2016/17 MPhil ACS / CST Part III
Category Theory and Logic (L108)
Exercise Sheet 2 – Solution Notes

Question 1

(a) For $i = 1, 2$ we have $\pi_i \circ (\langle g_1, g_2 \rangle \circ f) = (\pi_i \circ \langle g_1, g_2 \rangle) \circ f = g_i \circ f = \pi_i \circ \langle g_1 \circ f, g_2 \circ f \rangle$ and hence by the uniqueness part of the universal property for the product $Z_1 \times Z_2$, it is the case that $\langle g_1, g_2 \rangle \circ f = \langle g_1 \circ f, g_2 \circ f \rangle$.

(b)
$$\begin{aligned}
(f_1 \times f_2) \circ \langle g_1, g_2 \rangle &\triangleq \langle f_1 \circ \pi_1, f_2 \circ \pi_2 \rangle \circ \langle g_1, g_2 \rangle \\
&= \langle (f_1 \circ \pi_1) \circ \langle g_1, g_2 \rangle, (f_2 \circ \pi_2) \circ \langle g_1, g_2 \rangle \rangle \quad \text{(by part (a))} \\
&= \langle f_1 \circ (\pi_1 \circ \langle g_1, g_2 \rangle), f_2 \circ (\pi_2 \circ \langle g_1, g_2 \rangle) \rangle \\
&= \langle f_1 \circ g_1, f_2 \circ g_2 \rangle
\end{aligned}$$

(c)
$$\begin{aligned}
(h_1 \times h_2) \circ (k_1 \times k_2) &\triangleq (h_1 \times h_2) \circ \langle k_1 \circ \pi_1, k_2 \circ \pi_2 \rangle \\
&= \langle h_1 \circ (k_1 \circ \pi_1), h_2 \circ (k_2 \circ \pi_2) \rangle \quad \text{(by part (b))} \\
&= \langle (h_1 \circ k_1) \circ \pi_1, (h_2 \circ k_2) \circ \pi_2 \rangle \\
&\triangleq (h_1 \circ k_1) \times (h_2 \circ k_2)
\end{aligned}$$

For the second identity, note that $\mathrm{id}_X \times \mathrm{id}_Y \triangleq \langle \mathrm{id}_X \circ \pi_1, \mathrm{id}_Y \circ \pi_2 \rangle = \langle \pi_1, \pi_2 \rangle$. Since $\pi_i \circ \mathrm{id}_{X \times Y} = \pi_i = \pi_i \circ \langle \pi_1, \pi_2 \rangle$, by the uniqueness part of the universal property for the product $X \times Y$, we have $\mathrm{id}_{X \times Y} = \langle \pi_1, \pi_2 \rangle$. Therefore $\mathrm{id}_X \times \mathrm{id}_Y = \langle \pi_1, \pi_2 \rangle = \mathrm{id}_{X \times Y}$.

Question 2 Define

$$\begin{aligned}
\alpha_{X,Y,Z} &\triangleq \langle \mathrm{id}_X \times \pi_1, \pi_2 \circ \pi_2 \rangle & \alpha^{-1}_{X,Y,Z} &\triangleq \langle \pi_1 \circ \pi_1, \pi_2 \times \mathrm{id}_Z \rangle \\
\lambda_X &\triangleq \pi_2 & \lambda^{-1}_X &\triangleq \langle \langle \rangle_X, \mathrm{id}_X \rangle \\
\rho_X &\triangleq \pi_1 & \rho^{-1}_X &\triangleq \langle \mathrm{id}_X, \langle \rangle_X \rangle \\
\tau_{X,Y} &\triangleq \langle \pi_2, \pi_1 \rangle & \tau^{-1}_{X,Y} &\triangleq \langle \pi_2, \pi_1 \rangle
\end{aligned}$$

Then we have:

$$\begin{aligned}
\alpha_{X,Y,Z} \circ \alpha^{-1}_{X,Y,Z} &\triangleq \langle \mathrm{id}_X \times \pi_1, \pi_2 \circ \pi_2 \rangle \circ \langle \pi_1 \circ \pi_1, \pi_2 \times \mathrm{id}_Z \rangle \\
&= \langle (\mathrm{id}_X \times \pi_1) \circ \langle \pi_1 \circ \pi_1, \pi_2 \times \mathrm{id}_Z \rangle, \pi_2 \circ \pi_2 \circ \langle \pi_1 \circ \pi_1, \pi_2 \times \mathrm{id}_Z \rangle \rangle && \text{by (1)} \\
&= \langle \langle \pi_1 \circ \pi_1, \pi_1 \circ (\pi_2 \times \mathrm{id}_Z) \rangle, \pi_2 \circ \pi_2 \circ \langle \pi_1 \circ \pi_1, \pi_2 \times \mathrm{id}_Z \rangle \rangle && \text{by (3)} \\
&= \langle \langle \pi_1 \circ \pi_1, \pi_2 \circ \pi_1 \rangle, \pi_2 \rangle && \text{by (2)} \\
&= \langle \langle \pi_1, \pi_2 \rangle \circ \pi_1, \pi_2 \rangle && \text{by (1)}
\end{aligned}$$

