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Efficient Generic Search with Effect Handlers

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Programming Language Interest Group

(Joint work with Sam Lindley and John Longley)

A new complexity result for control operators

The crux of this work is to establish a new complexity result for control operators

Lay person's version of the result

There is a class of problems for which a language with control operators provides asymptotically more efficient solutions than a language without control operators ($\mathcal{O}(2^n)$ vs $\Omega(n2^n)$).

To establish the existence of this class, we use *generic search* as an example program and effect handlers as our control operator.

This talk is high-level walk-through of how we establish this result

(The possibility of the existence of this result can be traced back to Longley (2009))

Methodology

The plan of attack

- Define a pure functional language $\mathcal{L}$, and an extension thereof $\mathcal{L}_{\mathit{eff}}$ with effect handlers.
- Provide a specification (type signature) of *generic search* problem
- Implement an efficient version of generic search in $\mathcal{L}_{\mathit{eff}}$
- ...and prove that it is indeed efficient
- Finally show that *any* implementation of generic search in $\mathcal{L}$ has worse complexity

There is a single rule of engagement:

No change of types is allowed! (Longley and Normann 2015)

This rules out tricks such as

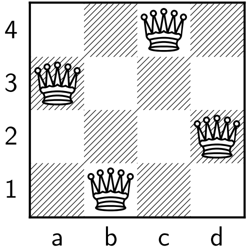
- CPS conversion (Hillerström et al. 2017)
- Implementing an interpreter for $\mathcal{L}_{\mathit{eff}}$ in $\mathcal{L}$

Generic search

Given a search problem P, a generic search algorithm finds solutions of P.

Applications include (Daniels 2016)

- n-Queens
- Sudoku
- Finding Nash equilibria
- Graph-colouring
- Exact real number integration



A solution to 4-Queens problem

Rather than finding solutions of P, we *count* the number of solutions of P

First instance of efficient generic search with effect handlers

Somewhat related is work on exhaustive search on infinite spaces

- Berger (1990): exhaustive search on the Cantor space $2^{\mathbb{N}}$
- Escardó (2007): characterisation of searchable infinite sets
- Bauer (2011): efficient search on infinite sets with effect handlers

Fine-grain call-by-value PCF (Levy et al. 2003)

The core of a "pure" functional programming language $\mathcal{L}$

$$
\begin{array}{lrcl}
\text{Types} & A, B, C, D & ::= & \langle\rangle \mid \mathsf{Bool} \mid \mathsf{Nat} \mid A \times B \mid A + B \mid A \to B \\[1em]
\text{Values} & V, W \in \mathrm{Val} & ::= & x \mid b \in \mathbb{B} \mid n \in \mathbb{N} \mid \mathsf{Plus} \mid \langle\rangle \mid \langle V; W\rangle \\
& & \mid & (\mathbf{inl}\, V)^B \mid (\mathbf{inr}\, W)^A \mid \lambda x^A.\, M \mid \mathbf{rec}\, f^A\, x.M \\[1em]
\text{Computations} & M, N \in \mathrm{Comp} & ::= & V\, W \\
& & \mid & \mathbf{let}\ \langle x; y\rangle = V\ \mathbf{in}\ N \\
& & \mid & \mathbf{if}\ V\ \mathbf{then}\ M\ \mathbf{else}\ N \\
& & \mid & \mathbf{case}\ V\ \{\mathbf{inl}\ x \mapsto M; \mathbf{inr}\ y \mapsto N\} \\
& & \mid & \mathbf{return}\ V \\
& & \mid & \mathbf{let}\ x \leftarrow M\ \mathbf{in}\ N \\[1em]
\text{Eval. contexts} & \mathcal{E} \in \mathsf{Eval} & ::= & [\,] \mid \mathbf{let}\ x \leftarrow \mathcal{E}\ \mathbf{in}\ N
\end{array}
$$

The static and dynamic semantics are completely standard.

