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Controllability and Hedgibility of Black-Scholes Equations with N Stocks

K. Sakthivel · J.-H. Kim

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Abstract This paper is to investigate the controllability and observability properties of linear and certain nonlinear Black-Scholes (B-S) type equations consisting of N stocks in an appropriate bounded domain I of $\mathbb{R}_+^N$. In this model both the stock volatility and interest rate are influenced by the stock prices and the control which is related to the hedging ratio in option pricing of finance is distributed over a subdomain of I. The proof of the controllability result for the linear B-S equations relies on the suitable observability inequality for the associated adjoint problem, and for the nonlinear model, fixed point technique is applied. Our result leads to that the dynamic hedgibility in finance is proved in the context of controllability theory.

Keywords Black-Scholes equations · Volatility · Controllability · Observability · Hedgibility

1 Introduction

A call option is the right to buy a security at a specified price (called the exercise or strike price) during a specified period of time. European options can only be exercised on the day of expiration of the option. American options can be exercised at any time up to and including the day of expiration of the option. In option pricing theory, Black-Scholes (B-S) equation is one of the most effective models because it is easy to use and understand. In the most celebrated work [11], the stochastic behavior of the underlying asset S is modeled by a geometric Brownian motion given by the stochastic differential equation

$$dS(t) = \beta S dt + \sigma S dB(t),$$

where $B(t)$ denotes a standard Wiener process. The parameters β and σ are called the drift rate and volatility of the underlying asset. By Ito's rule [19], the stochastic behavior of a

K. Sakthivel (✉) · J.-H. Kim
Department of Mathematics, Yonsei University, Seoul 120-749, Korea
e-mail: pktsakthi@gmail.com

J.-H. Kim
e-mail: jhkim96@yonsei.ac.kr

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derivative security $Y(S,t)$ is governed by the stochastic differential equation

$$dY(S,t) = \left(Y_t + \frac{1}{2}\sigma^2 S^2 Y_{SS} + \beta S Y_S\right)dt + \sigma S Y_S dB(t). \tag{1.1}$$

In their seminal paper, Black and Scholes constructed a dynamic portfolio consisting of a derivative security and a variable amount of the underlying asset completely eliminating risk in (1.1). In the absence of arbitrage, the instantaneous return of this portfolio equals the return of a riskless investment, denoted by the interest rate r, which together with (1.1) yields the famous B-S equation

$$Y_t + \frac{1}{2}\sigma^2 S^2 Y_{SS} + rSY_S - rY = 0. \tag{1.2}$$

As an interesting consequence of no-arbitrage arguments (see, [29]), the drift rate β does not enter in (1.2). In general, for European options, the B-S equation results in a boundary value problem of a diffusion equation whereas for American options, the B-S equation results in a free boundary value problem. There are usually two ways to solve these kind of option pricing problem: the analytic and numerical approaches (see, [1, 2, 12, 14, 18, 27]).

An important characteristic of the model (1.2) is the assumption that the volatility of the underlying security is constant. However, practitioners have observed the so-called volatility smile effect; namely options written on the same underlying asset usually trade, in B-S term, with different implied volatilities. Deep-in-the-money and deep-out-of-the-money options are traded at higher implied volatility than at the money options. This evidence is not consistent with the constant volatility assumption made in B-S model [11].

In this paper, we discuss the controllability and observability properties of the more general B-S model with N stocks allowing the volatility dependence over the stock prices. It should be emphasized that the external input (controller) in the option pricing model is related to the hedging ratio in option pricing of finance; since the option pricing problem is basically equivalent to finding the portfolio (or controller, say, u) so that the wealth becomes the option payoff at the terminal date. More precisely, the common factor of all these contracts is that they all are completely defined in terms of the underlying asset S, which makes it natural to call them as derivative instruments or contingent claims. Therefore a contingent claim with date of maturity T is an option payoff value, say, X is reachable or hedgeable, if there exists a self-financing portfolio u such that $Y^u(T) = X$. In this case we say that u is a hedge against X, which is also called a hedging portfolio (see, [10]). Moreover, if every contingent claim is reachable, we say that the market is complete. Thus importance of our article is that the dynamic hedgibility which is generally assumed in B-S theory, is proved in the context of controllability theory. Predicting the behavior of a financial market from the knowledge of its present state and financial laws is referred as direct problem of option pricing. On the other hand, the procedure of marking to market requires users to calibrate model parameters to match current market prices of benchmark options is called inverse problem of option pricing. To the best of our knowledge, there is no work that appeared regarding the controllability analysis of general option pricing problems in finance, though, there are some papers (see, for example, [13, 15]) for inverse option pricing problem.

2 Problem Formulation

First let us describe some notations and function spaces which will be used throughout the paper. Let I be a bounded domain(which is described below) in $\mathbb{R}^N_+$ and $0 < T < \infty$ be an arbitrary but fixed moment of time. Let us denote $z_t = \partial z/\partial t$, $z_{s_i} = \partial z/\partial s_i$ and $z_{s_i s_j} = \partial^2 z/\partial s_i \partial s_j$ for all $i, j = 1, 2, \ldots, N$ and for each $\mathbf{s} = (s_1, s_2, \ldots, s_N)$. We define $\mathcal{A}(z\mathbf{s}, \nu\mathbf{s}) = \sum_{i,j=1}^{N} a_{ij}(\mathbf{s})(s_i z_i)(s_j \nu_j)$, $\forall z \in \mathbb{R}^N$, where a_{ij} is the $N \times N$ matrix and ν is the outward unit normal vector to the boundary of I and $\mathcal{A}(\nabla\psi\mathbf{s}, \nabla\psi\mathbf{s})$ is analogously defined for some smooth function ψ.

Now for simplicity set $Q_T = (0, T) \times I$ and $\Sigma_T = (0, T) \times \partial I$. For each positive integer m and $p > 1$, or $p = \infty$, we denote as usual by $W^{m,p}(I)$, the Sobolev space of functions in $L^p(I)$ whose weak derivatives of order less than or equal to m are also in $L^p(I)$. When $p = 2$ instead of $W^{m,2}(I)$, we shall write $H^m(I)$. Further, we need the space $L^2(0, T; H^1(I))$ of all equivalence classes of square integrable functions from $(0, T)$ to $H^1(I)$. The space $L^2(0, T; L^2(I))$ is analogously defined. Besides, the space $H^1(0, T; L^2(I))$ contains those functions in $L^2(0, T; L^2(I))$ whose first order weak derivative in t belongs to $L^2(0, T; L^2(I))$. We also need the space $W_q^{2,1}(Q_T)$ which contains those functions in $L^q(Q_T)$ whose weak derivatives $D_t^m D_{s_i}^n$ belong to $L^q(Q_T)$, for any $2m + n \leq 2, i = 1, 2, \ldots, N$. For more details about these spaces, one can refer to Barbu [6] and Ladyzhenskaya et al. [23].

Now we are ready to state the problem. Consider the market consisting of the riskless asset and N stocks. Let $s_0(t)$ be the price of one share of the riskless asset at time t and $s_i(t), i = 1, 2, \ldots, N$ denote the prices of the stocks at time t. Further we suppose assume that the stock prices are driven by a standard Brownian motion. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space on which is given a standard Brownian motion $\mathbf{B}(t) = (B_1(t), B_2(t), \ldots, B_N(t)), 0 \leq t \leq T$ with $\mathbf{B}(0) = 0$ a.s. Now we define $\mathcal{F}^{\mathbf{B}}(t) = \sigma\{\mathbf{B}(\tau); 0 \leq \tau \leq t\}$, for all $t \in [0, T]$ and let $\mathcal{N}$ denote the P-null subsets of $\mathcal{F}^{\mathbf{B}}(T)$. Suppose $\mathcal{F}(t) = \sigma(\mathcal{F}^{\mathbf{B}}(t) \cup \mathcal{N})$, for all $t \in [0, T]$.

Then we assume that the value of the riskless asset increase in value at the same rate as the interest rate so that

$$ds_0(t) = r(t, \mathbf{s}(t))s_0(t)dt \tag{2.1}$$

and the change in the stock prices following a standard Brownian motion satisfy the stochastic differential equations

$$ds_i(t) = s_i(t)\left[b_i(t)dt + \sum_{j=1}^{N} \sigma_{ij}(t, \mathbf{s}(t))dB_j(t)\right] \quad \text{for all } t \in [0, T],\ i = 1, 2, \ldots, N. \tag{2.2}$$

Further, we suppose make the following assumptions:

- The underlying stocks pay dividends with rates $\delta_i(t, \mathbf{s}(t))$ during the life of the option.
- The interest rate $r(t, \mathbf{s}(t))$, volatility coefficient $\sigma_{ij}(t, \mathbf{s}(t))$ and the dividends $\delta_i(t, \mathbf{s}(t))$ are assumed to be continuous and bounded in the region $[0, T] \times I$. The volatility matrix σ is invertible with uniformly bounded inverse.
- The functions $b_i(t)$ are assumed to be bounded measurable with respect to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$.

Therefore the value of a call option, say v, on dividend paying stock returns depends on several factors; the current price of the stock $\mathbf{s}$, the time until expiration t, the interest rate r, the volatility of the stock prices σ, and the dividend during the life of the option δ. Now with the above assumptions and applying Ito's lemma for the option v together with (2.1), (2.2), one indeed can remove stochastic terms and obtain the following "*final value problem*" for the value of the option [11, 19]

$$v_t(t,\mathbf{s}) + \frac{1}{2}\sum_{i,j=1}^{N} s_i s_j a_{ij}(t,\mathbf{s}) v_{s_i s_j}(t,\mathbf{s}) + \sum_{i=1}^{N}(r(t,\mathbf{s}) - \delta_i(t,\mathbf{s}))s_i v_{s_i}(t,\mathbf{s}) - r(t,\mathbf{s})v(t,\mathbf{s}) = 0 \quad \text{for all } (t,\mathbf{s}) \in Q_T,$$

where coefficient matrix a_{ij} is given by the relation with volatility matrix σ as

$$a_{ij}(t,\mathbf{s}) = \sum_{k=1}^{N} \sigma_{ik}(t,\mathbf{s})\sigma_{jk}(t,\mathbf{s}).$$

In a real world we do not have enough market prices in the time direction, we simplify the problem assuming that volatility is time-independent and using only option prices with different strikes and a fixed maturity date. Indeed, the local volatility is defined as instantaneous time-independent variance of expected stock returns. Moreover, it depends on the stock prices, which in turn changes over time.

Thus by the time inversion $t \mapsto T - t$, we consider the following "*initial boundary value problem*" with suitable control of the form

$$\begin{cases} y_t + \mathcal{L}y = \chi(\mathbf{s})u & \text{in } Q_T \\ y(0,\mathbf{s}) = y_0(\mathbf{s}) & \text{in } I \\ y(t,\mathbf{s}) = 0 & \text{on } \Sigma_T, \end{cases} \tag{2.3}$$

where the second order operator $\mathcal{L}$ is given by

$$\mathcal{L}y = -\frac{1}{2}\sum_{i,j=1}^{N}(s_i s_j a_{ij}(\mathbf{s}) y_{s_i})_{s_j} + \frac{1}{2}\sum_{i,j=1}^{N}(s_i s_j a_{ij}(\mathbf{s}))_{s_j} y_{s_i} + \sum_{i=1}^{N} s_i(\delta_i(\mathbf{s}) - r(\mathbf{s}))y_{s_i} + r(\mathbf{s})y.$$

Here, the bounded domain $I \subset \mathbb{R}_+^N$ is defined by $I = \prod_{i=1}^{N}(\underline{s}_i, \overline{s}_i)$ with each $\underline{s}_i > 0$, $1 \le i \le N$ consisting of the smooth boundary ∂I, where $\underline{s}_i$ and $\overline{s}_i$ respectively denote the minimum and maximum prices over each stock. Further $0 \le t < T$, where t is the time to expiry and T is the time of expiry and $y(t,\mathbf{s})$ is the value of the option at time t if the price of the underlying stock at time t is $\mathbf{s}$. The initial value of the option y_0 is sufficiently regular(for instance $y_0 \in H_0^1(I)$) and χ is the usual characteristic function defined over the suitable subdomain I_1 of I.

