UiO : Department of informatics
University of Oslo

Adaptive Methods for Data-based Decision Making *Lecture 2*

IN-STK 5000 / 9000

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Today's Goal

Introducing basic notation and concepts in probability theory and statistics at the example of statistical decision making

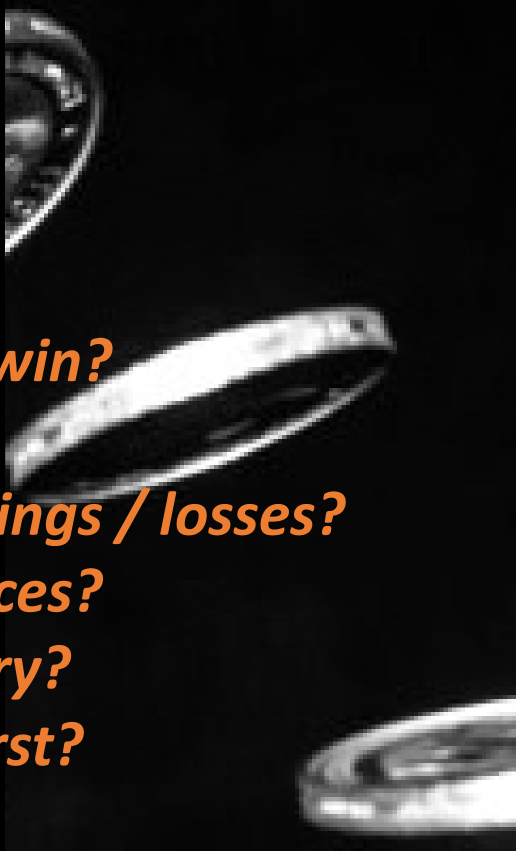
What we talk about today

- **Rewards & Utilities**
- **Probability Distributions (Lotteries)**
- **Expected Utilities**
- **Conditional Probabilities, Beliefs & Observations**
- **Posterior Distributions**
- **Decision Rules**

The Gamble

- Imagine you can place a bet on a coin throw.
- You know the coin has a bias, but you don't know what it is.

On what do you bet?
How much do you bet on it?



Depends:
What do we get when we win?
Or loose?
How do we value the winnings / losses?
How are our winning chances?
How do we value the lottery?
Can we observe the coin first?
Can we play again?

Rewards & Utilities

How do we value the winnings / losses?

- Let ***R* be a set of rewards**: $R = \{-10\ Kr, 0\ Kr, 100\ Kr, 500\ Kr\}$
  or $R = \{$  $\}$
- **A utility function** assigns every reward a real value: $U: R \to \mathbb{R}$
- Define a **relation over rewards** based on utility function U:
  "a is better than b" $\Leftrightarrow\ a \succcurlyeq b\ \Leftrightarrow \mathrm{U}(a) \geq U(b)$ for all $a, b \in R$.
- Example:

Reward	- 100 NOK	- 10 NOK	0 NOK	10 NOK	100 NOK	1.000.000
Utility	- 1	- 0.1	0	0.01	0.2	5000

Probability Distributions

How are our winning chances?

- **A probability distribution** is a function that assigns every outcome of a random variable a probability in $[0,1]$.
- <u>Examples</u>:
  - Fair coin: When tossing the coin $P(x = heads) = 0.5$ and $P(x = tails) = 0.5$
  - Distributions over rewards:

Distribution	Money	Probability
p_1	50.000 Kr	100%
p'_1	1.000.000 Kr 50.000 Kr 0 Kr	10% 89% 1%

Which distribution(s) do you prefer?

Distribution	Item	Probability
p_2	  	80% 15% 5%
p'_2	 	90% 10%

Expected Utility as Relation over Probability Distributions

How do we value the lottery?

- Let R be a set of rewards.
- Let p_1, p_2 be probability distributions over R.
- Let $U: R \to \mathbb{R}$ be a utility function.
- Let r be a real random variable with outcomes R.
- The **expected utilities** given p_1 and p_2 are defined as

R *discrete* set: $\mathbb{E}_{p_1}[U] = \sum_{r \in R} p_1(r) \cdot U(r)$ and $\mathbb{E}_{p_2}[U] = \sum_{r \in R} p_2(r) \cdot U(r)$.

R *continuous* set: $\mathbb{E}_{p_1}[U] = \int_{r \in R} U(r) dp_1(r)$ and $\mathbb{E}_{p_2}[U] = \int_{r \in R} U(r) dp_2(r)$.

Expected Utility as Relation over Probability Distributions

How do we value the lottery?

- Let R be a set of rewards and p_1, p_2 probability distributions over R.
- Let $U: R \to \mathbb{R}$ be a utility function.
- Define a relation $\succcurlyeq$ on probability distributions over rewards by:

$$p_1 \succcurlyeq p_2 \text{ if and only if } \mathbb{E}_{p_1}[U] \geq \mathbb{E}_{p_2}[U]$$

The expected utility of the rewards given by p_1 is higher than for p_2.

- <u>Example</u>:

r	$U(r)$	$p_{not\ play}$	p_{play}
Not play	0	1	0
Play & lose	-1	0	0.99
Play & win	9	0	0.01
$\mathbb{E}(U)$		0	-0.9

$$p_{not\ play} \succ p_{play} \quad !$$

How do we value the lottery?