Service announcement: Syntactic sugar

I shall permit myself to use regular call-by-value syntax, e.g. for $f, g, h, a \in \mathrm{Val}$

$$f\ (h\, a) + g\ \langle\rangle$$

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$$\begin{aligned}[\![f\ (h\, a) + g\ \langle\rangle]\!] = {} & \textbf{let}\ x \leftarrow h\, a\ \textbf{in} \\ & \textbf{let}\ y \leftarrow f\ x\ \textbf{in} \\ & \textbf{let}\ z \leftarrow g\ \langle\rangle\ \textbf{in} \\ & \mathsf{Plus}\, \langle y; z\rangle \end{aligned}$$

FGCB PCF with effect handlers

The language $\mathcal{L}_{eff}$

Handler types	$F ::= C \Rightarrow D$
Signatures	$\Sigma ::= \cdot \mid \{\ell : A \to B\} \uplus \Sigma$
Labels	$\ell \in \mathcal{L}$
Computations	$M, N \in \mathrm{Comp} ::= \cdots \mid \textbf{do}\ \ell\ V \mid \textbf{handle}\ M\ \textbf{with}\ H$
Handlers	$H ::= \{\textbf{val}\ x \mapsto M\} \mid \{\ell\ p\ r \mapsto N\} \uplus H$
Eval. contexts	$\mathcal{E} \in \mathsf{Eval} ::= \cdots \mid \textbf{handle}\ \mathcal{E}\ \textbf{with}\ H$

FGCB PCF with effect handlers (dynamic semantics)

S-Ret $\quad$ $\textbf{handle}\ (\textbf{return}\ V)\ \textbf{with}\ H$

$\quad\quad \rightsquigarrow N[V/x], \qquad$ where $H^{\mathrm{val}} = \{\textbf{val}\ x \mapsto N\}$

S-Op $\quad$ $\textbf{handle}\ \mathcal{E}[\textbf{do}\ \ell\ V]\ \textbf{with}\ H$

$\quad\quad \rightsquigarrow N[V/p, \lambda y.\, \textbf{handle}\ \mathcal{E}[\textbf{return}\ y]\ \textbf{with}\ H/r]$, where $H^{\ell} = \{\ell\ p\ r \mapsto N\}$

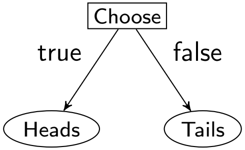
Example: Coin tossing (nondeterminism)

Fix $\Sigma = \{\mathsf{Choose} : \mathsf{Bool}\}$

A coin toss model

$$\begin{aligned} &\mathit{toss} : \langle\rangle \to \mathsf{Toss} \\ &\mathit{toss} = \textbf{if do } \mathsf{Choose} \textbf{ then } \mathsf{Heads} \\ &\qquad\quad \textbf{else } \mathsf{Tails} \end{aligned}$$

Computation tree



A possible handler for Choose

$$\begin{aligned} &\mathit{allChoices} : (\langle\rangle \to \mathsf{Toss}) \to [\mathsf{Toss}] \\ &\mathit{allChoices} = \lambda m.\, \textbf{handle } m\,\langle\rangle \textbf{ with} \\ &\qquad\qquad\qquad \textbf{val } x \mapsto [x] \\ &\qquad\qquad\qquad \mathsf{Choose}\ r \mapsto r\ \mathsf{true} +\!\!+\ r\ \mathsf{false} \end{aligned}$$

Enumerating all possible outcomes

$$\mathit{allChoices}\ \mathit{toss} \leadsto^{+} ??$$

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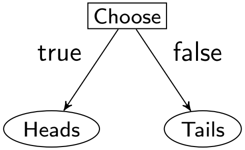

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Enumerating all possible outcomes

$$\mathit{allChoices}\ \mathit{toss} \leadsto^{+} [\mathsf{Heads}, \mathsf{Tails}]$$

