

A 2-Norm Condition Number for Bézier Curve Intersection

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Abstract

We present a condition number of the intersection of two Bézier curves.

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1. Introduction

The problem of intersecting two planar Bézier curves is an important one in Computer Aided Geometric Design (CAGD). Many intersection algorithms have been described in the literature, both geometric (Sederberg and Parry [1986]; Sederberg and Nishita [1990]; Kim et al. [1998]) and algebraic (Manocha and Demmel [1992]). Though the general convergence properties of these algorithms have been studied (e.g. Schulz [2009]), no condition number for the intersection problem has been described in the CAGD literature¹.

There are more generic condition numbers for rational polynomial systems (Herman and Tsigaridas [2015]) or nonlinear algebraic systems ([Higham, 2002, Chapter 25, Section 25.4]). However, the condition numbers with an algebraic focus (rather than an analytic one) often require too much computation to be useful. The numerical analytic condition numbers are in some ways too general to be useful for planar Bézier curve intersection.

In this paper, we describe a simple relative root condition number for this intersection problem. Since tangent intersections are to transversal intersections as multiple roots are to simple roots of a function, this condition number is infinite for non-transversal intersections. We present a few examples verifying that the condition number increases as a family of intersections approach an ill-behaved intersection.

2. Preliminaries

Throughout, we will refer to a parametric polynomial plane curve given by

$$b(s) = \sum_{j=0}^n \mathbf{b}_j B_{j,n}(s) \quad (2.1)$$

as a *Bézier curve*, where $B_{j,n}(s) = \binom{n}{j}(1-s)^{n-j}s^j$ is a Bernstein polynomial. When the parameter $s \in [0, 1]$, the coefficients $B_{j,n}(s) \in [0, 1]$ as well and the evaluation is a convex combination of the *control points* $\mathbf{b}_j \in \mathbf{R}^2$.

An intersection of two planar curves $b_0(s)$ and $b_1(t)$ corresponds to a root $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ of the function

$$F(s, t) = b_0(s) - b_1(t). \quad (2.2)$$

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¹As far as the author has been able to tell. In many CAGD textbooks, there is a long review of methods for intersecting two planar Bézier curves (e.g. Farin [2001]; Sederberg [2016]) but no mention of conditioning.

Note that $F : \mathbf{R}^2 \rightarrow \mathbf{R}^2$.

Each component $x(s)$ and $y(s)$ of a Bézier curve is a polynomial in Bernstein form. For such a polynomial

$$p(s) = \sum_{j=0}^n p_j B_{j,n}(s) \quad (2.3)$$

the (absolute) condition number of evaluation is $\tilde{p}(s) = \sum_{j=0}^n |p_j| B_{j,n}(s)$ (Farouki and Rajan [1987]) when the parameter s is in the unit interval.

3. Conditioning of Generic Root-finding

Consider a smooth function $F : \mathbf{R}^n \rightarrow \mathbf{R}^n$ with Jacobian $F_{\mathbf{x}} = J$. We want to consider a special class of functions of the form $F(\mathbf{x}) = \sum_j c_j \phi_j(\mathbf{x})$ where the basis functions ϕ_j are also smooth functions $\mathbf{R}^n \rightarrow \mathbf{R}^n$ and each $c_j \in \mathbf{R}$. We want to consider the effects on a root $\alpha \in \mathbf{R}^n$ of a perturbation in one of the coefficients c_j . We examine the perturbed function

$$G(\mathbf{x}, \delta) = F(\mathbf{x}) + \delta \phi_j(\mathbf{x}). \quad (3.1)$$

Since $G(\alpha, 0) = \mathbf{0}$, if J^{-1} exists at $\mathbf{x} = \alpha$ then the implicit function theorem tells us that we can define \mathbf{x} via

$$G(\mathbf{x}(\delta), \delta) = \mathbf{0}. \quad (3.2)$$

Taking the derivative with respect to δ we find that $\mathbf{0} = G_{\mathbf{x}} \mathbf{x}' + G_{\delta}$. Plugging in $\delta = 0$ we find that $\mathbf{0} = J(\alpha) \mathbf{x}' + \phi_j(\alpha)$, hence we conclude that

$$\mathbf{x}(\delta) = \alpha - J^{-1}(\alpha) \phi_j(\alpha) \delta + \mathcal{O}(\delta^2). \quad (3.3)$$

