Monte Carlo Methods Spring 2025
Homework 08 - Ising Model

Due: Tuesday, Mar 25, 2024, 11:59 PM

1. (20 points) The Ising model, even for a small number of spins, is a complex system. We will analyze it for a simple graph with three vertices. The goal is to derive the conditional probability of one spin given the other two spins for Gibbs sampling.

   Consider the following triangular graph:

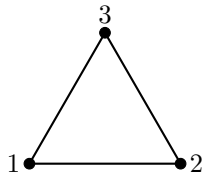

Figure 1: Triangular Ising model with three spins.

   Let X_1, X_2, X_3 be the spins at vertices $1, 2, 3$, respectively.

   (a) Write down the Hamiltonian function $H(\sigma)$ for the system as a function of the spins $\sigma = (X_1, X_2, X_3)$, the temperature T, and the coupling constant J.

   (b) Write down the expression for the probability of the system being in a particular state σ (up to a normalizing constant).

   (c) Derive the joint marginal distribution of the spins X_2, X_3, i.e.,

$$\mathbb{P}(X_2, X_3) = \mathbb{P}(X_1 = 1, X_2, X_3) + \mathbb{P}(X_1 = -1, X_2, X_3).$$

   (d) Derive the conditional probability $\mathbb{P}(X_1 = 1 \mid X_2, X_3)$.

   (e) Provide a Gibbs sampling algorithm to sample from the joint distribution of the spins X_1, X_2, X_3.

2. (30 points) Jupyter Notebook on Canvas.