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1-Dimensional Scattering Deterministic Chaos

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ABSTRACT

In this study, we examine the chaotic motion of the 1-Dimensional classical scattering problem using numerical techniques. In this example, the goal is to calculate the state of two point particles of differing masses and how they relate to one another. Here, we will test three cases of differing masses. Given certain initial conditions for each particle that result in purely elastic collisions between the particles and the ground, the resulting motion has the capability of becoming chaotic. To visualize such a system, Poincaré sections are taken at the time of the collision between the two particles. Using the Poincaré section plots in conjunction with simple position plots for each particle, it can be seen that changing the particle mass will change the "degree of chaos" of the system. In addition, the autocorrelation function is used to determine that the similarity decreases as the "degree of chaos" of the system increases.

PACs: 5.45.-a, 95.10.Fh

I. INTRODUCTION

In the mid 1800s, a man by the name of Henri Poincaré was born. As Poincaré aged, he grew into a world-renown French mathematician, physicist, engineer, and philosopher. Late in the 19th century, Poincaré became instrumental in the development of chaos theory through his study of the 3-body problem. In order to understand chaos theory, chaos must first be defined. A chaotic system is defined as a system that produces a radically different result by slightly altering the initial conditions. Another synonymous term with chaos is the idea of a "random system." A random set is wholly unsystematic where no trends or patterns can be ascertained. By definition, a chaotic system is a nonlinear system with at least three degrees of freedom.

In this report, the case of a 1-Dimensional classical scattering system is examined. Specifically, there are two masses (m_1 *and* m_2) with some predefined initial conditions denoted as x_{1i}, x_{2i}, v_{1i}, and v_{2i}. that are put into free fall where they can collide with each other and bounce off the "ground" located at $x=0$. This allows for multiple collisions between the masses to occur throughout the duration of the simulation.

The first step in modeling this system is to normalize the classical energy (obtained by summing the kinetic and gravitational energies of both masses) by obtaining normalized values for the initial positions and velocities. The given normalized values for the initial conditions can be assumed by taking the following definitions: $x_0 = E/mg$, and $v_0 = \sqrt{E/mg}$. Here, $M=m_1+m_2$. Now, plugging these definitions into our normalizing equations: $\overline{x_1} = x_1/x_0, \overline{x_2} = x_2/x_0, \overline{v_1} = v_1/v_0$, and $\overline{v_2} = v_2/v_0$ we can obtain a normalized equation for the energy of the system:

$$\bar{E} = \frac{m_1}{2M}\overline{v_1}^2 + \frac{m_2}{2M}\overline{v_2}^2 + \frac{m_1}{M}\overline{x_1} + \frac{m_2}{M}\overline{x_2} \quad (1)$$

This process normalizes g to 1.

To understand how this system could be chaotic in nature, one must see that the system is nonlinear. The motion of the masses can be described by deriving Newton's Second Law ($F=ma$) for a constant force equal to $-g$. This leads to the simple kinematic equation that describes the position of the masses:

$$x_n(t) = x_{n0} + v_{n0}t - \frac{gt^2}{2} \quad (2)$$

However, the nonlinearity arises from the discontinuities found when the masses collide with either each other or the ground. Since the collisions for the problem in question are purely elastic, energy and momentum are conserved as shown in Equation 3:

$$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2 \quad (3)$$

Where v is defined as the initial velocity and u is defined to be the final velocity.

By simply rearranging (3), expressions for u_1 *and* u_2 can be obtained.

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Since the system is nonlinear as apparent from the discontinuities at the collisions and has four degrees of freedom ($x_1(t)$, $x_2(t)$, $v_1(t)$, and $v_2(t)$), it is capable of exhibiting chaotic behavior. Throughout the rest of this report, one can expect to see the methodology for the numerical evaluation of this system. Then the concept of chaos will be examined through the lens of the fundamental Poincaré sections. Finally, there will be an examination of the autocorrelation function to study the self-similarity of the system from one timestep to the next. This is another approach to studying the concept of chaos.

