2016/17 MPhil ACS / CST Part III
Category Theory and Logic (L108)
Exercise Sheet 4 – Solution Notes

Question 1

(a)

$$
\begin{array}{lll}
\mathbf{V}(L,L) = \{\mathrm{id}_L\} & \mathbf{V}(P,L) = \{p\} & \mathbf{V}(R,L) = \emptyset \\
\mathbf{V}(L,P) = \emptyset & \mathbf{V}(P,P) = \{\mathrm{id}_P\} & \mathbf{V}(R,P) = \emptyset \\
\mathbf{V}(L,R) = \emptyset & \mathbf{V}(P,R) = \{q\} & \mathbf{V}(R,R) = \{\mathrm{id}_R\}
\end{array}
$$

By inspection, if two $\mathbf{V}$-morphisms f and g can be composed, that is, satisfy $\mathrm{cod}\, f = \mathrm{dom}\, g$, then either f is an identity morphism and the composition $g \circ f$ has to be g, or g is an identity morphism and the composition $g \circ f$ has to be f. So composition is uniquely determined, given the above sets of morphisms.

(b) $\mathbf{V}$ does not have a terminal object (L and P are not terminal because there are no morphisms to them from R; R is not terminal, because there is no morphism to it from L). $\mathbf{V}^{\mathrm{op}}$ does have a terminal object, namely P (there is a unique morphism from P to each of L, P and R in $\mathbf{V}$ – so it is initial in $\mathbf{V}$ and hence terminal in $\mathbf{V}^{\mathrm{op}}$).

(c) Note that $\mathbf{V}$ (and hence also $\mathbf{V}^{\mathrm{op}}$) is a category arising from a poset. Furthermore in this poset every pair of elements has a greatest lower bound (in particular the glb of L and R is P). So $\mathbf{V}$ has binary products. However, $\mathbf{V}^{\mathrm{op}}$ does not have them, since in the poset $\mathbf{V}^{\mathrm{op}}$, $\{L, R\}$ has no lower bound, let alone a greatest one.

Question 2

(a) For the eight choices of (x, y, z) in $\Sigma \times \Sigma \times \Sigma$ one can check that $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ and $(x \otimes y) \otimes z = x \otimes (y \otimes z)$. So the two binary operations are associative. Furthermore $e_M \triangleq b$ is a unit for $\oplus$ and $e_N \triangleq a$ is a unit for $\otimes$. So $M \triangleq (\Sigma, \oplus, b)$ and $N \triangleq (\Sigma, \otimes, a)$ are monoids.

(b) M and N are not isomorphic in **Mon**. For if there was an isomorphism $i : M \cong N$, then since i is in particular a monoid homomorphism, we would have $i(a \oplus a) = i(a) \otimes i(a)$ and $i(e_M) = e_N$, that is, $i(b) = a$; and since monoid isomorphisms are in particular bijective functions, the latter implies that we must also have $i(a) = b$. Hence $b = i(a) = i(a \oplus a) = i(a) \otimes i(a) = b \otimes b = a$, contradicting the fact that $a \neq b$. So no such isomorphism i can exist.

Question 3

(a) For $i = 1, 2$ we have

$$\pi_i \circ (\delta_X \circ f) = \pi_i \circ \langle \mathrm{id}_Y, \mathrm{id}_Y \rangle \circ f = \mathrm{id}_Y \circ f = f$$

and

$$\pi_i \circ ((f \times f) \circ \delta_X) = \pi_i \circ \langle f \circ \pi_1, f \circ \pi_2 \rangle \circ \delta_X = f \circ \pi_i \circ \delta_X = f \circ \mathrm{id}_X = f$$

and therefore $\delta_X \circ f = (f \times f) \circ \delta_X$, by the uniqueness part of the universal property of the product $Y \xleftarrow{\pi_1} Y \times Y \xrightarrow{\pi_2} Y$.