Further we assume that the symmetric condition $a_{ij} = a_{ji}, i, j = 1, 2, \ldots, N$ and it follows from the fact that the positive definiteness of the elements $\delta_{ik}(\mathbf{s})\delta_{jk}(\mathbf{s})$ of the invertible volatility matrix σ, the coefficient matrix a_{ij} is positive definite with smallest eigenvalue greater than or equal to $\theta > 0$ for all $\mathbf{s} \in I$, that is,

$$\sum_{i,j=1}^{N} a_{ij}(\mathbf{s}) p_i p_j \ge \theta \sum_{i=1}^{N} p_i^2 = \theta |p|^2, \quad \text{a.e. } \mathbf{s} \in I, \quad \text{for all } p \in \mathbb{R}^N. \tag{2.4}$$

Let us assume the inequality over the prices

$$\underline{s}_i \le s_i \le \overline{s}_i, \quad \forall i = 1, 2, \ldots, N \quad \text{and} \quad \underline{S} = \min_{1 \le i \le N} \underline{s}_i^2$$

and also we set

$$\begin{aligned} D_i &= \sup_{\mathbf{s}\in I} |\delta_i(\mathbf{s})|, \qquad D_{1,i} = \sup_{\mathbf{s}\in I} |\delta_i(\mathbf{s})_{s_i}|, \\ R &= \sup_{\mathbf{s}\in I} |r(\mathbf{s})| \quad \text{and} \quad R_1 = \sup_{\mathbf{s}\in I} |r(\mathbf{s})_{s_i}|, \quad \text{for all } i = 1, 2, \dots, N. \end{aligned} \tag{2.5}$$

Remark 2.1 The condition $\underline{s}_i > 0$, $1 \le i \le N$ is not restrictive. Indeed, the degeneracy of the operator $\mathcal{L}$ at $\mathbf{s} = 0$ can be removed by using the change variable $x_i = \ln(s_i)$, $i = 1, 2, \dots, N$.

Remark 2.2 The option price is typically increasing (or decreasing) with volatility. It is however bounded by the stock price. When the volatility obtains its highest (or lowest) value the option price tends to become constant. The appropriate boundary condition to use at this stage is the classical Neumann boundary data. The results of this paper can be extended to this case too.

We have to determine the option value y and controller u which indeed applied through the subdomain I_1. Thus we have following definitions.

Definition 2.1 In B-S model (2.3), an initial data $y_0 \in L^2(I)$ is null controllable or exactly null controllable in the expiry time T if there is a control $u \in L^2(0, T; L^2(I_1))$ so that its solution y satisfies $y \in C([0, T]; L^2(I)) \cap L^2(0, T; H_0^1(I))$ and $y(T, \mathbf{s}) \equiv 0$. Further, global exact null controllability at time T for (2.3) holds if any initial data $y_0 \in L^2(I)$ is exactly null controllable in time T.

In order to prove the existence of such a control we follow the methodology developed for some parabolic equations in Barbu [8] and Fursikov and Imanuvilov [17]. It should be said that the nature of the B-S model is completely different from some general parabolic equations in [17]; in fact, one has to do some careful computations for the more general form of (2.3) with several coefficients: volatility, interest rate, dividends and in particular the strike prices $s_i s_j$, $i, j = 1, 2, \dots, N$ which indeed play a crucial role in the controllability of the model. We have also computed the exact values of the constants arising in the estimates through certain parameters τ and μ. Besides, the nonlinearity treated in [17] is of globally Lipschitz, that is, f is a function with sublinear behavior at infinity, whereas in our paper, we assume that it is of superlinear behavior. To the best of our knowledge, as far as the controllability of general B-S model is concerned, there is no work appeared in the literature.

Before proceeding to discuss the main results, we recall some of the results mainly associated with some parabolic equations from the literature. For the first time, Lebeau and Robbiano [24] studied the null controllability of the linear heat equation in a bounded domain $\Omega \subset \mathbb{R}^N$ by a localized control force which acts on a subdomain $\omega \subset \Omega$ by using the spectral decomposition of the solutions. More interestingly, in the popular work of Fursikov and Imanuvilov [17] proved these results for semilinear equation

$$y_t - \Delta y + f(y) = 1_\omega u \quad \text{in } (0, T) \times \Omega$$

and also they proved the exact controllability of more general semilinear parabolic equations with variable coefficients when the nonlinearity is of globally Lipschitz continuous by establishing a global Carleman estimate for linearized problem. Barbu [7] (see, also [8, 30] for a related result) generalized those results of [17] for heat equation with some superlinear nonlinearities using a classical fixed point method and a Carleman estimate for associated linear heat type equations. Anita and Barbu [3] studied some interesting results on null controllability of nonlinear convective heat equations for $n = 1, 2, 3$. Recently, Sakthivel et al. [25] proved the null controllability of one dimensional Black-Scholes equations with globally Lipschitz nonlinearities. Apart from the exact controllability of parabolic equations, Anita and Barbu [4] proved the exact controllability of the reaction-diffusion system by two control forces while for related results on the similar model with Neumann boundary data by one control force (and also for other interesting results in this direction) one can refer to Wang and Zhang [28]. Besides, Barbu [9] proved the null controllability of the phase field system by two control forces. Moreover, the monograph by Klamka [21] discusses some controllability problems for distributed parameter systems.

Remark 2.3 It is worth noting that the applicability and the nature of B-S model are different from the above explained literature on either classical heat type equations or some general parabolic equations. It also gives an insight into the controllability of B-S model with higher order nonlinearities occur in option pricing problems (see, for example, [1, 14]) and the related inverse problem of recovering volatility coefficient(which is the only coefficient not known with certainty) in B-S model (2.3) is one of the interesting problems of inverse problem of option pricing and there is no satisfactory result for practice appeared in this direction as pointed out in Isakov [13].

A priori estimates of a weighted norm of the solution of partial differential equations and its derivatives that are the so-called Carleman estimates, which we have used in our paper to the theory of control problems, have been of enormous interest in the numerical methods for solving inverse problems. The book by Klibanov and Timonov [22] (and the cited references of there in) gives the insight that the weight functions associated with the Carleman estimates can also be used to construct the globally convergent numerical algorithms for coefficient inverse problems. A simple numerical study of illustrating the theoretical considerations given in the present paper by the method of Carleman estimates would be of great interest in option pricing theory.

This paper is arranged as follows. In Sect. 3 we prove the global null controllability of the general B-S equations in a bounded domain I of R_+^N by establishing a suitable observability estimate for associated adjoint system and certain energy estimates for linear B-S equations. In Sect. 4, we prove the local controllability of certain nonlinear B-S type equations in I for $1 \leq N < 6$, when the nonlinear term is of superlinear behavior at infinity. The results are obtained using the regularity of solutions, classical compactness arguments and Kakutani's fixed point theorem.

3 Controllability of Linear B-S Equations

In order to solve the global controllability problem (2.3), we approximate it by a family of optimal control problems for the same linear system. The estimates that we need to prove the convergence of the approximation procedure are obtained from certain energy estimates for linear system and an observability inequality for the adjoint equations which is in turn usually derived from a Carleman inequality for the same equations.

Now we state our first main result of this section.

Theorem 3.1 *Let I be an open bounded subset of $\mathbb{R}_+^N$ and I_1 be the suitable subset of I. Assume that the coefficient matrix $a_{ij} \in C^2(\bar{I})$ satisfies (2.4) and $r, \delta \in C^1(\bar{I})$. Then for all*

$y_0 \in L^2(I)$, *there exists* $(u, y) \in L^2(Q_T) \times C([0, T]; L^2(I)) \cap L^2(0, T; H_0^1(I))$ *satisfying the B-S model* (2.3) *and the terminal condition* $y(T, \mathbf{s}) \equiv 0, a.e.\ \mathbf{s} \in I$.

Proof Let $T > 0$ be fixed and suppose assume that $y_0 \in L^2(I)$. Consider the optimal control problem for any $\varepsilon > 0$, $\min\{J_\varepsilon(u) : u \in L^2(0, T; L^2(I_1))\}$, where the functional J_ε is defined by

$$J_\varepsilon(u) = \frac{1}{2}\int_0^T \int_{I_1} |u|^2 d\mathbf{s}dt + \frac{1}{2\varepsilon}\int_I |y(T, \mathbf{s})|^2 d\mathbf{s}, \tag{3.1}$$

where y is the solution of (2.3) associated with the control u. Now we divide the proof into three steps.

1. Characterization of the control. We shall follow here the classical calculus of variations technique. It is easy to see that the functional J_ε is continuous on $L^2((0, T) \times I)$ and strictly convex, and it is coercive, that is, $J_\varepsilon(u) \to \infty$ as $\|u\|_{L^2(0,T;L^2(I_1))} \to \infty$. Thus, the minimization problem (3.1) has a unique solution $(u^\varepsilon, y^\varepsilon)$ for all $\varepsilon > 0$. The control u^ε is characterized by $u^\varepsilon = -\chi(\mathbf{s})p^\varepsilon$, where p^ε is the solution of the following adjoint system associated with (2.3):

$$\begin{cases} -p_t^\varepsilon + \mathcal{L}^* p^\varepsilon = 0 & \text{in } Q_T \\ p^\varepsilon(T, \mathbf{s}) = \frac{1}{\varepsilon} y^\varepsilon(T, \mathbf{s}) & \text{in } I \\ p^\varepsilon(t, \mathbf{s}) = 0 & \text{on } \Sigma_T, \end{cases} \tag{3.2}$$

where the formal adjoint operator $\mathcal{L}^*$ is given by

$$\begin{aligned} \mathcal{L}^* p^\varepsilon = &-\frac{1}{2}\sum_{i,j=1}^N (s_i s_j a_{ij}(\mathbf{s}) p_{s_j}^\varepsilon)_{s_i} - \frac{1}{2}\sum_{i,j=1}^N ((s_i s_j a_{ij}(\mathbf{s}))_{s_j} p^\varepsilon)_{s_i} \\ &- \sum_{i=1}^N (s_i(\delta_i(\mathbf{s}) - r(\mathbf{s}))p^\varepsilon)_{s_i} + r(\mathbf{s})p^\varepsilon. \end{aligned}$$

Indeed, one can obtain the characterization as follows. Let $y = z + q$, where q is the solution of the homogeneous system associated with (2.3), then z satisfies the system

$$\begin{cases} z_t + \mathcal{L}z = \chi(\mathbf{s})u & \text{in } Q_T \\ z(0, \mathbf{s}) = 0 & \text{in } I \\ z(t, \mathbf{s}) = 0 & \text{on } \Sigma_T. \end{cases} \tag{3.3}$$

Now for any control $v \in L^2(0, T; L^2(I_1))$, we have $\langle J_\varepsilon'(u), v\rangle = 0$ and by the classical computations

$$\langle J_\varepsilon'(u), v\rangle = \int_0^T \int_{I_1} u^\varepsilon v d\mathbf{s}dt + \frac{1}{\varepsilon}\int_I y^\varepsilon(T, \mathbf{s})z(T, \mathbf{s})d\mathbf{s},$$

where z, together with the control v, is the solution of (3.3). On the other hand from (3.2) and (3.3), we have

$$\int_0^T \int_{I_1} p^\varepsilon v d\mathbf{s}dt = \int_I p^\varepsilon(T, \mathbf{s})z(T, \mathbf{s})d\mathbf{s} = \frac{1}{\varepsilon}\int_I y^\varepsilon(T, \mathbf{s})z(T, \mathbf{s})d\mathbf{s}. \tag{3.4}$$

Thus for all $v \in L^2(0,T;L^2(I_1))$, we have

$$\int_0^T \int_{I_1} u^\varepsilon v d\mathbf{s}dt + \int_0^T \int_{I_1} p^\varepsilon v d\mathbf{s}dt = 0,$$

whence it follows that $u^\varepsilon = -\chi(\mathbf{s})p^\varepsilon$.

