Qualitative Analysis of Invariant Tori in a Dynamical System

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Abstract

Invariant tori are examples of invariant manifolds in dynamical systems. Usual tools in dynamical systems such as analysis and numerical simulations alone are often not sufficient to understand the complicated mechanisms that cause changes in these manifolds. Computer-graphical visualization is a natural and powerful addition to these tools used for the qualitative study of dynamical systems, especially for the study of invariant manifolds.

The dynamics of two linearly coupled oscillators is the focus of this case study. With little or no coupling between the oscillators, an invariant torus is present but it breaks down for strong coupling. Visualization has been employed to gain a qualitative understanding of this breakdown process. The visualization has allowed key features of the tori to be recognized, and it has proven to be indispensable in developing and testing hypotheses about the tori.

1 Introduction

Dynamical systems are used to describe problems from a wide array of subjects with diverse origins, including engineering, biology, physics and economics. A dynamical system is a set of differential equations with some additional parameters. One of the questions arising in the study of dynamical systems is how "typical" behaviour of the system changes if the parameters are varied. Such a change in behaviour is called a bifurcation. A simple example of a bifurcation is the evolution of an attracting fixed point into an attracting periodic orbit under change of the parameters. Further changes of the parameters can lead to bifurcations of more complex structures like invariant tori or other invariant manifolds. Here, invariance means that any solution of the dynamical system that starts on the manifold will remain on the manifold for all time.

In the study of invariant manifolds and their bifurcations, people use several tools: analysis of the equations defining the dynamical system, numerical methods to compute and follow solutions of the system and further analysis of the numerically observed behaviour. As numerical methods become more sophisticated and computers more powerful, problems displaying increasingly complex behaviour can be studied. As a consequence, it is difficult to interpret the computational results without a good means of visualization. Therefore, computer-graphical techniques become indispensable to gain a qualitative understanding of dynamical systems.

At the moment there are a growing number of mathematical software packages to simulate dynamical systems, to calculate bifurcations, and to follow solutions. Some examples are AUTO [6] and DSTOOL [3]. These programs can follow certain types of solutions (fixed points, periodic orbits, homo/heteroclinic connections, etc.) and indicate their bifurcations, but they are generally only capable of dealing with one-dimensional invariant manifolds. For this reason there is a need for a package that can follow higher-dimensional invariant manifolds and that can assist in the analysis of their more complex qualitative behaviour. A natural extension of the one-dimensional invariant manifold, formed by a periodic orbit, is the important case of the two-dimensional torus. To understand the changes in the behaviour of an invariant torus, it is crucial to have access to a graphical package that visualizes this two-dimensional object in an appropriate way. A challenge for this visualization is that the torus is often embedded in a space with dimension larger than three.

An early example of the application of computer graphics to the study of invariant tori is the 1971 paper by Baxter *et al.* [4]. Even with very crude graphical capabilities, the authors cite the graphics as being crucial. A much more sophisticated application of computer graphics is given by Koçak *et al.* [9]. They study and visualize the foliation of tori within an energy level set of a four-dimensional completely integrable linear Hamiltonian system. As indicated by the authors, a small non-Hamiltonian perturbation drastically changes the behaviour of the system, and most tori will break up and disappear. A similar bifurcation process can be observed in other systems (e.g. dissipative systems) which have invariant tori. The general mechanism of this disintegration is not yet understood. By using computation and visualization to follow invariant tori towards breakdown, progress is being made in understanding this process.

2 Two coupled oscillators

The subject of this case study is a dynamical system which describes a pair of linearly coupled oscillators. The problem has physical significance in a number of ways. One of the motivating applications comes from bio-chemistry: the concentrations of two types of cells in living tissue are independently oscillating if there is no diffusion between those cell types. However, if there is diffusion present, the concentrations of the two types of cells start influencing each other and more complex behaviour occurs (see e.g. [2] and [10]).

To derive a mathematical model for this phenomenon, two phase variables corresponding to concentration and concentration flux are associated with each cell type. Without diffusion the model gives two oscillators, each with a two-dimensional phase space. By taking diffusion into account, those two oscillators become coupled and a four-dimensional space is required to analyze the problem. Convenient coordinates for the oscillators are polar coordinates: $(r_1, r_2, \theta_1, \theta_2) \in \mathbf{R}^2 \times S^2$. (The original concentrations are related to $r_1 \cos\theta_1$ and $r_2 \cos\theta_2$.)

