碩士學位 請求論文
指導教授 裵 鍾 植

Invariance principle for the stochastic process generated by the chaotic tent map

成均館大學校 一般大學院

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ABSTRACT

Invariance principle for the stochastic process generated by the chaotic tent map

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The chaotic tent map is an ergodic map defined on the unit interval. In this thesis, we study the asymptotic behaviors of the various processes generated by the chaotic tent map. We get the uniform version of the law of large numbers for the sequential process generated by the chaotic tent map. We also get the invariance principle of Donsker's type for the sequential process generated by the chaotic tent map

Chapter 1

The tent map

1.1 The general tent map

Let μ be a positive real constant. The general tent map φ_μ is an iterated function, in the shape of a tent, forming a discrete-time dynamical system. It takes a point x on the unit interval $[0,1]$ and maps it to another point:

$$\varphi_\mu(x) = \begin{cases} \mu x & \text{for } 0 \leq x < \frac{1}{2} \\ \mu(1-x) & \text{for } \frac{1}{2} \leq x \leq 1. \end{cases}$$

This dynamic system can also be written as a recurrence relation. Let x_0 be a point, called the seed, in $[0,1]$. Iterating the seed we get a sequence $\{x_n\}$:

$$x_{n+1} = \varphi_\mu(x_n) = \begin{cases} \mu x_n & \text{for } 0 \leq x_n < \frac{1}{2} \\ \mu(1-x_n) & \text{for } \frac{1}{2} \leq x_n \leq 1. \end{cases}$$

When $\mu = 2$, the tent map demonstrates a chaotic dynamical behavior. See Wikipedia, the free encyclopedia, the English edition[15]. More specifically, if $\mu = 2$ then the system maps the interval $[0,1]$ to itself. There are now periodic points with every orbit length within this interval, as well as non-periodic points. The periodic points are dense in $[0,1]$, so the map φ_2 has become chaotic. We name φ_2 as *the chaotic tent map* and written simply as φ.

In this work, we obtain uniform version of the law of large numbers(LLN) and the central limit theorem(CLT) for the process generated by the chaotic tent map.

1.2 The chaotic tent map

We illustrate the chaotic tent map. Let $\Omega = [0,1]$ be the sample space, $\mathcal{A}$ be the Borel sets and P be the Lebesgue measure. The chaotic tent map on the unit interval is defined by

$$\varphi(y) = \begin{cases} 2y, & \text{for } 0 \le y < \frac{1}{2} \\ 2(1-y), & \text{for } \frac{1}{2} \le y \le 1. \end{cases}$$

The chaotic tent map is an iterated function, in the shape of a tent. More specifically, if you plot $\varphi(y)$ versus y, it has two linear sections which rise to meet at $(1/2, 1)$ - it looks like a tent.

The chaotic tent map φ preserves Lebesgue measure and is equivalent to a shift and flip map τ on $\{0,1\}^{\{0,1,2,\dots\}}$:

$$\tau(\omega_0, \omega_1, \omega_2, \dots) = \begin{cases} (\omega_1, \omega_2, \dots) & \text{if } \omega_0 = 0 \\ (1-\omega_1, 1-\omega_2, \dots) & \text{if } \omega_0 = 1. \end{cases}$$

We can think about $(\omega_0, \omega_1, \omega_2, \dots) \in \{0,1\}^{\{0,1,2,\dots\}}$ as a point y in the unit interval $[0,1]$ by putting

$$y = \sum_{i=0}^{\infty} \frac{\omega_i}{2^{i+1}}.$$

It is known that the map φ is ergodic. See Durret[4].

Chapter 2

The sequential law of large numbers

2.1 Introduction

The tent map is an iterated function forming a discrete-time dynamical system. The tent map demonstrates a chaotic dynamical behavior. See Wikipedia, the free encyclopedia, the English edition[15].

In this chapter, we obtain uniform versions of the law of large numbers(LLN) for the process generated by the chaotic tent map.

In obtaining the uniform LLN, we observe the role played by bracketing of the indexed class of functions of the process. Then we employ the idea of DeHardt[3] of the bracketing method to the process generated by the chaotic tent map.

Let X be a random variable defined on a probability space $(\Omega, \mathcal{A}, P)$ whose distribution function is F. Consider a sequence $\{X_i : i \geq 1\}$ of independent copies of X. Given a Borel measurable function $f : \mathbb{R} \to \mathbb{R}$, we see that $\{f(X_i) : i \geq 1\}$ forms a sequence of independent and identically distributed random variables that are more flexible in applications than the sequence $\{X_i : i \geq 1\}$. Consider a class $\mathcal{F}$ of real-valued Borel measurable functions defined on $\mathbb{R}$. Introduce the usual empirical distribution function F_n defined by $F_n(x) = n^{-1} \sum_{i=1}^{n} 1_{\{X_i \leq x\}}$ for $x \in \mathbb{R}$. Define a function indexed integral process S_n by

$$S_n(t) = \frac{1}{n} \sum_{i=1}^{[nt]} (f(X_i) - Ef(X)) \text{ for } t \in [0, 1], \tag{2.1}$$

where $[x]$ is the integer part of x.

Remark 1 *The process given in (2.1) can be written as the following integral form:*

$$S_n(t) = \frac{[nt]}{n} \int_\Omega f(x) d(F_{[nt]} - F)(x) \text{ for } t \in [0,1].$$

Developing a uniform LLN for the function-indexed process such as $S_n(f)$ in (2.1) is usually meant that $\sup_{f\in\mathcal{F}} |S_n(f)|$ converges to zero in a certain sense under a certain entropy condition on the class $\mathcal{F}$. The process is indexed by $\mathcal{F}$ and is considered as random elements in $B(\mathcal{F})$, the space of the bounded real-valued functions on $\mathcal{F}$, taken with the sup norm $||\cdot||_{\mathcal{F}}$. It is known that $(B(\mathcal{F}), ||\cdot||_{\mathcal{F}})$ forms a Banach space.

Developing a uniform LLN for the random fields such as $S_n(f,s)$ in (3.2) is usually meant that $\sup_{(f,s)\in\mathcal{F}\otimes[0,1]} |S_n(f,s)|$ converges to zero in a certain sense under a certain entropy condition on the class $\mathcal{S} := \mathcal{F} \otimes [0,1]$. The process is indexed by $\mathcal{S}$ and is considered as random elements in $B(\mathcal{S})$, the space of the bounded real-valued functions on $\mathcal{S}$, taken with the sup norm $||\cdot||_{\mathcal{S}}$. The result may be used in the nonparametric statistical inference. See Van de Geer[13] for applications.