(b) We have

$$\pi_1 \circ (\tau_X \circ \delta_X) = \pi_1 \circ \langle \pi_2, \pi_1 \rangle \circ \delta_X = \pi_2 \circ \delta_X = \mathrm{id}_X$$

and similarly $\pi_2 \circ ((\tau_X \circ \delta_X) = \mathrm{id}_X$. Therefore $\tau_X \circ \delta_X = \langle \mathrm{id}_X, \mathrm{id}_X \rangle = \delta_X$, by the uniqueness part of the universal property of the product $X \xleftarrow{\pi_1} X \times X \xrightarrow{\pi_2} X$.

(c) We have

$$\pi_1 \circ (\tau_X \circ \tau_X) = \pi_1 \circ \langle \pi_2, \pi_1 \rangle \circ \tau_X = \pi_2 \circ \tau_X = \pi_1$$

and similarly $\pi_2 \circ (\tau_X \circ \tau_X) = \pi_2$. Therefore $\tau_X \circ \tau_X = \langle \pi_1, \pi_2 \rangle = \mathrm{id}_{X \times X}$, by the uniqueness part of the universal property of the product $X \xleftarrow{\pi_1} X \times X \xrightarrow{\pi_2} X$.

Question 4

(a) Given $k_1, k_2 : Z \rightrightarrows X$ with $e \circ k_1 = e \circ k_2$, we have to show $k_1 = k_2$. Putting $h \triangleq e \circ k_1 = e \circ k_2$, we have $f \circ h = (f \circ e) \circ k_1 = (g \circ e) \circ k_1 = g \circ h$ and $e \circ k = h$ for both $k = k_1$ and $k = k_2$; so by the uniqueness part of the property of being an equalizer, $k_1 = k_2$.

(b) The morphism f has equal compositions with both $f \circ g$ and id_Y, since $(f \circ g) \circ f = f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f$. If for some h we have $(f \circ g) \circ h = id_Y \circ h$, then $h = f \circ (g \circ h)$; and $g \circ h$ is the unique such morphism, because if $k : Z \to X$ also satisfies $h = f \circ k$, then $k = \mathrm{id}_X \circ k = (g \circ f) \circ k = g \circ (f \circ k) = g \circ h$.

(c) The equalizer of $f, g \in \mathbf{Set}(X, Y)$ is the inclusion $e : E \triangleq \{x \in X \mid f\,x = g\,x\} \hookrightarrow X$; in other words $e \in \mathbf{Set}(E, X)$ is the function $\{(x, x) \mid x \in E\}$.

For if $h \in \mathbf{Set}(Z, X)$ satisfies $f \circ h = g \circ h$, then for all $z \in Z$, $h\,z \in E$; so h factors through the inclusion $e : E \hookrightarrow X$, that is $h = e \circ k$, where $k \in \mathbf{Set}(Z, E)$ is the function $\{(z, h\,z) \mid z \in Z\}$; and it does so uniquely because inclusions, being injective functions, are monomorphisms in **Set**.

Question 5

(a) (X, id_X) is a terminal object in $\mathbf{C}/X$, because for any object (A, p) we have

$$p \in \mathbf{C}/X((A, p), (X, \mathrm{id}_X))$$

(since $\mathrm{id}_X \circ p = p$); and for any $q \in \mathbf{C}/X((A, p), (X, \mathrm{id}_X))$ we have $\mathrm{id}_X \circ q = p$ (by definition of morphisms in $\mathbf{C}/X$) and hence $q = p$.

