

High-order Solution Transfer between Curved Meshes and Ill-conditioned Bézier Curve Intersection

Danny Hermes

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dhermes@berkeley.edu
UC Berkeley



Outline

1. Introduction and motivation
2. Curved Elements
3. Solution Transfer
4. Ill-conditioned Bézier Curve Intersection
5. Compensated Evaluation
6. Modified Newton's for Intersection

Introduction and motivation

Method of Characteristics

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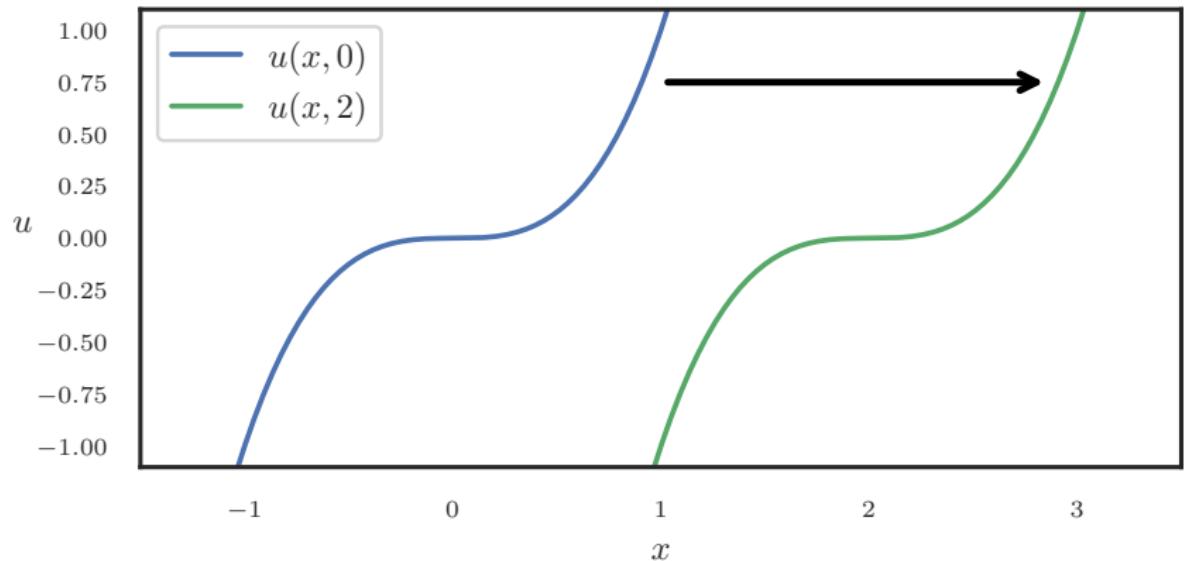
Divide physical domain

$$x(t) = x_0 + ct$$

PDE becomes a (trivial) ODE

$$\frac{d}{dt}u(x(t), t) = 0.$$

Method of Characteristics



Lagrangian Methods

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- Carry value (e.g. heat, pressure, density) along characteristic curve

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- Transform PDE to family of ODEs

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 - Resolve sensitive features

Remeshing Example

Consider

$$u_t + \begin{bmatrix} y^2 \\ 1 \end{bmatrix} \cdot \nabla u + F(u, \nabla u) = 0$$

Remeshing Example

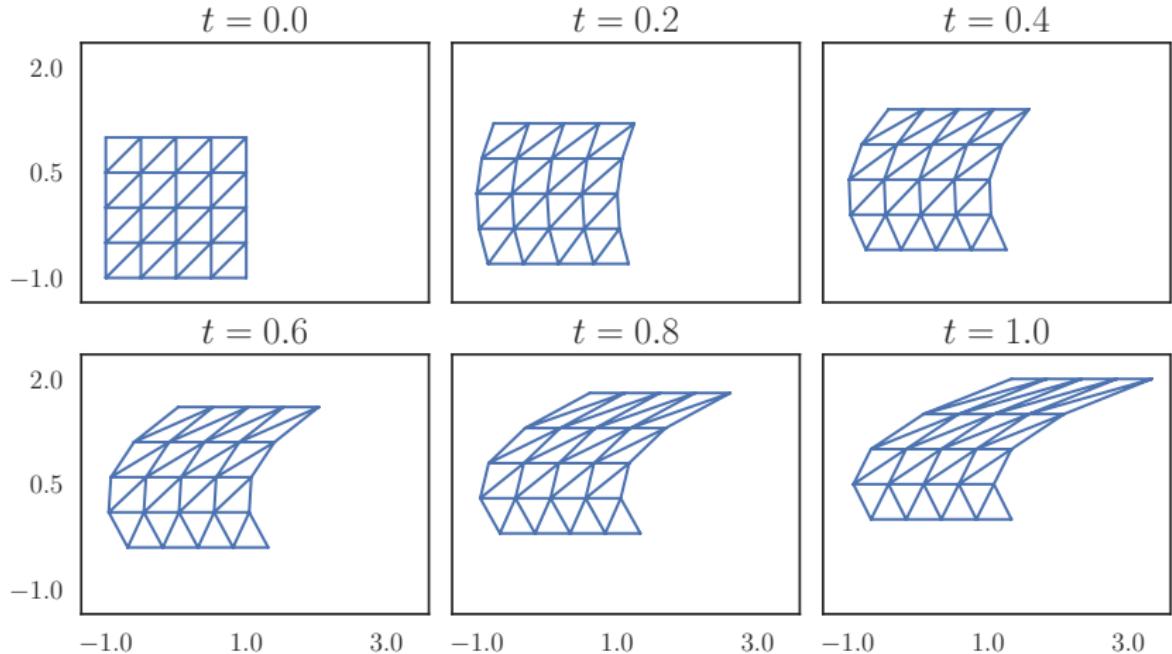
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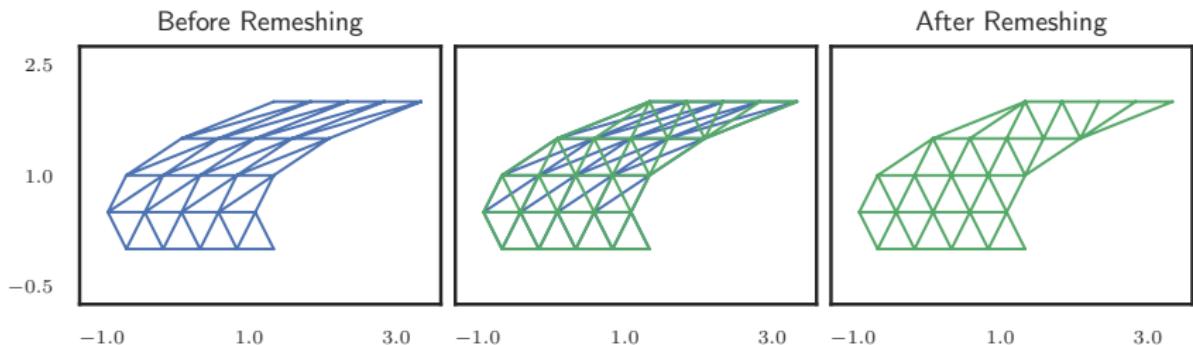
with cubic characteristics

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} (y_0 + t)^3 - y_0^3 \\ 3t \end{bmatrix}.$$

Remeshing Example



Remeshing Example



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Curved Meshes

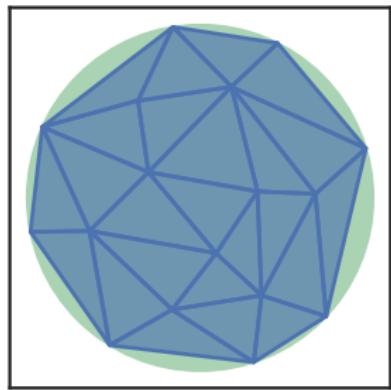
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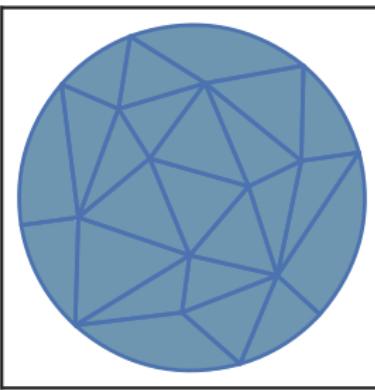
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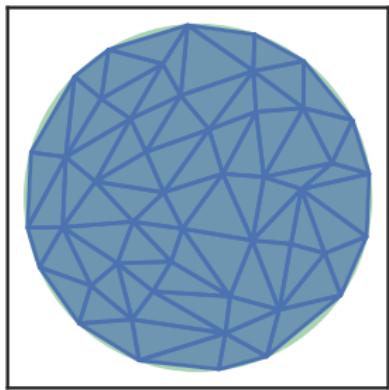
Linear Mesh, 24 elements



Quadratic Mesh, 24 elements



Linear Mesh, 74 elements



-1 0 1 -1 0 1 -1 0 1

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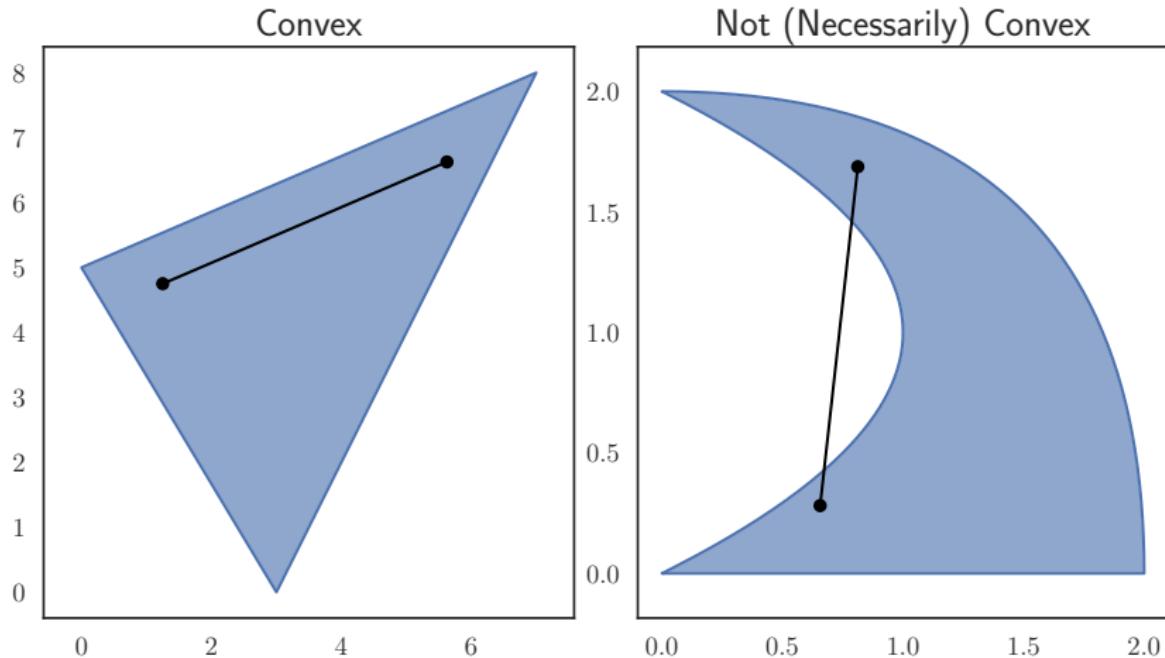
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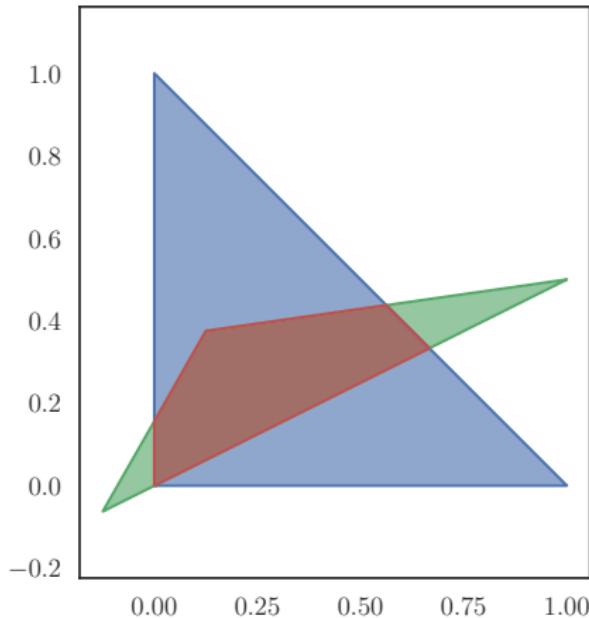
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 - Greater geometric flexibility
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- Drawbacks
 - Harder to implement
 - Loss of accuracy in high degree (e.g. Runge's phenomenon)
 - More challenging geometry

Curved Meshes

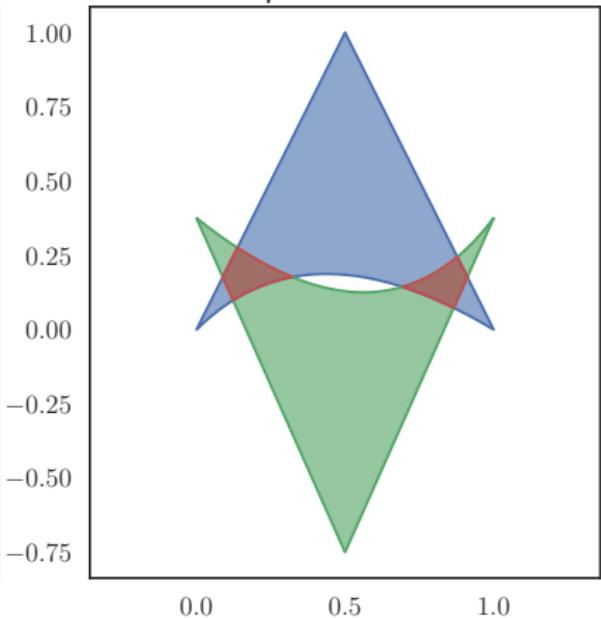


Curved Meshes

Convex Intersection



Multiple Intersections



Curved Elements

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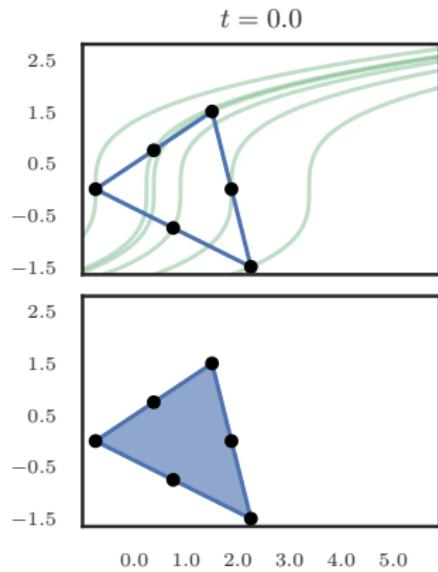
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- With non-linear shape functions (i.e. not straight sided), non-vertex nodes used

