2015/16 MPhil ACS / CST Part III
Category Theory and Logic (L108)
Exercise Sheet 5 – Solution Notes

Question 1 For each function $f \in \mathbf{Set}(\Sigma, \Sigma')$ there is a function $\mathrm{Pow}\, f \in \mathbf{Set}(\mathrm{Pow}\,\Sigma, \mathrm{Pow}\,\Sigma')$ defined by

$$(\mathrm{Pow}\, f)\, S \triangleq \{f\, x \mid x \in S\} \tag{2}$$

Note that this is a monoid homomorphism $(\mathrm{Pow}\,\Sigma, \cup, \emptyset) \to (\mathrm{Pow}\,\Sigma', \cup, \emptyset)$, because

- $\mathrm{Pow}\, f\, \emptyset = \{f\, x \mid x \in \emptyset\} = \emptyset$
- $\mathrm{Pow}\, f\, (S \cup S') = \{f\, x \mid x \in S \cup S'\} = \{f\, x \mid x \in S \ \vee\ x \in S'\} = \{f\, x \mid x \in S\} \cup \{f\, x \mid x \in S'\} = (\mathrm{Pow}\, f\, S) \cup (\mathrm{Pow}\, f\, S')$.

So we get $P\, f \in \mathbf{Mon}((\mathrm{Pow}\,\Sigma, \cup, \emptyset), (\mathrm{Pow}\,\Sigma', \cup, \emptyset))$. Furthermore

- $(\mathrm{Pow}\, \mathrm{id}_\Sigma)\, S = \{\mathrm{id}_\Sigma x \mid x \in S\} = S$ for all $S \in \mathrm{Pow}\,\Sigma$; and hence $P\, \mathrm{id}_\Sigma = \mathrm{id}_{P\,\Sigma}$.
- $\mathrm{Pow}\, g(\mathrm{Pow}\, f\, S) = \{g\, y \mid y \in \mathrm{Pow}\, f\, S\} = \{g(f\, x) \mid x \in S\} = \{(g \circ f) x \mid x \in S\} = \mathrm{Pow}(g \circ f)\, S$ for all $S \in \mathrm{Pow}\,\Sigma$; and hence $P(g \circ f) = (P\, g) \circ (P\, f)$.

So P is a functor $\mathbf{Set} \to \mathbf{Mon}$.

Question 2 For each set Σ, define $\theta_\Sigma \in \mathbf{Set}(\mathrm{List}\,\Sigma, \mathrm{Pow}\,\Sigma)$ by recursion on the length of lists:

$$\begin{aligned}\theta_\Sigma(\mathrm{nil}) &= \emptyset\\ \theta_\Sigma(a :: \ell) &= \{a\} \cup \theta_\Sigma(\ell)\end{aligned}$$

Then one can prove $(\forall \ell, \ell' \in \mathrm{List}\,\Sigma)\ \theta_\Sigma(\ell \mathbin{@} \ell') = \theta_\Sigma(\ell) \cup \theta_\Sigma(\ell')$ by induction on the length of ℓ. So we get that θ_Σ is in $\mathbf{Mon}(F\Sigma, P\Sigma)$. To show that these morphisms form a natural transformation $\theta : F \to P$, we have to show for each $f \in \mathbf{Set}(\Sigma, \Sigma')$ that $P\, f \circ \theta_\Sigma = \theta_{\Sigma'} \circ F\, f$; and by definition of F and P, this means proving $(\forall \ell \in \mathrm{List}\,\Sigma)\ \mathrm{Pow}\, f(\theta_\Sigma(\ell)) = \theta_{\Sigma'}(\mathrm{List}\, f\, \ell)$, which follows easily from the definitions of $\mathrm{Pow}\, f$, $\mathrm{List}\, f$ and θ_Σ, by induction on the length of ℓ.

Here is another proof, which uses the universal property of the free monoid $F\Sigma$ instead of recursion/induction on lists.