Setting up generic search

The secret of generic search is *higher-order* functions

$$\mathsf{Predicate} \doteq (\mathsf{Nat} \to \mathsf{Bool}) \to \mathsf{Bool}$$

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Some (silly) example predicates

$$\begin{aligned} \mathsf{tt}_n &\doteq \lambda p. p\,0; \cdots ; p\,(n-1); \textbf{return}\ \mathsf{true} \\ \mathsf{div}_n &\doteq \textbf{rec}\, d\ p. \textbf{if}\ p\,(n-1)\ \textbf{then}\ d\ p\ \textbf{else}\ \textbf{return}\ \mathsf{false} \\ \mathsf{odd}_n &\doteq \lambda p. \mathsf{reduce\ xor\ false}\,[p\,0, \ldots, p\,(n-1)] \end{aligned}$$

A pure generic search procedure

A possible implementation of generic search in $\mathcal{L}$

$$
\begin{aligned}
&\mathit{count}_n : (\mathsf{Predicate} \to \mathsf{Bool}) \to \mathsf{Nat} \\
&\mathit{count}_n \doteq \lambda \mathit{pred}.\mathit{count}'\, n\, (\lambda i.\bot) \\
&\qquad\qquad \mathbf{where} \\
&\qquad\qquad\qquad \mathit{count}'\, 0 \qquad\quad\;\; p \doteq \mathbf{if}\ \mathit{pred}\ p\ \mathbf{then}\ 1\ \mathbf{else}\ 0 \\
&\qquad\qquad\qquad \mathit{count}'\, (n+1)\ p \doteq \quad \mathit{count}'\, n\, (\lambda i.\mathbf{if}\ i = n\ \mathbf{then}\ \mathsf{true}\ \mathbf{else}\ p\, i) \\
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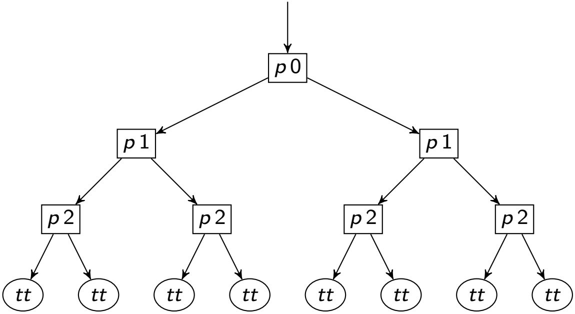
Example $\mathit{count}_3\ \mathit{tt}_3$:



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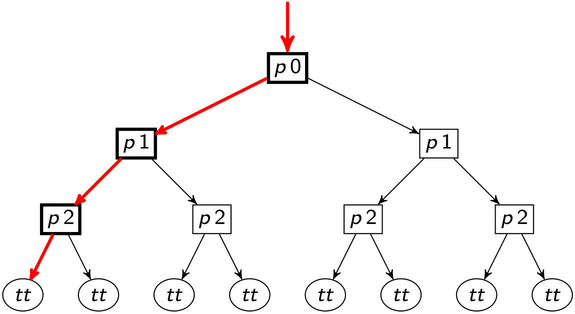
Example $count_3\ tt_3$: reaches the first leaf



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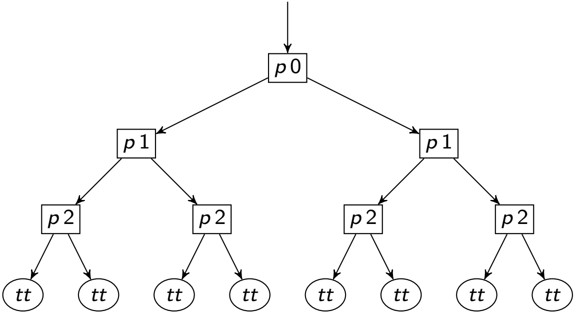
Example $count_3\ tt_3$: computation restarts



