A 2-Norm Condition Number for Bézier Curve Intersection

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Abstract

We present a condition number of the intersection of two Bézier curves.

Keywords: Bézier curve, Curve intersection, Condition number, Numerical analysis

1. Introduction

The problem of intersecting two planar Bézier curves is an important one in Computer Aided Geometric Design (CAGD). Many intersection algorithms have been described in the literature, both geometric (Sederberg and Parry [1986]; Sederberg and Nishita [1990]; Kim et al. [1998]) and algebraic (Manocha and Demmel [1992]). Though the general convergence properties of these algorithms have been studied (e.g. Schulz [2009]), no condition number for the intersection problem has been described in the CAGD literature[1].

There are more generic condition numbers for rational polynomial systems (Herman and Tsigaridas [2015]) or nonlinear algebraic systems ([Higham, 2002, Chapter 25, Section 25.4]). However, the condition numbers with an algebraic focus (rather than an analytic one) often require too much computation to be useful. The numerical analytic condition numbers are in some ways too general to be useful for planar Bézier curve intersection.

In this paper, we describe a simple relative root condition number for this intersection problem. Since tangent intersections are to transversal intersections as multiple roots are to simple roots of a function, this condition number is infinite for non-transversal intersections. We present a few examples verifying that the condition number increases as a family of intersections approach an ill-behaved intersection.

2. Preliminaries

Throughout, we will refer to a parametric polynomial plane curve given by

$$b(s) = \sum_{j=0}^{n} \boldsymbol{b}_j B_{j,n}(s) \tag{2.1}$$

as a *Bézier curve*, where $B_{j,n}(s) = \binom{n}{j}(1-s)^{n-j}s^j$ is a Bernstein polynomial. When the parameter $s \in [0,1]$, the coefficients $B_{j,n}(s) \in [0,1]$ as well and the evaluation is a convex combination of the *control points* $\boldsymbol{b}_j \in \mathbf{R}^2$.

An intersection of two planar curves $b_0(s)$ and $b_1(t)$ corresponds to a root $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ of the function

$$F(s,t) = b_0(s) - b_1(t). \tag{2.2}$$

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[1] As far as the author has been able to tell. In many CAGD textbooks, there is a long review of methods for intersecting two planar Bézier curves (e.g. Farin [2001]; Sederberg [2016]) but no mention of conditioning.

Note that $F : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$.

Each component $x(s)$ and $y(s)$ of a Bézier curve is a polynomial in Bernstein form. For such a polynomial

$$p(s) = \sum_{j=0}^{n} p_j B_{j,n}(s) \tag{2.3}$$

the (absolute) condition number of evaluation is

$$\widetilde{p}(s) = \sum_{j=0}^{n} |p_j| \, B_{j,n}(s) \tag{2.4}$$

(Farouki and Rajan [1987]) when the parameter s is in the unit interval.

3. Conditioning of Generic Root-finding

Consider a smooth function $F : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ with Jacobian $F_{\boldsymbol{x}} = J$. We want to consider a special class of functions of the form $F(\boldsymbol{x}) = \sum_j c_j \phi_j(\boldsymbol{x})$ where the basis functions ϕ_j are also smooth functions $\mathbf{R}^n \longrightarrow \mathbf{R}^n$ and each $c_j \in \mathbf{R}$. We want to consider the effects on a root $\boldsymbol{\alpha} \in \mathbf{R}^n$ of a perturbation in one of the coefficients c_j. We examine the perturbed function

$$G(\boldsymbol{x}, \delta) = F(\boldsymbol{x}) + \delta \phi_j(\boldsymbol{x}). \tag{3.1}$$

Since $G(\boldsymbol{\alpha}, 0) = \mathbf{0}$, if J^{-1} exists at $\boldsymbol{x} = \boldsymbol{\alpha}$, the implicit function theorem tells us that we can define $\boldsymbol{x}$ via

$$G(\boldsymbol{x}(\delta), \delta) = \mathbf{0}. \tag{3.2}$$

Taking the derivative with respect to δ we find that $\mathbf{0} = G_{\boldsymbol{x}} \boldsymbol{x}' + G_\delta$. Plugging in $\delta = 0$ we find that $\mathbf{0} = J(\boldsymbol{\alpha}) \boldsymbol{x}' + \phi_j(\boldsymbol{\alpha})$, hence we conclude that

$$\boldsymbol{x}(\delta) = \boldsymbol{\alpha} - J^{-1}(\boldsymbol{\alpha}) \phi_j(\boldsymbol{\alpha}) \delta + \mathcal{O}(\delta^2). \tag{3.3}$$

