Deterministic Chaos in Classical 1D Scattering

Consider a one-dimensional system that consists of two balls moving along the x-axis in response to gravity, as illustrated in the figure below:

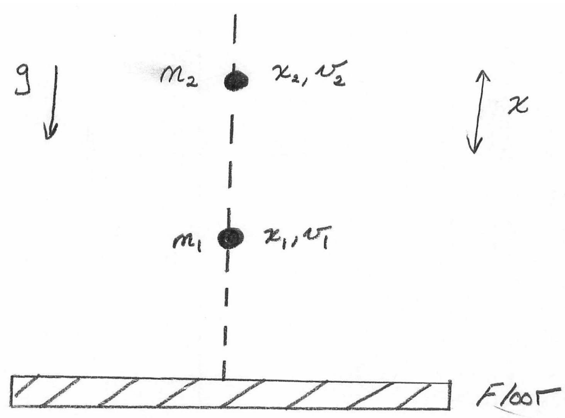

Figure 1: A basic dynamical system with two degrees of freedom. Two balls of mass m_1 and m_2 are constrained to move along the vertical dashed line in one dimension under the downward force of gravity.

The collisions between the balls and with the floor are perfectly elastic and do not deform the balls in any way. Due to these collisions, the velocities in the system exhibit discontinuities with respect to time. In addition, this is a conservative Hamiltonian system.

1. Normalize the dynamical equations of this system the same as is done in the *Whelan et al., 1990* reference given at the end of this description. The energy of the system is: $E = m_1 v_1^2/2 + m_2 v_2^2/2 + m_1 g x_1 + m_2 g x_2$. The normalization values of position, velocity and time are: $x_0 = E/mg$, $v_0 = \sqrt{E/m}$, and $t_0 = \sqrt{E/mg^2}$, where $m = m_1 + m_2$. Note that the E used in the definition of x_0, v_0, and t_0 is the energy *with units*. Using these definitions, the normalized position and velocities are: $\bar{x}_1 = x_1/x_0$, $\bar{x}_2 = x_2/x_0$, $\bar{v}_1 = v_1/v_0$, and $\bar{v}_2 = v_2/v_0$. Using these normalizations, the energy equation becomes:

$$\bar{E} = \frac{E}{E} = 1 = \frac{1}{2}\frac{m_1}{m}\bar{v}_1^2 + \frac{1}{2}\frac{m_2}{m}\bar{v}_2^2 + \frac{m_1}{m}\bar{x}_1 + \frac{m_2}{m}\bar{x}_2$$

   - Clearly describe your normalization and the relevant equations of motion in your text.
   - Convince yourself that $\bar{g} = 1$ for this choice of normalization.
   - Be aware that in this normalization, you should always choose your initial conditions such that the normalized energy = 1.

2. Write a program to follow the motion of both balls, where you have to pay special attention to the collision events at which the motion changes abruptly. When the collision with the floor occurs, you should reverse the velocity of the lower ball. On the other hand, when the balls collide with each other, you should use the formula for elastic collisions of two objects (derived in elementary classical mechanics) to find their velocities after the collision.

   There are two options for determining the dynamics of the balls between collisions in order to determine when the next collision will occur:

(a) **Suggested for PHYS460 students:** Follow the motion of the balls by using the finite difference midpoint method, which actually yields exact (but discretized) solutions for motions with constant acceleration):

$$\begin{aligned} v_{1,i+1} &= v_{1,i} - g\,\Delta t \\ x_{1,i+1} &= x_{1,i} + \frac{v_{1,i+1} + v_{1,i}}{2}\,\Delta t \\ v_{2,i+1} &= v_{2,i} - g\,\Delta t \\ x_{2,i+1} &= x_{2,i} + \frac{v_{2,i+1} + v_{2,i}}{2}\,\Delta t \end{aligned}$$

(b) **Required for PHYS660 students:** Calculate exactly the motion of the balls between collisions/bounces. In other words, after each collision calculate the exact time and position of the next collision. *One helpful comment:* In your program logic, it is probably helpful to use the fact that once the balls have collided, they will always bounce before they collide again.

3. Construct Poincaré sections by plotting v_2 against x_2 at times when the two balls collide. Take $m_1 = 1\,\text{kg}$ and use the initial conditions $x_1 = 1\,\text{m}$, $v_1 = 0\,\text{m/s}$, $x_2 = 3\,\text{m}$, and $v_2 = 0\,\text{m/s}$. *Warning:* Don't forget to convert these initial conditions with units into your normalized units. Do this for the three cases below:

   - $m_2 = 0.5\,m_1$
   - $m_2 = m_1$
   - $m_2 = 9\,m_1$

   Which, if any of these plots indicate that the system is chaotic?

4. Find the position and velocities of the two balls at equal time intervals. Plot x_1 and x_2 against time t for the three cases described in part 3.

5. The autocorrelation function measures the *self-similarity* of a signal with itself in different time intervals and is defined as:

$$C(\tau) = \int_0^\infty \left[x(t) - \bar{x}\right]\left[x(t+\tau) - \bar{x}\right]\,dt,$$

   where $\bar{x}$ is the average value of the signal. Use the autocorrelation function to analyze the "signals" $x_1(t)$ and $x_2(t)$ generated by the motion of two balls. In general, we expect the autocorrelation function to be constant or oscillating for *regular* motions, while it decays exponentially fast to zero for *chaotic* motions.

   - Comment on similarities and differences in the three cases in part 3.
   - For the $m_2 = 9\,m_1$ case, change the initial conditions (keeping $\bar{E} = 1$ until you find a Poincaré section that doesn't look chaotic). How is the autocorrelation function different in this case versus the original initial condition?

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6. Repeat the Poincaré section computation in part 3 for at least 11 additional sets of initial conditions. For example, you can decrease $x_2 = 3$ in steps of 0.1, while adjusting initial velocities so that the total energy remains the same for each set. Plot all 12 Poincaré sections on the same graph and compare with a single Poincaré section you obtained in part 3 using just a single set of initial conditions.

- Make sure that your Poincare sections reveal some of the complex underlying structure if the system you are plotting is chaotic. Don't just choose initial conditions that are all on the same phase space trajectory.

7. For the $m_2 = 9\,m_1$ case, study the Lyapunov exponent for both a chaotic and non-chaotic case.

   - Take your initial condition and run it as before, generating x_2 as a function of time. Then, generate a new x_2 versus time by changing the initial value of x_2 by only a factor of 10^{-6}. Examine how the difference between the two solutions changes in time.
   - Do this for both a chaotic and non-chaotic case and compare the difference. If appropriate, calculate the approximate Lyapunov exponent.

References

1. N. D. Whelan, D. A. Goodings, and J. K. Cannizzo, "Two balls in one dimension with gravity," *Physical Review A,* Vol. 42, No. 2, p. 742, 1990.
   Available at: `http://journals.aps.org.udel.idm.oclc.org/pra/abstract/10.1103/PhysRevA.42.742`

2. Lecture notes on the autocorrelation function of discrete and finite data sets, by W. Gekelman, 2004, UCLA.
   Available on on class wiki page here.

3. Wikipedia article on elastic collisions. Contains the elementary velocity formulas for elastic collisions:
   Available at: `http://en.wikipedia.org/wiki/Elastic_collision`