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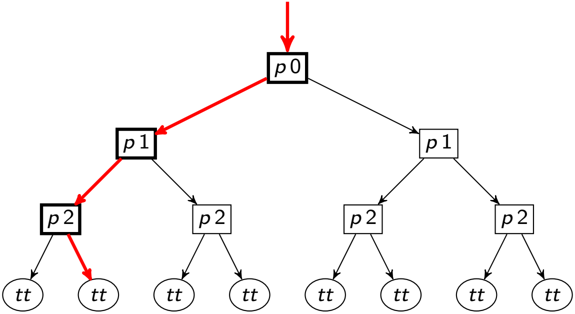
Example $count_3\ tt_3$: reaches the second leaf



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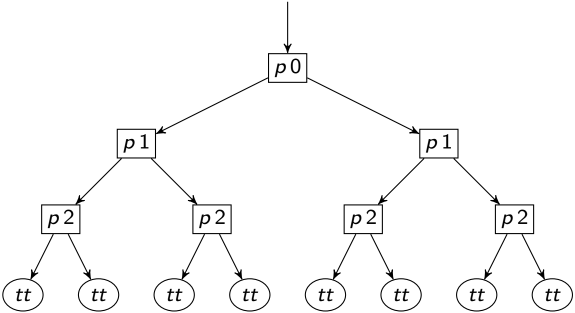

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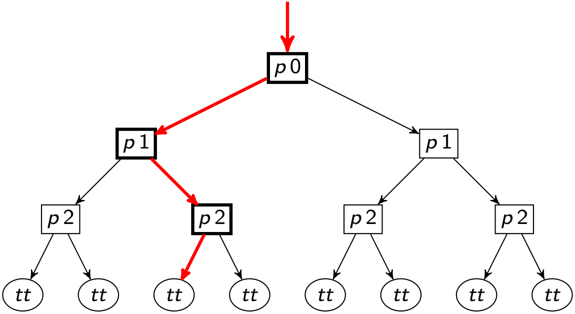
Example $count_3\, tt_3$: reaches the third leaf, etc...



The effectful generic search procedure

For the efficient implementation of generic search in $\mathcal{L}_{\mathit{eff}}$, we require one operation; fix $\Sigma \doteq \{\mathsf{Branch} : \mathsf{Bool}\}$

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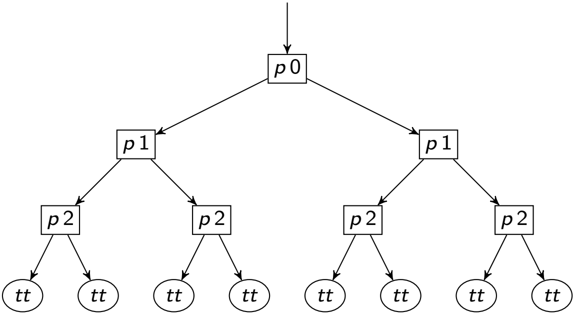

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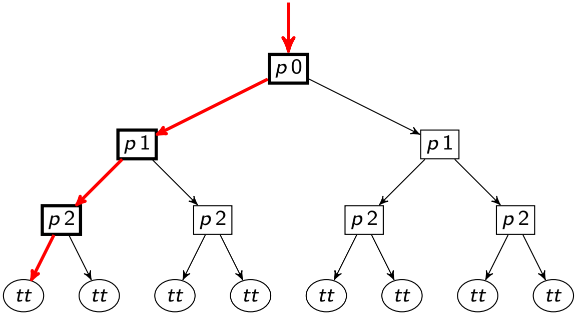
Example genericSearch tt_3: reaches the first leaf



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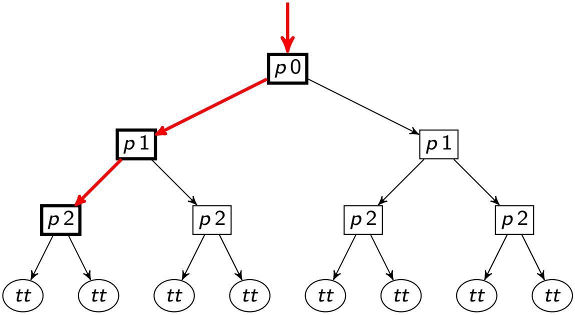
Example genericSearch tt_3: computation backtracks