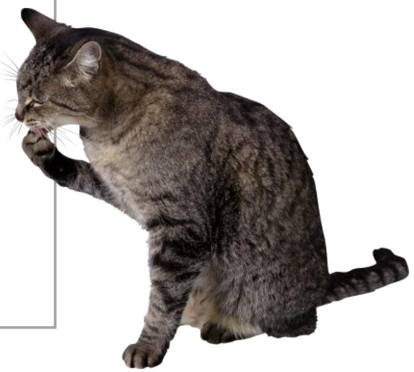
Expected Utility Hypothesis

- Let R be a set of rewards and p_1, p_2 probability distributions over R.
- Let $U: R \to \mathbb{R}$ be a utility function.
- <u>Expected Utility Hypothesis</u>: Prefer p_1 to p_2 iff $\mathbb{E}_{p_1}[U] \geq \mathbb{E}_{p_2}[U]$, where $\mathbb{E}_{p_i}[U] = \sum_{r \in R} U(r) P_i(r)$.
- <u>Example</u>:

Distribution	Money	Probability
A_1	50	100%
B_1	100 50 -20	10% 89% 1%
A_2	50 -20	11% 89%
B_2	100 -20	10% 90%

Which distribution(s) do you prefer?

Expected Utility Hypothesis would imply: $A_1 \preccurlyeq B_1$, if and only if $A_2 \preccurlyeq B_2$.



Expected Utility Hypothesis

How do we value the lottery?

- Let R be a set of rewards and p_1, p_2 probability distributions over R.
- Let $U: R \to \mathbb{R}$ be a utility function.
- Expected Utility Hypothesis: Prefer p_1 to p_2 iff $\mathbb{E}_{p_1}[U] \geq \mathbb{E}_{p_2}[U]$, where $\mathbb{E}_{p_i}[U] = \sum_{r \in R} U(r) P_i(r)$.
- Example:

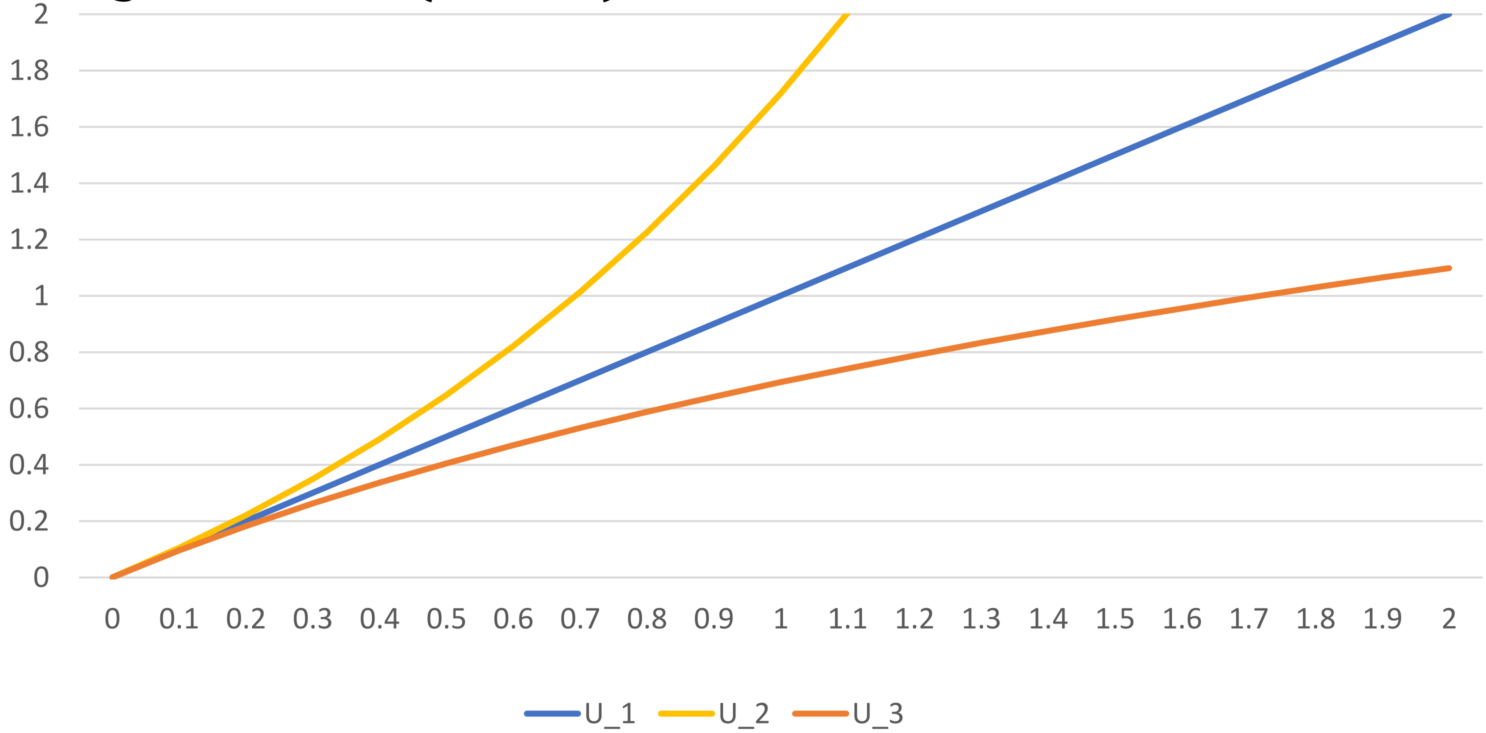
Distribution	Money	Probability
A_1	50	100%
B_1	100 50 -20	10% 89% 1%
A_2	50 -20	11% 89%
B_2	100 -20	10% 90%

From now on:
We always assume that the Expected Utility Hypothesis holds!
Even if this might not always model real human behavior…

Utility Functions: Examples

- Linear: $U_1(r) = r$
- Convex: $U_2(r) = e^r - 1$
- Concave: $U_3(r) = \ln(r + 1)$

Utility $\mathrm{U}(r)$



Reward r

Utility Functions and Risk Taking

Let the reward space be continuous $R = \mathbb{R}$. Assume the utility function U is …

- **Linear**: $U(r) = a \cdot r + b$ for some $a, b \in \mathbb{R}$

  → Risk Neutral: Any lottery is valued as much as its expected utility.