This gives

$$\frac{\|J(\alpha)^{-1} \phi_j(\alpha)\|}{\|\alpha\|}. \quad (3.4)$$

as the relative condition number for a perturbation in c_j . By considering perturbations in *all* of the coefficients: $|\delta_j| \leq \varepsilon |c_j|$, a similar analysis gives a root function

$$\mathbf{x}(\delta_0, \dots, \delta_n) = \alpha - J(\alpha)^{-1} \sum_{j=0}^n \delta_j \phi_j(\alpha) + \mathcal{O}(\varepsilon^2). \quad (3.5)$$

With this, we can define a root condition number

Definition 3.1. For a smooth function $F(\mathbf{x}) = \sum_j c_j \phi_j(\mathbf{x})$ parameterized by the coefficients c_j with root α and Jacobian $J(\alpha)$, we define a relative root condition number

$$\kappa_{\alpha} = \limsup_{\varepsilon \rightarrow 0} \frac{\|\delta \alpha\| / \varepsilon}{\|\alpha\|} = \lim_{\varepsilon \rightarrow 0} \left(\sup_{|\delta_j| \leq \varepsilon |c_j|} \frac{\|J(\alpha)^{-1} \sum_j \delta_j \phi_j(\alpha)\| / \varepsilon}{\|\alpha\|} \right). \quad (3.6)$$

In [Higham, 2002, Chapter 25, Section 25.4] a similar definition is given. Instead of bounding the perturbations component-wise, it bounds the entire perturbation vector

$$\lim_{\varepsilon \rightarrow 0} \left(\sup_{\|\delta\| \leq \varepsilon \|c\|} \frac{\|J(\alpha)^{-1} \sum_j \delta_j \phi_j(\alpha)\| / \varepsilon}{\|\alpha\|} \right) = \|J^{-1} F_c\| \frac{\|c\|}{\|\alpha\|}. \quad (3.7)$$

where $F_c = [\phi_0(\alpha) \ \dots \ \phi_n(\alpha)]^T$ is the vector of weight functions. While this has the benefit of having a closed form that is straightforward to compute, it may be less useful than the condition number given in Definition 3.1 since the larger coefficients can dominate the rest.

3.1. One-dimensional Case

When $n = 1$, due to the triangle inequality:

$$|\delta\alpha| = \left| J^{-1} \sum_{j=0}^n \delta_j \phi_j(\alpha) \right| \leq \frac{1}{|F'(\alpha)|} \sum_{j=0}^n |\delta_j \phi_j(\alpha)|. \quad (3.8)$$

The sign and magnitude of each δ_j can be chosen to make $\delta_j \phi_j(\alpha) = |c_j \phi_j(\alpha)| \varepsilon$, hence for these values equality holds in the triangle inequality:

$$\left| \sum_{j=0}^n \delta_j \phi_j(\alpha) \right| = \varepsilon \sum_{j=0}^n |c_j \phi_j(\alpha)|. \quad (3.9)$$

Thus we get a root condition number for a polynomial given in Bernstein form

$$\kappa_\alpha = \frac{1}{|\alpha F'(\alpha)|} \sum_{j=0}^n |c_j \phi_j(\alpha)|. \quad (3.10)$$

that agrees with the common definition ([Farouki, 2008, Equation 12.33]) for any polynomial basis ϕ_j .

4. Conditioning of Bézier Curve Intersection

To define a condition number for the intersection of two planar Bézier curves, we write the difference as

$$F(s, t) = \begin{bmatrix} x_0(s) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y_0(s) \end{bmatrix} - \begin{bmatrix} x_1(t) \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ y_1(t) \end{bmatrix}. \quad (4.1)$$

We can show that there is a closed form for the condition number given by Definition 3.1, specialized to the 2-norm.