This gives

$$\frac{\left\| J^{-1}(\boldsymbol{\alpha}) \phi_j(\boldsymbol{\alpha}) \right\|}{\|\boldsymbol{\alpha}\|}. \tag{3.4}$$

as the relative condition number for a perturbation in c_j. By considering perturbations in *all* of the coefficients: $|\delta_j| \leq \varepsilon |c_j|$, a similar analysis gives a root function

$$\boldsymbol{x}(\delta_0, \ldots, \delta_n) = \boldsymbol{\alpha} - J^{-1}(\boldsymbol{\alpha}) \sum_{j=0}^{n} \delta_j \phi_j(\boldsymbol{\alpha}) + \mathcal{O}(\varepsilon^2). \tag{3.5}$$

With this, we can define a root condition number

Definition 3.1. For a smooth function $F(\boldsymbol{x}) = \sum_j c_j \phi_j(\boldsymbol{x})$ parameterized by the coefficients c_j with root $\boldsymbol{\alpha}$ and Jacobian $J(\boldsymbol{\alpha})$, we define a relative root condition number

$$\kappa_{\boldsymbol{\alpha}} = \limsup_{\varepsilon \to 0} \frac{\|\delta \boldsymbol{\alpha}\| / \varepsilon}{\|\boldsymbol{\alpha}\|} = \lim_{\varepsilon \to 0} \left(\sup_{|\delta_j| \leq \varepsilon |c_j|} \frac{\left\| J^{-1}(\boldsymbol{\alpha}) \sum_j \delta_j \phi_j(\boldsymbol{\alpha}) \right\| / \varepsilon}{\|\boldsymbol{\alpha}\|} \right) \tag{3.6}$$

where $\boldsymbol{\alpha} + \delta \boldsymbol{\alpha}$ is a root of the perturbed function $\sum_j (c_j + \delta_j) \phi_j(\boldsymbol{x})$.

In [Higham, 2002, Chapter 25, Section 25.4] a similar definition is given. Instead of bounding the perturbations component-wise, it bounds the entire perturbation vector $\boldsymbol{\delta}$

$$\lim_{\varepsilon \to 0} \left(\sup_{\|\boldsymbol{\delta}\| \leq \varepsilon \|\boldsymbol{c}\|} \frac{\left\| J^{-1}(\boldsymbol{\alpha}) \sum_j \delta_j \phi_j(\boldsymbol{\alpha}) \right\| / \varepsilon}{\|\boldsymbol{\alpha}\|} \right). \tag{3.7}$$

It is possible to rewrite $\sum_j \delta_j \phi_j(\boldsymbol{\alpha}) = F_{\boldsymbol{c}} \boldsymbol{\delta}$ where $F_{\boldsymbol{c}} = \left[\frac{\partial F_i}{\partial c_j} \right]_{ij} = \left[\begin{array}{ccc} \phi_0(\boldsymbol{\alpha}) & \cdots & \phi_n(\boldsymbol{\alpha}) \end{array} \right]$ is the Jacobian of F with respect to the coefficients $\boldsymbol{c}$. With this representation, the Higham condition number has a closed form since

$$\left\| J^{-1} F_{\boldsymbol{c}} \boldsymbol{\delta} \right\| / \varepsilon \leq \left\| J^{-1} F_{\boldsymbol{c}} \right\| \|\boldsymbol{\delta}\| / \varepsilon \leq \left\| J^{-1} F_{\boldsymbol{c}} \right\| \|\boldsymbol{c}\| \tag{3.8}$$

for any matrix norm that is compatible with the vector norm used on $\boldsymbol{\delta}$. The Frobenius matrix norm and vector 2-norm can be combined to give a condition number that is straightforward to compute:

$$\kappa_H = \left\| J^{-1} F_{\boldsymbol{c}} \right\|_F \frac{\|\boldsymbol{c}\|_2}{\|\boldsymbol{\alpha}\|_2}. \tag{3.9}$$