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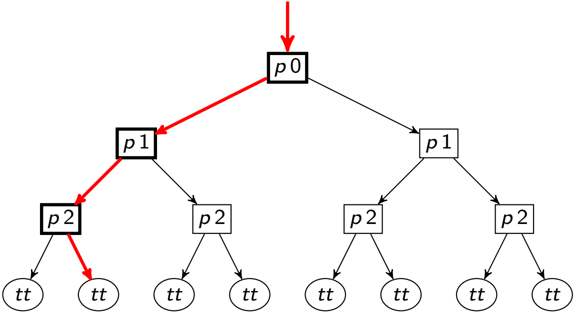
Example genericSearch tt_3: reaches the second leaf



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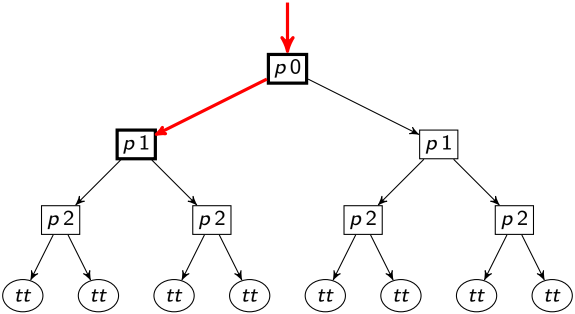

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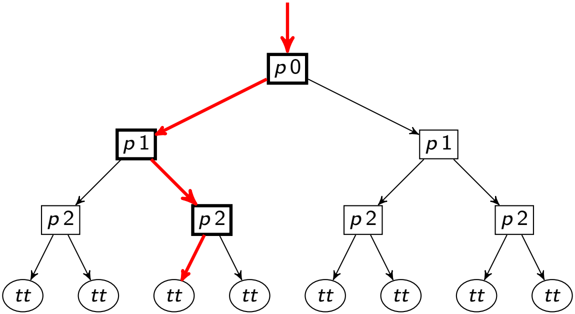


Semantics for predicates

Definition (The label set)

The set Lab consists of queries parameterised by a natural number and answers parameterised by a boolean, i.e. $\mathsf{Lab} \doteq \{!tt, !f\!f\} \cup \{?n \mid n \in \mathbb{N}\}$

Definition (Decision tree)

A decision tree is a partial function $t : \mathbb{B}^* \to \mathsf{Lab} \times \mathsf{Eval} \times \mathbb{N}$ from lists of booleans to node labels with the following properties:

- The domain of t, $dom(t)$, is prefix closed.
- For any boolean, $b \in \mathbb{B}$, and list, $bs \in \mathbb{B}^*$, of booleans, if $t_\ell(bs) = \ !b$ is an answer node then bs is a leaf of t.

Notation: write t_ℓ and t_s for the projection of the first and third components of $t(-)$, respectively.

Model construction

Definition

We implement the decision tree semantics as a partial function parameterised by an abstract point p, $\mathcal{T}_p : \mathrm{Comp} \rightharpoonup (\mathbb{B}^* \rightharpoonup \mathsf{Lab} \times \mathsf{Eval} \times \mathbb{N})$, that given a predicate, *pred*, constructs a function, that given a list of booleans, *bs*, returns the corresponding node label in model of *pred* p, where p is an "abstract point".

