2016/17 MPhil ACS / CST Part III
Category Theory and Logic (L108)
Exercise Sheet 3 – Solution Notes

Question 1 Recalling the isomorphism $\top \times X \cong X$ from question 2 on Exercise Sheet 2, define

$$\ulcorner f \urcorner = \mathrm{cur}(\top \times X \xrightarrow[\cong]{\pi_2} X \xrightarrow{f} Y)$$
$$\overline{g} = X \xrightarrow[\cong]{\langle\langle\rangle,\mathrm{id}_X\rangle} \top \times X \xrightarrow{e\times\mathrm{id}_X} Y^X \times X \xrightarrow{\mathrm{app}} Y$$

Thus

$$\begin{aligned}
\ulcorner \overline{g} \urcorner &= \mathrm{cur}(\mathrm{app} \circ (g \times \mathrm{id}_X) \circ \langle\langle\rangle, \mathrm{id}_X\rangle \circ \pi_2) \\
&= \mathrm{cur}(\mathrm{app} \circ (g \times \mathrm{id}_X)) \quad \text{since } \pi_2 : \top \times X \to X \text{ is an iso with inverse } \langle\langle\rangle, \mathrm{id}_X\rangle \\
&= g \quad \text{by the uniqueness part of the universal property of exponentials}
\end{aligned}$$

and

$$\begin{aligned}
\overline{\ulcorner f \urcorner} &= \mathrm{app} \circ (\mathrm{cur}(f \circ \pi_2) \times \mathrm{id}_X) \circ \langle\langle\rangle, \mathrm{id}_X\rangle \\
&= f \circ \pi_2 \circ \langle\langle\rangle, \mathrm{id}_X\rangle \quad \text{by definition of } \mathrm{cur}(f \circ \pi_2) \\
&= f \quad \text{since } \pi_2 : \top \times X \to X \text{ is an iso with inverse } \langle\langle\rangle, \mathrm{id}_X\rangle
\end{aligned}$$

Question 2 By definition, $\mathrm{cur}(\mathrm{app})$ is the unique morphism $f \in \mathbf{C}(Y^X, Y^X)$ satisfying $\mathrm{app} \circ (f \times \mathrm{id}_X) = \mathrm{app}$. But from Exercise Sheet 2 question 1c, we have $\mathrm{id}_{Y^X} \times \mathrm{id}_X = \mathrm{id}_{Y^X \times X}$ and hence $\mathrm{app} \circ (f \times \mathrm{id}_X) = \mathrm{app}$ also holds when $f = \mathrm{id}_{Y^X}$. Therefore $\mathrm{id}_{Y^X} = \mathrm{cur}(\mathrm{app})$.

Question 3 Note that

$$\begin{aligned}
\mathrm{app} \circ (((\mathrm{cur}\, f) \circ g) \times \mathrm{id}_X) &= \mathrm{app} \circ (\mathrm{cur}\, f \times \mathrm{id}_X) \circ (g \times \mathrm{id}_X) && \text{by Ex. Sh. 2, question 1c} \\
&= f \circ (g \times \mathrm{id}_X) && \text{by definition of cur } f
\end{aligned}$$

and therefore $(\mathrm{cur}\, f) \circ g = \mathrm{cur}(f \circ (g \times \mathrm{id}_X))$, by the uniqueness part of the universal property of exponentials.

Question 4

(a) $(\mathrm{id}_Y)^X \triangleq \mathrm{cur}(\mathrm{id}_Y \circ \mathrm{app}) = \mathrm{cur}(\mathrm{app}) = \mathrm{id}_{Y^X}$, by question 2.

(b)
$$\begin{aligned}
\mathrm{app} \circ ((g^X \circ \mathrm{cur}\, f) \times \mathrm{id}_X) &= \mathrm{app} \circ (g^X \times \mathrm{id}_X) \circ (\mathrm{cur}\, f \times \mathrm{id}_X) && \text{by Ex. Sh. 2, question 1c} \\
&= g \circ \mathrm{app} \circ (\mathrm{cur}\, f \times \mathrm{id}_X) && \text{by definition of } g^X \\
&= g \circ f && \text{by definition of cur } f
\end{aligned}$$

and therefore $g^X \circ \mathrm{cur}\, f = \mathrm{cur}(g \circ f)$, by the uniqueness part of the universal property of exponentials.

