4 Denotational Semantics (MPF)

(*a*) (*i*) Define the contextual-equivalence relation $\Gamma \vdash M \cong_{\mathrm{ctx}} M' : \tau$ for pairs of PCF terms M, M', PCF types τ, and PCF type environments Γ. [3 marks]

(*ii*) For PCF terms M and N with respective typings $\Gamma \vdash M : \tau \to \alpha$ and $\Gamma \vdash N : \alpha \to \sigma$, let $N \circ M$ be the PCF term $\mathbf{fn}\ x : \tau.\ N\,(M\,x)$, where $x \notin \mathrm{dom}(\Gamma)$, with typing $\Gamma \vdash N \circ M : \tau \to \sigma$.

State whether or not if $\Gamma \vdash M \cong_{\mathrm{ctx}} M' : \tau \to \alpha$ and $\Gamma \vdash N \cong_{\mathrm{ctx}} N' : \alpha \to \sigma$ then $\Gamma \vdash N \circ M \cong_{\mathrm{ctx}} N' \circ M' : \tau \to \sigma$. Justify your answer. [5 marks]

(*b*) By considering the countable chain of functions $(P_n)_{n \in \mathbb{N}}$ in the function domain $(\mathbb{N}_\perp \to \mathbb{B}_\perp)$ given by

$$P_n(k) \stackrel{\text{def}}{=} \begin{cases} \mathit{false} & \text{if } k \in \mathbb{N} \text{ and } k < n \\ \perp & \text{otherwise} \end{cases} \qquad (k \in \mathbb{N}_\perp)$$

or otherwise, show that the function ε from $(\mathbb{N}_\perp \to \mathbb{B}_\perp)$ to $\mathbb{B}_\perp$ given by

$$\varepsilon(P) \stackrel{\text{def}}{=} \begin{cases} \mathit{true} & \text{if } \exists\, n \in \mathbb{N}.\ P(n) = \mathit{true} \\ \mathit{false} & \text{if } \forall\, n \in \mathbb{N}.\ P(n) = \mathit{false} \\ \perp & \text{otherwise} \end{cases} \qquad \big(P \in (\mathbb{N}_\perp \to \mathbb{B}_\perp)\big)$$

is not continuous. Argue as to whether or not ε is definable by a closed term of type $(\mathit{nat} \to \mathit{bool}) \to \mathit{bool}$ in both PCF and PCF+por. [5 marks]

(*c*) Let M be the PCF+por term

$$\begin{array}{l} \mathbf{fn}\ f : (\mathit{nat} \to \mathit{bool}) \to \mathit{bool}\,. \\ \quad \mathbf{fn}\ P : \mathit{nat} \to \mathit{bool}\,. \\ \qquad \mathbf{por}\Big(P\,\mathbf{0}\,,\ f\,\big(\mathbf{fn}\ n : \mathit{nat}.\ P\,(\mathbf{succ}(n))\big)\Big) \end{array}$$

Give an explicit description of $[\![\, \mathbf{fix}(M) \,]\!] \in \big((\mathbb{N}_\perp \to \mathbb{B}_\perp) \to \mathbb{B}_\perp\big)$. [7 marks]