Theorem 4.1. Let $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ be a transversal intersection of two planar Bézier curves $b_0(s)$ and $b_1(t)$. Define the vectors

$$\tilde{F}_+ = \begin{bmatrix} \tilde{x}_0(\alpha) + \tilde{x}_1(\beta) \\ \tilde{y}_0(\alpha) + \tilde{y}_1(\beta) \end{bmatrix} \quad \text{and} \quad \tilde{F}_- = \begin{bmatrix} \tilde{x}_0(\alpha) + \tilde{x}_1(\beta) \\ -\tilde{y}_0(\alpha) - \tilde{y}_1(\beta) \end{bmatrix}. \quad (4.2)$$

Then the root condition number of the intersection is the greater of two vector norms:

$$\kappa_{\alpha, \beta} = \frac{1}{\sqrt{\alpha^2 + \beta^2}} \max \left\{ \left\| J(\alpha, \beta)^{-1} \tilde{F}_+ \right\|_2, \left\| J(\alpha, \beta)^{-1} \tilde{F}_- \right\|_2 \right\}. \quad (4.3)$$

Proof. Let the curve $b_0(s)$ be degree m and $b_1(t)$ be degree n . Then $F(s, t)$ can be written as $2(m+1)+2(n+1)$ terms:

$$F(s, t) = \sum_{i=0}^m c_i^{(1)} \begin{bmatrix} B_{i,m}(s) \\ 0 \end{bmatrix} + \sum_{i=0}^m c_i^{(2)} \begin{bmatrix} 0 \\ B_{i,m}(s) \end{bmatrix} + \sum_{j=0}^n c_j^{(3)} \begin{bmatrix} -B_{j,n}(t) \\ 0 \end{bmatrix} + \sum_{j=0}^n c_j^{(4)} \begin{bmatrix} 0 \\ -B_{j,n}(t) \end{bmatrix}. \quad (4.4)$$

Since $F(s, t) = b_0(s) - b_1(t)$ we have Jacobian $J(s, t) = \begin{bmatrix} b'_0(s) & -b'_1(t) \end{bmatrix}$. Since a transversal intersection, we have $\det J(\alpha, \beta) \neq 0$. In a perturbation, we replace each $c_j^{(k)}$ with a $\delta_j^{(k)}$ bounded by $|\delta_j^{(k)}| \leq \varepsilon |c_j^{(k)}|$. Writing $J^{-1} = \begin{bmatrix} \mathbf{v} & \mathbf{w} \end{bmatrix}$, we have

$$J(\alpha)^{-1} \sum_j \delta_j \phi_j(\alpha) = \left[\sum_{i=0}^m \delta_i^{(1)} B_{i,m}(\alpha) + \sum_{j=0}^n \delta_j^{(3)} B_{j,n}(\beta) \right] \mathbf{v}$$

$$+ \left[\sum_{i=0}^m \delta_i^{(2)} B_{i,m}(\alpha) + \sum_{j=0}^n \delta_j^{(4)} B_{j,n}(\beta) \right] \mathbf{w} = \nu_1 \mathbf{v} + \nu_2 \mathbf{w} \quad (4.5)$$

where

$$|\nu_1|/\varepsilon \leq \sum_{i=0}^m |c_i^{(1)}| B_{i,m}(\alpha) + \sum_{j=0}^n |c_j^{(3)}| B_{j,n}(\beta) = \tilde{x}_0(\alpha) + \tilde{x}_1(\beta) = \mu_1 \quad (4.6)$$

$$|\nu_2|/\varepsilon \leq \sum_{i=0}^m |c_i^{(2)}| B_{i,m}(\alpha) + \sum_{j=0}^n |c_j^{(4)}| B_{j,n}(\beta) = \tilde{y}_0(\alpha) + \tilde{y}_1(\beta) = \mu_2. \quad (4.7)$$