II. METHODOLOGY

To begin, consider the famous finite difference midpoint method to aid in the evaluation of this system.

2.1. FINITE DIFFERENCE MIDPOINT METHOD

In this study, the finite difference midpoint method is employed to evaluate the positions and velocities of the two masses for any given timestep. In this system, collisions can only occur between the two masses and the ground. An exact state of the masses can be obtained by using the data from the previous timestep, or collision. When (2) breaks down at a collision, the values of x_1, x_2, v_1, and v_2 are calculated for all timesteps up until that point. Then, by using the conservation of energy and conservation of momentum principles which is shown in (3), equations of state can be obtained through simple algebra. This manipulation leads to:

$$u_1 = \frac{v_1(m_1 - m_2) + 2m_2v_2}{M} \qquad (4)$$

$$u_2 = \frac{v_2(m_2 - m_1) + 2m_1v_1}{M} \qquad (5)$$

(4) and (5) are used to calculate the state of the system at the next timestep. More trivially, when a mass hits the floor, all that needs to be done is change the sign of the velocity to reflect a complete flip in direction.

Using the given formulae, the velocities of both masses at the next timestep can be calculated by taking the current velocity and subtracting the product of g and dt. To find the next positions, one would take the sum of the current position and the average velocity at that time and the velocity at the next step multiplied by dt.

By systematically moving forward and counting the discrete values at each timestep, a complete set of information for the system can eventually be obtained. Using the finite difference midpoint method will prove instrumental in the evaluation of the system and in determining the degree of chaos within the system through the Poincaré sections.

III. RESULTS AND CHAOS

Now that a method of evaluation has been proven, it is time to review the system for different mass values. Specifically, three cases will be examined to see the when the system will become chaotic. The degree of chaos can be ascertained by multiple different means. One way to determine how chaotic a system is is by examining the associated Poincaré section graph. Another way is to look at the simple position vs. time plot to see the motion of the two masses. One other method that will be used to examine chaos is the autocorrelation function.

3.1 COMPARING MASSES

In the following sections, three different cases will be examined to see if this dynamical system is chaotic. The first case will be when $m_2=0.5m_1$. The next case that will be looked at is when $m_2=m_1$. The final case in this report will be when $m_2=9m_1$. In addition, there are initial conditions for x_1, v_1, x_2, and v_2 that do not change throughout these three cases. These initial conditions take $x_1=1m$, $x_2=3m$, $v_1=0m/s$, and $v_2=0m/s$.

3.1.1 $m_1 < m_2$ CASE

In the first case, m_2 is taken to be $0.5m_1$. This indicates that m_2, or the top ball mass, is only half as massive as m_1, or the bottom ball mass. Now, as explained in Sec. II, the finite difference midpoint method will be used to evaluate the system. Figure 1 shows the simple position vs. time plot of both masses with respect to the ground.

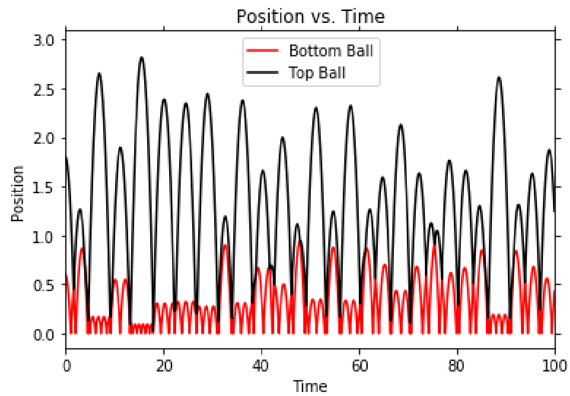

Figure 1: This plot shows the motion of the top and bottom balls as a function of time for the m_2=0,5m_1 case. Note the lack of any distinct patterns or periodic motion. This implies that this system may be chaotic.

There seems to be no obvious pattern or periodic motion leading one to believe that in this case, the system is chaotic. The top and bottom ball seem to be colliding with each other and the floor in a chaotic-like way. Note that the other cases will need to be examined before one can confidently say that this system is chaotic in nature rather than random.