While this closed form for κ_H is useful, it provides a less sharp measure than the condition number $\kappa_{\boldsymbol{\alpha}}$ given in Definition 3.1 since the ball $\|\boldsymbol{\delta}\|_2 \leq \varepsilon \|\boldsymbol{c}\|_2$ can allow larger perturbations of a single coefficient than the box determined by $|\delta_j| \leq \varepsilon |c_j|$ and allows perturbations in zero coefficients. When specialized to planar Bézier curves, we'll show in Theorem 4.1 that κ_α has a closed form as well[2]. In addition, this closed form shows that κ_α is a natural extension of the one-dimensional equivalent given in (3.12) below.

3.1. One-dimensional Case

When $n = 1$, due to the triangle inequality:

$$|\delta\alpha| = \left| J^{-1} \sum_{j=0}^{n} \delta_j \phi_j(\alpha) \right| \leq \frac{1}{|F'(\alpha)|} \sum_{j=0}^{n} |\delta_j \phi_j(\alpha)|. \tag{3.10}$$

The sign and magnitude of each δ_j can be chosen to make $\delta_j \phi_j(\alpha) = |c_j \phi_j(\alpha)| \varepsilon$, hence for these values equality holds in the triangle inequality:

$$\left| \sum_{j=0}^{n} \delta_j \phi_j(\alpha) \right| = \varepsilon \sum_{j=0}^{n} |c_j \phi_j(\alpha)|. \tag{3.11}$$

Thus we get a root condition number for a polynomial given in Bernstein form

$$\kappa_\alpha = \frac{1}{|\alpha F'(\alpha)|} \sum_{j=0}^{n} |c_j \phi_j(\alpha)| = \frac{\widetilde{F}(\alpha)}{|\alpha F'(\alpha)|} \tag{3.12}$$

that agrees with the common definition ([Farouki, 2008, Equation 12.33]) for any polynomial basis ϕ_j.

4. Conditioning of Bézier Curve Intersection

To define a condition number for the intersection of two planar Bézier curves, we write the difference as

$$F(s,t) = \left[\begin{array}{c} x_0(s) \\ 0 \end{array} \right] + \left[\begin{array}{c} 0 \\ y_0(s) \end{array} \right] - \left[\begin{array}{c} x_1(t) \\ 0 \end{array} \right] - \left[\begin{array}{c} 0 \\ y_1(t) \end{array} \right]. \tag{4.1}$$

We can show that there is a closed form for the condition number given by Definition 3.1, specialized to the 2-norm.

[2] When the 2-norm is used

Theorem 4.1. Let $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ be the parameter vector $\begin{bmatrix} s \\ t \end{bmatrix}$ at which two planar Bézier curves $b_0(s)$ and $b_1(t)$ have a transversal intersection. Then the root condition number of the intersection is

$$\kappa_{\alpha,\beta} = \sqrt{\frac{\mu_1^2\,(\boldsymbol{v}\cdot\boldsymbol{v}) + 2\mu_1\mu_2\,|\boldsymbol{v}\cdot\boldsymbol{w}| + \mu_2^2\,(\boldsymbol{w}\cdot\boldsymbol{w})}{\alpha^2+\beta^2}} \tag{4.2}$$

where

$$J^{-1}(\alpha,\beta) = \begin{bmatrix} \boldsymbol{v} & \boldsymbol{w} \end{bmatrix} \qquad \mu_1 = \widetilde{x}_0(\alpha) + \widetilde{x}_1(\beta) \qquad \mu_2 = \widetilde{y}_0(\alpha) + \widetilde{y}_1(\beta). \tag{4.3}$$

Here $b_0(s) = \begin{bmatrix} x_0(s) & y_0(s) \end{bmatrix}^T$, $b_1(t) = \begin{bmatrix} x_1(t) & y_1(t) \end{bmatrix}^T$ and the $\widetilde{x}_i, \widetilde{y}_j$ are as defined in (2.4).