As in (3.8), the bound can be attained by choosing the sign and magnitude of each perturbation so that $\delta_j^{(k)} B_{j,d} = \varepsilon |c_j^{(k)}| B_{j,d}$. The factor of ε can be cancelled to give condition number

$$\kappa_{\alpha,\beta} = \frac{1}{\sqrt{\alpha^2 + \beta^2}} \sup_{|\nu_k| \leq \mu_k} \|\nu_1 \mathbf{v} + \nu_2 \mathbf{w}\|_2 \quad (4.8)$$

$$= \sqrt{\frac{\sup_{|\nu_k| \leq \mu_k} \nu_1^2 (\mathbf{v} \cdot \mathbf{v}) + 2\nu_1 \nu_2 (\mathbf{v} \cdot \mathbf{w}) + \nu_2^2 (\mathbf{w} \cdot \mathbf{w})}{\alpha^2 + \beta^2}}. \quad (4.9)$$

Now we seek to maximize the objective function $\theta(\nu_1, \nu_2) = \nu_1^2 (\mathbf{v} \cdot \mathbf{v}) + 2\nu_1 \nu_2 (\mathbf{v} \cdot \mathbf{w}) + \nu_2^2 (\mathbf{w} \cdot \mathbf{w})$ in the rectangle $[-\mu_1, \mu_1] \times [-\mu_2, \mu_2]$.

To find interior critical points, we solve the system $\theta_{\nu_1} = \theta_{\nu_2} = 0$:

$$\begin{bmatrix} 2\mathbf{v} \cdot \mathbf{v} & 2\mathbf{v} \cdot \mathbf{w} \\ 2\mathbf{v} \cdot \mathbf{w} & 2\mathbf{w} \cdot \mathbf{w} \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (4.10)$$

This system has the unique solution $\nu_1 = \nu_2 = 0$ unless $\|\mathbf{v}\|_2 \|\mathbf{w}\|_2 = |\mathbf{v} \cdot \mathbf{w}|$. By the Cauchy-Schwarz inequality, this can only occur if \mathbf{v} and \mathbf{w} are parallel; since J^{-1} is invertible, we know they are not. Hence $\theta(0, 0) = 0$ is the only interior critical point and it is the global minimum.

Along the boundary of the rectangle, we fix one of ν_1 or ν_2 and the resulting univariate function is an up-opening parabola. For example, fixing $\nu_2 = c$ gives $\theta(\nu_1, c) = \nu_1^2 (\mathbf{v} \cdot \mathbf{v}) + \nu_1 [2c (\mathbf{v} \cdot \mathbf{w})] + c^2 (\mathbf{w} \cdot \mathbf{w})$ which has positive lead coefficient $\|\mathbf{v}\|_2^2$. The lead coefficient cannot be 0 since if \mathbf{v} were the zero vector we'd have $\det J = 0$. Since θ is an up-opening parabola along the boundary, any critical point must be a local minimum.

Thus we know the maximum occurs at one of the four corners of the rectangle. Due to sign cancellation, this leads to one of two values $\theta = \mu_1^2 (\mathbf{v} \cdot \mathbf{v}) \pm 2\mu_1 \mu_2 (\mathbf{v} \cdot \mathbf{w}) + \mu_2^2 (\mathbf{w} \cdot \mathbf{w})$, the largest of which is $\mu_1^2 (\mathbf{v} \cdot \mathbf{v}) + 2\mu_1 \mu_2 |\mathbf{v} \cdot \mathbf{w}| + \mu_2^2 (\mathbf{w} \cdot \mathbf{w})$. Thus

$$\kappa_{\alpha,\beta} = \sqrt{\frac{\mu_1^2 (\mathbf{v} \cdot \mathbf{v}) + 2\mu_1 \mu_2 |\mathbf{v} \cdot \mathbf{w}| + \mu_2^2 (\mathbf{w} \cdot \mathbf{w})}{\alpha^2 + \beta^2}}. \quad (4.11)$$

