

# High-order Solution Transfer between Curved Meshes and Ill-conditioned Bézier Curve Intersection

---

Danny Hermes

August 9, 2018

[dhermes@berkeley.edu](mailto:dhermes@berkeley.edu)  
UC Berkeley



# Outline

---

1. Introduction and motivation
2. Curved Elements
3. Solution Transfer
4. Ill-conditioned Bézier Curve Intersection
5. Compensated Evaluation
6. Modified Newton's for Intersection

## Introduction and motivation

---

# Method of Characteristics

# Method of Characteristics

Solve simple transport equation

$$u_t + cu_x = 0, \quad u(x, 0) = u_0(x).$$

# Method of Characteristics

Solve simple transport equation

$$u_t + cu_x = 0, \quad u(x, 0) = u_0(x).$$

Divide physical domain

$$x(t) = x_0 + ct$$

# Method of Characteristics

Solve simple transport equation

$$u_t + cu_x = 0, \quad u(x, 0) = u_0(x).$$

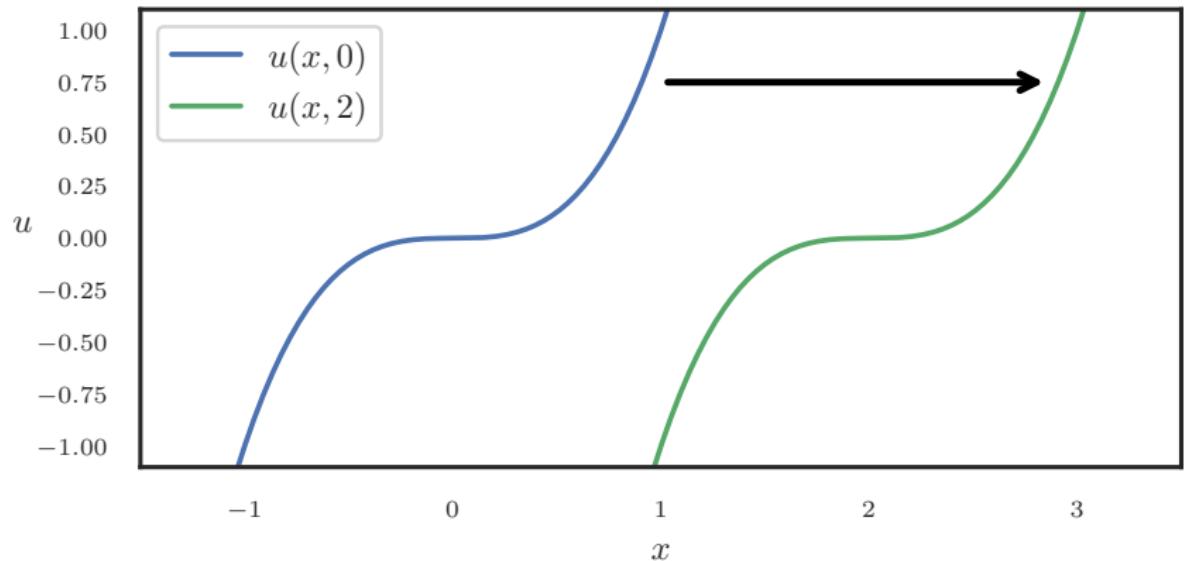
Divide physical domain

$$x(t) = x_0 + ct$$

PDE becomes a (trivial) ODE

$$\frac{d}{dt}u(x(t), t) = 0.$$

# Method of Characteristics



# Lagrangian Methods

# Lagrangian Methods

- Each point in physical domain is a **particle**

# Lagrangian Methods

- Each point in physical domain is a **particle**
- Carry value (e.g. heat, pressure, density) along characteristic curve

# Lagrangian Methods

- Each point in physical domain is a **particle**
- Carry value (e.g. heat, pressure, density) along characteristic curve
- Transform PDE to family of ODEs

# Lagrangian Methods

Add viscosity term to transport equation

$$u_t + cu_x - \varepsilon u_{xx} = 0.$$

# Lagrangian Methods

Add viscosity term to transport equation

$$u_t + cu_x - \varepsilon u_{xx} = 0.$$

Same characteristics used, **but** solution no longer constant

# Lagrangian Methods

Add viscosity term to transport equation

$$u_t + cu_x - \varepsilon u_{xx} = 0.$$

Same characteristics used, **but** solution no longer constant

$$\frac{d}{dt}u(x(t), t) = \varepsilon u_{xx}.$$

# Remeshing and Adaptivity

- Problems caused by flow-based mesh changes

# Remeshing and Adaptivity

- Problems caused by flow-based mesh changes
  - Distortion

# Remeshing and Adaptivity

- Problems caused by flow-based mesh changes
  - Distortion
  - Tangling

# Remeshing and Adaptivity

- Problems caused by flow-based mesh changes
  - Distortion
  - Tangling
  - Travel outside relevant physical domain

# Remeshing and Adaptivity

- Problems caused by flow-based mesh changes
  - Distortion
  - Tangling
  - Travel outside relevant physical domain
- Adaptivity

# Remeshing and Adaptivity

- Problems caused by flow-based mesh changes
  - Distortion
  - Tangling
  - Travel outside relevant physical domain
- Adaptivity
  - Dynamically focus computational effort

# Remeshing and Adaptivity

- Problems caused by flow-based mesh changes
  - Distortion
  - Tangling
  - Travel outside relevant physical domain
- Adaptivity
  - Dynamically focus computational effort
  - Resolve sensitive features

## Remeshing Example

Consider

$$u_t + \begin{bmatrix} y^2 \\ 1 \end{bmatrix} \cdot \nabla u + F(u, \nabla u) = 0$$

## Remeshing Example

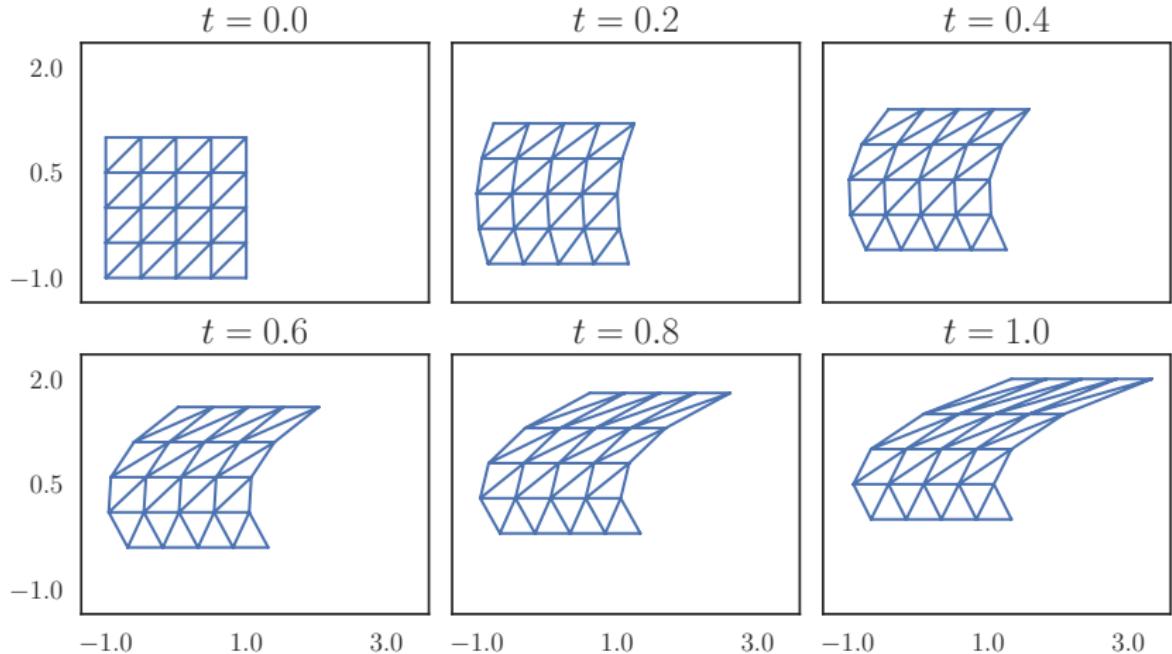
Consider

$$u_t + \begin{bmatrix} y^2 \\ 1 \end{bmatrix} \cdot \nabla u + F(u, \nabla u) = 0$$

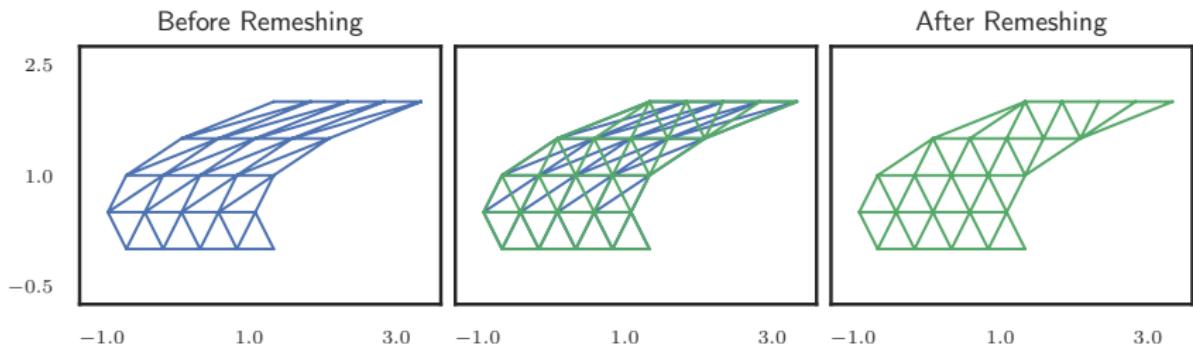
with cubic characteristics

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} (y_0 + t)^3 - y_0^3 \\ 3t \end{bmatrix}.$$

# Remeshing Example



# Remeshing Example



# Curved Meshes

- Benefits

# Curved Meshes

- Benefits
  - High-order shape functions, highly accurate solutions

# Curved Meshes

- Benefits
  - High-order shape functions, highly accurate solutions
  - Low dissipation and dispersion error

# Curved Meshes

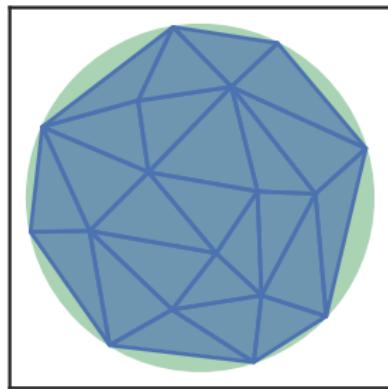
- Benefits
  - High-order shape functions, highly accurate solutions
  - Low dissipation and dispersion error
  - Greater geometric flexibility

# Curved Meshes

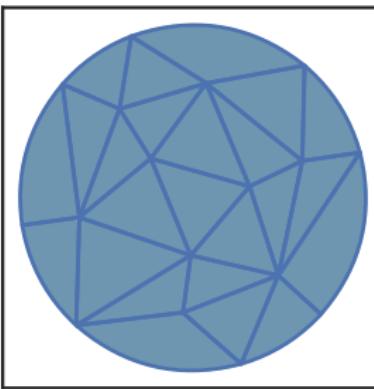
- Benefits
  - High-order shape functions, highly accurate solutions
  - Low dissipation and dispersion error
  - Greater geometric flexibility
  - Fewer elements

# Curved Meshes

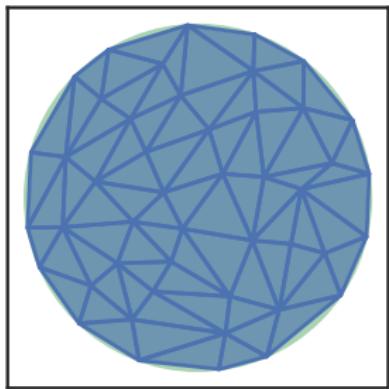
Linear Mesh, 24 elements



Quadratic Mesh, 24 elements



Linear Mesh, 74 elements



-1 0 1 -1 0 1 -1 0 1

# Curved Meshes

- Benefits
  - High-order shape functions, highly accurate solutions
  - Low dissipation and dispersion error
  - Greater geometric flexibility
  - Fewer elements

# Curved Meshes

- Benefits
  - High-order shape functions, highly accurate solutions
  - Low dissipation and dispersion error
  - Greater geometric flexibility
  - Fewer elements
- Drawbacks

# Curved Meshes

- Benefits
  - High-order shape functions, highly accurate solutions
  - Low dissipation and dispersion error
  - Greater geometric flexibility
  - Fewer elements
- Drawbacks
  - Harder to implement

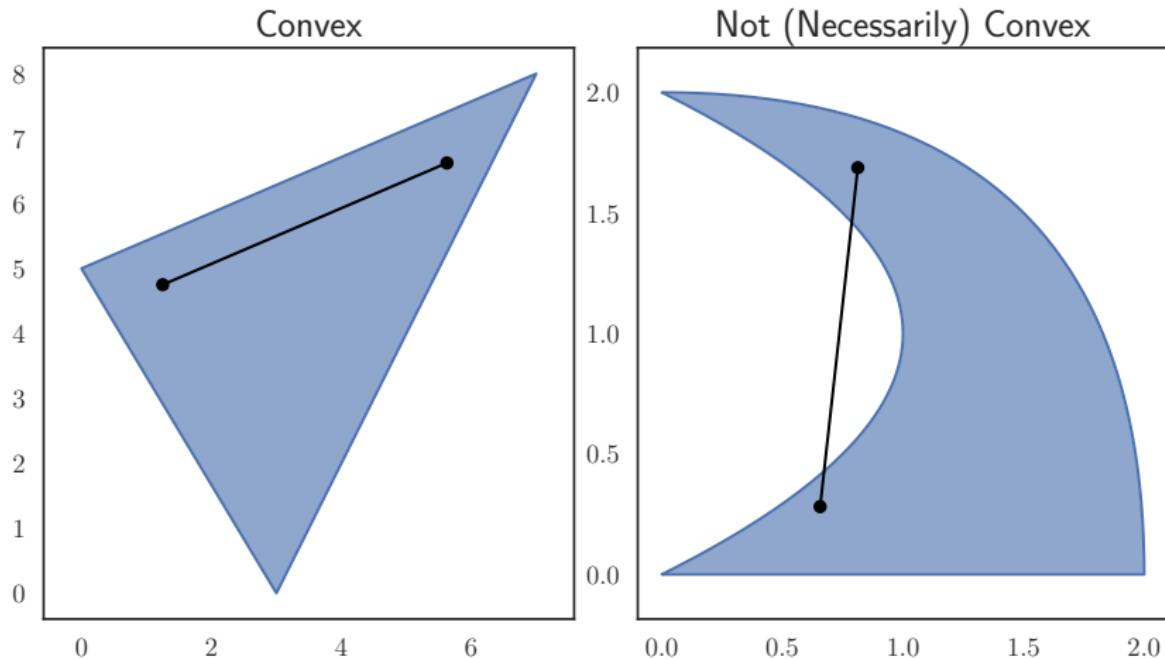
# Curved Meshes

- Benefits
  - High-order shape functions, highly accurate solutions
  - Low dissipation and dispersion error
  - Greater geometric flexibility
  - Fewer elements
- Drawbacks
  - Harder to implement
  - Loss of accuracy in high degree (e.g. Runge's phenomenon)

# Curved Meshes

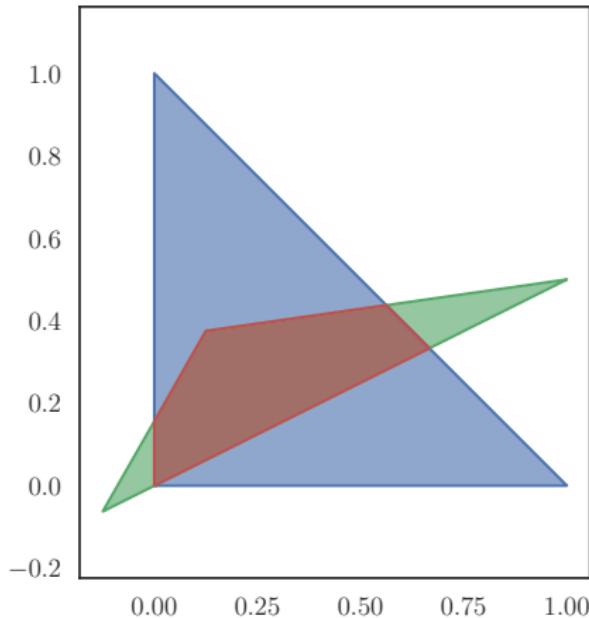
- Benefits
  - High-order shape functions, highly accurate solutions
  - Low dissipation and dispersion error
  - Greater geometric flexibility
  - Fewer elements
- Drawbacks
  - Harder to implement
  - Loss of accuracy in high degree (e.g. Runge's phenomenon)
  - More challenging geometry

