High-order Solution Transfer between Curved Meshes and Ill-conditioned Bézier Curve Intersection

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Outline

Introduction and motivation

Method of Characteristics

Solve simple transport equation

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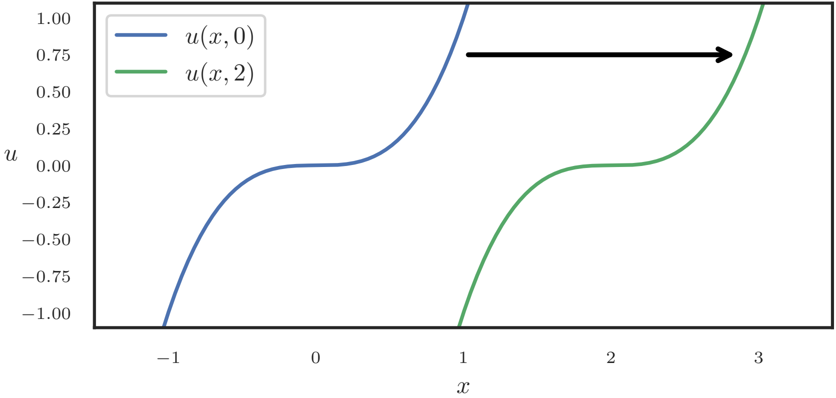
PDE becomes a (trivial) ODE

$$\frac{d}{dt}u(x(t),t) = 0.$$

Method of Characteristics



Lagrangian Methods

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- Transform PDE to family of ODEs

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    - Resolve sensitive features

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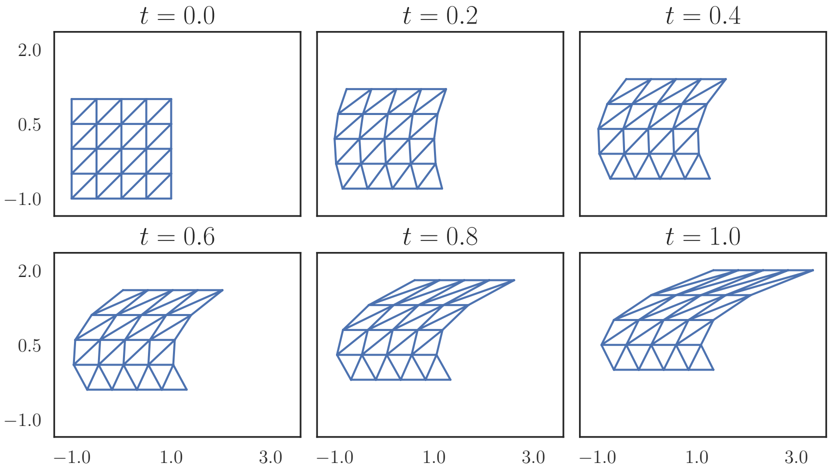
with cubic characteristics

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} (y_0 + t)^3 - y_0^3 \\ 3t \end{bmatrix}.$$

Remeshing Example



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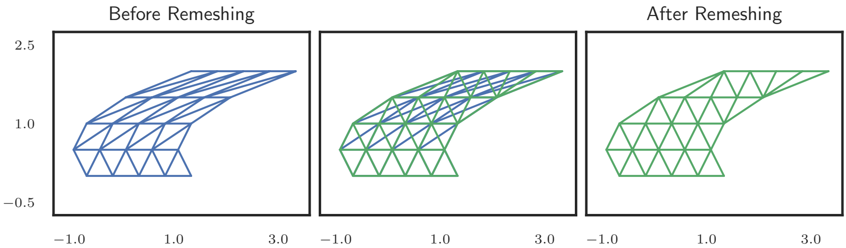

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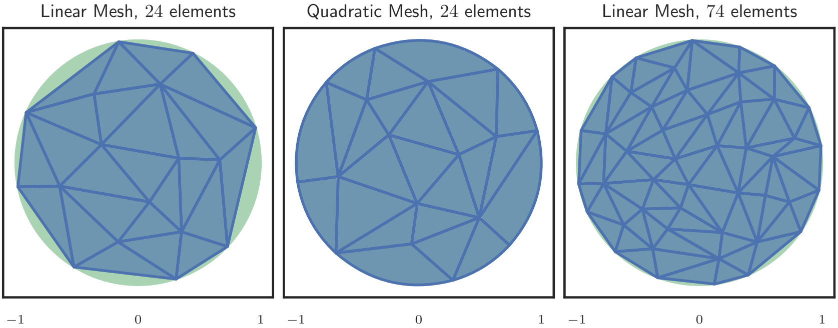