We use the following definition of almost sure convergence and convergence in the mean. See, for a recent reference, Van der Vaart and Wellner[14]. See also Hoffmann-Jörgensen[7].

Definition 1 *A sequence of $B(\mathcal{F})$-valued random functions $\{Y_n\}$ converges almost surely to a constant c if $P^*\{\sup_{f\in\mathcal{F}} Y_n(f) \to c\} = P\{\sup_{f\in\mathcal{F}} Y_n(f)^* \to c\} = 1$. $\{Y_n\}$ converges in the mean to a constant c if $E^* \sup_{f\in\mathcal{F}} Y_n(f) = E \sup_{f\in\mathcal{F}} Y_n(f)^* \to c$. Here E^* denotes the upper expectation with respect to the outer probability P^*, and $\sup_{f\in\mathcal{F}} Y_n(f)^*$ is the measurable cover function of $\sup_{f\in\mathcal{F}} Y_n(f)$.*

Remark 2 *If the measurability of the process Y_n is guaranteed then the convergence in the above definition boils down to the usual almost sure convergence and convergence in the mean.*

In 1971, DeHardt[3] obtained the uniform LLN for the sequence of independent and identically distributed random variables under bracketing entropy. DeHardt's result states that if $\mathcal{F}$ has a bracketing entropy then $\sup_{\mathcal{F}} |S_n| \to 0$ almost surely.

The aim of our work is to develop the LLN, and the uniform LLN for the process generated by the tent map by employing DeHardt's idea of bracketing method. Our results will be stated with respect to the mean convergence and with respect to the almost sure convergence. In section 2.2, we describe a stationary sequence of random variables generated by the chaotic tent map, and state the main results. In section 2.3, we restate the main results in a more general setting of function indexed processes and we provide the proofs for the main results. Finally, in section 2.4, we provide an application to Monte Carlo integration.

2.2 The main results

We now consider a series of stationary processes generated by the tent map φ.

Firstly, we start with $f_{\frac{1}{2}}(y) = 1_{[0,\frac{1}{2})}(y)$. Then $\{f_{\frac{1}{2}}(\varphi^{m-1}(y)) : m \geq 1\}$ are identically distributed random variables which have uniform distribution with

$$P(f_{\frac{1}{2}}(\varphi^{m-1}(y)) = 0) = \tfrac{1}{2}$$
$$P(f_{\frac{1}{2}}(\varphi^{m-1}(y)) = 1) = \tfrac{1}{2}.$$

Observe that $Ef_{\frac{1}{2}}(y) = \frac{1}{2}$ and $Var(f_{\frac{1}{2}}(y)) = \frac{1}{4}$. Define

$$S_{n1} = \frac{1}{n}\sum_{m=1}^{n} 2\left(f_{\frac{1}{2}}(\varphi^{m-1}(y)) - \frac{1}{2}\right) \text{ and}$$

$$S_{n1}(t) = \frac{1}{n}\sum_{m=1}^{[nt]} 2\left(f_{\frac{1}{2}}(\varphi^{m-1}(y)) - \frac{1}{2}\right) \text{ for } t \in [0,1].$$

Observe that

$$\begin{aligned}
S_{n1}(t) &= \frac{1}{n}\sum_{m=1}^{[nt]} 2\left(f_{\frac{1}{2}}(\varphi^{m-1}(y)) - \frac{1}{2}\right) \\
&= \frac{[nt]}{n}\frac{1}{[nt]}\sum_{m=1}^{[nt]} 2\left(f_{\frac{1}{2}}(\varphi^{m-1}(y)) - \frac{1}{2}\right) \\
&= \frac{1}{[nt]}\sum_{m=1}^{[nt]} \frac{[nt]}{n} 2\left(f_{\frac{1}{2}}(\varphi^{m-1}(y)) - \frac{1}{2}\right) \\
&= \frac{1}{[nt]}\sum_{m=1}^{[nt]} f_{nt}(\varphi^{m-1}(y))
\end{aligned}$$

where

$$f_{nt}(\varphi^{\cdot-1}(y)) = \frac{[nt]}{n} 2\left(f_{\frac{1}{2}}(\varphi^{\cdot-1}(y)) - \frac{1}{2}\right).$$

Observe that

$$f_{nt}(\varphi^{\cdot-1}(y)) \to f_t(\varphi^{\cdot-1}(y)) = t\ 2\left(f_{\frac{1}{2}}(\varphi^{\cdot-1}(y)) - \frac{1}{2}\right) \text{ almost surely.}$$

Then we have the following

Theorem 1

1. $\sup_{0\le t\le 1} |S_{n1}(t)| \to 0$ *almost surely.*

2. *In particular,* $S_{n1} \to 0$ *almost surely.*

Secondly, for fixed $j \in \mathbb{N}$ and for fixed $i = 1, 2, \ldots, 2^j$, we look at $f_{i,j}(y) = 1_{[\frac{i-1}{2^j}, \frac{i}{2^j})}(y)$. Then $\{f_{i,j}(\varphi^{m-1}(y)) : m \ge 1\}$ are identically distributed random variables with

$$P(f_{i,j}(\varphi^{m-1}(y)) = 0) = 1 - \tfrac{1}{2^j}$$
$$P(f_{i,j}(\varphi^{m-1}(y)) = 1) = \tfrac{1}{2^j}.$$

Observe that $Ef_{i,j}(y) = \frac{1}{2^j}$ and $Var(f_{i,j}(y)) = \frac{1}{2^j}(1 - \frac{1}{2^j})$. Define

$$S_{n2} = \frac{1}{n}\sum_{m=1}^{n} \frac{\left(f_{i,j}(\varphi^{m-1}(y)) - \frac{1}{2^j}\right)}{\sqrt{\frac{1}{2^j}\left(1 - \frac{1}{2^j}\right)}} \text{ and}$$

$$S_{n2}(t) = \frac{1}{n}\sum_{m=1}^{[nt]} \frac{\left(f_{i,j}(\varphi^{m-1}(y)) - \frac{1}{2^j}\right)}{\sqrt{\frac{1}{2^j}\left(1 - \frac{1}{2^j}\right)}} \text{ for } t \in [0, 1].$$

Then we have the following

Theorem 2

1. $\sup_{0\le t\le 1} |S_{n2}(t)| \to 0$ *almost surely.*

2. *In particular,* $S_{n2} \to 0$ *almost surely.*

Thirdly, we consider, for each fixed $j \in \mathbb{N}$, $f_j(y) = \sum_{i=1}^{2^j} \frac{\left(f_{i,j}(y)-\frac{1}{2^j}\right)}{\sqrt{\frac{1}{2^j}\left(1-\frac{1}{2^j}\right)}}$. Then, for fixed $j \in \mathbb{N}$, being a sequence of identically distributed random variables, $\{f_j(\varphi^{m-1}(y)) : m \geq 1\}$ is stationary and ergodic process.

