High-order Solution Transfer between Curved Meshes and Ill-conditioned Bézier Curve Intersection

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Outline

1. Introduction and motivation
2. Curved Elements
3. Solution Transfer
4. Ill-conditioned Bézier Curve Intersection
5. Compensated Evaluation
6. Modified Newton's for Intersection

Introduction and motivation

Method of Characteristics

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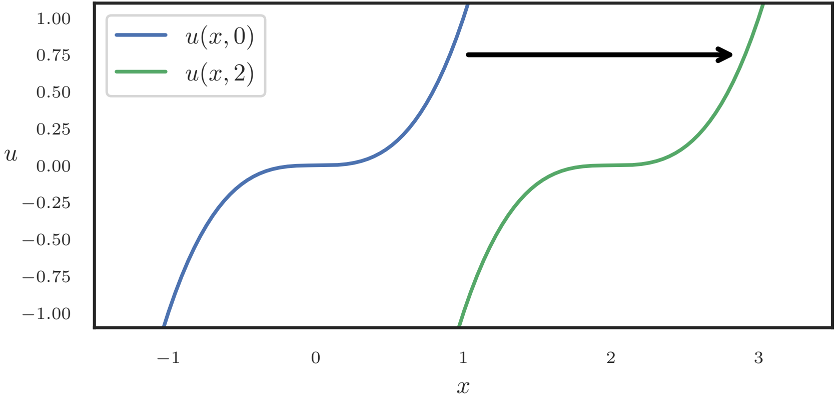
PDE becomes a (trivial) ODE

$$\frac{d}{dt} u(x(t), t) = 0.$$

Method of Characteristics



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- Transform PDE to family of ODEs

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    - Distortion

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Remeshing Example

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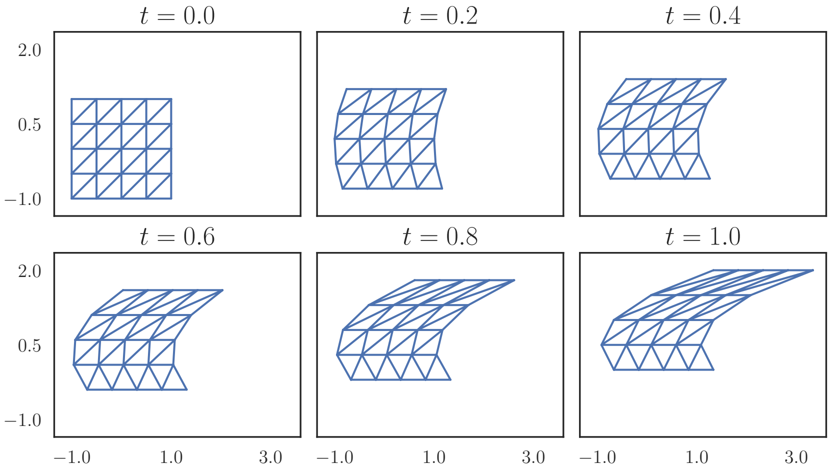
with cubic characteristics

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} (y_0 + t)^3 - y_0^3 \\ 3t \end{bmatrix}.$$

Remeshing Example



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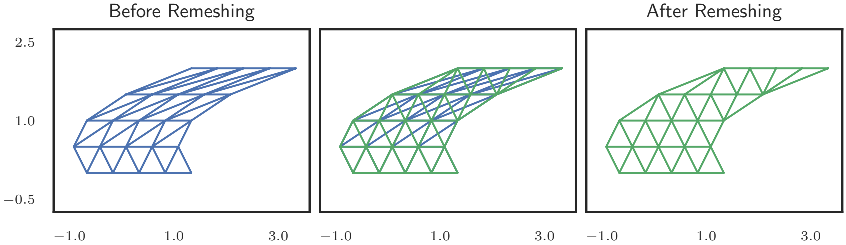

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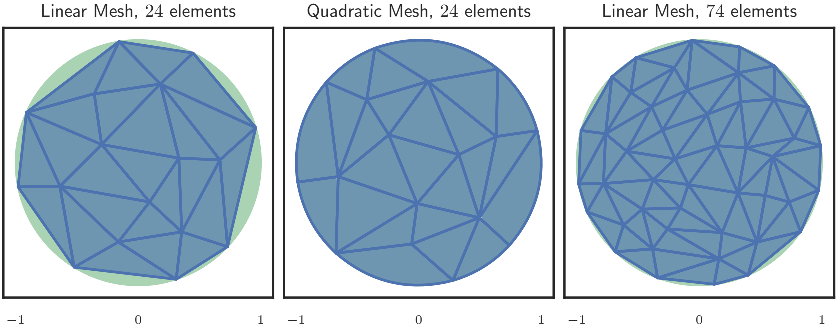

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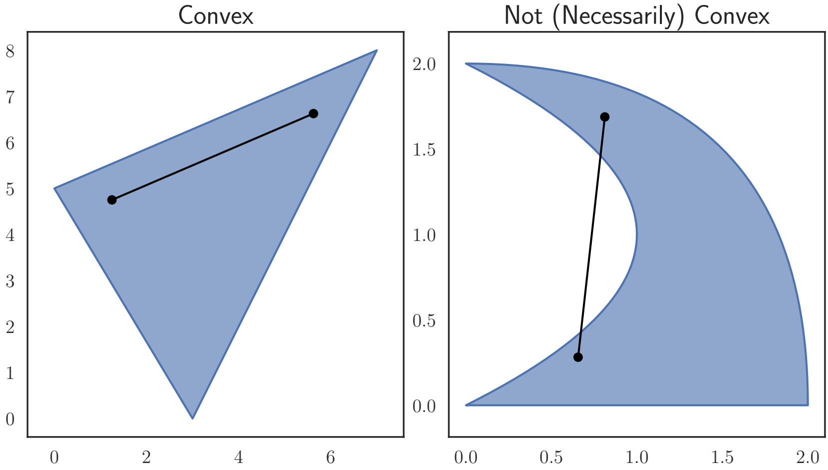

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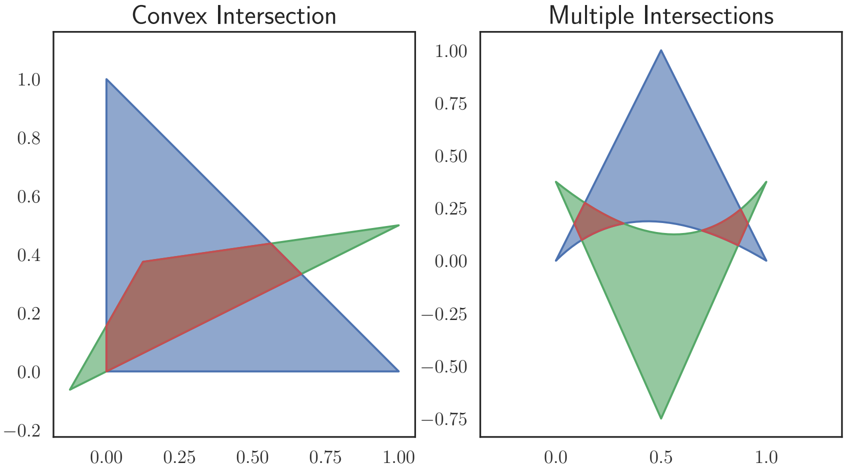

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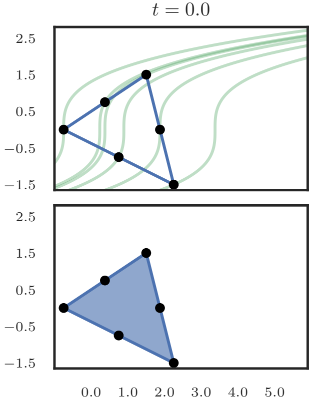
- Lagrangian method must either curve mesh or information about flow of geometry will be lost

Curved Elements: Necessary for High-order



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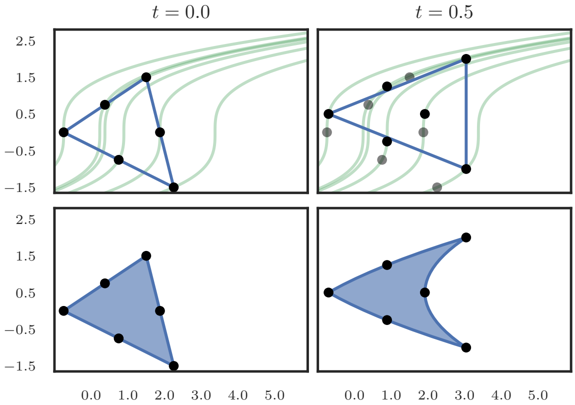

