High-order Solution Transfer between Curved Meshes and Ill-conditioned Bézier Curve Intersection

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Outline

1. Introduction and motivation
2. Curved Elements
3. Solution Transfer
4. Compensated Evaluation
5. Modified Newton's for Intersection

Introduction and motivation

Method of Characteristics

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Solve simple transport equation

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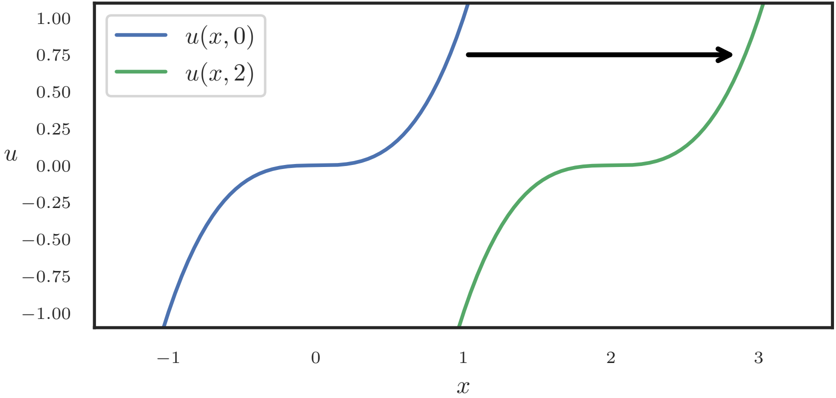
PDE becomes a (trivial) ODE

$$\frac{d}{dt}u(x(t),t) = 0.$$

Method of Characteristics



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- Transform PDE to family of ODEs

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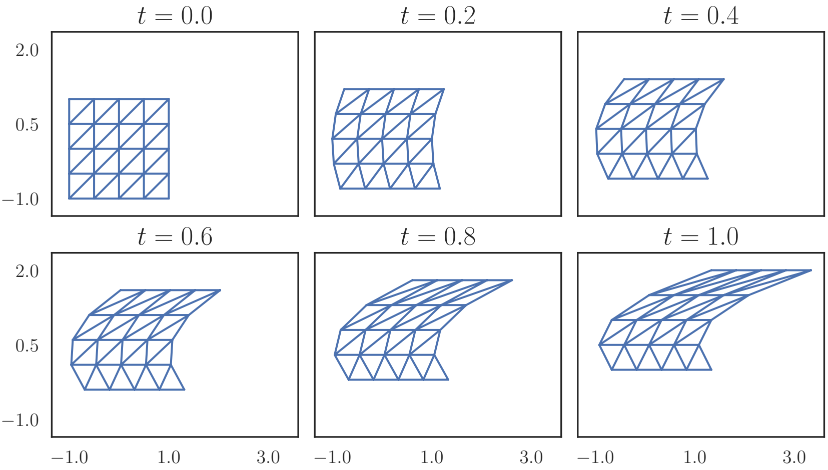
with cubic characteristics

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} (y_0 + t)^3 - y_0^3 \\ 3t \end{bmatrix}.$$

Remeshing Example



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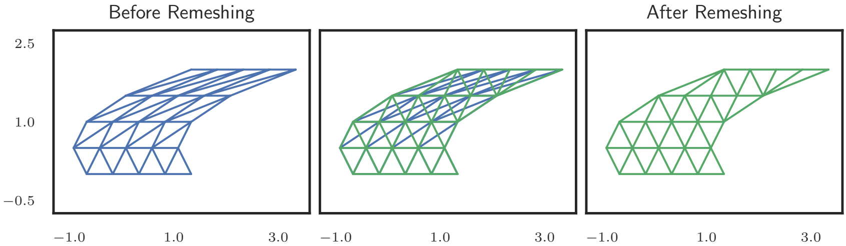