Notice that $\sum_{i=1}^{2^j} \frac{\left(f_{i,j}(y)-\frac{1}{2^j}\right)}{\sqrt{\frac{1}{2^j}\left(1-\frac{1}{2^j}\right)}} - \sum_{i=1}^{2^j-1} \frac{\left(f_{i,j}(y)-\frac{1}{2^j}\right)}{\sqrt{\frac{1}{2^j}\left(1-\frac{1}{2^j}\right)}} = \frac{\left(f_{2^j,j}(y)-\frac{1}{2^j}\right)}{\sqrt{\frac{1}{2^j}\left(1-\frac{1}{2^j}\right)}}$. We simply denote

$$d(y) := \frac{\left(f_{2^j,j}(y) - \frac{1}{2^j}\right)}{\sqrt{\frac{1}{2^j}\left(1 - \frac{1}{2^j}\right)}}. \tag{2.2}$$

Then, for fixed $j \in \mathbb{N}$, $\{d(\varphi^{m-1}(y)) : m \geq 1\}$ is a stationary martingale-difference sequence.

Define

$$S_{n3} := \frac{1}{n} \sum_{m=1}^{n} d(\varphi^{m-1}(y)) \text{ and}$$

where $\{d(\varphi^{m-1}(y)) : m \geq 1\}$ are stationary processes given in (2.2). We also define the corresponding sequential process

$$S_{n3}(t) := \frac{1}{n} \sum_{m=1}^{[nt]} d(\varphi^{m-1}(y)) \text{ for } t \in [0,1].$$

Then we have the following

Theorem 3

1. $\sup_{0 \leq t \leq 1} |S_{n3}(t)| \to 0$ *almost surely.*
2. *In particular,* $S_{n3} \to 0$ *almost surely.*

2.3 A general setting and proofs

Observe that

$$\begin{aligned}
&\sum_{m=1}^{[nt]} d(\varphi^{m-1}(y)) \\
&= d(y) + d(\varphi(y)) + \cdots + d(\varphi^{[nt]-1}(y)) \\
&= d(y)1_{[1,\ [nt]]}(1) + d(\varphi(y))1_{[1,\ [nt]]}(2) + \cdots + d(\varphi^{[nt]-1}(y))1_{[1,\ [nt]]}([nt])
\end{aligned}$$

$$
\begin{aligned}
&+d(\varphi^{[nt]}(y))1_{[1,\ [nt]]}([nt]+1)+\cdots+d(\varphi^{n-1}(y))1_{[1,\ [nt]]}(n)\\
=&\sum_{m=1}^{n} d(\varphi^{m-1}(y))1_{[1,\ [nt]]}(m)\\
=&\sum_{m=1}^{n} f_t(\varphi^{m-1}(y))
\end{aligned}
$$

where

$$f_t(\varphi^{\cdot-1}(y)) = d(\varphi^{\cdot-1}(y))1_{[1,\ [nt]]}(\cdot)$$

Then the process $\{S_{n3}(t) : t \in [0,1]\}$ can be regarded as the function indexed process $\{T_n(f) : f \in \mathcal{F}\}$ where $\mathcal{F} = \{f_t : t \in [0,1]\}$ and

$$T_n(f) = \frac{1}{n}\sum_{m=1}^{n} f(\varphi^{m-1}(y)),\ \ f \in \mathcal{F} \tag{2.3}$$

where $Ef(y) = 0$ for each $f \in \mathcal{F}$.

We firstly state the following law of large numbers that will turn out to be a special case of an ergodic theorem. See p.296 in Durrett[4].

Theorem 4 *For each fixed $f \in \mathcal{F} = \{f_t : t \in [0,1]\} \subseteq L^1$, as $n \to \infty$, $T_n(f) \to 0$ almost surely and in the mean.*

Proof. Denote $\mathcal{I}_\varphi$ the σ-field $\{A \in \mathcal{A} : \varphi^{-1}A = A\}$ of invariant sets under the tent map φ. Since $f \in L^1$, by the ergodic theorem, see for example p.296 in Durrett[**?**], $T_n(f) \to E[f(y)|\mathcal{I}_\varphi]$ almost surely and in the mean. Because the tent map φ is ergodic, we get $E[f(y)|\mathcal{I}_\varphi] = E[f(y)] = 0$. The proof is completed.

We are mainly interested in the uniform limiting behavior of the processes $\{T_n(f) : f \in \mathcal{F}\}$ in (2.3).

In order to measure the size of the function space, we define the following version of metric entropy with bracketing. See, for example, Van der Vaart and Wellner[14] for the recent reference.

Definition 2 *Given two functions l and u, the bracket $[l,u]$ is the set of all functions f with $l \le f \le u$. An ϵ-bracket is a bracket $[l,u]$ with $||u-l|| < \epsilon$. The bracketing number $N_{[\ ]}(\epsilon) := N_{[\ ]}(\epsilon, \mathcal{F}, ||\cdot||)$ is the minimum number of ϵ-brackets needed to cover $\mathcal{F}$.*

Remark 3 *It is known that for given $\epsilon > 0$ there exist finitely many ϵ-brackets $[l_k, u_k]$ whose union contains $\mathcal{F} = \{f_t : t \in [0,1]\}$. This is possible because the cardinality of $\mathcal{F}$ is the same as that of $[0,1]$. See Van der Vaart and Wellner[14].*

Let F be the distribution function of the random variable y. We introduce the empirical distribution function F_n for the random variables $y, \varphi(y)$, $\varphi^2(y), \ldots, \varphi^{n-1}(y)$ defined by $F_n(x) = n^{-1}\sum_{m=1}^{n} 1_{\{\varphi^{m-1}(y)\le x\}}$ for $x \in [0,1]$.