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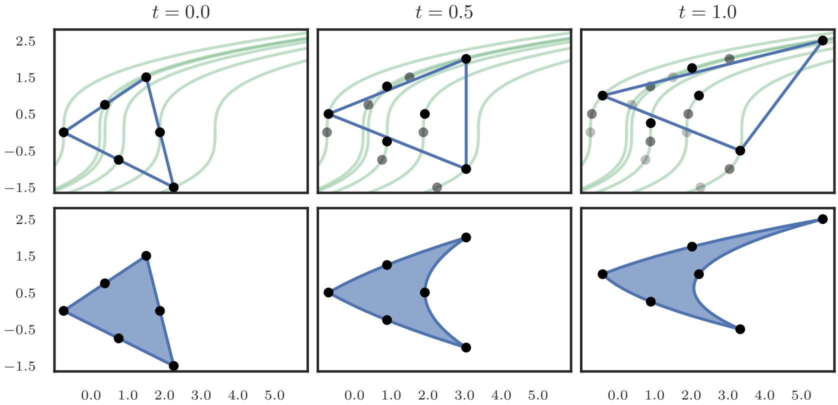

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- Convex combination of control points

$$b(s, t) = \sum_{\substack{i+j+k=n \\ i,j,k \geq 0}} \binom{n}{i, j, k} \lambda_1^i \lambda_2^j \lambda_3^k \, \boldsymbol{p}_{i,j,k}$$

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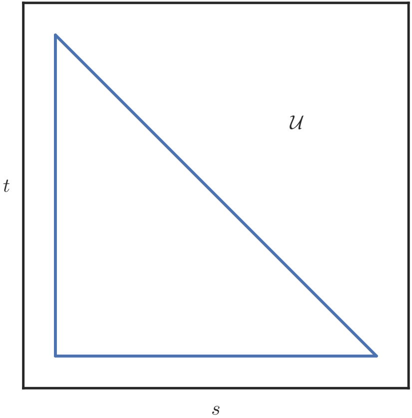

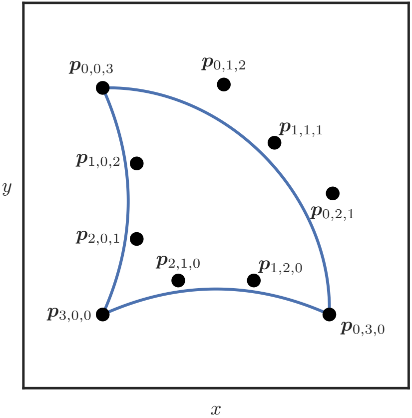

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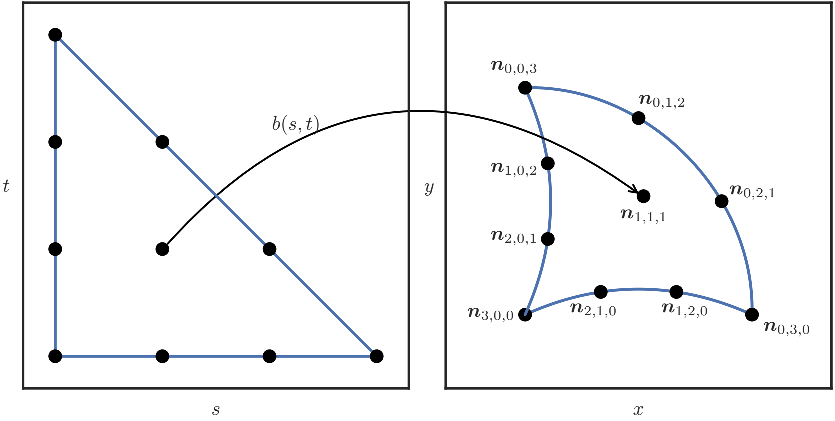

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- Conversion between $\boldsymbol{n}_{i,j,k}$ and $\boldsymbol{p}_{i,j,k}$ has condition number exponential in n

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- $\det(Db)$ positive, preserves orientation

Inverted Element

Consider element given by map

$$b(s,t) = \lambda_1^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_2^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \lambda_3^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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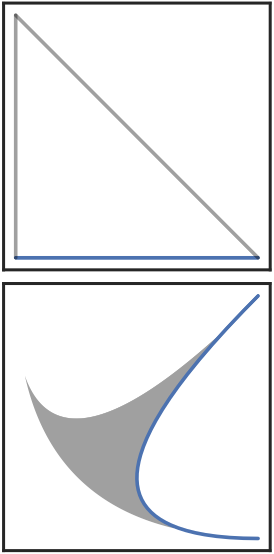

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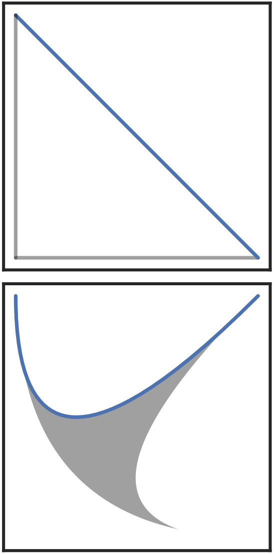

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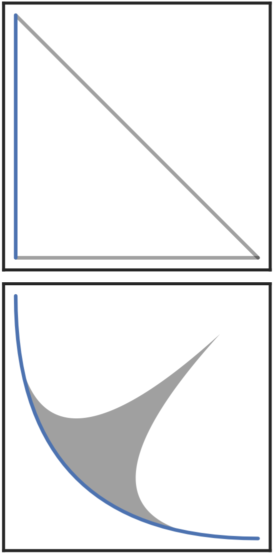

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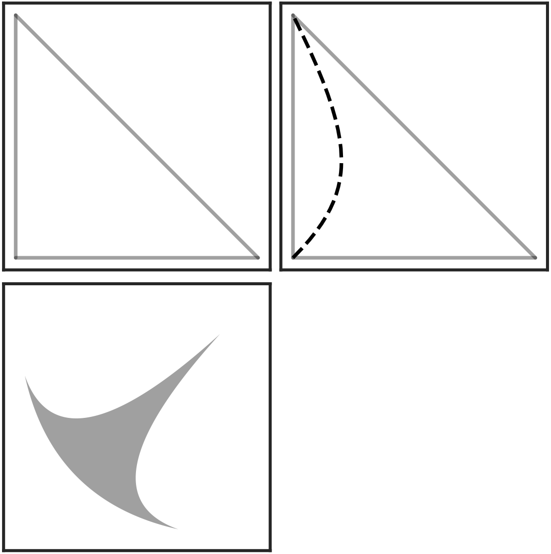

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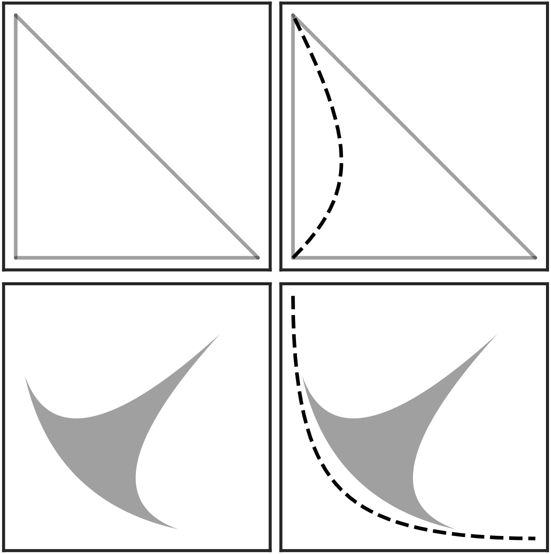

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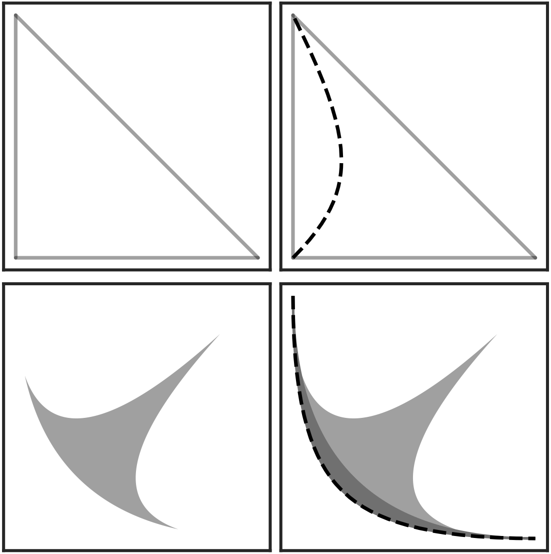