$$
\begin{aligned}
\mathcal{T}_p(\textbf{return}\ \mathsf{true})\,[\,] &= (!\mathsf{true}, [\,], 0)\\
\mathcal{T}_p(\textbf{return}\ \mathsf{false})\,[\,] &= (!\mathsf{false}, [\,], 0)\\
\mathcal{T}_p(\mathcal{E}[p\ n])\,[\,] &= (?n, \mathcal{E}, 0)\\
\mathcal{T}_p(\mathcal{E}[p\ n])(b :: bs) &\simeq \mathcal{T}_p(\mathcal{E}[\textbf{return}\ b])(bs)\\
\text{If } M \rightsquigarrow N \text{ then } \mathcal{T}_p(M)(bs) &\simeq \mathcal{I}(\mathcal{T}_p(N)(bs))\\
&\quad \text{where } \mathcal{I}(\ell, \mathcal{E}, i) = (\ell, \mathcal{E}, i+1)
\end{aligned}
$$

Define Model $\doteq \mathrm{Comp} \rightharpoonup (\mathbb{B}^* \times \mathsf{Eval} \times \mathbb{N})$.

Standard decision trees

We are interested in predicates whose models are complete binary trees, and query each component of a provided point exactly once.

Definition (n-standard trees)

For any $n > 0$ a decision tree t is said to be n-standard whenever

- The domain of t consists of all the lists whose length is at most n, i.e.

$$dom(t) = \{bs : \mathbb{B}^* \mid |bs| \leq n\}$$

- Every leaf node in t is an answer node, i.e. for all $bs \in dom(t)$

$$\text{if } t_\ell(bs) = \; !b \text{ then } |bs| = n$$

- There are no repeated queries in t, i.e. for all $bs, bs' \in dom(t), j \in \mathbb{N}$

$$\text{if } bs \sqsubseteq bs' \text{ and } t_\ell(bs) = t_\ell(bs') = ?j \text{ then } bs = bs'$$

  where $bs \sqsubseteq bs'$ means bs is a prefix of bs'.

Main theorem

Theorem

1. *For every n-standard predicate pred, the generic search procedure has at most time complexity*

$$\mathrm{Time}(\mathrm{genericSearch}\ pred) = \sum_{bs \in \mathbb{B}^*, |bs| \leq n} t_s(bs) + \mathcal{O}(2^n)$$

2. *Every generic counting function count $\in \mathcal{L}$ has for every n-standard predicate pred at least time complexity*

$$\mathrm{Time}(count\ pred) = \sum_{bs \in \mathbb{B}^*, |bs| \leq n} 2^{n-|bs|} t_s(bs) + \mathcal{O}(n2^n)$$

Proving the positive result

Define suitable evaluation state computing functions

$$\mathit{start}, \mathit{end} : \mathbb{B}^* \times \mathsf{Model} \to \mathsf{Comp}$$

Lemma

Suppose t is a model of a n-standard predicate, then for every boolean list bs $\in \mathbb{B}^*$

$$\begin{aligned}
& \mathit{start}(bs, t) \\
\longrightarrow^+ \ & \mathit{start}(\mathsf{true} :: bs, t) \longrightarrow^{\sum_{|bs|+1 \le n} t_s(\mathsf{true}::bs) + 2^{n-(|bs|+1)}} \mathit{end}(\mathsf{true} :: bs, t) \\
\longrightarrow^+ \ & \mathit{start}(\mathsf{false} :: bs, t) \longrightarrow^{\sum_{|bs|+1 \le n} t_s(\mathsf{false}::bs) + 2^{n-(|bs|+1)}} \mathit{end}(\mathsf{false} :: bs, t) \\
\longrightarrow^+ \ & \mathit{end}(bs, t)
\end{aligned}$$

Proof.

Proof by downward induction on the list of booleans *bs*. □

Proving the negative result

Suppose that we have an arbitrary implementation of generic search $\mathit{count} \in \mathcal{L}$. Pick any n-standard predicate *pred* and look at the computation arising from *count pred*. Now we need to show that

Lemma (Every leaf is visited (A))

The computation $(\mathit{count}\ \mathit{pred})$ visits every leaf in the model of pred.