For each set Σ, let $s_\Sigma \in \mathbf{Set}(\Sigma, \mathrm{Pow}\,\Sigma)$ be the function mapping each $x \in \Sigma$ to $s_\Sigma(x) \triangleq \{x\} \in \mathrm{Pow}\,\Sigma$. Using the universal property of the free monoid $i_\Sigma : \Sigma \to \mathrm{List}\,\Sigma$, there is a unique monoid homomorphism $\widehat{s_\Sigma} \in \mathbf{Mon}(F\,\Sigma, P\,\Sigma)$ with $\widehat{s_\Sigma} \circ i_\Sigma = s_\Sigma$. We take θ_Σ to be $\widehat{s_\Sigma}$ and show that these functions together give a natural transformation $\theta : F \to P$.

So we have to show for each $f \in \mathbf{Set}(\Sigma, \Sigma')$ that $\theta_{\Sigma'} \circ F\, f = P\, f \circ \theta_\Sigma \in \mathbf{Mon}(F\Sigma, P\Sigma')$. By the uniqueness part of the universal property of the free monoid $i_\Sigma : \Sigma' \to \mathrm{List}\,\Sigma$, for this it suffices to show that the two monoid homomorphisms $\theta_{\Sigma'} \circ F\, f$ and $P\, f \circ \theta_\Sigma$, when composed with the function i_Σ, give equal functions in $\mathbf{Set}(\Sigma, \mathrm{Pow}\,\Sigma')$. But

$$\begin{aligned}(P\, f \circ \theta_\Sigma) \circ i_\Sigma \triangleq ((\mathrm{Pow}\, f) \circ \widehat{s_\Sigma}) \circ i_\Sigma &= (\mathrm{Pow}\, f) \circ (\widehat{s_\Sigma} \circ i_\Sigma)\\ &= (\mathrm{Pow}\, f) \circ s_\Sigma \qquad \text{by definition of } \widehat{s_\Sigma}\end{aligned}$$

whereas

$$
\begin{aligned}
(\theta_{\Sigma'} \circ F\,f) \circ i_\Sigma \triangleq (\widehat{s_{\Sigma'}} \circ F\,f) \circ i_\Sigma &= \widehat{s_{\Sigma'}} \circ (F\,f \circ i_\Sigma) \\
&= \widehat{s_{\Sigma'}} \circ (i_{\Sigma'} \circ f) && \text{since } i \text{ is a natural transformation} \\
&= (\widehat{s_{\Sigma'}} \circ i_{\Sigma'}) \circ f \\
&= s_{\Sigma'} \circ f && \text{by definition of } \widehat{s_{\Sigma'}}
\end{aligned}
$$

So it suffices to prove that $(\mathrm{Pow}\,f) \circ s_\Sigma = s_{\Sigma'} \circ f \in \mathbf{Set}(\Sigma, \mathrm{Pow}\,\Sigma')$. But for all $x \in \Sigma$, we have $((\mathrm{Pow}\,f) \circ s_\Sigma)\,x = \mathrm{Pow}\,f\,(s_\Sigma\,x) = \mathrm{Pow}\,f\,\{x\} = \{f\,y \mid y \in \{x\}\} = \{f\,x\} = s_{\Sigma'}(f\,x) = (s_{\Sigma'} \circ f)\,x$.

Question 3 If $\theta \in \mathbf{D}^{\mathbf{C}}(F, G)$ is an isomorphism, then there is a natural transformation $\theta^{-1} \in \mathbf{D}^{\mathbf{C}}(G, F)$ with $\theta^{-1} \circ \theta = \mathrm{id}_F$ and $\theta \circ \theta^{-1} = \mathrm{id}_G$. By definition of identity and composition for natural transformations, that means that for all $X \in \mathrm{obj}\,\mathbf{C}$ we have $(\theta^{-1})_X \circ \theta_X = \mathrm{id}_{F\,X}$ and $\theta_X \circ (\theta^{-1})_X = \mathrm{id}_{G\,X}$. Therefore each $\theta_X \in \mathbf{D}(F\,X, G\,X)$ is an isomorphism in $\mathbf{D}$ with inverse $(\theta^{-1})_X$.

