array} \right], \phi_{-1,0,2} = \left[\begin{array}{c} 0 \\ -B_{0,n}(t) \end{array} \right], \cdots, \\ \phi_{-1,n,1} = \left[\begin{array}{c} -B_{n,n}(t) \\ 0 \end{array} \right], \phi_{-1,n,2} = \left[\begin{array}{c} 0 \\ -B_{n,n}(t) \end{array} \right]. \end{aligned} \tag{2.12}$$

Since $F(s,t) = b_1(s) - b_2(t)$ we have Jacobian $J(s,t) = \left[\begin{array}{cc} b_1'(s) & -b_2'(t) \end{array} \right]$. We'll consider a transversal intersection $F(\alpha, \beta) = \mathbf{0}$ with $\det J(\alpha, \beta) \neq 0$. Since each of the ϕ_j is just a scalar multiple of the standard basis vectors, writing $J^{-1} = \left[\begin{array}{cc} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{array} \right]$, we have

$$\begin{aligned} J\left(\alpha, \beta\right)^{-1} \sum_{\boldsymbol{j}} \delta_{\boldsymbol{j}} \phi_{\boldsymbol{j}}\left(\alpha, \beta\right) &= \left[\sum_{i=0}^{m} \delta_{i,-1,1} B_{i,m}\left(\alpha\right) + \sum_{j=0}^{n} \delta_{-1,j,1} B_{j,n}\left(\beta\right) \right] \boldsymbol{v}_1 \\ &+ \left[\sum_{i=0}^{m} \delta_{i,-1,2} B_{i,m}\left(\alpha\right) + \sum_{j=0}^{n} \delta_{-1,j,2} B_{j,n}\left(\beta\right) \right] \boldsymbol{v}_2 = \nu_1 \boldsymbol{v}_1 + \nu_2 \boldsymbol{v}_2. \end{aligned} \tag{2.13}$$

where

$$|\nu_k| / \varepsilon \leq \sum_{i=0}^{m} |c_{i,-1,k}| \, B_{i,m}(\alpha) + \sum_{j=0}^{n} |c_{-1,j,k}| \, B_{j,n}(\beta) = \mu_k \tag{2.14}$$

and the bound can be attained for both $k = 1, 2$ by making the signs of the $\delta_{\boldsymbol{j}}$ agree. If we name the components of each curve via $b_1(s) = \begin{bmatrix} x_1(s) & y_1(s) \end{bmatrix}^T$ and $b_2(t) = \begin{bmatrix} x_2(t) & y_2(t) \end{bmatrix}^T$ then we see that $\mu_1 = \widetilde{x}_1(\alpha) + \widetilde{x}_2(\beta)$ and $\mu_2 = \widetilde{y}_1(\alpha) + \widetilde{y}_2(\beta)$. Thus we have condition number

$$\kappa_{\alpha,\beta} = \frac{1}{\sqrt{\alpha^2 + \beta^2}} \max_{|\nu_k| \leq \mu_k} \|\nu_1 \boldsymbol{v}_1 + \nu_2 \boldsymbol{v}_2\|_2 \tag{2.15}$$

$$= \sqrt{\frac{\max_{|\nu_k| \leq \mu_k} \nu_1^2 (\boldsymbol{v}_1 \cdot \boldsymbol{v}_1) + 2\nu_1\nu_2 (\boldsymbol{v}_1 \cdot \boldsymbol{v}_2) + \nu_2^2 (\boldsymbol{v}_2 \cdot \boldsymbol{v}_2)}{\alpha^2 + \beta^2}}. \tag{2.16}$$

Since J^{-1} is invertible, we know $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are not parallel which can be used to show that the only internal critical point of the function to be maximimized in (2.16) is $\nu_1 = \nu_2 = 0$, which is the global minimum. Along the boundary of the rectangle $[-\mu_1, \mu_1] \times [-\mu_2, \mu_2]$, we fix one of ν_1 or ν_2 and the resulting univariate function is an up-opening parabola, hence any critical point must be a local minimum. Thus we know the maximum occurs at two of the four corners of the rectangle:

$$\kappa_{\alpha,\beta} = \sqrt{\frac{\mu_1^2 (\boldsymbol{v}_1 \cdot \boldsymbol{v}_1) + 2\mu_1\mu_2 |\boldsymbol{v}_1 \cdot \boldsymbol{v}_2| + \mu_2^2 (\boldsymbol{v}_2 \cdot \boldsymbol{v}_2)}{\alpha^2 + \beta^2}}. \tag{2.17}$$

As far as the author can tell, a condition number for the intersection of two planar Bézier curves has not been described in the Computer Aided Geometric Design (CAGD) literature. In [Hig02, Chapter 25, Equation 25.11] a more generic condition number is defined for the root of a nonlinear algebraic system that is similar to the definition above.

For an example, consider the line $b_1(s) = \begin{bmatrix} 2s & 2s \end{bmatrix}^T$ and improperly parameterized line $b_2(t) = \begin{bmatrix} 4t^2 & 2 - 4t^2 \end{bmatrix}^T$ which intersect at $\alpha = \beta = 1/2$. At the intersection we have $J^{-1} = \frac{1}{8} \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$, so that $\boldsymbol{v}_1 \cdot \boldsymbol{v}_1 = \boldsymbol{v}_2 \cdot \boldsymbol{v}_2 = 5/64$ and $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = 3/64$. Since the x-component of $F(s,t)$ can be written as $2s - 4t^2 = 2B_{1,1}(s) - 4B_{2,2}(t)$ and the y-component as $2s + 4t^2 - 2 = 2B_{1,1}(s) - 2B_{0,2}(t) - 2B_{1,2}(t) + 2B_{2,2}(t)$ we have

$$\mu_1 = 2B_{1,1}(\alpha) \qquad\qquad\qquad\qquad + 4B_{2,2}(\beta) = 2 \tag{2.18}$$

$$\mu_2 = 2B_{1,1}(\alpha) + 2B_{0,2}(\beta) + 2B_{1,2}(\beta) + 2B_{2,2}(\beta) = 3. \tag{2.19}$$

Following (2.17), this gives $\kappa_{\alpha,\beta} = \sqrt{202}/8 \approx 1.78$.

References

[Hig02] Nicholas J. Higham. *Accuracy and Stability of Numerical Algorithms.* Society for Industrial and Applied Mathematics, Jan 2002.