Monte Carlo Methods (Spring 2025)
Homework 09: Metropolis–Hastings

Due: Tuesday, April 1, 2025, 11:59 PM

1. (10 points) In class, we proved the reversibility of the Metropolis algorithm. In this exercise, we will prove the reversibility of the Metropolis–Hastings algorithm. Assume that the target distribution $p(x)$ is discrete with state space Ω. This time, the proposal distribution $q(y|x)$ is not necessarily symmetric.

   (a) Derive the transition probability $\mathbb{P}(X = b|X = a)$ for the Metropolis–Hastings algorithm.

   (b) Show that this defines a valid Markov chain by verifying that $\sum_{b \in \Omega} \mathbb{P}(X = b|X = a) = 1$.

   (c) Prove that the Metropolis–Hastings algorithm satisfies the detailed balance equations, ensuring its reversibility with respect to $p(x)$.

2. (10 points) The Metropolis–Hastings algorithm is a general framework from which many other algorithms arise as special cases.

   (a) Show that a random walk on graphs using a uniform proposal distribution is a special case of the Metropolis–Hastings algorithm.

   (b) Show that a single coordinate update step in Gibbs sampling is a special case of the Metropolis–Hastings algorithm.

   In this problem, all you need to do is provide the appropriate proposal distribution $q(y|x)$ and verify that the acceptance criterion agrees with the one used in the respective algorithms.

3. (5 points) Consider the Ising model on a square lattice with N^2 particles. We will now use the Metropolis–Hastings algorithm to generate samples. Using the proposal distribution:

$$q(\sigma'|\sigma) = \begin{cases} N^{-2}, & \text{if } \sigma' \text{ and } \sigma \text{ differ at exactly one site,} \\ 0, & \text{otherwise.} \end{cases}$$

   compute the acceptance probability $\alpha(\sigma'|\sigma)$ in the Metropolis–Hastings framework.

4. (25 points) Complete the Jupyter Notebook assignment on Canvas.

5. **Final Project Information (REQUIRED)**:

   (a) Provide a project title and a short description (one paragraph).

   (b) List any relevant references. **(REQUIRED)**

   (c) If applicable, specify the name of a collaborator for the final project.

   (d) Provide your final exam schedule. The official schedule is available here: `https://studentaffairs.jhu.edu/registrar/wp-content/uploads/sites/23/2025/01/Spring2025_FinalExamSchedule.pdf`.