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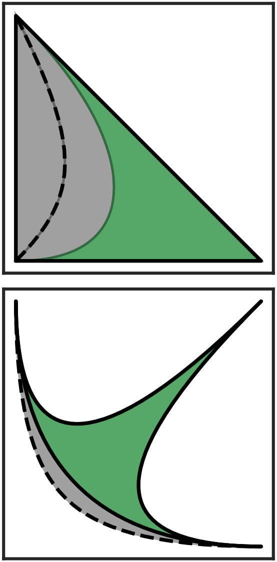

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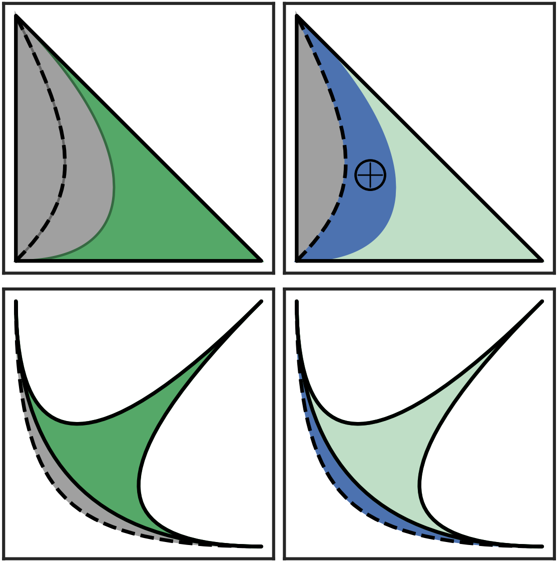

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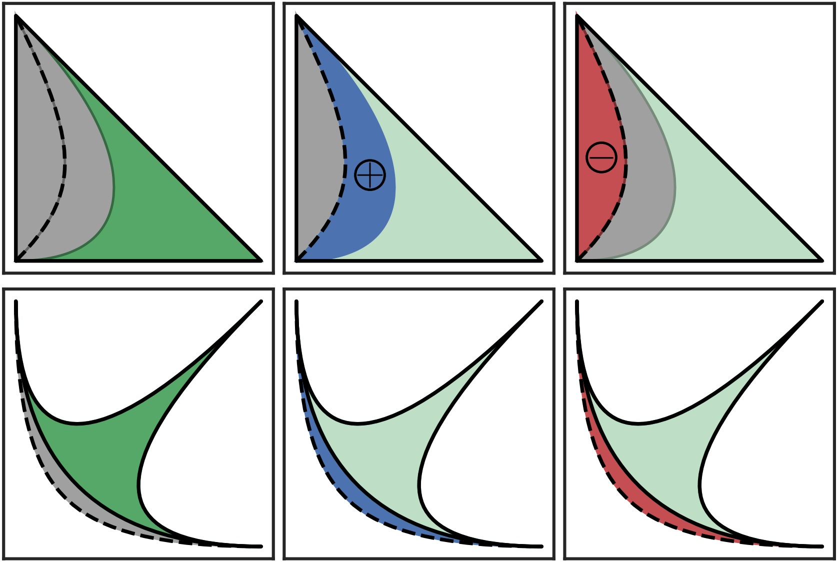

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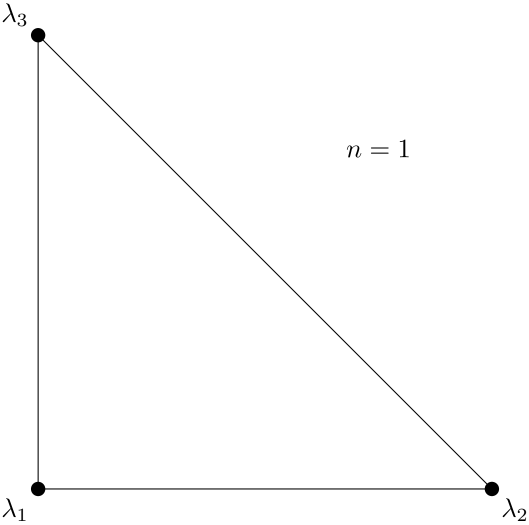

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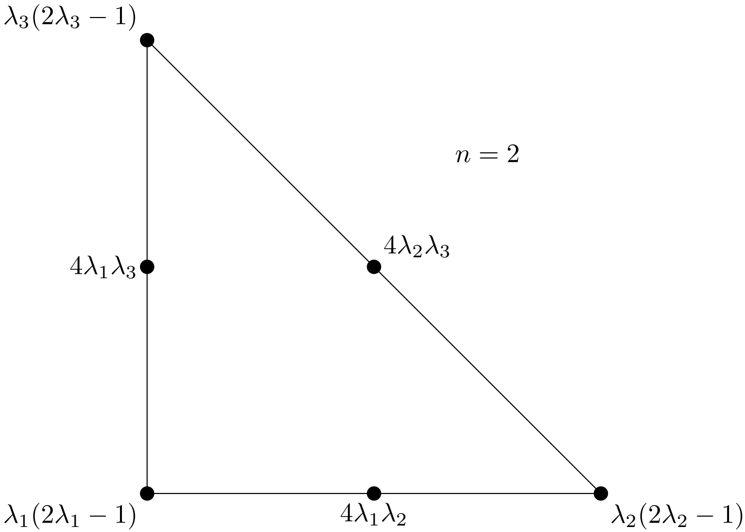

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Solution Transfer

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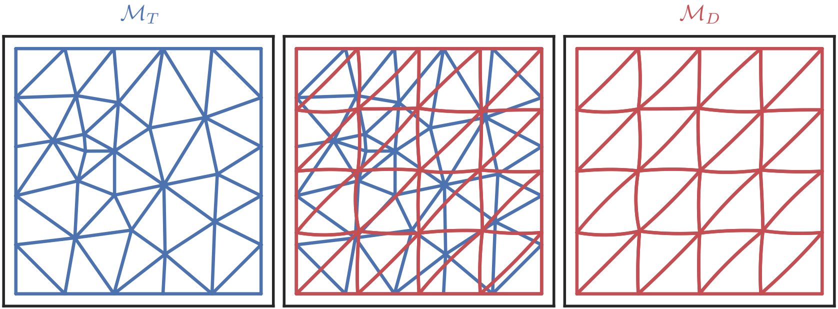
  - Known discrete field $\boldsymbol{q}_D = \sum_j d_j \phi_D^{(j)}$
- **Want**: L_2-optimal interpolant $\boldsymbol{q}_T = \sum_j t_j \phi_T^{(j)}$:

$$\|\boldsymbol{q}_T - \boldsymbol{q}_D\|_2 = \min_{\boldsymbol{q} \in \mathcal{V}_T} \|\boldsymbol{q} - \boldsymbol{q}_D\|_2$$

Galerkin Projection



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Differentiating w.r.t. each t_j in $\boldsymbol{q}_T = \sum_j t_j \phi_T^{(j)}$ gives **weak form**

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If $(\boldsymbol{x} \mapsto 1) \in \mathcal{V}_T$, then $\boldsymbol{q}_T$ is globally **conservative**

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Linear System

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$$M_T \boldsymbol{t} = M_{TD} \boldsymbol{d}.$$

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Integrate via substitution for $F = \phi_T^{(i)} \phi_T^{(j)}$

$$\int_{b(\mathcal{U})} F(x, y) \, dx \, dy = \int_{\mathcal{U}} \det(Db) F\left(x(s, t), y(s, t)\right) \, ds \, dt$$

and then use quadrature rule on $\mathcal{U}$.

