QB

② $\int_0^\infty e^{2t} \cdot t \cos dt$

→ we will find $L\{t \cos t\}$

$$L\{\cos t\} = \frac{s}{s^2+1^2} = \frac{s}{s^2+1}$$

$$L\{t \cos t\} = \frac{-d}{ds} \frac{s}{s^2+1} \quad \cdots \frac{u}{v}$$

$$= -\frac{\left[(s^2+1)\frac{d}{ds} s - s\frac{d}{ds}(s^2+1)\right]}{(s^2+1)^2}$$

$$= -\frac{[(s^2+1)(1) - s(2s)]}{(s^2+1)^2}$$

$$= -\frac{[s^2+1-2s^2]}{(s^2+1)^2}$$

$$= -\frac{[-s^2+1]}{(s^2+1)^2}$$

$$L\{t \cos t\} = \frac{s^2-1}{(s^2+1)^2}$$

By defn of LT

$$\int_0^\infty e^{-st} t \cos t \, dt = \frac{s^2-1}{(s^2+1)^2}$$

Put $s = -2$

$$\int_0^\infty e^{-(-2)t} t \cos t \, dt = \frac{(-2)^2-1}{((-2)^2+1)^2}$$

$$\int_0^\infty e^{2t} t \cos t \, dt = \frac{4-1}{(4+1)^2}$$

$$= \frac{3}{(5)^2} = \frac{3}{25}$$

③ $L\{t^3 \cosh t\}$

$\rightarrow L\left\{t^3\left(\frac{e^t+e^{-t}}{2}\right)\right\}$

$= \frac{1}{2} L\{t^3(e^t+e^{-t})\}$

$= \frac{1}{2} L\{t^3 e^t + t^3 e^{-t}\}$

$= \frac{1}{2}\left[L\{e^t t^3\} + L\{e^{-t} \cdot t^3\}\right]$ .... $L\{t^3\} = \frac{3!}{S^{3+1}} = \frac{3\times 2\times 1}{S^4}$

shifting $S \rightarrow S-1$ $\quad S \rightarrow S+1$ $\quad \therefore L\{t^3\} = \frac{6}{S^4}$

$= \frac{1}{2}\left[\frac{6}{(S-1)^4} + \frac{6}{(S+1)^4}\right]$

$= \frac{6}{2}\left[\frac{1}{(S-1)^4} + \frac{1}{(S+1)^4}\right]$

$= 3\left[\frac{1}{(S-1)^4} + \frac{1}{(S+1)^4}\right]$

(4) $L\{e^{-2t} + t^{3/2} + \sin^2 t\}$

$= L\{e^{-2t}\} + L\{t^{3/2}\} + L\{\sin^2 t\}$ -- $L\{t^n\} = \frac{\Gamma n+1}{s^{n+1}}$

$= \frac{1}{s+2} + \frac{\Gamma\frac{3}{2}+1}{s^{\frac{3}{2}+1}} + L\left\{\frac{1-\cos 2t}{2}\right\}$

$= \frac{1}{s+2} + \frac{\Gamma 5/2}{s^{5/2}} + \frac{1}{2} L\{1-\cos 2t\}$

$= \frac{1}{s+2} + \frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{s^{5/2}} + \frac{1}{2}\left[L\{1\} - L\{\cos 2t\}\right]$

$= \frac{1}{s+2} + \frac{\frac{3}{4}\cdot\sqrt{\pi}}{s^{5/2}} + \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+2^2}\right]$

$= \frac{1}{s+2} + \frac{3\sqrt{\pi}}{4\cdot s^{5/2}} + \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$

⑤ $L\left\{\int_0^t \frac{e^{-u}\sin 2u}{u}\,du\right\}$

$= L\left\{\int_0^t \frac{e^{-t}\sin 2t}{t}\,dt\right\}$

$L\{\sin 2t\} = \frac{2}{s^2+2^2}$

$L\{\sin 2t\} = \frac{2}{s^2+4}$

$L\left\{\frac{\sin 2t}{t}\right\} = \int_s^\infty \frac{2}{s^2+4}\,ds$

$= 2\int_s^\infty \frac{1}{s^2+2^2}\,ds$

$= \cancel{2}\left[\frac{1}{\cancel{2}}\tan^{-1}\frac{s}{2}\right]_s^\infty$

$L\left\{\frac{\sin 2t}{t}\right\} = \left[\tan^{-1}\frac{s}{2}\right]_s^\infty$

$= \tan^{-1}\left(\frac{\infty}{2}\right) - \tan^{-1}\left(\frac{s}{2}\right)$

$= \tan^{-1}\infty - \tan^{-1}\left(\frac{s}{2}\right)$

$L\left\{\frac{\sin 2t}{t}\right\} = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{2}\right)$