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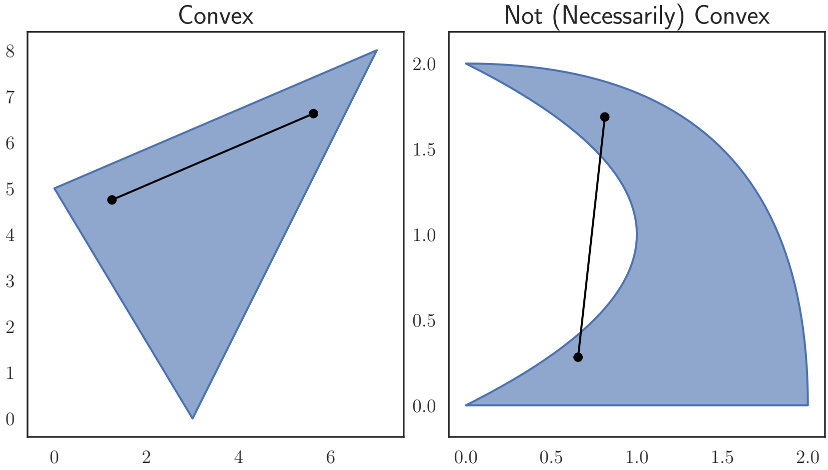
  - More challenging geometry

Curved Meshes: Drawbacks



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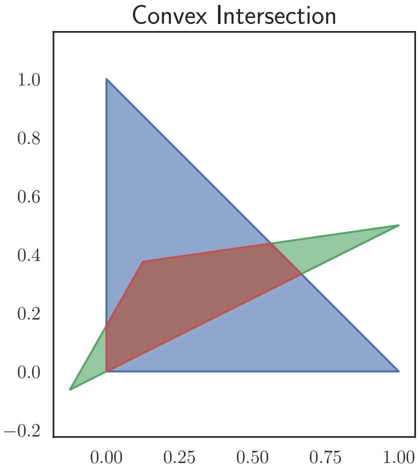

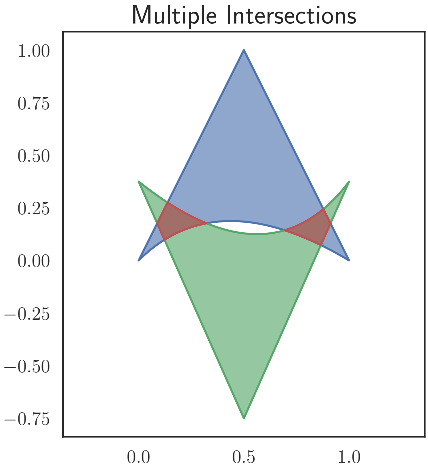

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- A Lagrangian method must either curve the mesh or information about the flow of the geometry will be lost.

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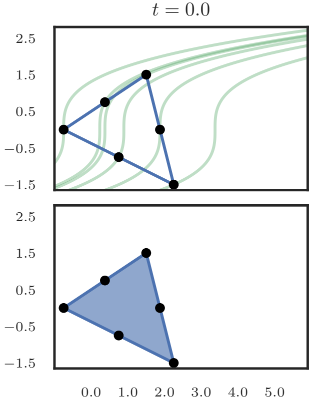

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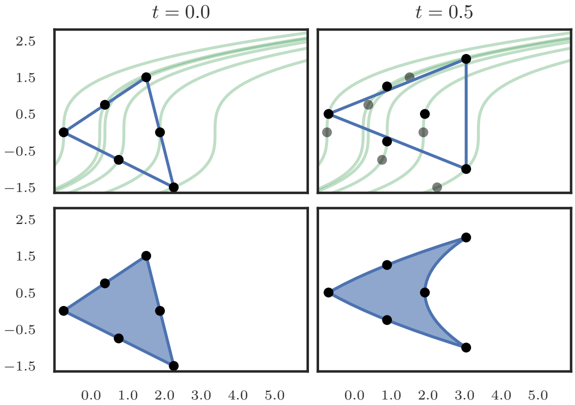

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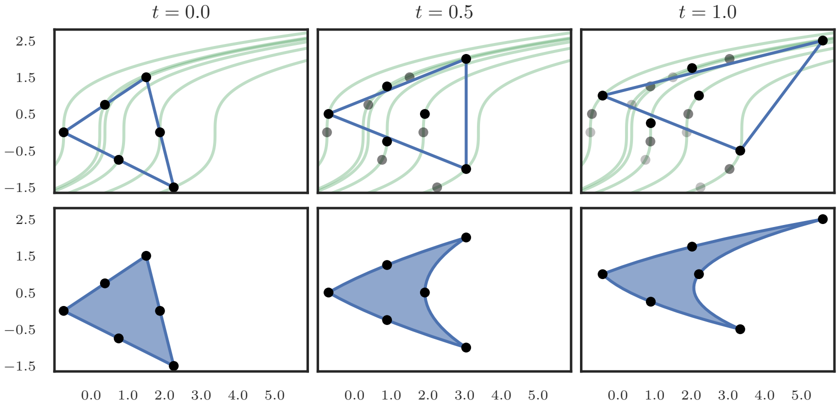

Solution Transfer

Compensated Evaluation

Modified Newton’s for Intersection