

**2016/17 MPhil ACS / CST Part III**  
**Category Theory and Logic (L108)**  
**Exercise Sheet 4 [GRADED]**

1. [8/45 marks] Let  $\mathbf{V}$  be the category with three distinct objects  $L, P, R$  and whose only non-identity morphisms are  $p : P \rightarrow L$  and  $q : P \rightarrow R$ .
  - (a) Complete the definition of  $\mathbf{V}$  by giving the nine sets  $\mathbf{V}(X, Y)$  of morphisms between pairs of objects  $X, Y \in \{L, P, R\}$  and defining the composition operations.
  - (b) Do either of  $\mathbf{V}$  or  $\mathbf{V}^{\text{op}}$  have a terminal object?
  - (c) Do either of  $\mathbf{V}$  or  $\mathbf{V}^{\text{op}}$  have binary products? [Hint: recall that in a pre-ordered set regarded as a category, products are given by greatest lower bounds.]
2. [8/45 marks] Let  $\Sigma = \{a, b\}$  be a two-element set ( $a \neq b$ ) and let  $_{-} \oplus _{-} : \Sigma \times \Sigma \rightarrow \Sigma$  and  $_{-} \otimes _{-} : \Sigma \times \Sigma \rightarrow \Sigma$  be binary operations on  $\Sigma$  defined by the following tables:

$\oplus$	$a$	$b$	$\otimes$	$a$	$b$
$a$	$a$	$a$	$a$	$a$	$b$
$b$	$a$	$b$	$b$	$b$	$a$

(so that  $a \oplus b = a, a \otimes b = b$ , etc.)

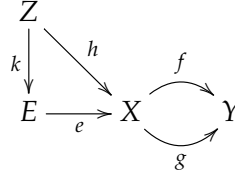
- (a) Show that for suitable choices of elements  $e_M, e_N \in \Sigma$ , there are monoids  $M = (\Sigma, \oplus, e_M)$  and  $N = (\Sigma, \otimes, e_N)$ .
  - (b) Show that  $M$  and  $N$  are not isomorphic in the category **Mon** of monoids and monoid homomorphisms. [Hint: assume they are isomorphic and derive a contradiction.]
3. [6/45 marks] Let  $\mathbf{C}$  be a category with binary products. Given a  $\mathbf{C}$ -object  $X$ , the *diagonal* morphism  $\delta_X \in \mathbf{C}(X, X \times X)$  and the *twist* morphism  $\tau_X \in \mathbf{C}(X \times X, X \times X)$  are defined by:
 

$$\delta_X \triangleq \langle \text{id}_X, \text{id}_X \rangle \tag{1}$$

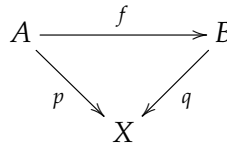
$$\tau_X \triangleq \langle \pi_2, \pi_1 \rangle \tag{2}$$

  - (a) For each  $f \in \mathbf{C}(X, Y)$ , show that  $\delta_Y \circ f = (f \times f) \circ \delta_X \in \mathbf{C}(X, Y \times Y)$  (where  $f \times f$  denotes the product of morphisms introduced in Ex. Sh. 2, question 1b).
  - (b) Show that  $\tau_X \circ \delta_X = \delta_X$ .
  - (c) Show that  $\tau_X \circ \tau_X = \text{id}_{X \times X}$ .
4. [8/45 marks] Let  $\mathbf{C}$  be a category. Given  $\mathbf{C}$ -objects  $X$  and  $Y$  and morphisms  $f, g \in \mathbf{C}(X, Y)$ , an *equalizer* for  $f$  and  $g$  is by definition a  $\mathbf{C}$ -object  $E$  and a morphism  $e \in \mathbf{C}(E, X)$  such that
  - $f \circ e = g \circ e \in \mathbf{C}(E, Y)$  and

- for all  $\mathbf{C}$ -objects  $Z$  and morphisms  $h \in \mathbf{C}(Z, X)$ , if  $f \circ h = g \circ h \in \mathbf{C}(Z, Y)$ , then there exists a unique morphism  $k \in \mathbf{C}(Z, E)$  satisfying  $e \circ k = h$ .



- Show that every equalizer is a monomorphism (see Ex. Sh. 1, question 4).
  - Suppose that  $f \in \mathbf{C}(X, Y)$  is a split monomorphism, that is, there is a morphism  $g \in \mathbf{C}(Y, X)$  with  $g \circ f = \text{id}_X$  (see Ex. Sh. 1, question 4). Show that  $f : X \rightarrow Y$  is the equalizer of the morphisms  $f \circ g$  and  $\text{id}_Y$ .
  - Show that the category **Set** of sets and functions possesses equalizers for all parallel pairs of morphisms.
5. [6/45 marks] Let  $X$  be an object of a category  $\mathbf{C}$ . The *slice category*  $\mathbf{C}/X$  is defined by:
- The objects of  $\mathbf{C}/X$  are pairs  $(A, p)$  where  $A \in \text{obj } \mathbf{C}$  and  $p \in \mathbf{C}(A, X)$ .
  - Given two such objects  $(A, p)$  and  $(B, q)$ , a morphism  $f : (A, p) \rightarrow (B, q)$  in  $\mathbf{C}/X$  is a  $\mathbf{C}$ -morphism  $f \in \mathbf{C}(A, B)$  such that  $q \circ f = p$



- Composition and identities in  $\mathbf{C}/X$  are given by those in  $\mathbf{C}$ .
- Show that  $\mathbf{C}/X$  always has a terminal object.
  - When  $\mathbf{C} = \mathbf{Set}$ , the category of sets and functions, show that  $\mathbf{Set}/X$  has binary products. [Hint: given  $(A, p), (B, q) \in \text{obj}(\mathbf{Set}/X)$ , consider a suitable subset of  $\{(a, b) \mid a \in A \wedge b \in B\}$ .]
6. [5/45 marks] Let  $\mathbf{C} = \mathbf{Set}^{\text{op}}$  be the opposite category of the category **Set** of sets and functions.
- State, without proof, what is the product in  $\mathbf{C}$  of two objects  $X$  and  $Y$ .
  - Show by example that there are objects  $X$  and  $Y$  in  $\mathbf{C}$  for which there is no exponential and hence that  $\mathbf{C}$  is not a cartesian closed category.
7. [4/45 marks] Call a term  $t$  of the Simply Typed Lambda Calculus (STLC) *pure* if it does not contain any constant symbols. Using facts about the semantics of STLC in cartesian closed categories, explain why there is no pure term  $t$  such that  $\diamond \vdash t : ((G \rightarrow G') \rightarrow G) \rightarrow G$  holds, where  $G$  and  $G'$  are distinct ground types. [Hint: consider the partially ordered set  $(\{X \in \mathbb{R} \mid 0 \leq X \leq 1\}, \leq)$  regarded as a cartesian closed category and recall that the exponential  $Y^X$  of two objects  $X$  and  $Y$  in this category satisfies  $Y^X = 1$  if  $X \leq Y$  and  $Y^X = Y$  if  $Y < X$ .]