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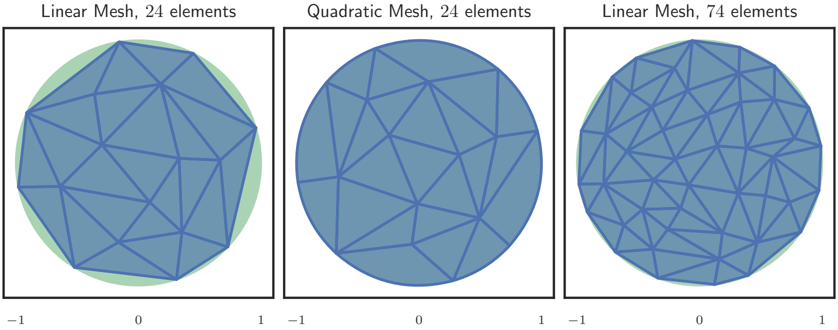

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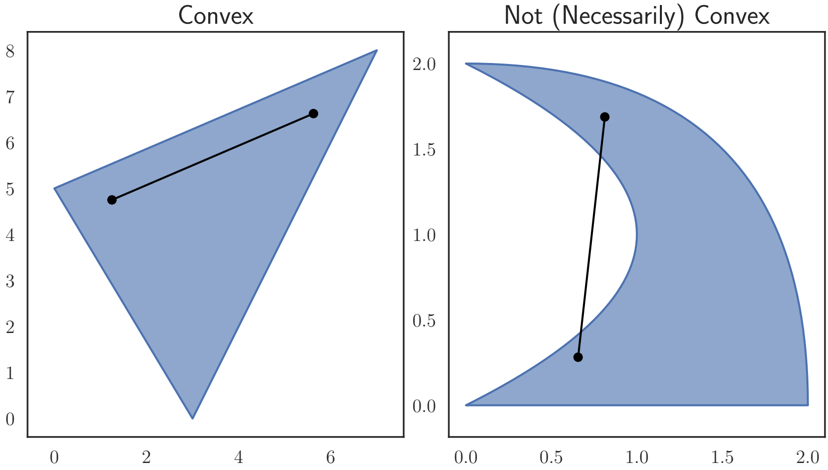
    - More challenging geometry

Curved Meshes



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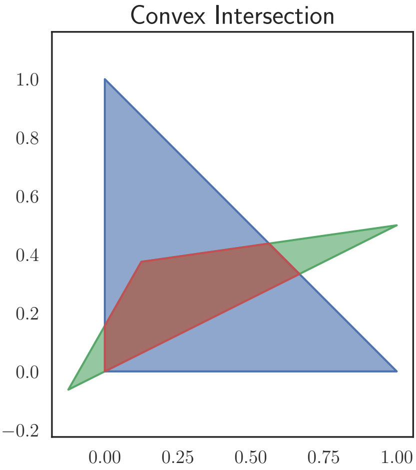

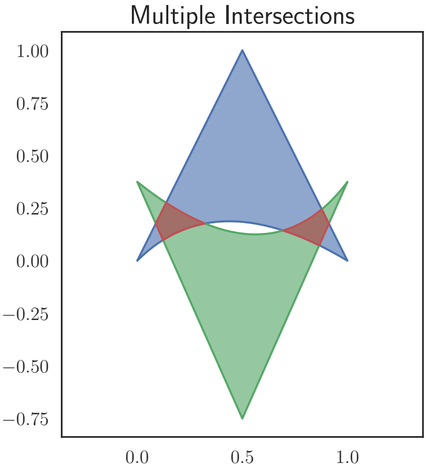

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- Lagrangian method must either curve mesh or information about flow of geometry will be lost

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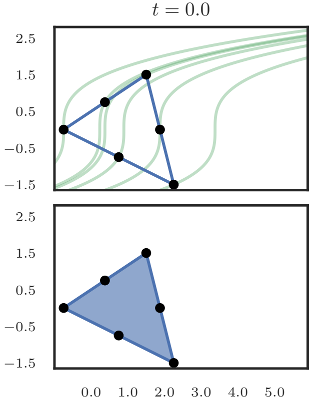