Utility Functions and Risk Taking

Let the reward space be continuous $R = \mathbb{R}$. Assume the utility function U is …

- **Convex**: For $\lambda \in [0,1]$ and all $x, y \in R$,

$$U(\lambda \cdot x + (1-\lambda) \cdot y) \leq \lambda \cdot \mathrm{U}(x) + (1-\lambda) \cdot U(y)$$

→ <u>Risk Affine</u>: Prefer a lottery over a certain outcome.

Ex.: Utility of getting $100\ \mathrm{Kr} = 0.3 \cdot 100\ \mathrm{Kr} + 0.5 \cdot 140\ \mathrm{Kr} + 0.2 \cdot 0\ \mathrm{Kr}$ for sure

is lower than the expected utility of getting $100\ \mathrm{Kr}$ w.p. 0.3 and $140\ \mathrm{Kr}$ w.p. 0.5.

Utility Functions and Risk Taking

Let the reward space be continuous $R = \mathbb{R}$. Assume the utility function U is …

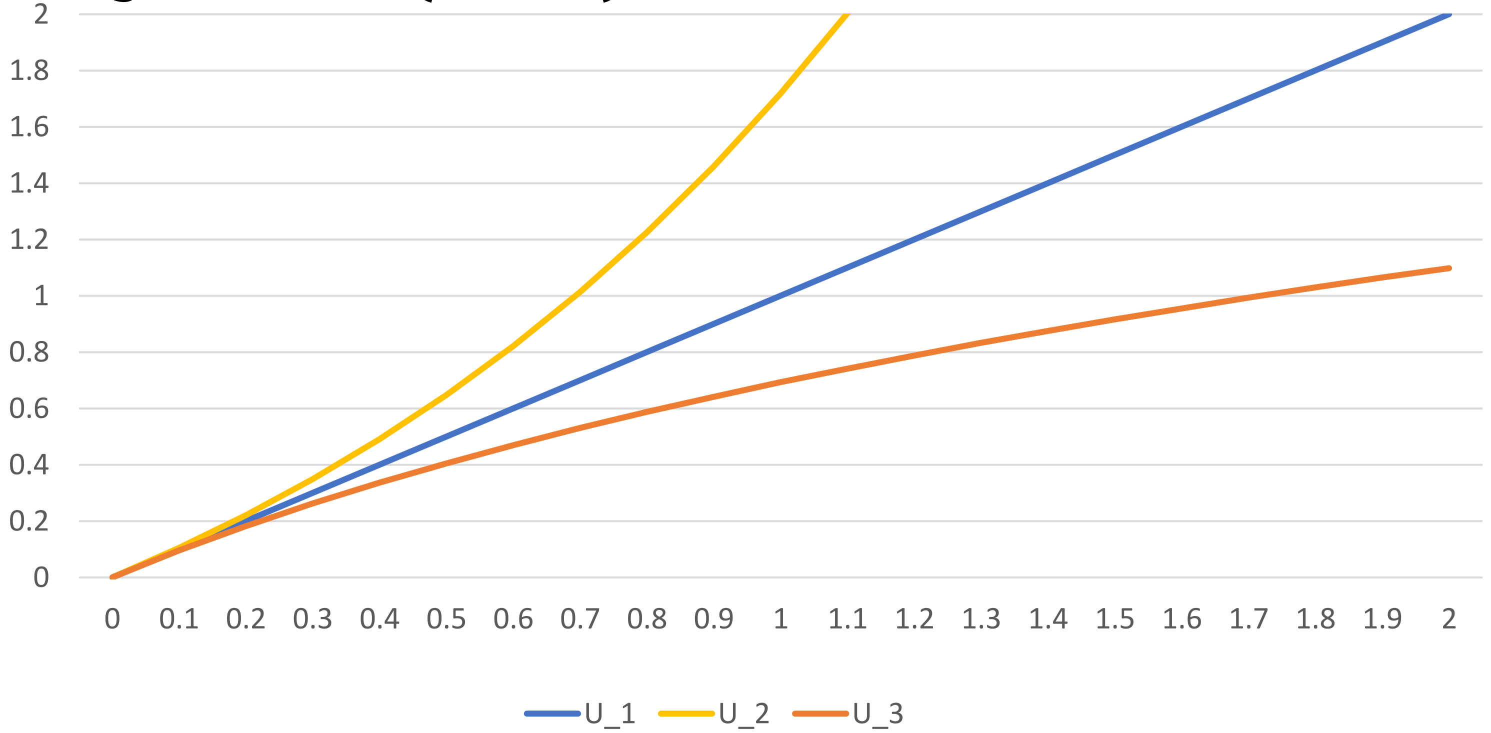
- **Concave**: For $\lambda \in [0,1]$ and all $x, y \in R$,

$$U(\lambda \cdot x + (1-\lambda) \cdot y) \geq \lambda \cdot \mathrm{U}(x) + (1-\lambda) \cdot U(y)$$

→ <u>Risk Averse</u>: Prefer a certain outcome over a lottery.

Utility Functions: Examples

- Linear: $U_1(r) = r$ → risk neutral
- Convex: $U_2(r) = e^r - 1$ → risk affine
- Concave: $U_3(r) = \ln(r + 1)$ → risk averse



Choosing Utility Maximising Actions

Business Example:
Investing *(… Kr)* into a new building.

- ➢ Does the price stay as initially calculated?
- ➢ What is the possible revenue?
- ➢ What are the risks?
  - ➢ Discovery of quick clay
  - ➢ Political decisions
  - ➢ …

Source: [Visits to the construction site - UiO:Life Science]

Utility Maximising Actions (Bets/Approx.)