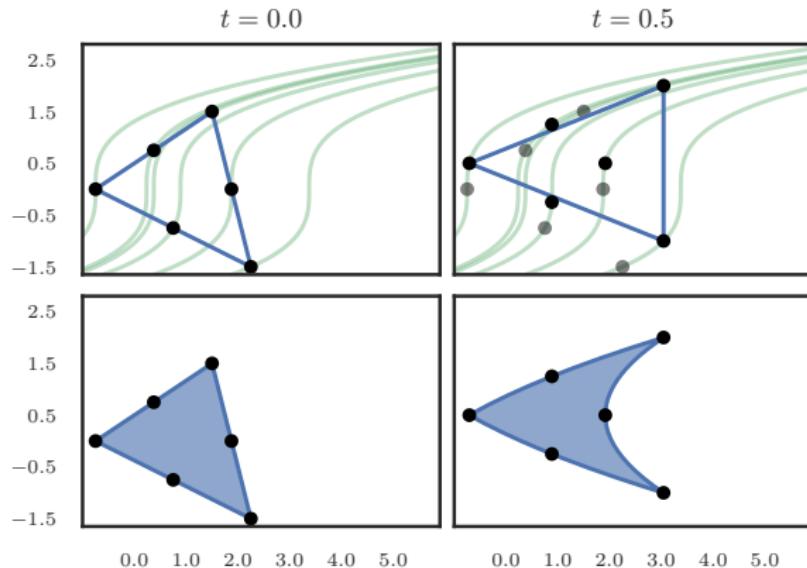
Curved Elements

- Necessary for High-order
- With non-linear shape functions (i.e. not straight sided), non-vertex nodes used
- Lagrangian method must either curve mesh or information about flow of geometry will be lost

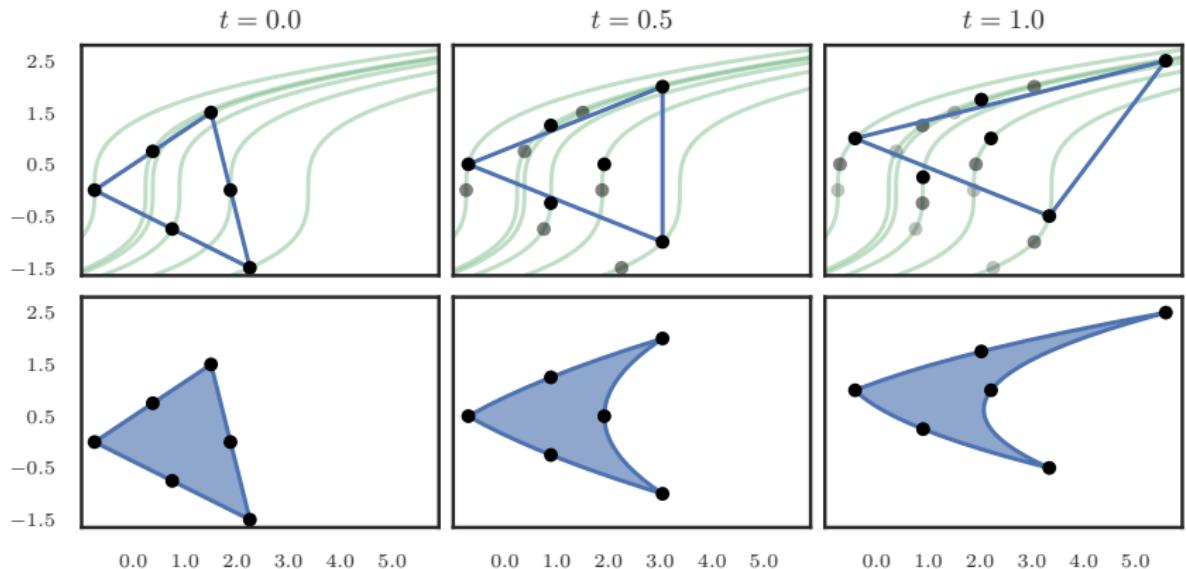
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Curved Elements

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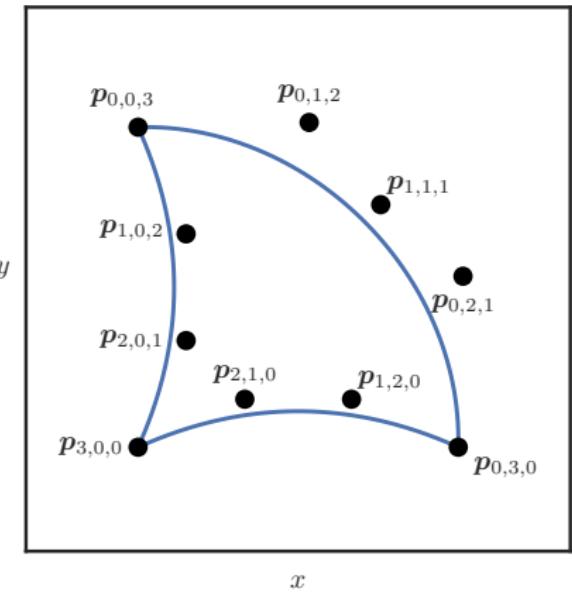
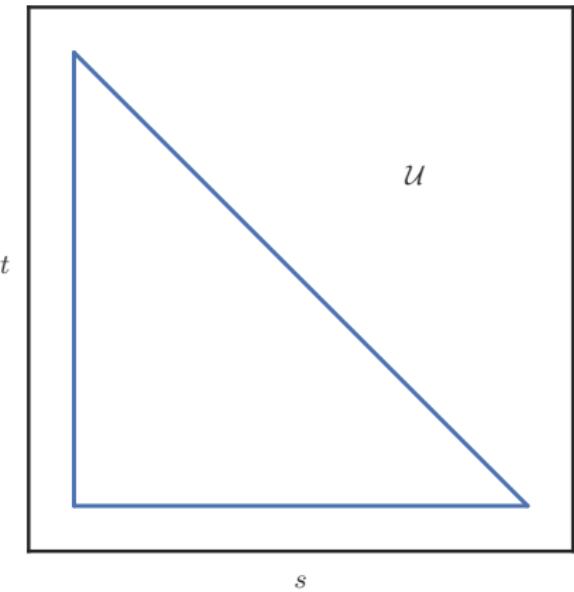
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- Convex combination of control points

$$b(s, t) = \sum_{\substack{i+j+k=n \\ i,j,k \geq 0}} \binom{n}{i,j,k} \lambda_1^i \lambda_2^j \lambda_3^k \mathbf{p}_{i,j,k}$$

Bézier Triangles



Bézier Triangles

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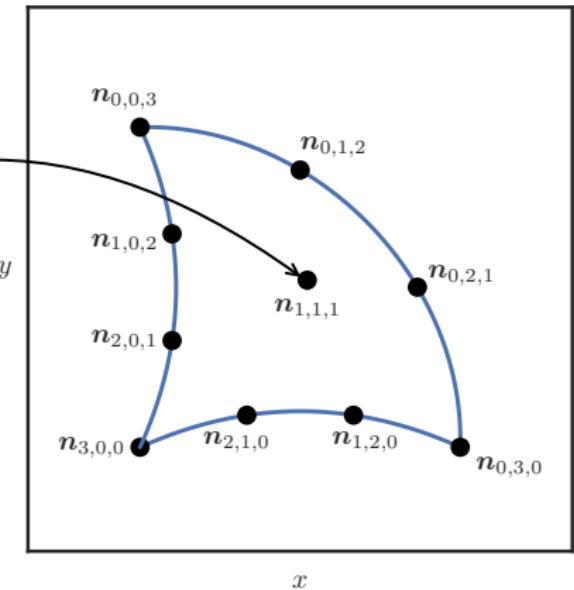
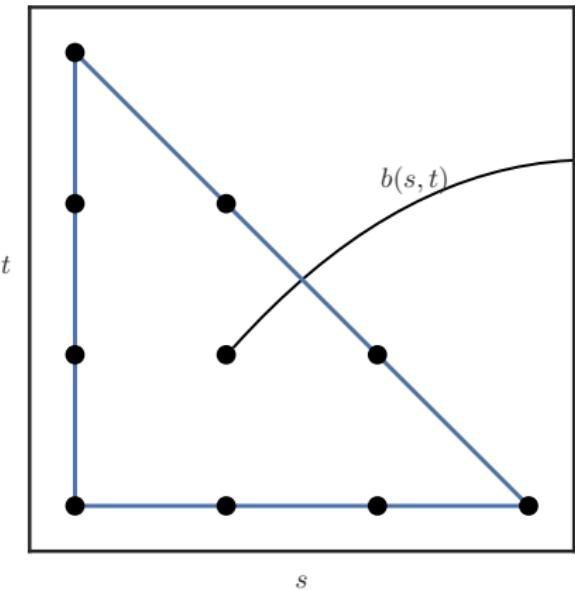
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- For example, taking $\mathbf{n}_{i,j,k} = \delta_{(i,j,k)} (i_0, j_0, k_0)$ gives degree n shape functions on \mathcal{U}
- Conversion between $\mathbf{n}_{i,j,k}$ and $\mathbf{p}_{i,j,k}$ has condition number exponential in n

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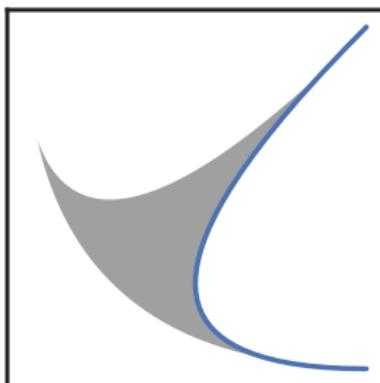
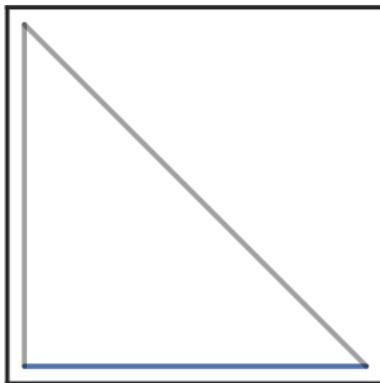
- Element \mathcal{T} is **valid** if diffeomorphic to \mathcal{U}
- $b(s, t)$ bijective, i.e. Jacobian Db is everywhere invertible
- $\det(Db)$ positive, preserves orientation

Inverted Element

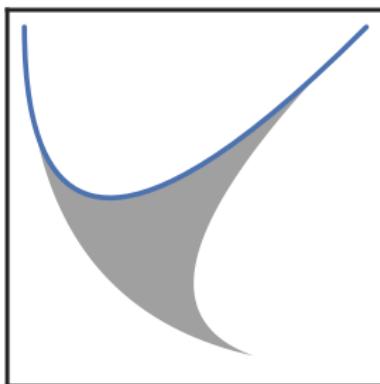
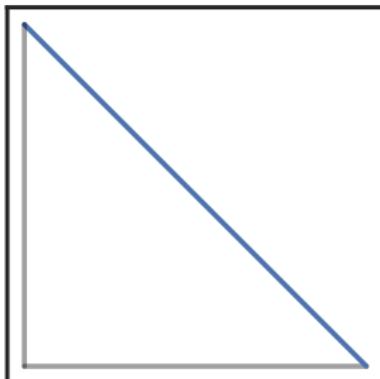
Consider element given by map

$$b(s, t) = \lambda_1^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_2^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \lambda_3^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

