2016/17 MPhil ACS / CST Part III
Category Theory and Logic (L108)
Exercise Sheet 6 – Solution Notes

Question 1

(a) There is at most one natural transformation $F \to G$; and there is one iff $(\forall x \in P)\ F\,x \leq_Q G\,x$.

(b) Given monotone functions $F, G : (P, \leq_P) \to (Q, \leq_Q)$, G is right adjoint to F iff $(\forall x \in P)(\forall y \in Q)\ F\,x \leq_Q y \;\Leftrightarrow\; x \leq_P G\,y$.

Question 2

(a) If $B \subseteq B'$, then for all $x \in f^{-1}B$, $f\,x \in B$, so $f\,x \in B'$, so $x \in f^{-1}B'$; therefore $f^{-1}B \subseteq f^{-1}B'$.

(b) It is immediate from the definitions of $\exists_f(A)$ and $\forall_f(A)$ that

$$A \subseteq A' \;\Rightarrow\; \exists_f A \subseteq \exists_f A' \;\wedge\; \forall_f A \subseteq \forall_f A'$$

so that $\exists_f$ and $\forall_f$ are monotone functions. So to see that $\exists_f$ and $\forall_f$ are respectively left and right adjoints to f^{-1}, by the answer to question 1b we have to prove for all $A \in \mathrm{Pow}\,X$ and $B \in \mathrm{Pow}\,Y$

$$\exists_f A \subseteq B \;\Leftrightarrow\; A \subseteq f^{-1}B \tag{4}$$
$$f^{-1}B \subseteq A \;\Leftrightarrow\; B \subseteq \forall_f A \tag{5}$$

For (4):
$$\begin{aligned}
\exists_f A \subseteq B &\Leftrightarrow (\forall y \in Y)\ ((\exists x \in X)\ f\,x = y \wedge x \in A) \;\Rightarrow\; y \in B\ . \\
&\Leftrightarrow (\forall y \in Y, x \in X)\ f\,x = y \;\wedge\; x \in A \;\Rightarrow\; y \in B \\
&\Leftrightarrow (\forall x \in X)\ x \in A \;\Rightarrow\; (\forall y \in Y)\ f\,x = y \;\Rightarrow\; y \in B \\
&\Leftrightarrow (\forall x \in X)\ x \in A \;\Rightarrow f\,x \in B \\
&\Leftrightarrow (\forall x \in X)\ x \in A \;\Rightarrow\; x \in f^{-1}B \\
&\Leftrightarrow A \subseteq f^{-1}B
\end{aligned}$$

For (5):
$$\begin{aligned}
B \subseteq \forall_f A &\Leftrightarrow (\forall y \in Y)\ y \in B \;\Rightarrow\; (\forall x \in X)\ f\,x = y \;\Rightarrow\; x \in A. \\
&\Leftrightarrow (\forall y \in Y, x \in X)\ y \in B \;\wedge\; f\,x = y \;\Rightarrow\; x \in A \\
&\Leftrightarrow (\forall x \in X)\ f\,x \in B \;\Rightarrow\; x \in A \\
&\Leftrightarrow (\forall x \in X)\ x \in f^{-1}B \;\Rightarrow\; x \in A \\
&\Leftrightarrow f^{-1}B \subseteq A
\end{aligned}$$

Question 3

(a) The universal property of (1) in $\mathbf{C}$ is exactly the same as the universal property for a product of (Y, f) and (Z, g) in the slice category $\mathbf{C}/X$.

(b) If 1 is terminal in **C**, then a pullback for $X \xrightarrow{\langle\rangle} 1 \xleftarrow{\langle\rangle} Y$ has the same universal property as a product for X and Y in **C**.

(c) Given $h : (Z, g) \to (Z', g')$ in $\mathbf{C}/X$ (so that $g' \circ h = g$ in **C**), using the universal property of the pullback $Y \xleftarrow{p'} Y {}_f\times_{g'} Z' \xrightarrow{q'} Z'$, let $f^*h : Y {}_f\times_g Z \to Y {}_f\times_{g'} Z'$ be the unique morphism making commute

$$\begin{tikzcd}
Y {}_f\times_g Z \arrow[dr, dashed, "f^*h"] \arrow[drr, bend left, "h\circ q"] \arrow[ddr, bend right, "p"'] & & \\
& Y {}_f\times_{g'} Z' \arrow[r, "q'"] \arrow[d, "p'"'] & Z \arrow[d, "g'"] \\
& Y \arrow[r, "f"'] & X
\end{tikzcd}$$

(Note that the outer square commutes, because $g' \circ (h \circ q) = (g' \circ h) \circ q = g \circ q = f \circ p$.) Since $p' \circ (f^*h) = p$, we get $f^*h : (Y {}_f\times_g Z, p) \to (Y {}_f\times_{g'} Z', p')$ in $\mathbf{C}/Y$, in other words $f^*h \in \mathbf{C}/Y(f^*(Z, g), f^*(Z', g'))$. So we have the action of f^* on both object and morphisms. The uniqueness part of the universal property for pullbacks implies that f^* respects composition ($f^*(h \circ h') = (f^*h') \circ (f^*h)$) and identities ($f^*(\mathrm{id}_{(Z,g)}) = \mathrm{id}_{f^*(Z,g)}$).