Note that $\tilde{F}_{\pm} = \begin{bmatrix} \mu_1 \\ \pm \mu_2 \end{bmatrix}$, so

$$\|J(\alpha, \beta)^{-1} \tilde{F}_{\pm}\|_2 = \left\| \begin{bmatrix} \mathbf{v} & \mathbf{w} \end{bmatrix} \tilde{F}_{\pm} \right\|_2 = \sqrt{\mu_1^2 (\mathbf{v} \cdot \mathbf{v}) \pm 2\mu_1 \mu_2 (\mathbf{v} \cdot \mathbf{w}) + \mu_2^2 (\mathbf{w} \cdot \mathbf{w})} \quad (4.12)$$

gives each of the two values that could produce the maximum value. ■

5. Condition Number in Practice

5.1. Transversal Intersection

Consider the line $b_0(s) = \begin{bmatrix} 2s \\ 2s \end{bmatrix}$ and improperly parameterized line $b_1(t) = \begin{bmatrix} 4t^2 \\ 2 - 4t^2 \end{bmatrix}$ which intersect at $\alpha = \beta = 1/2$. At the intersection we have $J^{-1} = \frac{1}{8} \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$, so that $\mathbf{v} \cdot \mathbf{v} = \mathbf{w} \cdot \mathbf{w} = 5/64$ and $\mathbf{v} \cdot \mathbf{w} = 3/64$. Since the x -component of $F(s, t)$ can be written as $2s - 4t^2 = 2B_{1,1}(s) - 4B_{2,2}(t)$ and the y -component as $2s + 4t^2 - 2 = 2B_{1,1}(s) - 2B_{0,2}(t) - 2B_{1,2}(t) + 2B_{2,2}(t)$ we have

$$\mu_1 = 2B_{1,1}(\alpha) + 4B_{2,2}(\beta) = 2 \quad (5.1)$$

$$\mu_2 = 2B_{1,1}(\alpha) + 2B_{0,2}(\beta) + 2B_{1,2}(\beta) + 2B_{2,2}(\beta) = 3. \quad (5.2)$$

Following (4.11), this gives $\kappa_{\alpha,\beta} = \sqrt{202}/8 \approx 1.78$. This low condition number is expected from a geometric point of view; i.e. the intersection is a transversal intersection of two lines. However, when using the resultant to eliminate each parameter, one of the two roots is a double root:

$$\text{Res}_t(x_0(s) - x_1(t), y_0(s) - y_1(t)) = 64(2s - 1)^2 \quad (5.3)$$

$$\text{Res}_s(x_0(s) - x_1(t), y_0(s) - y_1(t)) = 4(2t - 1)(2t + 1). \quad (5.4)$$

so an algebraic approach may lead to an incorrect conclusion that the intersection is ill-conditioned.

5.2. Line-line Intersection with Poorly Behaved Coefficients

Consider the intersection of the lines $y = x$ and $y = 1 - x$ when $x \in [0, 1]$. These correspond to the Bézier curves

$$b_0(s) = \begin{bmatrix} s \\ s \end{bmatrix}, \quad b_1(t) = \begin{bmatrix} t \\ 1 - t \end{bmatrix}. \quad (5.5)$$

By adding a scalar $D > 0$ to each component, we leave $F(s, t)$ and hence the solution unchanged. However, the coefficients of the curves change:

$$b_0(s) = \begin{bmatrix} D(1-s) + (1+D)s \\ D(1-s) + (1+D)s \end{bmatrix}, \quad b_1(t) = \begin{bmatrix} D(1-t) + (1+D)t \\ (1+D)(1-t) + Dt \end{bmatrix}. \quad (5.6)$$

At the solution $\alpha = \beta = 1/2$, we have

$$\tilde{F}_{\pm} = \begin{bmatrix} 2D + 1 \\ \pm(2D + 1) \end{bmatrix} \quad (5.7)$$

and in either case $\|J^{-1}\tilde{F}_{\pm}\|_2 = 2D + 1$.

So, we see the condition number $\kappa_{\alpha,\beta} = \sqrt{2}(2D + 1)$ increases towards infinity as D does. This is what we expect as the coefficients grow so large that their ratio $(1 + D)/D$ approaches 1.

5.3. Family of Lines Approaching Coincident

Consider a family of intersections in which one of the lines approaches the other:

$$b_0(s) = \begin{bmatrix} s \\ 1 \end{bmatrix}, \quad b_1(t) = \begin{bmatrix} t \\ (1+r)(1-t) + t \end{bmatrix}. \quad (5.8)$$

These lines $y = 1$ and $rx + y = 1 + r$ intersect when $\alpha = \beta = 1$. However as $r \rightarrow 0^+$, the lines become coincident: if $r = 0$ the single intersection becomes infinitely many.