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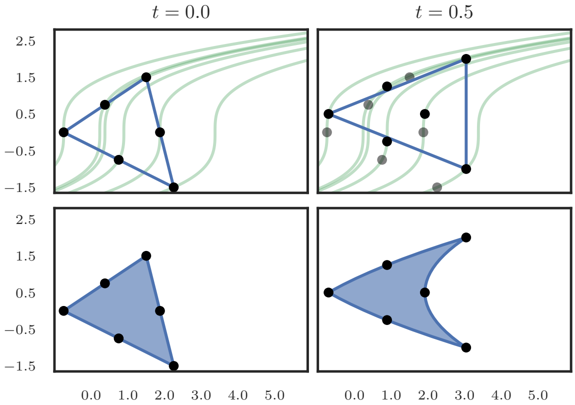

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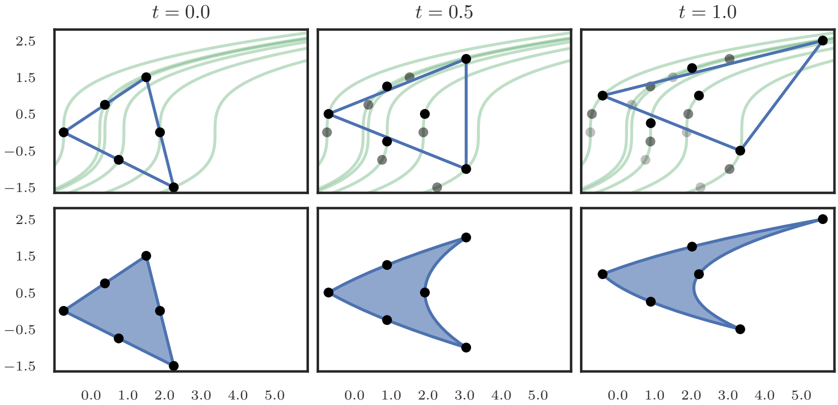

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- Convex combination of control points

$$b(s,t) = \sum_{\substack{i+j+k=n\\ i,j,k\geq 0}} \binom{n}{i,\,j,\,k} \lambda_1^i \lambda_2^j \lambda_3^k \, \boldsymbol{p}_{i,j,k}$$

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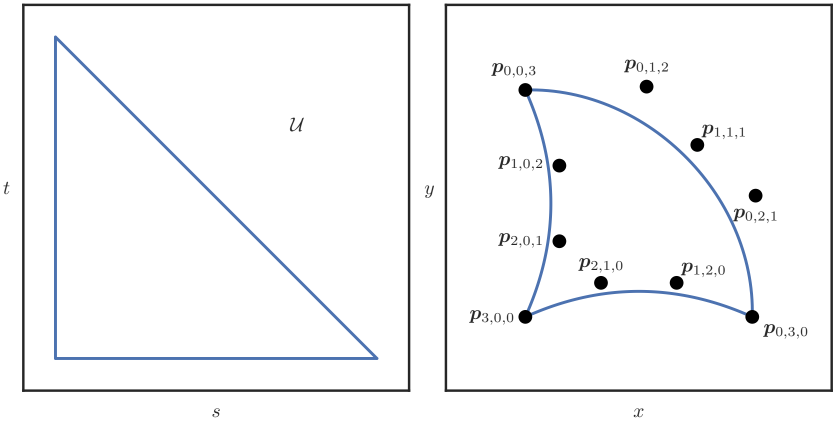

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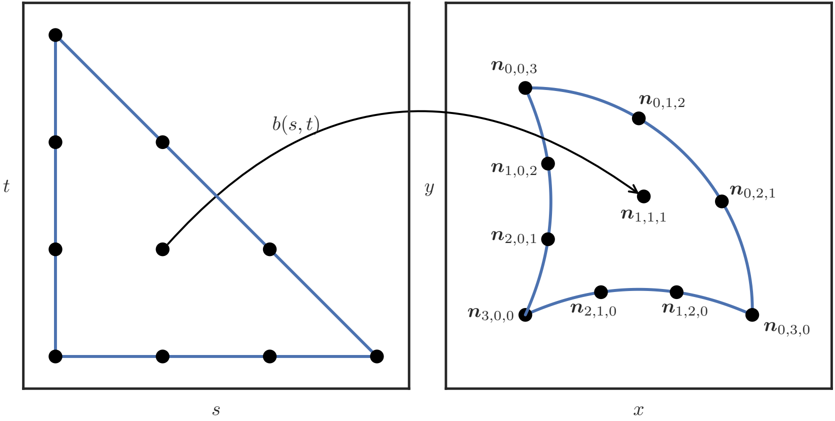