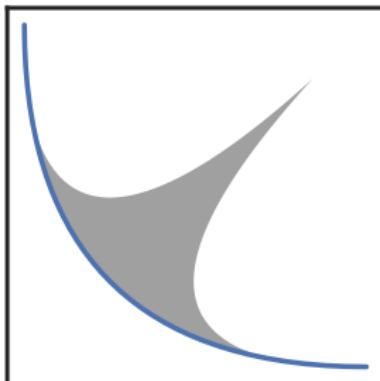
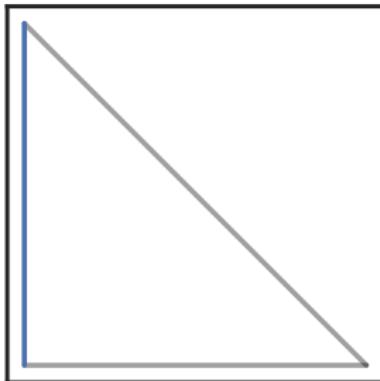
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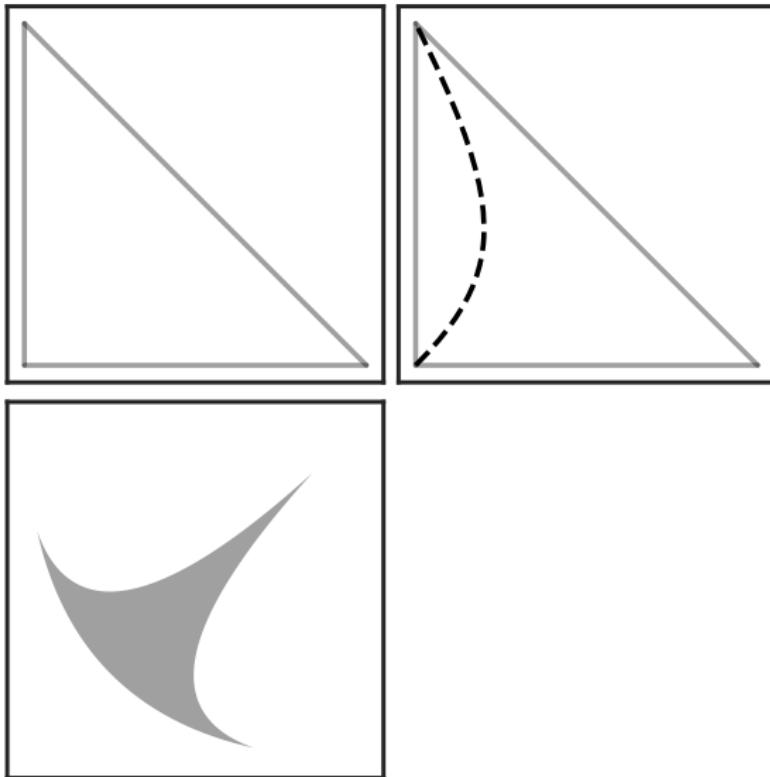
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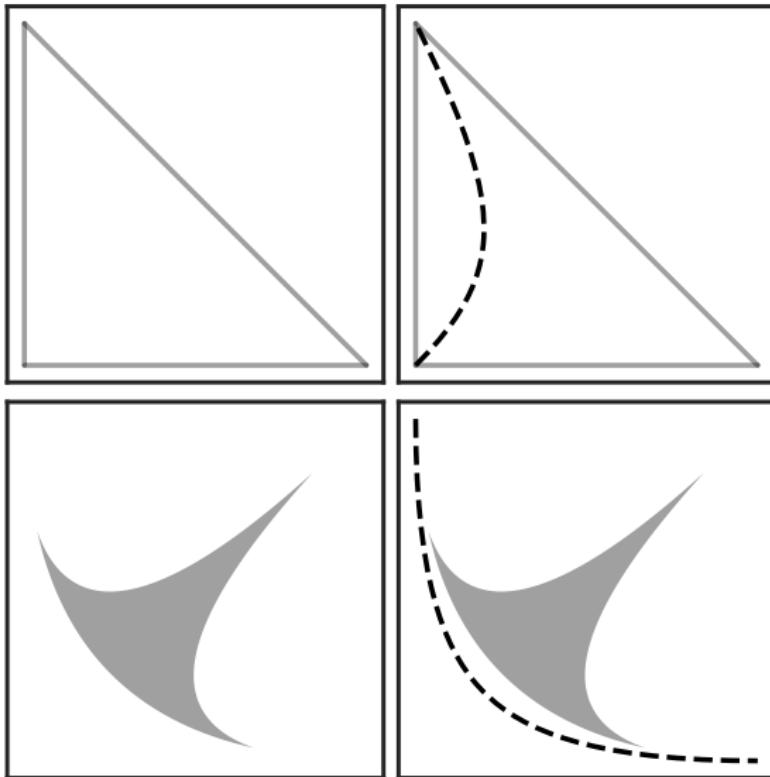
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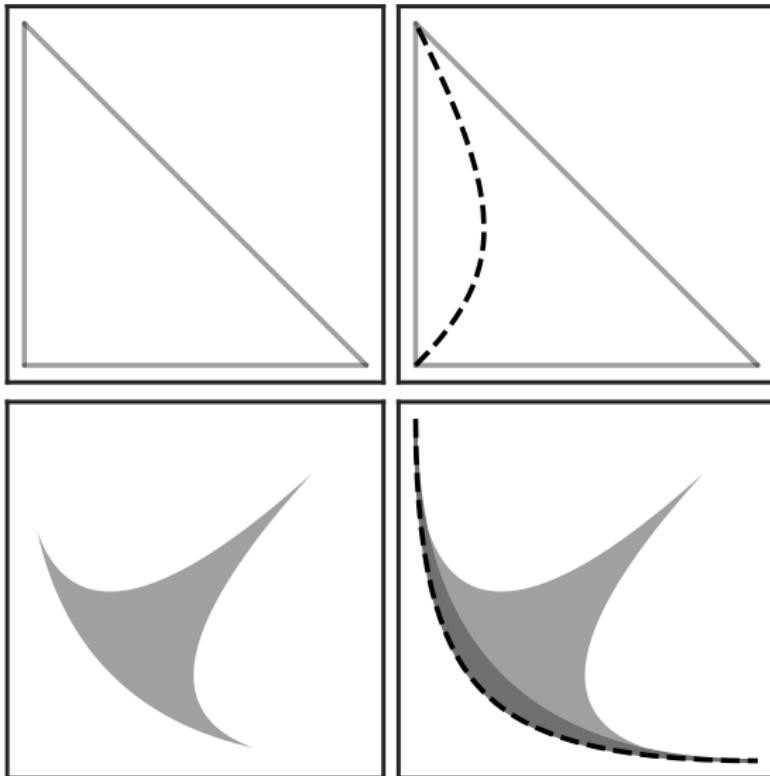
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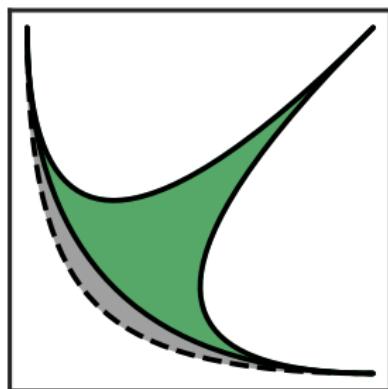
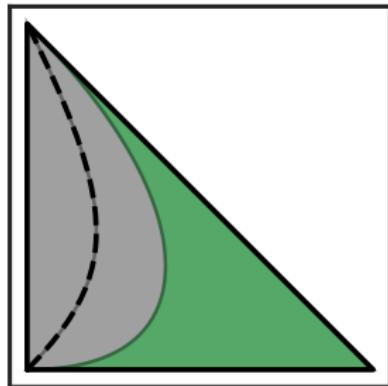
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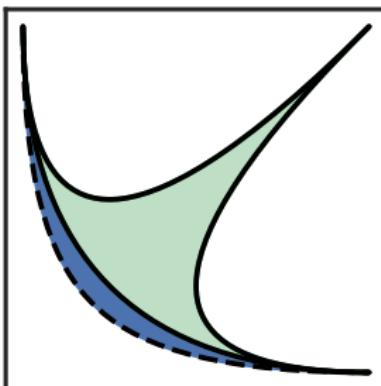
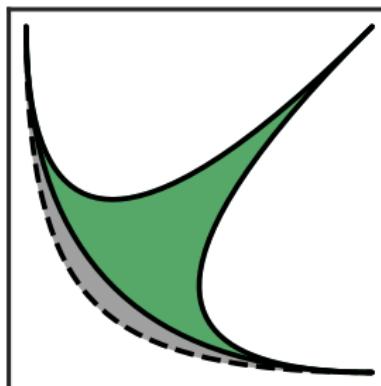
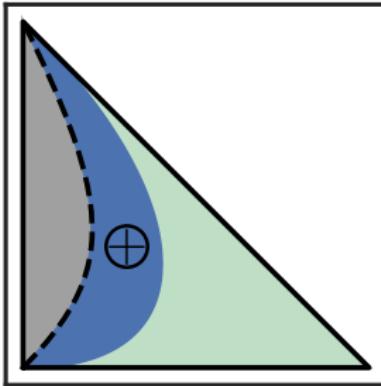
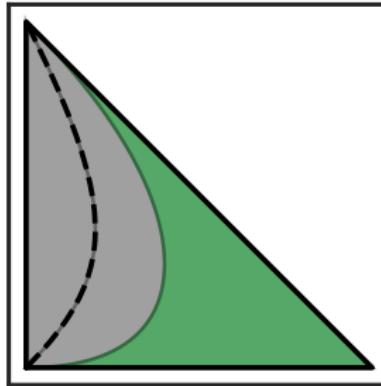
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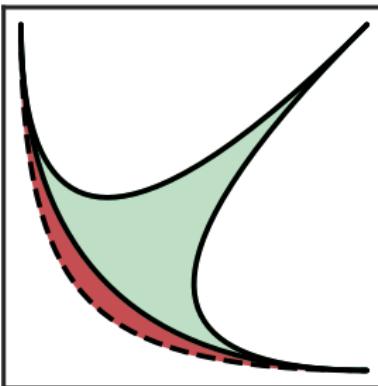
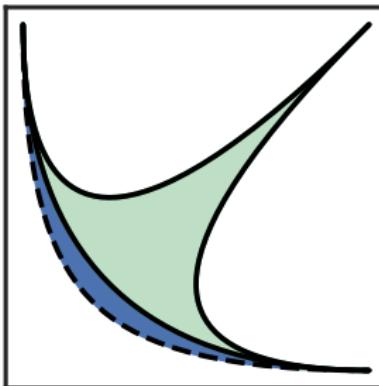
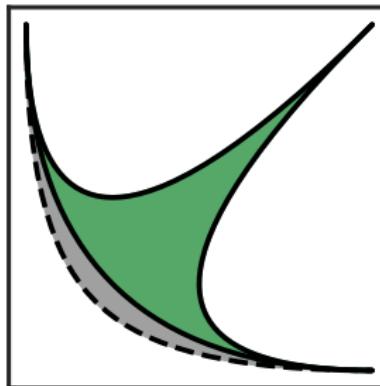
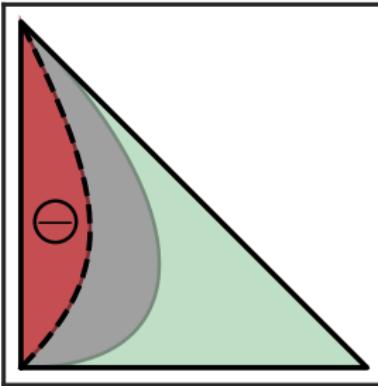
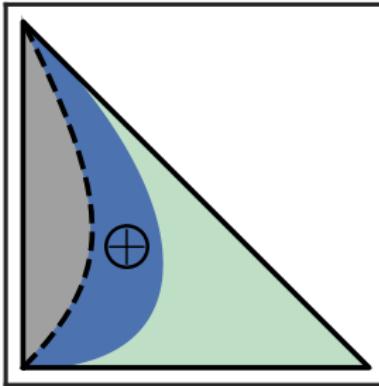
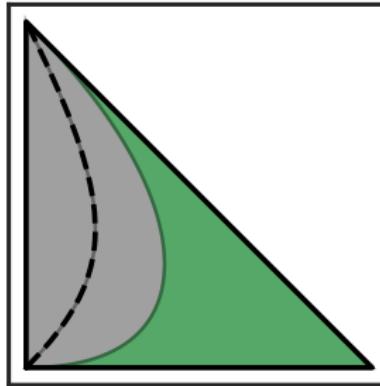
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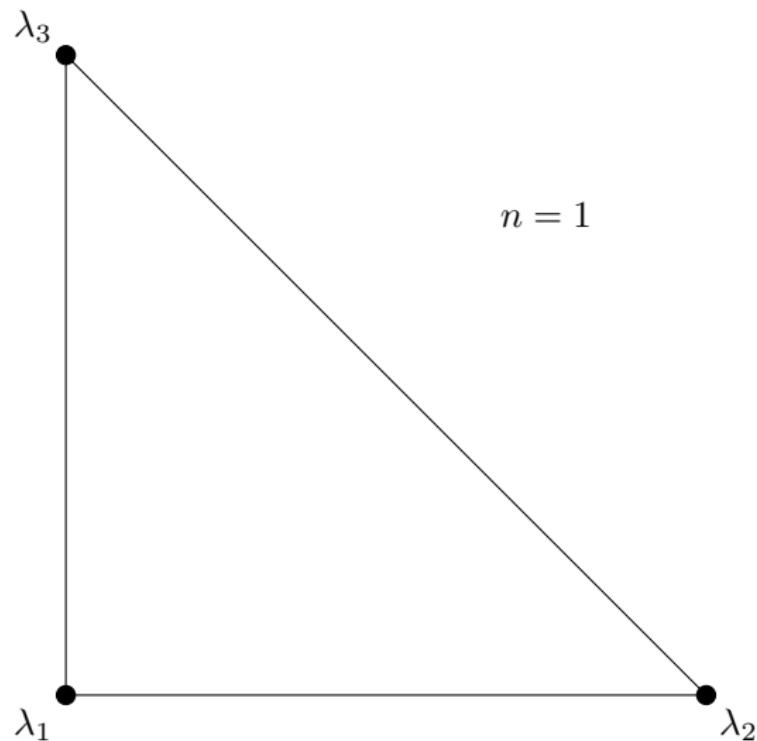
Shape Functions

- Based on $u_\alpha \in \mathcal{U}$ or $n_\alpha \in \mathbf{R}^2$ (α is a multi-index)

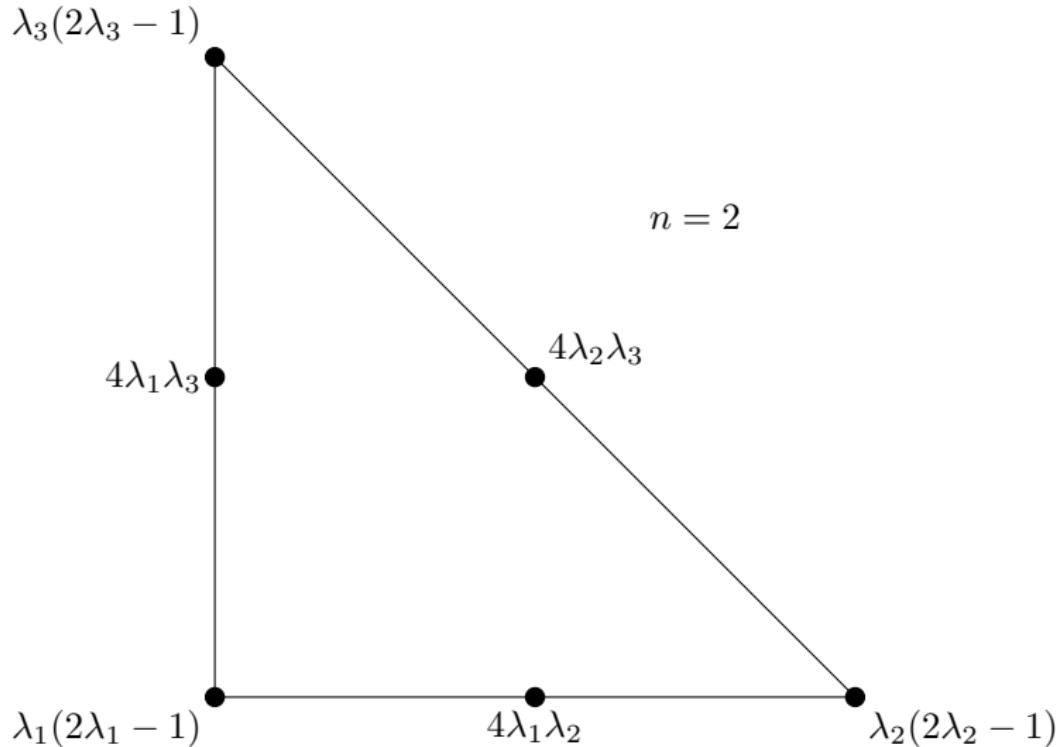
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- $\text{supp}(\phi) = \mathcal{T}$

Solution Transfer

Galerkin Projection

- Given:

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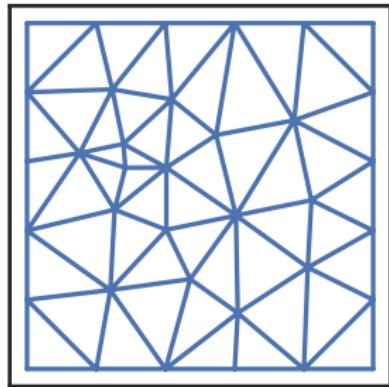
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 - Known discrete field $\mathbf{q}_D = \sum_j d_j \phi_D^{(j)}$
- Want: L_2 -optimal interpolant $\mathbf{q}_T = \sum_j t_j \phi_T^{(j)}$:

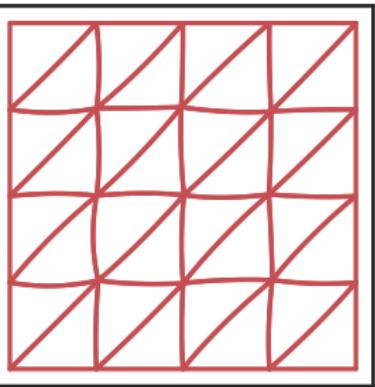
$$\|\mathbf{q}_T - \mathbf{q}_D\|_2 = \min_{\mathbf{q} \in \mathcal{V}_T} \|\mathbf{q} - \mathbf{q}_D\|_2$$

Galerkin Projection

\mathcal{M}_T



\mathcal{M}_D



Galerkin Projection

Differentiating w.r.t. each t_j in $\mathbf{q}_T = \sum_j t_j \phi_T^{(j)}$ gives **weak form**

$$\int_{\Omega} \mathbf{q}_D \phi_T^{(j)} dV = \int_{\Omega} \mathbf{q}_T \phi_T^{(j)} dV, \quad \text{for all } j.$$

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If $(x \mapsto 1) \in \mathcal{V}_T$, then \mathbf{q}_T is globally **conservative**

$$\int_{\Omega} \mathbf{q}_D dV = \int_{\Omega} \mathbf{q}_T dV.$$

Linear System

Weak form gives rise to a linear system in coefficients \mathbf{d} and t .

Linear System

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$$M_T \mathbf{t} = M_{TD} \mathbf{d}.$$

Linear System

M_T is (symmetric) mass matrix for target mesh

$$(M_T)_{ij} = \int_{\Omega} \phi_T^{(i)} \phi_T^{(j)} dV.$$

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Integrate via substitution for $F = \phi_T^{(i)} \phi_T^{(j)}$

$$\int_{b(\mathcal{U})} F(x, y) dx dy = \int_{\mathcal{U}} \det(Db) F(x(s, t), y(s, t)) ds dt$$

and then use quadrature rule on \mathcal{U} .

Linear System

Mixed mass matrix M_{TD}

$$(M_{TD})_{ij} = \int_{\Omega} \phi_T^{(i)} \phi_D^{(j)} dV.$$

Linear System

Mixed mass matrix M_{TD}

$$(M_{TD})_{ij} = \int_{\Omega} \phi_T^{(i)} \phi_D^{(j)} dV.$$

Not symmetric, nor even square; rows correspond to shape functions on target mesh and columns to donor mesh.