We secondly state the following uniform law of large numbers for the process generated by the chaotic tent map.

Theorem 5 *As $n \to \infty$, $\sup_{f\in\mathcal{F}} |T_n(f)| \to 0$ almost surely and in the mean.*

Proof. Observe that $T_n(f)$ can be transformed into the following integral form: $T_n(f) = \int f(x)dF_n(x)$ for $f \in \mathcal{F}$. Fix $\epsilon > 0$. Choose finitely many ϵ-brackets $[l_k, u_k]$ whose union contains $\mathcal{F}$ and such that $||u_k - l_k|| < \epsilon$ for every $k = 1, ..., N_{[\,]}(\epsilon)$. Then, for every $f \in \mathcal{F}$, there is a bracket such that

$$T_n(f) \le \int u_k dF_n + \int (u_k - l_k)dF \le \int u_k dF_n + \epsilon.$$

Consequently, $\sup_{f\in\mathcal{F}} T_n(f) \le \max_{1\le k\le N_{[\,]}(\epsilon)} \int u_k dF_n + \epsilon$. The right hand side converges almost surely and in the mean to ϵ by Theorem 4. Combination with a similar argument for $\inf_{f\in\mathcal{F}} T_n(f)$ yields that $\limsup_{n\to\infty} \sup_{f\in\mathcal{F}}$ $|T_n(f)|^* \le \epsilon$, almost surely, for every $\epsilon > 0$. Take a sequence $\epsilon_m \downarrow 0$ to see that the limsup must actually be zero almost surely. The proof is completed.

Finally, the proofs of Theorem 1 and Theorem 2 can be done similarily.

2.4 An application to Monte Carlo integration

Consider the observable random variable $d(y) := \frac{\left(f_{2^j,j}(y) - \frac{1}{2^j}\right)}{\sqrt{\frac{1}{2^j}\left(1-\frac{1}{2^j}\right)}}$ for a fixed $j \in \mathbb{N}$. Let F be the distribution function of $d(y)$. Let h be a known function with $\int |h(x)|dF(x) < \infty$.

It is often necessary to approximate the integral $\int h(x)dF(x)$. The usual "Monte Carlo" technique is still applied here for the stationary data $d(y)$, $d(\varphi(y)), d(\varphi^2(y)), \ldots$.

We may restate the law of large numbers and the sequential law of large numbers generated by the chaotic tent map as follow: Let h be a function with $\int |h(x)|dF(x) < \infty$, and let $d(y)$, $d(\varphi(y))$, $d(\varphi^2(y)), \ldots$ be stationary random variables obtained from the chaotic tent map. Then, as $n \to \infty$, $\frac{1}{n}\sum_{m=1}^{n} h(d(\varphi^{m-1}(y))) \to \int h(x)dF(x)$ almost surely and in the mean. Moreover, as $n \to \infty$, $\sup_{t\in[0,1]} |\frac{1}{n}\sum_{m=1}^{[nt]} h(d(\varphi^{m-1}(y))) - \int h(x)dF(x)| \to 0$ almost surely and in the mean.

Chapter 3

The invariance principle for the chaotic tent map

3.1 Introduction and the main results

In this chapter, we obtain the invariance principle of Donsker-type for the process generated by the chaotic tent map. In obtaining the results, we employ the method of uniformly integrable entropy of Ziegler[12] to the process.

Let X be a *uniformly bounded* random variable defined on $([0,1], \mathcal{B}[0,1], P$ $=$ Lebesgue measure) whose distribution function is F. Consider a sequence $\{X_i : i \geq 1\}$ of independent copies of X. Given a Borel measurable function $f : \mathbb{R} \to [-1,1]$, we see that $\{f(X_i) : i \geq 1\}$ forms a sequence of independent identically distributed(IID) random variables that are more flexible in applications than the sequence $\{X_i : i \geq 1\}$. Consider a class $\mathcal{F}$ of real-valued Borel measurable functions defined on $\mathbb{R}$. Introduce the usual empirical distribution function F_n defined by $F_n(x) = n^{-1}\sum_{i=1}^{n} 1_{\{X_i \leq x\}}$ for $x \in \mathbb{R}$. Define a function indexed integral process S_n by $S_n(f) = n^{1/2}\int f(x)d(F_n - F)(x)$ for $f \in \mathcal{F}$. Given a class $\mathcal{F}$ of functions, establishing a uniform central limit theorem(CLT) means showing that $\mathcal{L}(S_n(f) : f \in \mathcal{F}) \to \mathcal{L}(G(f) : f \in \mathcal{F})$, where the processes are considered as random elements of the Banach space, $B(\mathcal{F}) := \{z : \mathcal{F} \to \mathbb{R} : ||z||_{\mathcal{F}} := \sup_{f\in\mathcal{F}} |z(f)| < \infty\}$, the space of the bounded real-valued functions on $\mathcal{F}$, taken with the sup norm. The limiting process $G = (G(f) : f \in \mathcal{F})$ is a Gaussian process whose sample paths are contained in $U_B(\mathcal{F}, \rho) := \{z \in B(\mathcal{F}) : z$ is uniformly continuous with respect to $\rho\}$, where ρ is a metric on

$\mathcal{F}$. Notice that $(B(\mathcal{F}), ||\cdot||_{\mathcal{F}})$ is a Banach space and $U_B(\mathcal{F}, \rho)$ is a closed subspace of $(B(\mathcal{F}), ||\cdot||_{\mathcal{F}})$ and hence is a Banach space. In particular $U_B(\mathcal{F}, \rho)$ is separable if and only if $(\mathcal{F}, \rho)$ is totally bounded.

In 1997, Ziegler[12] obtained the uniform CLT for the sequence of independent random variables under the uniformly integrable entropy. Ziegler's result states that if $\mathcal{F}$ has a uniformly integrable entropy then S_n is asymptotically Gaussian.

We use the following weak convergence due to Hoffmann-Jörgensen[7].

