

## 6 Denotational Semantics (MPF)

For partially ordered sets  $(P, \sqsubseteq_P)$  and  $(Q, \sqsubseteq_Q)$ , define the set

$$(P \Rightarrow Q) = \{f \mid f \text{ is a monotone function from } (P, \sqsubseteq_P) \text{ to } (Q, \sqsubseteq_Q)\}$$

and, for all  $f, g \in (P \Rightarrow Q)$ , let

$$f \sqsubseteq_{(P \Rightarrow Q)} g \iff \forall p \in P. f(p) \sqsubseteq_Q g(p)$$

(a) Let  $(P, \sqsubseteq_P)$  and  $(Q, \sqsubseteq_Q)$  be partially ordered sets.

(i) Prove that  $((P \Rightarrow Q), \sqsubseteq_{(P \Rightarrow Q)})$  is a partially ordered set. [4 marks]

(ii) Prove that if  $(Q, \sqsubseteq_Q)$  is a domain then so is  $((P \Rightarrow Q), \sqsubseteq_{(P \Rightarrow Q)})$ . [6 marks]

(b) For  $\mathbb{N}$  the set of natural numbers partially ordered by the equality relation and for  $S_\perp$  the flat domain determined by a set  $S$ , consider the domain  $((\mathbb{N} \Rightarrow S_\perp), \sqsubseteq_{(\mathbb{N} \Rightarrow S_\perp)})$ .

(i) A function  $f \in (\mathbb{N} \Rightarrow S_\perp)$  is said to be *finite* whenever the subset of  $\mathbb{N}$  given by  $\{n \mid f(n) \neq \perp\}$  is finite.

Show that every function in  $(\mathbb{N} \Rightarrow S_\perp)$  is the least upper bound of a countable chain of finite functions. [4 marks]

(ii) For a domain  $(D, \sqsubseteq)$ , an element  $d \in D$  is said to be *isolated* (with respect to  $\sqsubseteq$ ) whenever, for all countable chains  $(x_0 \sqsubseteq \dots \sqsubseteq x_n \sqsubseteq \dots)$  in  $D$  with  $d \sqsubseteq \bigsqcup_{n \geq 0} x_n$ , there exists  $m \geq 0$  with  $d \sqsubseteq x_m$ .

Prove that a function in  $(\mathbb{N} \Rightarrow S_\perp)$  is isolated (with respect to  $\sqsubseteq_{(\mathbb{N} \Rightarrow S_\perp)}$ ) iff it is finite. [6 marks]