2016/17 MPhil ACS / CST Part III
Category Theory and Logic (L108)
Exercise Sheet 6

1. [5/30 marks] Recall (from Lecture 2) that a pre-ordered set $(P, \leq_P)$ can be regarded as a category whose objects are the elements of P and whose morphism sets $P(x, x')$ contain at most one element and do so iff $x \leq_P x'$. Note that given two pre-ordered sets $(P, \leq_P)$ and $(Q, \leq_Q)$, a functor $F : P \to Q$ is the same thing as a monotone function.

   (a) Given two such functors $F, G : P \to Q$, how many natural transformations are there from F to G?

   (b) Given monotone functions $F : P \to Q$ and $G : Q \to P$, give a property of F and G which ensures that, regarding them as functors, G is right adjoint to F.

2. [5/30 marks] Recall that **Pre** denotes the category of pre-ordered sets and monotone functions. For each set X, let $(\mathrm{Pow}\, X, \subseteq) \in \mathrm{obj}\,\mathbf{Pre}$ be the set of all subsets of X partially ordered by inclusion. Given a function $f : X \to Y$, let $f^{-1} : \mathrm{Pow}\, Y \to \mathrm{Pow}\, X$ be the function that maps each subset $B \subseteq Y$ to the subset $f^{-1}B \triangleq \{x \in X \mid f(x) \in B\} \subseteq X$.

   (a) Show that f^{-1} is a monotone function and hence gives a morphism $(\mathrm{Pow}\, Y, \subseteq) \to (\mathrm{Pow}\, X, \subseteq)$ in **Pre**.

   (b) Regarding f^{-1} as a functor as in question (1), show that it has both left and right adjoints, given on objects by the following 'generalized quantifiers'

$$\exists_f A \triangleq \{y \in Y \mid (\exists x \in X)\ f(x) = y \ \wedge\ x \in A\}$$
$$\forall_f A \triangleq \{y \in Y \mid (\forall x \in X)\ f(x) = y \ \Rightarrow\ x \in A\}$$

   (for all $A \in \mathrm{Pow}\, X$). [Hint: use your answer to question 1b.]

3. [10/30 marks] A category **C** *has pullbacks* if for every pair of **C**-morphisms with a common codomain, $Y \xrightarrow{f} X \xleftarrow{g} Z$, there is an object $Y\,{}_f\!\times_g Z$ and morphisms p, q making the following diagram commute in **C** (that is, $f \circ p = g \circ q$)

$$\begin{tikzcd} Y\,{}_f\!\times_g Z \arrow[r, "q"] \arrow[d, "p"'] & Z \arrow[d, "g"] \\ Y \arrow[r, "f"'] & X \end{tikzcd} \tag{1}$$

   and with the following universal property:

   For all $Y \xleftarrow{h} W \xrightarrow{k} Z$ in **C** with $f \circ h = g \circ k$, there is a unique morphism $\ell \in \mathbf{C}(W, Y\,{}_f\!\times_g$

Z) satisfying $p \circ \ell = h$ and $q \circ \ell = k$

$$\begin{tikzcd}
W \arrow[dr, dashed, "\ell"] \arrow[drr, bend left, "k"] \arrow[ddr, bend right, "h"'] & & \\
& Y {}_f\!\times_g Z \arrow[r, "q"] \arrow[d, "p"'] & Z \arrow[d, "g"] \\
& Y \arrow[r, "f"'] & X
\end{tikzcd} \tag{2}$$

(a) Show that $\mathbf{C}$ has pullbacks iff for all $X \in \operatorname{obj} \mathbf{C}$ the slice category $\mathbf{C}/X$ has binary products (cf. Exercise Sheet 4, question 6).

(b) Show that if $\mathbf{C}$ has a terminal object and pullbacks, then it has binary products.

(c) Suppose $\mathbf{C}$ has pullbacks. Given $f \in \mathbf{C}(Y, X)$, show that the mapping

$$f^* : \begin{array}{c} Z \\ \downarrow g \\ X \end{array} \quad \mapsto \quad \begin{array}{c} Y {}_f\!\times_g Z \\ \downarrow p \\ Y \end{array}$$

is the object part of a functor $f^* : \mathbf{C}/X \to \mathbf{C}/Y$ between slice categories.

(d) Show that the functor f^* in part (c) always has a left adjoint $f_! : \mathbf{C}/Y \to \mathbf{C}/X$, which on objects sends $(W, h) \in \operatorname{obj}(\mathbf{C}/Y)$ to $f_!(W, h) \triangleq (W, f \circ h) \in \operatorname{obj}(\mathbf{C}/X)$.

4. [10/30 marks] A functor $F : \mathbf{C} \to \mathbf{D}$ is said to *preserve binary products* if whenever

$$X \xleftarrow{\pi_1} P \xrightarrow{\pi_2} Y \tag{3}$$

is a product in $\mathbf{C}$ for the objects X and Y, then

$$F(X) \xleftarrow{F(\pi_1)} F(P) \xrightarrow{F(\pi_2)} F(Y)$$

is a product in $\mathbf{D}$ for the objects $F(X)$ and $F(Y)$.

(a) Show that the Yoneda functor $y : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ (from Lectures 15 & 16) preserves binary products.

(b) Give an example of categories $\mathbf{C}, \mathbf{D}$ and a functor $F : \mathbf{C} \to \mathbf{D}$ for which $\mathbf{C}$ has all binary products, but F does not preserve all of them.