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Instead, compute entire RHS

$$(M_{TD}\boldsymbol{d})_j = \int_\Omega \phi_T^{(j)} \boldsymbol{q}_D \, dV.$$

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Given ϕ supported on $\mathcal{T}$

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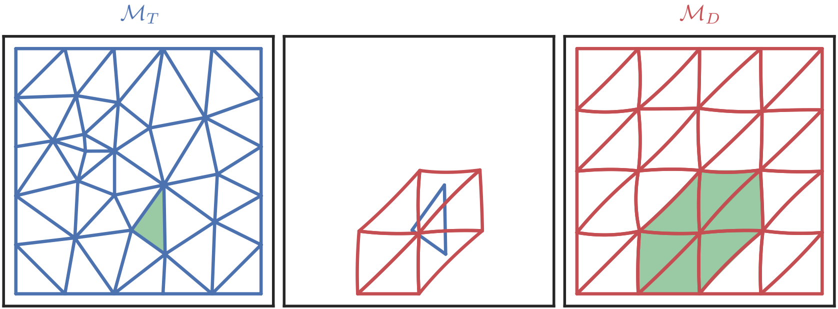
In CG, $\boldsymbol{q}_D$ need not be differentiable across elements and in DG $\boldsymbol{q}_D$ need not even be continuous

Common Refinement



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    - Intersecting Curved Elements: $\mathcal{T} \cap \mathcal{T}'$
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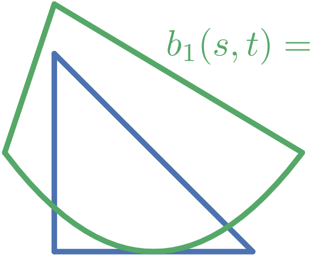
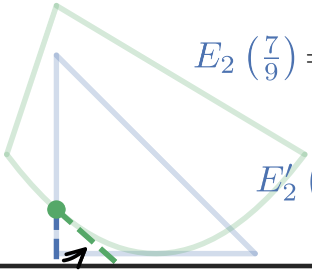
Intersecting Curved Elements



$$b_1(s,t) = \begin{bmatrix} 2(6s + t - 1) \\ 2(8s^2 + 8st - 8s + 3t + 2) \end{bmatrix}$$

$$b_0(s,t) = \begin{bmatrix} 8s \\ 8t \end{bmatrix}$$

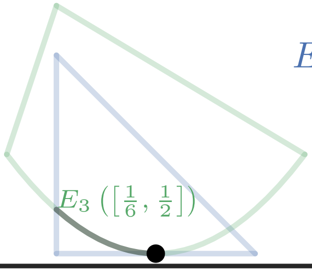
Intersecting Curved Elements



$$E_2\left(\tfrac{7}{9}\right) = E_3\left(\tfrac{1}{6}\right) = \frac{16}{9}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$E_2'\left(\tfrac{7}{9}\right) \times E_3'\left(\tfrac{1}{6}\right) = 96 > 0$$

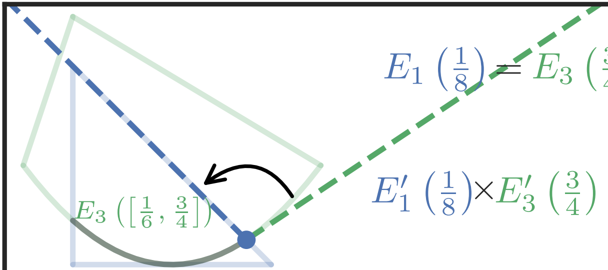
Intersecting Curved Elements



$$E_0\left(\frac{1}{2}\right) = E_3\left(\frac{1}{2}\right) = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$E_0'\left(\frac{1}{2}\right) \times E_3'\left(\frac{1}{2}\right) = 0$$

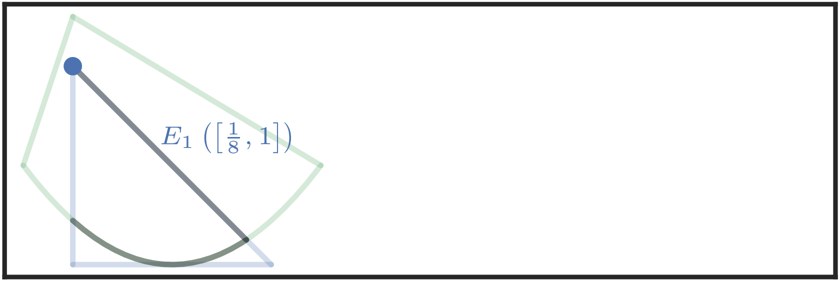
Intersecting Curved Elements



$$E_1\left(\frac{1}{8}\right) = E_3\left(\frac{3}{4}\right) = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

$$E_1'\left(\frac{1}{8}\right) \times E_3'\left(\frac{3}{4}\right) = -160 < 0$$

Intersecting Curved Elements



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- $\partial \mathcal{P} = E_3\left(\left[\frac{1}{6}, \frac{3}{4}\right]\right) \cup E_1\left(\left[\frac{1}{8}, 1\right]\right) \cup E_2\left(\left[0, \frac{7}{9}\right]\right)$

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- Find all pairs $\mathcal{T}, \mathcal{T}'$ of intersecting target and donor elements

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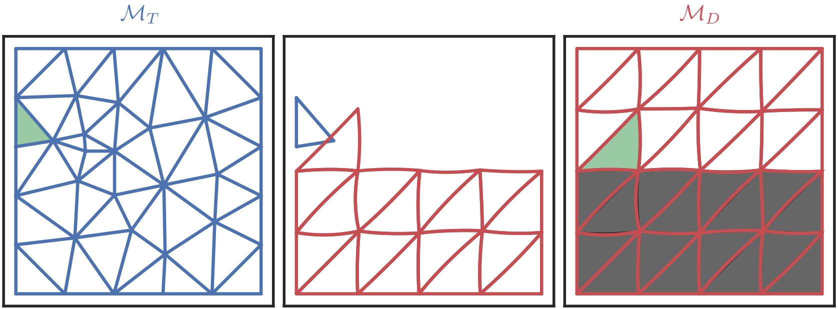

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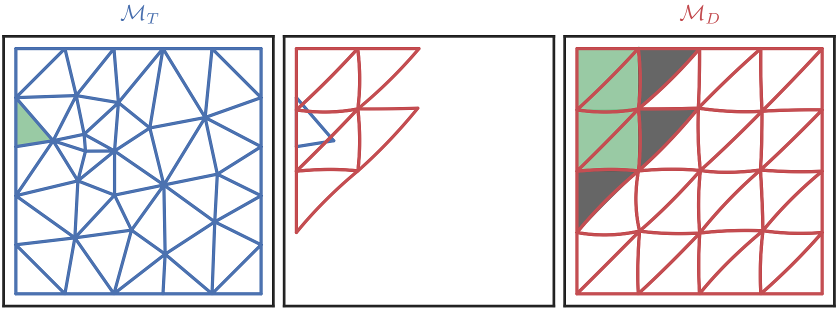

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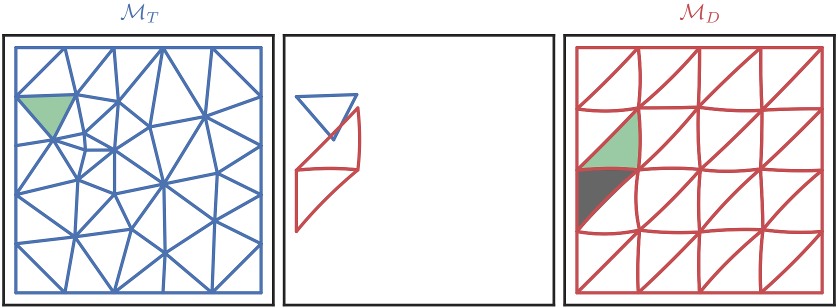

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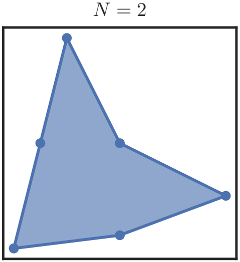
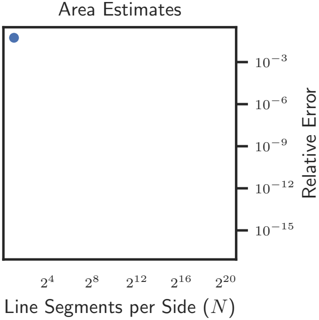
- Total $\mathcal{O}\left(|\mathcal{M}_D| + |\mathcal{M}_T|\right)$

Integration over Curved Polygons

$$\int_{\mathcal{P}} F(x, y)\, dV, \qquad \mathcal{P} = \mathcal{T}_0 \cap \mathcal{T}_1, \qquad F = \phi_0\, \phi_1$$

Integrate via Polygonal Approximation





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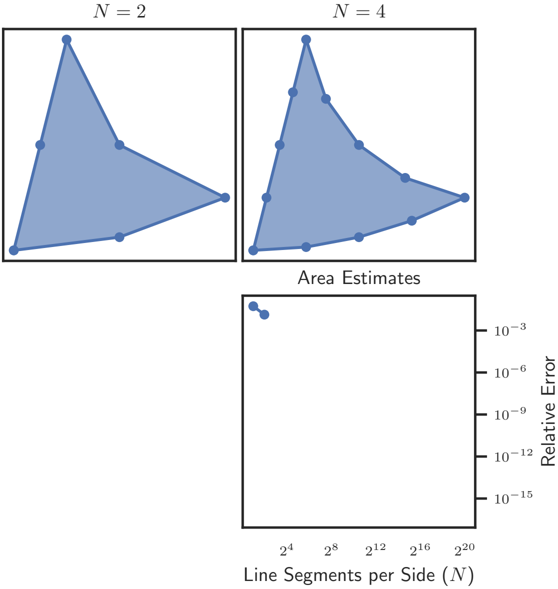