# Curved Meshes

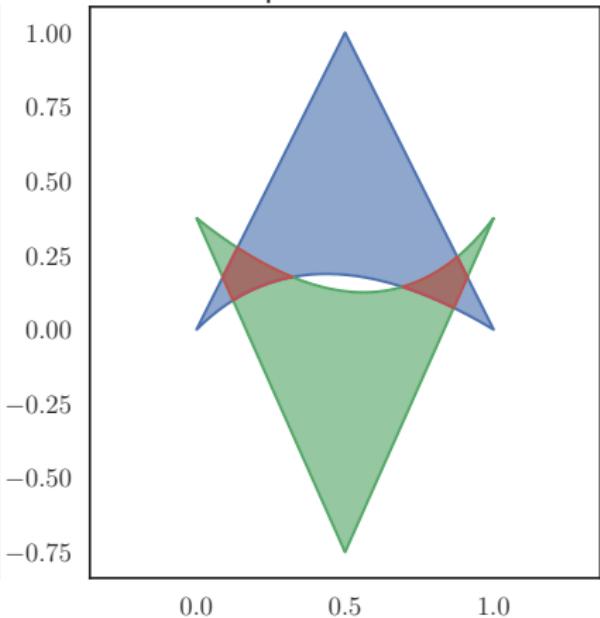


# Curved Meshes

Convex Intersection



Multiple Intersections



# Curved Elements

- Necessary for High-order

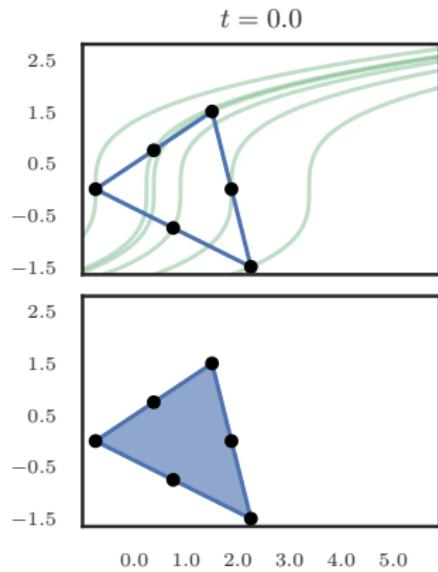
## Curved Elements

- Necessary for High-order
- With non-linear shape functions (i.e. not straight sided), non-vertex nodes used

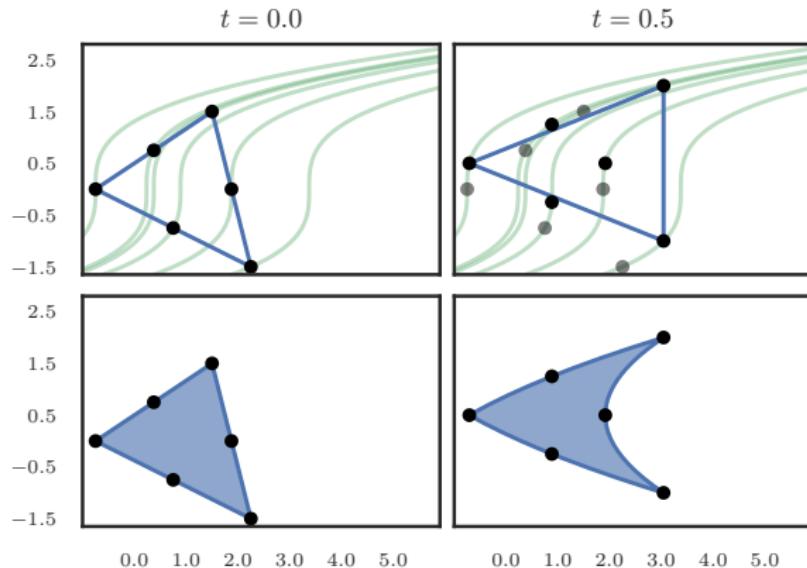
## Curved Elements

- Necessary for High-order
- With non-linear shape functions (i.e. not straight sided), non-vertex nodes used
- Lagrangian method must either curve mesh or information about flow of geometry will be lost

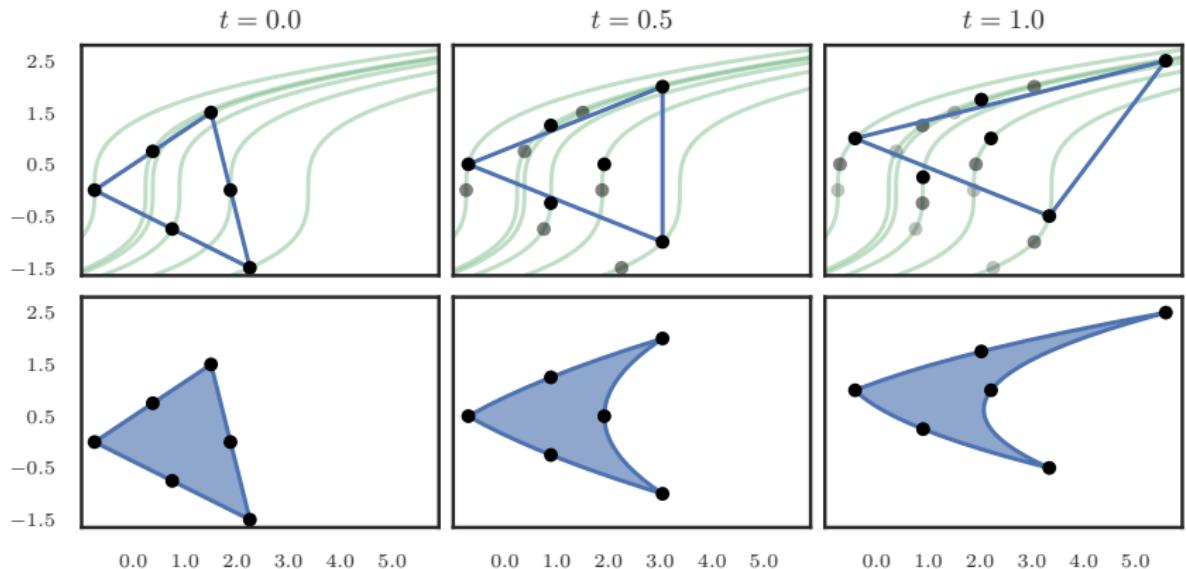
# Curved Elements: Necessary for High-order



# Curved Elements: Necessary for High-order



# Curved Elements: Necessary for High-order



## Curved Elements

---

# Bézier Triangles

- Image  $\mathcal{T} = b(\mathcal{U})$  of reference triangle under polynomial map  
 $b(s, t)$

# Bézier Triangles

- Image  $\mathcal{T} = b(\mathcal{U})$  of reference triangle under polynomial map  $b(s, t)$
- Barycentric coordinates  $\lambda_1 = 1 - s - t, \lambda_2 = s, \lambda_3 = t$

## Bézier Triangles

- Image  $\mathcal{T} = b(\mathcal{U})$  of reference triangle under polynomial map  $b(s, t)$
- Barycentric coordinates  $\lambda_1 = 1 - s - t, \lambda_2 = s, \lambda_3 = t$
- Bernstein basis via trinomial expansion:

$$1 = (\lambda_1 + \lambda_2 + \lambda_3)^n$$

# Bézier Triangles

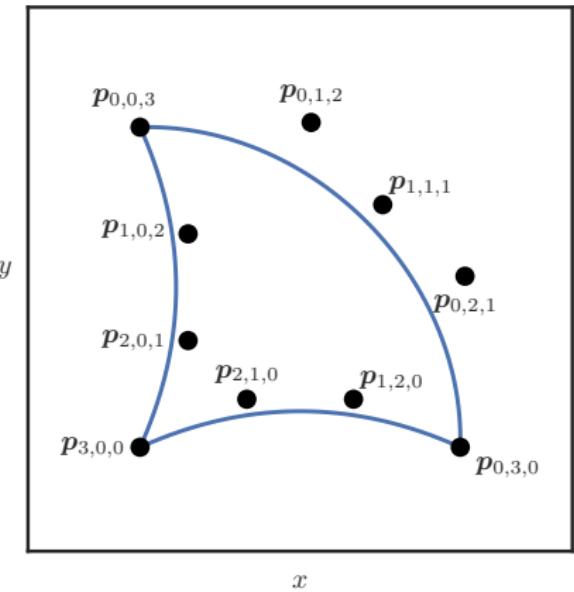
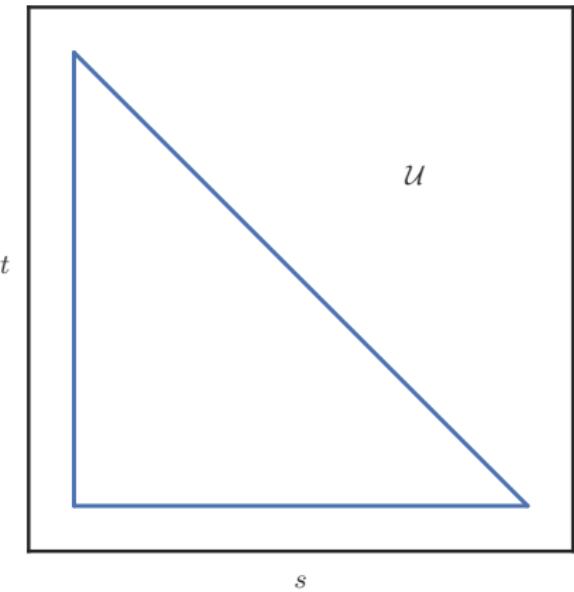
- Image  $\mathcal{T} = b(\mathcal{U})$  of reference triangle under polynomial map  $b(s, t)$
- Barycentric coordinates  $\lambda_1 = 1 - s - t, \lambda_2 = s, \lambda_3 = t$
- Bernstein basis via trinomial expansion:

$$1 = (\lambda_1 + \lambda_2 + \lambda_3)^n$$

- Convex combination of control points

$$b(s, t) = \sum_{\substack{i+j+k=n \\ i,j,k \geq 0}} \binom{n}{i,j,k} \lambda_1^i \lambda_2^j \lambda_3^k \mathbf{p}_{i,j,k}$$

# Bézier Triangles



## Bézier Triangles

- $b(s, t)$  can be defined by data other than control net

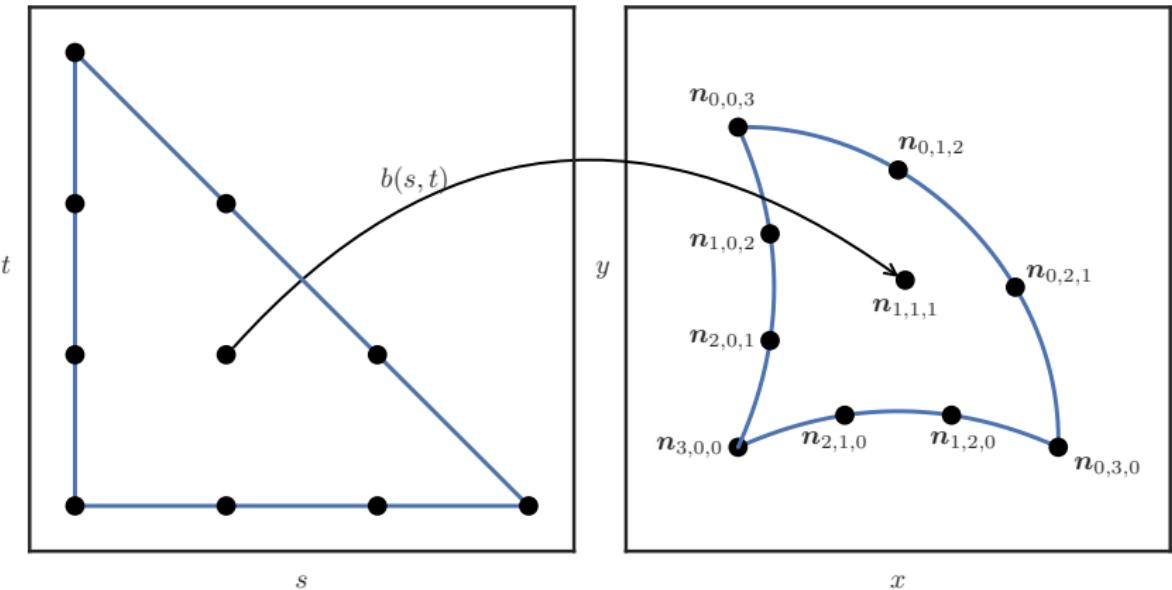
# Bézier Triangles

- $b(s, t)$  can be defined by data other than control net
- Regular grid in  $\mathcal{U}$ ,  $\mathbf{u}_{i,j,k} = \left( \frac{j}{n}, \frac{k}{n} \right)$

# Bézier Triangles

- $b(s, t)$  can be defined by data other than control net
- Regular grid in  $\mathcal{U}$ ,  $\mathbf{u}_{i,j,k} = \left( \frac{j}{n}, \frac{k}{n} \right)$
- $b(\mathbf{u}_{i,j,k}) = \mathbf{n}_{i,j,k}$ ; refer to  $\mathbf{n}_{i,j,k}$  as **standard nodes**

# Bézier Triangles



# Bézier Triangles

- $b(s, t)$  can be defined by data other than control net
- Regular grid in  $\mathcal{U}$ ,  $\mathbf{u}_{i,j,k} = \left( \frac{j}{n}, \frac{k}{n} \right)$
- $b(\mathbf{u}_{i,j,k}) = \mathbf{n}_{i,j,k}$ ; refer to  $\mathbf{n}_{i,j,k}$  as **standard nodes**

# Bézier Triangles

- $b(s, t)$  can be defined by data other than control net
- Regular grid in  $\mathcal{U}$ ,  $\mathbf{u}_{i,j,k} = \left( \frac{j}{n}, \frac{k}{n} \right)$
- $b(\mathbf{u}_{i,j,k}) = \mathbf{n}_{i,j,k}$ ; refer to  $\mathbf{n}_{i,j,k}$  as **standard nodes**
- For example, taking  $\mathbf{n}_{i,j,k} = \delta_{(i,j,k)} (i_0, j_0, k_0)$  gives degree  $n$  shape functions on  $\mathcal{U}$

# Bézier Triangles

- $b(s, t)$  can be defined by data other than control net
- Regular grid in  $\mathcal{U}$ ,  $\mathbf{u}_{i,j,k} = \left( \frac{j}{n}, \frac{k}{n} \right)$
- $b(\mathbf{u}_{i,j,k}) = \mathbf{n}_{i,j,k}$ ; refer to  $\mathbf{n}_{i,j,k}$  as **standard nodes**
- For example, taking  $\mathbf{n}_{i,j,k} = \delta_{(i,j,k)}(i_0, j_0, k_0)$  gives degree  $n$  shape functions on  $\mathcal{U}$
- Conversion between  $\mathbf{n}_{i,j,k}$  and  $\mathbf{p}_{i,j,k}$  has condition number exponential in  $n$

## Valid Element

- Element  $\mathcal{T}$  is **valid** if diffeomorphic to  $\mathcal{U}$

## Valid Element

- Element  $\mathcal{T}$  is **valid** if diffeomorphic to  $\mathcal{U}$
- $b(s, t)$  bijective, i.e. Jacobian  $D_b$  is everywhere invertible

## Valid Element

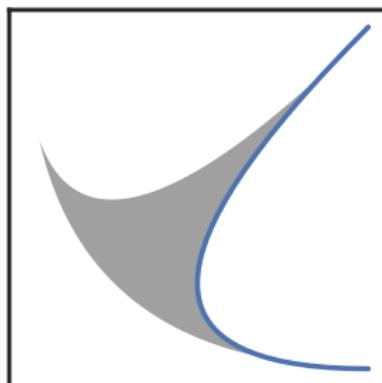
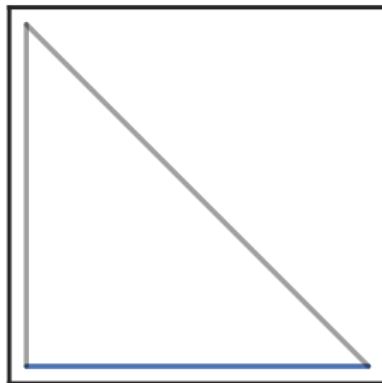
- Element  $\mathcal{T}$  is **valid** if diffeomorphic to  $\mathcal{U}$
- $b(s, t)$  bijective, i.e. Jacobian  $Db$  is everywhere invertible
- $\det(Db)$  positive, preserves orientation