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Instead, compute entire RHS

$$(M_{TD}\mathbf{d})_j = \int_{\Omega} \phi_T^{(j)} \mathbf{q}_D dV.$$

Common Refinement

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Common Refinement

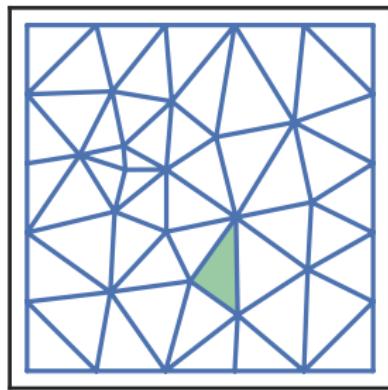
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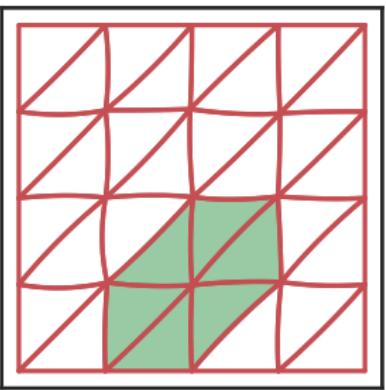
In CG, \mathbf{q}_D need not be differentiable across elements and in DG \mathbf{q}_D need not even be continuous

Common Refinement

\mathcal{M}_T



\mathcal{M}_D



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- Three subproblems:

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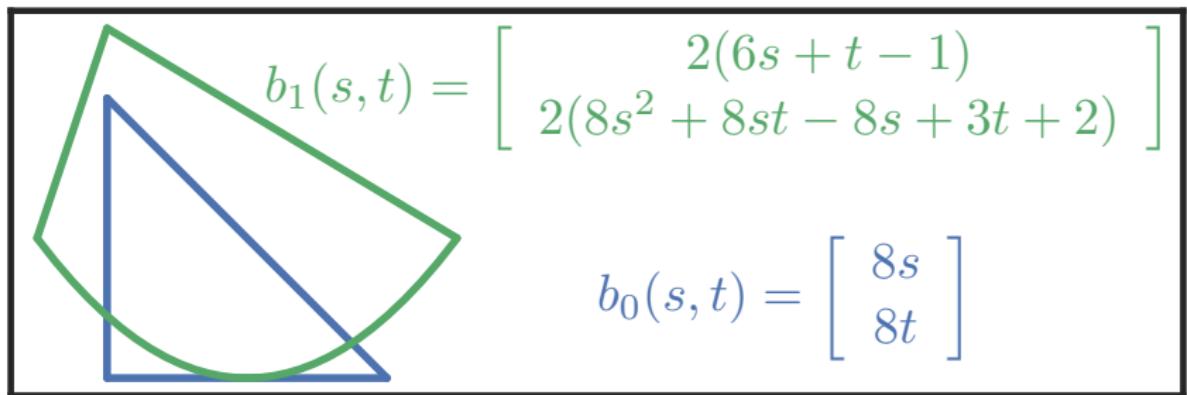
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- $\mathcal{P} = \mathcal{T}_0 \cap \mathcal{T}_1$, $\partial\mathcal{P}$ defined by segments of edges from \mathcal{T}_0 and \mathcal{T}_1

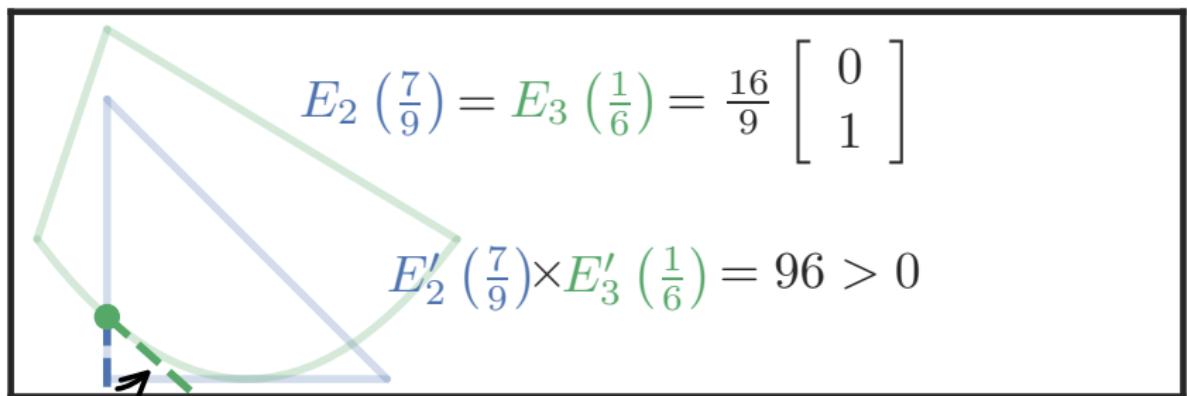
Intersecting Curved Elements



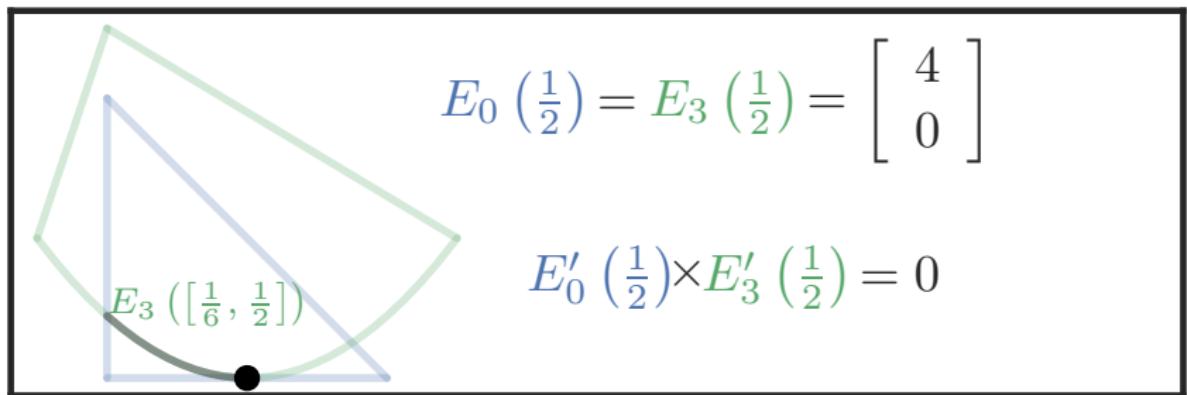
Intersecting Curved Elements

$$E_2 \left(\frac{7}{9} \right) = E_3 \left(\frac{1}{6} \right) = \frac{16}{9} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

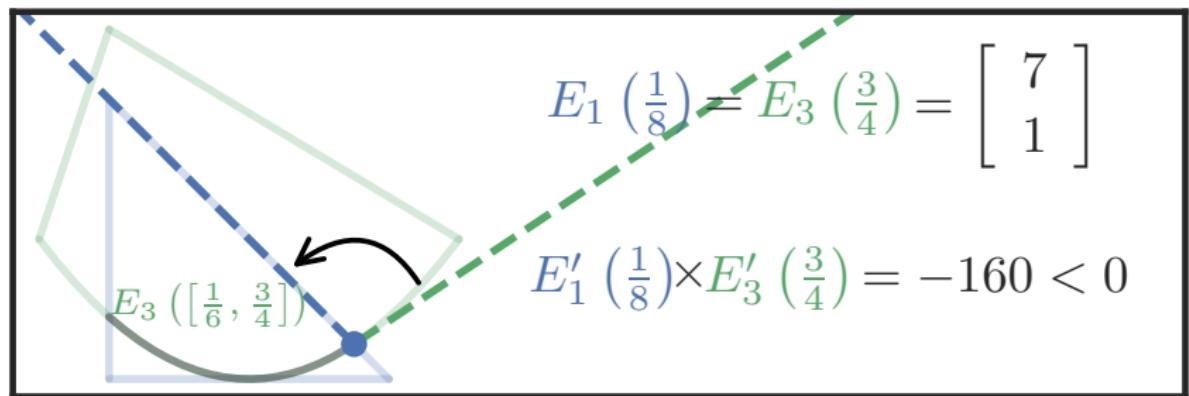
$$E'_2 \left(\frac{7}{9} \right) \times E'_3 \left(\frac{1}{6} \right) = 96 > 0$$



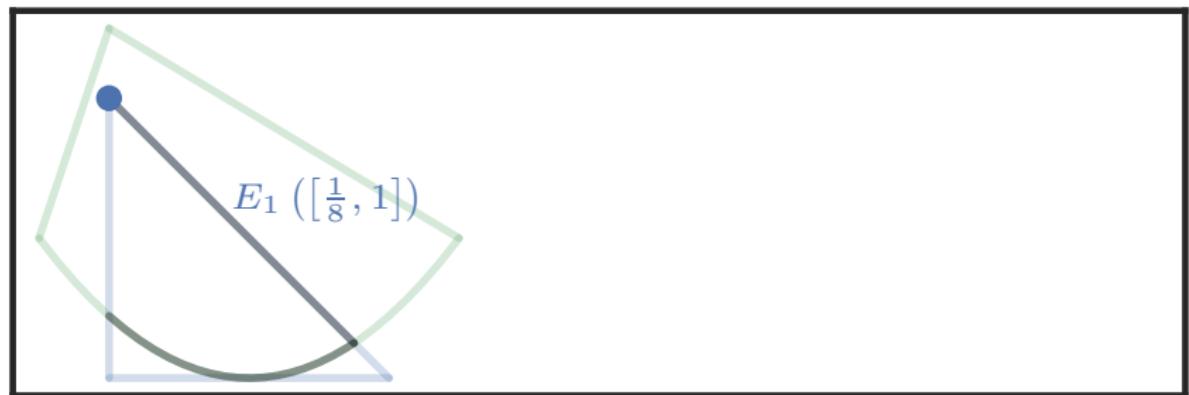
Intersecting Curved Elements



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Intersecting Curved Elements



Intersecting Curved Elements

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Advancing Front

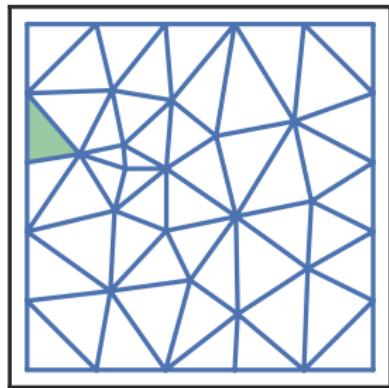
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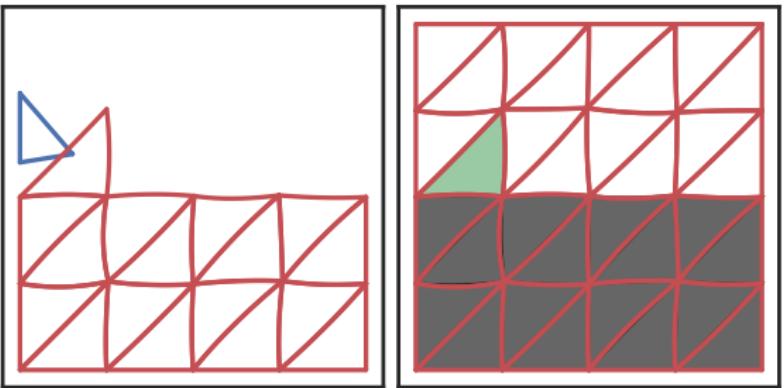
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Advancing Front

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\mathcal{M}_D



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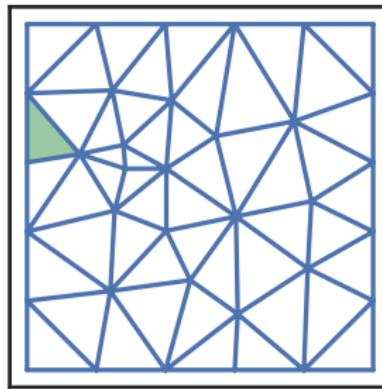
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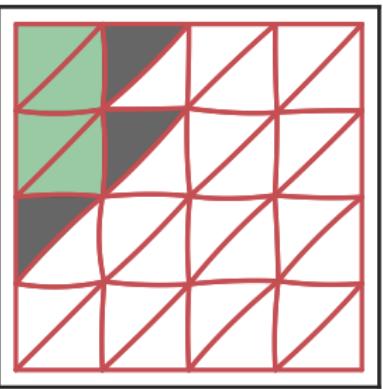
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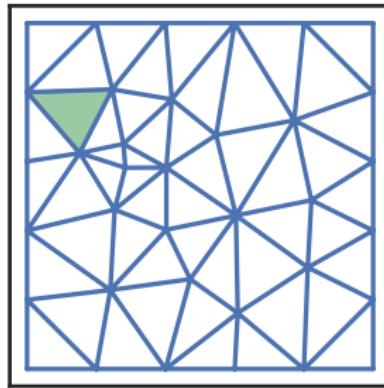
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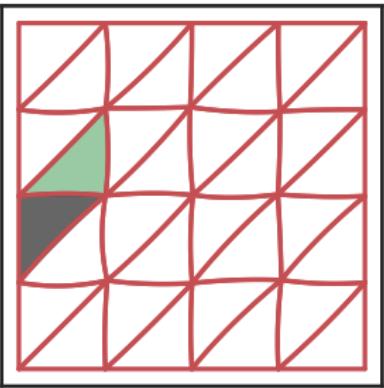
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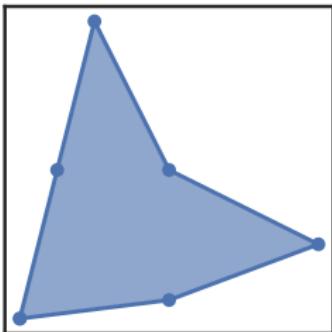
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- Total $\mathcal{O}(|\mathcal{M}_D| + |\mathcal{M}_T|)$

Integration over Curved Polygons

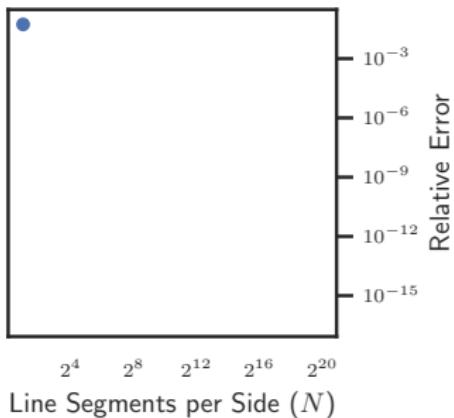
$$\int_{\mathcal{P}} F(x, y) \, dV, \quad \mathcal{P} = \mathcal{T}_0 \cap \mathcal{T}_1, \quad F = \phi_0 \phi_1$$