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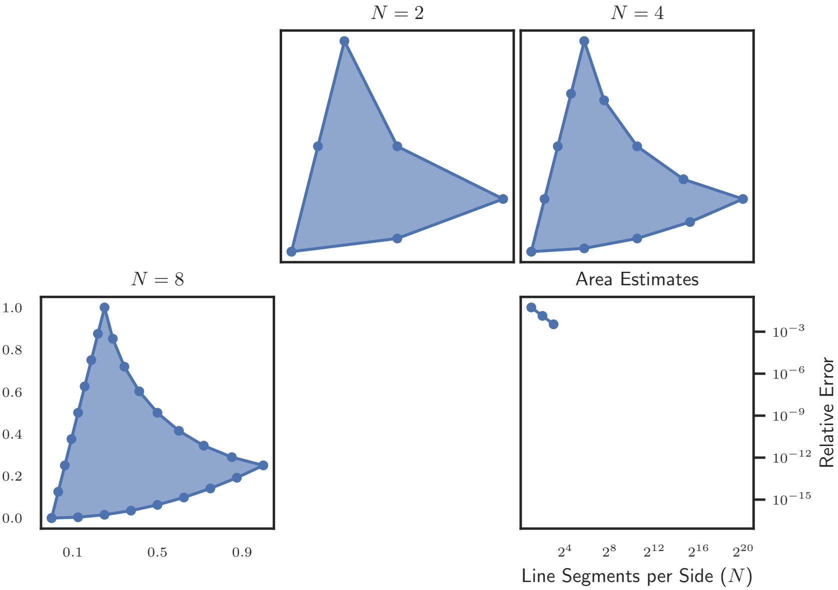

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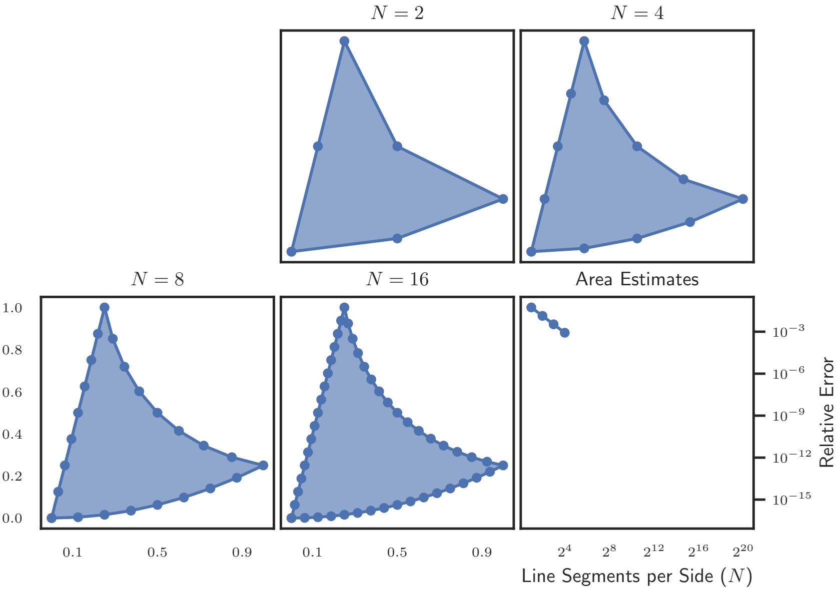

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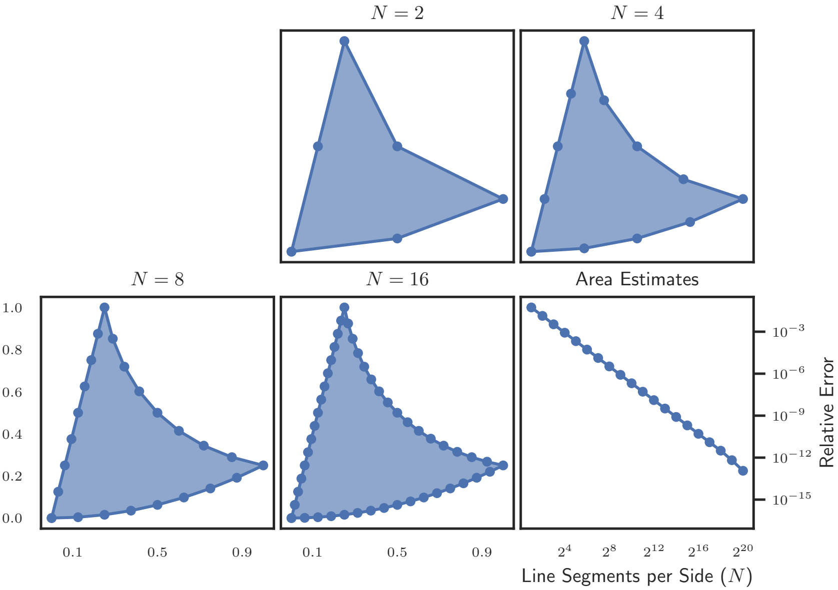

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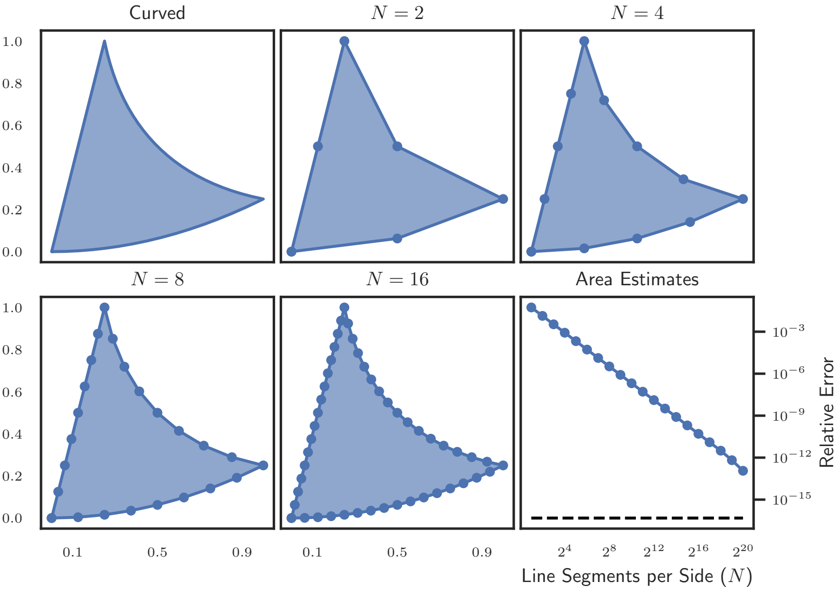

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  - Maps from (straight sided) reference domain, but increase degree or are not bijective

Integrate via Tessellation

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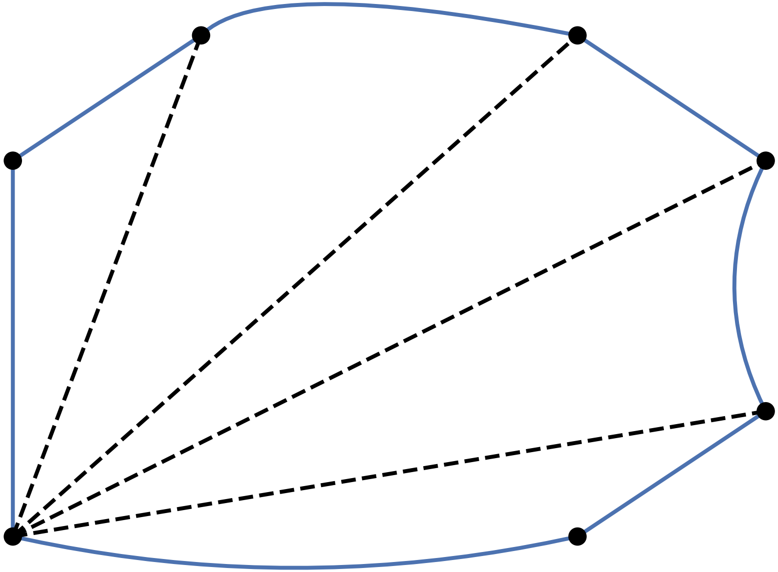
- Tessellate $\mathcal{P}$ into disjoint union of Bézier triangles

