Control Variates

Apurva Nakade

2025-04-11

Table of contents

Control Variates

Control random variables are examples of random variables that are positively correlated. In this case, the difference

$$\mathrm{var}(X-Y) = \mathrm{var}(X) + \mathrm{var}(Y) - 2\mathrm{cov}(X,Y)$$

will have lower variance. Let $X \sim p(x)$. Suppose we want to estimate $\ell = \mathbb{E}[f(x)]$ for some function $f(x)$. We can use a control variate $Y = h(X)$ for some function $h(x)$ such that

1. $\mathbb{E}[h(X)]$ is known, say $\mathbb{E}[h(X)] = h_0$.
2. $h(x)$ is strongly positively correlated with $f(x)$, i.e., $\mathrm{cov}(f(X), h(X)) \gg 0$.

Then we can use the control variate estimator

$$\hat{\ell}_{\mathrm{CV}} = \frac{1}{n}\sum_{i=1}^{n}\left[f(X_i) - \beta(h(X_i) - h_0)\right]$$

where β is a constant. It is easy to see that $\hat{\ell}_{\mathrm{CV}}$ is an unbiased estimator of ℓ. The variance of the control variate estimator is given by

$$\begin{aligned}\mathrm{var}(\hat{\ell}_{\mathrm{CV}}) &= \frac{1}{n}\left[\mathrm{var}(f(X)) + \beta^2\mathrm{var}(h(X)) - 2\beta\mathrm{cov}(f(X), h(X))\right] \\ &= \frac{1}{n}\left[\mathrm{var}(f(X)) + \beta\mathrm{var}(h(X))\left[\beta - 2\frac{\mathrm{cov}(f(X), h(X))}{\mathrm{var}(h(X))}\right]\right]\end{aligned}$$

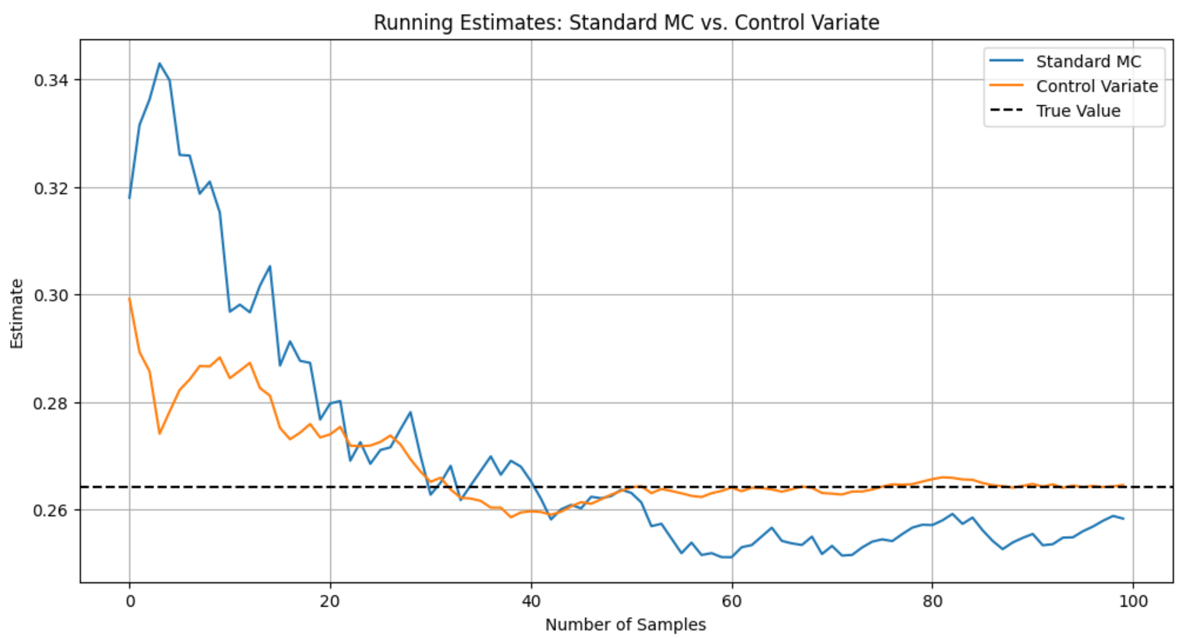
By choosing $\beta < 2\frac{\text{cov}(f(X),h(X))}{\text{var}(h(X))}$, we can reduce the variance of the control variate estimator. In practice, it is not easy to calculate the covariance between $f(X)$ and $h(X)$, so we

1. Pick a control variate $h(X)$ that is strongly correlated with $f(X)$, and whose expectation is known.
2. Experiment with different values of β to find the one that minimizes the variance of the control variate estimator.

Example 0.1. In the example below, we estimate the integral of xe^{-x} over the interval $[0, 1]$ using control function $g(x) = x$. We know that $\mathbb{E}[g(X)] = \frac{1}{2}$ so the estimator is given by

$$\hat{\ell}_{\text{CV}} = \frac{1}{n}\sum_{i=1}^{n}\left[f(X_i) - \beta(g(X_i) - \frac{1}{2})\right]$$

where β is a constant and $X_i \sim \text{Uniform}(0, 1)$ are i.i.d. We choose $\beta \approx 0.35$ which minimizes the variance of the control variate estimator. This results in an 8-fold reduction in variance compared to the naive estimator.




```
Beta: 0.358238
True Integral Value: 0.264241
Standard MC Estimate: 0.258290, Variance: 1.18e-04
Control Variate Estimate: 0.264526, Variance: 1.26e-05
Variance Reduction Factor: 9.36x
```