The differential equations for the coupled oscillators are

$$\begin{aligned}
\dot{r}_1 &= r_1(1-r_1^2) - \delta\,[\,r_1\,(1-\sin(2\theta_1)) + r_2\,D(\theta_1,\theta_2)\,] \\
\dot{r}_2 &= r_2(1-r_2^2) - \delta\,[\,r_2\,(1-\sin(2\theta_2)) + r_1\,D(\theta_2,\theta_1)\,] \\
\dot{\theta}_1 &= \beta + \delta\,[\,\cos(2\theta_1) - \frac{r_2}{r_1}\,D(\theta_1,-\theta_2)\,] \\
\dot{\theta}_2 &= \beta + \delta\,[\,\cos(2\theta_2) - \frac{r_1}{r_2}\,D(\theta_2,-\theta_1)\,]
\end{aligned}$$

with

$$D(\theta_1,\theta_2) = \sin(\theta_1+\theta_2) - \cos(\theta_1-\theta_2).$$

In these equations, δ gives the strength of the coupling (i.e. diffusion) and β gives the angular velocity of the uncoupled oscillators.

Without coupling (i.e. $\delta = 0$), each oscillator has a stable, attracting, periodic limit cycle which is a circle. The Cartesian product of these two circles forms a stable invariant torus, filled with periodic solutions. Under weak coupling, the stable invariant torus itself persists as an invariant manifold but only two periodic solutions on the torus survive. One is stable and the other one is π radians apart and unstable. Now, most solutions on the torus are curves flowing from the unstable periodic orbit to the stable one. A detailed analysis of several kinds bifurcations of the unstable periodic solution on the torus can be found in Aronson *et al.* [1].

Larger changes in the parameter δ, corresponding to stronger coupling, cause dramatic changes in the shape of the torus. Beyond a certain point, the torus breaks down and only the stable solution persists. The way this breakdown occurs depends on the second parameter β. The goal of this study is to gain a qualitative understanding of the deformation of the torus as the strength of coupling increases to the point of breakdown. For a discussion of the computational methods used to calculate the invariant tori, the reader is referred to Dieci *et al.* [5] and Edoh *et al.* [7].

3 Visualization

Although each invariant torus is embedded in a four-dimensional space, it is possible to visualize the tori in $\mathbf{R}^3$ with appropriate transformations and projections. Since a torus is parameterized by θ_1 and θ_2, an effective representation is constructed by plotting the two angles with each of the radii. Two sheets can be defined by using the coordinates $(r_1, \theta_1, \theta_2)$ and $(r_2, \theta_1, \theta_2)$ (see Figure 1). The periodicity of the angles allows a cylinder and torus to be constructed for each radius, with simple transformations. These representations have been successfully used in the visualization, but extra care must be taken (as Koçak *et al.* [9] also note) when interpreting them because they do not fully express the four-dimensional nature of the tori.

In the original polar coordinates, the periodic stable and unstable solutions wrap around the torus and their behaviour is obscured. To create more comprehensible representations, a shear transformation and modulo arithmetic is used to untwist the torus and realign the periodic solutions. This transformation is illustrated for the sheet form in Figure 2. Two possible ways of making tori from the sheets after this transformation are depicted in Figure 3.

A further projection to $\mathbf{R}^2$ is employed to more closely examine local deformations and symmetries of tori. It involves plotting the individual cross-sections of the torus and is illustrated in Figure 4.

If the torus is invariant, the trajectories of all solutions which begin on the torus will remain there. A visual means of verifying this property is to plot the vector field on the torus. If the torus is indeed invariant, all vectors will lie tangent to the surface. Beyond this simple test, the direction and magnitude of the vector field can quickly indicate the dynamics on the torus. A further tool in studying the local behaviour on the torus is the interactive selection of starting points for integration. Plotting individual trajectories can add a great deal to the qualitative understanding of the problem.

