#### DIVERSION

Exercise Sheet 2, question 4
updated to make it more do-able
(explains what is a "monoid object"
in a category with finite products)

If  $\mathbb{C}$  is a category with a terminal object T and products  $X \leftarrow X \times Y \rightarrow Y$  for all  $X, Y \in Sbj \mathbb{C}$ ,

then a monoid in  $\mathbb{C}$  is given by  $M \in \mathcal{O}$  by  $M \in \mathcal{O}$   $M \in \mathbb{C}$  ( $M \times M, M$ ),  $u : \mathbb{C}(T, M)$  such that these diagrams in  $\mathbb{C}$ 

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 $(M \times M) \times M \xrightarrow{M \times id} M \times M \xrightarrow{M} M$   $\langle \pi_i \pi_i, \langle \pi_j \pi_i, \pi_j \rangle \stackrel{\cong}{\downarrow} id$   $M \times (M \times M) \xrightarrow{id \times M} M \times M \xrightarrow{M} M$ 

 $\left[c.f. \ \forall x,y,z. \ m\left(m(x,y),z\right) = m(x,m(y,z))\right]$ 

If C is a category with a terminal object T and products  $X \leftarrow X \times Y \rightarrow Y$  for all  $X, Y \in Sbj C$ ,

then a monoid in  $\mathbb{C}$  is given by  $M \in \mathcal{O}$  by  $\mathbb{C}$ ,  $M \in \mathbb{C}(M \times M, M)$ ,  $u : \mathbb{C}(T, M)$  such that these diagrams in  $\mathbb{C}$ 

TxM  $\xrightarrow{u \times id}$   $\xrightarrow{m}$   $\xrightarrow{m}$ 

7.4

If C is a category with a terminal object T and products  $X \leftarrow X \times Y \rightarrow Y$  for all  $X, Y \in Sbj C$ ,

then a monoid in  $\mathbb{C}$  is given by  $M \in \mathcal{G}$ bj  $\mathbb{C}$ ,  $M \in \mathbb{C}(M \times M, M)$ ,  $u : \mathbb{C}(T, M)$  such that these diagrams in  $\mathbb{C}$ 

7.5

#### END OF DIVERSION

NEXT UP Simply Typed \lambda - Calculus Intuitionistic Propositional Logic

$$\frac{\Phi \vdash \varphi \quad \Phi, \varphi \vdash \psi}{\Phi \vdash \psi} (\text{Cut})$$

$$\overline{\mathbb{D}}, \varphi \vdash \varphi^{(A\times)}$$

$$\frac{\Phi \vdash \varphi}{\Phi, + \vdash \varphi}(Wk)$$

$$\overline{\underline{\oplus}} \vdash \overline{\top}^{(r)}$$

$$\frac{\Phi, \psi + \psi}{\Phi + \varphi \Rightarrow \psi} (\Rightarrow I)$$

$$\frac{\overline{\Phi} \vdash \varphi \not + \varphi}{\overline{\Phi} \vdash \varphi} (\wedge \xi_i)$$

Recall the derivation of  $\varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$ :

$$\frac{\varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \psi \Rightarrow \theta}{(wk)} \xrightarrow{(wk)} \frac{(Ax)}{\bigoplus \vdash \psi \Rightarrow \psi} \xrightarrow{\bigoplus \vdash \varphi} (Ax)$$

$$\frac{\Phi \vdash \psi \Rightarrow \theta}{\bigoplus \vdash \psi} \xrightarrow{(\Rightarrow E)}$$

$$\frac{\Phi \vdash \psi \Rightarrow \theta}{(\Rightarrow E)}$$

$$\frac{\Phi \vdash \psi \Rightarrow \varphi}{(\Rightarrow E)}$$

$$\frac{\Phi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \varphi}{(\Rightarrow E)}$$

$$\frac{\Phi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \varphi}{(\Rightarrow E)}$$

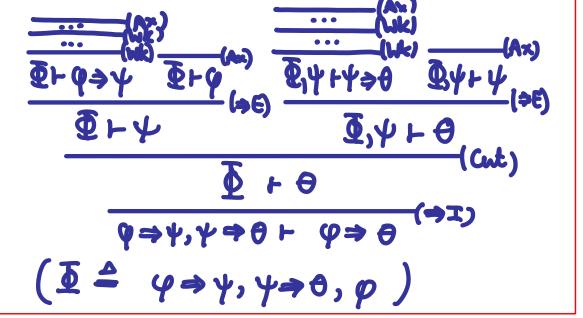
Another derivation of  $\varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$ :

# Proof Theony

$$\frac{\varphi \Rightarrow \psi, \psi \Rightarrow \Theta \vdash \psi \Rightarrow \varphi}{\Phi \vdash \psi \Rightarrow \varphi} \frac{(Ax)}{\Phi \vdash \psi \Rightarrow \psi} \frac{(Ax)}{\Phi \vdash \varphi} \frac{(Ax)}{\Phi$$

Why is this proof Simpler than?

FACT: if It of is derivable, it is derivable WITHOUT USING THE CUT RULE ("Cut elimination")

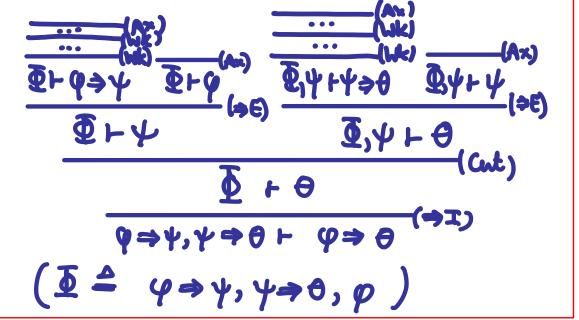


# Proof Theony

$$\frac{\varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \psi \Rightarrow \psi}{(M\psi)} = \frac{(M\psi)}{(M\psi)} = \frac{(M\psi)}{(M\psi$$

Why is
this proof
Simpler than
this 2 one?