Definition 3 *A sequence of $B(\mathcal{F})$-valued random functions $\{Y_n : n \geq 1\}$ converges in law to a $B(\mathcal{F})$-valued Borel measurable random function Y whose law concentrates on a separable subset of $B(\mathcal{F})$, denoted by $Y_n \Rightarrow Y$, if $Eg(Y) = \lim_{n\to\infty} E^*g(Y_n)$ for all $g \in C(B(\mathcal{F}), ||\cdot||_{\mathcal{F}})$, where $C(B(\mathcal{F}), ||\cdot||_{\mathcal{F}})$ is the set of real bounded, continuous functions.*

Let X be a random variable defined on a probability space $(\Omega, \mathcal{A}, P)$ whose distribution function is F. Consider a sequence $\{X_i : i \geq 1\}$ of independent copies of X. Given a Borel measurable function $f : \mathbb{R} \to \mathbb{R}$, we see that $\{f(X_i) : i \geq 1\}$ forms a sequence of IID random variables that are more flexible in applications than the sequence $\{X_i : i \geq 1\}$. Define a $[0,1]$ indexed process S_n by

$$S_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} (f(X_i) - Ef(X)) \text{ for } t \in [0,1], \tag{3.1}$$

where $[x]$ is the integer part of x. Introduce the usual empirical distribution function F_n defined by $F_n(x) = n^{-1} \sum_{i=1}^{n} 1_{\{X_i \leq x\}}$ for $x \in \mathbb{R}$. This process S_n can be written as the following integral

$$S_n(t) = n^{1/2} \frac{[nt]}{n} \int f(x) d(F_{[nt]} - F)(x) \text{ for } t \in [0,1]. \tag{3.2}$$

Establishing a uniform CLT, or an invariance principle, for the process such as $S_n(t)$ in (3.2) means showing that $\mathcal{L}(S_n(t) : t \in [0,1]) \to \mathcal{L}(U(t) : t \in [0,1])$, where the processes are considered as random elements of $D[0,1]$ the space of all real-valued functions on $[0,1]$ that are right continuous at each point of $[0,1)$ with left limits existing at each point of $(0,1]$. The limiting process $U = (U(t) : t \in [0,1])$ is the Gaussian process with continuous sample paths with the mean zero and the covariance $EU(s)U(t) = \lim_{n\to\infty} Cov(S_n(s), S_n(t))$.

We now consider a series of stationary processes generated by the tent map φ.

Firstly, we start with $f_{\frac{1}{2}}(y) = 1_{[0,\frac{1}{2})}(y)$. Then $\{f_{\frac{1}{2}}(\varphi^{m-1}(y)) : m \geq 1\}$ are identically distributed random variables which have uniform distribution with

$$P(f_{\frac{1}{2}}(\varphi^{m-1}(y)) = 0) = \tfrac{1}{2}$$
$$P(f_{\frac{1}{2}}(\varphi^{m-1}(y)) = 1) = \tfrac{1}{2}.$$

Therefore $\{f_{\frac{1}{2}}(\varphi^{m-1}(y)) : m \geq 1\}$ is a sequence of stationary random variables. Observe that $Ef_{\frac{1}{2}}(y) = \frac{1}{2}$ and $Var(f_{\frac{1}{2}}(y)) = \frac{1}{4}$. Define

$$S_{n1} = n^{-1/2} \sum_{m=1}^{n} 2\left(f_{\frac{1}{2}}(\varphi^{m-1}(y)) - \frac{1}{2}\right) \text{ and}$$

$$S_{n1}(t) = n^{-1/2} \sum_{m=1}^{[nt]} 2\left(f_{\frac{1}{2}}(\varphi^{m-1}(y)) - \frac{1}{2}\right) \text{ for } t \in [0,1].$$

Observe that

$$\begin{aligned}
S_{n1}(t) &= \frac{1}{\sqrt{n}} \sum_{m=1}^{[nt]} 2\left(f_{\frac{1}{2}}(\varphi^{m-1}(y)) - \frac{1}{2}\right) \\
&= \frac{\sqrt{[nt]}}{\sqrt{n}} \frac{1}{\sqrt{[nt]}} \sum_{m=1}^{[nt]} 2\left(f_{\frac{1}{2}}(\varphi^{m-1}(y)) - \frac{1}{2}\right) \\
&= \frac{1}{\sqrt{[nt]}} \sum_{m=1}^{[nt]} \frac{\sqrt{[nt]}}{\sqrt{n}} 2\left(f_{\frac{1}{2}}(\varphi^{m-1}(y)) - \frac{1}{2}\right) \\
&= \frac{1}{\sqrt{[nt]}} \sum_{m=1}^{[nt]} f_{nt}(\varphi^{m-1}(y))
\end{aligned}$$

where

$$f_{nt}(\varphi^{\cdot-1}(y)) = \frac{\sqrt{[nt]}}{\sqrt{n}} 2\left(f_{\frac{1}{2}}(\varphi^{\cdot-1}(y)) - \frac{1}{2}\right).$$

Observe that

$$f_{nt}(\varphi^{\cdot-1}(y)) \to f_t(\varphi^{\cdot-1}(y)) = \sqrt{t}\; 2\left(f_{\frac{1}{2}}(\varphi^{\cdot-1}(y)) - \frac{1}{2}\right) \text{ almost surely.}$$

Then we have the following

Theorem 6

1. *As* $n \to \infty$, $S_{n1} \Rightarrow \mathcal{N}(0,1)$.

2. *For each fixed* $t_1, t_2, \ldots, t_k$ *in* $[0,1]$, $(S_{n1}(t_1), S_{n1}(t_2), \ldots, S_{n1}(t_k)) \Rightarrow \mathcal{N}_k(\mathbf{0}, \Sigma)$ *as random vectors of* $\mathbb{R}^k$. *The variance covariance matrix is*

$$\Sigma = \begin{pmatrix} t_1 & t_1 \wedge t_2 & \cdots & t_1 \wedge t_k \\ & t_2 & \cdots & t_2 \wedge t_k \\ & & \cdots & \\ Sym. & & & t_k \end{pmatrix}$$

where $t_1 \wedge t_2 = min\{t_1, t_2\}$

Proof.

1. By the central limit theorem for stationary processes, see Gordin[6], S_{n1} converges in distribution to a standard normal random variable.