Conversely, if each $\theta_X \in \mathbf{D}(F\,X, G\,X)$ is an isomorphism in $\mathbf{D}$, then the inverse morphisms $(\theta_X)^{-1}$ are natural in X because for any $f \in \mathbf{C}(X, Y)$ we have

$$
\begin{aligned}
F\,f \circ (\theta_X)^{-1} &= (\theta_Y)^{-1} \circ \theta_Y \circ F\,f \circ (\theta_X)^{-1} && \text{because } (\theta_Y)^{-1} \circ \theta_Y = \mathrm{id}_{F\,Y} \\
&= (\theta_Y)^{-1} \circ G\,f \circ \theta_X \circ (\theta_X)^{-1} && \text{because } \theta_X \text{ is natural in } X \\
&= (\theta_Y)^{-1} \circ G\,f && \text{because } \theta_X \circ (\theta_X)^{-1} = \mathrm{id}_{G\,X}
\end{aligned}
$$

and so determine a natural transformation $\theta \in \mathbf{D}^{\mathbf{C}}(G, F)$ with $(\theta^{-1})_X \triangleq (\theta_X)^{-1}$ for each $X \in \mathrm{obj}\,\mathbf{C}$. This gives an inverse for θ. For $(\theta^{-1} \circ \theta)_X = (\theta^{-1})_X \circ \theta_X = (\theta_X)^{-1} \circ \theta_X = \mathrm{id}_{F\,X} = (\mathrm{id}_F)_X$, so that $\theta^{-1} \circ \theta = \mathrm{id}_F$; and similarly, $\theta \circ \theta^{-1} = \mathrm{id}_G$.

Question 4 If ch_X were natural in X, then taking $X = 2 = \{0, 1\}$ and letting τ be as in the hint, there would be a commutative square in **Set**:

$$
\begin{tikzcd}
P^+2 \arrow[r, "ch_2"] \arrow[d, "P^+\tau"'] & 2 \arrow[d, "\tau"] \\
P^+2 \arrow[r, "ch_2"'] & 2
\end{tikzcd}
\tag{3}
$$

Consider $\{0, 1\} \in P^+2$. We have

$$
P^+\tau\,\{0, 1\} = \{\tau\,0, \tau\,1\} = \{1, 0\} = \{0, 1\} \tag{4}
$$

Since $ch_2(\{0, 1\}) \in \{0, 1\}$, either $ch_2(\{0, 1\}) = 0$, or $ch_2(\{0, 1\}) = 1$. In the first case we get

$$
\begin{aligned}
1 = \tau\,0 = \tau(ch_2\{0, 1\}) &= ch_2(P^+\tau\,\{0, 1\}) && \text{by (3)} \\
&= ch_2\{0, 1\} && \text{by (4)} \\
&= 0 && \text{by assumption}
\end{aligned}
$$

which is a contradiction; and in the second case we get a similar contradiction. So (3) cannot commute and in particular ch_X cannot be natural in X.

Question 5

(a) Define $(I\,\alpha)_X \triangleq I(\alpha_X) : I(F\,X) \to I(G\,X)$. Since α_X is natural in $X \in \operatorname{obj}\mathbf{C}$, we have $G\,f \circ \alpha_X = \alpha_Y \circ F\,f$; and then since I is a functor, we get $I(G\,f) \circ I(\alpha_X) = I(\alpha_Y) \circ I(F\,f)$. So $(I\,\alpha)_X$ is natural in X.

(b) Define $(\gamma_F)_X \triangleq \gamma_{(F\,X)} : I(F\,X) \to J(F\,X)$. Since γ_Y is natural in $Y \in \operatorname{obj}\mathbf{D}$, $(\gamma_F)_X$ is natural in $X \in \operatorname{obj}\mathbf{C}$.

(c) Define $(\beta \circ \alpha)_X \triangleq \beta_X \circ \alpha_X : F\,X \to H\,X$. Since α_X and β_X are natural in $X \in \operatorname{obj}\mathbf{C}$, so is $(\beta \circ \alpha)_X$.