Need a language & calculus of proofs [for IPL] to answer questions like this...



# Simply Typed Lambda (alculus (sp.c) (with finite products)

ground types Simple types:
A,B,C,... Gr.G...
unit type AXB = product type A > B \ function type

# Simply Typed Lambda (alculus (STLC) (with finite products)

Terms: s,t,r,...

7

Constants each with a given type

Variables (Countably many) 7x()
(sit)
fst t 5x + 1 5x + 1 5x + 1 5x + 1

Simple types:

 $A_1B_1C_2...$  ::=  $G_1G_2'...$ unit  $A \times B$   $A \rightarrow B$ 

\-abstraction

application

## Alpha Equivalence

STLC terms ove abstract syntax trees modulo renaming  $\lambda$ -bound variables. E.g.  $\lambda f: A \rightarrow B$ .  $\lambda x: A$ . fx &  $\lambda x: A \rightarrow B$ .  $\lambda y: A$ . xy

are the same term.

### Alpha Equivalence

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are the same term.

Formally, we quotient syntax trees by the equivalence relation of exequivalence  $=_{\infty}$  (or use a nameless' (de Bruijn) representation).

## Alpha Equivalence

$$x = x$$

$$S = {}_{\alpha} S' \quad t = {}_{\alpha} t'$$
 $(s, t) = {}_{\alpha} (s', t')$ 

$$\frac{t = \alpha t'}{fstt} = \frac{t = \alpha t'}{sndt}$$

$$S = \alpha S' + \alpha t'$$

$$S + \alpha S' + \alpha t'$$

$$S + \alpha S' + \alpha$$

result of replacing all occurrences of 2 with y in term t

$$(yx)\cdot t =_{\alpha} (yx')\cdot t'$$

y does not occur in {x,x,t,t}

$$\lambda x : A. t =_{\alpha} \lambda x' : A. t'$$

## Simply Typed Lambda Calculus (st.c)

Typing relation

t: A term type

is inductively defined by the following rules...

Tok means: no variable occurs more than once in T dom! = finite set of variables occurring in T Typing rules for variables

> Tok x ∉ dom[ T,x:A + x:A (var)

Γrx: A x' ∉ domΓ Γ, x': A'rx: A Tok means: no variable occurs more than once in T

Typing rules for constants & unit value

#### Typing rules for pairing and projections

#### Typing rules for function application & abstraction

$$\frac{\Gamma_{F} + E : A \rightarrow A'}{\Gamma_{F} + E' : A} \xrightarrow{(\alpha p p)} \Gamma_{F} \times A'$$

$$\frac{\Gamma_{F} \times A + E : A'}{\Gamma_{F} \times A \times A'} \xrightarrow{(\lambda)} \Gamma_{F} \times A'$$

Typing rules for function application & abstraction

$$\frac{\Gamma_{F} + E : A \rightarrow A'}{\Gamma_{F} + E' : A} \xrightarrow{(\alpha p p)} (\alpha p p)}$$

$$\frac{\Gamma_{F} \times A + E : A'}{\Gamma_{F} \times A : A \rightarrow A'} \xrightarrow{(\lambda)}$$

N.B. when using rule  $(\lambda)$  "bottom-up" to seach for a proof of  $\Gamma \vdash \lambda x : A \cdot t : A \rightarrow A'$ , since terms are syntax trees mod =  $\chi$ , can always assume or  $\neq$  dom  $\Gamma$ 

#### Example typing derivation

$$\frac{\Gamma_{+}g:B\rightarrow C}{\Gamma_{,}x:A+f:A\rightarrow B} \frac{(var)}{\Gamma_{,}x:A+1:A} \frac{(var)}{\Gamma_{,}x:A+g:B\rightarrow C} \frac{\Gamma_{,}x:A+f:A\rightarrow B}{\Gamma_{,}x:A+f:A\rightarrow B} \frac{(var)}{\Gamma_{,}x:A+g:B} \frac{(var)}{(app)}$$

$$\Gamma_{,}x:A+g(fx):C$$

$$\frac{\Gamma_{1}x:A\vdash g(fx):C}{\Gamma\vdash \lambda x:A.g(fx):A\Rightarrow C}$$

(where 
$$\Gamma = 0, f : A \rightarrow B, g : B \rightarrow C$$
)

N.B. typing rules are "syntax-directed" (by the Structure of t & then T. for variables)

## Semantics of STLC bypes in a CCC C Given a function M ground types G > objects M(G) E C

we extend it to a function

types A > objects M[A] EC

by recursion on the structure of A:

M[G] = M(G)-terminal object M [ unit ] = 1 W[AXB] = M[A] XM[B] M[A>B] = M[B] (M[A]) exponential

## Semantics of STLC types in a ccc C

```
M[G] = M(G)
M[unit] = 1
M[A \times B] = M[A] \times M[B]
M[A \rightarrow B] = M[B]
```

extend this to typing environments:

$$M[0] = 1$$
 $M[\Gamma, x: A] = M[\Gamma] \times M[A]$