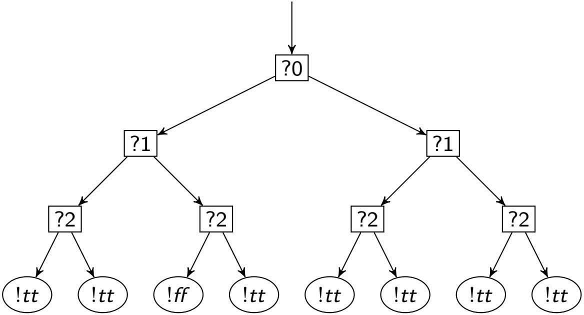
Lemma (No shared computation (B))

If p and p' are distinct points then their subcomputations are disjoint.

Since each subcomputation has length at least $\Omega(n)$ the entire computation must have at least length $\Omega(n2^n)$.

Threads and sections

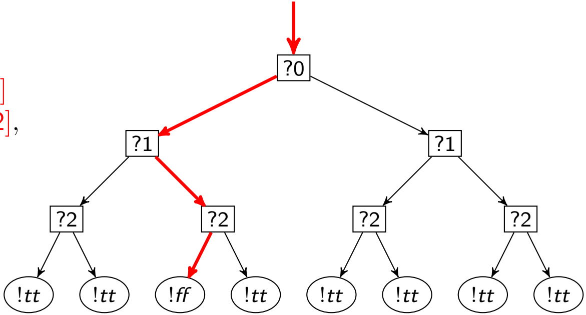
Consider a 3-standard predicate *seven* (has seven true leaves)



Threads and sections

Consider a 3-standard predicate *seven* (has seven true leaves)

$$\mathit{Thread} \doteq \{\ \mathit{pred}\ p \leadsto^* \mathcal{E}_0[p\,0],\ \mathcal{E}_0[\text{true}] \leadsto^* \mathcal{E}_1[p\,1]\ \mathcal{E}_1[\text{false}] \leadsto^* \mathcal{E}_2[p\,2],\ \mathcal{E}_2[\text{true}] \to \text{false}\ \}$$



Any n-standard predicate has 2^n threads, and every thread consists of $n+1$ sections.

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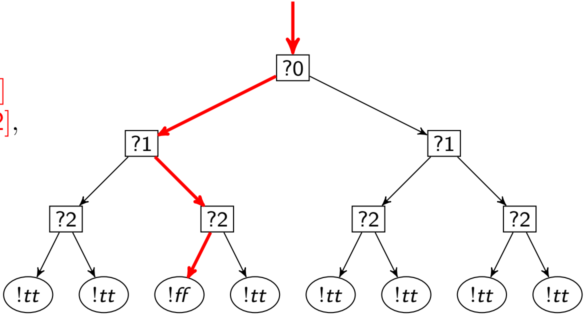


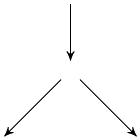
Any n-standard predicate has 2^n threads, and every thread consists of $n+1$ sections.

Proof of Lemma A.

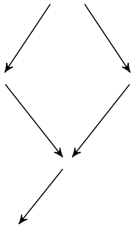
By contradiction: pick a leaf that has no thread; negate the value at the leaf; tweak the predicate accordingly; observe a wrong result. □

No shared computation

Every section has a unique successor



Every section has a single predecessor



Proof.

Follows by definition of section and the semantics being deterministic. □

Proof.

By direct calculation on the reduction sequence induced by a section. □

Summary and future work

In summary

- We have defined two languages $\mathcal{L}$ and $\mathcal{L}_{\mathit{eff}}$
- We have demonstrated that $\mathcal{L}_{\mathit{eff}}$ provides strictly more efficient implementations of generic search than $\mathcal{L}$ ($\mathcal{O}(2^n)$ vs $\Omega(n2^n)$)
- ...which establish a new complexity result for control operators

Future considerations

- Perform empirical experiments to observe the result in practice (Daniels 2016)
- Study the robustness of the result, i.e. what feature(s) can we add to $\mathcal{L}$ whilst retaining an efficiency gap between $\mathcal{L}$ and $\mathcal{L}_{\mathit{eff}}$?
- Generalise the result to all conceivable effective models of computations

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