and since $\langle \pi_1, \pi_2 \rangle = \mathrm{id}$, we get $\alpha_{X,Y,Z} \circ \alpha^{-1}_{X,Y,Z} = \langle \mathrm{id}_{X\times Y} \circ \pi_1, \pi_2 \rangle = \mathrm{id}_{(X\times Y)\times Z}$. Similar tedious calculations using the properties from question 1 give

$$\begin{aligned}
\alpha^{-1} \circ \alpha &= \mathrm{id} \\
\lambda \circ \lambda^{-1} &= \mathrm{id} \\
\lambda^{-1} \circ \lambda &= \mathrm{id} \\
\rho \circ \rho^{-1} &= \mathrm{id} \\
\rho^{-1} \circ \rho &= \mathrm{id} \\
\tau \circ \tau^{-1} &= \mathrm{id} \\
\tau^{-1} \circ \tau &= \mathrm{id}
\end{aligned}$$

Question 3 Regarding M as a category with a single object, $*$ say, it suffices to show that $* \xleftarrow{p_1} * \xrightarrow{p_2} *$ is a product in M, that is: for all $x, y \in M$, there is a unique $z \in M$ with $p_1 \cdot z = x$ and $p_2 \cdot z = y$. But $\langle x, y \rangle$ is such a z; and it is unique since if $p_1 \cdot z = x$ and $p_2 \cdot z = y$,then $z = e \cdot z = \langle p_1, p_2 \rangle \cdot z = \langle p_1 \cdot z, p_2 \cdot z \rangle = \langle x, y \rangle$.

Question 4

(a) Recall that the product monoid $(M, \cdot_M, e_M) \times (M, \cdot_M, e_M)$ is $(M \times M, \cdot, (e_M, e_M))$ where the binary operation $\_ \cdot \_ : (M \times M) \times (M \times M) \to (M \times M)$ is given by:

$$(x, y) \cdot (x', y') = (x \cdot_M x', y \cdot_M y')$$

Thus for all $x, x', y, y' \in M$ we have

$$\begin{aligned}
m((x,y) \cdot (x',y')) &= m(x \cdot_M x', y \cdot_M y') \\
&\triangleq (x \cdot_M x') \cdot_M (y \cdot_M y') \\
&= x \cdot_M ((x' \cdot_M y) \cdot_M y') && \text{since } \cdot_M \text{ is associative} \\
&= x \cdot_M ((y \cdot_M x') \cdot_M y') && \text{since } \cdot_M \text{ is commutative} \\
&= (x \cdot_M y) \cdot_M (x' \cdot_M y') && \text{since } \cdot_M \text{ is associative} \\
&\triangleq m(x,y) \cdot_M m(x',y') \\
m(e_M, e_M) &\triangleq e_M \cdot_M e_M \\
&= e_M && \text{since } e_M \text{ is a unit for } \cdot_M
\end{aligned}$$

so m is a monoid morphism; and u is one too because $u(0 \cdot 0) = u(0) = e_M = e_M \cdot_M e_M = u(0) \cdot_M u(0)$.

To see that m and u make M into a monoid object in **Mon**, just note that diagram (14) commutes because $(\forall x, y, z \in M)\ x \cdot_M (y \cdot_M z) = (x \cdot_M y) \cdot_M z$, (15) commutes because $(\forall x \in M)\ e_M \cdot_M x = x$ and (16) commutes because $(\forall x \in M)\ x \cdot_M e_M = x$.

(b) Suppose we are given monoid morphisms $m \in \mathbf{Mon}(M \times M, M)$ and $u \in \mathbf{Mon}(\top, M)$ that make (14)–(16) commute. Since u is a monoid morphism we have $u(0) = e_M$ and therefore from the commutation of (15) and (16) we deduce that for all $x \in M$

$$m(e_M, x) = x = m(x, e_M) \tag{20}$$

Now by definition of the monoid multiplication operation for the product monoid $(M, \cdot_M, e_M) \times (M, \cdot_M, e_M)$ we have

$$(x, e_M) \cdot (e_M, y) = (x \cdot_M e_M, e_M \cdot_M y) = (x, y) = (e_M \cdot_M x, y \cdot_M e_M) = (e_M, y) \cdot (x, e_M)$$

Therefore since m is a monoid homomorphism, we have

$$\begin{aligned} m(x, e_M) \cdot_M m(e_M, y) = m((x, e_M) \cdot (e_M, y)) = m(x, y) = \\ m((e_M, y) \cdot (e_M, x)) = m(e_M, y) \cdot_M m(x, e_M) \end{aligned}$$

and hence from (20) we get $x \cdot y = m(x, y) = y \cdot x$. Therefore $(M, \cdot_M, e_M)$ is abelian and the monoid object $((M, \cdot_M, e_M), m, u)$ in **Mon** coincides with the one from part (a).

