2016/17 MPhil ACS / CST Part III Category Theory and Logic (L108) Exercise Sheet 6

- 1. [5/30 marks] Recall (from Lecture 2) that a pre-ordered set (P, \leq_P) can be regarded as a category whose objects are the elements of P and whose morphism sets P(x, x') contain at most one element and do so iff $x \leq_P x'$. Note that given two pre-ordered sets (P, \leq_P) and (Q, \leq_Q) , a functor $F: P \to Q$ is the same thing as a monotone function.
 - (a) Given two such functors $F, G : P \rightarrow Q$, how many natural transformations are there from F to G?
 - (b) Given monotone functions $F: P \to Q$ and $G: Q \to P$, give a property of F and G which ensures that, regarding them as functors, G is right adjoint to F.
- 2. [5/30 marks] Recall that **Pre** denotes the category of pre-ordered sets and monotone functions. For each set X, let $(\operatorname{Pow} X, \subseteq) \in \operatorname{obj} \operatorname{Pre}$ be the set of all subsets of X partially ordered by inclusion. Given a function $f: X \to Y$, let $f^{-1}: \operatorname{Pow} Y \to \operatorname{Pow} X$ be the function that maps each subset $B \subseteq Y$ to the subset $f^{-1}B \triangleq \{x \in X \mid f(x) \in B\} \subseteq X$.
 - (a) Show that f^{-1} is a monotone function and hence gives a morphism $(\text{Pow } Y, \subseteq) \to (\text{Pow } X, \subseteq)$ in **Pre**.
 - (b) Regarding f^{-1} as a functor as in question (1), show that it has both left and right adjoints, given on objects by the following 'generalized quantifiers'

$$\exists_f A \triangleq \{ y \in Y \mid (\exists x \in X) \ f(x) = y \land x \in A \}$$

$$\forall_f A \triangleq \{ y \in Y \mid (\forall x \in X) \ f(x) = y \Rightarrow x \in A \}$$

(for all $A \in \text{Pow } X$). [Hint: use your answer to question 1b.]

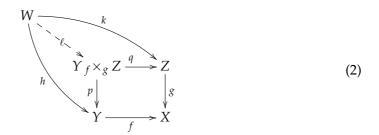
3. [10/30 marks] A category **C** *has pullbacks* if for every pair of **C**-morphisms with a common codomain, $Y \xrightarrow{f} X \xleftarrow{g} Z$, there is an object $Y_f \times_g Z$ and morphisms p,q making the following diagram commute in **C** (that is, $f \circ p = g \circ q$)

$$\begin{array}{ccc}
Y_f \times_g Z \xrightarrow{q} & Z \\
\downarrow p & & \downarrow g \\
Y \xrightarrow{f} & X
\end{array}$$
(1)

and with the following universal property:

For all $Y \xleftarrow{h} W \xrightarrow{k} Z$ in **C** with $f \circ h = g \circ k$, there is a unique morphism $\ell \in \mathbf{C}(W, Y_f \times_g X_g)$

Z) satisfying $p \circ \ell = h$ and $q \circ \ell = k$



- (a) Show that **C** has pullbacks iff for all $X \in \text{obj } \mathbf{C}$ the slice category \mathbf{C}/X has binary products (cf. Exercise Sheet 4, question 6).
- (b) Show that if **C** has a terminal object and pullbacks, then it has binary products.
- (c) Suppose C has pullbacks. Given $f \in C(Y, X)$, show that the mapping

$$\begin{array}{ccc}
Z & Y_f \times_g Z \\
f^* \colon \bigvee_{g} & \mapsto & \bigvee_{f} p \\
X & Y
\end{array}$$

is the object part of a functor $f^* : \mathbb{C}/X \to \mathbb{C}/Y$ between slice categories.

- (d) Show that the functor f^* in part (c) always has a left adjoint $f_!: \mathbb{C}/Y \to \mathbb{C}/X$, which on objects sends $(W,h) \in \text{obj}(\mathbb{C}/Y)$ to $f_!(W,h) \triangleq (W,f \circ h) \in \text{obj}(\mathbb{C}/X)$.
- 4. [10/30 marks] A functor $F : \mathbf{C} \to \mathbf{D}$ is said to *preserve binary products* if whenever

$$X \stackrel{\pi_1}{\longleftarrow} P \stackrel{\pi_2}{\longrightarrow} Y \tag{3}$$

is a product in **C** for the objects *X* and *Y*, then

$$F(X) \stackrel{F(\pi_1)}{\longleftarrow} F(P) \xrightarrow{F(\pi_2)} F(Y)$$

is a product in **D** for the objects F(X) and F(Y).

- (a) Show that the Yoneda functor $y: \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ (from Lectures 15 & 16) preserves binary products.
- (b) Give an example of categories C, D and a functor $F : C \to D$ for which C has all binary products, but F does not preserve all of them.