(d) First note that $f_!$ acts trivially on morphisms: given $w : (W, h) \to (W', h')$ in $\mathbf{C}/Y$, since $h' \circ w = h$ in **C** we also have $(f \circ h') \circ w = f \circ h$ and hence $w : f_!(W, h) \to f_!(W'h')$. So we can define $f_!(w) \triangleq w$ and in this way $f_!$ is a functor $\mathbf{C}/Y \to \mathbf{C}/X$. To see that it gives a left adjoint to f^*, first note that morphisms in $\mathbf{C}/X(f_!(W, h), (Z, g))$ are just morphisms $k : W \to Z$ in **C** making the outer square in (2) commute; and therefore the universal property of pullbacks says that there is a bijection

$$\theta_{(W,h),(Z,g)} : \mathbf{C}/X(f_!(W, h), (Z, g)) \cong \mathbf{C}/Y((W, h), f^*(Z, g))$$

The uniqueness part of the universal property can be used to show that this bijection is natural in (W, h) and (Z, g) (details omitted). So we get an adjunction as required.

Question 4

(a) Given a product (3) in **C** and morphisms $y(X) \xleftarrow{\alpha} F \xrightarrow{\beta} y(Y)$ in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$, we have to show that there is a unique morphism $\gamma : F \to y(P)$ satisfying

$$y(\pi_1) \circ \gamma = \alpha \quad \text{and} \quad y(\pi_2) \circ \gamma = \beta \tag{6}$$

Uniqueness of γ: If there is such a γ, then for each **C**-object Z and element $c \in F(Z)$, by definition of $y(\pi_1)$ and $y(\pi_2)$ we have that $\alpha_Z(c) \in \mathbf{C}(Z, X)$ and $\beta_Z(c) \in \mathbf{C}(Z, Y)$ satisfy

$$\alpha_Z(c) = (y(\pi_1) \circ \gamma)_Z(c) = y(\pi_1)_Z(\gamma_Z(c)) = \pi_1 \circ (\gamma_Z(c))$$
$$\beta_Z(c) = (y(\pi_2) \circ \gamma)_Z(c) = y(\pi_2)_Z(\gamma_Z(c)) = \pi_2 \circ (\gamma_Z(c))$$

so that $\gamma_Z(c) = \langle \alpha_Z(c), \beta_Z(c) \rangle \in \mathbf{C}(Z, P)$, the unique $\mathbf{C}$-morphism whose compositions with π_1 and π_2 are $\alpha_Z(c)$ and $\beta_Z(c)$ respectively (using the universal property of the product (3)). So γ is uniquely determined by (6).

Existence of γ: For each $\mathbf{C}$-object Z, using the universal property of the product (3), we define $\gamma_Z : F(Z) \to \mathbf{C}(Z, P)$ to be the function mapping each $c \in F(Z)$ to

$$\gamma_Z(c) \triangleq \langle \alpha_Z(c), \beta_Z(c) \rangle \tag{7}$$

This gives a natural transformation $\gamma : F \to y(P)$, since for any $f \in \mathbf{C}(Z', Z)$

$$\begin{tikzcd}
F(Z) \arrow[r, "\gamma_Z"] \arrow[d, "F(f)"'] & \mathbf{C}(Z, P) = y(P)(Z) \arrow[d, "f^* = y(P)(f)"] \\
F(Z') \arrow[r, "\gamma_{Z'}"'] & \mathbf{C}(Z', P) = y(P)(Z')
\end{tikzcd}$$

commutes in **Set**, because for any $c \in F(Z)$

$$\begin{aligned}
\gamma_{Z'}(F(f)(c)) &= \langle \alpha_Z(F(f)(c)), \beta_Z(F(f)(c)) \rangle \\
&= \langle \alpha_Z(c) \circ f, \beta_Z(c) \circ f \rangle && \text{since } \alpha \text{ and } \beta \text{ are natural transformations} \\
&= \langle \alpha_Z(c), \beta_Z(c) \rangle \circ f && \text{by Exercise Sheet 2, question 1(a)} \\
&= y(P)(f)(\gamma_Z(c)) && \text{by definition of the Yoneda functor } y
\end{aligned}$$

Furthermore (6) holds since for all $Z \in \operatorname{obj} \mathbf{C}$ and $c \in (Z)$

$$\begin{aligned}
(y(\pi_1) \circ \gamma)_Z(c) &= y(\pi_1)_Z(\gamma_Z(c)) = \pi_1 \circ (\gamma_Z(c)) \triangleq \pi_1 \circ \langle \alpha_Z(c), \beta_Z(c) \rangle = \alpha_Z(c) \\
(y(\pi_2) \circ \gamma)_Z(c) &= y(\pi_2)_Z(\gamma_Z(c)) = \pi_2 \circ (\gamma_Z(c)) \triangleq \pi_2 \circ \langle \alpha_Z(c), \beta_Z(c) \rangle = \beta_Z(c)
\end{aligned}$$

(b) For example, consider the category $\mathbf{V}$ from Exercise Sheet 4, question 1, which does have all binary products. Let $F : \mathbf{V} \to \mathbf{Set}$ be the functor mapping the object P to the empty set $\emptyset$ and objects L and R to a one-element set $1 = \{0\}$. (Clearly this is a functor.) Note that F maps the product $L \leftarrow P \rightarrow R$ in $\mathbf{V}$ to $1 \leftarrow \emptyset \rightarrow 1$ (uniquely determined functions), which is not a product for 1 and 1 in **Set**.