- Action space: A $\quad$ $A = [0,1]$ invested money
- State space: S $\quad$ $S = \{10\%, 30\%, -50\%\}$ increase of investment
- Probabilities: $P: S \to [0,1]$ $\quad$ $P(\ \ 10\%) = 0.5$, $P(\ \ 30\%) = 0.1$, $P(-50\%) = 0.4$
- Utility function: $U: A \times S \to \mathbb{R}$ $\quad$ $U(a,s) = 0.98 \cdot ((1-a) + (1+s) \cdot a) = 0.98 \cdot (1 + s \cdot a)$ the money after investment and after loss through inflation (2%)
- Objective: $\max\limits_{a \in A} \mathbb{E}[U|a] = \max\limits_{a \in A} \sum_{s \in S} U(a,s) \cdot P(s)$,

  i.e., choose action a that maximises the expected utility.

How much should you invest?

What do we need to calculate?

$$\max_{a \in A} \sum_{s \in S} U(a,s) \cdot P(s)$$

$$= \max_{a \in [0,1]} \begin{matrix} 0.98 \cdot (1 + 0.1 \cdot a) \cdot 0.5 \\ +0.98 \cdot (1 + 0.3 \cdot a) \cdot 0.1 \\ +0.98 \cdot (1 - 0.5 \cdot a) \cdot 0.4 \end{matrix}$$

$$= \max_{a \in [0,1]} \begin{matrix} 0.98 + a \cdot 0.98 \cdot (0.1 \cdot 0.5 \\ +0.3 \cdot 0.1 \\ -0.5 \cdot 0.4) \end{matrix}$$

$$= \max_{a \in [0,1]} 0.98 - a \cdot 0.1176$$

Choose $a = 0$, i.e., invest no money!

What if we don't know the probabilities $P(s)$?

Let’s take a break…

Back on in 5 min!

Remember what we talked about?

Reminder

Utility Maximising Actions (Bets/Approx.)

How much should you invest?
What do we need to calculate?

- Action space: A — $A = [0,1]$ invested money
- State space: S — $\mathrm{S} = \{10\%, 30\%, -50\%\}$ increase of investment
- Probabilities: $P: \mathrm{S} \rightarrow [0,1]$ —
  $$P(\ 10\%) = ?$$
  $$P(\ 30\%) = ?$$
  $$P(-50\%) = ?$$
  Unknown → need more info!
- Utility function: $U: A \times \mathrm{S} \rightarrow \mathbb{R}$ —
  $$U(a, s) = 0.98 \cdot ((1 - a) + (1 + s) \cdot a) = 0.98 \cdot (1 + s \cdot a)$$
  the money after investment and after loss through inflation (2%)
- Objective:
  $$\max_{a \in A} \mathbb{E}[U|a] = \max_{a \in A} \sum_{s \in \mathrm{S}} U(a, s) \cdot P(s),$$
  i.e., choose action a that maximises the expected utility.

The Gamble

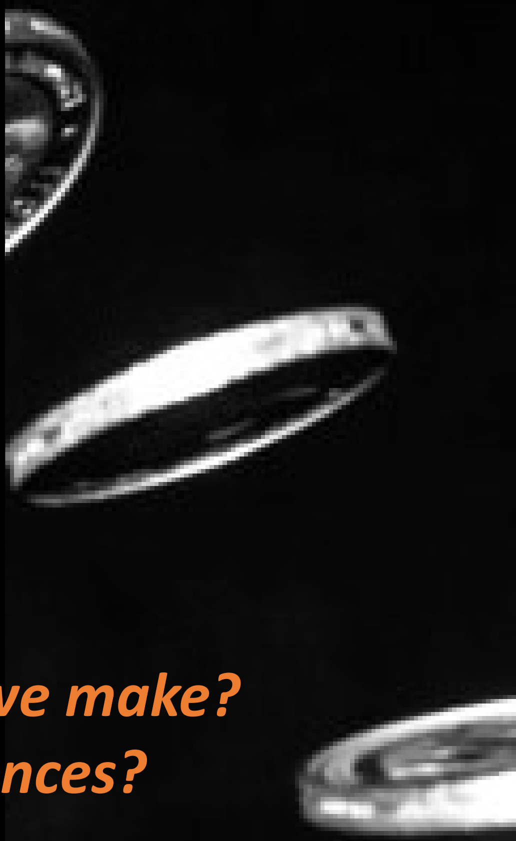
- Imagine you can place a bet on a coin throw.
- You know the coin has a bias, but you don't know what it is.
- You observe some throws with that same coin.

→ DECISION BASED ON DATA

What is your bet?

<u>Depends</u>:

Reward, Utility?
Which observations did we make?
How are our winning chances?



Choosing Utility Maximising Actions: Example

- A action space, S state space, $U: A \times \mathrm{S} \to \mathbb{R}$ utility function.
- Objective: $\max_{a \in \mathrm{A}} \mathbb{E}[U|a] = \max_{a \in A} \sum_{s \in S} U(a, s) \cdot P(s)$,
  i.e., choose action that maximises expected utility.
- **<u>Assumptions</u>**:
  - True $P(s)$ unknown,
    but we know some candidate distributions: **Model family $\boldsymbol{P}$**.
  - Observe some data. → Which model fits the data best?