Integrate via Polygonal Approximation

$$N = 2$$

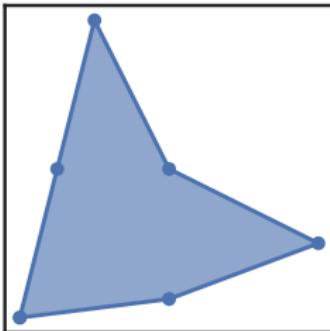


Area Estimates

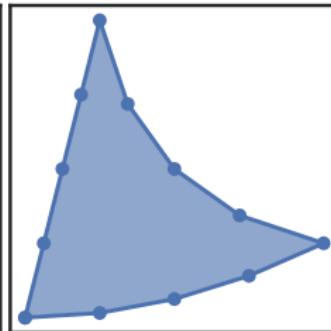


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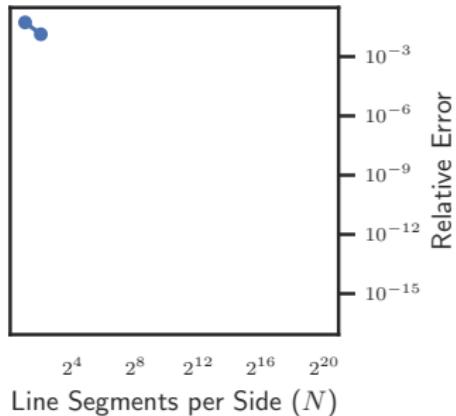
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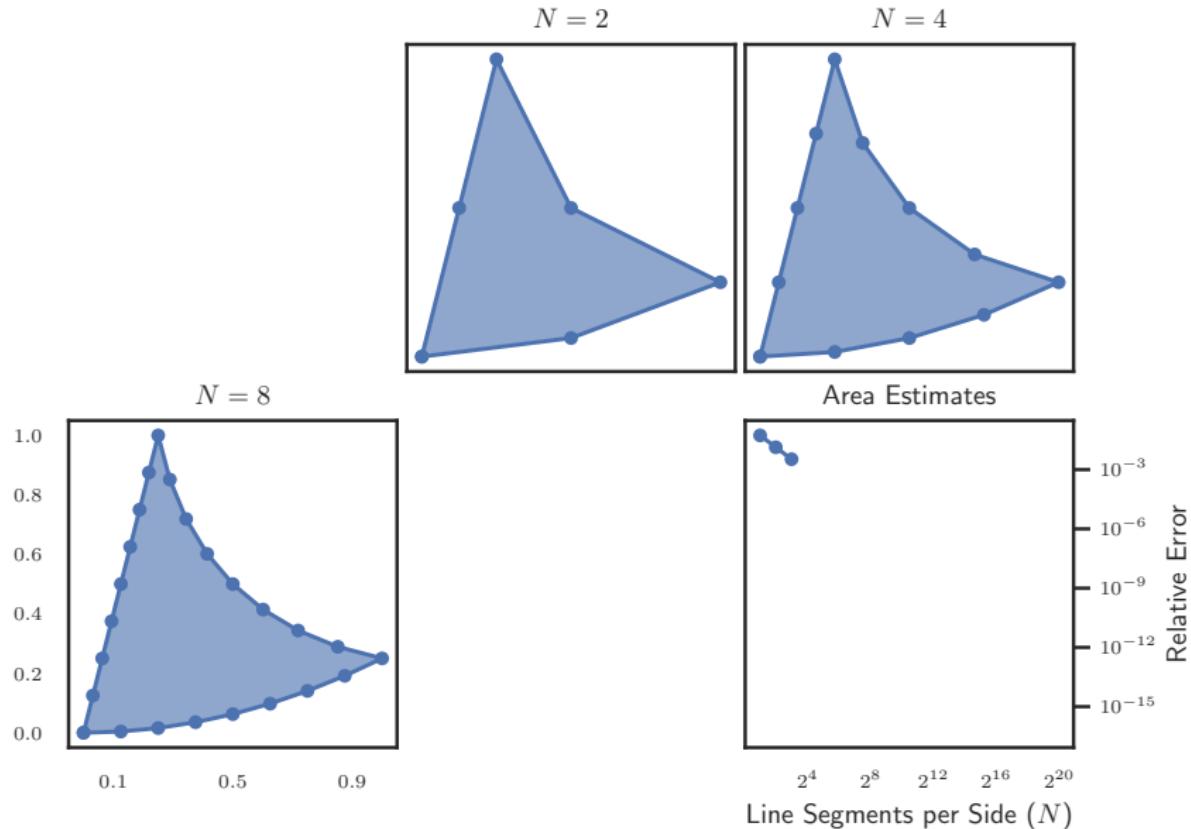
$N = 4$



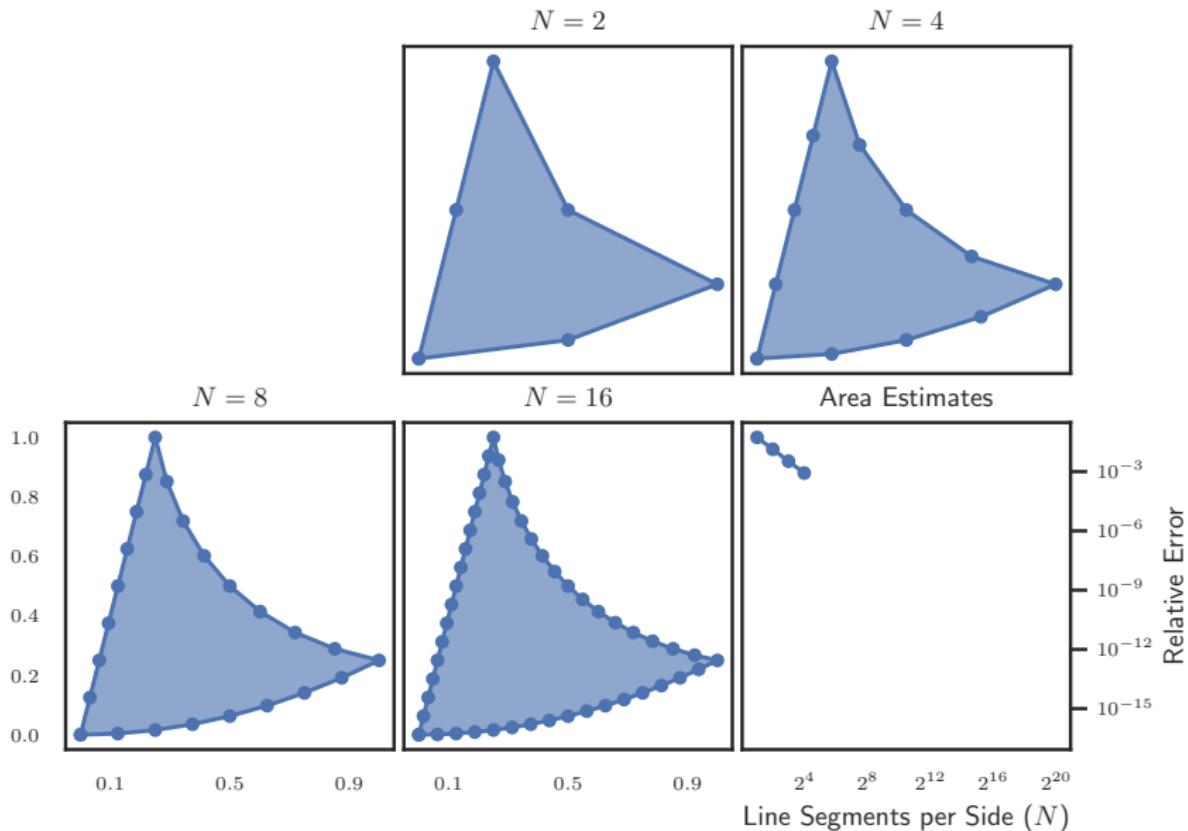
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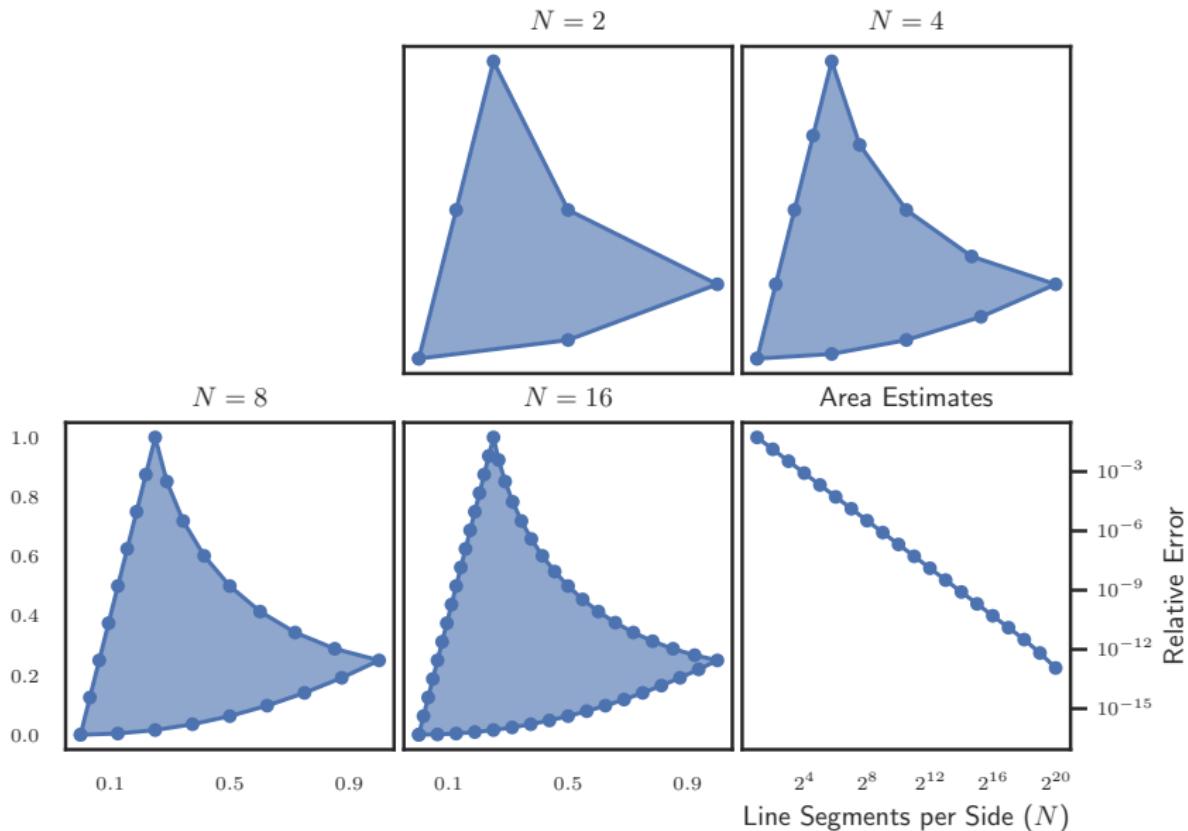
Integrate via Polygonal Approximation



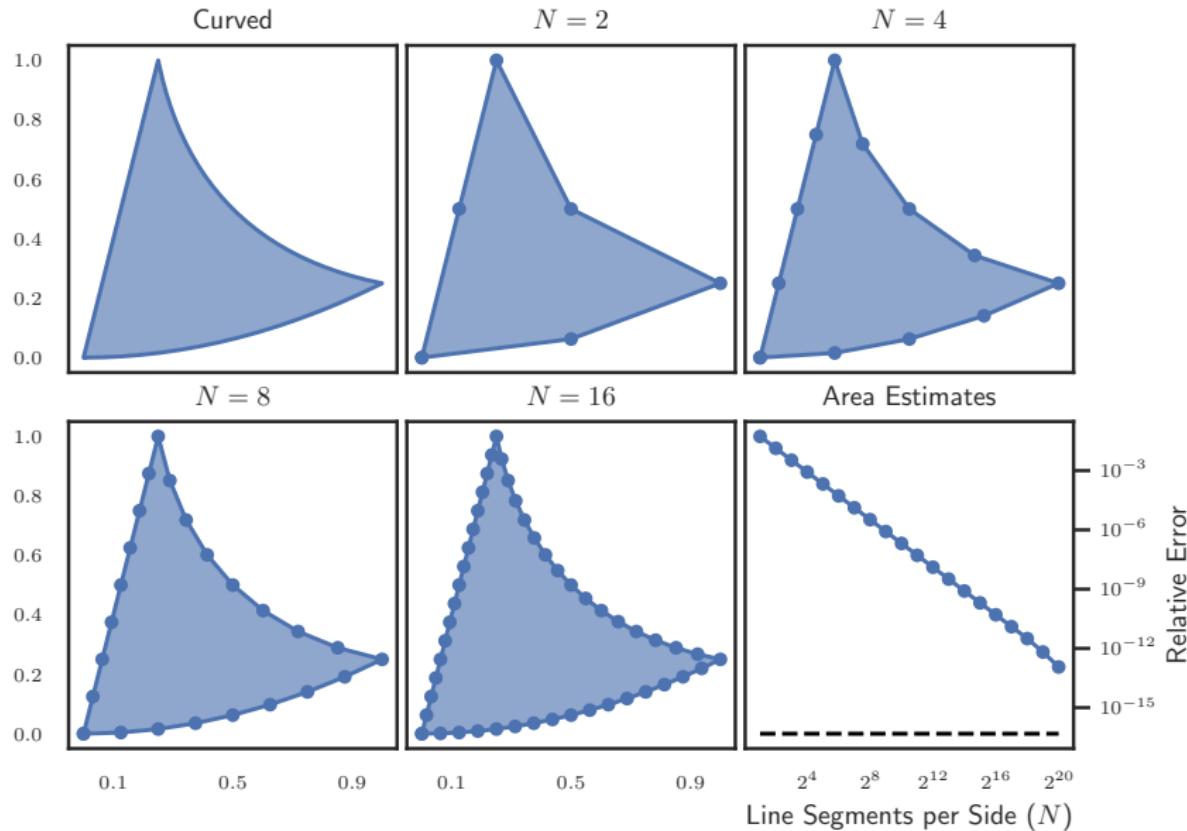
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- Transfinite interpolation or mean value coordinates
 - Maps from (straight sided) reference domain, but increase degree or are not bijective