Proof. Let the curve $b_0(s)$ be of degree m and $b_1(t)$ be of degree n. Then $F(s,t)$ can be written as a sum of $2(m+1)+2(n+1)$ terms:

$$F(s,t) = \sum_{i=0}^{m} c_i^{(1)} \begin{bmatrix} B_{i,m}(s) \\ 0 \end{bmatrix} + \sum_{i=0}^{m} c_i^{(2)} \begin{bmatrix} 0 \\ B_{i,m}(s) \end{bmatrix} + \sum_{j=0}^{n} c_j^{(3)} \begin{bmatrix} -B_{j,n}(t) \\ 0 \end{bmatrix} + \sum_{j=0}^{n} c_j^{(4)} \begin{bmatrix} 0 \\ -B_{j,n}(t) \end{bmatrix}. \tag{4.4}$$

Since $F(s,t) = b_0(s) - b_1(t)$ we have Jacobian $J(s,t) = \begin{bmatrix} b_0'(s) & -b_1'(t) \end{bmatrix}$. Since we are considering a transversal intersection, we have $\det J(\alpha,\beta) \neq 0$. In a perturbed F, we replace each $c_j^{(k)}$ with $c_j^{(k)} + \delta_j^{(k)}$ where $\left|\delta_j^{(k)}\right| \leq \varepsilon \left|c_j^{(k)}\right|$. By writing $J^{-1} = \begin{bmatrix} \boldsymbol{v} & \boldsymbol{w} \end{bmatrix}$, we have

$$\begin{aligned} J^{-1}(\boldsymbol{\alpha}) \sum_j \delta_j \phi_j(\boldsymbol{\alpha}) = {} & \left[\sum_{i=0}^{m} \delta_i^{(1)} B_{i,m}(\alpha) + \sum_{j=0}^{n} \delta_j^{(3)} B_{j,n}(\beta) \right] \boldsymbol{v} \\ & + \left[\sum_{i=0}^{m} \delta_i^{(2)} B_{i,m}(\alpha) + \sum_{j=0}^{n} \delta_j^{(4)} B_{j,n}(\beta) \right] \boldsymbol{w} = \nu_1 \boldsymbol{v} + \nu_2 \boldsymbol{w} \end{aligned} \tag{4.5}$$

where

$$|\nu_1|/\varepsilon \leq \sum_{i=0}^{m} \left|c_i^{(1)}\right| B_{i,m}(\alpha) + \sum_{j=0}^{n} \left|c_j^{(3)}\right| B_{j,n}(\beta) = \widetilde{x}_0(\alpha) + \widetilde{x}_1(\beta) = \mu_1 \tag{4.6}$$

$$|\nu_2|/\varepsilon \leq \sum_{i=0}^{m} \left|c_i^{(2)}\right| B_{i,m}(\alpha) + \sum_{j=0}^{n} \left|c_j^{(4)}\right| B_{j,n}(\beta) = \widetilde{y}_0(\alpha) + \widetilde{y}_1(\beta) = \mu_2. \tag{4.7}$$

As in (3.10), the bound can be attained by choosing the sign and magnitude of each perturbation so that $\delta_j^{(k)} B_{j,d} = \varepsilon \left|c_j^{(k)}\right| B_{j,d}$. The factor ε can be cancelled to give the relative root condition number

$$\kappa_{\alpha,\beta} = \frac{1}{\sqrt{\alpha^2+\beta^2}} \sup_{|\nu_k| \leq \mu_k} \left\| \nu_1 \boldsymbol{v} + \nu_2 \boldsymbol{w} \right\|_2 \tag{4.8}$$

$$= \sqrt{\frac{\sup_{|\nu_k| \leq \mu_k} \nu_1^2\,(\boldsymbol{v}\cdot\boldsymbol{v}) + 2\nu_1\nu_2\,(\boldsymbol{v}\cdot\boldsymbol{w}) + \nu_2^2\,(\boldsymbol{w}\cdot\boldsymbol{w})}{\alpha^2+\beta^2}}. \tag{4.9}$$

Now we seek to maximize the objective function $\theta(\nu_1,\nu_2) = \nu_1^2\,(\boldsymbol{v}\cdot\boldsymbol{v}) + 2\nu_1\nu_2\,(\boldsymbol{v}\cdot\boldsymbol{w}) + \nu_2^2\,(\boldsymbol{w}\cdot\boldsymbol{w})$ in the rectangle $[-\mu_1,\mu_1] \times [-\mu_2,\mu_2]$.