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- Conversion between $\boldsymbol{n}_{i,j,k}$ and $\boldsymbol{p}_{i,j,k}$ has condition number exponential in n

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- $\det(Db)$ positive, preserves orientation

Inverted Element

Consider element given by map

$$b(s,t) = \lambda_1^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_2^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \lambda_3^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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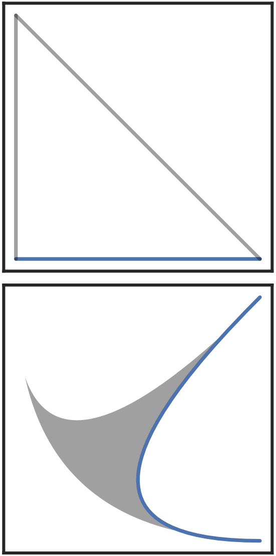

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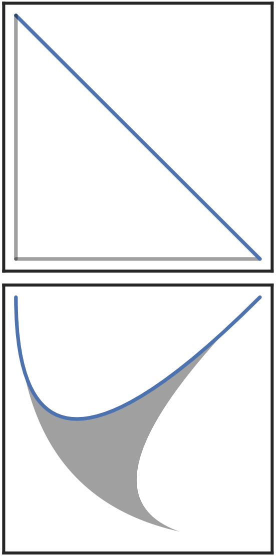

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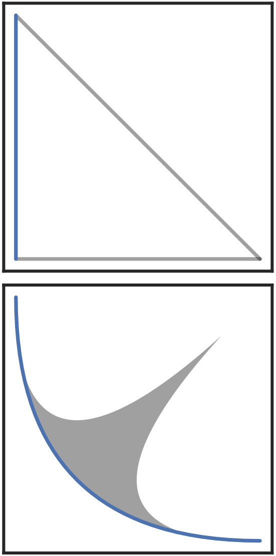

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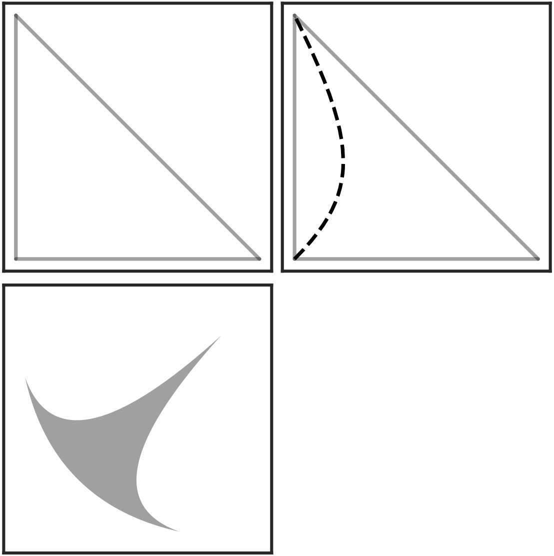

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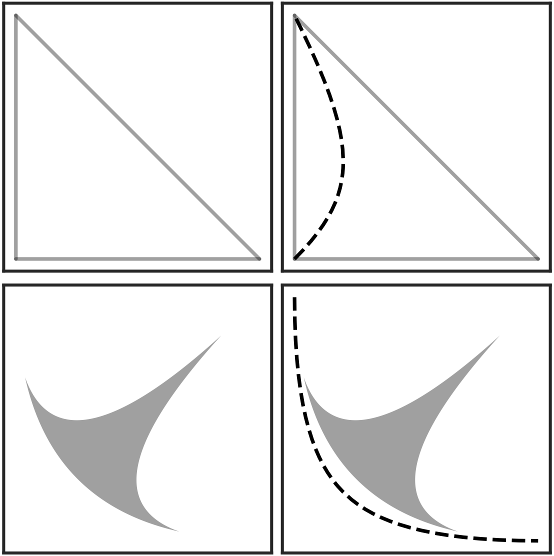