(b) The product of (A, p) and (B, q) in $\mathbf{Set}/X$ is

$$(A, p) \xleftarrow{\pi_1} (P, r) \xrightarrow{\pi_2} (B, q)$$

where $P \triangleq \{(a,b) \in A \times B \mid p\,a = q\,b\}$ and for all $(a,b) \in P$

$$r(a,b) \triangleq p\,a = q\,b \qquad \pi_1(a,b) \triangleq a \qquad \pi_2(a,b) \triangleq b$$

For if we have $(A,p) \xleftarrow{f} (Y,s) \xrightarrow{g} (B,q)$ in $\mathbf{Set}/X$, then $\langle f,g\rangle : Y \to A \times B$ factors through the subset $P \subseteq A \times B$ (since for all $y \in Y$, $p(f\,y) = s\,y = q(g\,y)$) to give a morphism $\langle f,g\rangle : (Y,s) \to (P,r)$ with $\pi_1 \circ \langle f,g\rangle = f$ and $\pi_2 \circ \langle f,g\rangle = g$. It is unique with this property, since if $h : (Y,s) \to (P,r)$ also satisfies $\pi_1 \circ h = f$ and $\pi_2 \circ h = g$, then for all $y \in Y$, $h\,y = (f\,y, g\,y) = \langle f,g\rangle\, y$, so that $h = \langle f,g\rangle$.

Question 6

(a) The product of X and Y in $\mathbf{C}$ is their coproduct in $\mathbf{Set}$, which is the disjoint union

$$X \uplus Y = \{(x,0) \mid x \in X\} \cup \{(y,1) \mid y \in Y\}$$

together with the functions $\mathrm{inl} \in \mathbf{Set}(X, X \uplus Y)$ and $\mathrm{inr} \in \mathbf{Set}(Y, X \uplus Y)$ that respectively map $x \in X$ to $(x,0) \in X \uplus Y$ and $y \in Y$ to $(y,1) \in X \uplus Y$.

(b) Consider the one-element set $1 = \{0\}$ as an object of $\mathbf{C}$. If the exponential 1^1 existed in $\mathbf{C}$, there would be a bijection $\mathbf{C}(1 \times 1, 1) \cong \mathbf{C}(1, 1^1)$. But from part (a)

$$\mathbf{C}(1 \times 1, 1) \triangleq \mathbf{Set}(1, 1 \uplus 1)$$

is a two-element set, whereas

$$\mathbf{C}(1, 1^1) \triangleq \mathbf{Set}(1^1, 1)$$

has exactly one element no matter what set 1^1 is. Thus for any set X, the sets $\mathbf{C}(1 \times 1, 1)$ and $\mathbf{C}(1, X)$ cannot be in bijection and therefore the exponential 1^1 of 1 and 1 in $\mathbf{C}$ cannot exist.

Question 7 Recall that the semantics of STLC types and terms in a ccc depends upon giving an interpretation function M mapping ground types to objects and constants to global sections. Since a pure term contains no constants, its meaning in the ccc only depends on how the ground types involved are mapped to objects in the ccc. If there were a pure term t satisfying $\diamond \vdash t : ((G \to G') \to G) \to G$, then for any interpretation M of the ground types in a cartesian closed category $\mathbf{C}$, we would get a morphism

$$M[\![\diamond \vdash t : ((G \to G') \to G) \to G]\!] \in \mathbf{C}(\top, X^{(X^{(Y^X)})}) \tag{3}$$

where $X = M(G)$ and $Y = M(G')$.

But consider when $\mathbf{C}$ is the cartesian closed preorder given by the unit interval $[0,1]$ with the usual order relation. Recall that in this ccc, the terminal object $\top$ is $1 \in [0,1]$; and given $X, Y \in [0,1]$ their exponential (Heyting implication) Y^X is

$$Y^X = \begin{cases} 1 & \text{if } X \leq Y \\ Y & \text{otherwise} \end{cases}$$

If (3) holds in this $\mathbf{C}$, then $1 \leq X^{(X^{(Y^X)})}$, that is $X^{(X^{(Y^X)})} = 1$. But we can take M to map G to $\frac{1}{2}$ and G' to 0, in which case we get $Y^X = 0$, so $X^{(Y^X)} = 1$ and hence $X^{(X^{(Y^X)})} = \frac{1}{2} \neq 1$. Therefore there can be no such pure term t.