$= \cot^{-1}\left(\frac{s}{2}\right)$

$$L\left\{e^{-t}\,\frac{\sin 2t}{t}\right\} = \cot^{-1}\left(\frac{s+1}{2}\right)$$

$s \to s+1$

$$L\left\{\int_0^t e^{-t}\,\frac{\sin 2t}{t}\,dt\right\} = \frac{1}{s}\cot^{-1}\left(\frac{s+1}{2}\right)$$

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(6) Find $L\{t\,f(2t)\}$ if $L\{f(t)\} = \frac{s}{s^2+1}$

$$L\{f(t)\} = \frac{s}{s^2+1} = \bar{f}(s)$$

if $L\{f(t)\} = \bar{f}(s)$ Then
$L\{f(at)\} = \frac{1}{a}\,\bar{f}\left(\frac{s}{a}\right)$

$$\therefore L\{f(2t)\} = \frac{1}{2}\,\bar{f}\left(\frac{s}{2}\right)$$

$$= \frac{1}{2}\left[\frac{\frac{s}{2}}{\left(\frac{s}{2}\right)^2+1}\right]$$

$$= \frac{\frac{s}{4}}{\frac{s^2}{4}+1} = \frac{s/4}{\frac{s^2+4}{4}}$$

$$\therefore L\{f(2t)\} = \frac{s}{s^2+4}$$

$$L\{t\,f(2t)\} = -\frac{d}{ds}\,\frac{s}{s^2+4}$$

- 1 derivative

$$= -\frac{\left[(s^2+4)\frac{d}{ds}s - s\frac{d}{ds}(s^2+4)\right]}{(s^2+4)^2}$$

$$= -\frac{\left[(s^2+4)(1) - s(2s)\right]}{(s^2+4)^2}$$

$$= -\frac{\left[s^2+4-2s^2\right]}{(s^2+4)^2}$$

$$= \frac{-\left[-s^2+4\right]}{(s^2+4)^2}$$

$$\therefore L\{t f(2t)\} = \frac{(s^2-4)}{(s^2+4)^2}$$

(9) $L^{-1}\left\{\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\}$ ∵ $\log\frac{a}{b} = \log a - \log b$

$= L^{-1}\left\{\log(s^2+a^2) - \log(s^2+b^2)\right\}$

$= \frac{-1}{t} L^{-1}\left\{\frac{d}{ds}\log(s^2+a^2) - \frac{d}{ds}\log(s^2+b^2)\right\}$ ∵ $\frac{d}{dx}\log u = \frac{1}{u}\frac{du}{dx}$

$= \frac{-1}{t} L^{-1}\left\{\frac{1}{s^2+a^2}\frac{d}{ds}(s^2+a^2) - \frac{1}{(s^2+b^2)}\frac{d}{ds}(s^2+b^2)\right\}$

$= \frac{-1}{t} L^{-1}\left\{\frac{1}{s^2+a^2}(2s+0) - \frac{1}{s^2+b^2}(2s+0)\right\}$

$= \frac{-1}{t} L^{-1}\left\{\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right\}$

$= \frac{-1}{t}\left[2L^{-1}\left\{\frac{s}{s^2+a^2}\right\} - 2L^{-1}\left\{\frac{s}{s^2+b^2}\right\}\right]$

$= \frac{-1}{t}\left[2\cos at - 2\cos bt\right]$ ∵ $L\{\cos at\} = \frac{s}{s^2+a^2}$

$\cos at \leftarrow L^{-1}\left\{\frac{s}{s^2+a^2}\right\}$

$= \frac{-2}{t}\left[\cos at - \cos bt\right]$

$= \frac{2}{t}\left[\cos bt - \cos at\right]$

(10) $L^{-1}\left\{\frac{s}{(s+1)(s+5)}\right\}$

let $\frac{s}{(s+1)(s+5)} = \frac{A}{(s+1)} + \frac{B}{(s+5)}$

$A \left| \begin{pmatrix} s+1=0 \\ s=-1 \end{pmatrix} \right. = \frac{-1}{(-1+5)} = \frac{-1}{4}$

$B \left| \begin{pmatrix} s+5=0 \\ s=-5 \end{pmatrix} \right. = \frac{-5}{(-5+1)} = \frac{-5}{-4} = \frac{5}{4}$

$\frac{s}{(s+1)(s+5)} = \frac{-1/4}{s+1} + \frac{5/4}{s+5}$

$L^{-1}\left\{\frac{s}{(s+1)(s+5)}\right\} = L^{-1}\left\{\frac{-1/4}{s+1} + \frac{5/4}{s+5}\right\}$