The Poincaré section plot will yield more information that can aid in the determination of this case's chaotic nature. Figure 2 shows the Poincaré section for this case.

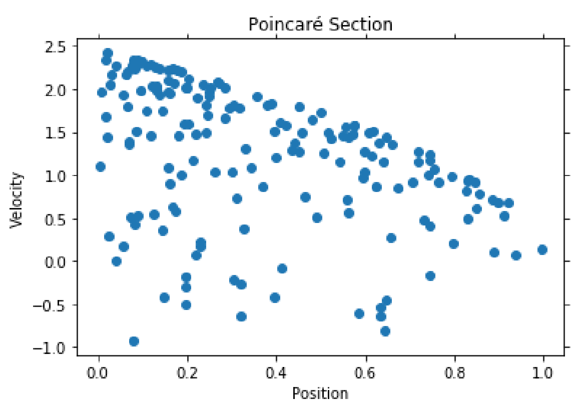

Figure 2: This plot shows the Poincaré section of the top ball for the case when m_2 is less than m_1. There is little definitive structure implying that the system is chaotic.

Figure 2 shows the Poincaré section for the top ball where velocity and position are plotted against one another. As seen from the plot above, there is a slight overall downward trend which simply is the ball losing velocity and energy as it undergoes more collisions with m_1 (bottom ball) and the ground. Other than this trivial point, the system seems to have no inherent structure. This lack of structure in the Poincaré section suggests that this case is chaotic.

3.1.2 $m_1 > m_2$ CASE

In the next case, m_2 is taken to equal $9m_1$. Now, rather than the top ball being less massive than the bottom ball, it is much more massive than its counterpart. Again, using the same evaluation method, Figure 3 shows the position vs. time plot of the two balls.

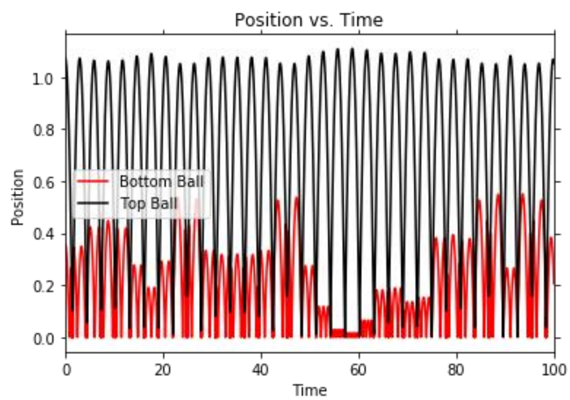

Figure 3: This plot shows the motion of the top and bottom balls as a function of time for the m_2=9m_1 case. Here, the bottom ball, or m_1 seems to have a chaotic motion while the top ball undergoes a distinct periodic motion. This plot also suggests that the system may be chaotic due to the top ball's effect on the bottom ball's motion.

In contrast from Figure 1, Figure 3 now seems to show a fairly periodic motion for the top ball while the bottom ball still follows no distinctive pattern of motion. The bottom ball is still undergoing chaotic motion due to the top ball being so much more massive. During these collisions, the major change in motion occurs in the bottom ball due to the conservation of momentum principle.

Turning to the Poincaré section for this case, it will again yield valuable insight into determining the degree of chaos in the system. Figure 4 reflects as such.

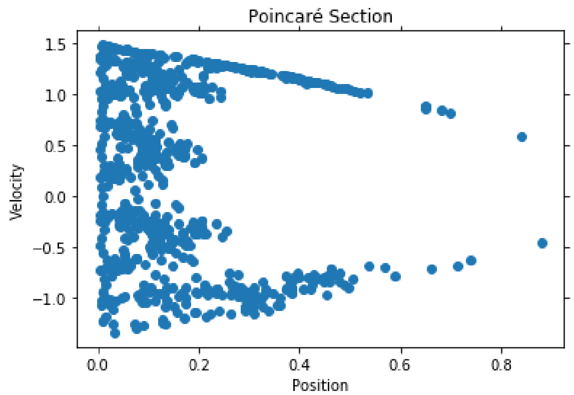

Figure 4: Here is a plot of the Poincaré section when $m_2=9m_1$. This emphasizes the top ball. This plot shows little pattern implying that this system is also chaotic in nature.