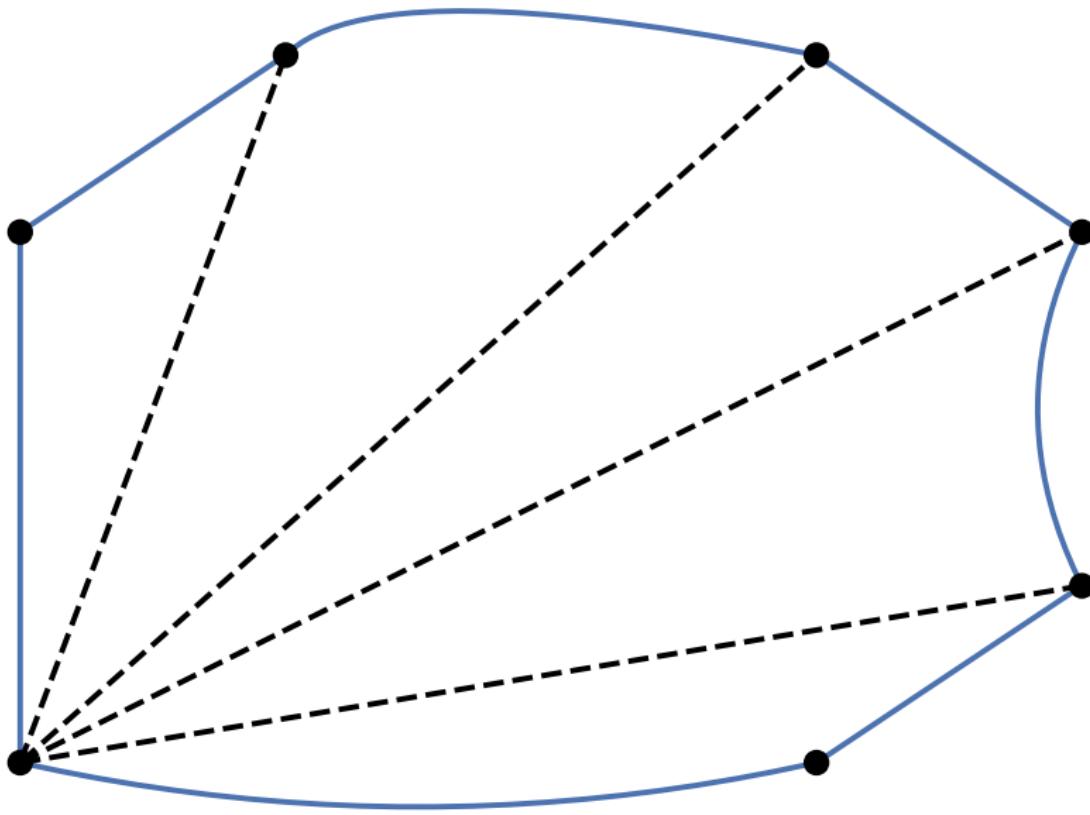
Integrate via Tessellation

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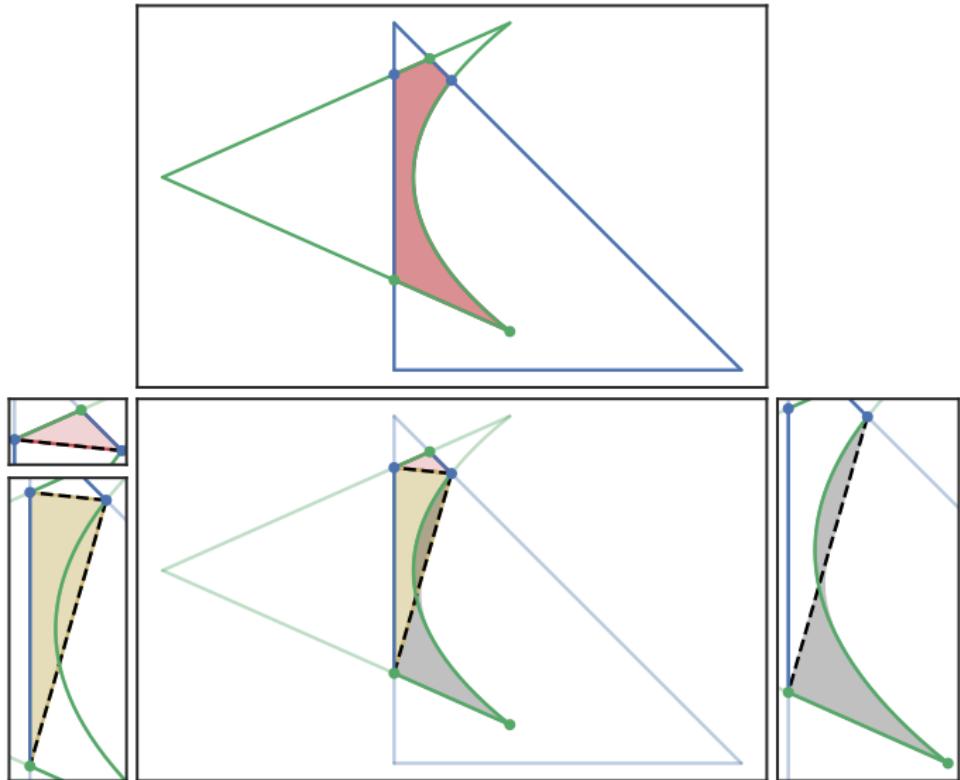
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- If diagonals are valid, high degree Bézier triangles need interior control points introduced that don't cause triangle to invert

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$$G(r) = H(x(r), y(r))y'(r) - V(x(r), y(r))x'(r)$$

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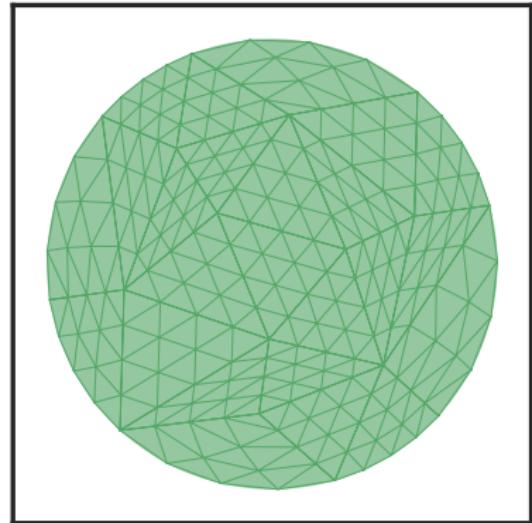
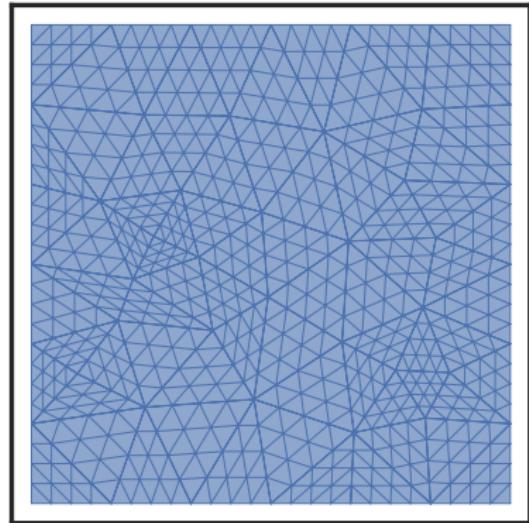
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Numerical Experiments

- Three meshes ($p = 1, 2, 3$)

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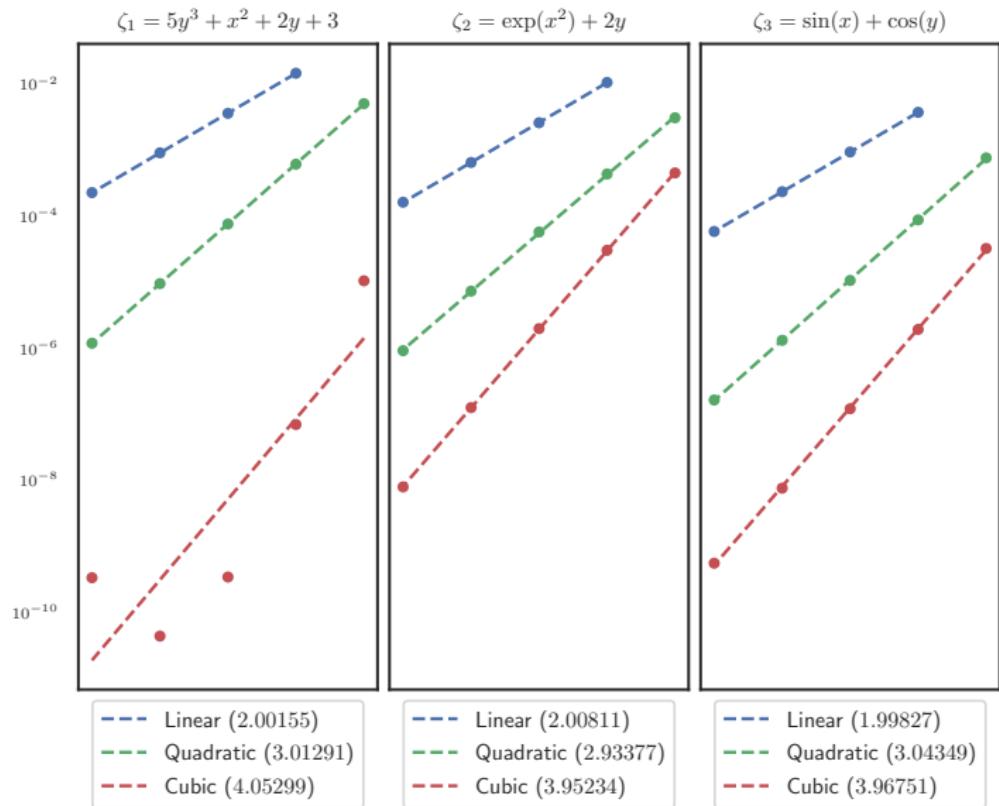
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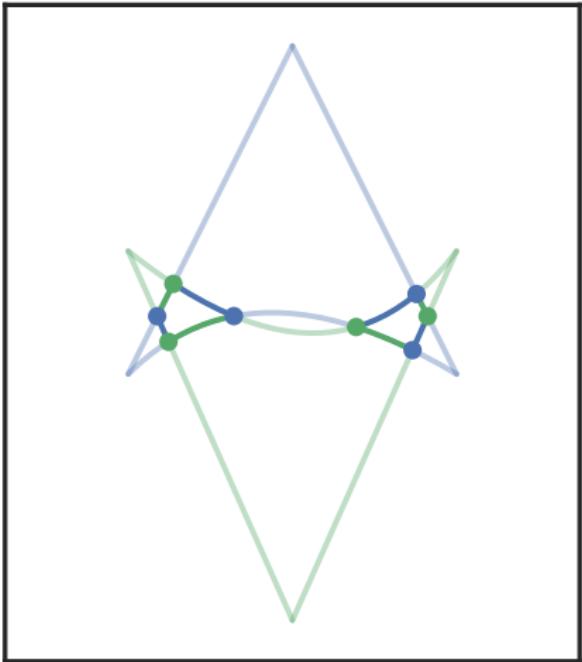
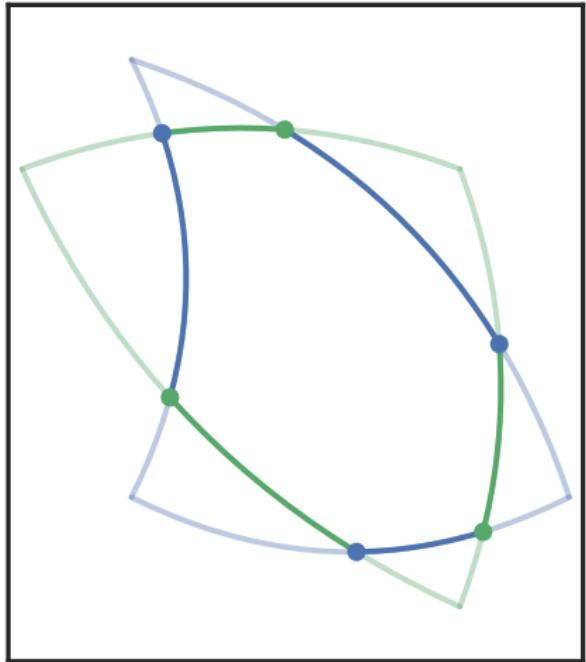
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Numerical Experiments

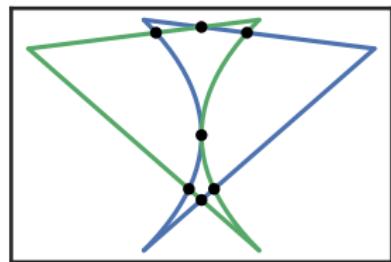
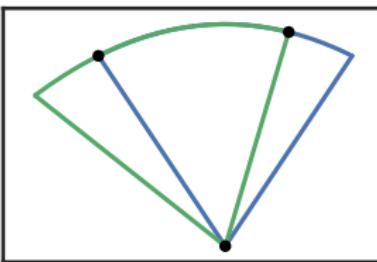
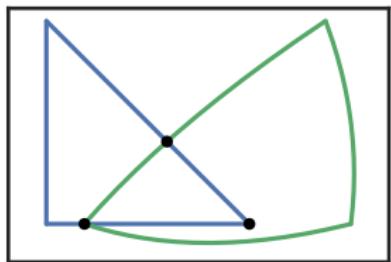


Ill-conditioned Bézier Curve Intersection

III-Conditioned Intersections



Ill-Conditioned Intersections



Ill-Conditioned Intersections

- Edges are Bézier curves, e.g. $b(r, 0) = \sum_{j=0}^n \binom{n}{j} (1-r)^{n-j} r^j \mathbf{p}_{n-j,j,0}$

Ill-Conditioned Intersections

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Ill-Conditioned Intersections

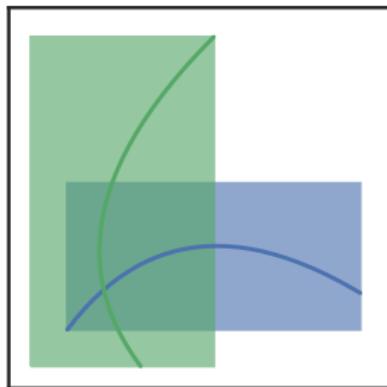
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- Random pair of meshes, “almost tangent” intersections increasingly frequent as $h \rightarrow 0^+$

Intersection Algorithm

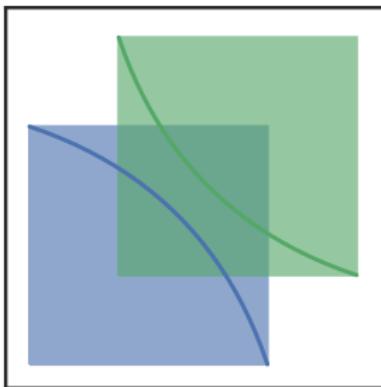
- Bounding box check

Intersection Algorithm

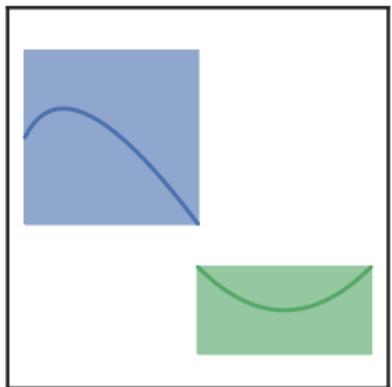
MAYBE



MAYBE



NO



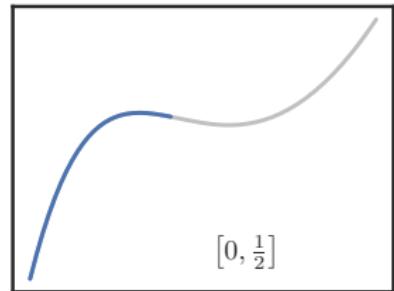
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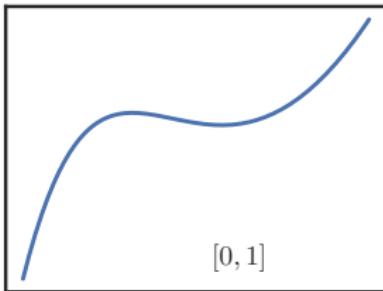
Intersection Algorithm

- Bounding box check
- Curve subdivision

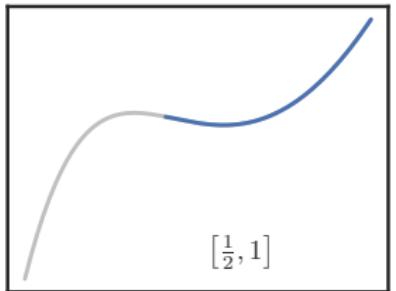
Intersection Algorithm



$$\left[0, \frac{1}{2}\right]$$



$$\left[0, 1\right]$$



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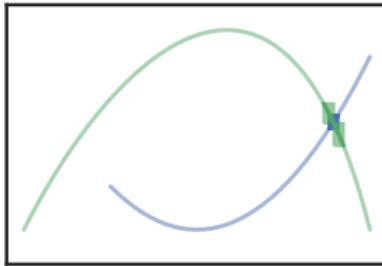
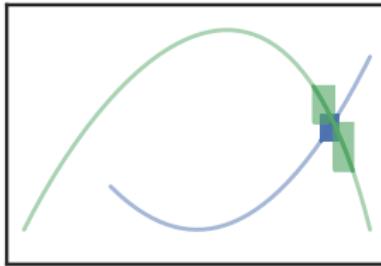
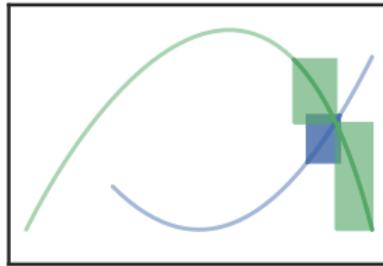
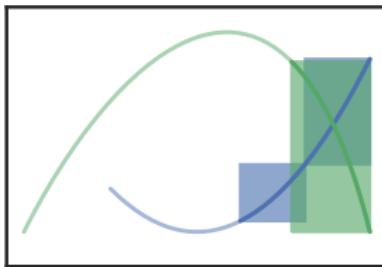
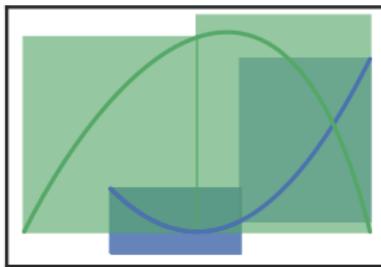
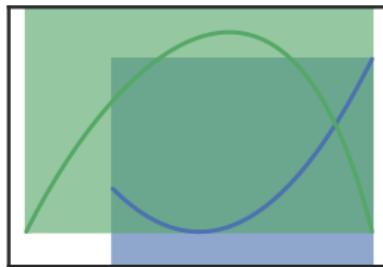
Intersection Algorithm

