# 2016/17 MPhil ACS / CST Part III
# *Category Theory and Logic* (L108)
# Exercise Sheet 4 [GRADED]

1. [8/45 marks] Let $\mathbf{V}$ be the category with three distinct objects $L, P, R$ and whose only non-identity morphisms are $p : P \to L$ and $q : P \to R$.

   (a) Complete the definition of $\mathbf{V}$ by giving the nine sets $\mathbf{V}(X, Y)$ of morphisms between pairs of objects $X, Y \in \{L, P, R\}$ and defining the composition operations.

   (b) Do either of $\mathbf{V}$ or $\mathbf{V}^{\mathrm{op}}$ have a terminal object?

   (c) Do either of $\mathbf{V}$ or $\mathbf{V}^{\mathrm{op}}$ have binary products? [Hint: recall that in a pre-ordered set regarded as a category, products are given by greatest lower bounds.]

2. [8/45 marks] Let $\Sigma = \{a, b\}$ be a two-element set ($a \neq b$) and let $\_ \oplus \_ : \Sigma \times \Sigma \to \Sigma$ and $\_ \otimes \_ : \Sigma \times \Sigma \to \Sigma$ be binary operations on $\Sigma$ defined by the following tables:

| $\oplus$ | $a$ | $b$ |
|---|---|---|
| $a$ | $a$ | $a$ |
| $b$ | $a$ | $b$ |

| $\otimes$ | $a$ | $b$ |
|---|---|---|
| $a$ | $a$ | $b$ |
| $b$ | $b$ | $a$ |

   (so that $a \oplus b = a$, $a \otimes b = b$, etc.)

   (a) Show that for suitable choices of elements $e_M, e_N \in \Sigma$, there are monoids $M = (\Sigma, \oplus, e_M)$ and $N = (\Sigma, \otimes, e_N)$.

   (b) Show that $M$ and $N$ are not isomorphic in the category **Mon** of monoids and monoid homomorphisms. [Hint: assume they are isomorphic and derive a contradiction.]

3. [6/45 marks] Let $\mathbf{C}$ be a category with binary products. Given a $\mathbf{C}$-object $X$, the *diagonal* morphism $\delta_X \in \mathbf{C}(X, X \times X)$ and the *twist* morphism $\tau_X \in \mathbf{C}(X \times X, X \times X)$ are defined by:

$$\delta_X \triangleq \langle \mathrm{id}_X, \mathrm{id}_X \rangle \tag{1}$$

$$\tau_X \triangleq \langle \pi_2, \pi_1 \rangle \tag{2}$$

   (a) For each $f \in \mathbf{C}(X, Y)$, show that $\delta_Y \circ f = (f \times f) \circ \delta_X \in \mathbf{C}(X, Y \times Y)$ (where $f \times f$ denotes the product of morphisms introduced in Ex. Sh. 2, question 1b).

   (b) Show that $\tau_X \circ \delta_X = \delta_X$.

   (c) Show that $\tau_X \circ \tau_X = \mathrm{id}_{X \times X}$.

4. [8/45 marks] Let $\mathbf{C}$ be a category. Given $\mathbf{C}$-objects $X$ and $Y$ and morphisms $f, g \in \mathbf{C}(X, Y)$, an *equalizer* for $f$ and $g$ is by definition a $\mathbf{C}$-object $E$ and a morphism $e \in \mathbf{C}(E, X)$ such that

   - $f \circ e = g \circ e \in \mathbf{C}(E, Y)$ and

- for all $\mathbf{C}$-objects $Z$ and morphisms $h \in \mathbf{C}(Z, X)$, if $f \circ h = g \circ h \in \mathbf{C}(Z, Y)$, then there exists a unique morphism $k \in \mathbf{C}(Z, E)$ satisfying $e \circ k = h$.

$$\begin{tikzcd} Z \arrow[d, "k"'] \arrow[dr, "h"] & & \\ E \arrow[r, "e"'] & X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Y \end{tikzcd}$$

(a) Show that every equalizer is a monomorphism (see Ex. Sh. 1, question 4).

(b) Suppose that $f \in \mathbf{C}(X, Y)$ is a split monomorphism, that is, there is a morphism $g \in \mathbf{C}(Y, X)$ with $g \circ f = \mathrm{id}_X$ (see Ex. Sh. 1, question 4). Show that $f : X \to Y$ is the equalizer of the morphisms $f \circ g$ and $\mathrm{id}_Y$.

(c) Show that the category **Set** of sets and functions possesses equalizers for all parallel pairs of morphisms.

5. [6/45 marks] Let $X$ be an object of a category $\mathbf{C}$. The *slice category* $\mathbf{C}/X$ is defined by:

- The objects of $\mathbf{C}/X$ are pairs $(A, p)$ where $A \in \mathrm{obj}\,\mathbf{C}$ and $p \in \mathbf{C}(A, X)$.
- Given two such objects $(A, p)$ and $(B, q)$, a morphism $f : (A, p) \to (B, q)$ in $\mathbf{C}/X$ is a $\mathbf{C}$-morphism $f \in \mathbf{C}(A, B)$ such that $q \circ f = p$

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "p"'] & & B \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

- Composition and identities in $\mathbf{C}/X$ are given by those in $\mathbf{C}$.
  (a) Show that $\mathbf{C}/X$ always has a terminal object.
  (b) When $\mathbf{C} = \mathbf{Set}$, the category of sets and functions, show that $\mathbf{Set}/X$ has binary products. [Hint: given $(A, p), (B, q) \in \mathrm{obj}(\mathbf{Set}/X)$, consider a suitable subset of $\{(a, b) \mid a \in A \wedge b \in B\}$.]

6. [5/45 marks] Let $\mathbf{C} = \mathbf{Set}^{\mathrm{op}}$ be the opposite category of the category **Set** of sets and functions.

   (a) State, without proof, what is the product in $\mathbf{C}$ of two objects $X$ and $Y$.

   (b) Show by example that there are objects $X$ and $Y$ in $\mathbf{C}$ for which there is no exponential and hence that $\mathbf{C}$ is not a cartesian closed category.

7. [4/45 marks] Call a term $t$ of the Simply Typed Lambda Calculus (STLC) *pure* if it does not contain any constant symbols. Using facts about the semantics of STLC in cartesian closed categories, explain why there is no pure term $t$ such that $\diamond \vdash t : ((G \mathbin{\texttt{->}} G') \mathbin{\texttt{->}} G) \mathbin{\texttt{->}} G$ holds, where $G$ and $G'$ are distinct ground types. [Hint: consider the partially ordered set $(\{X \in \mathbb{R} \mid 0 \leq X \leq 1\}, \leq)$ regarded as a cartesian closed category and recall that the exponential $Y^X$ of two objects $X$ and $Y$ in this category satisfies $Y^X = 1$ if $X \leq Y$ and $Y^X = Y$ if $Y < X$.]