At the solution, we have $\tilde{F}_{\pm} = \begin{bmatrix} 2 & \pm 2 \end{bmatrix}^T$ and again have a condition number

$$\kappa_{\alpha,\beta} = \sqrt{\frac{4}{r^2} + \frac{4}{r} + 2} = \frac{2}{r} + 1 + \frac{r}{4} + \mathcal{O}(r^2). \quad (5.9)$$

that increases towards infinity as the parameter $r \rightarrow 0^+$.

6. Conclusion and Future Work

The author hopes that this can be useful for evaluating and comparing the performance of curve intersection implementations. By establishing a straightforward and easy to compute closed form, the condition number can be used more often to differentiate between cases where the algorithm or computer code is at fault for loss in accuracy and cases where the conditioning of the intersection itself is the cause. The framework set forth in Section 3 can be applied in future work to compute the condition number of higher order intersections such as surface-surface intersections in \mathbf{R}^3 .

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References

- Farin, G., 2001. Curves and Surfaces for CAGD, Fifth Edition: A Practical Guide (The Morgan Kaufmann Series in Computer Graphics). Morgan Kaufmann.
URL <https://www.amazon.com/Curves-Surfaces-CAGD-Fifth-Practical/dp/1558607374>
- Farouki, R., Rajan, V., Nov 1987. On the numerical condition of polynomials in Bernstein form. Computer Aided Geometric Design 4 (3), 191–216.
URL [https://dx.doi.org/10.1016/0167-8396\(87\)90012-4](https://dx.doi.org/10.1016/0167-8396(87)90012-4)
- Farouki, R. T., Feb 2008. Pythagorean-Hodograph Curves: Algebra and Geometry Inseparable. Springer.
URL <https://www.amazon.com/dp/B00FC7KYVG>
- Herman, A., Tsigaridas, E., 2015. Bounds for the condition number of polynomials systems with integer coefficients. In: Computer Algebra in Scientific Computing. Springer International Publishing, pp. 210–219.
URL https://doi.org/10.1007/978-3-319-24021-3_16
- Higham, N. J., Jan 2002. Accuracy and Stability of Numerical Algorithms. Society for Industrial and Applied Mathematics.
URL <https://doi.org/10.1137/1.9780898718027>
- Kim, D.-S., Lee, S.-W., Shin, H., Nov 1998. A cocktail algorithm for planar Bézier curve intersections. Computer-Aided Design 30 (13), 1047–1051.
URL [https://dx.doi.org/10.1016/S0010-4485\(98\)00052-9](https://dx.doi.org/10.1016/S0010-4485(98)00052-9)
- Manocha, D., Demmel, J. W., Aug 1992. Algorithms for Intersecting Parametric and Algebraic Curves. Tech. Rep. UCB/CSD-92-698, EECS Department, University of California, Berkeley.
URL <https://www2.eecs.berkeley.edu/Pubs/TechRpts/1992/6266.html>
- Schulz, C., Jan 2009. Bézier clipping is quadratically convergent. Computer Aided Geometric Design 26 (1), 61–74.
URL <https://doi.org/10.1016/j.cagd.2007.12.006>
- Sederberg, T., Nishita, T., Nov 1990. Curve intersection using Bézier clipping. Computer-Aided Design 22 (9), 538–549.
URL [https://dx.doi.org/10.1016/0010-4485\(90\)90039-F](https://dx.doi.org/10.1016/0010-4485(90)90039-F)
- Sederberg, T. W., Sep 2016. Lecture notes: Computer aided geometric design.
URL <http://tom.cs.byu.edu/~557/text/cagd.pdf>
- Sederberg, T. W., Parry, S. R., Jan 1986. Comparison of three curve intersection algorithms. Computer-Aided Design 18 (1), 58–63.
URL [https://dx.doi.org/10.1016/S0010-4485\(86\)80013-6](https://dx.doi.org/10.1016/S0010-4485(86)80013-6)