To find interior critical points, we solve the system $\theta_{\nu_1} = \theta_{\nu_2} = 0$:

$$\begin{bmatrix} 2\boldsymbol{v}\cdot\boldsymbol{v} & 2\boldsymbol{v}\cdot\boldsymbol{w} \\ 2\boldsymbol{v}\cdot\boldsymbol{w} & 2\boldsymbol{w}\cdot\boldsymbol{w} \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4.10}$$

This system has the unique solution $\nu_1 = \nu_2 = 0$ unless $\|\boldsymbol{v}\|_2 \|\boldsymbol{w}\|_2 = |\boldsymbol{v} \cdot \boldsymbol{w}|$. By the Cauchy-Schwarz inequality, this can only occur if $\boldsymbol{v}$ and $\boldsymbol{w}$ are parallel; since J^{-1} is invertible, we know they are not. Hence $\theta(0, 0) = 0$ is the only interior critical point and it is the global minimum.

Along the boundary of the rectangle, we fix one of ν_1 or ν_2 and the resulting univariate function is an up-opening parabola. For example, fixing $\nu_2 = c$ gives $\theta(\nu_1, c) = \nu_1^2 (\boldsymbol{v} \cdot \boldsymbol{v}) + \nu_1 [2c (\boldsymbol{v} \cdot \boldsymbol{w})] + c^2 (\boldsymbol{w} \cdot \boldsymbol{w})$ which has positive lead coefficient $\|\boldsymbol{v}\|_2^2$. The lead coefficient cannot be 0 since if $\boldsymbol{v}$ were the zero vector we would have $\det J = 0$. Since θ is an up-opening parabola along the boundary, any critical point must be a local minimum.

Thus we know the maximum occurs at one of the four corners of the rectangle. Due to sign cancellation, this leads to one of two values $\theta = \mu_1^2 (\boldsymbol{v} \cdot \boldsymbol{v}) \pm 2\mu_1\mu_2 (\boldsymbol{v} \cdot \boldsymbol{w}) + \mu_2^2 (\boldsymbol{w} \cdot \boldsymbol{w})$, the largest of which is $\mu_1^2 (\boldsymbol{v} \cdot \boldsymbol{v}) + 2\mu_1\mu_2 |\boldsymbol{v} \cdot \boldsymbol{w}| + \mu_2^2 (\boldsymbol{w} \cdot \boldsymbol{w})$. Thus

$$\kappa_{\alpha,\beta} = \sqrt{\frac{\mu_1^2 (\boldsymbol{v} \cdot \boldsymbol{v}) + 2\mu_1\mu_2 |\boldsymbol{v} \cdot \boldsymbol{w}| + \mu_2^2 (\boldsymbol{w} \cdot \boldsymbol{w})}{\alpha^2 + \beta^2}} \tag{4.11}$$

as desired. ■

With this closed form $\kappa_{\alpha,\beta}$ in hand, we can now compare to the Higham condition number κ_H from (3.9). We'll show that $\kappa_{\alpha,\beta} \le \kappa_H$ by comparing $\kappa_{\alpha,\beta}^2 \left(\alpha^2 + \beta^2\right)$ to $\kappa_H^2 \left(\alpha^2 + \beta^2\right) = \left\|J^{-1} F_{\boldsymbol{c}}\right\|_F^2 \|\boldsymbol{c}\|_2^2$. In the case of planar curves with basis as in (4.4),

$$\left\|J^{-1} F_{\boldsymbol{c}}\right\|_F^2 = \left[\|\boldsymbol{v}\|_2^2 + \|\boldsymbol{w}\|_2^2\right] W \quad \text{where} \quad W = \sum_{i=0}^{m} B_{i,m}^2(\alpha) + \sum_{j=0}^{n} B_{j,n}^2(\beta) \tag{4.12}$$

