

7 Denotational Semantics (MPF)

(a) (i) Define the notion of continuous function between domains. [2 marks]

(ii) Let $\mathcal{P}(\mathbb{N}^2)$ be the domain of all subsets of pairs of natural numbers ordered by inclusion. Show that the function $f : \mathcal{P}(\mathbb{N}^2) \rightarrow \mathcal{P}(\mathbb{N}^2)$ given by

$$f(S) = \{ (1, 1) \} \cup \{ (x + 1, x \cdot y) \in \mathbb{N}^2 \mid (x, y) \in S \} \quad (S \subseteq \mathbb{N}^2)$$

is continuous. [3 marks]

(b) (i) State Tarski's fixed point theorem for a continuous endofunction on a domain. [2 marks]

(ii) Give a concrete explicit description of the fixed point $\text{fix}(f) \subseteq \mathbb{N}^2$ of the continuous function f in Part (a)(ii). Briefly justify your answer. [3 marks]

(c) (i) Define the notion of an admissible subset of a domain. [2 marks]

(ii) Let $P \subseteq \mathcal{P}(\mathbb{N}^2)$ be defined as $P = \{ S \subseteq \mathbb{N}^2 \mid \forall (x, y) \in S. \log y \leq x \cdot \log x \}$. Show that P is an admissible subset of the domain $\mathcal{P}(\mathbb{N}^2)$. [3 marks]

(d) (i) State Scott's fixed point induction principle. [2 marks]

(ii) Use Scott's fixed point induction principle to show that $\text{fix}(f) \in P$ for f the continuous function in Part (a)(ii) and P the admissible subset of the domain $\mathcal{P}(\mathbb{N}^2)$ in Part (c)(ii). [3 marks]