2. The one dimensional result and Cramer-Wold device give the finite dimensional convergence to a multivariate normal distribution and the covariance structure is given by

$$\lim_{n\to\infty} ES_{ni}(t_i)S_{n1}(t_j) = t_i \wedge t_j \quad \text{for } i, j = 1, \ldots, k.$$

Secondly, for fixed $j \in \mathbb{N}$ and for fixed $i = 1, 2, \ldots, 2^j$, we look at $f_{i,j}(y) = 1_{[\frac{i-1}{2^j}, \frac{i}{2^j})}(y)$. Then $\{f_{i,j}(\varphi^{m-1}(y)) : m \geq 1\}$ are identically distributed random variables with

$$P(f_{i,j}(\varphi^{m-1}(y)) = 0) = 1 - 2^{-j}$$
$$P(f_{i,j}(\varphi^{m-1}(y)) = 1) = 2^{-j}.$$

Observe that $Ef_{i,j}(y) = \frac{1}{2^j}$ and $Var(f_{i,j}(y)) = \frac{1}{2^j}(1 - \frac{1}{2^j})$.

Define

$$S_{n2} = n^{-1/2} \sum_{m=1}^{n} \frac{f_{i,j}(\varphi^{m-1}(y)) - 2^{-j}}{\{2^{-j}(1 - 2^{-j})\}^{1/2}} \text{ and}$$

$$S_{n2}(t) = n^{-1/2} \sum_{m=1}^{[nt]} \frac{f_{i,j}(\varphi^{m-1}(y)) - 2^{-j}}{\{2^{-j}(1 - 2^{-j})\}^{1/2}} \text{ for } t \in [0,1].$$

Observe that

$$S_{n2}(t) \quad = \quad \frac{1}{\sqrt{[nt]}} \sum_{m=1}^{[nt]} g_{nt}(\varphi^{m-1}(y))$$

where

$$g_{nt}(\varphi^{\cdot-1}(y)) = \frac{\sqrt{[nt]}}{\sqrt{n}} \frac{f_{i,j}(\varphi^{\cdot-1}(y)) - 2^{-j}}{\{2^{-j}(1-2^{-j})\}^{1/2}}.$$

Observe that

$$g_{nt}(\varphi^{\cdot-1}(y)) \to g_t(\varphi^{\cdot-1}(y)) = \sqrt{t}\, \frac{f_{i,j}(\varphi^{m-1}(y)) - 2^{-j}}{\{2^{-j}(1-2^{-j})\}^{1/2}} \text{ almost surely.}$$

Then we have the following

Theorem 7

1. *As $n \to \infty$, $S_{n2} \Rightarrow \mathcal{N}(0,1)$.*

2. *For each fixed $t_1, t_2, \ldots, t_k$ in $[0,1]$, $(S_{n2}(t_1), S_{n2}(t_2), \ldots, S_{n2}(t_k)) \Rightarrow \mathcal{N}_k(\mathbf{0}, \Sigma)$ as random vectors of $\mathbb{R}^k$. The variance covariance matrix is*

$$\Sigma = \begin{pmatrix} t_1 & t_1 \wedge t_2 & \cdots & t_1 \wedge t_k \\ & t_2 & \cdots & t_2 \wedge t_k \\ & & \cdots & \\ Sym. & & & t_k \end{pmatrix}.$$

Proof. Similar as above.

Thirdly, for each fixed $j \in \mathbb{N}$, we consider the sum

$$f_j(y) = \sum_{i=1}^{2^j} \frac{f_{i,j}(y) - 2^{-j}}{\{2^{-j}(1-2^{-j})\}^{1/2}}$$

of the random variables

$$\frac{f_{1,j}(y) - 2^{-j}}{\{2^{-j}(1-2^{-j})\}^{1/2}}, \frac{f_{2,j}(y) - 2^{-j}}{\{2^{-j}(1-2^{-j})\}^{1/2}}, \ldots, \frac{f_{2^j,j}(y) - 2^{-j}}{\{2^{-j}(1-2^{-j})\}^{1/2}}.$$

Then, for fixed $j \in \mathbb{N}$, being a sequence of identically distributed random variables, $\{f_j(\varphi^{m-1}(y)) : m \geq 1\}$ is stationary and ergodic process. Consider the equation

$$\sum_{i=1}^{2^j} \frac{f_{i,j}(y) - 2^{-j}}{\{2^{-j}(1-2^{-j})\}^{1/2}} - \sum_{i=1}^{2^j-1} \frac{f_{i,j}(y) - 2^{-j}}{\{2^{-j}(1-2^{-j})\}^{1/2}} = \frac{f_{2^j,j}(y) - 2^{-j}}{\{2^{-j}(1-2^{-j})\}^{1/2}}.$$

We simply denote

$$d(y) := \frac{f_{2^j,j}(y) - 2^{-j}}{\{2^{-j}(1-2^{-j})\}^{1/2}}.$$

Recall that $\Omega = [0,1]$ is the sample space, $\mathcal{A}$ is the Borel sets and P is the Lebesgue measure. Then $\varphi : \Omega \to \Omega$ is a P-preserving measurable transformation. Assume that $\mathcal{F}_0 := \{\emptyset, \Omega\}$ is the φ-invariant σ-field (i.e. $\varphi^{-1}\mathcal{F}_0 \subset \mathcal{F}_0$), set $\mathcal{F}_n = \varphi^{-n}\mathcal{F}_0$, and denote by E_n the conditional expectation operator with respect to the σ-algebra $\mathcal{F}_n$. Notice that $E_{m-2}d(\varphi^{m-1}(y)) = 0$. Indeed,

$$E_{m-2}d(\varphi^{m-1}(y)) = \frac{E_{m-2}1_{[1-2^{-j},1)}(\varphi^{m-1}(y)) - 2^{-j}}{\{2^{-j}(1-2^{-j})\}^{1/2}}$$

and

$$\begin{aligned}
E_{m-2}1_{[1-2^{-j},1)}(\varphi^{m-1}(y)) &= E\left(1_{[1-2^{-j},1)}(\varphi^{m-1}(y)) \mid \mathcal{F}_{m-2}\right) \\
&= P\left(\varphi^{m-1}(y) \in \left[1-2^{-j},1\right) \mid \mathcal{F}_{m-2}\right) \\
&= P\left(\varphi(y) \in \left[1-2^{-j},1\right) \mid \mathcal{F}_0\right) \\
&= P\left(\varphi(y) \in \left[1-2^{-j},1\right)\right) \\
&= P\left(y \in \left[1-2^{-j},1\right)\right) \\
&= 2^{-j}.
\end{aligned}$$

This shows that, for fixed $j \in \mathbb{N}$, the sequence $\{d(\varphi^{m-1}(y)) : m \geq 1\}$ is an ergodic, stationary sequence of martingale-differences.