## Inverted Element

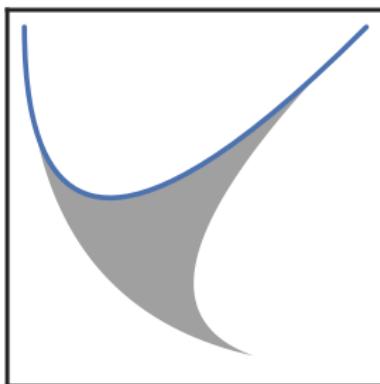
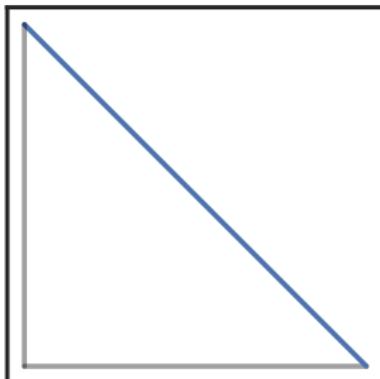
Consider element given by map

$$b(s, t) = \lambda_1^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_2^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \lambda_3^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

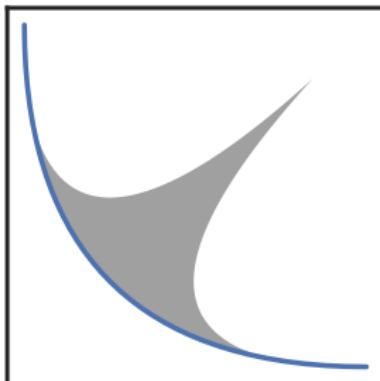
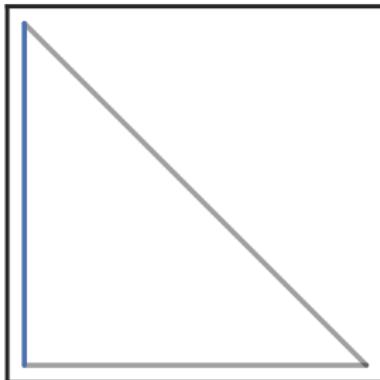
# Inverted Element



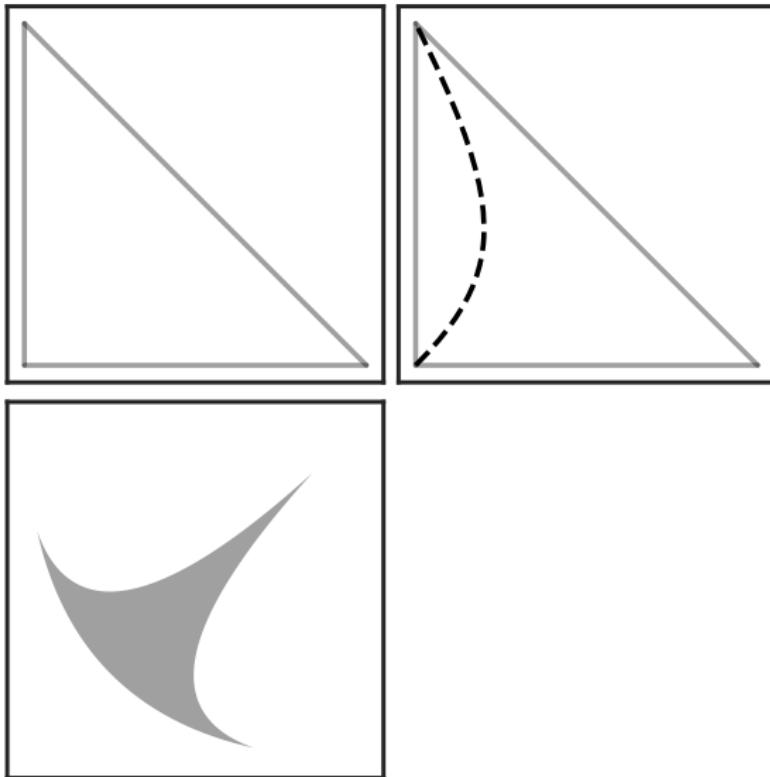
# Inverted Element



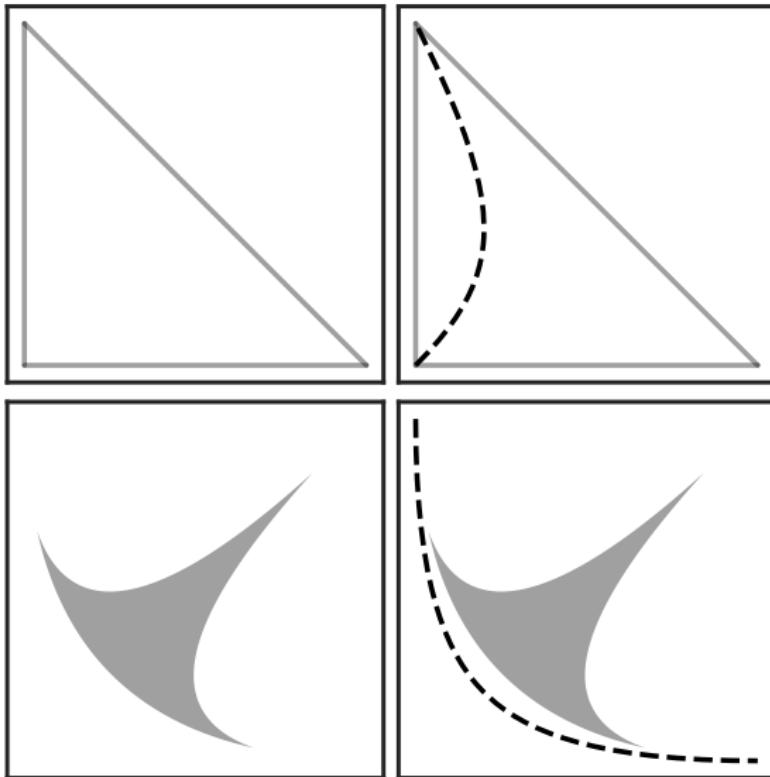
# Inverted Element



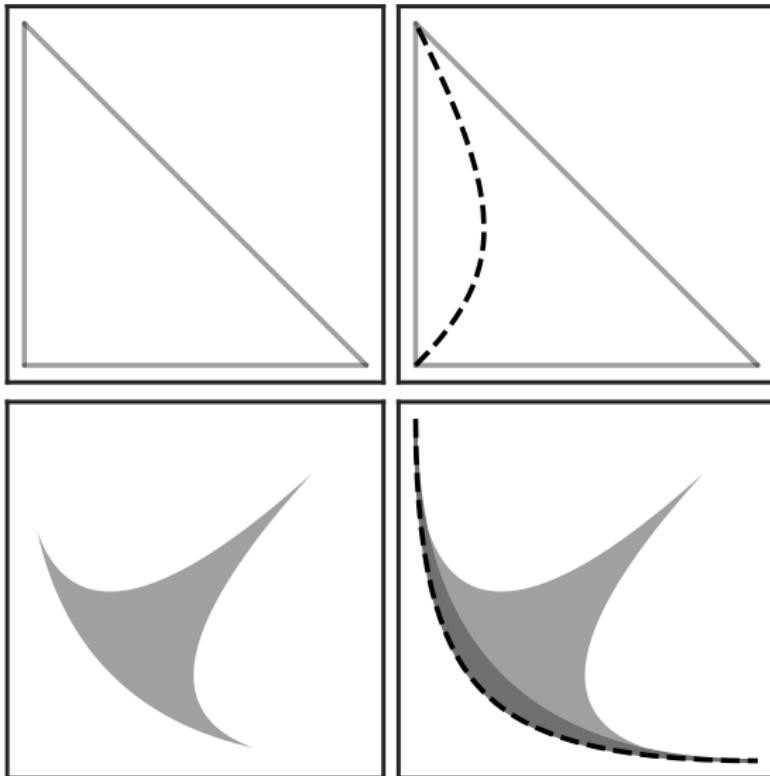
# Inverted Element



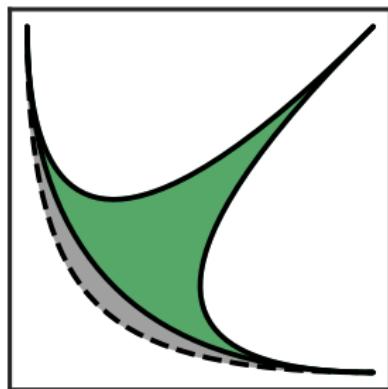
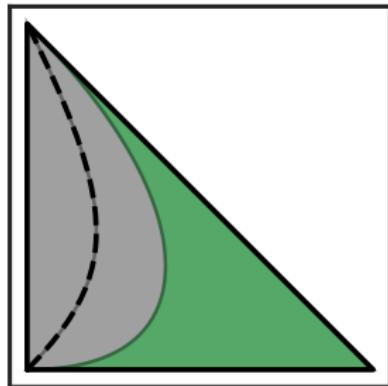
# Inverted Element



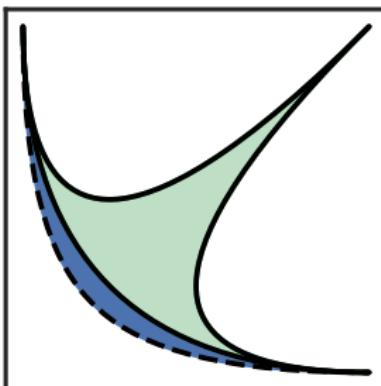
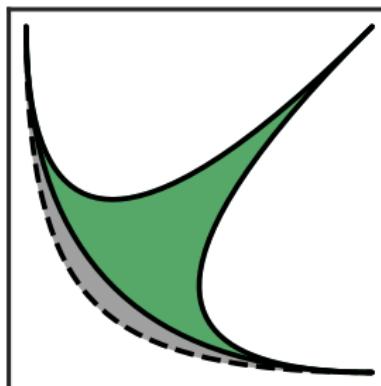
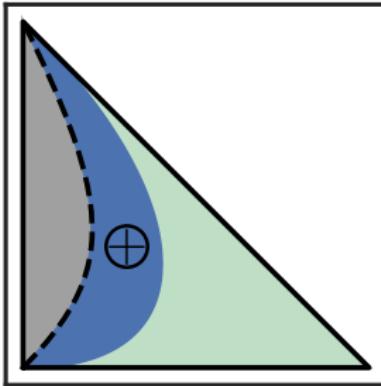
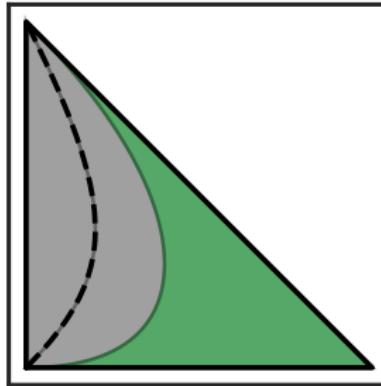
# Inverted Element



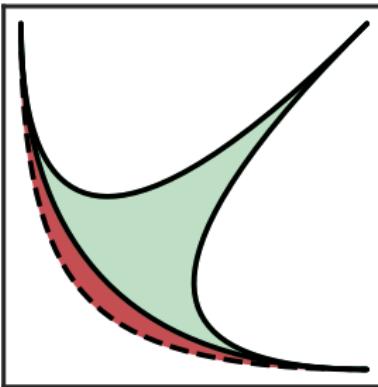
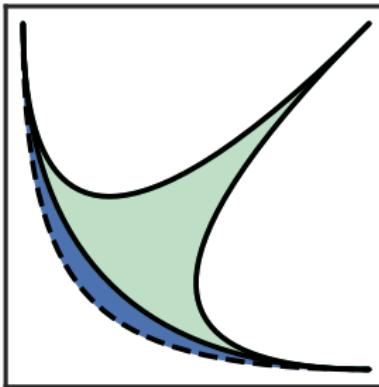
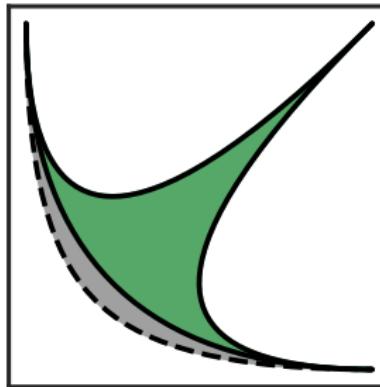
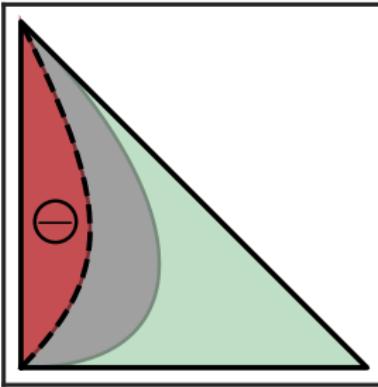
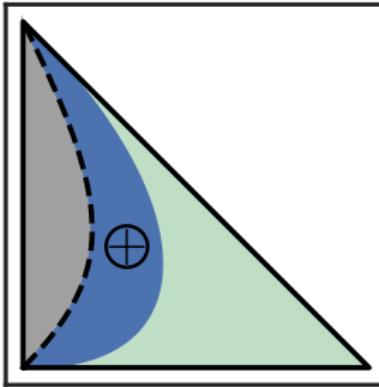
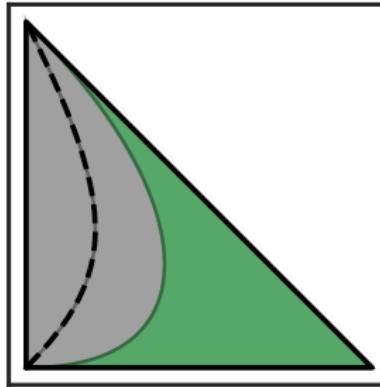
# Inverted Element



# Inverted Element



# Inverted Element



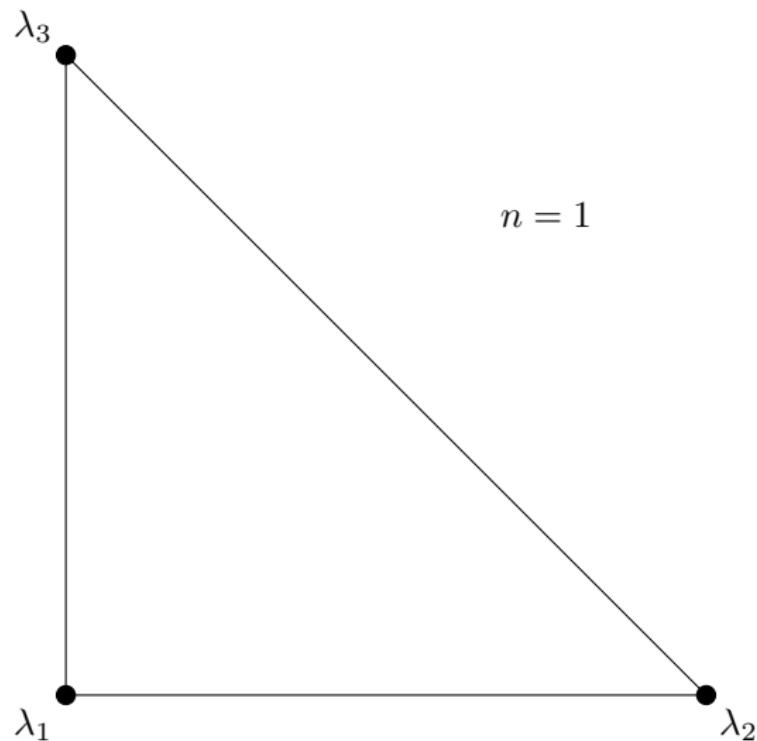
## Shape Functions

- Based on  $u_\alpha \in \mathcal{U}$  or  $n_\alpha \in \mathbf{R}^2$  ( $\alpha$  is a multi-index)