Model Families

- A **family of models** $P = \{P_\omega | \omega \in \Omega\}$ is a set of probability distributions, that is parameterized by parameters Ω.
- <u>Example</u>:

  Biased coin with unknown bias

  - $\Omega = \{0.2, 0.4, 0.5, 0.6\}$ possible biases of the coin
  - Bernoulli distribution: $X_{Coin} \sim Bernoulli(\omega)$
  - $P = \{Bernoulli(\omega) \mid \omega \in \Omega\}$ is a model family

Heads = 1 Tails = 0

→ $\Pr(X_{Coin} = 1) = \omega$ $\Pr(X_{Coin} = 0) = 1 - \omega$

Maximum Likelihood Model

- **Model family** $P = \{P_\omega | \omega \in \Omega\}$. **Observed data** x.

- The **maximum likelihood model** is defined as: $\omega^*_{ML}(x) = \arg\max_{\omega} P_\omega(x)$

- <u>Example</u>: Biased coin with bias in $\Omega = [0,1]$ unknown

  $P = \{Bernoulli(\omega) \mid \omega \in \Omega\}$

Flip the coin repeatedly: Heads (1), Heads (1), Tails (0), Heads (1), Tails (0), Heads (1), Tails (0), …

Maximum likelihood model $\omega^*_{ML}(x)$: $1,\quad 1,\quad 2/3,\quad 3/4,\quad 3/5,\quad 4/6,\quad 4/7, \ldots$

$$\boldsymbol{\omega^*_{ML}(x = (1,1,0))} = \arg\max_{\omega\in[0,1]} P_\omega(x = (1,1,0))$$

$$= \arg\max_{\boldsymbol{\omega\in[0,1]}} \omega^2(1-\omega) = 2/3$$

Maximum Likelihood Approach

- **Model family** $P = \{P_\omega | \omega \in \Omega\}$. **Observed data** x.
- A action space, S state space, $U: A \times \mathrm{S} \to \mathbb{R}$ utility function.
- The **maximum likelihood model** is defined as: $\omega^*_{ML}(x) = \arg\max_\omega P_\omega(x)$

- Deciding based on the maximum likelihood model:

$$\max_{a \in A} \sum_{s \in \mathcal{S}} U(a, s) \cdot P_{\omega^*_{ML}(x)}(s) = \max_{a \in \mathrm{A}} \mathbb{E}_{P_{\omega^*_{ML}(x)}}[U | a],$$

i.e., choose the action that maximizes the expected utility w.r.t. $\omega^*_{ML}(x)$.

Maximum Likelihood Approach

- Deciding based on maximum likelihood model: $\max_{a \in A} \sum_{s \in \mathcal{S}} U(a,s) \cdot P_{\omega^*_{ML}(x)}(s)$,
  where $\omega^*_{ML}(x) = \arg\max_{\omega} P_{\omega}(x)$
- Example: Coin bias in $\Omega = [0,1]$ unknown, $P = \{Bernoulli(\omega) \mid \omega \in \Omega\}$
  Utility = 1 for win, = 0 for loss.

Trying to maximise the number of wins:

Flip the coin repeatedly	:	Heads (1),	Heads (1),	Tails (0),	Heads (1),	Tails (0),	Heads (1), …
Maximum likelihood model $\omega^*_{ML}(x)$	:	1,	1,	$2/3$,	$3/4$,	$3/5$,	$4/6$, …
Best next bet	:	Heads,	Heads,	Heads,	Heads,	Heads,	Heads, …
Number of wins	:	+ 0 ,	+ 1 ,	+ 0 ,	+ 1 ,	+ 0 ,	+ 1 , …

The Meteorologist

- Let $P = \{P_\omega | \omega \in \Omega\}$ be a model family for predicting the weather.
- Assume we have a <u>prior belief ξ</u> of which models might be good.
  "It is probably >16 °C in Oslo in September."
- Assume we observe the weather a few times, i.e., have some data x.

How can we update our belief ξ over the models?

Believes: Meteorology Example

- Let $P = \{P_\omega | \omega \in \Omega\}$ be a model family for predicting the weather.
- Assume we have a prior belief ξ of which models might be good:
  - ξ is a probability distribution over the parameters in Ω.
  - ξ gives us, for every weather model P_ω,
    a probability that this model is the correct one $\xi(\omega)$
- Assume we observe the weather a few times, i.e., have some data x.

How can we update the belief ξ on which of the models is correct?

Posterior belief: $\xi(\omega | x)$ probability conditioned on observations

Conditional Probability & Marginalisation

- Let $A, D \subseteq \mathcal{E}$ be events.
- Let $P(A|D) \triangleq \frac{P(A \cap D)}{P(D)}$ be the probability of A given D happened.
- Then $P(A \cap D) = P(A|D) \cdot P(D)$.
- Marginalisation: $P(D) = P(D|A) \cdot P(A) + P(D|A^C) \cdot P(A^C)$
  Marginalising the probability of D by A.
- More generally: $P(D) = \sum_{j=1,\ldots,n} P\left(D \middle| A_j\right) \cdot P(A_j)$
  for any events $A_1, \ldots, A_n \subseteq \mathcal{E}$ with $\bigcup_{j=1,\ldots,n} A_j = \mathcal{E}$.

Bayes Theorem

<u>Bayes Theorem</u>: Let $D \subseteq \mathcal{E}$ and $A_1, \dots, A_n \subseteq \mathcal{E}$ with $\bigcup_{j=1,\dots,n} A_j = \mathcal{E}$.

Observed data Weather models

Likelihood of data

… under specified weather model

Then $P(A_i|D) = \frac{P(D|A_i) \cdot P(A_i)}{\sum_{j=1,\dots,n} P(D|A_j) \cdot P(A_j)}$.

Prior

Probability of weather model before observing data (belief)

Posterior

Probability of weather model after observing data (updated belief)

What has that to do with updating beliefs over (weather) models?