Figure 4 shows the Poincaré section for the top ball where its position and velocity are plotted against each other. Again, there seems to be very little inherent structure to this plot leading one to believe that this system is also chaotic. The more massive ball's effect on its much less massive partner also seems to be chaotic in nature.

3.1.3 $m_1 = m_2$ CASE

The third and final case in this report is when the masses are equal. Now, $m_1=m_2=1$. Figure 5 shows the relationship between the top and bottom ball's motions. Here, the masses are equal in value.

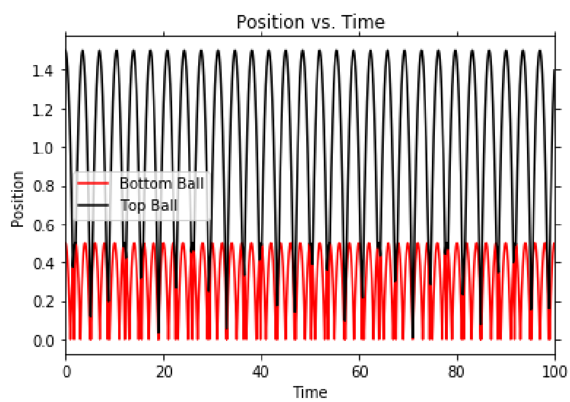

Figure 5: This plot shows the motions of the top and bottom balls as a function of time. Notice how these motions follow an obviously periodic pattern of motion. This obvious pattern could suggest that this case is not chaotic in nature.

In stark contrast to the other two cases, the case when the masses are equal exhibits a highly periodic motion in both balls. Just by examining Figure 5, one may immediately think that this case is *not* chaotic. Unlike the other cases, these motions do not deviate much from a regular periodic motion. Since this motion stays effective constant in nature and does not change over time, the implication may be that it is not chaotic in nature.

Just to be sure, the Poincaré section is again plotted to note the velocity vs. position of the top ball. Figure 6 shows this.

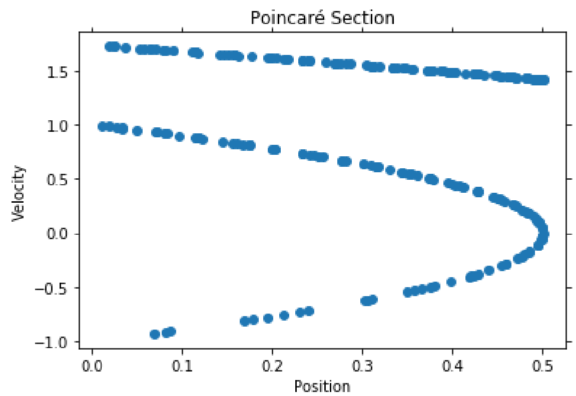

Figure 6: This plot is of the Poincaré section for the top ball in the cases where the masses of the two balls are equal in magnitude. Here, an obvious pattern is visible between the velocity and position. The data does not deviate from

Notice how in Figure 6, there is a distinct pattern that the data follows. It is broken up between a "straight line" and a "sideways parabola." These fairly distinct and defined shape with only slight deviations suggest that this case of the system is nonchaotic.

In summary, the two extreme cases where m_2 is less than m_1 and when m_2 is much greater than m_1 exhibit signs of chaos. This intuitively makes sense solely due to the fact that if the masses are not similar in magnitude, one mass will simply overpower the other and force it into a chaotic state.

3.2 CHANGING THE INITIAL CONDITIONS

In addition to manipulating the masses to induce or not induce chaos, one may also consider changing the velocities. In this section, the case where $m_2=9m_1$ will be examined in an attempt to eliminate the chaotic nature of this case while keeping the energy normalized to a standard value of 1.0.

In order to achieve a non-chaotic system without altering the masses, one must change the initial positions, velocities or both for the masses.