(d) Define $(\gamma * \alpha)_X \triangleq \gamma_{G\,X} \circ I(\alpha_X) : I(F\,X) \to J(G\,X)$. This is natural in X, because for any $f \in \mathbf{C}(X, Y)$

$$
\begin{aligned}
J(G\,f) \circ (\gamma * \alpha)_X &\triangleq J(G\,f) \circ \gamma_{G\,X} \circ I(\alpha_X) \\
&= J(G\,f) \circ J(\alpha_X) \circ \gamma_{F\,X} && \text{by natuality for } \gamma \\
&= J(G\,f \circ \alpha_X) \circ \gamma_{F\,X} && \text{by functoriality for } J \\
&= J(\alpha_Y \circ F\,f) \circ \gamma_{F\,X} && \text{by natuality for } \alpha \\
&= J(\alpha_Y) \circ J(F\,f) \circ \gamma_{F\,X} && \text{by functoriality for } J \\
&= J(\alpha_Y) \circ \gamma_{F\,Y} \circ I(F\,f) && \text{by natuality for } \gamma \\
&= \gamma_{G\,Y} \circ I(\alpha_Y) \circ I(F\,f) && \text{by natuality for } \gamma \\
&\triangleq (\gamma * \alpha)_Y \circ I(F\,f)
\end{aligned}
$$

(e)
$$
\begin{aligned}
((\delta * \beta) \circ (\gamma * \alpha))_X &\triangleq (\delta * \beta)_X \circ (\gamma * \alpha)_X \\
&\triangleq \delta_{H\,X} \circ J(\beta_X) \circ \gamma_{G\,X} \circ I(\alpha_X) \\
&= \delta_{H\,X} \circ \gamma_{H\,X} \circ I(\beta_X) \circ I(\alpha_X) && \text{by naturality for } \gamma \\
&\triangleq (\delta \circ \gamma)_{H\,X} \circ I(\beta_X) \circ I(\alpha_X) \\
&= (\delta \circ \gamma)_{H\,X} \circ I(\beta_X \circ \alpha_X) && \text{by functoriality for } I \\
&\triangleq (\delta \circ \gamma)_{H\,X} \circ I((\beta \circ \alpha)_X) \\
&\triangleq ((\delta \circ \gamma) * (\beta \circ \alpha))_X
\end{aligned}
$$

Question 6

(a) We use the notation $\overline{g} \triangleq \theta_{X,Y}(g)$ and $\overline{f} \triangleq \theta^{-1}_{X,Y}(f)$ from Lecture 14.

Define $\eta_X \triangleq \overline{\mathrm{id}_{F\,X}} \in \mathbf{C}(X, G(F\,X))$. This is natural in $X \in \operatorname{obj}\mathbf{C}$, because using naturality for θ (twice) we have

$$
G(F\,f) \circ \eta_X \triangleq G(F\,f) \circ \overline{\mathrm{id}_{F\,X}} = \overline{F\,f \circ \mathrm{id}_{F\,X}} = \overline{\mathrm{id}_{F\,Y} \circ F\,f} = \overline{\mathrm{id}_{F\,Y}} \circ f \triangleq \eta_Y \circ f
$$

Dually, define $\varepsilon_Y \triangleq \overline{\mathrm{id}_{G\,Y}} \in \mathbf{D}(F(G\,Y), Y)$ and prove it is natural in $Y \in \operatorname{obj}\mathbf{D}$ by a similar calculation.

(b) $$\begin{aligned}(\varepsilon_F \circ F\,\eta)_X &\triangleq (\varepsilon_F)_X \circ (F\,\eta)_X \\ &\triangleq \varepsilon_{F\,X} \circ F(\eta_X) \\ &\triangleq \overline{\mathrm{id}_{G(F\,X)}} \circ F(\eta_X) \\ &= \overline{\mathrm{id}_{G(F\,X)} \circ \eta_X} && \text{by naturality of } \theta \\ &= \overline{\eta_X} \\ &\triangleq \overline{\overline{id_{F\,X}}} \\ &= \mathrm{id}_{F\,X} && \text{since } \theta \text{ is an isomorphism} \\ &\triangleq (\mathrm{id}_F)_X\end{aligned}$$

The proof that $(G\,\varepsilon \circ \eta_G)_Y = (\mathrm{id}_G)_Y$ is dual to the one above.

Question 7 This is a standard result; see for example Proposition 10.1 on page 254 of Awodey's *Category Theory* book.