From stationarity, using the Kolmogorov consistency theorem, we may assume that the process, for fixed $j \in \mathbb{N}$, $\{d(\varphi^{m-1}(y)) : m \in \mathbb{Z}\}$ is double sided.

Define

$$S_{n3} := n^{-1/2} \sum_{m=1}^{n} d(\varphi^{m-1}(y)) \text{ and} \tag{3.3}$$

$$S_{n3}(t) := n^{-1/2} \sum_{m=1}^{[nt]} d(\varphi^{m-1}(y)) \text{ for } t \in [0,1] \tag{3.4}$$

Observe that

$$S_{n3}(t) \quad = \quad \frac{1}{\sqrt{[nt]}} \sum_{m=1}^{[nt]} h_{nt}(\varphi^{m-1}(y))$$

where

$$h_{nt}(\varphi^{\cdot-1}(y)) = \frac{\sqrt{[nt]}}{\sqrt{n}}\, d(\varphi^{\cdot-1}(y)).$$

Observe that

$$h_{nt}(\varphi^{\cdot-1}(y)) \to h_t(\varphi^{\cdot-1}(y)) = \sqrt{t}\, d(\varphi^{\cdot-1}(y)) \text{ almost surely.}$$

Then we have the following

Theorem 8

1. *As* $n \to \infty$, $S_{n3} \Rightarrow \mathcal{N}(0,1)$.

2. *As* $n \to \infty$, $S_{n3} \Rightarrow U$ *as random elements of* $D[0,1]$. *The limiting process* U *is a Gaussian process with continuous sample paths and* $EU(t) = 0$ *and*
$$EU(s)U(t) = s \wedge t.$$

In this paper, we obtain the uniform version of the central limit theorems(CLT) for the process T_n in (3.4). In obtaining the uniform CLT, we employ the method of uniformly integrable entropy of Ziegler[12] to the process.

Identify the points $t \in [0,1]$ with the class of indicator functions $\{h_t : t \in [0,1]\}$.

We are interested in developing the uniform limiting behavior of the process S_{n3} in (3.4) by using a method of uniformly integrable entropy. We firstly state the following central limit theorem for the sequence of random variables generated by the tent map.

The aim of our work is to develop the CLT and the uniform CLT for the process generated by the tent map by employing Ziegler's idea of the uniformly integrable entropy method.

In section 3.2, we restate the main results in a more general setting of function indexed processes and provide the proofs for the main results. Finally, in section 3.3, we provide an application to Kolmogorov-Smirnov type result.

3.2 A general setting and proofs

We use the following set up to state problem in a concrete fashion. From stationarity, using the Kolmogorov consistency theorem, we may assume that the process, for fixed $j \in \mathbb{N}$, $\{d(\varphi^{m-1}(y))\}$ is double sided. We choose $(\Omega = [0,1]^{\mathbb{Z}}, \mathcal{T} = (\mathcal{B}[0,1])^{\mathbb{Z}}, P)$. We know that the Lebesgue measure P is invariant under φ, that is, $P\varphi^{-1} = P$. We also know that φ is an ergodic map. Define for $m \in \mathbb{Z}$ a σ-field $M_{m-1} = \sigma(\varphi^n(y) : n \leq m-1)$ and $H_{m-1} = \{f : \Omega \to [-1,1] : f$ is M_{m-1} measurable and $f \in L^2(\Omega)\}$. For each $f \in L^2(\Omega)$ we simply denote $E_{m-1}f$ to mean $E(f|M_{m-1})$ and $H_0 \ominus H_{-1} = \{f \in H_0 : E(fg) = 0$ for each $g \in H_{-1}\}$. On $L^2(\Omega)$ we define a metric d by $d(f,g) = [E(f-g)^2]^{1/2}$. Let $\mathcal{F} \subseteq H_0 \ominus H_{-1}$. Consider the function indexed processes defined by

$$T_n(f) = n^{-1/2} \sum_{m=1}^{n} f(\varphi^{m-1}(y)),\ f \in \mathcal{F}$$

where $Ef(y) = 0$ for each $f \in \mathcal{F}$.

In order to measure the size of the function space, we define the following version of metric entropy condition. See, for example, Van der Vaart and Wellner[14] for the recent reference.

Definition 4 *Let $\mathcal{F}$ be a class of measurable functions defined on a measurable space $(\mathbf{X}, \mathcal{X})$ such that $\sup_{f\in\mathcal{F}} |f(x)| \leq 1$ for $x \in \mathbf{X}$. The covering number $N(\epsilon, \mathcal{F}, ||\cdot||)$, simply denote $N(\epsilon)$ when there is no risk of ambiguity, is the minimum number of balls $\{g : ||g-h|| < \epsilon\}$ of radius ϵ needed to cover $\mathcal{F}$.*

Definition 5 *Let $M(\mathbf{X})$ be the set of all finite measures γ on $(\mathbf{X}, \mathcal{X})$. We say $\mathcal{F}$ has uniformly integrable entropy if $\int_0^\infty \sup_{\gamma\in M(\mathbf{X})}[\ln N(\epsilon, \mathcal{F}, d_\gamma^{(2)})]^{1/2}\, d\epsilon < \infty$, where $d_\gamma^{(2)}(f,g) := [\int_{\mathbf{X}}(f-g)^2 d\gamma]^{1/2}$.*

Remark 4 *It is known that $\mathcal{F} = \{h_t : t \in [0,1]\}$ has a uniformly integrable entropy. This is possible because the cardinality of $\mathcal{F}$ is the same as that of $[0,1]$. See the section 2.6 of Van der Vaart and Wellner[14].*

We firstly state the central limit theorem for the sequence of the random variables generated by the chaotic tent map.