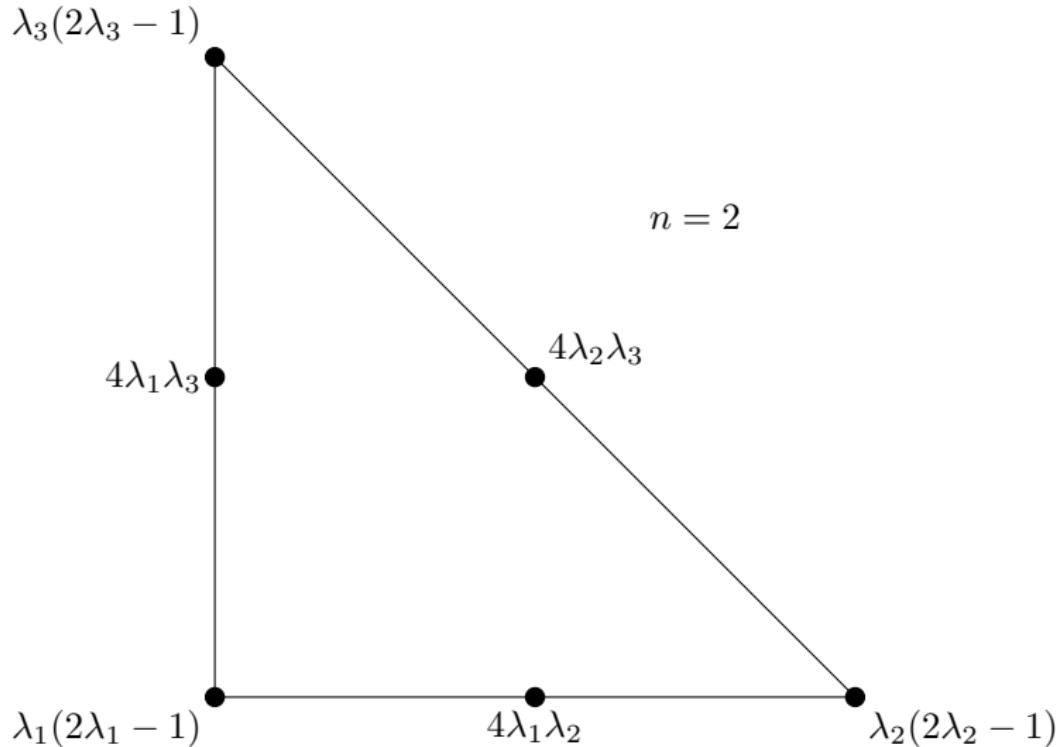
# Shape Functions

- Based on  $u_\alpha \in \mathcal{U}$  or  $n_\alpha \in \mathbf{R}^2$  ( $\alpha$  is a multi-index)
- Pre-Image Basis:  $\phi_\alpha(n_\beta) = \hat{\phi}_\alpha(u_\beta) = \hat{\phi}_\alpha(b^{-1}(n_\beta))$

# Shape Functions



# Shape Functions



# Shape Functions

- Based on  $u_\alpha \in \mathcal{U}$  or  $n_\alpha \in \mathbf{R}^2$  ( $\alpha$  is a multi-index)
- Pre-Image Basis:  $\phi_\alpha(n_\beta) = \hat{\phi}_\alpha(u_\beta) = \hat{\phi}_\alpha(b^{-1}(n_\beta))$

# Shape Functions

- Based on  $u_\alpha \in \mathcal{U}$  or  $n_\alpha \in \mathbf{R}^2$  ( $\alpha$  is a multi-index)
- Pre-Image Basis:  $\phi_\alpha(n_\beta) = \hat{\phi}_\alpha(u_\beta) = \hat{\phi}_\alpha(b^{-1}(n_\beta))$
- Global Coordinates Basis:  $\phi_\alpha(n_\beta) = \delta_{\alpha\beta}$

# Shape Functions

- Based on  $u_\alpha \in \mathcal{U}$  or  $n_\alpha \in \mathbf{R}^2$  ( $\alpha$  is a multi-index)
- Pre-Image Basis:  $\phi_\alpha(n_\beta) = \hat{\phi}_\alpha(u_\beta) = \hat{\phi}_\alpha(b^{-1}(n_\beta))$
- Global Coordinates Basis:  $\phi_\alpha(n_\beta) = \delta_{\alpha\beta}$
- Element is **isoparametric** when numerical solution expressed in span of shape functions

# Shape Functions

- Based on  $u_\alpha \in \mathcal{U}$  or  $n_\alpha \in \mathbf{R}^2$  ( $\alpha$  is a multi-index)
- Pre-Image Basis:  $\phi_\alpha(n_\beta) = \hat{\phi}_\alpha(u_\beta) = \hat{\phi}_\alpha(b^{-1}(n_\beta))$
- Global Coordinates Basis:  $\phi_\alpha(n_\beta) = \delta_{\alpha\beta}$
- Element is **isoparametric** when numerical solution expressed in span of shape functions
- $\text{supp}(\phi) = \mathcal{T}$

## Solution Transfer

---

# Galerkin Projection

- Given:

# Galerkin Projection

- Given:
  - Donor mesh  $\mathcal{M}_D$  and target mesh  $\mathcal{M}_T$

# Galerkin Projection

- Given:
  - Donor mesh  $\mathcal{M}_D$  and target mesh  $\mathcal{M}_T$
  - Shape function bases  $\phi_D^{(j)}$  and  $\phi_T^{(j)}$

# Galerkin Projection

- Given:
  - Donor mesh  $\mathcal{M}_D$  and target mesh  $\mathcal{M}_T$
  - Shape function bases  $\phi_D^{(j)}$  and  $\phi_T^{(j)}$
  - Known discrete field  $\mathbf{q}_D = \sum_j d_j \phi_D^{(j)}$

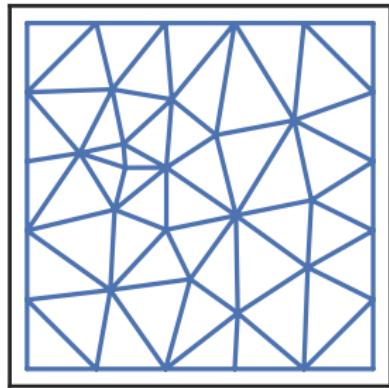
# Galerkin Projection

- Given:
  - Donor mesh  $\mathcal{M}_D$  and target mesh  $\mathcal{M}_T$
  - Shape function bases  $\phi_D^{(j)}$  and  $\phi_T^{(j)}$
  - Known discrete field  $\mathbf{q}_D = \sum_j d_j \phi_D^{(j)}$
- Want:  $L_2$ -optimal interpolant  $\mathbf{q}_T = \sum_j t_j \phi_T^{(j)}$ :

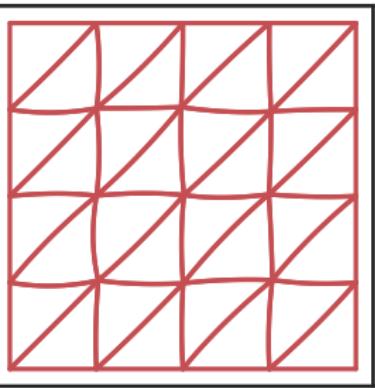
$$\|\mathbf{q}_T - \mathbf{q}_D\|_2 = \min_{\mathbf{q} \in \mathcal{V}_T} \|\mathbf{q} - \mathbf{q}_D\|_2$$

# Galerkin Projection

$\mathcal{M}_T$



$\mathcal{M}_D$



## Galerkin Projection

Differentiating w.r.t. each  $t_j$  in  $\mathbf{q}_T = \sum_j t_j \phi_T^{(j)}$  gives **weak form**

$$\int_{\Omega} \mathbf{q}_D \phi_T^{(j)} dV = \int_{\Omega} \mathbf{q}_T \phi_T^{(j)} dV, \quad \text{for all } j.$$

## Galerkin Projection

Differentiating w.r.t. each  $t_j$  in  $\mathbf{q}_T = \sum_j t_j \phi_T^{(j)}$  gives **weak form**

$$\int_{\Omega} \mathbf{q}_D \phi_T^{(j)} dV = \int_{\Omega} \mathbf{q}_T \phi_T^{(j)} dV, \quad \text{for all } j.$$

If  $(x \mapsto 1) \in \mathcal{V}_T$ , then  $\mathbf{q}_T$  is globally **conservative**

$$\int_{\Omega} \mathbf{q}_D dV = \int_{\Omega} \mathbf{q}_T dV.$$

# Linear System

Weak form gives rise to a linear system in coefficients  $\mathbf{d}$  and  $t$ .

# Linear System

Weak form gives rise to a linear system in coefficients  $\mathbf{d}$  and  $\mathbf{t}$ :

$$M_T \mathbf{t} = M_{TD} \mathbf{d}.$$

# Linear System

$M_T$  is (symmetric) mass matrix for target mesh

$$(M_T)_{ij} = \int_{\Omega} \phi_T^{(i)} \phi_T^{(j)} dV.$$

## Linear System

$M_T$  is (symmetric) mass matrix for target mesh

$$(M_T)_{ij} = \int_{\Omega} \phi_T^{(i)} \phi_T^{(j)} dV.$$

Since  $\text{supp}(\phi) = \mathcal{T}$ ,  $M_T$  is block diagonal in DG, sparse but globally coupled in CG.

# Linear System

$M_T$  is (symmetric) mass matrix for target mesh

$$(M_T)_{ij} = \int_{\Omega} \phi_T^{(i)} \phi_T^{(j)} dV.$$

Since  $\text{supp}(\phi) = \mathcal{T}$ ,  $M_T$  is block diagonal in DG, sparse but globally coupled in CG.

Integrate via substitution for  $F = \phi_T^{(i)} \phi_T^{(j)}$

$$\int_{b(\mathcal{U})} F(x, y) dx dy = \int_{\mathcal{U}} \det(Db) F(x(s, t), y(s, t)) ds dt$$

and then use quadrature rule on  $\mathcal{U}$ .

# Linear System

Mixed mass matrix  $M_{TD}$

$$(M_{TD})_{ij} = \int_{\Omega} \phi_T^{(i)} \phi_D^{(j)} dV.$$

# Linear System

Mixed mass matrix  $M_{TD}$

$$(M_{TD})_{ij} = \int_{\Omega} \phi_T^{(i)} \phi_D^{(j)} dV.$$

Not symmetric, nor even square; rows correspond to shape functions on target mesh and columns to donor mesh.

# Linear System

Mixed mass matrix  $M_{TD}$

$$(M_{TD})_{ij} = \int_{\Omega} \phi_T^{(i)} \phi_D^{(j)} dV.$$

Not symmetric, nor even square; rows correspond to shape functions on target mesh and columns to donor mesh.

Instead, compute entire RHS

$$(M_{TD}\mathbf{d})_j = \int_{\Omega} \phi_T^{(j)} \mathbf{q}_D dV.$$

## Common Refinement

Given  $\phi$  supported on  $\mathcal{T}$

$$\int_{\Omega} \phi \mathbf{q}_D dV$$

## Common Refinement

Given  $\phi$  supported on  $\mathcal{T}$

$$\int_{\Omega} \phi \mathbf{q}_D dV = \int_{\mathcal{T}} \phi \mathbf{q}_D dV$$

## Common Refinement

Given  $\phi$  supported on  $\mathcal{T}$

$$\int_{\Omega} \phi \mathbf{q}_D dV = \int_{\mathcal{T}} \phi \mathbf{q}_D dV = \sum_{\mathcal{T}' \in \mathcal{M}_D} \int_{\mathcal{T} \cap \mathcal{T}'} \phi |_{\mathcal{T}'} \mathbf{q}_D dV$$

## Common Refinement

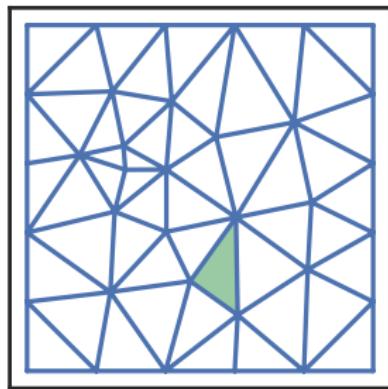
Given  $\phi$  supported on  $\mathcal{T}$

$$\int_{\Omega} \phi \mathbf{q}_D dV = \int_{\mathcal{T}} \phi \mathbf{q}_D dV = \sum_{\mathcal{T}' \in \mathcal{M}_D} \int_{\mathcal{T} \cap \mathcal{T}'} \phi |_{\mathcal{T}'} \mathbf{q}_D dV$$

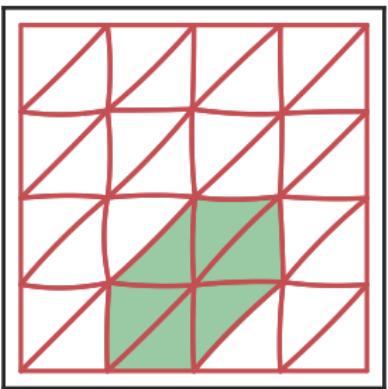
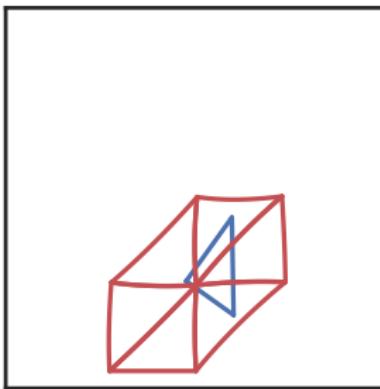
In CG,  $\mathbf{q}_D$  need not be differentiable across elements and in DG  $\mathbf{q}_D$  need not even be continuous

# Common Refinement

$\mathcal{M}_T$



$\mathcal{M}_D$



# Common Refinement

- Three subproblems:

# Common Refinement

- Three subproblems:
  - Intersecting Curved Elements:  $\mathcal{T} \cap \mathcal{T}'$

# Common Refinement

- Three subproblems:
  - Intersecting Curved Elements:  $\mathcal{T} \cap \mathcal{T}'$
  - Advancing Front:  $(\mathcal{T}, \mathcal{T}') \in \mathcal{M}_T \times \mathcal{M}_D$

# Common Refinement

- Three subproblems:
  - Intersecting Curved Elements:  $\mathcal{T} \cap \mathcal{T}'$
  - Advancing Front:  $(\mathcal{T}, \mathcal{T}') \in \mathcal{M}_T \times \mathcal{M}_D$
  - Integration over Curved Polygons:  $\int_{\mathcal{T} \cap \mathcal{T}'} F dV$

# Intersecting Curved Elements

## Intersecting Curved Elements

- Find all points where edges intersect

## Intersecting Curved Elements

- Find all points where edges intersect
- $\mathcal{T}_0 = b_0(\mathcal{U})$ ,  $\partial\mathcal{T}_0 = E_0 \cup E_1 \cup E_2$
- $\mathcal{T}_1 = b_1(\mathcal{U})$ ,  $\partial\mathcal{T}_1 = E_3 \cup E_4 \cup E_5$

## Intersecting Curved Elements

- Find all points where edges intersect
- $\mathcal{T}_0 = b_0(\mathcal{U})$ ,  $\partial\mathcal{T}_0 = E_0 \cup E_1 \cup E_2$
- $\mathcal{T}_1 = b_1(\mathcal{U})$ ,  $\partial\mathcal{T}_1 = E_3 \cup E_4 \cup E_5$
- Determine which curve is interior at each intersection

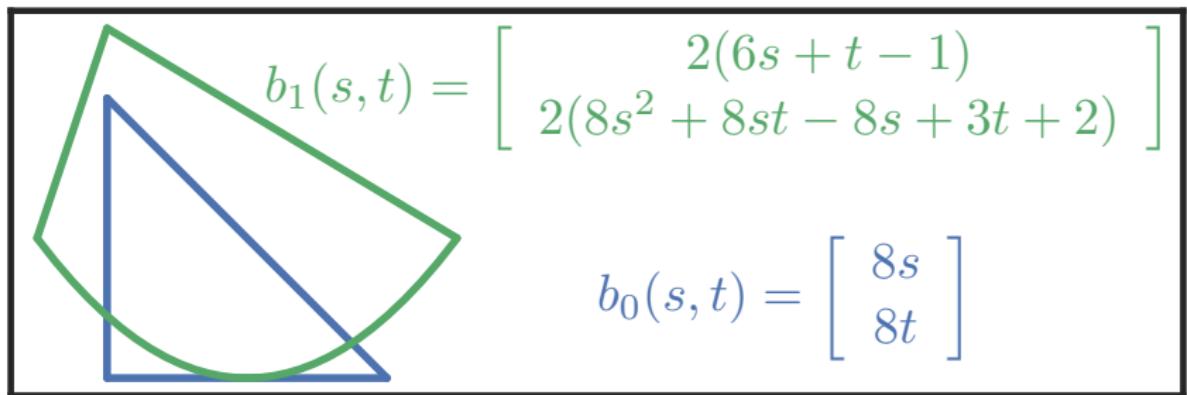
## Intersecting Curved Elements

- Find all points where edges intersect
- $\mathcal{T}_0 = b_0(\mathcal{U})$ ,  $\partial\mathcal{T}_0 = E_0 \cup E_1 \cup E_2$
- $\mathcal{T}_1 = b_1(\mathcal{U})$ ,  $\partial\mathcal{T}_1 = E_3 \cup E_4 \cup E_5$
- Determine which curve is interior at each intersection
- Track parameter values at each intersection and combine into boundary of curved polygon