is the sum of squared Bernstein weights. With two applications of the Cauchy-Schwarz inequality we know that

$$\kappa_{\alpha,\beta}^2 \left(\alpha^2 + \beta^2\right) \le \left(\mu_1 \|\boldsymbol{v}\|_2 + \mu_2 \|\boldsymbol{w}\|_2\right)^2 \le \left(\mu_1^2 + \mu_2^2\right) \left[\|\boldsymbol{v}\|_2^2 + \|\boldsymbol{w}\|_2^2\right]. \tag{4.13}$$

So it remains to show that $\mu_1^2 + \mu_2^2 \le W \|\boldsymbol{c}\|_2^2$, which can be done with another application of Cauchy-Schwarz to the terms in μ_1 and μ_2:

$$\mu_1^2 + \mu_2^2 \le \left(\sum_{i=0}^{m} \left|c_i^{(1)}\right|^2 + \sum_{j=0}^{n} \left|c_j^{(3)}\right|^2 \right) W + \left(\sum_{i=0}^{m} \left|c_i^{(2)}\right|^2 + \sum_{j=0}^{n} \left|c_j^{(4)}\right|^2 \right) W = W \|\boldsymbol{c}\|_2^2. \tag{4.14}$$

5. Condition Number in Practice

5.1. *Transversal Intersection*

Consider the line $b_0(s) = \begin{bmatrix} 2s \\ 2s \end{bmatrix}$ and quadratically parameterized line $b_1(t) = \begin{bmatrix} 4t^2 \\ 2 - 4t^2 \end{bmatrix}$ which intersect at $\alpha = \beta = 1/2$. At the intersection we have $J^{-1} = \frac{1}{8} \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$, so that $\boldsymbol{v} \cdot \boldsymbol{v} = \boldsymbol{w} \cdot \boldsymbol{w} = 5/64$ and $\boldsymbol{v} \cdot \boldsymbol{w} = 3/64$. Since the x-component of $F(s, t)$ can be written as $2s - 4t^2 = 2B_{1,1}(s) - 4B_{2,2}(t)$ and the y-component as $2s + 4t^2 - 2 = 2B_{1,1}(s) - 2B_{0,2}(t) - 2B_{1,2}(t) + 2B_{2,2}(t)$ we have

$$\mu_1 = 2B_{1,1}(\alpha) + 4B_{2,2}(\beta) = 2 \tag{5.1}$$

$$\mu_2 = 2B_{1,1}(\alpha) + 2B_{0,2}(\beta) + 2B_{1,2}(\beta) + 2B_{2,2}(\beta) = 3. \tag{5.2}$$

Following (4.11), this gives $\kappa_{\alpha,\beta} = \sqrt{202}/8 \approx 1.78$. This low condition number is expected from a geometric point of view; i.e. the intersection is a transversal intersection of two lines. However, when using the resultant to eliminate each parameter, one of the two roots is a double root:

$$\operatorname{Res}_t \left(x_0(s) - x_1(t), y_0(s) - y_1(t)\right) = 64(2s - 1)^2 \tag{5.3}$$

$$\operatorname{Res}_s \left(x_0(s) - x_1(t), y_0(s) - y_1(t)\right) = 4(2t - 1)(2t + 1). \tag{5.4}$$

so an algebraic approach may lead to an incorrect conclusion that the intersection is ill-conditioned.

5.2. Collapsing to One-dimensional Case

One key argument for choosing Definition 3.1 over the Highham condition number κ_H from (3.9) is that $\kappa_{\alpha,\beta}$ is a natural extension of the condition number for the equivalent one-dimensional problem. To see that this is so, we'll define a "trivial" example by starting with a polynomial $p(s)$ in Bernstein form and a simple root α.

