Part I for both PHYS460 and PHYS660 Students

Consider a radioactive decay problem involving two types of nuclei, A and B, with populations $N_A(t)$ and $N_B(t)$. Suppose that type A nuclei decay to form type B nuclei, which then also decay. This dual decay is described by the following differential equations:

$$\begin{aligned}\frac{dN_A}{dt} &= -\frac{N_A}{\tau_A},\\ \frac{dN_B}{dt} &= \frac{N_A}{\tau_A} - \frac{N_B}{\tau_B},\end{aligned}$$

where τ_A and τ_B are the decay time constants for each type of nucleus.

First, normalize this set of equations. You will find that the normalization is very simple. Use the Euler method to solve these coupled equations numerically for $N_A(t)$ and $N_B(t)$ as a function of time. To avoid having to assign too many numerical values, use $N_A(0) = 200, N_B(0) = 5$, and $\tau_A = 1$ as the unit of time. Then, obtain the analytical solutions for this system of differential equations by using either the "paper-and-pencil" method or Mathematica.

Compare your analytical solution with the numerical one. Explore what controls the stability and accuracy of your numerical solution. Describe your findings in your journal article with plots to back up your conclusions.

Examine the three different cases for relative $\tau_A$and τ_B.

(a) $\tau_A > \tau_B$

(b) $\tau_A = \tau_B$

(c) $\tau_A < \tau_B$

Explore the different behavior of these three cases and when possible highlight the physics controlling this behavior. In particular, try to interpret the short and long time behaviors for different values of the ratio τ_A/τ_B. Write up your findings in your journal article, using plots of these three cases to back up your findings. Make sure to include the exact solutions also in your plot to show that the numerical solutions are accurate. Remember that 1/3 of your grade is based on exploring the physics and numerics of this system using your code. Remember also that 1/3 of your grade is based on a clearly written journal article with good grammar and organization.

Part II for PHYS660 Students Only

Consider again the same problem as in Part I, but now suppose that nuclei of type A decay into the ones of type B, while nuclei of type B decay into the ones of type A. Strictly speaking, this is not a "decay," since it is possible for the type B nuclei to turn back into type A nuclei. A better analogy would be a resonance in which a system can tunnel or move back and forth between two states A and B which have equal energies. The corresponding rate equations are:

$$\frac{dN_A}{dt} = \frac{N_B}{2\tau} - \frac{N_A}{\tau},$$
$$\frac{dN_B}{dt} = \frac{N_A}{\tau} - \frac{N_B}{2\tau},$$

where for simplicity we assume that the two types of decay are characterized by the same constant τ. Solve this system of equations numerically for the numbers of nuclei $N_A(t)$ and $N_B(t)$, with initial conditions $N_A(0) = 150$ and $N_B(0) = 0$, and normalize the equations with $t_0 = \tau$. Show that your numerical results are consistent with the idea that the system reaches a steady state in which N_A and N_B are constant. In such a steady state, the time derivatives dN_A/dt and dN_B/dt should vanish.