## Intersecting Curved Elements

- Find all points where edges intersect
- $\mathcal{T}_0 = b_0(\mathcal{U})$ ,  $\partial\mathcal{T}_0 = E_0 \cup E_1 \cup E_2$
- $\mathcal{T}_1 = b_1(\mathcal{U})$ ,  $\partial\mathcal{T}_1 = E_3 \cup E_4 \cup E_5$
- Determine which curve is interior at each intersection
- Track parameter values at each intersection and combine into boundary of **curved polygon**
- $\mathcal{P} = \mathcal{T}_0 \cap \mathcal{T}_1$ ,  $\partial\mathcal{P}$  defined by segments of edges from  $\mathcal{T}_0$  and  $\mathcal{T}_1$

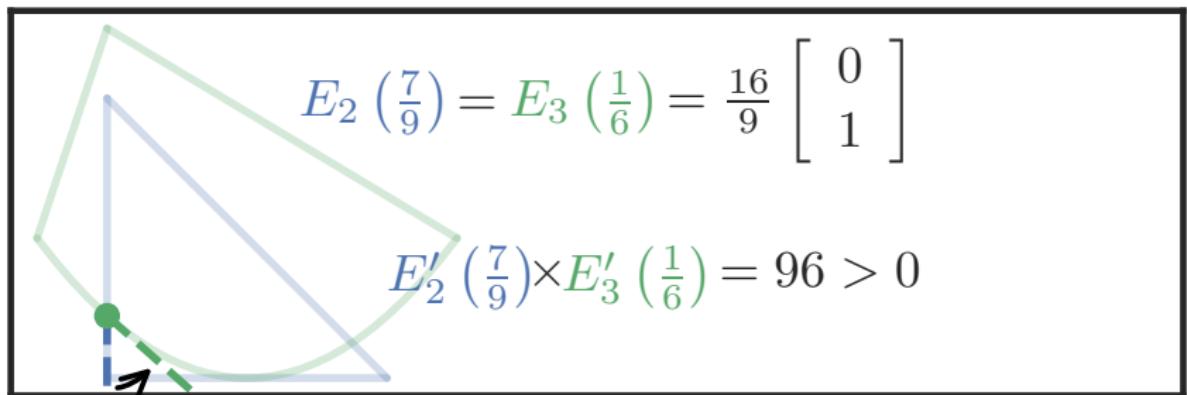
## Intersecting Curved Elements



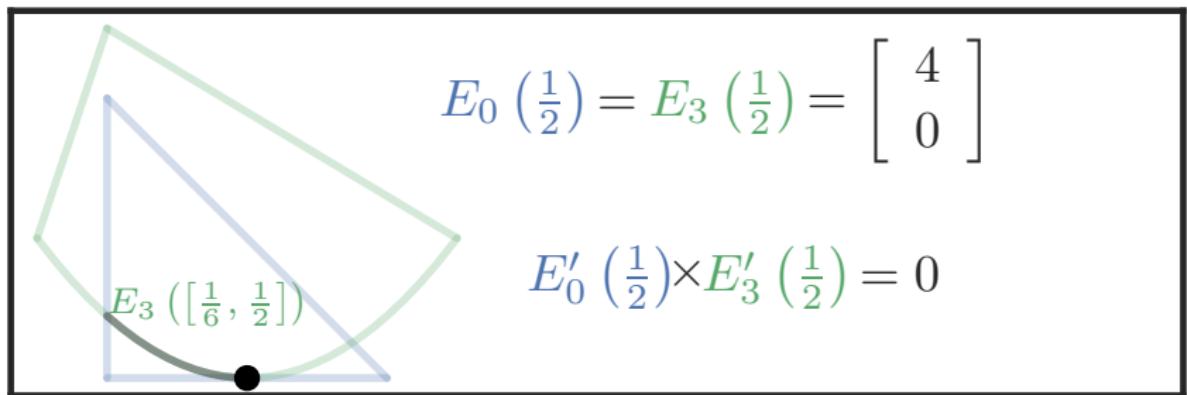
# Intersecting Curved Elements

$$E_2 \left( \frac{7}{9} \right) = E_3 \left( \frac{1}{6} \right) = \frac{16}{9} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

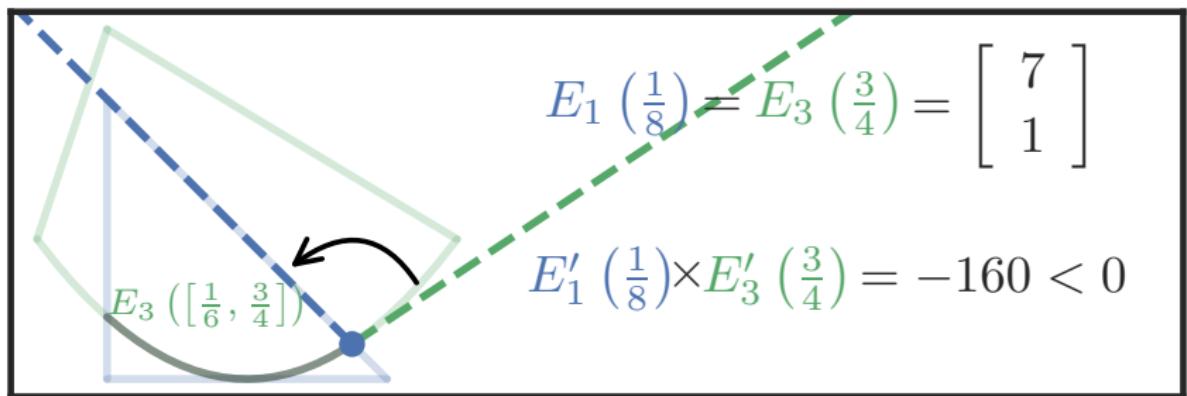
$$E'_2 \left( \frac{7}{9} \right) \times E'_3 \left( \frac{1}{6} \right) = 96 > 0$$



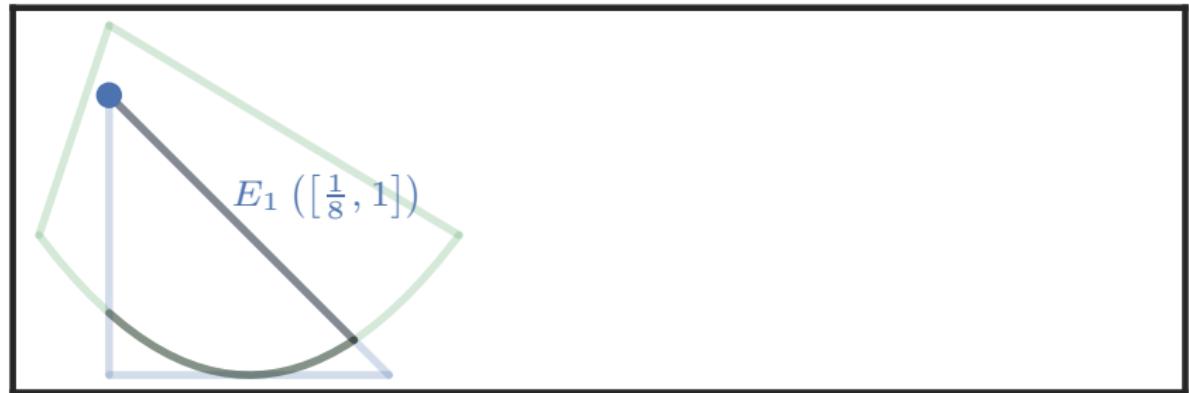
# Intersecting Curved Elements



# Intersecting Curved Elements



# Intersecting Curved Elements



# Intersecting Curved Elements



# Intersecting Curved Elements

- $\mathcal{P} = \mathcal{T}_0 \cap \mathcal{T}_1$
- $\partial\mathcal{P} = E_3 \left( \left[ \frac{1}{6}, \frac{3}{4} \right] \right) \cup E_1 \left( \left[ \frac{1}{8}, 1 \right] \right) \cup E_2 \left( \left[ 0, \frac{7}{9} \right] \right)$

## Advancing Front

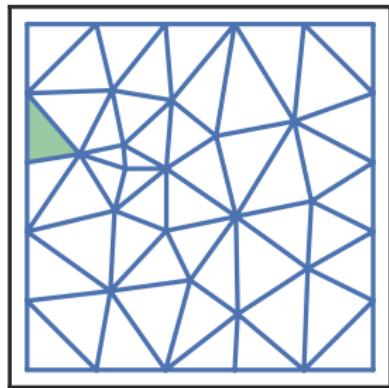
- Find all pairs  $\mathcal{T}, \mathcal{T}'$  of intersecting target and donor elements

## Advancing Front

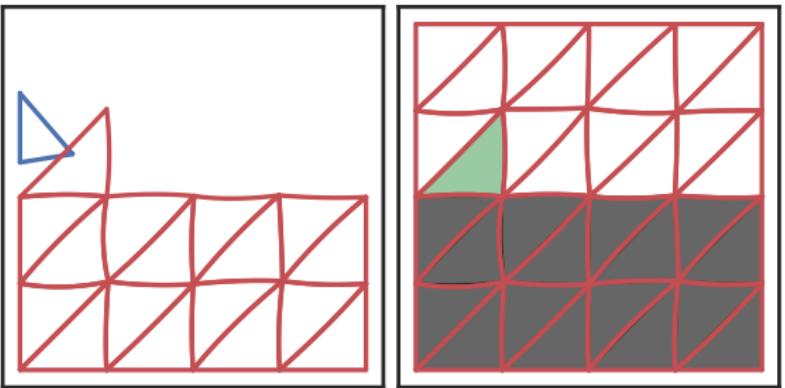
- Find all pairs  $\mathcal{T}, \mathcal{T}'$  of intersecting target and donor elements
- Fix target element  $\mathcal{T}$ , perform brute-force search

# Advancing Front

$\mathcal{M}_T$



$\mathcal{M}_D$



## Advancing Front

- Find all pairs  $\mathcal{T}, \mathcal{T}'$  of intersecting target and donor elements
- Fix target element  $\mathcal{T}$ , perform brute-force search

# Advancing Front

- Find all pairs  $\mathcal{T}, \mathcal{T}'$  of intersecting target and donor elements
- Fix target element  $\mathcal{T}$ , perform brute-force search
  - $\mathcal{O}(|\mathcal{M}_D|)$

## Advancing Front

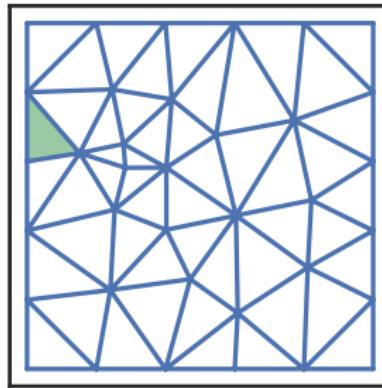
- Find all pairs  $\mathcal{T}, \mathcal{T}'$  of intersecting target and donor elements
- Fix target element  $\mathcal{T}$ , perform brute-force search
  - $\mathcal{O}(|\mathcal{M}_D|)$
  - $\mathcal{O}(1)$  in typical ALE, though acts as CFL

# Advancing Front

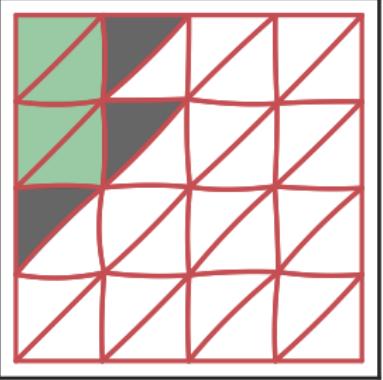
- Find all pairs  $\mathcal{T}, \mathcal{T}'$  of intersecting target and donor elements
- Fix target element  $\mathcal{T}$ , perform brute-force search
  - $\mathcal{O}(|\mathcal{M}_D|)$
  - $\mathcal{O}(1)$  in typical ALE, though acts as CFL
- Use donor mesh connectivity to perform breadth first search for neighbors  $\mathcal{N}_{\mathcal{T}}$  that intersect  $\mathcal{T}$  (as well as a one element buffer)

# Advancing Front

$\mathcal{M}_T$



$\mathcal{M}_D$



# Advancing Front

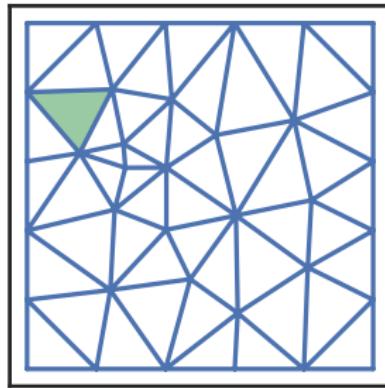
- Find all pairs  $\mathcal{T}, \mathcal{T}'$  of intersecting target and donor elements
- Fix target element  $\mathcal{T}$ , perform brute-force search
  - $\mathcal{O}(|\mathcal{M}_D|)$
  - $\mathcal{O}(1)$  in typical ALE, though acts as CFL
- Use donor mesh connectivity to perform breadth first search for neighbors  $\mathcal{N}_{\mathcal{T}}$  that intersect  $\mathcal{T}$  (as well as a one element buffer)

## Advancing Front

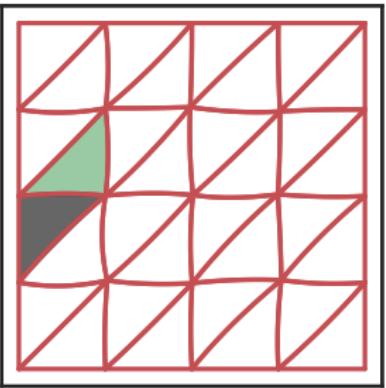
- Find all pairs  $\mathcal{T}, \mathcal{T}'$  of intersecting target and donor elements
- Fix target element  $\mathcal{T}$ , perform brute-force search
  - $\mathcal{O}(|\mathcal{M}_D|)$
  - $\mathcal{O}(1)$  in typical ALE, though acts as CFL
- Use donor mesh connectivity to perform breadth first search for neighbors  $\mathcal{N}_{\mathcal{T}}$  that intersect  $\mathcal{T}$  (as well as a one element buffer)
- Neighbors of  $\mathcal{T}$  can find one matching donor element among  $\mathcal{N}_{\mathcal{T}}$  in  $\mathcal{O}(|\mathcal{N}_{\mathcal{T}}|) = \mathcal{O}(1)$

# Advancing Front

$\mathcal{M}_T$



$\mathcal{M}_D$



## Advancing Front

- Find all pairs  $\mathcal{T}, \mathcal{T}'$  of intersecting target and donor elements
- Fix target element  $\mathcal{T}$ , perform brute-force search
  - $\mathcal{O}(|\mathcal{M}_D|)$
  - $\mathcal{O}(1)$  in typical ALE, though acts as CFL
- Use donor mesh connectivity to perform breadth first search for neighbors  $\mathcal{N}_{\mathcal{T}}$  that intersect  $\mathcal{T}$  (as well as a one element buffer)
- Neighbors of  $\mathcal{T}$  can find one matching donor element among  $\mathcal{N}_{\mathcal{T}}$  in  $\mathcal{O}(|\mathcal{N}_{\mathcal{T}}|) = \mathcal{O}(1)$

