6 Denotational Semantics (MPF)

(*a*) For monotone functions $f, f' : P \to Q$ between posets $(P, \sqsubseteq_P)$ and $(Q, \sqsubseteq_Q)$, let $f \sqsubseteq f' \overset{\text{def}}{\iff} \forall x \in P.\ f(x) \sqsubseteq_Q f'(x)$.

(*i*) Prove that the binary relation $\sqsubseteq$ is a partial order. [3 marks]

(*ii*) For monotone functions between posets $p : P' \to P$, $f, f' : P \to Q$, and $q : Q \to Q'$, prove that $f \sqsubseteq f' \implies q \circ f \circ p \sqsubseteq q \circ f' \circ p$. [1 mark]

(*b*) An *adjoint pair* $\big(f : P \to Q, g : Q \to P\big)$ is a pair of monotone functions between posets such that $\mathrm{id}_P \sqsubseteq g \circ f$ and $f \circ g \sqsubseteq \mathrm{id}_Q$.

(*i*) Let $f_1, f_2 : P \to Q$ and $g_1, g_2 : Q \to P$ be monotone functions between posets such that (f_1, g_1) and (f_2, g_2) are adjoint pairs. Prove that:

(A) $f_1 \sqsubseteq f_2 \iff g_2 \sqsubseteq g_1$ [4 marks]

(B) $f_1 = f_2 \iff g_1 = g_2$ [2 marks]

(*ii*) Let $\big(f : P \to Q, g : Q \to P\big)$ be an adjoint pair where the posets P and Q have least elements. Prove that the monotone function f is strict. [2 marks]

(*c*) (*i*) Define the notion of lub (least upper bound) of a countable increasing chain in a poset. [2 marks]

(*ii*) Let $\big(f : D \to E, g : E \to D\big)$ be an adjoint pair where each of the posets D and E is a cpo (chain complete poset). Prove that the monotone function f is continuous. [6 marks]