Theorem 9 *For each fixed $f \in \mathcal{F}$, as $n \to \infty$, the sequence of random variables $T_n(f)$ converges in distribution to a standard normal distribution.*

Proof. This is a one dimensional central limit theorem for a sequence of an ergodic stationary martingale differences. See for example p.375 in Durrett[4].

We secondly state the uniform central limit theorem for the process generated by the chaotic tent map. Define $\sigma_n^2(f,g) := n^{-1}\sum_{m=1}^{n} E_{m-2}[f(\varphi^{m-1}(y)) - g(\varphi^{m-1}(y))]^2$ for $f, g \in \mathcal{F}$.

Theorem 10 *Assume that there exists a constant L such that*

$$P^*\left\{\sup_{f,g\in\mathcal{F}} \frac{\sigma_n^2(f,g)}{d^2(f,g)} \geq L\right\} \to 0, \text{ as } n \to \infty. \tag{3.5}$$

Then, as $n \to \infty$, $T_n \Rightarrow G$ as random elements of $B(\mathcal{F})$. The limiting process G is a Gaussian process with $EG(f) = 0$ and $EG(f)G(g) = Efg$.

Proof. Observe that the finite dimensional distributions convergence of T_n to those of G is obtained from Theorem 9. The result is now a consequence of Proposition 1 below, taking $V_{n,m}(f) := n^{-1/2} f(\varphi^{m-1}(y))$, by applying Theorem 10.2 of Pollard[10] to the process $(T_n(f) : f \in \mathcal{F})$ indexed by the totally bounded pseudometric space $(\mathcal{F}, d)$.

Consider a process $\{S_n(f) : f \in \mathcal{F}\}$ defined by $S_n(f) := \sum_{m \leq m(n)} V_{n,m}(f)$ for $f \in \mathcal{F}$ where $\{V_{n,m}(f) : m \leq m(n), n \in \mathbb{N}, f \in \mathcal{F}\}$ is a martingale-difference array of $\mathcal{L}_2$-process indexed by the class $\mathcal{F}$ of measurable functions with respect to the σ-fields $\{\mathcal{E}_{n,m} : 0 \leq m \leq m(n), n \in \mathbb{N}\}$. Define $\sigma_n^2(f,g) := \sum_{m\leq m(n)} E_{n,m-1}[V_{n,m}(f) - V_{n,m}(g)]^2$ for $f, g \in \mathcal{F}$. Given random measures μ_n on $(\mathbf{X}, \mathcal{X})$, we define $d_{\mu_n}^{(2)}(f,g) := [\mu_n(f-g)^2]^{1/2}$. The proof of following Proposition 1 appears in Bae, J., et al[2].

Proposition 1 *Let* $\{V_{n,m}(f) : m \leq m(n), n \in \mathbb{N}, f \in \mathcal{F}\}$ *is a martingale-difference array of* $\mathcal{L}_2$*-process indexed by the class* $\mathcal{F}$ *of measurable functions with an envelope* M *on a measurable space* $(\mathbf{X}, \mathcal{X})$*. Suppose that* $\mathcal{F}$ *has uniformly integrable entropy. Let* μ_n*,* $n \in \mathbb{N}$*, be random measures such that*

$$P^* \left\{ \sup_{f,g\in\mathcal{F}} \frac{\sigma_n^2(f,g)}{(d_{\mu_n}^{(2)}(f,g))^2} \geq L \right\} \to 0, \text{ as } n \to \infty, \text{ for a constant } L.$$

Suppose

$$L_n(\delta) := \frac{6}{\delta} \sum_{m\leq m(n)} E[V_{n,m}^2(M) 1_{\{V_{n,m}(M)>\delta\}}] \to 0 \text{ for every } \delta > 0. \tag{3.6}$$

Then given $\epsilon > 0$ *and* $\beta > 0$ *there exists an* $\eta > 0$ *for which* $\limsup_{n\to\infty} P^*(\sup_{d_{\mu_n}^{(2)}(f,g)\leq\eta} \mid S_n(f) - S_n(g)| > 5\beta) \leq 3\epsilon$*.*

Remark 5 *The Lindeberg condition (3.6) is naturally satisfied when the envelop function* M *is constant.*

Finally we remain to finish the proof of Theorem 10. Observe first that $\mathcal{F} = \{h_t : t \in [0,1]\}$ has a uniformly integrable entropy. We need to verify the assumption (3.5) for $\mathcal{F} = \{h_t : t \in [0,1]\}$. Notice that $d^2(f,g) = E[h_t - h_s]^2 = |t-s|^2 E d^2(y) = |t-s|^2$ and $E_{m-2}[f(y_{m-1}) - g(y_{m-1})]^2 = |t-s|^2 E_{m-2} d^2(y_{m-1}) = |t-s|^2 E_{-1} d^2(y)$, where $y_{m-1} := \varphi^{m-1}(y)$. Then the assumption (3.5) boils down to the following lemma.

Lemma 1 *There exist* $L > 0$ *such that*

$$P\left(\sup_{s,t\in[0,1]} \frac{|t-s|^2 E_{-1} d^2(y)}{|t-s|^2} > L \right) \to 0 \text{ as } n \to \infty$$

whose proof is self evident because $d(y)$ is a uniformly bounded random variable.

Hence the proof of Theorem 10 is finished.

3.3 An application to Kolmogorov-Sminov type result

Consider the observable random variable $d(y) := (f_{2^j,j}(y) - 2^{-j})/\{2^{-j}(1 - 2^{-j})\}^{1/2}$ for $j \in \mathbb{N}$. The usual "Kolmogorov-Smirnov type" result is still applied here for the stationary data $d(y), d(\varphi(y)), d(\varphi^2(y)), \ldots$. We may apply the uniform central limit theorem for the process generated by the chaotic tent map to get: $\sup_{t\in[0,1]} n^{-1/2} \sum_{m=1}^{[nt]} d(\varphi^{m-1}(y)) \Rightarrow \sup_{t\in[0,1]} U(t)$.

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논문요약

Tent map에 의해 생성된 확률과정에 대한 불변원리

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Chaotic tent map은 단위구간에서 정의된 ergodic map이다. 본 논문에서는 Chaotic tent map에 의해 생성된 다양한 프로세스의 asymptotic behaviors를 연구한다. Chaotic tent map에 의해 생성된 sequential process의 대수의 법칙의 uniform version과 Donsker의 형태에 대한 불변원리를 연구한다.