# Advancing Front

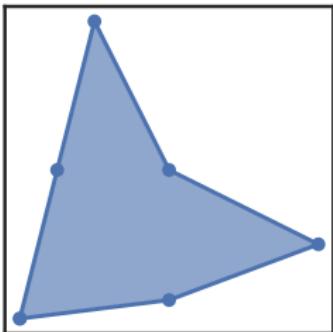
- Find all pairs  $\mathcal{T}, \mathcal{T}'$  of intersecting target and donor elements
- Fix target element  $\mathcal{T}$ , perform brute-force search
  - $\mathcal{O}(|\mathcal{M}_D|)$
  - $\mathcal{O}(1)$  in typical ALE, though acts as CFL
- Use donor mesh connectivity to perform breadth first search for neighbors  $\mathcal{N}_{\mathcal{T}}$  that intersect  $\mathcal{T}$  (as well as a one element buffer)
- Neighbors of  $\mathcal{T}$  can find one matching donor element among  $\mathcal{N}_{\mathcal{T}}$  in  $\mathcal{O}(|\mathcal{N}_{\mathcal{T}}|) = \mathcal{O}(1)$
- Total  $\mathcal{O}(|\mathcal{M}_D| + |\mathcal{M}_T|)$

# Integration over Curved Polygons

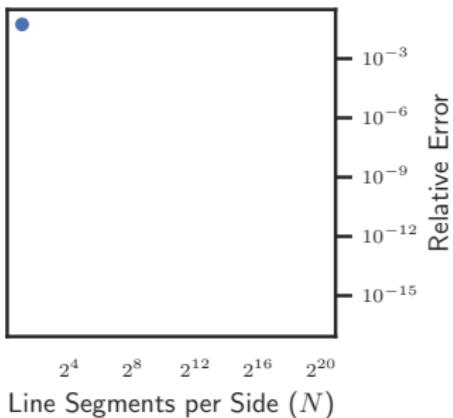
$$\int_{\mathcal{P}} F(x, y) \, dV, \quad \mathcal{P} = \mathcal{T}_0 \cap \mathcal{T}_1, \quad F = \phi_0 \phi_1$$

# Integrate via Polygonal Approximation

$$N = 2$$

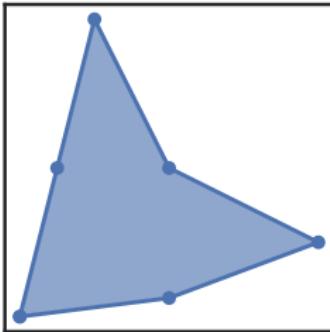


Area Estimates

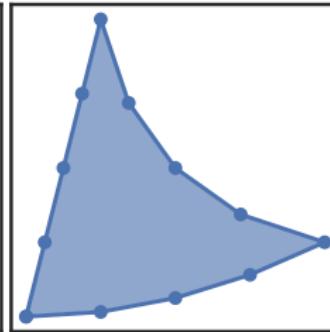


# Integrate via Polygonal Approximation

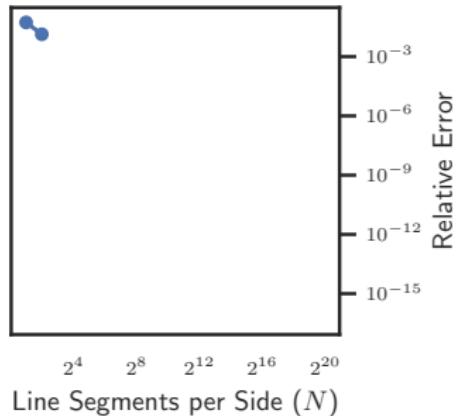
$N = 2$



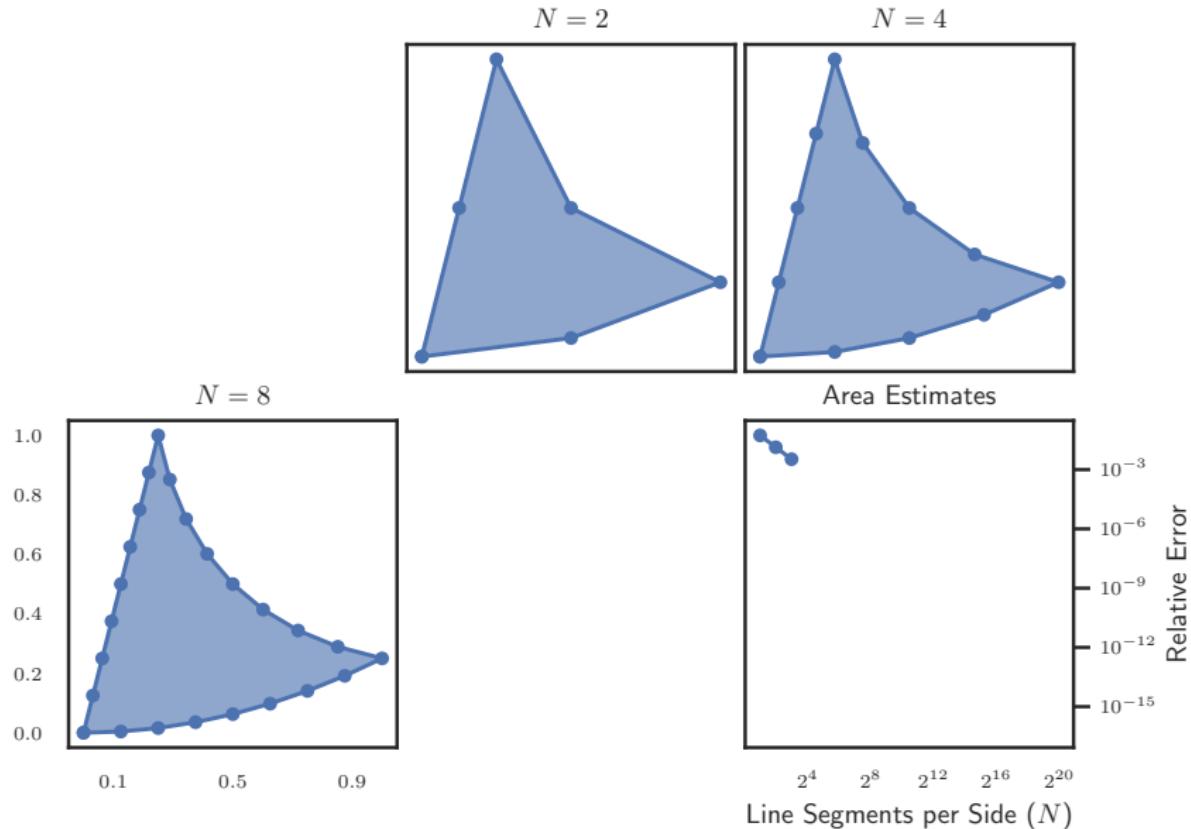
$N = 4$



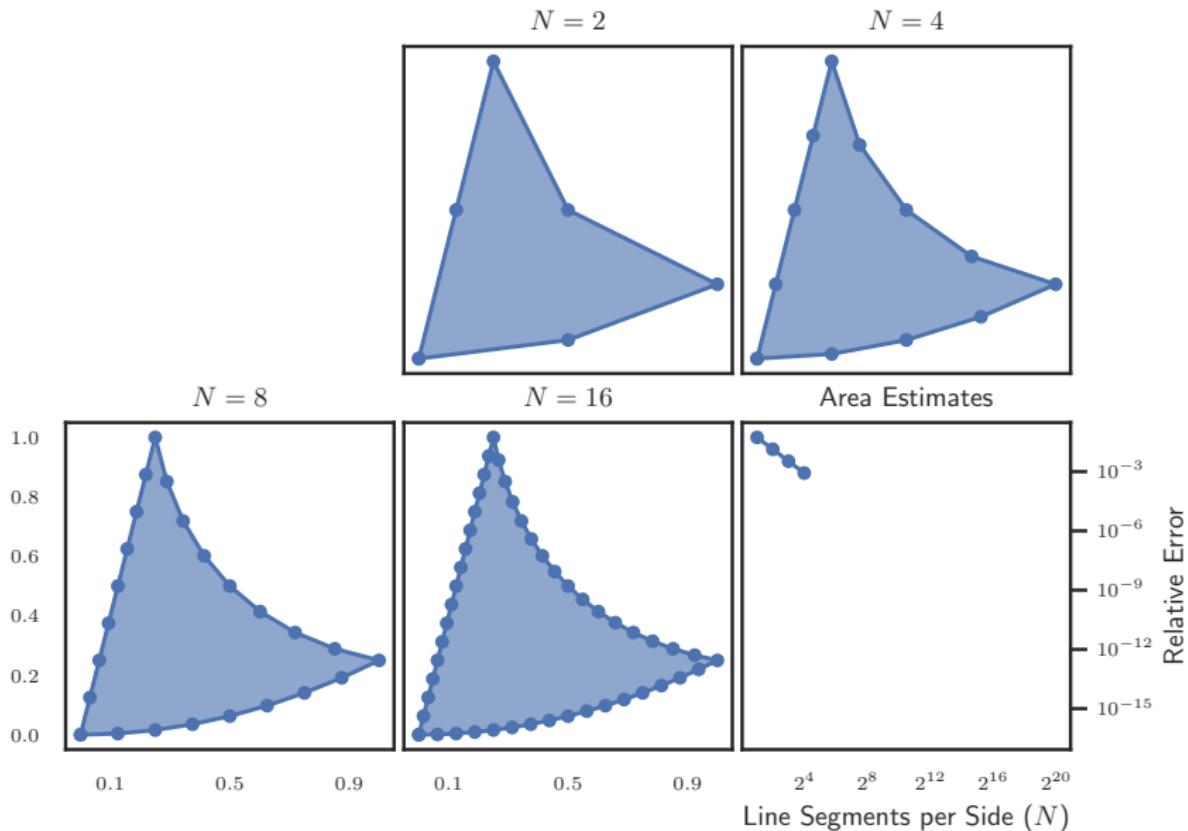
Area Estimates



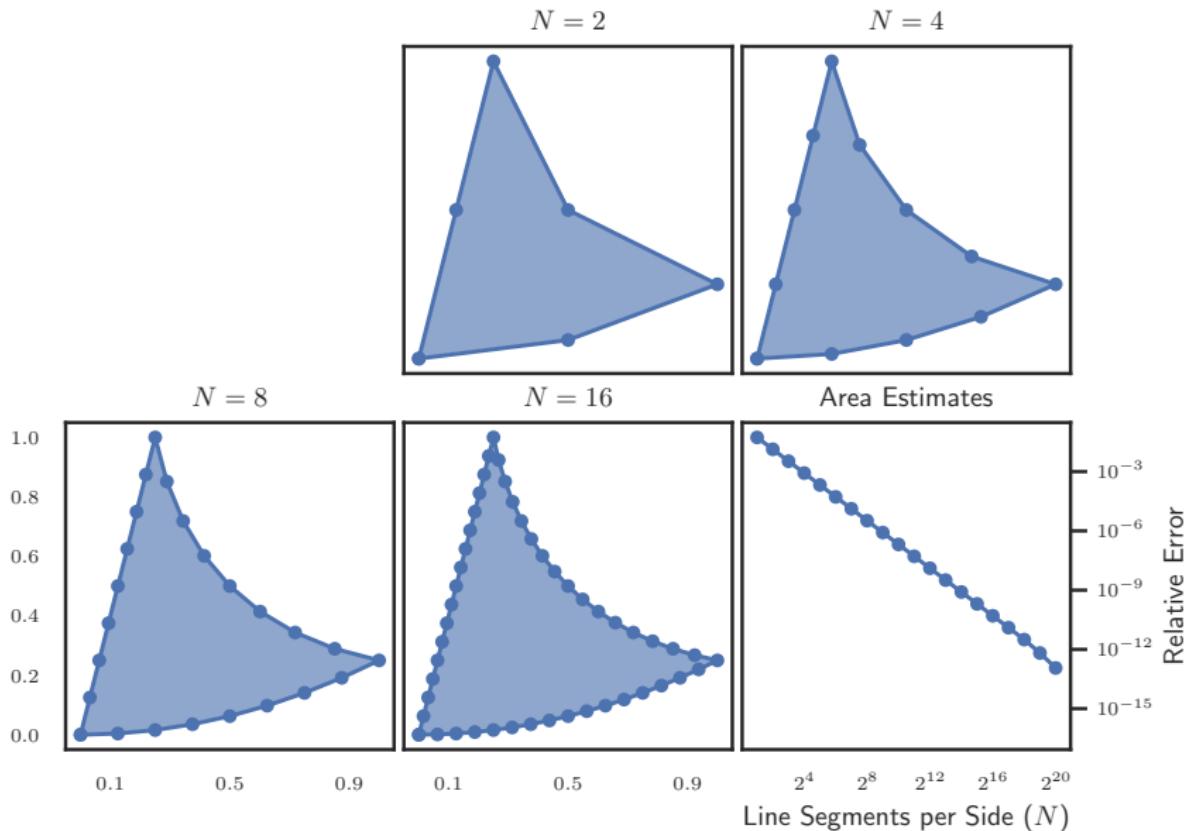
# Integrate via Polygonal Approximation



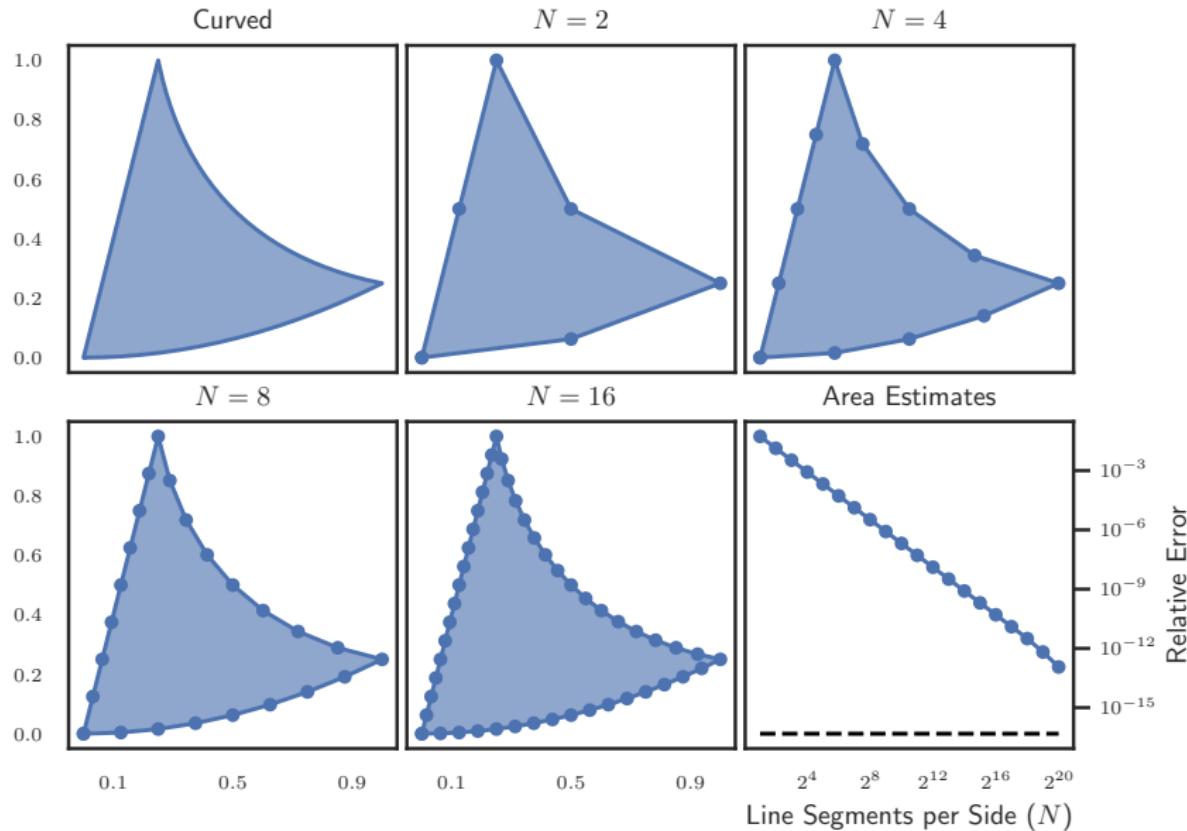
# Integrate via Polygonal Approximation



# Integrate via Polygonal Approximation



# Integrate via Polygonal Approximation



## Integrate via Polygonal Quadrature

- Polygonal quadrature rules not in wide use

## Integrate via Polygonal Quadrature

- Polygonal quadrature rules not in wide use
- $\mathcal{P}$  has curved edges

## Integrate via Polygonal Quadrature

- Polygonal quadrature rules not in wide use
- $\mathcal{P}$  has curved edges
- Transfinite interpolation or mean value coordinates