Beliefs: Meteorology Example

- Let $P = \{P_\omega | \omega \in \Omega\}$ be a model family for predicting the weather.
- Assume we have a prior belief ξ of which models might be good:
  - ξ is a probability distribution over the parameters in Ω.
  - ξ gives us, for every weather model P_ω,
    a probability that this model is the correct one $\xi(\omega)$
- Assume we observe the weather a few times, i.e., have some data x.

Posterior belief: $\xi(\omega|x) = \frac{P_\omega(x) \cdot \xi(\omega)}{\sum_{\omega'} P_{\omega'}(x) \cdot \xi(\omega')}$

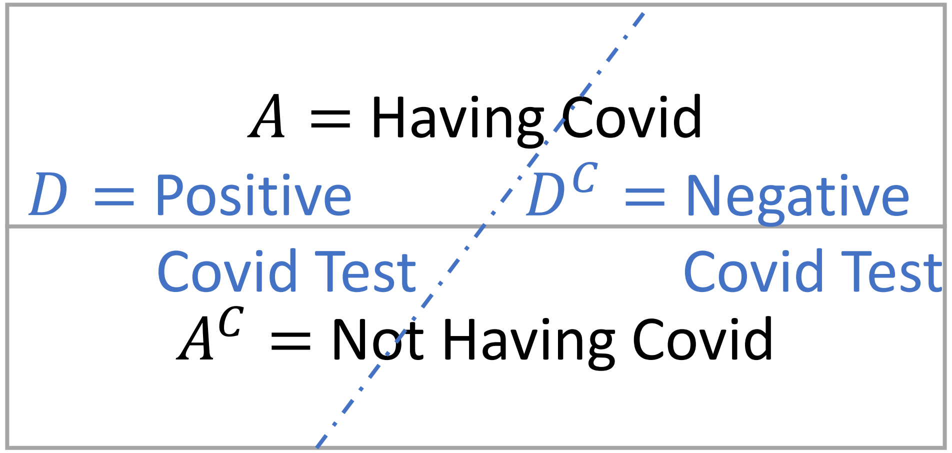
$= \xi(x\,|\omega)$ Probability to observe data x, given weather model (with parameter) ω

Probability weather model (with parameter) ω is correct, given we observe data x

… generally, not easy to compute
→ conjugate priors can help!

Example

- Let A = having Covid
  D = positive Covid test
  - $P(A) \quad = 90\%$
  - $P(D|A) \quad = 95\%$ true positive test
  - $P(D|A^C) =$ 5% false positive test



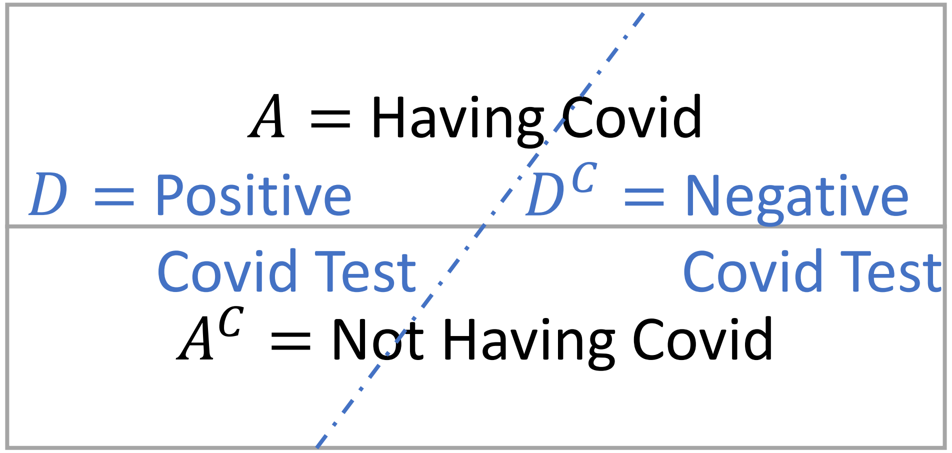
- <u>Bayes Theorem</u>: Let $D \subseteq \mathcal{E}$ and $A_1, \ldots, A_n \subseteq \mathcal{E}$ with $\bigcup_{j=1,\ldots,n} A_j = \mathcal{E}$.
  Then $P(A_i|D) = \frac{P(D|A_i) \cdot P(A_i)}{\sum_{j=1,\ldots,n} P\left(D|A_j\right) \cdot P(A_j)}$.

- Exercise (5-10 min, with your neighbor or alone):
  What is the probability of having Covid, when having a negative test result?

Example

- Let A = having Covid
  D = positive Covid test

  - $P(A) \quad = 90\%$
  - $P(D|A) \quad = 95\%$ true positive test
  - $P(D|A^C) = \quad 5\%$ false positive test



- Bayes Theorem: Let $D \subseteq \mathcal{E}$ and $A_1, \dots, A_n \subseteq \mathcal{E}$ with $\bigcup_{j=1,\dots,n} A_j = \mathcal{E}$.

$$\text{Then } P(A_i|D) = \frac{P(D|A_i) \cdot P(A_i)}{\sum_{j=1,\dots,n} P\left(D \middle| A_j\right) \cdot P(A_j)}.$$

Probability of having Covid when having a negative test result:

$$P(A|D^C) = \frac{P(D^C|A) \cdot P(A)}{P(D^C)} = \frac{P(D^C|A) \cdot P(A)}{P(D^C|A) \cdot P(A) + P(D^C|A^C) \cdot P(A^C)} = \frac{0.05 \cdot 0.9}{0.05 \cdot 0.9 + 0.95 \cdot 0.1} \approx 0.32$$

Conjugate Prior Example: Beta Distribution over Bernoulli Models

Biased coin with bias in $\Omega = [0,1]$, $P = \{Bernoulli(\omega) \mid \omega \in \Omega\}$

