2016/17 MPhil ACS / CST Part III
Category Theory and Logic (L108)
Exercise Sheet 3

1. Show that for any objects X and Y in a cartesian closed category $\mathbf{C}$, there are functions

$$f \in \mathbf{C}(X,Y) \mapsto \ulcorner f \urcorner \in \mathbf{C}(\top, Y^X)$$
$$g \in \mathbf{C}(\top, Y^X) \mapsto \overline{g} \in \mathbf{C}(X,Y)$$

that give a bijection between the set $\mathbf{C}(X,Y)$ of $\mathbf{C}$-morphisms from X to Y and the set $\mathbf{C}(\top, Y^X)$ of $\mathbf{C}$-morphisms from the terminal object $\top$ to the exponential Y^X. [Hint: use the isomorphism (7) from Exercise Sheet 2, question 2.]

2. Show that for any objects X and Y in a cartesian closed category $\mathbf{C}$, the morphism $\mathrm{app} : Y^X \times X \to Y$ satisfies $\mathrm{cur}(\mathrm{app}) = \mathrm{id}_{Y^X}$. [Hint: recall from equation (4) on Exercise Sheet 2 that $\mathrm{id}_{Y^X} \times \mathrm{id}_X = \mathrm{id}_{Y^X \times X}$.]

3. Suppose $f : Y \times X \to Z$ and $g : W \to Y$ are morphisms in a cartesian closed category $\mathbf{C}$. Prove that
$$\mathrm{cur}(f \circ (g \times \mathrm{id}_X)) = (\mathrm{cur}\, f) \circ g \in \mathbf{C}(W, Z^X) \tag{1}$$
[Hint: use Exercise Sheet 2, question 1c.]

4. Let $\mathbf{C}$ be a cartesian closed category. For each $\mathbf{C}$-object X and $\mathbf{C}$-morphism $f : Y \to Z$, define
$$f^X \triangleq \mathrm{cur}(Y^X \times X \xrightarrow{\mathrm{app}} Y \xrightarrow{f} Z) \in \mathbf{C}(Y^X, Z^X) \tag{2}$$
   (a) Prove that $(\mathrm{id}_Y)^X = \mathrm{id}_{Y^X}$.
   (b) Given $f \in \mathbf{C}(Y \times X, Z)$ and $g \in \mathbf{C}(Z, W)$, prove that
$$\mathrm{cur}(g \circ f) = g^X \circ \mathrm{cur}\, f \in \mathbf{C}(Y, W^X) \tag{3}$$
   (c) Deduce that if $u \in \mathbf{C}(Y,Z)$ and $v \in \mathbf{C}(Z,W)$, then $(v \circ u)^X = v^X \circ u^X \in \mathbf{C}(Y^X, W^X)$.

   [Hint: for part (4a) use question 2; for part (4b) use Exercise Sheet 2, question 1c.]

5. Let $\mathbf{C}$ be a cartesian closed category. For each $\mathbf{C}$-object X and $\mathbf{C}$-morphism $f : Y \to Z$, define
$$X^f \triangleq \mathrm{cur}(X^Z \times Y \xrightarrow{\mathrm{id} \times f} X^Z \times Z \xrightarrow{\mathrm{app}} X) \in \mathbf{C}(X^Z, X^Y) \tag{4}$$
   (a) Prove that $X^{\mathrm{id}_Y} = \mathrm{id}_{X^Y}$.
   (b) Given $g \in \mathbf{C}(W,X)$ and $f \in \mathbf{C}(Y \times X, Z)$, prove that
$$\mathrm{cur}(f \circ (\mathrm{id}_Y \times g)) = Z^g \circ \mathrm{cur}\, f \in \mathbf{C}(Y, Z^W) \tag{5}$$
   (c) Deduce that if $u \in \mathbf{C}(Y,Z)$ and $v \in \mathbf{C}(Z,W)$, then $X^{(v \circ u)} = X^u \circ X^v \in \mathbf{C}(X^W, X^Y)$.

   [Hint: for part (5a) use question 2; for part (5b) use Exercise Sheet 2, question 1c.]

6. Let $\mathbf{C}$ be a cartesian closed category in which every pair of objects X and Y possesses a binary coproduct $X \xrightarrow{\mathrm{inl}_{X,Y}} X + Y \xleftarrow{\mathrm{inr}_{X,Y}} Y$. For all objects $X, Y, Z \in \mathbf{C}$ construct an isomorphism $(Y + Z) \times X \cong (Y \times X) + (Z \times X)$. [Hint: you may find it helpful to use some of the properties from question 4.]

7. Using the natural deduction rules for Intuitionistic Propositional Logic (given in Lecture 6), give proofs of the following judgements. In each case write down a corresponding typing judgement of the Simply Typed Lambda Calculus.

   (a) $\psi \vdash (\varphi \Rightarrow \psi) \Rightarrow \psi$

   (b) $\varphi \vdash (\varphi \Rightarrow \psi) \Rightarrow \psi$

   (c) $((\varphi \Rightarrow \psi) \Rightarrow \psi) \Rightarrow \psi \vdash \varphi \Rightarrow \psi$

8. (a) Given simple types A, B, C, give terms s and t of the Simply Typed Lambda Calculus that satisfy the following typing and $\beta\eta$-equality judgements:

$$\diamond, x : (A \times B) \rightarrow C \vdash s : A \rightarrow (B \rightarrow C) \tag{6}$$
$$\diamond, y : A \rightarrow (B \rightarrow C) \vdash t : (A \times B) \rightarrow C \tag{7}$$
$$\diamond, x : (A \times B) \rightarrow C \vdash t[s/y] =_{\beta\eta} x : (A \times B) \rightarrow C \tag{8}$$
$$\diamond, y : A \rightarrow (B \rightarrow C) \vdash s[t/x] =_{\beta\eta} y : A \rightarrow (B \rightarrow C) \tag{9}$$

   (b) Explain why question (8a) implies that for any three objects X,Y and Z in a cartesian closed category $\mathbf{C}$, there are morphisms

$$f : Z^{(X\times Y)} \rightarrow (Z^Y)^X \tag{10}$$
$$g : (Z^Y)^X \rightarrow Z^{(X\times Y)} \tag{11}$$

   that give an isomorphism $Z^{(X\times Y)} \cong (Z^Y)^X$ in $\mathbf{C}$.

9. Make up and solve a question like question 8 ending with an isomorphism $X^\top \cong X$ for any object X in a cartesian closed category $\mathbf{C}$ (with terminal object $\top$).