## Integrate via Polygonal Quadrature

- Polygonal quadrature rules not in wide use
- $\mathcal{P}$  has curved edges
- Transfinite interpolation or mean value coordinates
  - Maps from (straight sided) reference domain, but increase degree or are not bijective

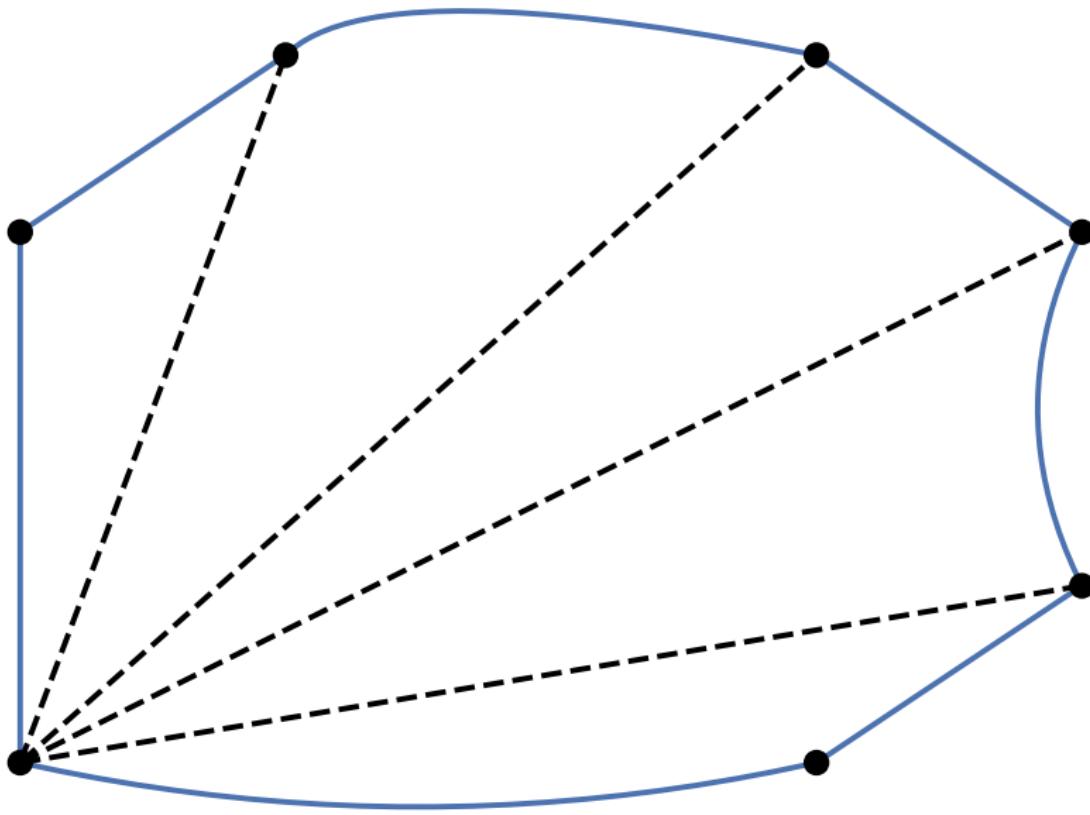
## Integrate via Tessellation

- Triangular quadrature rules simple and well established

## Integrate via Tessellation

- Triangular quadrature rules simple and well established
- Tessellate  $\mathcal{P}$  into disjoint union of Bézier triangles

## Integrate via Tessellation



## Integrate via Tessellation

- Not clear if an arbitrary curved polygon **can** be tessellated without introducing interior nodes

## Integrate via Tessellation

- Not clear if an arbitrary curved polygon **can** be tessellated without introducing interior nodes
- Need to check if diagonals cross edges; even worse if diagonals are also curved

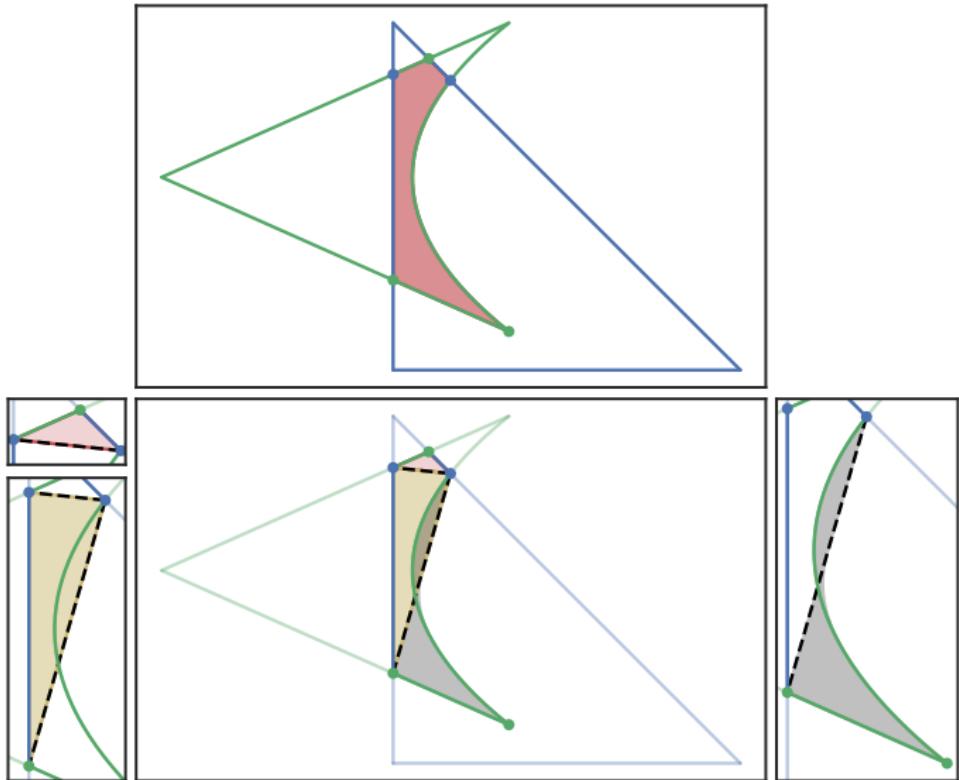
## Integrate via Tessellation

- Not clear if an arbitrary curved polygon **can** be tessellated without introducing interior nodes
- Need to check if diagonals cross edges; even worse if diagonals are also curved
- If diagonals are valid, high degree Bézier triangles need interior control points introduced that don't cause triangle to invert

## Integrate via Tessellation

- Tessellate with inverted Bézier triangles?

# Integrate via Tessellation



## Integrate via Tessellation

- Tessellate with inverted Bézier triangles?

## Integrate via Tessellation

- Tessellate with inverted Bézier triangles?
- E.g. tessellation of  $\mathcal{P}$  contains inverted  $\mathcal{T}_2 = b(\mathcal{U})$ , then numerically integrate

$$\int_{\mathcal{T}_2} \phi_0 \phi_1 \, dx \, dy = \int_{\mathcal{U}} |\det(Db)| (\phi_0 \circ b) (\phi_1 \circ b) \, ds \, dt$$

## Integrate via Tessellation

- Tessellate with inverted Bézier triangles?
- E.g. tessellation of  $\mathcal{P}$  contains inverted  $\mathcal{T}_2 = b(\mathcal{U})$ , then numerically integrate

$$\int_{\mathcal{T}_2} \phi_0 \phi_1 \, dx \, dy = \int_{\mathcal{U}} |\det(Db)| (\phi_0 \circ b) (\phi_1 \circ b) \, ds \, dt$$

- $|\det(Db)|$  non-smooth, bad for quadrature

## Integrate via Tessellation

- Tessellate with inverted Bézier triangles?
- E.g. tessellation of  $\mathcal{P}$  contains inverted  $\mathcal{T}_2 = b(\mathcal{U})$ , then numerically integrate

$$\int_{\mathcal{T}_2} \phi_0 \phi_1 \, dx \, dy = \int_{\mathcal{U}} |\det(Db)| (\phi_0 \circ b) (\phi_1 \circ b) \, ds \, dt$$

- $|\det(Db)|$  non-smooth, bad for quadrature
- If  $\phi_0$  from pre-image basis, then  $\phi_0 \circ b = \widehat{\phi}_0 \circ b_0^{-1} \circ b$

## Integrate via Tessellation

- Tessellate with inverted Bézier triangles?
- E.g. tessellation of  $\mathcal{P}$  contains inverted  $\mathcal{T}_2 = b(\mathcal{U})$ , then numerically integrate

$$\int_{\mathcal{T}_2} \phi_0 \phi_1 \, dx \, dy = \int_{\mathcal{U}} |\det(Db)| (\phi_0 \circ b) (\phi_1 \circ b) \, ds \, dt$$

- $|\det(Db)|$  non-smooth, bad for quadrature
- If  $\phi_0$  from pre-image basis, then  $\phi_0 \circ b = \widehat{\phi}_0 \circ b_0^{-1} \circ b$
- $\mathcal{T}_2$  inverted,  $b(s, t) \notin \mathcal{P}$  is possible but  $b_0^{-1}$  need not be defined outside of  $\mathcal{T}_0$

## Integrate via Green's Theorem

- Horizontal antiderivative  $H$  and vertical antiderivative  $V$  s.t.

$$H_x = V_y = F$$

## Integrate via Green's Theorem

- Horizontal antiderivative  $H$  and vertical antiderivative  $V$  s.t.  
$$H_x = V_y = F$$
- $H(0, y) \equiv V(x, 0) \equiv 0$  for uniqueness (needed to evaluate)

## Integrate via Green's Theorem

- Horizontal antiderivative  $H$  and vertical antiderivative  $V$  s.t.  
$$H_x = V_y = F$$
- $H(0, y) \equiv V(x, 0) \equiv 0$  for uniqueness (needed to evaluate)
- If  $\partial\mathcal{P} = C_1 \cup \dots \cup C_n$ , since  $2F = H_x + V_y$  Green's gives

$$\int_{\mathcal{P}} 2F dV = \oint_{\partial\mathcal{P}} H dy - V dx = \sum_j \int_{C_j} H dy - V dx$$

## Integrate via Green's Theorem

- Horizontal antiderivative  $H$  and vertical antiderivative  $V$  s.t.  
$$H_x = V_y = F$$
- $H(0, y) \equiv V(x, 0) \equiv 0$  for uniqueness (needed to evaluate)
- If  $\partial\mathcal{P} = C_1 \cup \dots \cup C_n$ , since  $2F = H_x + V_y$  Green's gives

$$\int_{\mathcal{P}} 2F dV = \oint_{\partial\mathcal{P}} H dy - V dx = \sum_j \int_{C_j} H dy - V dx$$

- Each  $C$  given by  $x(r), y(r)$ , 1D quadrature on unit interval of

$$G(r) = H(x(r), y(r))y'(r) - V(x(r), y(r))x'(r)$$

## Integrate via Green's Theorem

- How to evaluate  $H(x(r), y(r))$ ?

## Integrate via Green's Theorem

- How to evaluate  $H(x(r), y(r))$ ?
- Fundamental theorem of calculus

$$H(\alpha, \beta) = H(\alpha, \beta) - H(0, \beta) = \int_0^\alpha F(x, \beta) \ dx$$

## Integrate via Green's Theorem

- How to evaluate  $H(x(r), y(r))$ ?
- Fundamental theorem of calculus

$$H(\alpha, \beta) = H(\alpha, \beta) - H(0, \beta) = \int_0^\alpha F(x, \beta) \ dx$$

- Quadrature points  $F(x_j, \beta)$  may fall outside  $\mathcal{P}$ , so pre-image basis cannot be used

## Integrate via Green's Theorem

- How to evaluate  $H(x(r), y(r))$ ?
- Fundamental theorem of calculus

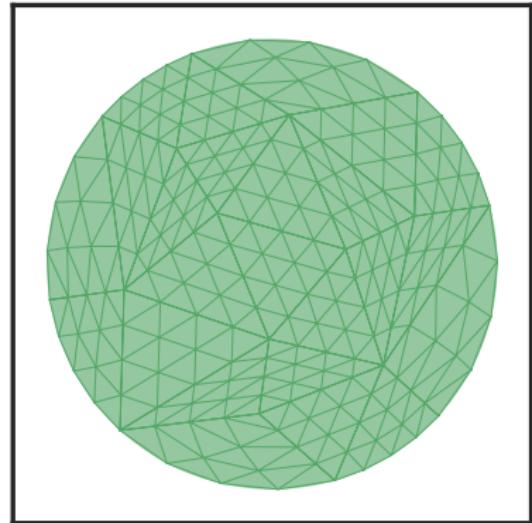
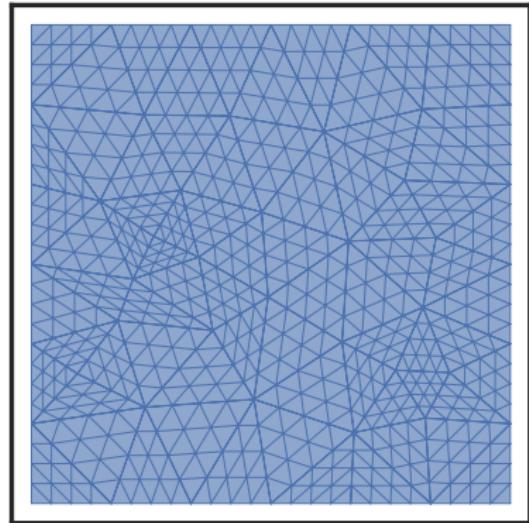
$$H(\alpha, \beta) = H(\alpha, \beta) - H(0, \beta) = \int_0^\alpha F(x, \beta) \, dx$$

- Quadrature points  $F(x_j, \beta)$  may fall outside  $\mathcal{P}$ , so pre-image basis cannot be used
- Global coordinates basis means  $F$  is a polynomial on  $\mathbf{R}^2$ , so quadrature is exact

# Numerical Experiments

- Three meshes ( $p = 1, 2, 3$ )

# Numerical Experiments



# Numerical Experiments

- Three meshes ( $p = 1, 2, 3$ )

# Numerical Experiments

- Three meshes ( $p = 1, 2, 3$ )
- Three functions

# Numerical Experiments

- Three meshes ( $p = 1, 2, 3$ )
- Three functions
  - $\zeta_1(x, y) = 5y^3 + x^2 + 2y + 3$

# Numerical Experiments

- Three meshes ( $p = 1, 2, 3$ )
- Three functions
  - $\zeta_1(x, y) = 5y^3 + x^2 + 2y + 3$
  - $\zeta_2(x, y) = \exp(x^2) + 2y$

# Numerical Experiments

- Three meshes ( $p = 1, 2, 3$ )
- Three functions
  - $\zeta_1(x, y) = 5y^3 + x^2 + 2y + 3$
  - $\zeta_2(x, y) = \exp(x^2) + 2y$
  - $\zeta_3(x, y) = \sin(x) + \cos(y)$

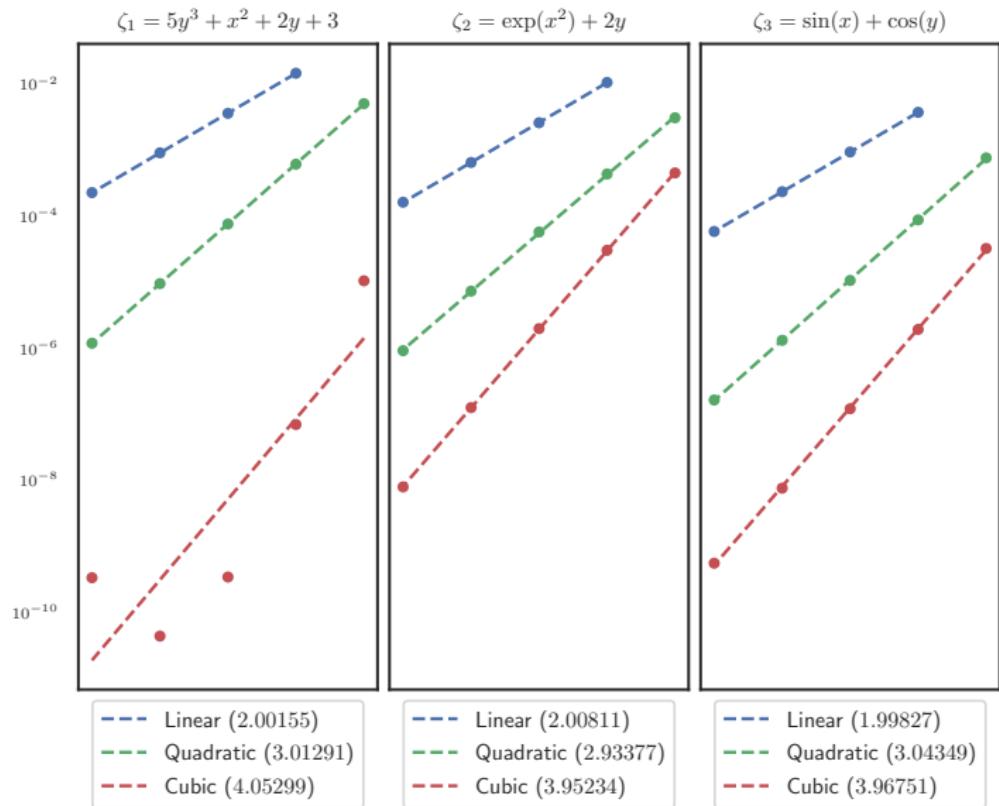
# Numerical Experiments

- Three meshes ( $p = 1, 2, 3$ )
- Three functions
  - $\zeta_1(x, y) = 5y^3 + x^2 + 2y + 3$
  - $\zeta_2(x, y) = \exp(x^2) + 2y$
  - $\zeta_3(x, y) = \sin(x) + \cos(y)$
- Nodal interpolant  $\mathbf{q}_D = \sum_j \zeta(\mathbf{n}_j) \phi_D^{(j)}$