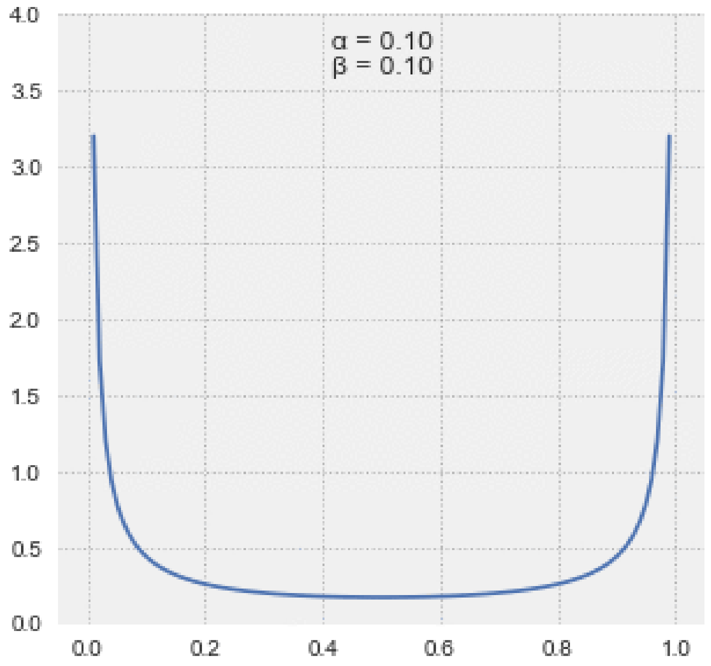
<u>Prior</u>:

- $\xi(\omega) = Beta(\alpha, \beta)$
  Beta distribution with parameters α, β
  with **expectation** $\mathbb{E}_\xi[\omega] = \frac{\alpha}{\alpha+\beta}$

<u>Posterior</u>:

- $\xi(\omega|x) = Beta\big(\alpha + x, \beta + (1-x)\big)$
  for **observations** $x \sim Bernoulli(\omega)$

→ **Conjugate prior**:
$\xi(\omega|x)$ is same type of distribution as $\xi(\omega)$



The Beta Distribution for different parameters (source link)

Maximum a Posteriori Model

- **Model family** $P = \{P_\omega | \omega \in \Omega\}$. **Prior belief** $\xi : \Omega \to [0,1]$. **Observed data** x.
- The **maximum a posteriori model** is defined as $\omega^*_{MAP}(x) = \arg\max_\omega \xi(\omega|x)$.
- Example: Biased coin with bias in $\Omega = \{0.3, 0.6., 0.9\}$ *(discrete set)* unknown,
  Initial belief uniform $\xi(\omega) = \frac{1}{3}$ for all $\omega \in \Omega$
  $P = \{Bernoulli(\omega) \mid \omega \in \Omega\}$

Flip the coin repeatedly: Heads (1), Heads (1), Tails (0), Heads (1), Tails (0), Heads (1), Tails (0), …

Posterior $\xi(0.3 \mid \dots), \xi(0.6 \mid \dots), \xi(0.9 \mid \dots)$: $[\frac{1}{6}, \frac{2}{6}, \frac{3}{6}]$, $[\frac{3}{42}, \frac{12}{42}, \frac{27}{42}]$, …

Maximum a posteriori model $\omega^*_{MAP}(x)$: 0.9, 0.9, …

$$\xi(0.3 \mid x = 1) = \frac{P_{0.3}(x = 1)\,\xi(0.3)}{\sum_{\omega'} P_{\omega'}(x)\,\xi(\omega')} = \frac{0.3 \cdot 1/3}{0.3 \cdot \frac{1}{3} + 0.6 \cdot \frac{1}{3} + 0.9 \cdot \frac{1}{3}} = \frac{1}{6}$$

$$\xi(0.9 \mid x = 1, 1) = \frac{P_{0.9}(x = 1)\,\xi(0.9 \mid 1)}{\sum_{\omega'} P_{\omega'}(x)\,\xi(\omega' \mid 1)} = \frac{0.9 \cdot 3/6}{0.3 \cdot \frac{1}{6} + 0.6 \cdot \frac{2}{6} + 0.9 \cdot \frac{3}{6}} = \frac{27}{42}$$

Maximum a Posteriori Approach

- **Model family** $P = \{P_\omega | \omega \in \Omega\}$. **Prior belief** $\xi \colon \Omega \to [0,1]$. **Observed data** x.
- The **maximum a posteriori model** is defined as $\omega^*_{MAP}(x) = \arg\max_\omega \xi(\omega|x)$.
- A action space, S state space, $U \colon A \times \mathrm{S} \to \mathbb{R}$ utility function.

- Deciding based on the maximum a posteriori model:

$$\max_{a \in A} \sum_{s \in \mathcal{S}} U(a, s) \cdot P_{\omega^*_{MAP}(x)}(s) = \max_{a \in \mathrm{A}} \mathbb{E}_{\omega^*_{MAP}(x)}[U|a],$$

i.e., choose the action that maximizes the expected utility w.r.t. $\omega^*_{MAP}(x)$.