Question 5

(a) If M and N are both abelian monoids, then the product operation of the monoid $M \times N$ satisfies for all $x, x' \in M$ and $y, y' \in N$

$$\begin{aligned} (x, y) \cdot (x', y') &\triangleq (x \cdot x', y \cdot y') \\ &= (x' \cdot x, y' \cdot y) && \text{since } M \text{ and } N \text{ are abelian} \\ &\triangleq (x', y') \cdot (x, y) \end{aligned}$$

so that $M \times N$ is also abelian. Therefore the universal property of $M \xleftarrow{\pi_1} M \times N \xrightarrow{\pi_2} N$ in **Mon** restricts to give the correct universal property for a product in **AbMon**.

(b) The functions

$$\begin{aligned} i(x) &\triangleq (x, e) \\ j(y) &\triangleq (e, y) \end{aligned}$$

clearly give morphisms $M \xrightarrow{i} M \times N \xleftarrow{j} N$ in **AbMon**. We show that it is a coproduct diagram. Given any morphisms $M \xrightarrow{f} P \xleftarrow{g} N$ in **AbMon**, consider the function $h : M \times N \to P$ defined by

$$h(x, y) \triangleq (f\,x) \cdot (g\,y)$$

It is a morphism in $\mathbf{AbMon}(M \times N, P)$ because $h(e, e) = (f\,e) \cdot (g\,e) = e \cdot e = e$ and

$$\begin{aligned} h((x, y) \cdot (x', y')) &\triangleq f(x \cdot x') \cdot g(y \cdot y') \\ &= (f\,x \cdot f\,x') \cdot (g\,y \cdot g\,y') && \text{since } f \text{ and } g \text{ are morphisms} \\ &= f\,x \cdot (f\,x' \cdot g\,y) \cdot g\,y' && \text{associativity} \\ &= f\,x \cdot (g\,y \cdot f\,x') \cdot g\,y' && \text{since } P \text{ is abelian} \\ &= (f\,x \cdot g\,y) \cdot (f\,x' \cdot g\,y') && \text{associativity} \\ &\triangleq h(x, y) \cdot h(x', y') \end{aligned}$$

Furthermore, since $h(i\,x) = h(x,e) = f\,x \cdot g\,e = f\,x \cdot e = f\,x$ and $h(j\,y) = h(e,y) = f\,e \cdot g\,y = e \cdot g\,y = g\,y$, we have that

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[dr, "f"'] & M \times N \arrow[d, "h"] & N \arrow[l, "j"'] \arrow[dl, "g"] \\ & P & \end{tikzcd}$$

commutes. Finally, h is the unique such morphism, since if $h' \in \mathbf{AbMon}(M \times N, P)$ also satisfies $h' \circ i = f$ and $h' \circ j = g$, then

$$h'(x,y) = h'((x,e) \cdot (e,y)) = h'(x,e) \cdot h'(e,y) = h'(i\,x) \cdot h'(j\,y) = f\,x \cdot g\,y \triangleq h(x,y).$$

so that $h' = h$.

Question 6

- *Terminal object* is $(\mathbb{N}, (\_)^+, |\_|)$, where for all $n \in \mathbb{N}$, $n^+ \triangleq n+1$ and $|n| \triangleq n$, which trivially have the required property (17). For each object $(X, (\_)^+, |\_|) \in \mathbf{Set}^\omega$, the unique morphism $(X, (\_)^+, |\_|) \to (\mathbb{N}, (\_)^+, |\_|)$ is given by $|\_|$.