# Numerical Experiments

- Three meshes ( $p = 1, 2, 3$ )
- Three functions
  - $\zeta_1(x, y) = 5y^3 + x^2 + 2y + 3$
  - $\zeta_2(x, y) = \exp(x^2) + 2y$
  - $\zeta_3(x, y) = \sin(x) + \cos(y)$
- Nodal interpolant  $\mathbf{q}_D = \sum_j \zeta(\mathbf{n}_j) \phi_D^{(j)}$
- Expect  $\mathcal{O}(h^{p+1})$  errors

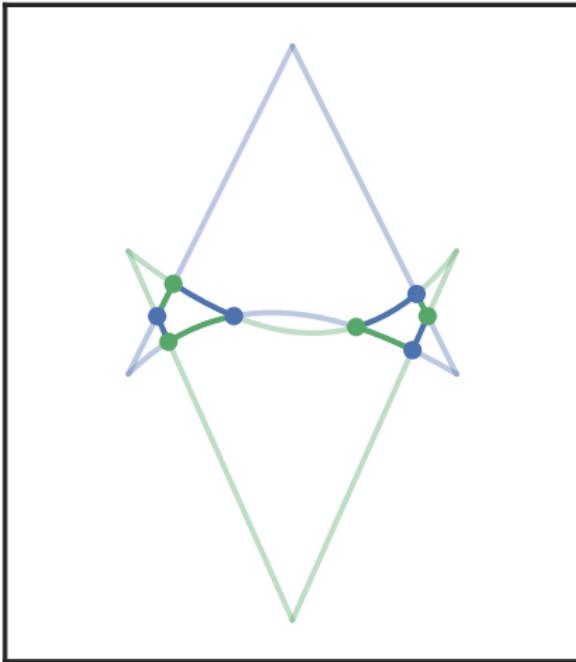
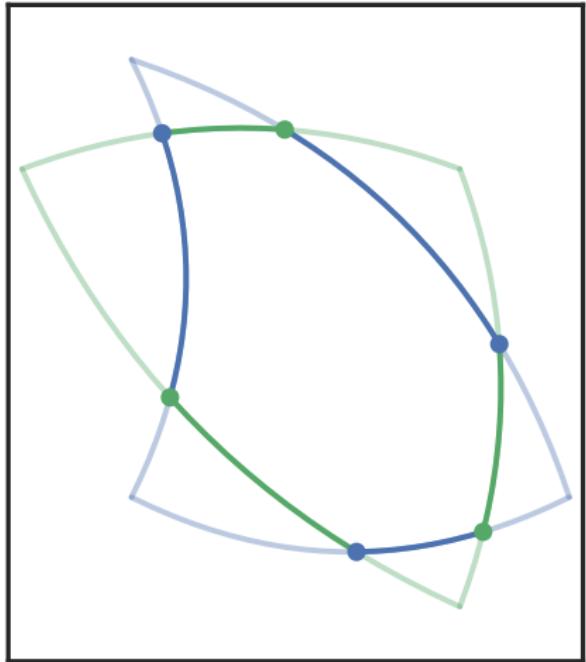
# Numerical Experiments



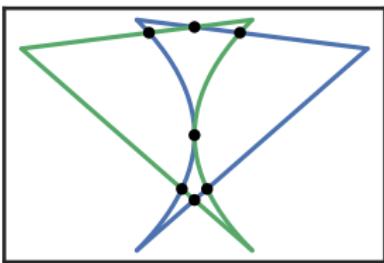
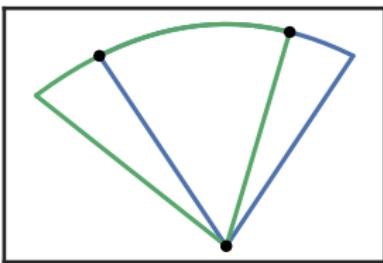
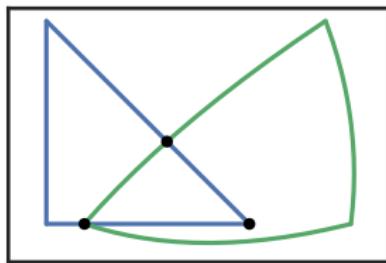
## Ill-conditioned Bézier Curve Intersection

---

## III-Conditioned Intersections



# Ill-Conditioned Intersections



## Ill-Conditioned Intersections

- Edges are Bézier curves, e.g.  $b(r, 0) = \sum_{j=0}^n \binom{n}{j} (1-r)^{n-j} r^j \mathbf{p}_{n-j,j,0}$

## Ill-Conditioned Intersections

- Edges are Bézier curves, e.g.  $b(r, 0) = \sum_{j=0}^n \binom{n}{j} (1-r)^{n-j} r^j \mathbf{p}_{n-j,j,0}$
- Tangent intersection equivalent to double root of polynomial,  
i.e. condition number is infinite

## Ill-Conditioned Intersections

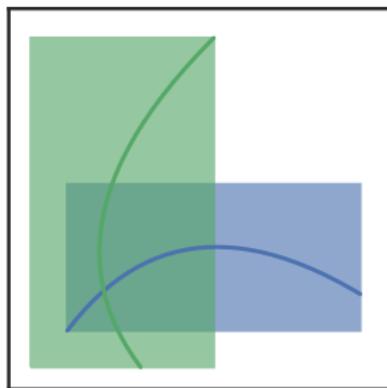
- Edges are Bézier curves, e.g.  $b(r, 0) = \sum_{j=0}^n \binom{n}{j} (1-r)^{n-j} r^j \mathbf{p}_{n-j,j,0}$
- Tangent intersection equivalent to double root of polynomial, i.e. condition number is infinite
- Random pair of meshes, “almost tangent” intersections increasingly frequent as  $h \rightarrow 0^+$

# Intersection Algorithm

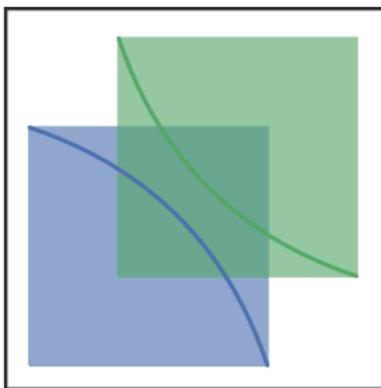
- Bounding box check

# Intersection Algorithm

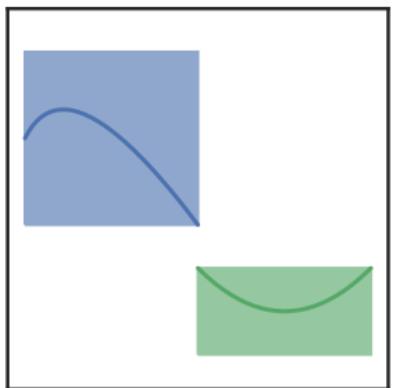
MAYBE



MAYBE



NO



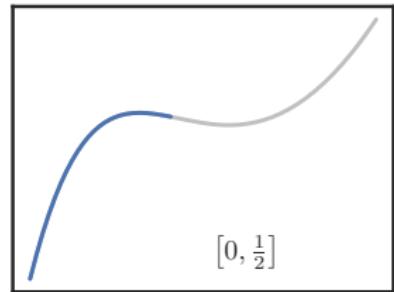
# Intersection Algorithm

- Bounding box check

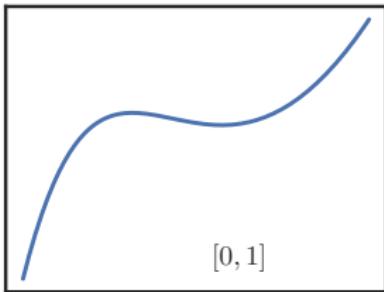
# Intersection Algorithm

- Bounding box check
- Curve subdivision

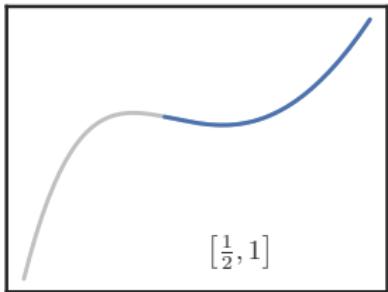
# Intersection Algorithm



$$\left[0, \frac{1}{2}\right]$$



$$\left[0, 1\right]$$



$$\left[\frac{1}{2}, 1\right]$$

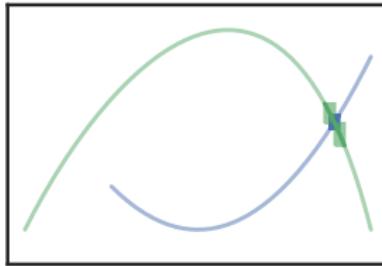
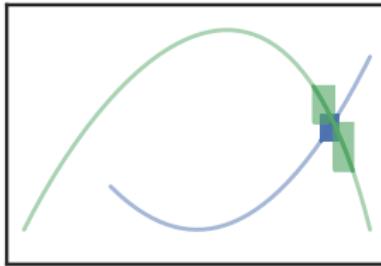
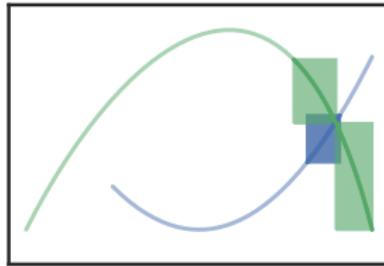
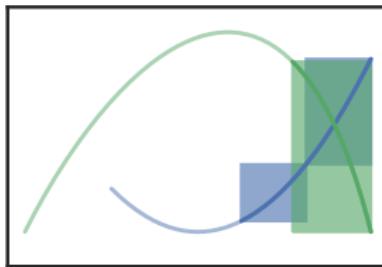
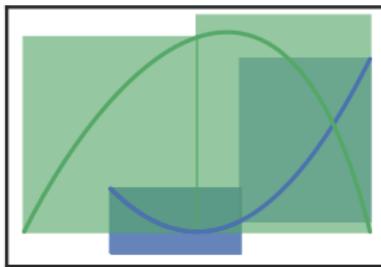
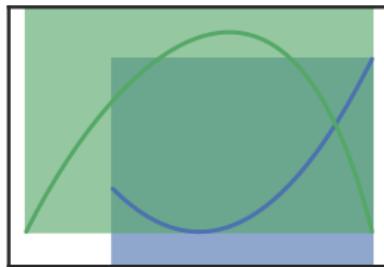
# Intersection Algorithm

- Bounding box check
- Curve subdivision

# Intersection Algorithm

- Bounding box check
- Curve subdivision
- Check subdivided pairs

# Intersection Algorithm



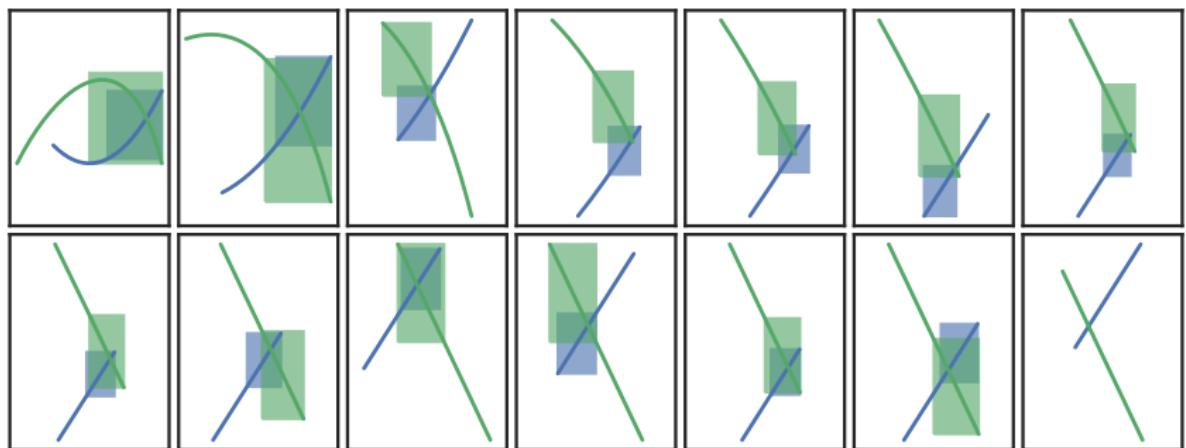
# Intersection Algorithm

- Bounding box check
- Curve subdivision
- Check subdivided pairs

# Intersection Algorithm

- Bounding box check
- Curve subdivision
- Check subdivided pairs
- Subdivide until “almost” linear

# Intersection Algorithm



# Intersection Algorithm

- Bounding box check
- Curve subdivision
- Check subdivided pairs
- Subdivide until “almost” linear

# Intersection Algorithm

- Bounding box check
- Curve subdivision
- Check subdivided pairs
- Subdivide until “almost” linear
- Use intersection of lines as seed for Newton’s method

## Newton's Method

- Bézier curves  $b_0(s), b_1(t)$  map into  $\mathbf{R}^2$

## Newton's Method

- Bézier curves  $b_0(s), b_1(t)$  map into  $\mathbf{R}^2$
- Residual  $F(s, t) = b_0(s) - b_1(t)$ , Jacobian  $J = \begin{bmatrix} b'_0(s) & -b'_1(t) \end{bmatrix}$

# Newton's Method

- Bézier curves  $b_0(s), b_1(t)$  map into  $\mathbf{R}^2$
- Residual  $F(s, t) = b_0(s) - b_1(t)$ , Jacobian  $J = \begin{bmatrix} b'_0(s) & -b'_1(t) \end{bmatrix}$
- Newton's:

$$\begin{bmatrix} s_{n+1} \\ t_{n+1} \end{bmatrix} = \begin{bmatrix} s_n \\ t_n \end{bmatrix} - J_n^{-1} F_n$$

# Newton's Method

- Bézier curves  $b_0(s), b_1(t)$  map into  $\mathbf{R}^2$
- Residual  $F(s, t) = b_0(s) - b_1(t)$ , Jacobian  $J = \begin{bmatrix} b'_0(s) & -b'_1(t) \end{bmatrix}$
- Newton's:  
$$\begin{bmatrix} s_{n+1} \\ t_{n+1} \end{bmatrix} = \begin{bmatrix} s_n \\ t_n \end{bmatrix} - J_n^{-1} F_n$$
- At tangent intersections,  $J$  is exactly singular

## Newton's Method

- Bézier curves  $b_0(s), b_1(t)$  map into  $\mathbf{R}^2$
- Residual  $F(s, t) = b_0(s) - b_1(t)$ , Jacobian  $J = \begin{bmatrix} b'_0(s) & -b'_1(t) \end{bmatrix}$
- Newton's:

$$\begin{bmatrix} s_{n+1} \\ t_{n+1} \end{bmatrix} = \begin{bmatrix} s_n \\ t_n \end{bmatrix} - J_n^{-1} F_n$$

- At tangent intersections,  $J$  is exactly singular
- At ill-conditioned intersections,  $J$  is almost singular and evaluation of  $F$  is typically ill-conditioned as well

## Compensated Evaluation

---

## Modified Newton's for Intersection

---