- Bounding box check
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Intersection Algorithm

- Bounding box check
- Curve subdivision
- Check subdivided pairs

Intersection Algorithm



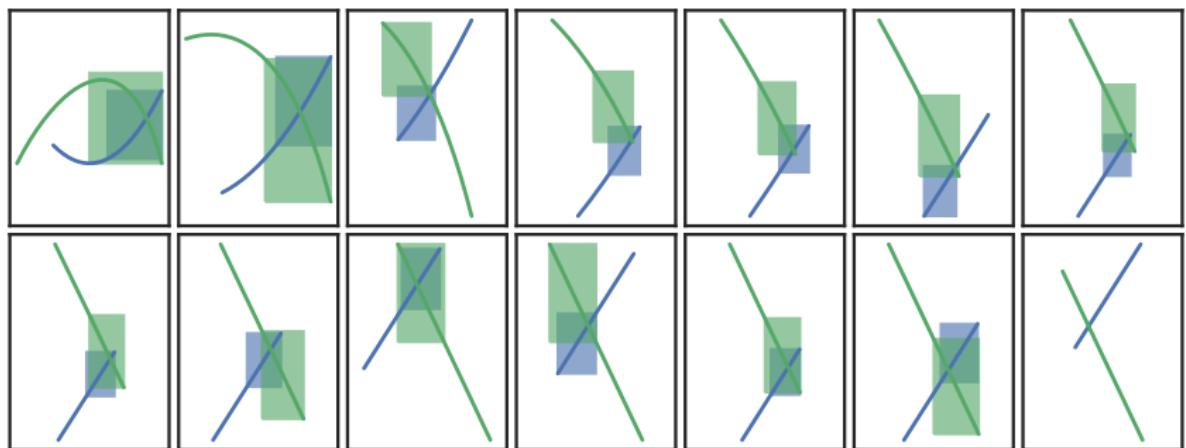
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Intersection Algorithm

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- Use intersection of lines as seed for Newton’s method

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- At ill-conditioned intersections, J is almost singular and evaluation of F is typically ill-conditioned as well

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 - Tisseur (2001) showed this generically and applied to iterative refinement for generalized eigenvalue problem

Compensated Evaluation

Compensated Algorithms

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 - Error is **exact**, hence “error-free”
- **Compensated** algorithm uses $\hat{f} \oplus \hat{e}$; better approximation of $f(x)$ than \hat{f} (usually)

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- **Not possible** to apply this to \emptyset

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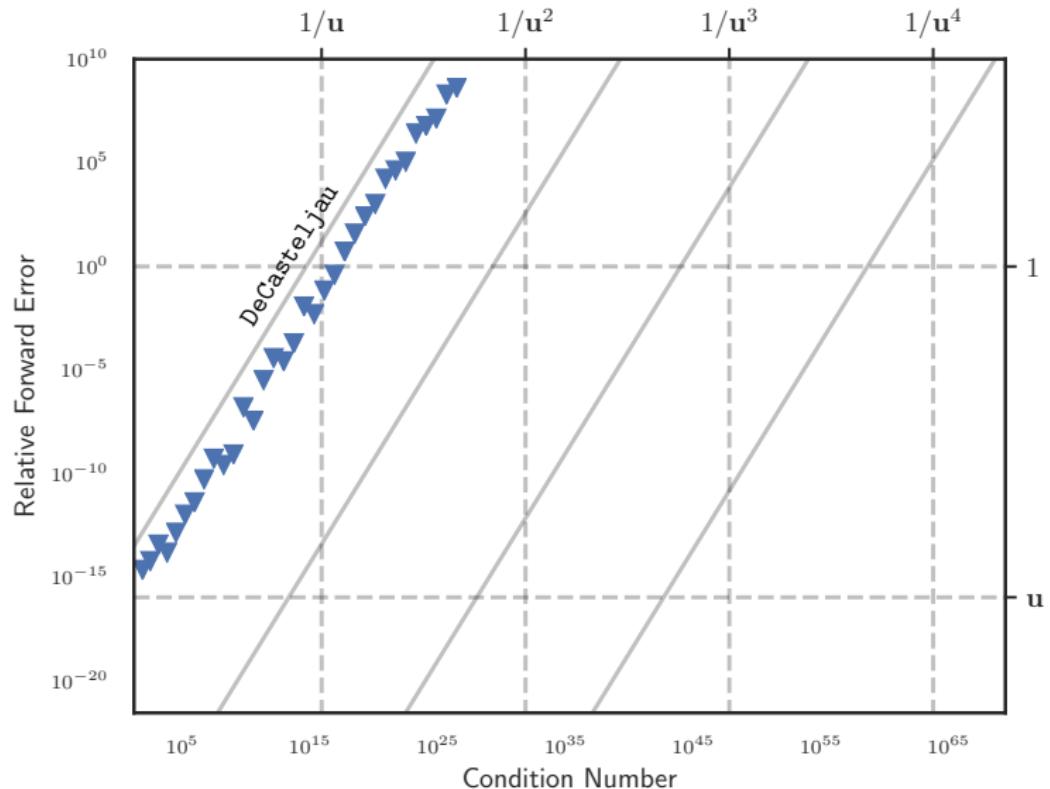
Compensated de Castlejau

- Only uses \oplus , \ominus and \otimes
- Track errors and combine them progressively
- Test polynomial $p(s) = (s - 1) \left(s - \frac{3}{4}\right)^7$; evaluation at $s = \frac{3}{4}$ has infinite condition, very ill-conditioned nearby

Compensated de Castlejau

$$\frac{|p(s) - \text{DeCasteljau}(p, s)|}{|p(s)|} \leq \text{cond}(p, s) \cdot \mathcal{O}(\mathbf{u})$$

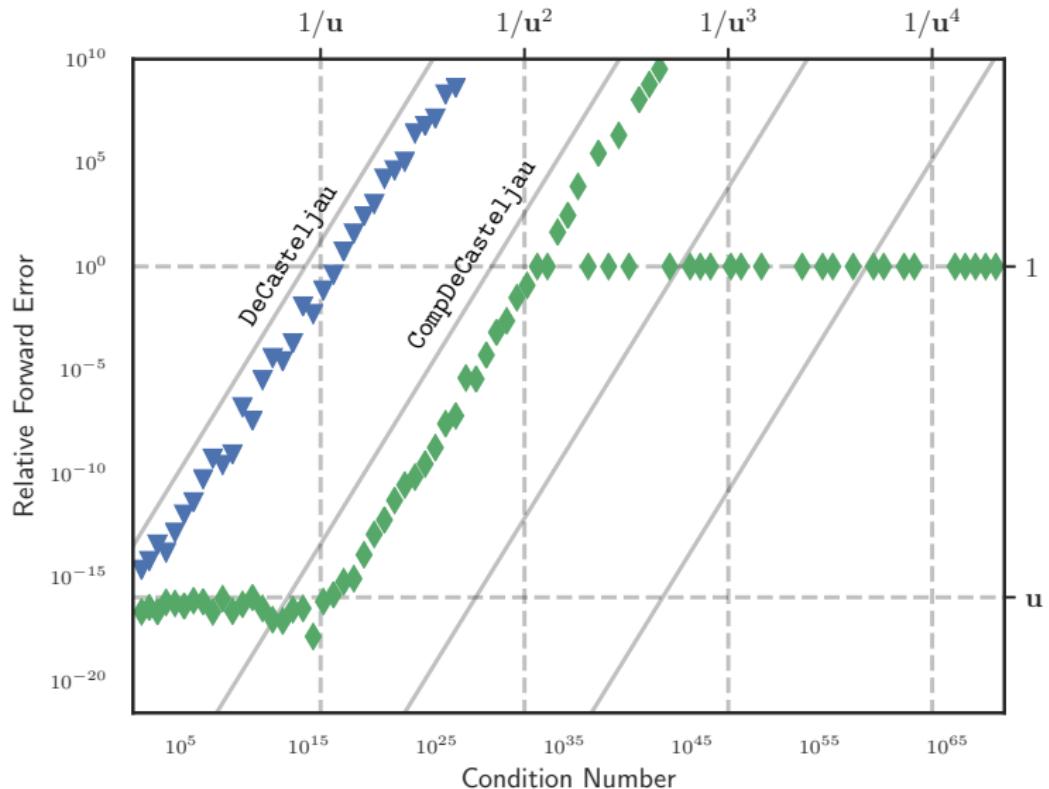
Compensated de Castlejau



Compensated de Castlejau

$$\frac{|p(s) - \text{CompDeCasteljau}(p, s)|}{|p(s)|} \leq \mathcal{O}(\mathbf{u}) + \text{cond}(p, s) \cdot \mathcal{O}(\mathbf{u}^2)$$

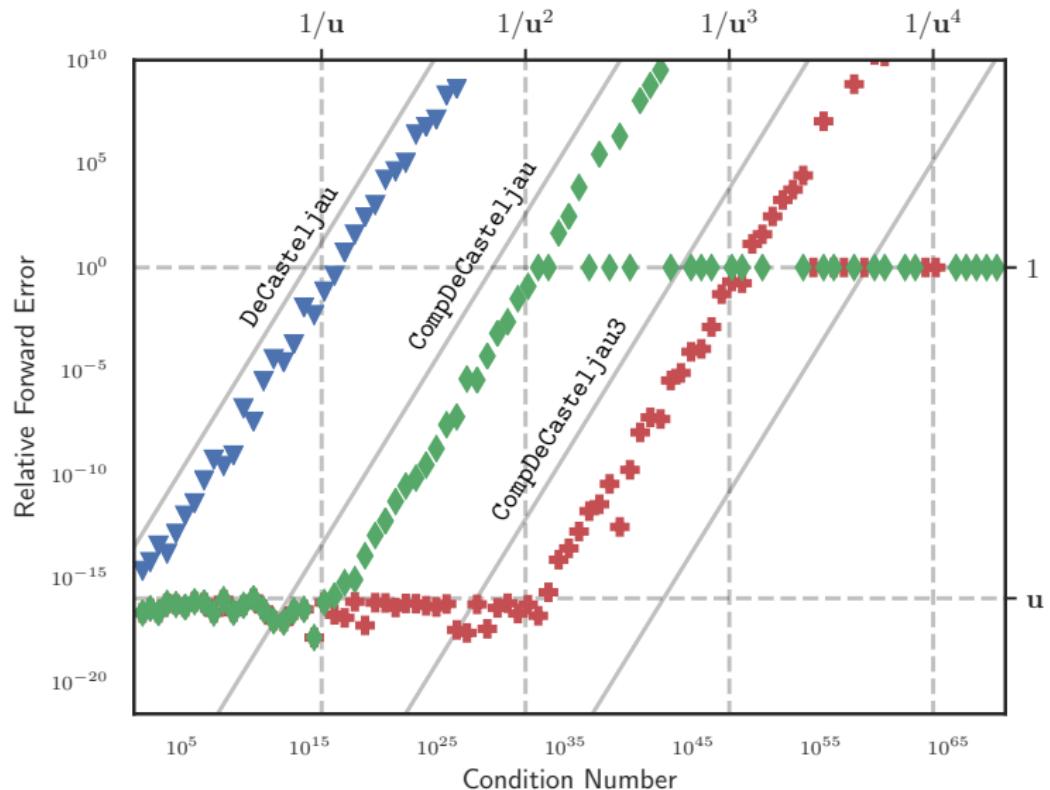
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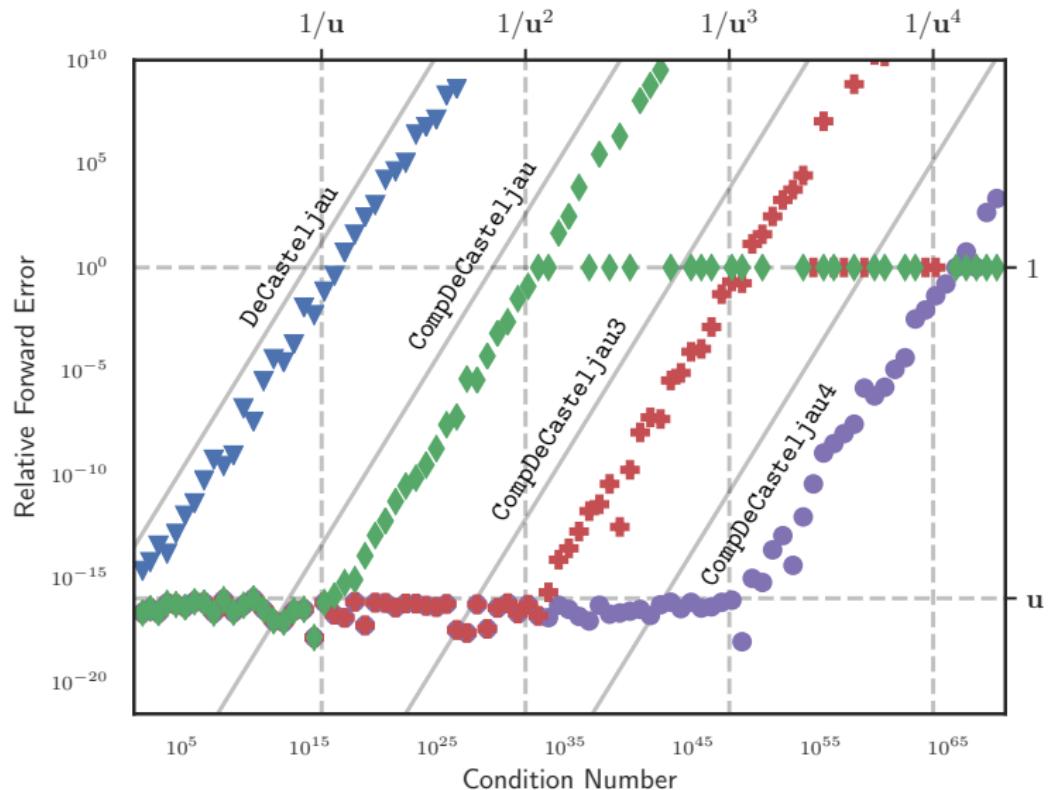
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$$\frac{|p(s) - \text{CompDeCasteljauK}(p, s)|}{|p(s)|} \leq \mathcal{O}(\mathbf{u}) + \text{cond}(p, s) \cdot \mathcal{O}(\mathbf{u}^K)$$

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Modified Newton's for Intersection

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DeCasteljau for Jacobian
 - **DNewtonFull**: CompDeCasteljau for residual and Jacobian

