

5 Denotational Semantics (MPF)

For all PCF types τ and all $M \in \text{PCF}_{\tau \rightarrow \text{bool}}$, let $M^\# \subseteq \llbracket \tau \rrbracket$ be defined as

$$M^\# = \{ d \in \llbracket \tau \rrbracket \mid \llbracket M \rrbracket(d) = \text{true} \}$$

Indicate whether the following statements are true or false, respectively providing a proof or a counterexample. You may use any standard results provided that you state them clearly.

- (a) For all PCF types τ and all $M, N \in \text{PCF}_{\tau \rightarrow \text{bool}}$, if $M^\# \subseteq N^\#$ then $\vdash M \leq_{\text{ctx}} N : \tau$. [5 marks]
- (b) For all PCF types τ and all $M, N \in \text{PCF}_{\tau \rightarrow \text{bool}}$, if $\vdash M \leq_{\text{ctx}} N : \tau$ then $M^\# \subseteq N^\#$. [5 marks]
- (c) For all PCF types τ and all $M, N \in \text{PCF}_{\tau \rightarrow \text{bool}}$, there exists $P \in \text{PCF}_{(\tau \rightarrow \text{bool}) \rightarrow ((\tau \rightarrow \text{bool}) \rightarrow (\tau \rightarrow \text{bool}))}$ such that $(P M N)^\# = M^\# \cap N^\#$. [5 marks]
- (d) For all PCF types τ and all $M, N \in \text{PCF}_{\tau \rightarrow \text{bool}}$, there exists $P \in \text{PCF}_{(\tau \rightarrow \text{bool}) \rightarrow ((\tau \rightarrow \text{bool}) \rightarrow (\tau \rightarrow \text{bool}))}$ such that $(P M N)^\# = M^\# \cup N^\#$.

Hint: Consider the fact, for which you need not provide a proof, that there is no PCF-definable function $f \in (\mathbb{B}_\perp \rightarrow (\mathbb{B}_\perp \rightarrow \mathbb{B}_\perp))$ such that $f \text{ true } \perp = f \perp \text{ true} = \text{true}$ and $f \text{ false false} \neq \text{true}$. [5 marks]