2. A priori estimates for u_ϵ and y_ϵ. We shall now show that (u_ϵ, y_ϵ) converges along a subsequence of $\{\epsilon\}$ in a certain topology. To attain this, we need a suitable estimate for (u_ϵ, y_ϵ); in particular, we obtain certain L^2-estimates. Now the duality between (2.3) and (3.2) gives

$$\begin{aligned}&\int_0^T \int_{I_1} |p^\varepsilon|^2 d\mathbf{s}dt + \frac{1}{\varepsilon}\int_I |y^\varepsilon(T,\mathbf{s})|^2 d\mathbf{s}\\ &\quad = \int_I y_0(\mathbf{s})p^\varepsilon(0,\mathbf{s})d\mathbf{s} \le \frac{\eta}{2}\int_I |y_0(\mathbf{s})|^2 d\mathbf{s} + \frac{1}{2\eta}\int_I |p^\varepsilon(0,\mathbf{s})|^2 d\mathbf{s}, \quad \text{for any } \eta > 0. \end{aligned} \tag{3.5}$$

It is now clear that to proceed with (3.5), we need to estimate the term $\|p_\epsilon(0,\cdot)\|^2_{L^2(I)}$, the so-called observability estimate for the adjoint problem (3.2), that is, an estimate for the initial state on I by means of the states taken on I_1 at all the subsequent moments. Thus we need the following lemma.

Lemma 3.1 *Suppose all the assumptions of Theorem 3.1 are satisfied. For any $p_T \in L^2(I)$, let p be the solution of the adjoint system (3.2). Then there exists a constant $C > 0$ depending only on $I, I_1, \underline{S}$ and $\overline{S}$ satisfying the following estimate*:

$$\int_I |p^\varepsilon(0,\mathbf{s})|^2 d\mathbf{s} \le \exp(C\Lambda_T)\int_0^T \int_{I_1} |p^\varepsilon|^2 d\mathbf{s}dt, \tag{3.6}$$

where the constant

$$\Lambda_T = 1 + \frac{1}{T}(1 + A_0 + A_1) + (1+T)\sum_{i=0}^{5} A_i + \sqrt{A_0} + \sqrt{A_4}$$

and the constants $\overline{S}$, A_i are defined in (3.10).

We postpone the proof of this lemma to the end of this section. Now making use of the estimate (3.6) in (3.5) with the choice of $\eta = \exp(C\Lambda_T)$ and then using the characterization of the control from the first step, we arrive at

$$\begin{aligned}&\frac{1}{2}\int_0^T \int_{I_1} |u^\varepsilon|^2 d\mathbf{s}dt + \frac{1}{\varepsilon}\int_I |y^\varepsilon(T,\mathbf{s})|^2 d\mathbf{s}\\ &\quad \le \exp(C\Lambda_T)\int_I |y_0(\mathbf{s})|^2 d\mathbf{s}, \quad \text{for any } \varepsilon > 0. \end{aligned} \tag{3.7}$$

Moreover, we have the following energy estimates for the solutions of (2.3). For any $y_0 \in L^2(I)$, multiplying (2.3) by y^ε and integrating on I, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_I |y^\varepsilon|^2 d\mathbf{s} + \frac{1}{2}\int_I \mathcal{A}(\nabla y^\varepsilon \mathbf{s}, \nabla y^\varepsilon \mathbf{s})d\mathbf{s}$$

$$
\begin{aligned}
&= \int_I \chi(\mathbf{s}) u^\varepsilon y^\varepsilon d\mathbf{s} - \frac{1}{2}\int_I \sum_{i,j=1}^N (s_i s_j a_{ij}(\mathbf{s}))_{s_j} y^\varepsilon_{s_i} y^\varepsilon d\mathbf{s} \\
&\quad + \int_I \sum_{i=1}^N s_i (r(\mathbf{s}) - \delta_i(\mathbf{s})) y^\varepsilon_{s_i} y^\varepsilon d\mathbf{s} - \int_I r(\mathbf{s}) y^{\varepsilon 2} d\mathbf{s} \\
&= \sum_{i=1}^4 \mathcal{J}_i.
\end{aligned}
$$

Applying Young's inequality, we get

$$
\mathcal{J}_2 \le \frac{1}{8\gamma}\int_I |y^\varepsilon|^2 d\mathbf{s} + \frac{\gamma}{2} C_1^2 \int_I |\nabla y^\varepsilon|^2 d\mathbf{s}
$$

where

$$
C_1 = \max_{1\le i\le N} \sum_{j=1}^N (\overline{s}_i \overline{s}_j |(a_{ij})_{s_j}| + (N+1)\overline{s}_i |a_{ij}|)
$$

and

$$
\mathcal{J}_3 + \mathcal{J}_4 \le \left(R + \frac{1}{8\gamma}\right)\int_I |y^\varepsilon|^2 d\mathbf{s} + \frac{\gamma}{2} C_2^2 \int_I |\nabla y^\varepsilon|^2 d\mathbf{s}
$$

where $C_2 = \max_{1\le i\le N}\{\overline{s}_i(D_i + R)\}$. Using positive definiteness of the matrix a_{ij}, we get

$$
\int_I \mathcal{A}(\nabla y^\varepsilon \mathbf{s}, \nabla y^\varepsilon \mathbf{s}) d\mathbf{s} \ge \theta \int_I \sum_{i=1}^N (s_i y^\varepsilon_{s_i})^2 d\mathbf{s}.
$$

Thus, choosing γ small enough such that $\gamma(C_1^2 + C_2^2) < (\theta/4)\underline{S}$ and applying Gronwall's inequality, we have

$$
\begin{aligned}
&\int_I |y^\varepsilon(t)|^2 d\mathbf{s} + \int_0^t \int_I |\nabla y^\varepsilon|^2 d\mathbf{s} dr \\
&\quad \le e^{CT}\left(\int_I |y_0(\mathbf{s})|^2 d\mathbf{s} + \int_0^t \int_{I_1} |u^\varepsilon|^2 d\mathbf{s} dr\right), \quad \forall t \in [0, T].
\end{aligned} \tag{3.8}
$$

3. Convergence arguments and conclusion. A priori estimates (3.7) and (3.8) allow us to pass to the weak limit in (2.3) and that gives a solution to the null controllability problem as $\varepsilon \to 0$. Indeed, the boundedness of u^ε in $L^2(Q_T)$(since $u^\varepsilon = 0$ outside of I_1) implies that of y^ε in $L^2(0, T; H_0^1(I))$ via the estimate (3.8). Hence we infer that there exists $(u, y) \in L^2(Q_T) \times L^2(0, T; H_0^1(I))$ such that on a subsequence of $\{\varepsilon\}$ still indexed by ε, we have the following convergence as $\varepsilon \to 0$:

$$
\begin{aligned}
&u^\varepsilon \to u \quad \text{weakly in } L^2(Q_T) \\
&y^\varepsilon \to y \quad \text{weakly in } L^2(0, T; H_0^1(I)) \quad \text{hence strongly in } C([0, T]; L^2(I)).
\end{aligned}
$$

Thus replacing (u, y) by $(u^\varepsilon, y^\varepsilon)$ in (2.3) and passing the limit $\varepsilon \to 0$, we obtain that (u, y) satisfies the system (2.3). Besides, from the a priori estimate (3.7), it is clear that

$$\int_I |y(T,\mathbf{s})|^2 d\mathbf{s} \le \liminf_{\varepsilon\to 0} \int_I |y^\varepsilon(T,\mathbf{s})|^2 d\mathbf{s} = 0, \tag{3.9}$$

which gives $y(T,\mathbf{s}) \equiv 0$, a.e. $\mathbf{s} \in I$. The proof of the theorem is thus completed. □

Now let us define precisely the constants which will be used in the following sections:

$$\begin{aligned} A_0 &= \max_{1\le i\le N} \sum_{j=1}^N |a_{ij}|, \qquad A_1 = \max_{1\le i\le N} \sum_{j=1}^N |(a_{ij})_{s_i}|, \\ A_2 &= \max_{1\le i\le N} \sum_{j=1}^N |(a_{ij})_{s_i s_j}|, \\ A_3 &= \max_{1\le k\le N} \sum_{l=1}^N |a_{kl}| \left(\max_{1\le i\le N} \sum_{j=1}^N \big(|a_{ij}| + |(a_{ij})_{s_l}| \big) \right) \end{aligned} \tag{3.10}$$

and for simplicity of the constant expressions, we also denote

$$A_4 = R + \max_{1\le i\le N} D_i, \qquad A_5 = R_1 + \max_{1\le i\le N} D_{1,i} \quad \text{and} \quad \overline{S} = \max_{1\le i,j\le N} (\overline{s}_i \overline{s}_j),$$

where the constants R, R_1 and D_i, $D_{1,i}$ are defined in (2.5). Now we prove Lemma 3.1.

3.1 Proof of Lemma 3.1

Proof The observability inequality (3.6) is a consequence of a suitable global Carleman estimate for the adjoint system (3.2). For simplicity we divide the proof into three steps. The proof follows the similar steps used in [8].

1. Change of variable and basic inequality. Let $I_2 \subset\subset I_1$ be a suitable subdomain and $\psi \in C^2(\bar{I})$ be a cut-off function satisfying $\psi(\mathbf{s}) > 0\ \forall \mathbf{s} \in I$, $\psi = 0$ on ∂I and $|\nabla\psi(\mathbf{s})| > 0\ \forall \mathbf{s} \in \bar{I}\backslash I_2$. The existence of such a function ψ can be found in [17]. Then for some parameter $\mu > 0$, we define a weight function $\alpha = \alpha(t,\mathbf{s})$ such that $\alpha = \varphi(\mathbf{s})/\rho(t)$, where $\varphi(\mathbf{s}) = (e^{\mu\psi} - e^{2\mu\|\psi\|_{C(I)}})$ and $\rho(t) = t(T-t)$.