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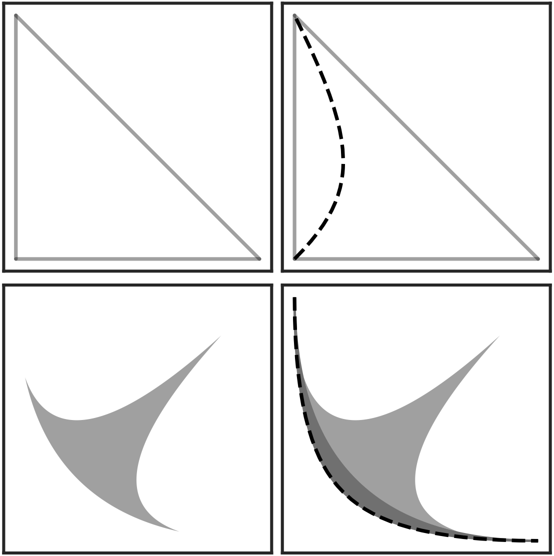

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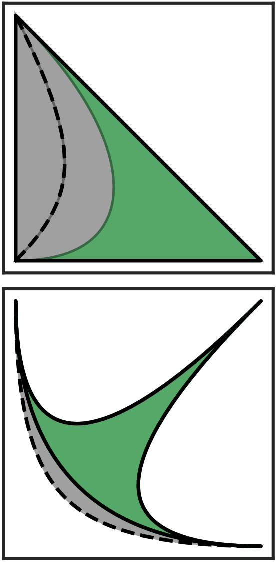

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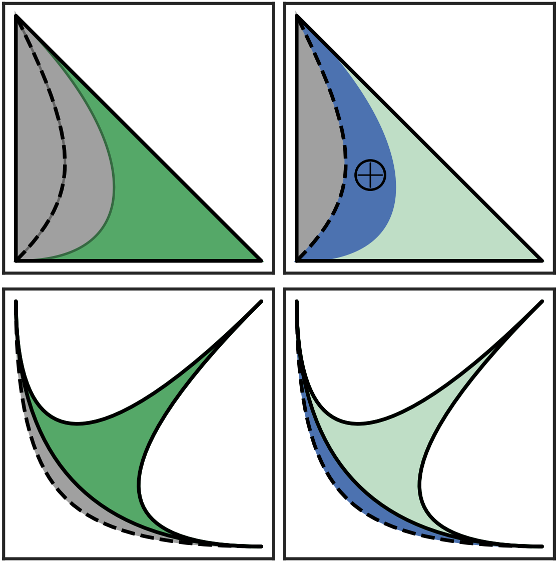

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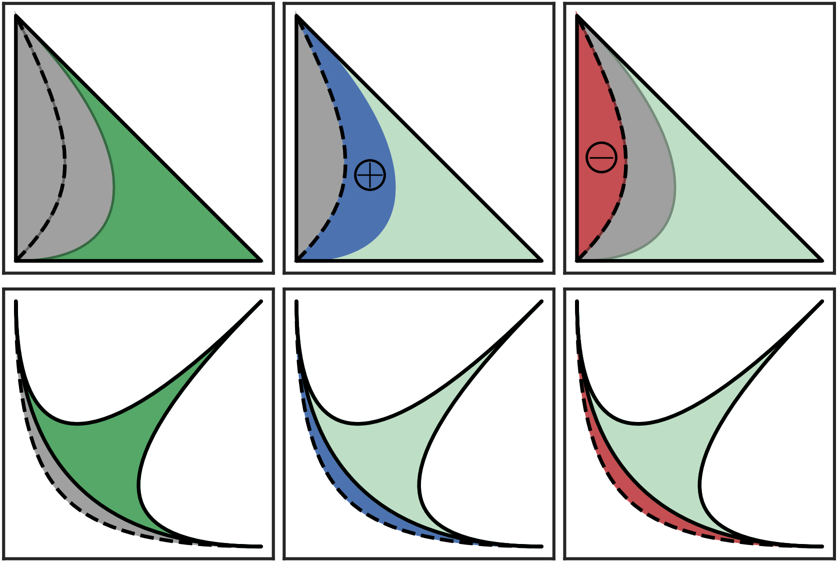