We define the Bézier curves $b_0(s) = \begin{bmatrix} p(s) \\ 0 \end{bmatrix}$ and $b_1(t) = \begin{bmatrix} 0 \\ t \end{bmatrix}$. These curves intersect when $\beta = 0$ and α is a root of $p(s)$. At such an intersection $\mu_1 = \widetilde{p}(\alpha) + 0$, $\mu_2 = 0 + \beta = 0$ and

$$J^{-1} = \begin{bmatrix} 1/p'(\alpha) & 0 \\ 0 & -1 \end{bmatrix} \tag{5.5}$$

so that $\boldsymbol{v} \cdot \boldsymbol{v} = 1/\left[p'(\alpha)\right]^2$, $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ and $\boldsymbol{w} \cdot \boldsymbol{w} = 1$. This produces

$$\kappa_{\alpha,0} = \sqrt{\frac{\mu_1^2 \left(\boldsymbol{v} \cdot \boldsymbol{v}\right) + 0 + 0}{\alpha^2 + 0}} = \frac{\widetilde{p}(\alpha)}{\left|\alpha p'(\alpha)\right|}, \tag{5.6}$$

the commonly used condition number presented in (3.12).

5.3. Line-line Intersection with Poorly Behaved Coefficients

Consider the intersection of the lines $y = x$ and $y = 1 - x$ when $x \in [0, 1]$. These correspond to the Bézier curves

$$b_0(s) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} s, \quad b_1(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} (1 - t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} t. \tag{5.7}$$

By adding a scalar $D > 0$ to each component, we leave $F(s, t)$ and hence the solution unchanged. However, the coefficients of the curves change:

$$b_0(s) = \begin{bmatrix} D(1 - s) + (1 + D)s \\ D(1 - s) + (1 + D)s \end{bmatrix}, \quad b_1(t) = \begin{bmatrix} D(1 - t) + (1 + D)t \\ (1 + D)(1 - t) + Dt \end{bmatrix}. \tag{5.8}$$

At the solution $\alpha = \beta = 1/2$, we have $\mu_1 = \mu_2 = 2D + 1$ and

$$J^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \tag{5.9}$$

so that $\boldsymbol{v} \cdot \boldsymbol{v} = \boldsymbol{w} \cdot \boldsymbol{w} = 1/2$ and $\boldsymbol{v} \cdot \boldsymbol{w} = 0$. So, we see the condition number $\kappa_{\alpha,\beta} = \sqrt{2}(2D + 1)$ increases towards infinity as D does. This is what we expect as the coefficients grow so large that their ratio $(1+D)/D$ approaches 1.

5.4. Family of Lines Approaching Coincidence

Consider a family of intersections in which one of the lines approaches the other:

$$b_0(s) = \begin{bmatrix} s \\ 1 \end{bmatrix}, \quad b_1(t) = \begin{bmatrix} t \\ (1 + r)(1 - t) + t \end{bmatrix}. \tag{5.10}$$

These lines $y = 1$ and $rx + y = 1 + r$ intersect when $\alpha = \beta = 1$. However as $r \longrightarrow 0^+$, the lines become coincident: if $r = 0$ the single intersection becomes infinitely many.

At the solution, we have $\mu_1 = \mu_2 = 2$ and

$$J^{-1} = \frac{1}{r} \begin{bmatrix} r & 1 \\ 0 & 1 \end{bmatrix} \tag{5.11}$$

so that $\boldsymbol{v} \cdot \boldsymbol{v} = 1$, $\boldsymbol{v} \cdot \boldsymbol{w} = 1/r$ and $\boldsymbol{w} \cdot \boldsymbol{w} = 2/r^2$. Again we have a condition number

$$\kappa_{\alpha,\beta} = \sqrt{\frac{4}{r^2} + \frac{4}{r} + 2} = \frac{2}{r} + 1 + \frac{r}{4} + \mathcal{O}\left(r^2\right). \tag{5.12}$$

that increases towards infinity as the parameter $r \longrightarrow 0^+$.

6. Conclusion and Future Work

The author hopes that this can be useful for evaluating and comparing the performance of curve intersection implementations. By establishing a straightforward and easy to compute closed form, the condition number can be used more often to differentiate between cases where the algorithm or computer code is at fault for loss in accuracy and cases where the conditioning of the intersection itself is the cause. The framework set forth in Section 3 can be applied in future work to compute the condition number of higher order intersections such as surface-surface intersections in $\mathbf{R}^3$.

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