Now let us set $p = e^{-\tau\alpha} q$, where τ, μ are some positive parameters which enable us to handle the arbitrarily large coefficients in the coupling terms. Now the system (3.2) can equivalently be written after some computation as (ignoring the superscript ε)

$$\begin{cases} q_t - \mathcal{L}^1(t)q + \mathcal{L}^2(t)q = \mathcal{L}^3(t)q & \text{in } Q_T \\ q(t,\mathbf{s}) = 0 & \text{on } \Sigma_T \\ q(0,\mathbf{s}) = q(T,\mathbf{s}) = 0, & \forall \mathbf{s} \in I, \end{cases} \tag{3.11}$$

where the operators $\mathcal{L}^1, \mathcal{L}^2$ and $\mathcal{L}^3$ with $\tilde{\alpha} = e^{\mu\psi}/\rho(t)$ stand for

$$\mathcal{L}^1(t)q = -\frac{1}{2} \sum_{i,j=1}^N s_i s_j a_{ij}(\mathbf{s}) [q_{s_i s_j} + (\tau\tilde{\alpha} + \tau^2\tilde{\alpha}^2)\mu^2 \psi_{s_i}\psi_{s_j} q] + \tau\alpha_t q,$$

$$\mathcal{L}^2(t)q = -\tau\tilde{\alpha}\sum_{k,l=1}^{N} s_k s_l a_{kl}(\mathbf{s})[\mu\psi_{s_k}q_{s_l} + \mu^2\psi_{s_k}\psi_{s_l}q],$$

$$\mathcal{L}^3(t)q = \frac{1}{2}\sum_{i,j=1}^{N}[s_i s_j a_{ij}(\mathbf{s})\tau\mu\tilde{\alpha}\psi_{s_i s_j}q - (s_i s_j a_{ij}(\mathbf{s}))_{s_i s_j}q - 2(s_i s_j a_{ij}(\mathbf{s}))_{s_j}(q_{s_i} - \tau\mu\tilde{\alpha}\psi_{s_i}q)]$$

$$+\sum_{i=1}^{N}[(s_i(r(\mathbf{s}) - \delta_i(\mathbf{s}))_{s_i}q + (s_i(r(\mathbf{s}) - \delta_i(\mathbf{s}))(q_{s_i} - \tau\mu\tilde{\alpha}\psi_{s_i}q) + r(\mathbf{s})q].$$

Now we consider the identity

$$\frac{d}{dt}\int_I(\mathcal{L}^1(t)q)q\,d\mathbf{s} = 2\int_I(\mathcal{L}^1(t)q)[\mathcal{L}^3(t)q - \mathcal{L}^2(t)q + \mathcal{L}^1(t)q]d\mathbf{s} + \int_I(\mathcal{L}^1_t(t)q)q\,d\mathbf{s},$$

where $\mathcal{L}^1_t(t)$ denotes the time derivative of the coefficients in $\mathcal{L}^1(t)$. Integrating over $(0,T)$ and using the boundary conditions, we get

$$2\int_{Q_T}(\mathcal{L}^1(t)q)^2 d\mathbf{s}dt + 2\mathcal{L}^4(t) = -2\int_{Q_T}(\mathcal{L}^1(t)q)(\mathcal{L}^3(t)q)d\mathbf{s}dt - \int_{Q_T}(\mathcal{L}^1_t(t)q)q\,d\mathbf{s}dt,$$

so that

$$\mathcal{L}^4(t) \le \frac{1}{4}\|\mathcal{L}^3(t)q\|^2_{L^2(Q_T)} + \frac{1}{2}\left|\int_{Q_T}(\mathcal{L}^1_t(t)q)q\,d\mathbf{s}dt\right| \tag{3.12}$$

where

$$\mathcal{L}^4(t) = -\int_{Q_T}(\mathcal{L}^1(t)q)(\mathcal{L}^2(t)q)d\mathbf{s}dt.$$

Thus the inequality (3.12) clearly gives a direction to proceed our computations.

2. Crucial calculations and key estimate. In this step we obtain the lower bound for the left hand side terms as well as the upper bound for the right hand side terms of (3.12) which will indeed give the key estimate containing the L^2-norm of the solution and it's gradient in the left hand side. It is clear from $\mathcal{L}^1(t)$ that

$$\int_{Q_T}(\mathcal{L}^1_t(t)q)q\,d\mathbf{s}dt = -\frac{1}{2}\int_{Q_T}\sum_{i,j=1}^{N}s_i s_j a_{ij}(\mathbf{s})(\tau\tilde{\alpha}_t + 2\tau^2\tilde{\alpha}\tilde{\alpha}_t)\mu^2\psi_{s_i}\psi_{s_j}q^2 d\mathbf{s}dt$$

$$+\int_{Q_T}\tau\alpha_{tt}q^2 d\mathbf{s}dt.$$

Note that the weight function satisfies the estimates

$$\begin{aligned} &|\tilde{\alpha}_t| \le C(I)T\tilde{\alpha}^2, \qquad |\alpha_t| \le C(I)T\tilde{\alpha}^2,\\ &|\alpha_{tt}| \le C(I)T^2\tilde{\alpha}^3 \quad \text{and} \quad \tilde{\alpha}^{-1} \le (T/2)^2, \end{aligned} \tag{3.13}$$

where $C > 0$ is a constant independent of μ and take

$$C_1 = \max_{1\le i\le N}\sup_{\mathbf{s}\in I}|\psi_{s_i}|, \qquad C_2 = \max_{1\le i\le N}\sup_{\mathbf{s}\in I}|\psi_{s_i}|^2 \quad \text{and} \quad C_3 = \max_{1\le i,j\le N}\sup_{\mathbf{s}\in I}|\psi_{s_i s_j}|.$$

However, we use the generic constant C which will depend only on I and I_1. Taking the preceding estimates into account, we obtain

$$\left|\int_{Q_T} (\mathcal{L}_t^1(t)q)q\,d\mathbf{s}dt\right| \le \int_{Q_T} \tau^2\mu^3\tilde{\alpha}^3|q|^2 d\mathbf{s}dt \tag{3.14}$$

for any $\mu \ge 1 + C(I,\overline{S})A_0T$ and $\tau \ge 1 + C(I,\overline{S})(T^2 + A_0T^3)$. Next we estimate the terms in $\mathcal{L}^3(t) = \mathcal{L}^{31}(t) + \mathcal{L}^{32}(t)$. It is easy to verify by simple computation that the term $\mathcal{L}^{31}(t)$ can be bounded as

$$\|\mathcal{L}^{31}(t)q\|^2_{L^2(Q_T)} \le \int_{Q_T} \tau^2\mu^3\tilde{\alpha}^3|q|^2 d\mathbf{s}dt + \frac{\eta_0}{2}\int_{Q_T} \mu^2\tau\tilde{\alpha}|\nabla q|^2 d\mathbf{s}dt \tag{3.15}$$

for some $\eta_0 > 0$, provided

$$\mu \ge 1 + \frac{1}{\eta_0}C(I,\overline{S})(T+T^2)\sum_{i=0}^{1} A_i \quad \text{and} \quad \tau \ge 1 + C(I,\overline{S})T^3\sum_{i=0}^{2} A_i.$$

Similarly we also obtain that

$$\|\mathcal{L}^{32}(t)q\|^2_{L^2(Q_T)} \le \int_{Q_T} \tau^2\mu^3\tilde{\alpha}^3|q|^2 d\mathbf{s}dt + \frac{\eta_0}{2}\int_{Q_T} \mu^2\tau\tilde{\alpha}|\nabla q|^2 d\mathbf{s}dt, \tag{3.16}$$

for the choice of

$$\mu \ge 1 + \frac{1}{\eta_0}C(I,\overline{S})(T+T^2)A_4 \quad \text{and} \quad \tau \ge 1 + C(I,\overline{S})T^3\sum_{i=4}^{5} A_i.$$

Next we proceed to get the lower bound for the integral $\mathcal{L}^4 = \sum_{i=1}^{3}\sum_{j=1}^{2}\mathcal{L}^4_{ij}$, where $\mathcal{L}^4_{ij}$ denotes the scalar products between the i-th term in $\mathcal{L}^1$ and the j-th term in $\mathcal{L}^2$. Integrating by parts with respect to space variable (in fact using Green's theorem) in $\mathcal{L}^4_{11}$, we obtain

$$\begin{aligned}
\mathcal{L}^4_{11} &= \frac{1}{2}\int_{Q_T} \mu\tau\tilde{\alpha}\sum_{i,j=1}^{N}(s_is_ja_{ij}(\mathbf{s}))_{s_j}q_{s_i}\sum_{k,l=1}^{N}s_ks_la_{kl}(\mathbf{s})\psi_{s_k}q_{s_l}\,d\mathbf{s}dt \\
&\quad + \frac{1}{2}\int_{Q_T} \mu\tau\tilde{\alpha}\sum_{i,j=1}^{N}s_is_ja_{ij}(\mathbf{s})q_{s_i}\sum_{k,l=1}^{N}(s_ks_la_{kl}(\mathbf{s})\psi_{s_k})_{s_j}q_{s_l}\,d\mathbf{s}dt \\
&\quad + \frac{1}{2}\int_{Q_T} \mu^2\tau\tilde{\alpha}\mathcal{A}(\nabla q\mathbf{s},\nabla\psi\mathbf{s})^2 d\mathbf{s}dt + \frac{1}{2}\int_{\Sigma_T} \tau\mu\tilde{\alpha}|\nabla\psi||\mathcal{A}(\nabla q\mathbf{s},\nu\mathbf{s})|^2 d\Sigma \\
&\quad + \frac{1}{2}\int_{Q_T} \mu\tau\tilde{\alpha}\sum_{i,j=1}^{N}s_is_ja_{ij}(\mathbf{s})q_{s_i}\sum_{k,l=1}^{N}s_ks_la_{kl}(\mathbf{s})\psi_{s_k}q_{s_ls_j}\,d\mathbf{s}dt \\
&= \sum_{i=1}^{5}\mathcal{D}_i,
\end{aligned} \tag{3.17}$$

where we used the fact that $\nu = -\nabla\psi/|\nabla\psi|$ since $\psi > 0$ in I and $\psi = 0$ on ∂I. We rewrite the integral $\mathcal{D}_5$ as

$$\mathcal{D}_5 = -\frac{1}{4}\int_{Q_T} \mu\tau\tilde{\alpha} \sum_{k,l=1}^{N} s_k s_l a_{kl}(\mathbf{s})\psi_{s_k} \sum_{i,j=1}^{N} (s_i s_j a_{ij}(\mathbf{s}))_{s_l} q_{s_i} q_{s_j} d\mathbf{s}dt$$
$$+\frac{1}{4}\int_{Q_T} \mu\tau\tilde{\alpha} \sum_{k,l=1}^{N} s_k s_l a_{kl}(\mathbf{s})\psi_{s_k} \sum_{i,j=1}^{N} (s_i s_j a_{ij}(\mathbf{s}) q_{s_i} q_{s_j})_{s_l} d\mathbf{s}dt$$

and then integrate by parts in the last integral to get

$$\mathcal{D}_5 = -\frac{1}{4}\int_{Q_T} \mu\tau\tilde{\alpha} \sum_{k,l=1}^{N} s_k s_l a_{kl}(\mathbf{s})\psi_{s_k} \sum_{i,j=1}^{N} (s_i s_j a_{ij}(\mathbf{s}))_{s_l} q_{s_i} q_{s_j} d\mathbf{s}dt$$
$$-\frac{1}{4}\int_{Q_T} \mu^2\tau\tilde{\alpha}\mathcal{A}(\nabla q\mathbf{s}, \nabla q\mathbf{s})\mathcal{A}(\nabla\psi\mathbf{s}, \nabla\psi\mathbf{s}) d\mathbf{s}dt$$
$$-\frac{1}{4}\int_{Q_T} \mu\tau\tilde{\alpha} \sum_{k,l=1}^{N} (s_k s_l a_{kl}(\mathbf{s})\psi_{s_k})_{s_l} \mathcal{A}(\nabla q\mathbf{s}, \nabla q\mathbf{s}) d\mathbf{s}dt$$
$$-\frac{1}{4}\int_{\Sigma_T} \tau\mu\tilde{\alpha}|\nabla\psi||\mathcal{A}(\nabla q\mathbf{s}, \nu\mathbf{s})|^2 d\Sigma.$$