(c)
$$\begin{aligned}
g^X \circ f^X &= g^X \circ \mathrm{cur}(f \circ \mathrm{app}) && \text{by definition of } f^X \\
&= \mathrm{cur}(g \circ f \circ \mathrm{app}) && \text{by part (b)} \\
&\triangleq (g \circ f)^X
\end{aligned}$$

Question 5

(a) $X^{\mathrm{id}_Y} \triangleq \mathrm{cur}(\mathrm{app} \circ (\mathrm{id}_{X^Y} \times \mathrm{id}_Y)) = \mathrm{cur}(\mathrm{app} \circ \mathrm{id}_{X^Y \times Y}) = \mathrm{cur}(\mathrm{app}) = \mathrm{id}_{X^Y}$, by question 2.

(b)
$$
\begin{aligned}
\mathrm{app} \circ ((Z^g \circ \mathrm{cur}\, f) \times \mathrm{id}_W) &= \mathrm{app} \circ (Z^g \times \mathrm{id}_W) \circ (\mathrm{cur}\, f \times \mathrm{id}_W) && \text{by Ex.Sh. 2, question 1c} \\
&= \mathrm{app} \circ (\mathrm{id}_Y \times g) \circ (\mathrm{cur}\, f \times \mathrm{id}_W) && \text{by definition of } Z^g \\
&= \mathrm{app} \circ (\mathrm{cur}\, f \times \mathrm{id}_X) \circ (\mathrm{id}_Y \times g) && \text{by Ex.Sh. 2, question 1c} \\
&= f \circ (\mathrm{id}_Y \times g) && \text{by definition of } \mathrm{cur}\, f
\end{aligned}
$$

and therefore $Z^g \circ \mathrm{cur}\, f = \mathrm{cur}(f \circ (\mathrm{id}_Y \times g))$, by the uniqueness part of the universal property of exponentials.

(c)
$$
\begin{aligned}
X^u \circ X^v &= X^u \circ \mathrm{cur}(\mathrm{app} \circ (\mathrm{id} \times v)) && \text{by definition of } X^v \\
&= \mathrm{cur}(\mathrm{app} \circ (\mathrm{id} \times v) \circ (\mathrm{id} \times u)) && \text{by part (b)} \\
&= \mathrm{cur}(\mathrm{app} \circ (\mathrm{id} \times (v \circ u))) && \text{by Ex.Sh. 2, question 1c} \\
&\triangleq X^{(v \circ u)}
\end{aligned}
$$

Question 6 The universal property of the coproduct $X + Y$ says that for all $f \in \mathbf{C}(X, Z)$ and $g \in \mathbf{C}(Y, Z)$ there is a unique morphism $[f, g] \in \mathbf{C}(X + Y, Z)$ with $[f, g] \circ \mathrm{inl}_{X,Y} = f$ and $[f, g] \circ \mathrm{inr}_{X,Y} = g$. Given objects $X, Y, Z \in \mathbf{C}$, from

$$
\begin{aligned}
\mathrm{cur}(\mathrm{inl}_{Y \times X, Z \times X}) &: Y \to ((Y \times X) + (Z \times X))^X \\
\mathrm{cur}(\mathrm{inr}_{Y \times X, Z \times X}) &: Z \to ((Y \times X) + (Z \times X))^X
\end{aligned}
$$

we get

$$[\mathrm{cur}(\mathrm{inl}_{Y \times X, Z \times X}), \mathrm{cur}(\mathrm{inr}_{Y \times X, Z \times X})] : Y + Z \to ((Y \times X) + (Z \times X))^X$$

and hence

$$i \triangleq \mathrm{app} \circ ([\mathrm{cur}(\mathrm{inl}_{Y \times X, Z \times X}), \mathrm{cur}(\mathrm{inr}_{Y \times X, Z \times X})] \times \mathrm{id}_X) \in \mathbf{C}((Y + Z) \times X, (Y \times X) + (Z \times X))$$

In the other direction, define

$$j \triangleq [\mathrm{inl}_{Y,Z} \times \mathrm{id}_X, \mathrm{inr}_{Y,Z} \times \mathrm{id}_X] \in \mathbf{C}((Y \times X) + (Z \times X), (Y + Z) \times X)$$