Maximum a Posteriori Approach

- Deciding based on the maximum a posteriori model: $\max_{a \in A} \sum_{s \in \mathcal{S}} U(a, s) \cdot P_{\omega^*_{MAP}(x)}(s)$

- <u>Example</u>: Biased coin with bias in $\Omega = \{0.3, 0.6., 0.9\}$ *(discrete set)* unknown,
  Initial belief uniform $\xi(\omega) = \frac{1}{3}$ for all $\omega \in \Omega$
  $P = \{Bernoulli(\omega) \mid \omega \in \Omega\}$

<u>Trying to maximise the number of wins</u>:

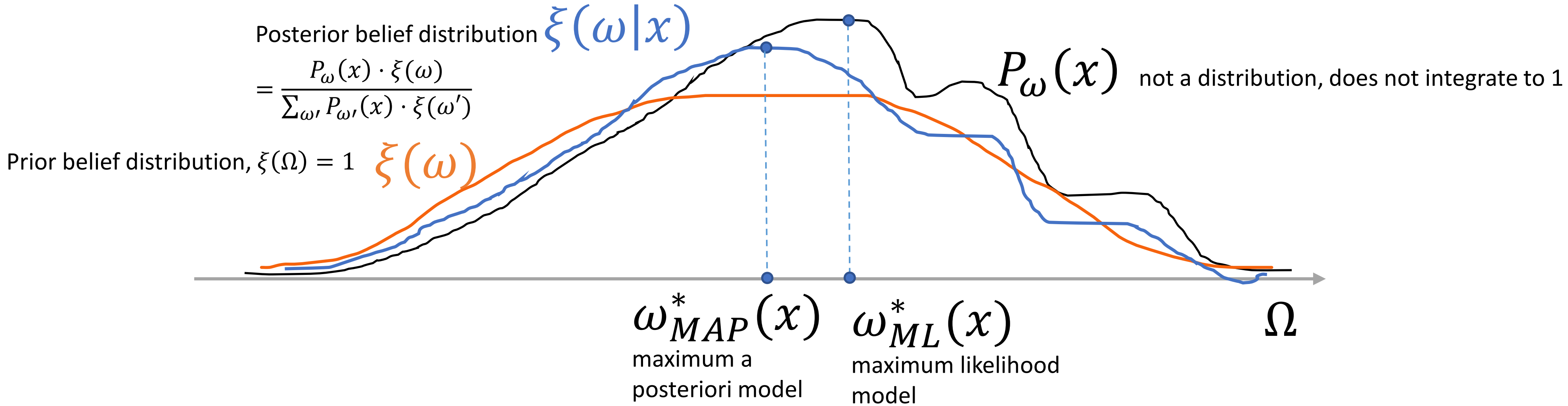
Flip the coin repeatedly	:	Heads (1),	Heads (1),	Tails (0),	Heads (1),	Tails (0),	Heads (1), …
Maximum a posteriori model $\omega^*_{MAP}(x)$	:	0.9,	0.9,	0.9,	0.9,	0.9,	0.9, …
Best next bet	:	Heads,	Heads,	Heads,	Heads,	Heads,	Heads, …
Number of wins	:	+ 0 ,	+ 1 ,	+ 0 ,	+ 1 ,	+ 0 ,	+ 1 , …

But why do we disregard our beliefs over the other models?

Bayesian Inference

- **Model family** $P = \{P_\omega | \omega \in \Omega\}$. **Prior belief** $\xi : \Omega \to [0,1]$. **Observed data** x.

- <u>Bayesian Inference</u>: Want to maintain full posterior distribution $\xi(\omega|x)$ rather than fixing one model.



Bayes Decision Rule

- **Model family** $P = \{P_\omega | \omega \in \Omega\}$. **Prior belief** $\xi : \Omega \rightarrow [0,1]$. **Observed data** x.
- A action space, S state space, $U: A \times \mathrm{S} \rightarrow \mathbb{R}$ utility function.

- Deciding based on **Bayes Rule**: $\max_{a \in \mathrm{A}} \mathbb{E}_{\omega \sim \xi(.|x)}[\mathbb{E}_{P_\omega}[U|a]]$

$$= \max_{a \in A} \sum_{\omega \in \Omega} \xi(\omega|x) \sum_{s \in \mathcal{S}} U(a, s) \cdot P_\omega(s)$$

$$= \max_{a \in A} \sum_{s \in \mathcal{S}} U(a, s) \sum_{\omega \in \Omega} \xi(\omega|x) \cdot P_\omega(s)$$

$$= \max_{a \in \mathrm{A}} \sum_{s \in \mathcal{S}} U(a, s) \cdot \mathbb{E}_{\omega \sim \xi(.|x)}[P_\omega(s)],$$

i.e., choose an action that maximises the expected utility w.r.t. the posterior distr. $\xi(.|x)$.

Summary

Bayes Theorem:
Let D be some data and A_i with $i = 1, \dots, n$ events such that $\Omega = \bigcup_{i=1,\dots,n} A_i$. Then

$$\boldsymbol{P(A_i|D)} = \frac{P(D|A_i) \cdot P(A_i)}{\sum_{j=1,\dots,n} P(D|A_j) \cdot P(A_j)}.$$

Decision Scenario:
A action space, $\mathcal{S}$ state space

$U: A \times \mathcal{S} \to \mathbb{R}$ utility function

$P = \{P_\omega | \omega \in \Omega\}$ family of models,
= parameterised distr. over states

x observed data,
ξ belief (distribution) over Ω

→ posterior distribution:

$$\xi(\omega|x) = \frac{P_\omega(x) \cdot \xi(\omega)}{\sum_{\omega'} P_{\omega'}(x) \cdot \xi(\omega')}$$

Decision Rules:

Maximum likelihood model
$\omega^*_{ML}(x) = \arg\max_\omega P_\omega(x)$

→ Objective $\max_{a \in A} \mathbb{E}_{\omega^*_{ML}(x)}[U|a]$

Maximum a posteriori model
$\omega^*_{MAP}(x) = \arg\max_\omega \xi(\omega|x)$

→ Objective $\max_{a \in A} \mathbb{E}_{\omega^*_{MAP}(x)}[U|a]$

Bayes Inference Objective:

→ $\max_{a \in A} \sum_{s \in \mathcal{S}} U(a, s) \cdot \mathbb{E}_{\omega \sim \xi(.|x)}[P_\omega(s)]$