Using the positive definiteness of a_{ij}, one can obtain that

$$\mathcal{D}_5 \geq -\frac{\mathcal{D}_4}{2} - \frac{\theta^2}{4}\int_{Q_T} \mu^2\tau\tilde{\alpha} \sum_{i=1}^{N} s_i^4 q_{s_i}^2 \psi_{s_i}^2 d\mathbf{s}dt - \frac{\eta_1}{2}\int_{Q_T} \mu^2\tau\tilde{\alpha}|\nabla q|^2 d\mathbf{s}dt$$

for any $\eta_1 > 0$ and for large enough $\mu \geq \frac{1}{\eta_1} C(I, \overline{S})A_3$. Noting $\mathcal{D}_3 \geq 0, \mathcal{D}_4 \geq 0$, (3.17) can now be estimated for the same choice of μ as

$$\mathcal{L}_{11}^4 \geq -\frac{\theta^2}{4}\int_{Q_T} \mu^2\tau\tilde{\alpha} \sum_{i=1}^{N} s_i^4 q_{s_i}^2 \psi_{s_i}^2 d\mathbf{s}dt - \eta_1\int_{Q_T} \mu^2\tau\tilde{\alpha}|\nabla q|^2 d\mathbf{s}dt. \tag{3.18}$$

Now we estimate the scalar product term $\mathcal{L}_{12}^4$. Integration by parts in space yields that

$$\mathcal{L}_{12}^4 = \frac{1}{2}\int_{Q_T} \mu^2\tau\tilde{\alpha} \sum_{i,j=1}^{N} (s_i s_j a_{ij}(\mathbf{s}))_{s_j} q_{s_i} \mathcal{A}(\nabla\psi\mathbf{s}, \nabla\psi\mathbf{s}) q d\mathbf{s}dt$$
$$+\frac{1}{2}\int_{Q_T} \mu^2\tau\tilde{\alpha}\mathcal{A}(\nabla q\mathbf{s}, \nabla q\mathbf{s})\mathcal{A}(\nabla\psi\mathbf{s}, \nabla\psi\mathbf{s}) d\mathbf{s}dt$$
$$+\frac{1}{2}\int_{Q_T} \mu^2\tau\tilde{\alpha} \sum_{i,j=1}^{N} s_i s_j a_{ij}(\mathbf{s}) q_{s_i} \mathcal{A}(\nabla\psi\mathbf{s}, \nabla\psi\mathbf{s})_{s_j} q d\mathbf{s}dt$$
$$+\frac{1}{2}\int_{Q_T} \mu^3\tau\tilde{\alpha}\mathcal{A}(\nabla q\mathbf{s}, \nabla\psi\mathbf{s})\mathcal{A}(\nabla\psi\mathbf{s}, \nabla\psi\mathbf{s}) q d\mathbf{s}dt$$
$$= \sum_{i=1}^{4} \mathcal{E}_i.$$

For any $\mu \geq \frac{C}{\eta_2} T^2$ and $\tau \geq C(I, \overline{S}) T^2 A_3$ with some $\eta_2 > 0$, one can get

$$\mathcal{E}_1 + \mathcal{E}_2 \geq \frac{\theta^2}{2} \int_{Q_T} \mu^2 \tau \tilde{\alpha} \sum_{i=1}^{N} s_i^4 q_{s_i}^2 \psi_{s_i}^2 d\mathbf{s} dt \\ - \frac{\eta_2}{2} \int_{Q_T} \mu^2 \tau \tilde{\alpha} |\nabla q|^2 d\mathbf{s} dt - \int_{Q_T} \mu^3 \tau^2 \tilde{\alpha}^3 |q|^2 d\mathbf{s} dt \tag{3.19}$$

and

$$\mathcal{E}_3 + \mathcal{E}_4 \geq -\frac{\eta_2}{2} \int_{Q_T} \mu^2 \tau \tilde{\alpha} |\nabla q|^2 d\mathbf{s} dt - C \int_{Q_T} (\mu^3 + \mu^4) \tau^2 \tilde{\alpha}^3 |q|^2 d\mathbf{s} dt. \tag{3.20}$$

Let us now estimate the integral $\mathcal{L}_{21}^4$. Again integrating by parts, we get

$$\mathcal{L}_{21}^4 = \frac{1}{4} \int_{Q_T} \mu^3 \mathcal{A}(\nabla \psi \mathbf{s}, \nabla \psi \mathbf{s})_{s_l} \sum_{k,l=1}^{N} s_k s_l a_{kl}(\mathbf{s}) \psi_{s_k} (\tau^2 \tilde{\alpha}^2 + \tau^3 \tilde{\alpha}^3) |q|^2 d\mathbf{s} dt \\ + \frac{1}{4} \int_{Q_T} \mu^3 \mathcal{A}(\nabla \psi \mathbf{s}, \nabla \psi \mathbf{s}) \sum_{k,l=1}^{N} (s_k s_l a_{kl}(\mathbf{s}) \psi_{s_k})_{s_l} (\tau^2 \tilde{\alpha}^2 + \tau^3 \tilde{\alpha}^3) |q|^2 d\mathbf{s} dt \\ + \frac{1}{4} \int_{Q_T} \mu^4 \mathcal{A}(\nabla \psi \mathbf{s}, \nabla \psi \mathbf{s})^2 (2\tau^2 \tilde{\alpha}^2 + 3\tau^3 \tilde{\alpha}^3) |q|^2 d\mathbf{s} dt.$$

Noting the addition of the term $\mathcal{L}_{22}^4$ with the(last term in the) above integrals in $\mathcal{L}_{21}^4$, further reduces it's lower bounds and so choosing $\tau \geq C(I, \overline{S}) T^2 A_3$, we obtain

$$\mathcal{L}_{21}^4 + \mathcal{L}_{22}^4 \geq \frac{\theta^2}{4} \int_{Q_T} \mu^4 \tau^3 \tilde{\alpha}^3 \sum_{i=1}^{N} s_i^4 \psi_{s_i}^4 |q|^2 d\mathbf{s} dt - C \int_{Q_T} \mu^3 \tau^3 \tilde{\alpha}^3 |q|^2 d\mathbf{s} dt. \tag{3.21}$$

Next we estimate the integral $\mathcal{L}_{31}^4$. Taking $\alpha_t = \alpha \frac{d}{dt}(\ln \rho^{-1}(t))$ into account, we have

$$\mathcal{L}_{31}^4 = -\frac{1}{2} \int_{Q_T} \mu^2 \tau^2 (\tilde{\alpha} \alpha_t + \tilde{\alpha}^2 d/dt (\ln \rho^{-1}(t))) \mathcal{A}(\nabla \psi \mathbf{s}, \nabla \psi \mathbf{s}) |q|^2 d\mathbf{s} dt \\ - \frac{1}{2} \int_{Q_T} \mu \tau^2 \tilde{\alpha} \alpha_t \sum_{k,l=1}^{N} (s_k s_l a_{kl}(\mathbf{s}) \psi_{s_k})_{s_l} |q|^2 d\mathbf{s} dt \\ \geq -\int_{Q_T} \mu^3 \tau^3 \tilde{\alpha}^3 |q|^2 d\mathbf{s} dt, \tag{3.22}$$

for any $\mu \geq 1$ and $\tau \geq C(I, \overline{S}) T (A_0 + A_1)$ since we note that $|\frac{d}{dt}(\ln \rho^{-1}(t))| \leq C T \tilde{\alpha}$. Finally we obtain that

$$\mathcal{L}_{32}^4 \geq -\int_{Q_T} \mu^3 \tau^3 \tilde{\alpha}^3 |q|^2 d\mathbf{s} dt, \quad \text{for } \tau \geq C(I, \overline{S}) T A_0. \tag{3.23}$$

Substituting the estimates (3.14)–(3.16) and (3.18)–(3.23) into the inequality (3.12), we get

$$\frac{\theta^2}{4} \int_{Q_T} \mu^4 \tau^3 \tilde{\alpha}^3 \sum_{i=1}^{N} s_i^4 \psi_{s_i}^4 |q|^2 d\mathbf{s} dt + \frac{\theta^2}{4} \int_{Q_T} \mu^2 \tau \tilde{\alpha} \sum_{i=1}^{N} s_i^4 q_{s_i}^2 \psi_{s_i}^2 d\mathbf{s} dt$$

$$\leq C\int_{Q_T}(\mu^3\tau^3+\mu^4\tau^2)\tilde{\alpha}^3|q|^2 d\mathbf{s}dt + C\sum_{i=0}^{2}\eta_i\int_{Q_T}\mu^2\tau\tilde{\alpha}|\nabla q|^2 d\mathbf{s}dt.$$

Choose $\eta_i, i=0,1,2$ sufficiently small such that $\frac{\theta^2}{4}\min_{1\leq i\leq N}\underline{s}_i^4 - C\sum_{i=0}^{2}\eta_i > 0$, to have

$$\int_{Q_T}\mu^4\tau^3\tilde{\alpha}^3|\nabla\psi|^4|q|^2 d\mathbf{s}dt + \int_{Q_T}\mu^2\tau\tilde{\alpha}|\nabla\psi|^2|\nabla q|^2 d\mathbf{s}dt$$
$$\leq C\int_{Q_T}(\mu^3\tau^3+\mu^4\tau^2)\tilde{\alpha}^3|q|^2 d\mathbf{s}dt,$$

where the constant $C>0$ depends on I,θ. Taking the properties of ψ into account, set $\zeta = \inf_{\mathbf{s}\in I\setminus I_2}\{|\nabla\psi|\} > 0$ so that we have the following lower bounds:

$$\int_{Q_T}\mu^4\tau^3\tilde{\alpha}^3|\nabla\psi|^4|q|^2 d\mathbf{s}dt \geq \zeta^4\left(\int_{Q_T}\mu^4\tau^3\tilde{\alpha}^3|q|^2 d\mathbf{s}dt - \int_0^T\int_{I_2}\mu^4\tau^3\tilde{\alpha}^3|q|^2 d\mathbf{s}dt\right)$$

and similar estimate holds true for the other integral as well. Choosing

$$\mu \geq \mu_0 = \max\left\{2C/\zeta^4, C(I,\overline{S})\left(1+T^2\left(1+\sum_{i=0}^{4}A_i\right)+A_3\right)\right\}$$

and then $\tau \geq 4C/\zeta^4$, we obtain

$$\int_{Q_T}(\mu^4\tau^3\tilde{\alpha}^3|q|^2+\mu^2\tau\tilde{\alpha}|\nabla q|^2)d\mathbf{s}dt$$
$$\leq C\int_0^T\int_{I_2}(\mu^4\tau^3\tilde{\alpha}^3|q|^2+\mu^2\tau\tilde{\alpha}|\nabla q|^2)d\mathbf{s}dt, \tag{3.24}$$

which indeed gives the key estimate to reduce the observability estimate.