It is possible that (periodic) solutions which are not part of the torus may influence its shape. Without computer-graphical techniques allowing one to depict these solutions together with the torus, such an influence would be extremely difficult to study. For example, it is not physically possible for one of these solutions to intersect the full torus, but a particular projection into $\mathbf{R}^3$ can suggest such an intersection. Therefore, other projections into $\mathbf{R}^3$ must be considered to reveal that such an apparent intersection is false.

It can be observed that the dynamical system for the two coupled oscillators has D_2-symmetry. Specifically, if $(r_1(t), r_2(t), \theta_1(t), \theta_2(t))$ is a solution of the dynamical system, then

$$\begin{aligned}&(r_2(t), r_1(t), \theta_2(t), \theta_1(t)),\\ &(r_1(t), r_2(t), \theta_1(t)+\pi, \theta_2(t)+\pi), \qquad \text{and}\\ &(r_2(t), r_1(t), \theta_2(t)+\pi, \theta_1(t)+\pi)\end{aligned}$$

are solutions, too. The two periodic solutions on the torus are invariant with respect to the action of this symmetry group. Using this fact, it can be argued that the torus has to be invariant under this symmetry group. It is straightforward to check this property using visualization. Furthermore, even if one is unaware of symmetries, such a feature becomes readily accessible through the visualization process. Computer graphics also becomes an ideal tool in the development of the computational methods used in this problem [8]. The existence of these symmetries provides an important way to quickly verify the reliability of computation, as it is straightforward to determine when the symmetry conditions are not met.

The evolution of invariant tori has been analysed as δ increases, with the parameter β fixed at 0.55. As δ increases from 0 towards a critical value around $\delta = 0.26$ (near the onset of breakdown), the torus deforms from a smooth doughnut-like shape into a configuration with some sharp cusps. This is depicted in Figure 5. From the visualization, it seems that in this case the formation of cusps is related to the breakdown.

A simulation with the bifurcation program AUTO shows that for $\delta \approx 0.2605$ there is a saddle-node like bifurcation of the unstable orbit and two other periodic orbits, which are not part of the torus. One could expect that this bifurcation is the driving force behind the breakdown of the torus. However, through visualization, no specific relationship between the cusps and these other periodic orbits is apparent. The ability to interact with the torus representations and follow selected solution trajectories provides a unique means to seek explanations for the breakdown. By examining the vector field and following solutions, one can see that most solutions on the torus tend to go around until they meet a cusp, follow the cusp towards the stable periodic orbit, and finally leave the cusp to follow this periodic orbit. This behaviour may suggest that there are some special solutions off the torus, very close to the cusps. These solutions may approach the torus

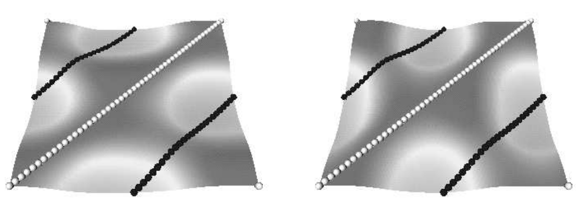

Figure 1: Sheet representations for $(r_1, \theta_1, \theta_2)$ and $(r_2, \theta_1, \theta_2)$ with $\delta = 0.2$. The unstable periodic solution is indicated by the light (white) dots, the unstable one by the dark (blue) dots. The sheets are shaded according to the magnitude of radius.

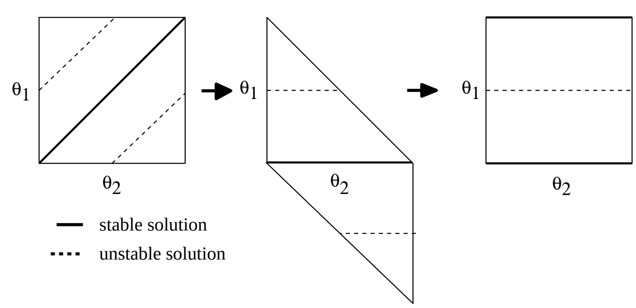

Figure 2: Sheet transformation to "untwist" the torus. A shear transformation is applied to the left image to arrive at the middle one and then the right image is derived by modulo arithmetic.