To see that $i \circ j = \mathrm{id}$, note that

$$
\begin{aligned}
i \circ j \circ \mathrm{inl} &= i \circ (\mathrm{inl} \times \mathrm{id}) && \text{by definition of } j \\
&= \mathrm{app} \circ ([\mathrm{cur}\,\mathrm{inl}, \mathrm{cur}\,\mathrm{inr}] \times \mathrm{id}) \circ (\mathrm{inl} \times \mathrm{id}) && \text{by definition of } i \\
&= \mathrm{app} \circ (([\mathrm{cur}\,\mathrm{inl}, \mathrm{cur}\,\mathrm{inr}] \circ \mathrm{inl}) \times \mathrm{id}) && \text{by Ex.Sh. 2, question 1c} \\
&= \mathrm{app} \circ (\mathrm{cur}\,\mathrm{inl} \times \mathrm{id}) && \text{by definition of } [\_, \_] \\
&= \mathrm{inl} && \text{by definition of } \mathrm{cur}\,\_ \\
&= \mathrm{id} \circ \mathrm{inl}
\end{aligned}
$$

and similarly, $i \circ j \circ \mathrm{inr} = \mathrm{id} \circ \mathrm{inr}$; therefore by the uniqueness part of the universal property for coproducts we have $i \circ j = \mathrm{id}$. To see that $j \circ i = \mathrm{id}$, note that

$$
\begin{aligned}
\mathrm{cur}(j \circ i) &= j^X \circ \mathrm{cur}\, i && \text{by (3)}\\
&= j^X \circ [\mathrm{cur}\,\mathrm{inl}, \mathrm{cur}\,\mathrm{inr}] && \text{by definition of } i\\
&= [j^X \circ \mathrm{cur}\,\mathrm{inl}, j^X \circ \mathrm{cur}\,\mathrm{inr}] && \text{by the dual of property (1) for products from Ex.Sh. 2}\\
&= [\mathrm{cur}(j \circ \mathrm{inl}), \mathrm{cur}(j \circ \mathrm{inr})] && \text{by (3)}\\
&= [\mathrm{cur}(\mathrm{inl} \times \mathrm{id}), \mathrm{cur}(\mathrm{inr} \times \mathrm{id})] && \text{by definition of } j\\
&= [(\mathrm{cur}\,\mathrm{id}) \circ \mathrm{inl}, (\mathrm{cur}\,\mathrm{id}) \circ \mathrm{inr}] && \text{by (1)}\\
&= (\mathrm{cur}\,\mathrm{id}) \circ [\mathrm{inl}, \mathrm{inr}] && \text{by the dual of property (1) for products from Ex.Sh. 2}\\
&= (\mathrm{cur}\,\mathrm{id}) \circ \mathrm{id} && \text{by uniqueness part of univ. property of coproducts}\\
&= \mathrm{cur}\,\mathrm{id}
\end{aligned}
$$

and hence $j \circ i = \mathrm{app}(\mathrm{cur}(j \circ i) \times \mathrm{id}) = \mathrm{app}(\mathrm{cur}\,\mathrm{id} \times \mathrm{id}) = \mathrm{id}$.

Question 7

(a) IPL proof tree

$$
\dfrac{\dfrac{\dfrac{}{\psi \vdash \psi}\,(\text{AX})}{\psi, \varphi \Rightarrow \psi \vdash \psi}\,(\text{WK})}{\psi \vdash (\varphi \Rightarrow \psi) \Rightarrow \psi}\,(\Rightarrow\text{I})
$$

STLC typing judgement $\quad y : \psi \vdash \lambda f : \varphi \Rightarrow \psi.\, y : (\varphi \Rightarrow \psi) \Rightarrow \psi$

(b) IPL proof tree

$$
\dfrac{\dfrac{\dfrac{}{\varphi, \varphi \Rightarrow \psi \vdash \varphi \Rightarrow \psi}\,(\text{AX}) \qquad \dfrac{\dfrac{}{\varphi \vdash \varphi}\,(\text{AX})}{\varphi, \varphi \Rightarrow \psi \vdash \varphi}\,(\text{WK})}{\varphi, \varphi \Rightarrow \psi \vdash \psi}\,(\Rightarrow\text{E})}{\varphi \vdash (\varphi \Rightarrow \psi) \Rightarrow \psi}\,(\Rightarrow\text{I})
$$