Integrate via Tessellation



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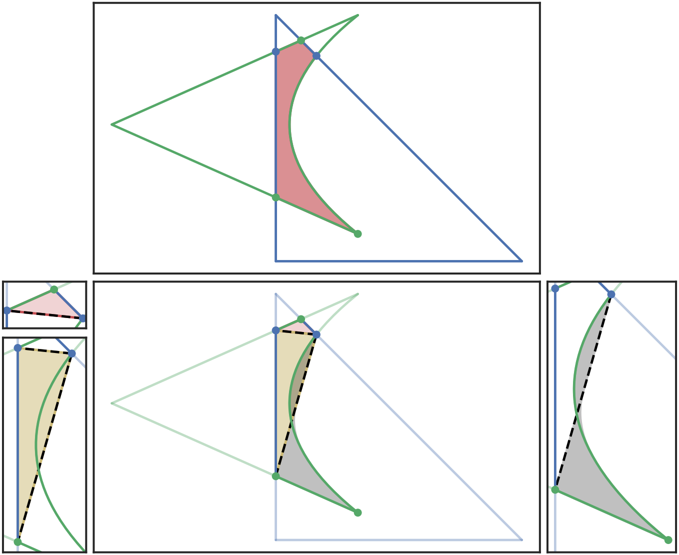
- If diagonals are valid, high degree Bézier triangles need interior control points introduced that don't cause triangle to invert

Integrate via Tessellation

- Tessellate with inverted Bézier triangles?

Integrate via Tessellation



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- $\mathcal{T}_2$ inverted, $b(s, t) \notin \mathcal{P}$ is possible but b_0^{-1} need not need be defined outside of $\mathcal{T}_0$

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- Each C given by $x(r), y(r)$, 1D quadrature on unit interval of

$$G(r) = H(x(r), y(r))\, y'(r) - V(x(r), y(r))\, x'(r)$$

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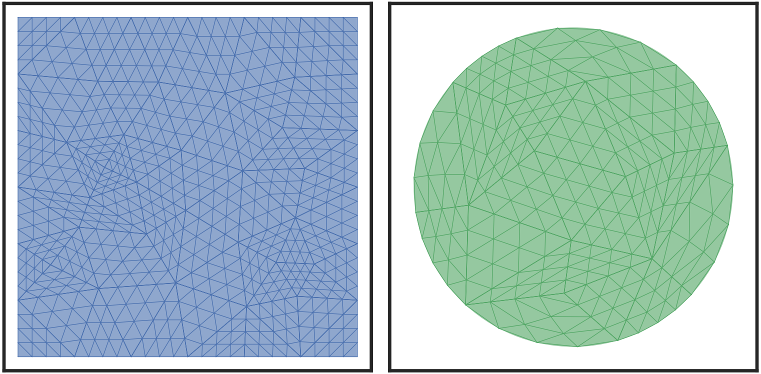
- Global coordinates basis means F is a polynomial on $\mathbf{R}^2$, so quadrature is exact

Numerical Experiments

- Three meshes ($p = 1, 2, 3$)

Numerical Experiments



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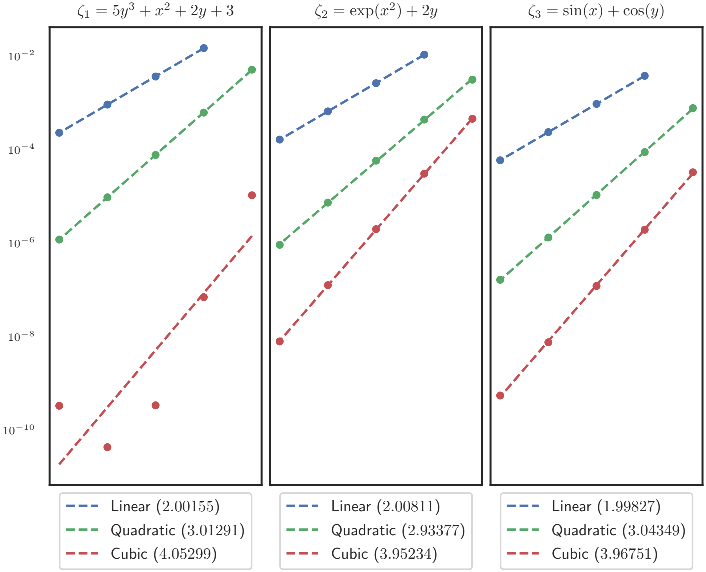
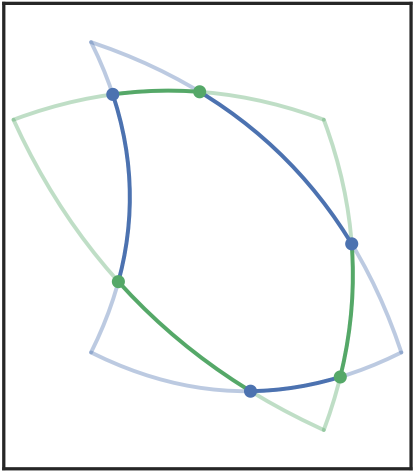
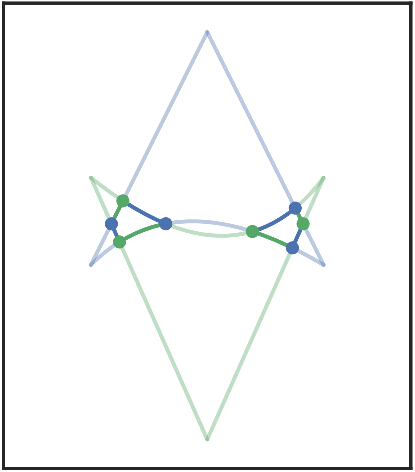
Numerical Experiments



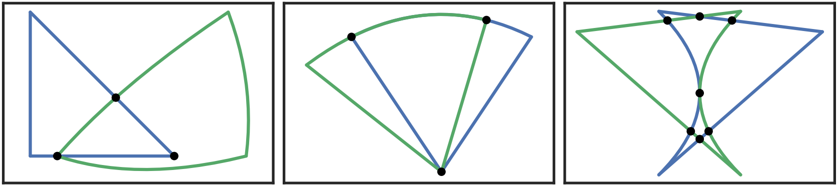
Ill-conditioned Bézier Curve Intersection

Ill-Conditioned Intersections





Ill-Conditioned Intersections



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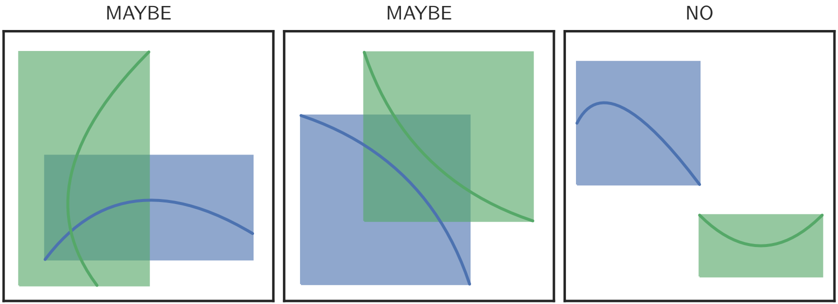
- Random pair of meshes, "almost tangent" intersections increasingly frequent as $h \longrightarrow 0^+$

Intersection Algorithm

- Bounding box check

Intersection Algorithm



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Intersection Algorithm

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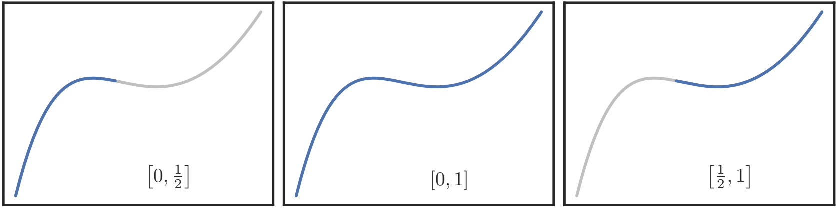
- Curve subdivision