3. Change of variable and reduction to observability estimate. Going back to the original variable $q = e^{\tau\alpha}p$(noting $\zeta_1 = \inf_{\mathbf{s}\in I}\{e^{\mu\psi}\} > 0$ and $\zeta_2 = \sup_{\mathbf{s}\in I}\{e^{\mu\psi}\} < +\infty$), we get

$$\int_{Q_T}e^{2\tau\alpha}(\mu^4\tau^3\rho^{-3}|p|^2+\mu^2\tau\rho^{-1}|\nabla p|^2)d\mathbf{s}dt$$
$$\leq C\int_0^T\int_{I_2}e^{2\tau\alpha}(\mu^4\tau^3\rho^{-3}|p|^2+\mu^2\tau\rho^{-1}|\nabla p|^2)d\mathbf{s}dt, \tag{3.25}$$

where the constant $C>0$ is larger than the one in (3.24) depending on ζ_1,ζ_2 as well. Next we estimate the gradient term on the right hand side in terms of p^2 over the larger domain $(0,T)\times I_1$(since $I_2\subset\subset I_1$), where the control is applied. To do this so, let us consider a truncating function $\vartheta = \vartheta(\mathbf{s})$ such that $\vartheta\in C_0^2(I_1)$, $\vartheta\equiv 1$ in I_2 with $0\leq\vartheta\leq 1$. Multiplying (3.2) by $e^{2\tau\alpha}\mu^2\tau\rho^{-1}\vartheta p$ and a lengthy but simple integration by parts with space and time yields

$$\frac{1}{2}\int_0^T\int_{I_1}e^{2\tau\alpha}\rho^{-1}\mu^2\tau\vartheta\mathcal{A}(\nabla p\mathbf{s},\nabla p\mathbf{s})d\mathbf{s}dt$$

$$
\begin{aligned}
&= -\frac{1}{2}\int_0^T \int_{I_1} (e^{2\tau\alpha}\rho^{-1})_t \mu^2\tau\vartheta p^2 d\mathbf{s}dt \\
&\quad + \frac{1}{4}\int_0^T \int_{I_1} \rho^{-1}\mu^2\tau \sum_{i,j=1}^N (s_i s_j a_{ij}(\mathbf{s})(e^{2\tau\alpha}\vartheta)_{s_i s_j} + (s_i s_j a_{ij}(\mathbf{s}))_{s_i s_j} e^{2\tau\alpha}\vartheta)|p|^2 d\mathbf{s}dt \\
&\quad + \frac{1}{2}\int_0^T \int_{I_1} \rho^{-1}\mu^2\tau \sum_{i=1}^N ((s_i\delta_i(\mathbf{s}) - r(\mathbf{s}))_{s_i} e^{2\tau\alpha}\vartheta - s_i(\delta_i(\mathbf{s}) - r(\mathbf{s}))(e^{2\tau\alpha}\vartheta)_{s_i})|p|^2 d\mathbf{s}dt \\
&\quad - \int_0^T \int_{I_1} \rho^{-1}\mu^2\tau e^{2\tau\alpha}\vartheta r(\mathbf{s}) p^2 d\mathbf{s}dt.
\end{aligned}
$$

It is easy to see that for any $\mu \geq 1$

$$
|\mu^2\tau(e^{2\tau\alpha}\rho^{-1})_t| = e^{2\tau\alpha}\mu^2\tau|2\tau\alpha_t\rho^{-1} + (2t - T)\rho^{-2}| \leq e^{2\tau\alpha}\mu^4\tau^3\rho^{-3}, \quad \tau \geq CT,
$$

and

$$
\rho^{-1}\mu^2\tau A_4|(e^{2\tau\alpha}\vartheta)_{s_i}| = e^{2\tau\alpha}\rho^{-1}\mu^2\tau A_4|2\tau\mu\tilde{\alpha}\psi_{s_i}\vartheta + \vartheta_{s_i}| \leq e^{2\tau\alpha}\mu^4\tau^3\rho^{-3},
$$

for $\tau \geq C(I)T^2(A_4 + \sqrt{A_4})$, for all $i, j = 1, 2, \dots, N$.

Similarly, it is easy to check that

$$
\rho^{-1}\mu^2\tau A_0|(e^{2\tau\alpha}\vartheta)_{s_i s_j}| \leq Ce^{2\tau\alpha}\mu^4\tau^3\rho^{-3}, \quad \tau \geq C(I)T^2(A_0 + \sqrt{A_0}).
$$

Using the above estimates, we obtain that

$$
\int_0^T \int_{I_2} e^{2\tau\alpha}\rho^{-1}\mu^2\tau \sum_{i=1}^N s_i^2 p_{s_i}^2 d\mathbf{s}dt \leq C \int_0^T \int_{I_1} e^{2\tau\alpha}\mu^4\tau^3\rho^{-3}|p|^2 d\mathbf{s}dt.
$$

Now taking $\underline{S} > 0$ into account, the estimate (3.25) reduces to

$$
\begin{aligned}
&\int_{Q_T} e^{2\tau\alpha}(\mu^4\tau^3\rho^{-3}|p|^2 + \mu^2\tau\rho^{-1}|\nabla p|^2) d\mathbf{s}dt \\
&\qquad \leq C \int_0^T \int_{I_1} e^{2\tau\alpha}\mu^4\tau^3\rho^{-3}|p|^2 d\mathbf{s}dt,
\end{aligned} \tag{3.26}
$$

for $\tau \geq \tau_0 = \max\{4C/\zeta^4, C(I, \overline{S})T^2\Lambda_T\}$, where Λ_T is defined in (3.6). Now we use the following sharp estimates

$$
e^{2\tau\alpha}\rho^{-3} \leq (2/T)^6 e^{-C\tau T^{-2}}, \quad \text{for all } \tau \geq \tau_1 = \max\{\tau_0, C(I)T^2\}, \ (t, \mathbf{s}) \in \bar{Q}_T
$$

and for $\tau \geq \tau_1$, we also have

$$
e^{2\tau\alpha}\rho^{-3} \geq (16/3T^2)^3 e^{-C\tau T^{-2}}, \quad \forall (t, \mathbf{s}) \in [T/4, 3T/4] \times \bar{I}.
$$

Using the preceding estimates, setting $\tau \geq \tau_2 = C(I, \overline{S})T^2\Lambda_T$, we can rewrite (3.26) as

$$
\int_{T/4}^{3T/4} \int_I |p|^2 d\mathbf{s}dt \leq \exp(C(I, \overline{S})\Lambda_T) \int_0^T \int_{I_1} |p|^2 d\mathbf{s}dt. \tag{3.27}
$$

Now we are ready to reduce the observability estimate. Multiplying the adjoint system (3.2) by p and integrating over I, we get

$$\frac{1}{2}\int_I \mathcal{A}(\nabla p\mathbf{s}, \nabla p\mathbf{s})d\mathbf{s} - \frac{1}{2}\frac{d}{dt}\int_I |p|^2 d\mathbf{s}$$
$$\leq C(I, \overline{S})(A_0 + A_1 + A_4)\int_I |\nabla p||p|d\mathbf{s} + R\int_I |p|^2 d\mathbf{s}.$$

Applying Cauchy's inequality with $\gamma > 0$ to the first integral on the right side and then choosing γ sufficiently small enough such that $C(I, \overline{S})(A_0 + A_1 + A_4)\gamma \leq (\theta/4)\underline{S}$, we get

$$-\frac{d}{dt}\int_I |p|^2 d\mathbf{s} \leq C(I, \overline{S})(A_0 + A_1 + A_4 + R)\int_I |p|^2 d\mathbf{s},$$

whence it follows that

$$-\frac{d}{dt}\left(\exp(Vt)\int_I |p|^2 d\mathbf{s}\right) \leq 0, \quad \text{where } V = C(I, \overline{S})(A_0 + A_1 + A_4 + R). \tag{3.28}$$

Integrating over $0 \leq t \leq T/4$, we get

$$\int_I |p(0, \mathbf{s})|^2 d\mathbf{s} \leq \exp(VT/4)\int_I |p(T/4, \mathbf{s})|^2 d\mathbf{s}.$$

Integrating (3.28) from $T/4$ to t with $t \in [T/4, 3T/4]$ and then combining with the preceding estimate, we get

$$\int_I |p(0, \mathbf{s})|^2 d\mathbf{s} \leq \exp(3VT/4)\int_I |p|^2 d\mathbf{s}. \tag{3.29}$$

Finally integrating the above inequality over $(T/4, 3T/4)$ and applying the estimate (3.27), one can conclude the proof. $\square$

4 Controllability of Nonlinear B-S Equations

In this section we discuss the following controlled nonlinear B-S type equations

$$\begin{cases} y_t + \mathcal{L}y + g(t, \mathbf{s}, y) = \chi(\mathbf{s})u & \text{in } Q_T \\ y(0, \mathbf{s}) = y_0(\mathbf{s}) & \text{in } I \\ y(t, \mathbf{s}) = 0 & \text{on } \Sigma_T, \end{cases} \tag{4.1}$$

where the second order operator $\mathcal{L}$ is given in (2.3). The given nonlinear function $g : Q_T \times \mathbb{R} \to \mathbb{R}$ is continuous in y, measurable in $(\mathbf{s}, t)$ and which has the following growth

$$|g(t, \mathbf{s}, p(t, \mathbf{s}))| \leq M|p(t, \mathbf{s})|[g_1(|p(t, \mathbf{s})|) + |g_2(\mathbf{s})|]$$
$$\text{for all } (t, \mathbf{s}, p) \in Q_T \times \mathbb{R}, \tag{4.2}$$

and satisfies the "good-sign" condition

$$g(t, \mathbf{s}, p(t, \mathbf{s})) \geq -m_0 p(t, \mathbf{s}) \quad \text{for all } (t, \mathbf{s}, p) \in Q_T \times \mathbb{R}, \tag{4.3}$$

where M, m_0 are positive constants, g_1 is a nonnegative, continuous and increasing function and the function $g_2 \in L^\infty(I)$. The nonlinear term has a financial interpretation as a prespecified cash flow that should be received or paid during the life of the contract (see, [20]) and more general higher order nonlinearities arise in B-S theory when the option price depends on the volatility coefficients of stochastic nature (see, [1, 2, 14]). The growth condition of the nonlinear term we assumed here is similar to the one in [7] for semilinear heat equation.

Recalling the literature, one of the main techniques to obtain the exact null controllability for nonlinear parabolic equations is to establish first the null controllability for the corresponding linearized equations and then to apply some kind of fixed point theorem [3, 7, 9, 17]. Now we use the similar method to obtain the exact null controllability for the system (4.1).

Let $1 \le N < 6$ and $\frac{N+2}{2} < q_N < 2\frac{N+2}{N-2}$ if $N \ge 3$ and $q_N \in (2, +\infty)$ if $N = 1, 2$. Note that for this choice of N and q_N, we have the embeddings (see, [23])

$$W^{2,1}_{q_N}(Q_T) \hookrightarrow L^\infty(Q_T) \quad \text{and} \quad H^{2,1}(Q_T) = (W^{2,1}_2(Q_T) \cap H^1(0, T; L^2(I))) \hookrightarrow L^{q_N}(Q_T).$$

Now let us introduce the function

$$h(t, \mathbf{s}, z) = \begin{cases} \frac{g(t,\mathbf{s},z)}{z} & \text{if } |z| > 0 \\ \lim_{\eta\to 0} \frac{g(t,\mathbf{s},\eta)}{\eta} & \text{if } z = 0, \end{cases}$$

and for any $y_0 \in Y_N(I) = H^1_0(I) \cap W^{2,q_N}(I)$ and for an arbitrary but fixed constant $\kappa > 0$, define the set

$$\Pi = \{z \in L^\infty(Q_T);\ \|z\|_{L^\infty(Q_T)} \le \kappa\}.$$

Then for $z \in \Pi$, consider the linearized system

$$\begin{cases} y_t + \mathcal{L}y + h(t, \mathbf{s}, z)y = \chi(\mathbf{s})u & \text{in } Q_T \\ y(0, \mathbf{s}) = y_0(\mathbf{s}) & \text{in } I \\ y(t, \mathbf{s}) = 0 & \text{on } \Sigma_T. \end{cases} \tag{4.4}$$