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Thinking about the system in a physical sense, what would need to happen in order to make a system with a much more massive ball less chaotic? Intuition may say to increase the velocity of the less massive ball, so let's start there.

Figures 7 and 8 reflect the Poincaré section and position plots with new initial conditions of $m_1=1$, $m_2=9m_1$, $x_1=1m$, $v_1=5m/s$, $x_2=3m$, and $v_2=0m/s$.

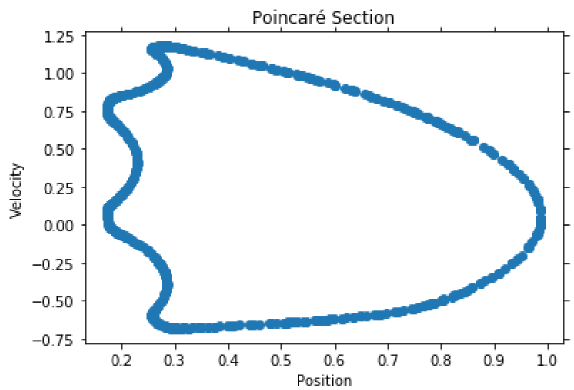

Figure 7: This figure shows the Poincaré section for the $m_2=9m_1$ case with the updated initial conditions. Comparing this to Figure 4, the data looks a lot smoother and more confined to a specified track. This can lead one to believe that the system is less chaotic than before.

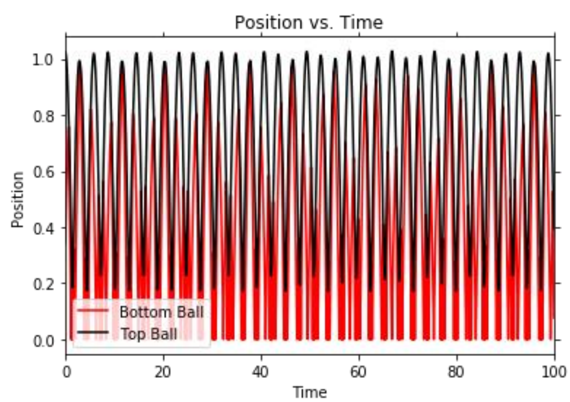

Figure 8: This plot shows the position vs. time graph for the top and bottom balls. Notice how this motion does not change in nature as radically as the motion presented in Figure 3. This is another good indication that the system is less chaotic.

It seems that intuition has proven correct. The presence of a well-defined track of data in Figure 7 implies that the data produced from this system is less chaotic in nature than before. The motion follows a set pattern. This is further emphasized in Figure 8. The periodic motion of the two masses looks much more uniform than before.

As is good in all scientific experiments, gathering as much data as possible helps increase accuracy. So, one more case will be tested to examine the nature of this system. Let's see what happens when the velocity of the more massive ball is changed. The new initial conditions will be $m_1=1$, $m_2=9m_1$, $x_1=1m$, $v_1=0m/s$, $x_2=3m$, and $v_2=5m/s$. Figures 9 and 10 reflect the Poincaré section and position vs. time plots for this case.

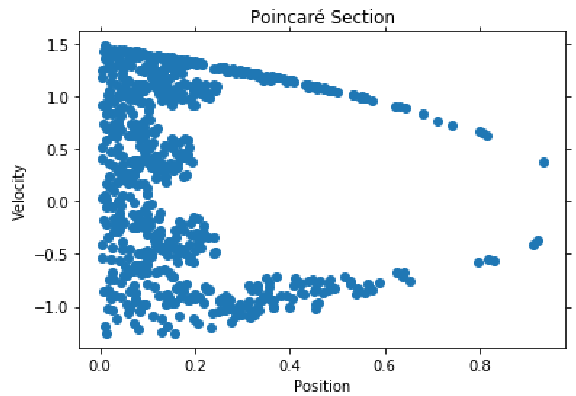

Figure 9: This plot shows the Poincaré section for the $m_2=9m_1$ case with the updated initial conditions for the more massive ball. Note that this looks very similar to Figure 4, implying that the system is chaotic again.