STLC typing judgement $\quad y : \psi \vdash \lambda f : \varphi \Rightarrow \psi.\, f\, x : (\varphi \Rightarrow \psi) \Rightarrow \psi$

(c) IPL proof tree, where $\theta \triangleq ((\varphi \Rightarrow \psi) \Rightarrow \psi) \Rightarrow \psi$

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\theta \vdash \theta}\,(\text{AX})}{\theta, \varphi \vdash \theta}\,(\text{WK}) \qquad \dfrac{\dfrac{\dfrac{}{\theta, \varphi, \varphi \Rightarrow \psi \vdash \varphi \Rightarrow \psi}\,(\text{AX}) \qquad \dfrac{\dfrac{}{\theta, \varphi \vdash \varphi}\,(\text{AX})}{\theta, \varphi, \varphi \Rightarrow \psi \vdash \varphi}\,(\text{WK})}{\theta, \varphi, \varphi \Rightarrow \psi \vdash \psi}\,(\Rightarrow\text{E})}{\theta, \varphi \vdash (\varphi \Rightarrow \psi) \Rightarrow \psi}\,(\Rightarrow\text{I})}{\theta, \varphi \vdash \psi}\,(\Rightarrow\text{E})}{\theta \vdash \varphi \Rightarrow \psi}\,(\Rightarrow\text{I})
$$

STLC typing judgement $\quad f : \theta \vdash \lambda x : \varphi.\, f(\lambda g : \varphi \Rightarrow \psi.\, g\, x) : \varphi \Rightarrow \psi$

Question 9

(a) $s \triangleq \lambda a : A.\, \lambda b : B.\, x\,(a, b)$

$t \triangleq \lambda c : A \times B.\, y\,(\mathrm{fst}\, c)\,(\mathrm{snd}\, c)$

Proof of (6), where $\Gamma \triangleq \diamond, x : (A \times B) \to C, a : A, b : B$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\cdots}\,(\text{VAR})}{\cdots}\,(\text{VAR}')}{\Gamma \vdash x : A \times B \to C}\,(\text{VAR}') \qquad \dfrac{\dfrac{\dfrac{}{\cdots}\,(\text{VAR})}{\Gamma \vdash a : A}\,(\text{VAR}') \qquad \dfrac{}{\Gamma \vdash b : B}\,(\text{VAR})}{\Gamma \vdash (a, b) : A \times B}\,(\text{PAIR})}{\dfrac{\Gamma \vdash x(a, b) : C}{\diamond, x : (A \times B) \to C \vdash s : A \to (B \to C)}\,(\lambda^2)}\,(\text{APP})$$

Proof of (7), where $\Gamma' \triangleq \diamond, y : A \to (B \to C), c : A \times B$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\cdots}\,(\text{VAR})}{\Gamma' \vdash y : A \to (B \to C)}\,(\text{VAR}') \qquad \dfrac{\dfrac{}{\Gamma' \vdash c : A \times B}\,(\text{VAR})}{\Gamma' \vdash \mathrm{fst}\, c : A}\,(\text{FST})}{\Gamma' \vdash y\,(\mathrm{fst}\, c) : B \to C}\,(\text{APP}) \qquad \dfrac{\dfrac{}{\Gamma' \vdash c : A \times B}\,(\text{VAR})}{\Gamma' \vdash \mathrm{snd}\, c : B}\,(\text{SND})}{\dfrac{\Gamma' \vdash y\,(\mathrm{fst}\, c)\,(\mathrm{snd}\, c) : C}{\diamond, y : A \to (B \to C) \vdash t : (A \times B) \to C}\,(\lambda)}\,(\text{APP})$$

Proof of (8) (not laid out as a tree):

$$\begin{aligned}
t[s/y] &\triangleq \lambda c : A \times B.\, (\lambda a : A. \lambda b : B.\, x\,(a, b))\,(\mathrm{fst}\, c)\,(\mathrm{snd}\, c) && \\
&=_{\beta\eta} \lambda c : A \times B.\, x\,(\mathrm{fst}\, c, \mathrm{snd}\, c) && \beta\text{-conversion, twice} \\
&=_{\beta\eta} \lambda c : A \times B.\, x\, c && \eta\text{-conv. at type } A \times B \\
&=_{\beta\eta} x && \eta\text{-conv. at type } (A \times B) \to C
\end{aligned}$$