- *Binary product* of $(X, (\_)^+, |\_|)$ and $(Y, (\_)^+, |\_|)$ is $(X, (\_)^+, |\_|) \xleftarrow{\pi_1} (P, (\_)^+, |\_|) \xrightarrow{\pi_2} (Y, (\_)^+, |\_|)$, where

$$\begin{aligned} P &\triangleq \{(x,y) \in X \times Y \mid |x| = |y|\} \\ (x,y)^+ &\triangleq (x^+, y^+) \\ |(x,y)| &\triangleq |x| (= |y|) \\ \pi_1(x,y) &\triangleq x \\ \pi_2(x,y) &\triangleq y \end{aligned}$$

Given morphisms $(X, (\_)^+, |\_|) \xleftarrow{f} (Z, (\_)^+, |\_|) \xrightarrow{g} (Y, (\_)^+, |\_|)$, the unique morphism $\langle f, g\rangle : (Z, (\_)^+, |\_|) \to (P, (\_)^+, |\_|)$ with $\pi_1 \circ \langle f, g\rangle = f$ and $\pi_2 \circ \langle f, g\rangle = g$ maps each $z \in Z$ to

$$\langle f, g\rangle\, z \triangleq (f\,z, g\,z)$$

(which does lie in P because $|f\,z| = |z| = |g\,z|$).

Question 7 I do not give the proof that a one-element poset is terminal in **Pre**, or that the binary product of $(P, \leq)$ and $(Q, \leq)$ in **Pre** is given by the cartesian product of underlying sets together with the partial order

$$(p_1, q_1) \leq (p_2, q_2) \triangleq p_1 \leq p_2 \wedge q_1 \leq q_2 \qquad \text{for all } p_1, p_2 \in P \text{ and } q_1, q_2 \in Q.$$

Let us show that the exponential of $(P, \leq)$ and $(Q, \leq)$ is given by:

$$P \to Q \triangleq \{f \in Q^P \mid (\forall p, p')\ p \leq p' \in P \Rightarrow f\,p \leq f\,p' \in Q\} \tag{21}$$

$$f \leq f' \in P \to Q \triangleq (\forall p \in P)\ f\,p \leq f'\,p \tag{22}$$

$$\mathrm{app}(f, p) \triangleq f\,p \tag{23}$$

Two things need checking (that we don't do here):

- (22) does define a partial order on the set (21), and
- (23) does give a monotone function.

So we have a morphism $\mathrm{app} : (P \to Q, \leq) \times (P, \leq) \to (Q, \leq)$ in **Pre** and we need to see that it has the universal property of the exponential of $(P, \leq)$ and $(Q, \leq)$.

Given $f : (R, \leq) \times (P, \leq) \to (Q, \leq)$ in **Pre**, since $f \in \mathbf{Set}(R \times P, Q)$ we have the function $\mathrm{cur}\, f \in \mathbf{Set}(R, Q^P)$, where as usual, $\mathrm{cur}\, f\, r\, p = f(r, p)$ for all $r \in R$ and $p \in P$. Note that

$$\begin{aligned} p \leq p' \in P &\Rightarrow (r, p) \leq (r, p') \in R \times P \\ &\Rightarrow \mathrm{cur}\ f\, r\, p = f(r, p) \leq f(r, p') = \mathrm{cur}\, f\, r\, p' \qquad \text{since } f \text{ is monotone} \end{aligned}$$

so that for each $r \in R$, we have $\mathrm{cur}\, f\, r \in P \to Q$. In other words, $\mathrm{cur}\, f \in \mathbf{Set}(R, P \to Q)$. Furthermore $\mathrm{cur}\, f$ is a monotone function, because

$$\begin{aligned} r \leq r' \in R &\Rightarrow (\forall p \in P)\ (r, p) \leq (r', p) \in R \times P \\ &\Rightarrow (\forall p \in P)\ \mathrm{cur}\, f\, r\, p = f(r, p) \leq f(r', p) = \mathrm{cur}\, f\, r' p \quad \text{since } f \text{ is monotone} \end{aligned}$$

Note that $\mathrm{app} \circ (\mathrm{cur}\, f \times \mathrm{id}_P) = f \in \mathbf{Pre}((R, \leq) \times (P, \leq), (Q, \leq))$, because for all $(r, p) \in R \times P$

$$(\mathrm{app} \circ (\mathrm{cur}\, f \times \mathrm{id}_P))(r, p) = \mathrm{app}((\mathrm{cur}\, f \times \mathrm{id}_P)(r, p)) = \mathrm{app}(\mathrm{cur}\, f\, r, p) = \mathrm{cur}\, f\, r\, p = f(r, p)$$

Finally, $\mathrm{cur}\, f$ is the only element $g \in \mathbf{Pre}((R, \leq), (P \to Q, \leq))$ satisfying $\mathrm{app} \circ (g \times \mathrm{id}_P) = f$, since the latter equation implies that $g\, r\, p = (\mathrm{app} \circ (g \times \mathrm{id}_P))(r, p) = f(r, p) = \mathrm{cur}\, f\, r\, p$ for all $(r, p) \in R \times P$. Hence for any $r \in R$, $g\, r$ and $\mathrm{cur}\, f\, r$ are equal functions from P to Q; and therefore $g = \mathrm{cur}\, f$.