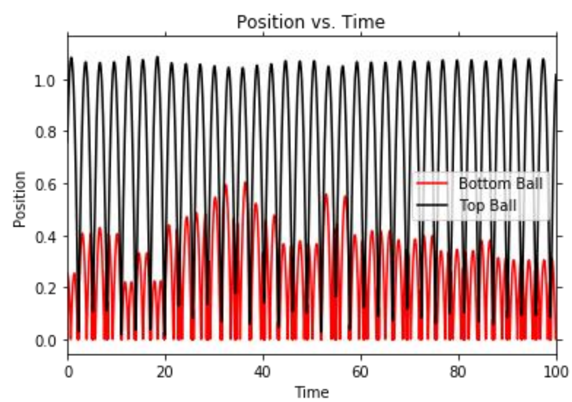

Figure 10: The position vs. time graph for the top and bottom ball with the updated velocity for the more massive ball shows that the motion does not follow a systematic pattern.

Figures 9 and 10 help to exemplify the definition of chaos. It only takes a slight change in initial conditions to return the system to a less chaotic regime. It also only takes a small change to make the system chaotic again. In cases like this, physically intuition provides a great starting point to figure out the best initial conditions to achieve chaos or a lack thereof.

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IV. THE AUTO CORRELATION FUNCTION

One other indicator of chaos is through the autocorrelation function. The autocorrelation function measures the amount of self-similarity that the system has as time progresses. Chaotic systems show almost no similarity with itself over time. The autocorrelation, which is evaluated by the following integral, accounts for this.

$$C(\tau) = \int_0^{\infty} [x(t) - \bar{x}][x(t+\tau) - \bar{x}]dt \qquad (6)$$

The next few plots will show the autocorrelation functions for the cases highlighted in sections *3.1.1*, *3.1.2*, *3.1.3*, and *3.2*. Note the start differences between the chaotic and non-chaotic cases. Chaotic cases should show no real pattern while the non-chaotic cases should show a clear periodic pattern for this problem.

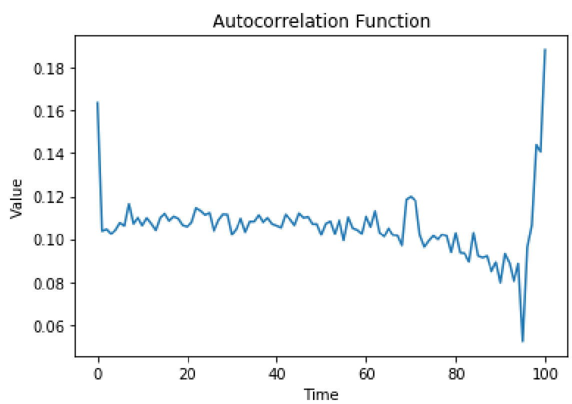

Figure 11: This plot shows the autocorrelation function from section 3.1.1, or when $m_2=0.5m_1$. From above, this case is anticipated to be chaotic. The autocorrelation function's lack of periodic nature further strengthens the conclusion that the system is chaotic.

Figure 11 shows no clear structure. This strengthens the argument from above that the case in section *3.1.1* is chaotic.

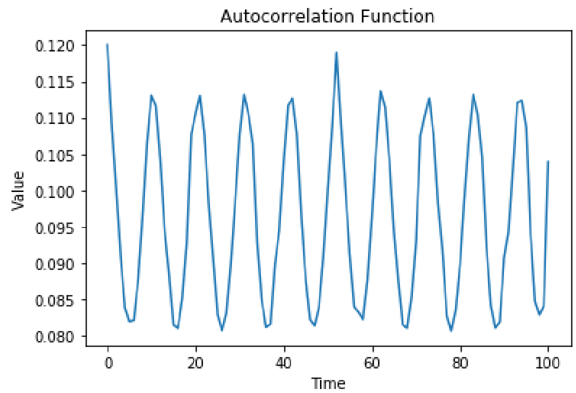

Figure 12: This figure emphasizes the case where $m_2=m_1$. As suggested above, this system is non-chaotic. See the clear periodic structure in this graph. This implies that this system is non-chaotic and has some periodic nature to it.