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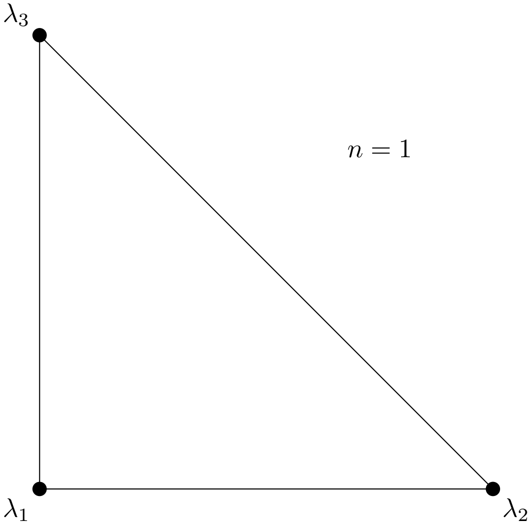

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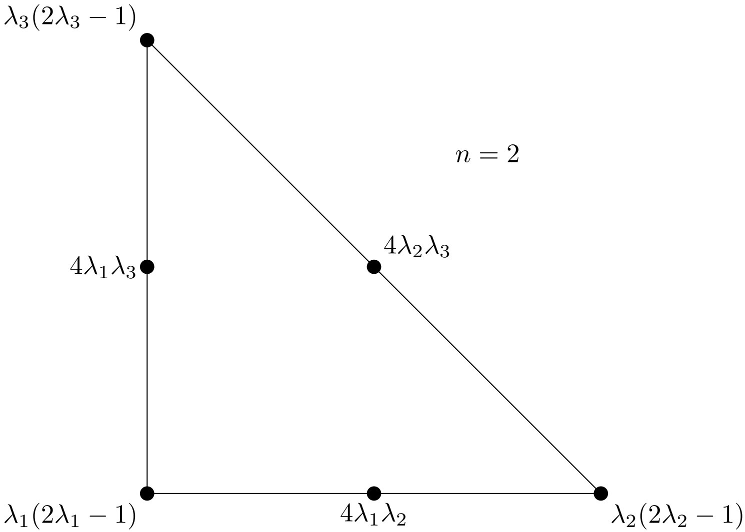

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Solution Transfer

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  - Known discrete field $\boldsymbol{q}_D = \sum_j d_j \phi_D^{(j)}$
- **Want**: L_2-optimal interpolant $\boldsymbol{q}_T = \sum_j t_j \phi_T^{(j)}$:

$$\|\boldsymbol{q}_T - \boldsymbol{q}_D\|_2 = \min_{\boldsymbol{q} \in \mathcal{V}_T} \|\boldsymbol{q} - \boldsymbol{q}_D\|_2$$

Galerkin Projection

Differentiating w.r.t. each t_j in $\boldsymbol{q}_T = \sum_j t_j \phi_T^{(j)}$ gives **weak form**

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If $(\boldsymbol{x} \mapsto 1) \in \mathcal{V}_T$, then $\boldsymbol{q}_T$ is globally **conservative**

$$\int_\Omega \boldsymbol{q}_D \, dV = \int_\Omega \boldsymbol{q}_T \, dV.$$

Linear System

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M_T is (symmetric) mass matrix for target mesh

$$(M_T)_{ij} = \int_\Omega \phi_T^{(i)} \phi_T^{(j)} \, dV.$$

Linear System

Each shape function ϕ has $\mathrm{supp}(\phi) = \mathcal{T}$ for some (curved) element, hence M_T is block diagonal in DG, sparse but globally coupled in CG.

Compensated Evaluation

Modified Newton's for Intersection