2016/17 MPhil ACS / CST Part III
Category Theory and Logic (L108)
Exercise Sheet 2

1. Let $\mathbf{C}$ be a category with binary products.

   (a) For morphisms $f \in \mathbf{C}(X, Y)$, $g_1 \in \mathbf{C}(Y, Z_1)$ and $g_2 \in \mathbf{C}(Y, Z_2)$, show that

   $$\langle g_1, g_2 \rangle \circ f = \langle g_1 \circ f, g_2 \circ f \rangle \in \mathbf{C}(X, Z_1 \times Z_2) \tag{1}$$

   (b) For morphisms $f_1 \in \mathbf{C}(X_1, Y_1)$ and $f_2 \in \mathbf{C}(X_2, Y_2)$, define

   $$f_1 \times f_2 \triangleq \langle f_1 \circ \pi_1, f_2 \circ \pi_2 \rangle \in \mathbf{C}(X_1 \times X_2, Y_1 \times Y_2) \tag{2}$$

   For any $g_1 \in \mathbf{C}(Z, X_1)$ and $g_2 \in \mathbf{C}(Z, X_2)$, show that

   $$(f_1 \times f_2) \circ \langle g_1, g_2 \rangle = \langle f_1 \circ g_1, f_2 \circ g_2 \rangle \in \mathbf{C}(Z, Y_1 \times Y_2) \tag{3}$$

   (c) Show that the operation $f_1, f_2 \mapsto f_1 \times f_2$ defined in part (1b) satisfies

   $$(h_1 \times h_2) \circ (k_1 \times k_2) = (h_1 \circ k_1) \times (h_2 \circ k_2) \tag{4}$$
   $$\mathrm{id}_X \times \mathrm{id}_Y = \mathrm{id}_{X \times Y} \tag{5}$$

2. Let $\mathbf{C}$ be a category with binary products ($\_ \times \_$) and a terminal object ($\top$). Given objects $X, Y, Z \in \mathbf{C}$, construct isomorphisms

   $$\alpha_{X,Y,Z} : X \times (Y \times Z) \cong (X \times Y) \times Z \tag{6}$$
   $$\lambda_X : \top \times X \cong X \tag{7}$$
   $$\rho_X : X \times \top \cong X \tag{8}$$
   $$\tau_{X,Y} : X \times Y \cong Y \times X \tag{9}$$

3. A *pairing* for a monoid $(M, \cdot, e)$ consists of elements $p_1, p_2 \in M$ and a binary operation $\langle \_, \_ \rangle : M \times M \to M$ satisfying for all $x, y, z \in M$

   $$p_1 \cdot \langle x, y \rangle = x \tag{10}$$
   $$p_2 \cdot \langle x, y \rangle = y \tag{11}$$
   $$\langle p_1, p_2 \rangle = e \tag{12}$$
   $$\langle x, y \rangle \cdot z = \langle x \cdot z, y \cdot z \rangle \tag{13}$$

   Given such a pairing, show that the monoid, when regarded as a one-object category, has binary products.

4. A monoid $(M, \cdot_M, e_M)$ is said to be *abelian* if its multiplication is commutative: $(\forall x, y \in M)\ x \cdot_M y = y \cdot_M x$.

(a) If $(M, \cdot_M, e_M)$ is an abelian monoid, show that the functions $m \in \mathbf{Set}(M\times, M, M)$ and $u \in \mathbf{Set}(\top, M)$ defined by

$$\begin{aligned} m(x,y) &= x \cdot_M y & \qquad\qquad (\text{all } x, y \in M) \\ u(0) &= e_M \end{aligned}$$

determine morphisms in the catgory **Mon** of monoids, $m \in \mathbf{Mon}(M \times M, M)$ and $u \in \mathbf{Mon}(\top, M)$ (where as usual we just write M for the monoid $(M, \cdot_M, e_M)$ and $\top$ for the terminal monoid $(\top, \cdot_\top, e_\top)$ with $\top$ a one-element set, $\{0\}$ say, $0 \cdot_\top 0 = 0$ and $e_\top = 0$).

Show futher that m and u make the monoid M into a "monoid object in the category **Mon**", in the sense that the following diagrams in **Mon** commute:

$$\begin{tikzcd} (M \times M) \times M \arrow[r, "m\times\mathrm{id}"] \arrow[d, "\langle \pi_1\circ\pi_1, \langle \pi_2\circ\pi_1, \pi_2\rangle\rangle"', "\cong"] & M \times M \arrow[r, "m"] & M \arrow[d, "\cong"', "\mathrm{id}"] \\ M \times (M \times M) \arrow[r, "\mathrm{id}\times m"'] & M \times M \arrow[r, "m"'] & M \end{tikzcd} \quad (\text{associativity}) \tag{14}$$

$$\begin{tikzcd} \top \times M \arrow[r, "u\times\mathrm{id}"] \arrow[d, "\pi_2"', "\cong"] & M \times M \arrow[r, "m"] & M \arrow[d, "\cong"', "\mathrm{id}"] \\ M \arrow[rr, "\mathrm{id}"'] & & M \end{tikzcd} \quad (\text{left unit}) \tag{15}$$

$$\begin{tikzcd} M \times \top \arrow[r, "\mathrm{id}\times u"] \arrow[d, "\pi_1"', "\cong"] & M \times M \arrow[r, "m"] & M \arrow[d, "\cong"', "\mathrm{id}"] \\ M \arrow[rr, "\mathrm{id}"'] & & M \end{tikzcd} \quad (\text{right unit}) \tag{16}$$

(b) Show that every monoid object in the category **Mon** (in the above sense) arises as in (4a). [Hint: if necessary, search the internet for "Eckmann-Hilton argument".]

5. Let **AbMon** be the category whose objects are abelian monoids (question 4) and whose morphisms, identity morphisms and composition are as in **Mon**.

(a) Show that the product in **Mon** of two abelian monoids gives their product in **AbMon**.

(b) Given $M, N \in \mathbf{AbMon}$ define morphisms $i \in \mathbf{AbMon}(M, M \times N)$ and $j \in \mathbf{AbMon}(N, M \times N)$ that make $M \times N$ into a *coproduct* in **AbMon**.

6. The category $\mathbf{Set}^\omega$ of 'sets evolving through discrete time' is defined as follows:

- Objects are triples $(X, (\_)^+, |\_|)$, where $X \in \mathbf{Set}$, $(\_)^+ \in \mathbf{Set}(X, X)$ and $|\_| \in \mathbf{Set}(X, \mathbb{N})$ satisfy for all $x \in X$

$$|x^+| = |x| + 1 \tag{17}$$

[Think of $|x|$ as the instant of time at which x exists and $x \mapsto x^+$ as saying how an element evolves from one instant to the next.]

- Morphisms $f : (X, (\_)^+, |\_|) \to (Y, (\_)^+, |\_|)$ are functions $f \in \mathbf{Set}(X, Y)$ satisfying for all $x \in X$

$$(f\,x)^+ = f(x^+) \tag{18}$$

$$|f\,x| = |x| \tag{19}$$

- Composition and identities are as in the category **Set**.

Show that $\mathbf{Set}^\omega$ has a terminal object and binary products.

7. Show that the category **Pre** of pre-ordered sets and monotone functions is a cartesian closed category.