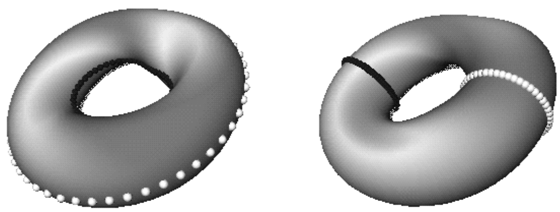

Figure 3: Two possible torus representations of the (transformed) sheets. On the left, the r_1 sheet is folded first around θ_1 and then around θ_2. The procedure is reversed on the right as the r_2 sheet is folded first around θ_2 and then around θ_1. Again, the two special periodic solutions are denoted by light (white) and dark (blue) dots respectively.

and trigger the breakdown. An analysis is currently underway in an attempt to prove the existence of special solutions related to these cusps.

4 Conclusions and future work

Traditionally, computation and analysis have both been necessary in the study of complex mathematical problems. When limits of analysis are reached, computation is used to lend insight into otherwise obscured features and consequently to direct further analytical study. For (nonlinear) dynamical systems, complexity is sufficiently great that visualization becomes an indispensable new partner in such investigations. Here is an example of how visualization of the torus provides insight into the reliability of numerical results and provides focus for what new numerical and analytical studies would be fruitful. We feel that the relatively few successful studies of dynamical systems in this way is due not to its lack of potential but rather to the difficulty in bringing together persons with expertise in the visualization, numerical methods and analytical complexities of dynamical systems.

The qualitative analysis capability provided by the visualization is of great help in the understanding of the global dynamical behaviour of the linearly coupled oscillators under study. This visualization has allowed hypotheses to be formulated and tested with regards to the breakdown of the invariant torus which would otherwise have been very difficult to consider.

It seems inevitable that visualization will play a crucial role in developing further understanding about this complex problem of torus breakdown. To facilitate this goal, further development of effective display techniques and representations will continue.

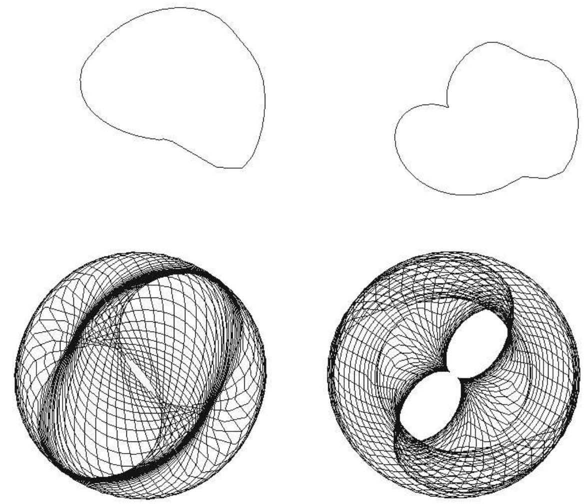

Figure 4: Cross sections of tori can be obtained by fixing the outer angle. The result of this technique is also sometimes known as a Poincaré section. The top of the figure shows a particular section for r_1 (left) and r_2 (right). In the bottom of the figure, the sections corresponding to all outer angle values in the data are overlayed. In this way, whole tori are condensed into single views. Notice the symmetry which is apparent.

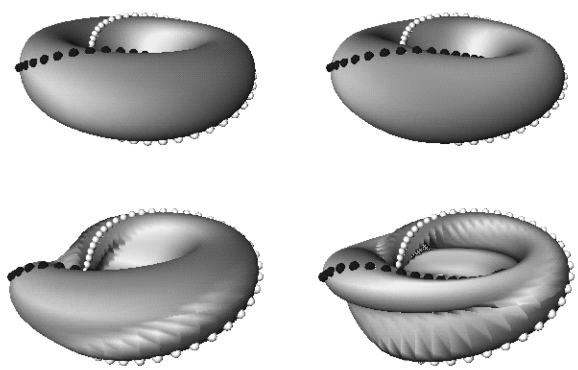

Figure 5: The two tori, $(r_1, \theta_1, \theta_2)$ and $(r_2, \theta_1, \theta_2)$, for two increasing values of the coupling at $\beta = 0.55$. At the top are displayed the tori representing the invariant torus for $\delta = 0.20$ and at the bottom are the tori representing the invariant torus for $\delta = 0.26$. Here the torus is very close to breakdown and the development of sharp cusps can be seen.

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