For any $\varepsilon > 0$ and $0 < \upsilon < 2$, consider the optimal control problem $\min\{J_\varepsilon(u) : u \in L^2(0, T; L^2(I_1))\}$, where the functional J_ε is defined by

$$J_\varepsilon(u) = \frac{1}{2}\int_0^T \int_{I_1} e^{-\upsilon\tau\alpha}|u|^2 d\mathbf{s}dt + \frac{1}{2\varepsilon}\int_I |y(T, \mathbf{s})|^2 d\mathbf{s}, \tag{4.5}$$

where y is the solution of (4.4) associated with the control u. Arguing similar to the first step of Theorem 3.1, there exists a unique solution $(u_\varepsilon, y_\varepsilon)$ for the optimal control problem and the control is characterized by $u_\varepsilon = -\chi(\mathbf{s})e^{\upsilon\tau\alpha}p^\varepsilon$, where p_ε is the solution of the adjoint system

$$\begin{cases} -p^\varepsilon_t + \mathcal{L}^* p^\varepsilon = f = -h(t, \mathbf{s}, z)p^\varepsilon & \text{in } Q_T \\ p^\varepsilon(T, \mathbf{s}) = \frac{1}{\varepsilon}y^\varepsilon(T, \mathbf{s}) & \text{in } I \\ p^\varepsilon(t, \mathbf{s}) = 0 & \text{on } \Sigma_T, \end{cases} \tag{4.6}$$

where the operator $\mathcal{L}^*$ is defined in (3.2). Following the proof of Lemma 3.1, we have from the estimate (3.26) that

$$\int_{Q_T} e^{2\tau\alpha}(\mu^4\tau^3\rho^{-3}|p^\varepsilon|^2 + \mu^2\tau\rho^{-1}|\nabla p^\varepsilon|^2)d\mathbf{s}dt$$

$$\leq C\int_0^T\int_{I_1} e^{2\tau\alpha}\mu^4\tau^3\rho^{-3}|p^\varepsilon|^2 d\mathbf{s}dt + C\int_{Q_T} e^{2\tau\alpha}|f|^2 d\mathbf{s}dt$$
$$\leq C\int_0^T\int_{I_1} e^{2\tau\alpha}\mu^4\tau^3\rho^{-3}|p^\varepsilon|^2 d\mathbf{s}dt$$
$$+ CT^6M^2(g_1(\kappa)+\|g_2\|_{L^\infty(I)})^2\int_{Q_T} e^{2\tau\alpha}\rho^{-3}|p^\varepsilon|^2 d\mathbf{s}dt.$$

Choosing sufficiently large

$$\tau \geq \tau_3 = \tau_2 + CT^2M^{2/3}(g_1(\kappa)+\|g_2\|_{L^\infty(I)})^{2/3}$$

and $\mu \geq \mu_0$, we obtain

$$\int_{Q_T} e^{2\tau\alpha}(\mu^4\tau^3\rho^{-3}|p^\varepsilon|^2 + \mu^2\tau\rho^{-1}|\nabla p^\varepsilon|^2)d\mathbf{s}dt$$
$$\leq C\int_0^T\int_{I_1} e^{2\tau\alpha}\mu^4\tau^3\rho^{-3}|p^\varepsilon|^2 d\mathbf{s}dt, \tag{4.7}$$

where C is independent of p, z and M. Observe that for any $\upsilon \in (0,2)$,

$$e^{(2-\upsilon)\tau\alpha}\rho^{-3} \leq (2/T)^6 e^{-C\tau T^{-2}}, \quad \text{for all } \tau \geq \tau_4 = \max\{\tau_3, C(I)T^2\},\ (t,\mathbf{s}) \in \bar{Q}_T,$$

and so

$$\int_{Q_T} e^{2\tau\alpha}\rho^{-3}|p^\varepsilon|^2 \leq \frac{C}{T^6}\int_0^T\int_{I_1} e^{\upsilon\tau\alpha}|p^\varepsilon|^2 d\mathbf{s}dt.$$

For $\tau \geq \tau_4$, we also have $e^{2\tau\alpha}\rho^{-3} \geq (16/3T^2)^3 e^{-C\tau T^{-2}}$, $\forall (t,\mathbf{s}) \in [T/4, 3T/4]\times \bar{I}$ and so

$$\int_{T/4}^{3T/4}\int_I |p^\varepsilon|^2 d\mathbf{s}dt \leq \exp(C\bar{U})\int_0^T\int_{I_1} e^{\upsilon\tau\alpha}|p^\varepsilon|^2 d\mathbf{s}dt,$$

for sufficiently large

$$\tau = T^2(C(I,\overline{S})\Lambda_T + CM^{2/3}(g_1(\kappa)+\|g_2\|_{L^\infty(I)})^{2/3}), \quad \text{where } \bar{U} = \frac{\tau}{T^2}.$$

Thanks to the assumption $h \geq -m_0$, it is not difficult to get the estimate (3.28) with the constant $\bar{V} = V_{m_0}$ and hence the observability estimate

$$\int_I |p^\varepsilon(0,\mathbf{s})|^2 d\mathbf{s} \leq \exp(C_{m_0}\kappa_1)\int_0^T\int_{I_1} e^{\upsilon\tau\alpha}|p^\varepsilon|^2 d\mathbf{s}dt,$$

whence it follows from the system (4.4) and (4.6) that

$$\frac{1}{2}\int_0^T\int_{I_1} e^{-\upsilon\tau\alpha}|u^\varepsilon|^2 d\mathbf{s}dt + \frac{1}{\varepsilon}\int_I |y^\varepsilon(T,\mathbf{s})|^2 d\mathbf{s}$$
$$\leq \exp(C\kappa_1)\int_I |y_0(\mathbf{s})|^2 d\mathbf{s}, \quad \text{for any } \varepsilon > 0, \tag{4.8}$$

where the constant κ_1 (with Λ_T as given in (3.6)) is defined as

$$\kappa_1 = \Lambda_T + M^{2/3}(g_1(\kappa) + \|g_2\|_{L^\infty(I)})^{2/3}.$$

In view of the estimate (3.8), taking again $h \geq -m_0$ into account, one can also get

$$\|y^\varepsilon(t)\|^2_{L^2(I)} + \|y^\varepsilon\|^2_{L^2(0,T;H^1_0(I))} \leq e^{C_{m_0}T}(\|y_0\|^2_{L^2(I)} + \|u^\varepsilon\|^2_{L^2(0,T;L^2(I_1))}). \tag{4.9}$$

Moreover, for any $y_0 \in H^1_0(I)$ and $u^\varepsilon \in L^2(0, T; L^2(I_1))$, multiplying (4.4) by y^ε_t, we get

$$\begin{aligned}
&\|y^\varepsilon_t\|^2_{L^2(I)} + \frac{1}{2}\int_I \sum_{i,j=1}^N a_{ij}(\mathbf{s})(s_i y^\varepsilon_{s_i})(s_j (y^\varepsilon_t)_{s_j}) d\mathbf{s} \\
&\quad\leq \eta \|y^\varepsilon_t\|^2_{L^2(I)} + \frac{R^2}{\eta}\|y^\varepsilon\|^2_{L^2(I)} \\
&\qquad + \frac{1}{\eta} C(\overline{S})(A_0 + A_1 + A_4)^2 \int_I \sum_{i=1}^N (y^\varepsilon)^2_{s_i} d\mathbf{s} + \frac{1}{\eta}\|h(t, \mathbf{s}, z) y^\varepsilon\|^2_{L^2(I)} + \frac{1}{\eta}\|u^\varepsilon\|^2_{L^2(I_1)}
\end{aligned}$$

for some $\eta > 0$. Here we note that

$$\begin{aligned}
\|h(t, \mathbf{s}, z) y^\varepsilon\|^2_{L^2(I)} &\leq \int_I |h(t, \mathbf{s}, z)|^2 |y^\varepsilon|^2 d\mathbf{s} \\
&\leq M^2 (g_1(\kappa) + \|g_2\|_{L^\infty(I)})^2 \|y^\varepsilon\|^2_{L^2(I)}
\end{aligned} \tag{4.10}$$

and a_{ij} $(i, j = 1, 2, \ldots, N)$ do not depend on t so that for $\eta = 3/4$, we get

$$\begin{aligned}
&\frac{1}{4}\|y^\varepsilon_t\|^2_{L^2(I)} + \frac{1}{4}\frac{d}{dt}\int_I \mathcal{A}(\nabla y^\varepsilon \mathbf{s}, \nabla y^\varepsilon \mathbf{s}) d\mathbf{s} \\
&\quad\leq C\kappa_2 \int_I \sum_{i=1}^N (y^\varepsilon)^2_{s_i} d\mathbf{s} + \frac{4}{3}\|u^\varepsilon\|^2_{L^2(I_1)},
\end{aligned} \tag{4.11}$$

where

$$\kappa_2 = C(\overline{S})(A_0 + A_1 + A_4 + R)^2 + M^2(g_1(\kappa) + \|g_2\|_{L^\infty(I)})^2.$$

Applying positive definiteness of a_{ij} and integrating over 0 to t for $0 \leq t \leq T$, we arrive at

$$\begin{aligned}
&\frac{1}{4}\|y^\varepsilon_t\|^2_{L^2(0,t;L^2(I))} + \frac{\theta}{4}\int_I \sum_{i=1}^N s_i^2 (y^\varepsilon_{s_i})^2 d\mathbf{s} \\
&\quad\leq C\kappa_2 \left(\int_I \sum_{i=1}^N s_i^2 y^2_{s_i}(0, \mathbf{s}) d\mathbf{s} + \int_0^t \int_I \sum_{i=1}^N (y^\varepsilon)^2_{s_i} d\mathbf{s} + \|u^\varepsilon\|^2_{L^2(0,t;L^2(I_1))} \right)
\end{aligned}$$

and noting $\underline{S} > 0$, we also have

$$\int_I |\nabla y|^2 d\mathbf{s} \leq \exp(CT\kappa_2)(\|y_0\|^2_{H^1_0(I)} + \|u^\varepsilon\|^2_{L^2(0,T;L^2(I_1))}).$$

Now let us rewrite (4.4) as $\mathcal{L}y^\varepsilon = G$, where $G := \chi u^\varepsilon - h(t,\mathbf{s},z)y^\varepsilon - y_t^\varepsilon$. Since $G \in L^2(I)$ for a.e. $0 \le t \le T$, using elliptic regularity result (see, [16], Chap. 6) we have $y^\varepsilon \in H^2(I)$, for a.e. $0 \le t \le T$. As a result, we get

$$\begin{aligned}
&\|y^\varepsilon\|^2_{H_0^1(I)} + \|y_t^\varepsilon\|^2_{L^2(0,T;L^2(I))} + \|y^\varepsilon\|^2_{L^2(0,T;H^2(I))}\\
&\quad \le C\exp(CT\kappa_2)(\|y_0\|^2_{H_0^1(I)} + \|u^\varepsilon\|^2_{L^2(0,T;L^2(I_1))}),
\end{aligned} \tag{4.12}$$

where C depends on the constants $\underline{S}$ and $\overline{S}$. Arguing as in the third step of Theorem 3.1, one can obtain on a subsequence of $\{\varepsilon\}$ as $\varepsilon \to 0: u^\varepsilon \to u$ weakly in $L^2(Q_T)$ and $y^\varepsilon \to y$ weakly in $L^2(0,T;H_0^1(I)\cap H^2(I))\cap H^1(0,T;L^2(I))$ and $y(T,\mathbf{s}) \equiv 0$.