Proof of (9) (not laid out as a tree):

$$\begin{aligned}
s[t/x] &\triangleq \lambda a : A. \lambda b : B.\, (\lambda c : A \times B.\, y\,(\mathrm{fst}\, c)\,(\mathrm{snd}\, c))\,(a, b) && \\
&=_{\beta\eta} \lambda a : A. \lambda b : B.\, y\,(\mathrm{fst}(a, b))\,(\mathrm{snd}(a, b)) && \beta\text{-conversion,} \\
&=_{\beta\eta} \lambda a : A. \lambda b : B.\, y\, a\, b && \beta\text{-conversion, twice} \\
&=_{\beta\eta} \lambda a : A.\, y\, a && \eta\text{-conv. at type } B \to C \\
&=_{\beta\eta} y && \eta\text{-conv. at type } A \to (B \to C)
\end{aligned}$$

(b) In part (9a), if we take A, B, C to be ground types that are interpreted in $\mathbf{C}$ by the objects X, Y, Z, then the interpretations of (6) and (7) give morphisms

$$f \triangleq \left(Z^{X \times Y} \cong \top \times Z^{X \times Y} \xrightarrow{M[\![\diamond, x:(A \times B) \to C \vdash s : A \to (B \to C)]\!]} (Z^Y)^X \right)$$

$$g \triangleq \left((Z^Y)^X \cong \top \times (Z^Y)^X \xrightarrow{M[\![\diamond, y:A \to (B \to C) \vdash t : (A \times B) \to C]\!]} Z^{X \times Y} \right)$$

with the required domains and codomains. Furthermore, by the semantics of substitution and the Soundness Theorem for STLC, (8) implies

$$\begin{aligned}
g \circ f &= \left(Z^{X \times Y} \cong \top \times Z^{X \times Y} \xrightarrow{M[\![\diamond, x:(A\times B)\text{->}C \vdash t[s/y]:(A\times B)\text{->}C]\!]} Z^{X \times Y} \right) \\
&= \left(Z^{X \times Y} \cong \top \times Z^{X \times Y} \xrightarrow{M[\![\diamond, x:(A\times B)\text{->}C \vdash x:(A\times B)\text{->}C]\!]} Z^{X \times Y} \right) \\
&= \left(Z^{X \times Y} \cong \top \times Z^{X \times Y} \xrightarrow{\pi_2} Z^{X \times Y} \right) \\
&= \mathrm{id}_{Z^{(X \times Y)}}
\end{aligned}$$

and similarly (9) implies $f \circ g = \mathrm{id}_{(Z^Y)^X}$.

For the record, f and g can be described using the structure of a cartesian closed category as follows:

$$\begin{aligned}
f &\triangleq \mathrm{cur}\left(\mathrm{cur}\left((Z^{(X\times Y)} \times X) \times Y \xrightarrow{\langle \pi_1 \circ \pi_1, \langle \pi_2 \circ \pi_1, \pi_2 \rangle \rangle} Z^{(X \times Y)} \times (X \times Y) \xrightarrow{\mathrm{app}} Z \right) \right) \\
g &\triangleq \mathrm{cur}\left((Z^Y)^X \times (X \times Y) \xrightarrow{\langle \langle \pi_1, \pi_1 \circ \pi_2 \rangle, \pi_2 \circ \pi_2 \rangle} ((Z^Y)^X \times X) \times Y \xrightarrow{\mathrm{app} \times \mathrm{id}_Y} Z^Y \times Y \xrightarrow{\mathrm{app}} Z \right)
\end{aligned}$$

However, it is quite tedious to use these descriptions to verify that f and g are mutually inverse.

Question 10 The STLC terms you need to use are

$$\begin{aligned}
\diamond, x : \texttt{unit -> } A &\vdash x\ \texttt{()} : A \\
\diamond, y : A &\vdash \lambda z : \texttt{unit}.\, y : \texttt{unit -> } A
\end{aligned}$$