Intersection Algorithm



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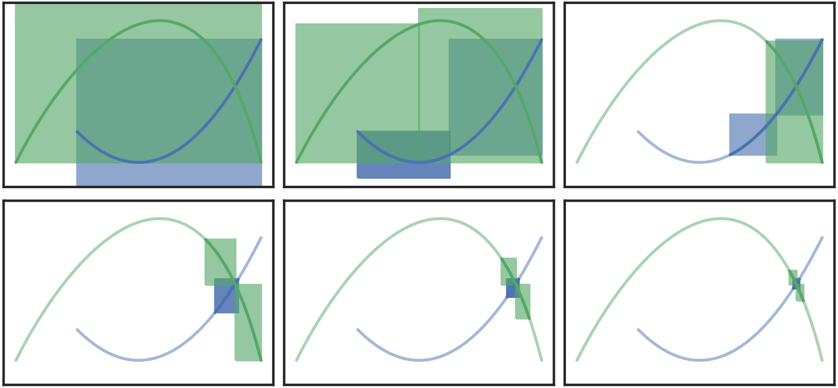
- Curve subdivision
- Check subdivided pairs

Intersection Algorithm



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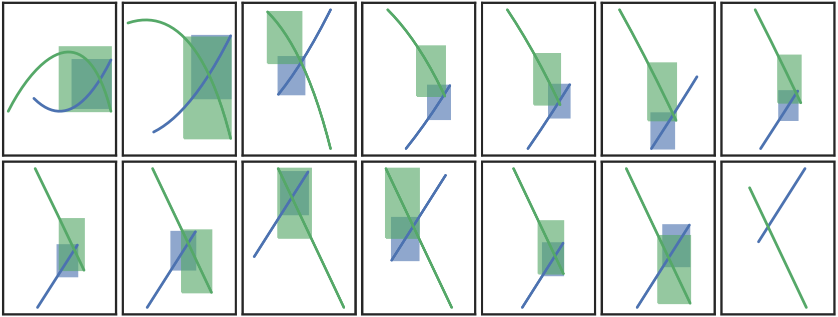
- Curve subdivision
- Check subdivided pairs
- Subdivide until “almost” linear

Intersection Algorithm



Intersection Algorithm

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Intersection Algorithm

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- Curve subdivision
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- Use intersection of lines as seed for Newton's method

Newton's Method

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- At tangent intersections, J is exactly singular
- At ill-conditioned intersections, J is almost singular and evaluation of F is typically ill-conditioned as well

Modified Newton's Method

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  - Tisseur (2001) showed this generically and applied to iterative refinement for generalized eigenvalue problem

Compensated Evaluation

Compensated Algorithms

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- **Compensated** algorithm uses $\widehat{f} \oplus \widehat{e}$; better approximation of $f(x)$ than $\widehat{f}$ (usually)

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- **Not possible** to apply this to $\oslash$

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- Test polynomial $p(s) = (s-1)\left(s - \frac{3}{4}\right)^7$; evaluation at $s = \frac{3}{4}$ has infinite condition, very ill-conditioned nearby

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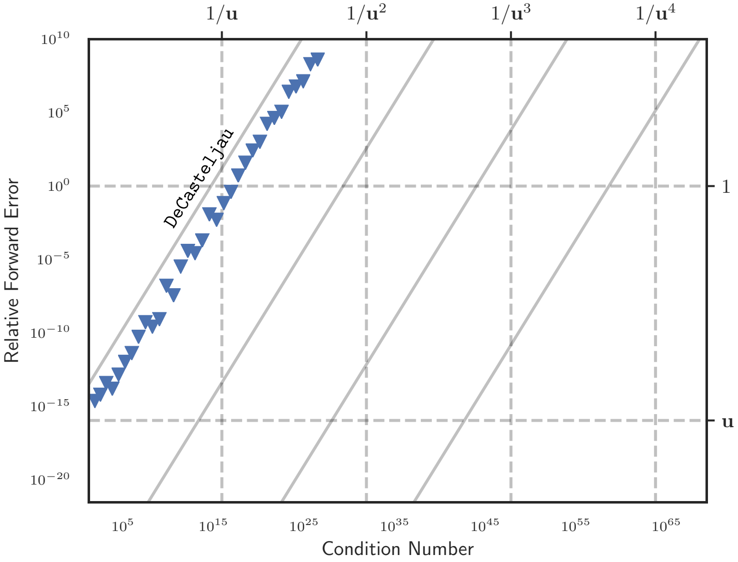
$$\frac{|p(s) - \texttt{DeCasteljau}\,(p, s)|}{|p(s)|} \leq \text{cond}\,(p, s) \cdot \mathcal{O}\,(\mathbf{u})$$

Compensated de Castlejau



Compensated de Castlejau

$$\frac{|p(s) - \texttt{CompDeCasteljau}\,(p, s)|}{|p(s)|} \leq \mathcal{O}\left(\mathbf{u}\right) + \mathrm{cond}\left(p, s\right) \cdot \mathcal{O}\left(\mathbf{u}^2\right)$$

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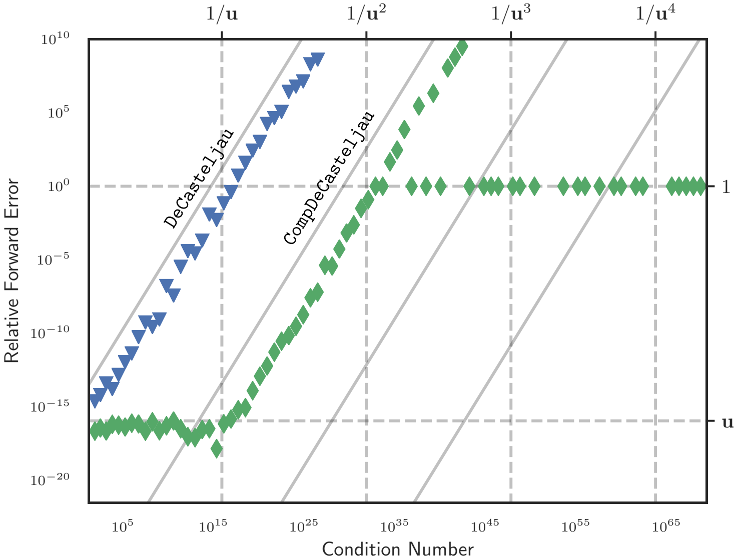

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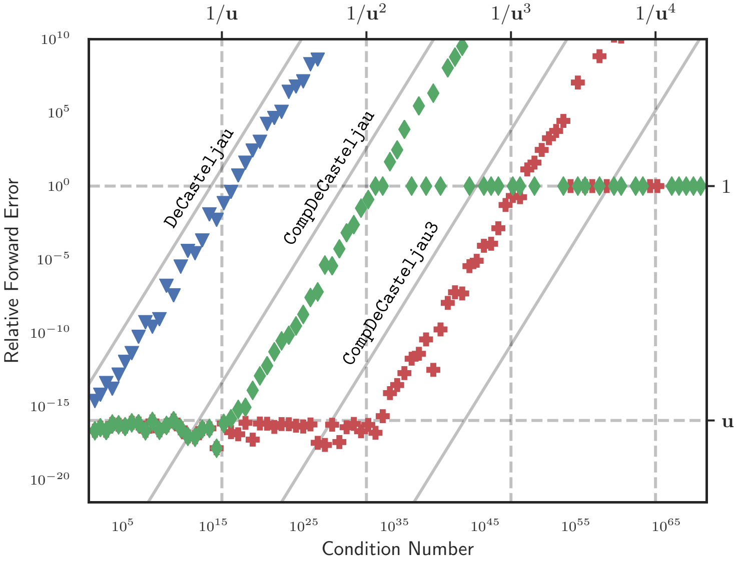

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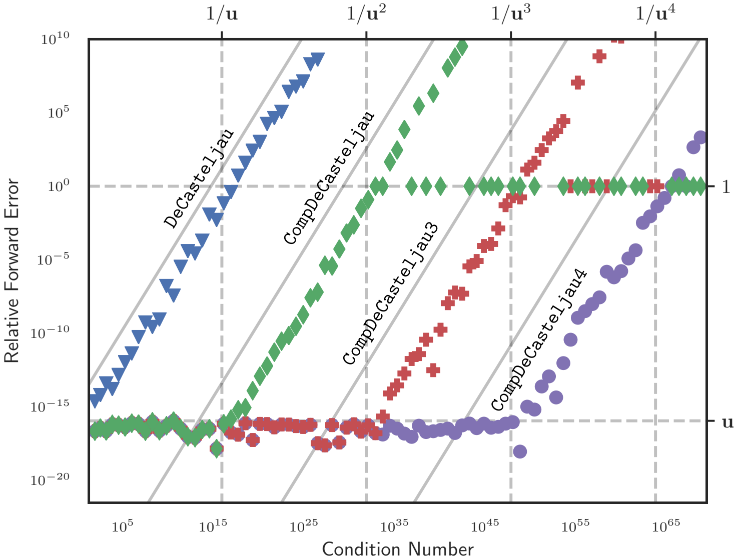