Polynomial Roots

- Test problem; need coefficients that can be exactly represented

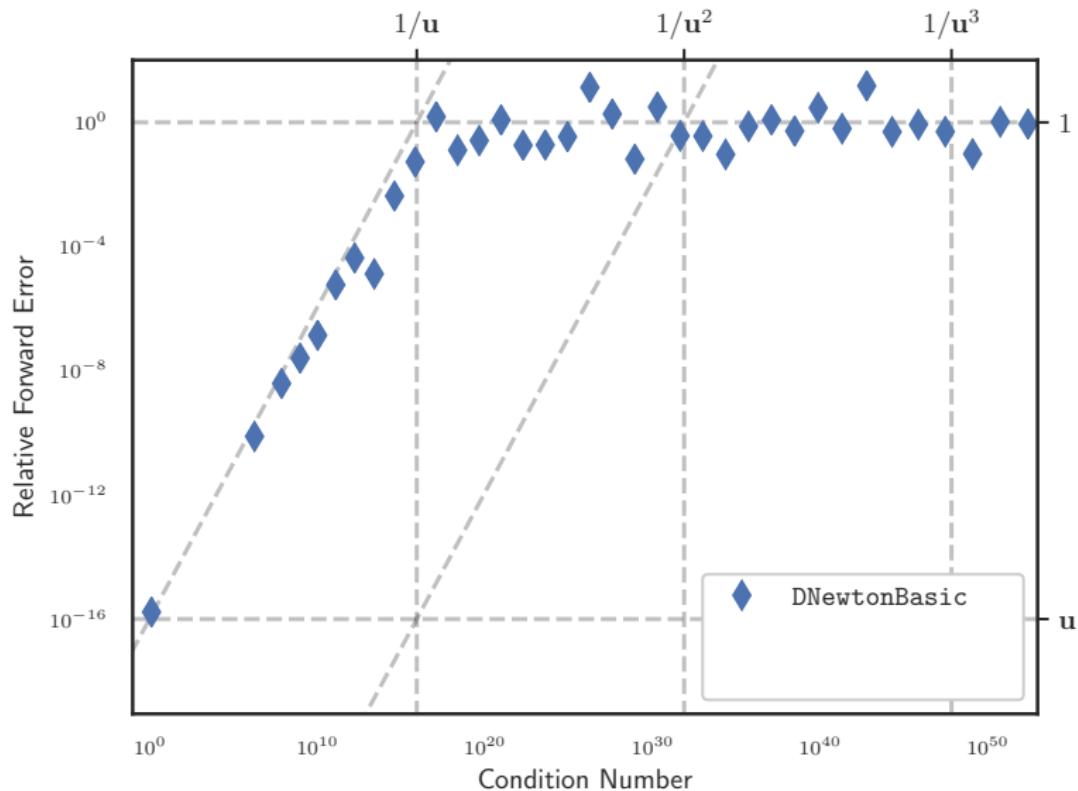
Polynomial Roots

- Test problem; need coefficients that can be exactly represented
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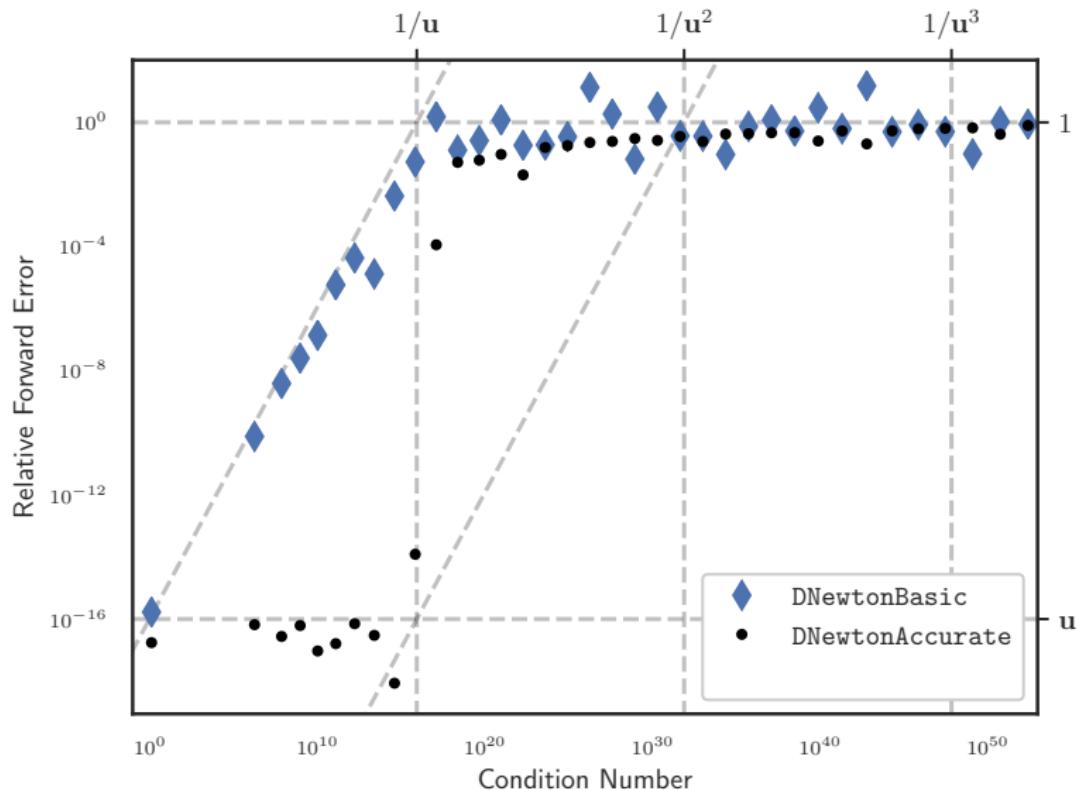
Polynomial Roots

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- Root $\alpha = \frac{1 + 2^{30/n}}{5 + 3 \cdot 2^{30/n}} \in \left[\frac{1}{4}, \frac{1}{3} \right]$

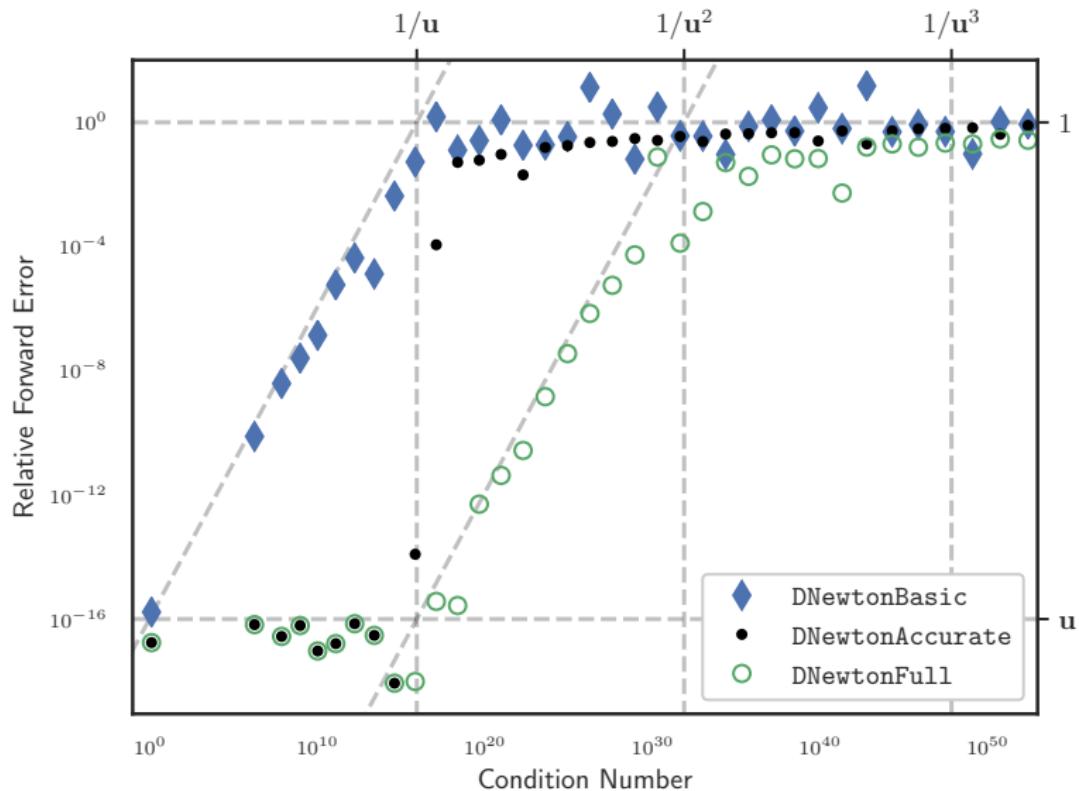
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- Evaluate $x_0(s), y_0(s), x_1(t), y_1(t)$ via de Casteljau
- Modify Newton's method by using “extended precision” (i.e. compensated method) for \hat{F}

Compensated Residual

$$\cdot \hat{F} = \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}$$

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- Computed residual $D \oplus \tau$

Modified Newton's in Practice

$$F(s, t) = \begin{bmatrix} 2(4s^2 - 1) \\ (2s - 1)^2 + 1 \end{bmatrix} - \begin{bmatrix} 4(4t^2 - 1) \\ 4(2t - 1)^2 + 1 \end{bmatrix}$$

Modified Newton's in Practice

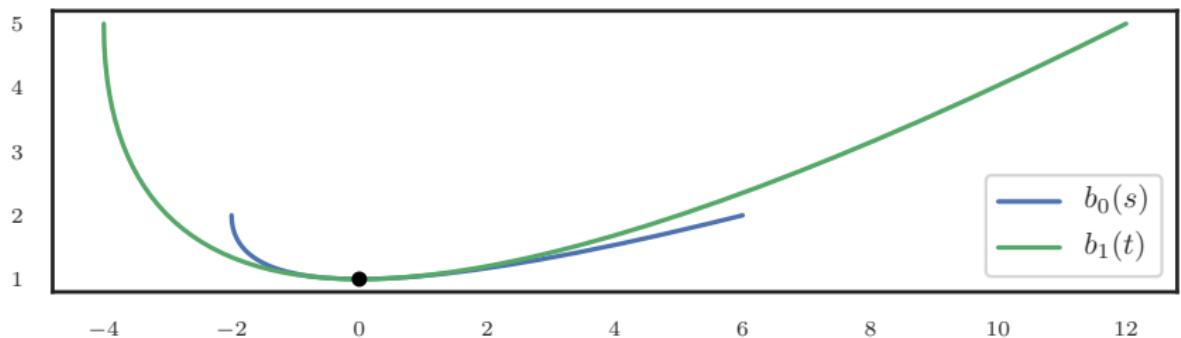
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Triple root at $F\left(\frac{1}{2}, \frac{1}{2}\right)$

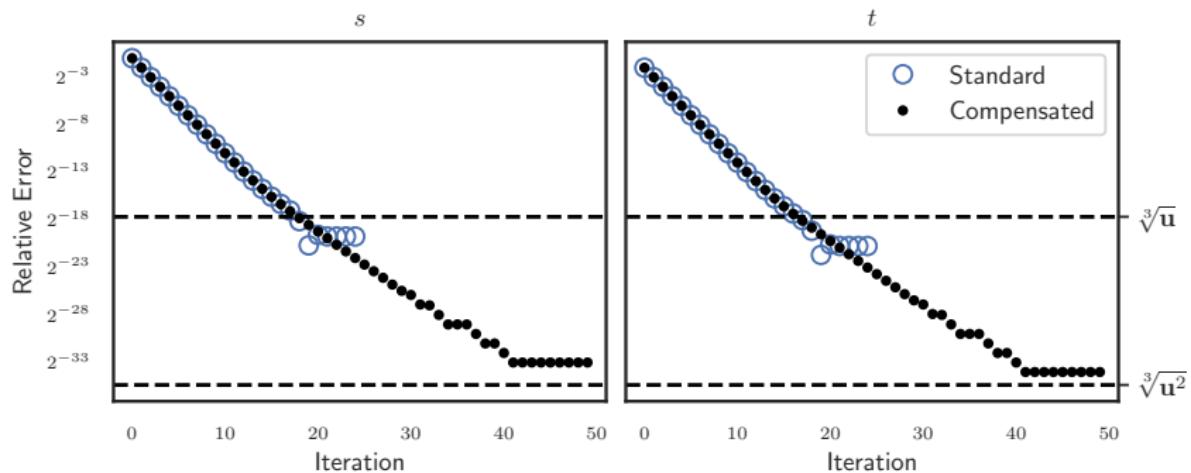
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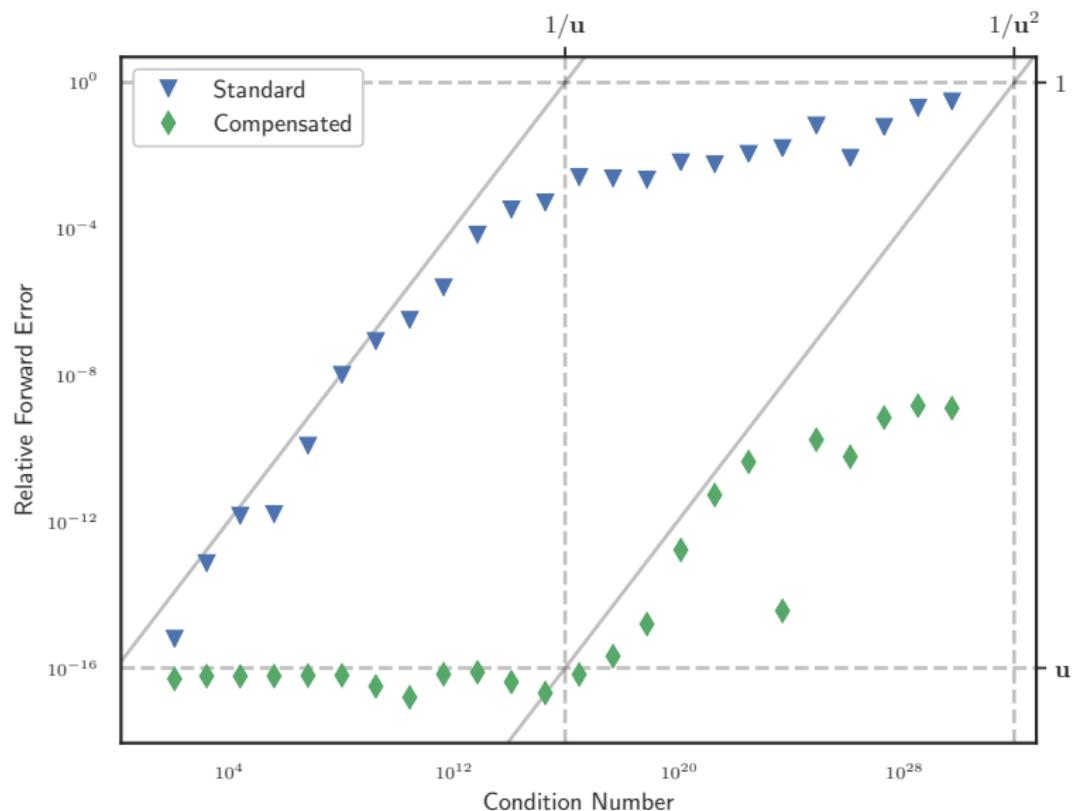
$$G(s, t) = \begin{bmatrix} x_0(s) - r \\ y_0(s) + \frac{1}{r} \end{bmatrix} - \begin{bmatrix} x_1(t) \\ y_1(t) + \frac{1}{r} \end{bmatrix}$$

Modified Newton's in Practice

$$G(s, t) = \begin{bmatrix} x_0(s) - r \\ y_0(s) + \frac{1}{r} \end{bmatrix} - \begin{bmatrix} x_1(t) \\ y_1(t) + \frac{1}{r} \end{bmatrix}$$

Approaches triple root as $r \rightarrow 0^+$, $F\left(\frac{1+\sqrt{r}}{2}, \frac{2+\sqrt{r}}{4}\right)$

Modified Newton's in Practice



Conclusion

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 - Conservative L_2 -optimal interpolation; split \int_{Ω} into $\sum \int_{\mathcal{T} \cap \mathcal{T}'}$ where integrand is smooth
 - Intersecting Curved Elements: $\mathcal{T} \cap \mathcal{T}'$

Conclusion

- **Solution Transfer on Curved Meshes**

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 - Modified Newton's method; computes residual $F(s, t)$ as if in extended precision

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- Solution transfer in \mathbf{R}^3 ; integrate via Stoke's theorem, geometry is greatest challenge