Figure 12 shows a clear periodic structure suggesting that the case in section *3.1.2* is not chaotic.
Figure 13 below shows the autocorrelation function for the case in section *3.1.3*. Again, notice the lack of an inherent structure to the plot leading one to believe that this system is chaotic, as stated above in the relevant section.

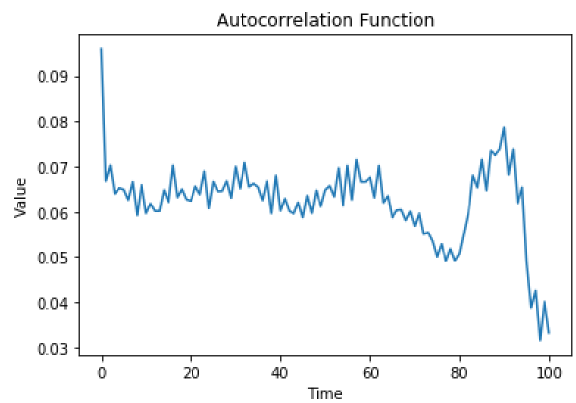

Figure 13: This plot shows the autocorrelation function for the case where $m_2=9m_1$. See that there is no real pattern in this plot leading one to believe that this case is chaotic. Again, this further strengthens the claim from section 3.1.3 above.

Figure 14 shows the autocorrelation function from section *3.2* where the initial conditions from the $m_2=9m_1$ case where changed to reflect a higher velocity for the less massive ball. Recall that the initial conditions were changed to $m_1=1$, $m_2=9m_1$, $x_1=1m$, $v_1=5m/s$, $x_2=3m$, and $v_2=0m/s$. Note the difference between Figure 14 and Figure 13. Figure 14 has a periodic structure to it, suggesting that now, the system is less chaotic. Again, slightly

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altering the initial conditions can have huge impacts on the nature of the system.

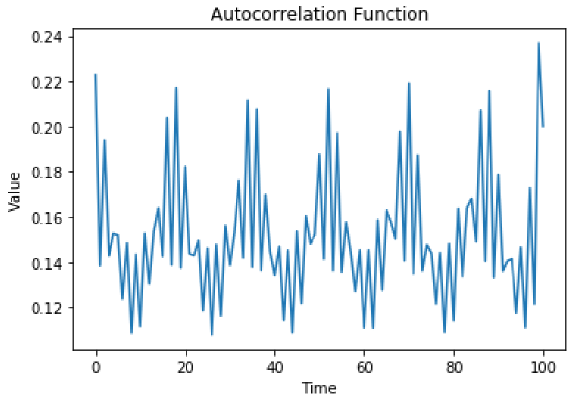

Figure 14: This plot shows the autocorrelation function for the updated initial conditions found in section 3.2: $m_1=1$, $m_2=9m_1$, $x_1=1m$, $v_1=5m/s$, $x_2=3m$, and $v_2=0m/s$. Notice how the data follows a periodic pattern. This further again suggests that this case is not chaotic. Only changing one initial condition can have quite an impact on the result.

V. CONCLUSION

To conclude, analyzing a simple chaotic system such as two masses being dropped in free-fall and constrained to collide with each other and the ground provides valuable insight into the nature of chaos and how it evolves. In this system, when m_2 is less massive or much more massive than m_1, the system turns out to be chaotic. However, when the masses are equal, the system is not chaotic.

Slight changes to the initial conditions can have significant impacts on the resulting degree of chaos within the system. Recalling from section *3.2*, only changing the velocity of the less massive ball to reflect a faster speed radically changed the nature of the system from chaotic to not chaotic. Poincaré sections, position vs. time plots, and the autocorrelation function are all good gauges for determining the degree of chaos within a system.

Chaos and the N-Body problem are fascinating subjects to study and can be studied in much more detail. But understanding the fundamental concepts in a simple situation such as this provides a pathway to the examination of much more complex systems found in areas of quantum and astrophysics.