Modified Newton's for Intersection

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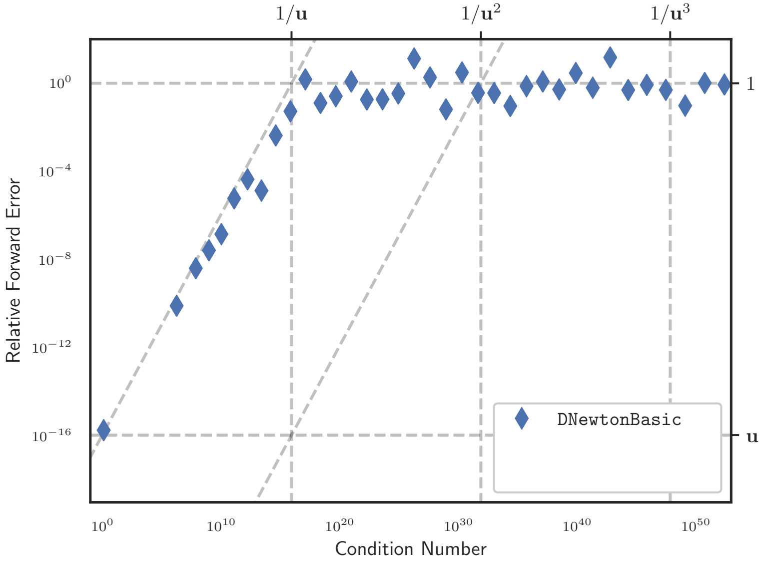
- Root $\alpha = \dfrac{1+2^{30/n}}{5+3\cdot 2^{30/n}} \in \left[\dfrac{1}{4}, \dfrac{1}{3}\right]$

Polynomial Roots



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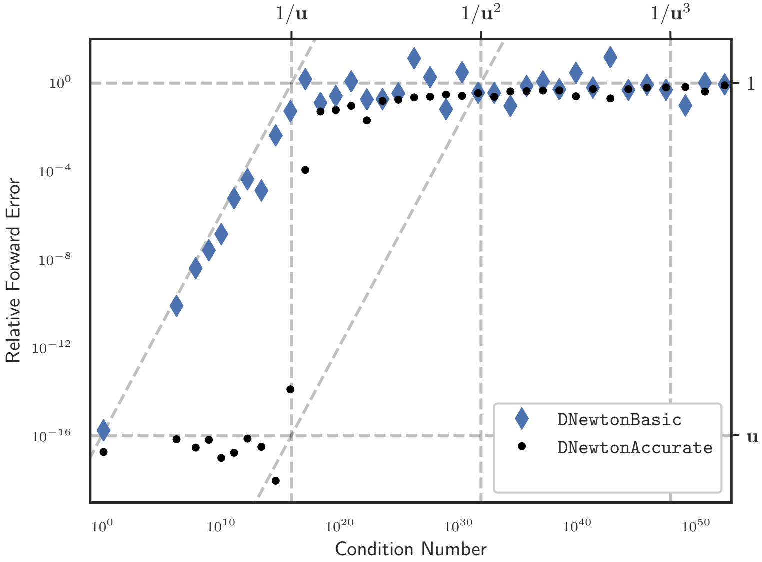

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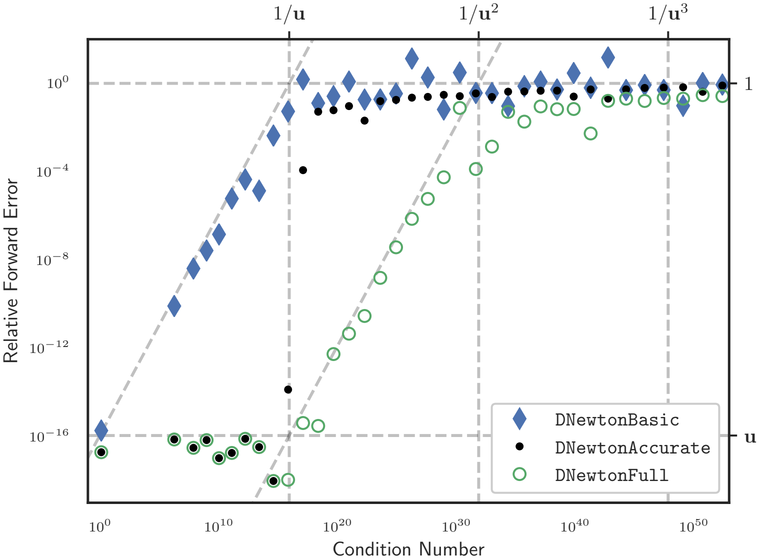

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- Modify Newton's method by using "extended precision" (i.e. compensated method) for $\widehat{F}$

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- Computed residual $D \oplus \tau$

Modified Newton's in Practice

$$F(s,t) = \begin{bmatrix} 2(4s^2-1) \\ (2s-1)^2+1 \end{bmatrix} - \begin{bmatrix} 4(4t^2-1) \\ 4(2t-1)^2+1 \end{bmatrix}$$

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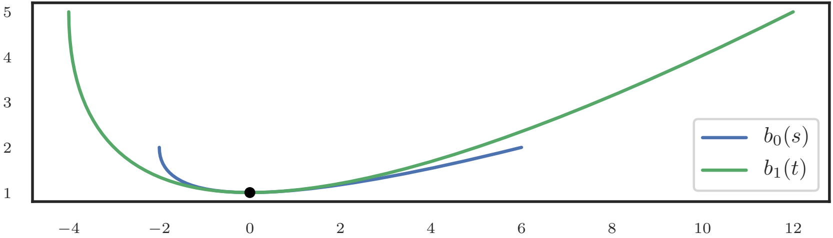

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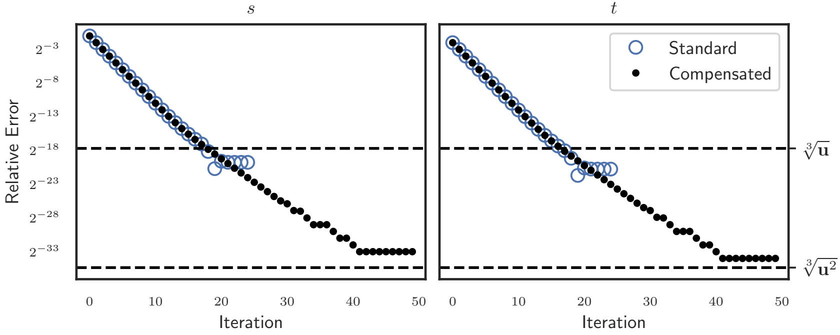

Modified Newton's in Practice

$$G(s,t) = \begin{bmatrix} x_0(s) - r \\ y_0(s) + \frac{1}{r} \end{bmatrix} - \begin{bmatrix} x_1(t) \\ y_1(t) + \frac{1}{r} \end{bmatrix}$$

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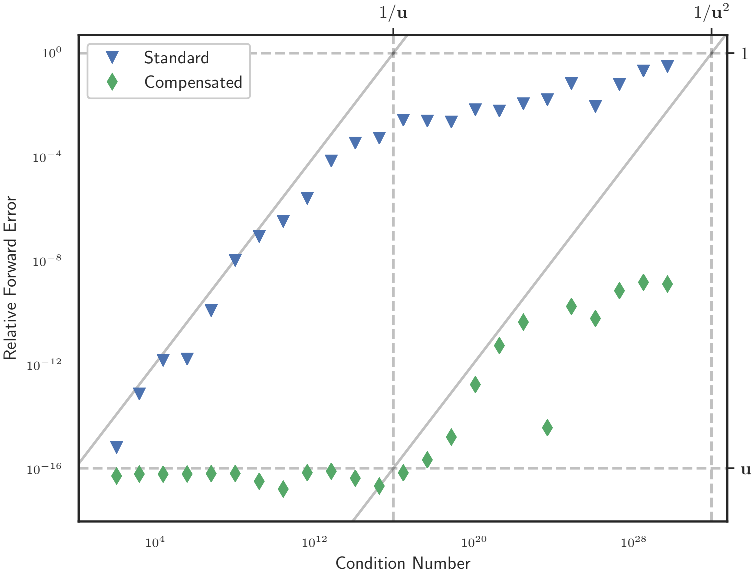
Approaches triple root as $r \longrightarrow 0^+$, $F\left(\frac{1+\sqrt{r}}{2}, \frac{2+\sqrt{r}}{4}\right)$

Modified Newton's in Practice



Conclusion