In order to prove the controllability of the nonlinear system, we need some regularity on the control. To attain this end, set $\tilde{p}^\varepsilon = e^{\upsilon\tau\alpha}p^\varepsilon$, to obtain from the system (4.6) that

$$\begin{cases}
\tilde{p}_t^\varepsilon + \frac{1}{2}\sum_{i,j=1}^N s_i s_j a_{ij}(\mathbf{s})\tilde{p}^\varepsilon_{s_i s_j} = F_{p^\varepsilon} & \text{in } Q_T\\
\tilde{p}^\varepsilon(T,\mathbf{s}) = 0 & \text{in } I\\
\tilde{p}^\varepsilon(t,\mathbf{s}) = 0 & \text{on } \Sigma_T,
\end{cases} \tag{4.13}$$

where

$$\begin{aligned}
F_{p^\varepsilon} &= \sum_{i,j=1}^N [s_i s_j a_{ij}(\mathbf{s})(e^{\upsilon\tau\alpha})_{s_i} - e^{\upsilon\tau\alpha}(s_i s_j a_{ij}(\mathbf{s}))_{s_i}]p_{s_j} - \sum_{i=1}^N e^{\upsilon\tau\alpha}s_i(\delta_i(\mathbf{s}) - r(\mathbf{s}))p_{s_i}\\
&\quad + \frac{1}{2}\sum_{i,j=1}^N [s_i s_j a_{ij}(\mathbf{s})(e^{\upsilon\tau\alpha})_{s_i s_j} - e^{\upsilon\tau\alpha}(s_i s_j a_{ij}(\mathbf{s}))_{s_i s_j}]p + (e^{\upsilon\tau\alpha})_t p\\
&\quad - \sum_{i=1}^N e^{\upsilon\tau\alpha}(s_i(\delta_i(\mathbf{s}) - r(\mathbf{s}))_{s_i} p + e^{\upsilon\tau\alpha}(r(\mathbf{s}) + h(t,\mathbf{s},z))p\\
&= J_1 + J_2 + J_3.
\end{aligned}$$

By the regularity of solutions of (4.13), we have

$$\|\tilde{p}^\varepsilon\|_{H^{2,1}(Q_T)} \le C\|F_{p^\varepsilon}\|_{L^2(Q_T)}.$$

For sufficiently large but fixed τ and μ, using (4.7), let us estimate the terms in F_{p^ε}:

$$\begin{aligned}
\|J_1\|^2_{L^2(Q_T)} &\le C(I,\overline{S})(A_0\|e^{2(\upsilon-1)\tau\alpha}\rho^{-1}\|_{L^\infty(Q_T)}\\
&\qquad + (A_0 + A_1 + A_4)\|e^{2(\upsilon-1)\tau\alpha}\rho\|_{L^\infty(Q_T)})\int_{Q_T} e^{2\tau\alpha}\rho^{-1}|\nabla p^\varepsilon|^2 d\mathbf{s}dt\\
&\le C(I,\overline{S})\left(\frac{A_0}{T^2} + (A_0 + A_1 + A_4)T^2\right)\int_0^T\int_{I_1} e^{\upsilon\tau\alpha}|p^\varepsilon|^2 d\mathbf{s}dt,
\end{aligned}$$

for any $\upsilon \ge 1$, where the constants A_i are defined in (3.10). By similar computation, we get

$$\begin{aligned}
\|J_2\|^2_{L^2(Q_T)} &\le C(I,\overline{S})(A_0\|e^{2(\upsilon-1)\tau\alpha}(\rho^{-1} + \rho)\|_{L^\infty(Q_T)}\\
&\quad + (A_0 + A_1 + A_2)\|e^{2(\upsilon-1)\tau\alpha}\rho^3\|_{L^\infty(Q_T)}
\end{aligned}$$

$$
\begin{aligned}
&+T^2\|e^{2(\upsilon-1)\tau\alpha}\rho^{-1}\|_{L^\infty(Q_T)}\Big)\int_{Q_T} e^{2\tau\alpha}\rho^{-3}|p^\varepsilon|^2 d\mathbf{s}dt\\
&\le C(I,\overline{S})\left(1+\frac{A_0}{T^2}+A_0T^2+(A_0+A_1+A_2)T^6\right)\int_0^T\int_{I_1} e^{\upsilon\tau\alpha}|p^\varepsilon|^2 d\mathbf{s}dt.
\end{aligned}
$$

Finally, using (4.10), we also have

$$
\|J_3\|^2_{L^2(Q_T)} \le C(I,\overline{S})T^6(A_4+A_5+M^2(g_1(\kappa)+\|g_2\|_{L^\infty(I)})^2)\int_0^T\int_{I_1} e^{\upsilon\tau\alpha}|p^\varepsilon|^2 d\mathbf{s}dt.
$$

Thus from the preceding estimates, we get

$$
\|\tilde{p}^\varepsilon\|_{H^{2,1}(Q_T)} \le C\kappa_3\int_0^T\int_{I_1} e^{\upsilon\tau\alpha}|p^\varepsilon|^2 d\mathbf{s}dt
$$

and so applying the embeddings and the definition of the control with (4.8), we arrive at

$$
\begin{aligned}
\|u^\varepsilon\|^2_{L^{q_N}(Q_T)} = \|\chi\tilde{p}^\varepsilon\|^2_{L^{q_N}(Q_T)} &\le C\kappa_3\int_0^T\int_{I_1} e^{\upsilon\tau\alpha}|p^\varepsilon|^2 d\mathbf{s}dt\\
&\le C\exp(\bar{C}(\kappa_1+\kappa_3))\|y_0\|^2_{L^2(I)},
\end{aligned}
\tag{4.14}
$$

where the positive constants C and $\bar{C}$ are independent of y_0,κ and

$$
\kappa_3 = 1+\frac{A_0}{T^2}+T^6\left(M^2(g_1(\kappa)+\|g_2\|_{L^\infty(I)})^2+\sum_{i=0}^{5}A_i\right).
$$

Now with the estimate (4.14) and the existence of solutions of parabolic boundary value problems [23], we have the following convergence on a subsequence of ε as $\varepsilon\to 0$:

$$
\begin{aligned}
&u^\varepsilon\to u \quad \text{weakly in } L^{q_N}(Q_T)\\
&y^\varepsilon\to y \quad \text{weakly in } L^2(0,T;H_0^1(I))\cap W^{2,1}_{q_N}(Q_T).
\end{aligned}
$$

Passing to the week limit in (4.4), we obtain that (u,y) satisfy the system (4.4) with $y(T,\mathbf{s})\equiv 0$ a.e. in I. Thus, we have proved the following theorem:

Theorem 4.1 *Suppose all the assumptions of Theorem 3.1 are satisfied. In addition, let the growth condition (4.2) and the assumption (4.3) on g be satisfied. Then for each $z\in\Pi$, there exist $(u,y)\in L^{q_N}(Q_T)\times\mathcal{H}$ satisfying the system (4.4) with $y(T,\mathbf{s})\equiv 0$ a.e. in I and*

$$
\|\chi u\|^2_{L^{q_N}(Q_T)} \le C\exp(\bar{C}(\kappa_1+\kappa_3))\|y_0\|^2_{L^2(I)},
\tag{4.15}
$$

where κ_1 and κ_3 are defined respectively in (4.8) and (4.14), and the space

$$
\mathcal{H} = L^2(0,T;H_0^1(I))\cap W^{2,1}_{q_N}(Q_T).
$$

Now we are ready to state and prove the main theorem of this section. The proof follows the classical arguments, used for instance in [7].

Theorem 4.2 *Suppose all the assumptions of Theorem 4.1 are satisfied. Then for* $1 \leq N < 6$ *and for all* $y_0 \in Y_N(I)$ *with* $\|y_0\|_{Y_N(I)} \leq \widetilde{R}$ *for some constant* $\widetilde{R} > 0$, *there exists a control* u *satisfying* $y(T, \mathbf{s}) \equiv 0,\ a.e.\ \mathbf{s} \in I$.

Proof For any $u \in L^{q_N}(Q_T)$, let $y \in \mathcal{H}$ be the solution of (4.4) with right hand side u. For each $z \in \Pi$, let us introduce the set

$$\mathcal{U}(z) = \left\{u \in L^{q_N}(Q_T) : y(T, \mathbf{s}) = 0,\ \|\chi u\|^2_{L^{q_N}(Q_T)} \leq C \exp(\bar{C}(\kappa_1 + \kappa_3))\|y_0\|^2_{L^2(I)}\right\}$$

and

$$\Phi(z) = \{y : u \in \mathcal{U}(z),\ y \in \mathcal{H}\}.$$

Now we can introduce the set valued mapping on Π by $z \mapsto \Phi(z)$. We will prove that this mapping possesses at least one fixed point y and to prove the existence of such a fixed point we use the classical theorem due to Kakutani (see, [5], Chap. 9).

Now we proceed to validate the conditions of this fixed point theorem. First let us note from Theorem 4.1 that $\Phi(z)$ is, for every $z \in \Pi$, a nonempty set. It is easy to check that $\Phi(z)$ is a bounded closed convex subset of $L^2(Q_T)$.

For some sufficiently large $\widetilde{R} > 0$, we now show that $\Phi(\Pi) \subset \Pi$. It is clear from the proof of Theorem 4.1 that for any $y \in \Phi(z)$,

$$\|y\|_{H^{2,1}(Q_T)} \leq C \exp(\bar{C}\kappa_2)(\|y_0\|_{H^1_0(I)} + \|u\|_{L^2(0,T;L^2(I_1))}),$$

where the constant C(independent of z) depends only on $\underline{S}$ and $\overline{S}$. With the estimate (4.15) for the control together with the regularity of the initial data, we have from the theory of parabolic boundary value problems [23] that

$$\|y\|^2_{W^{2,1}_{q_N}(Q_T)} \leq C \exp(\bar{C}(\kappa_1 + \kappa_2 + \kappa_3))\|y_0\|^2_{Y_N(I)}.$$

Applying the embeddings, we get

$$\|y\|^2_{L^\infty(Q_T)} \leq C\|y\|^2_{W^{2,1}_{q_N}(Q_T)} \leq C \exp(\bar{C}(\kappa_1 + \kappa_2 + \kappa_3))\|y_0\|^2_{Y_N(I)}, \tag{4.16}$$

where κ_1, κ_2 and κ_3 are defined respectively in (4.8), (4.11) and (4.14). It follows from the preceding estimate that if

$$\|y_0\|_{Y_N(I)} \leq C\kappa \exp(-\bar{C}(\kappa_1 + \kappa_2 + \kappa_3)),$$

then by the definition of Π, we have $\Phi(z) \subset \Pi, \forall z \in \Pi$. Moreover, by the estimate (4.16) and the classical compactness results [26], we can conclude that $\Phi(z)$ is a relatively compact subset of $L^2(Q_T)$.

Next we prove that the mapping $z \mapsto \Phi(z)$ is upper semicontinuous. Let $z_n \in \Pi$, $z_n \to z$ in $L^2(Q_T)$ and $y_n \in \Phi(z_n) \to y$ in $L^2(Q_T)$ and let u_n be the corresponding controls. Then by Theorem 4.1, it follows at least on a subsequence that

$$\begin{aligned} &u_n \to u \quad \text{weakly in } L^2(Q_T) \\ &y_n \to y \quad \text{weakly in } H^{2,1}(Q_T) \quad \text{and strongly in } C([0,T]; L^2(I)). \end{aligned}$$

Then by the continuity of g, we also have: $h(t,\mathbf{s},z_n)y_n \to h(t,\mathbf{s},z)y$ a.e. in Q_T. Note that (u_n, y_n) is the solution to the system

$$\begin{cases} (y_n)_t + \mathcal{L}y_n + h(t,\mathbf{s},z_n)y_n = \chi(\mathbf{s})u_n & \text{in } Q_T \\ y_n(0,\mathbf{s}) = y_0(\mathbf{s}) & \text{in } I \\ y_n(t,\mathbf{s}) = 0 & \text{on } \Sigma_T. \end{cases}$$

And therefore passing to the weak limit, we can conclude that $y \in \Phi(z)$. This proves that $z \mapsto \Phi(z)$ is upper semicontinuous.

Thus we have verified all the conditions of Kakutani's fixed point theorem. Hence applying the theorem in the space $L^2(Q_T)$, we deduce that there is at least one $z \in L^\infty(Q_T)$ such that $z \in \Phi(z)$. Therefore, by the definition of Φ, it is clear that there exists at least one pair (u, y) satisfying the system (4.1) such that $y(T,\mathbf{s}) \equiv 0$ a.e. in I. This completes the proof. □

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