$= \frac{-1}{4}L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{5}{4}L^{-1}\left\{\frac{1}{s+5}\right\}$

$= \frac{-1}{4}e^{-1t} + \frac{5}{4}e^{-5t} = \frac{\left(-e^{-t} + 5e^{-5t}\right)}{4}$

$= \frac{5e^{-5t} - e^{-t}}{4}$

(11) $L^{-1}\left\{\dfrac{s}{(s^2+9)(s^2+5)}\right\}$

Let $\dfrac{1}{(s^2+9)(s^2+5)}$ where $s^2 = x$

$$\frac{1}{(x+9)(x+5)} = \frac{A}{x+9} + \frac{B}{x+5}$$

$A \left| \begin{pmatrix} x+9=0 \\ x=-9 \end{pmatrix} \right. = \dfrac{1}{(-9+5)} = \dfrac{1}{-4} = \dfrac{-1}{4}$

$B \left| \begin{pmatrix} x+5=0 \\ x=-5 \end{pmatrix} \right. = \dfrac{1}{(-5+9)} = \dfrac{1}{4} = \dfrac{1}{4}$

$\dfrac{1}{(x+9)(x+5)} = \dfrac{-1/4}{x+9} + \dfrac{1/4}{x+5}$ .... Put $x = s^2$

$$\frac{1}{(s^2+9)(s^2+5)} = \frac{-1/4}{s^2+9} + \frac{1/4}{s^2+5}$$

Multiply both sides by s

$$\frac{s}{(s^2+9)(s^2+5)} = \frac{-1/4\, s}{s^2+9} + \frac{1/4\, s}{s^2+5}$$

$$L^{-1}\left\{\frac{s}{(s^2+9)(s^2+5)}\right\} = L^{-1}\left\{\frac{-1/4\, s}{s^2+9}\right\} + L^{-1}\left\{\frac{1/4\, s}{s^2+5}\right\}$$

$$= \frac{-1}{4} L^{-1}\left\{\frac{s}{s^2+3^2}\right\} + \frac{1}{4} L^{-1}\left\{\frac{s}{s^2+\sqrt{5}^2}\right.$$

$$= \frac{-1}{4}\cos 3t + \frac{1}{4}\cos\sqrt{5}\,t$$

$$= \frac{-\cos 3t + \cos\sqrt{5}\,t}{4}$$

$$= \frac{\cos\sqrt{5}\,t - \cos 3t}{4}$$

(12) $L^{-1}\left\{\dfrac{1}{s^2(s+3)}\right\}$

Let $\dfrac{1}{s^2(s+3)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s+3}$ ... (1)

$$\frac{1}{\cancel{s^2(s+3)}} = \frac{As(s+3) + B(s+3) + Cs^2}{\cancel{s^2(s+3)}}$$

$1 = As(s+3) + B(s+3) + Cs^2$ ... (2)

Put $s=0$ in (2)

$1 = A(0)(0+3) + B(0+3) + C(0)^2$

$1 = 0 + B(3) + 0$

$\therefore 1 = 3B$

$\therefore \boxed{B = \dfrac{1}{3}}$

Put $s+3=0$ i.e $s=-3$ in (2)

$1 = A(-3)\underbrace{(-3+3)}_{0} + B\underbrace{(-3+3)}_{0} + C(-3)^2$

$1 = 0 + 0 + C(9)$ $\therefore 1 = 9C$

$\therefore \boxed{\dfrac{1}{9} = C}$

Put $s=1$ in eqⁿ (2)

$$1 = A(1)(1+3) + B(1+3) + C(1)^2$$

$$1 = A(1)(4) + \frac{1}{3}(4) + \frac{1}{9}(1)$$

$$1 = 4A + \frac{4}{3} + \frac{1}{9}$$

$$\therefore 1 - \frac{4}{3} - \frac{1}{9} = 4A$$

$$\therefore \frac{-4}{9} = 4A$$

$$\frac{-1}{9} = A \qquad \therefore \boxed{A = \frac{-1}{9}}$$

$$L^{-1}\left\{\frac{s^2}{(s^2+1)(s^2+4)}\right\} = L^{-1}\left\{\frac{s}{s^2+1}\times\frac{s}{s^2+4}\right\}$$

$$= L^{-1}\{\bar f(s)\times \bar g(s)\}$$

where $\bar f(s) = \frac{s}{s^2+1}$, $\bar g(s) = \frac{s}{s^2+4}$

$f(t) = L^{-1}\left\{\frac{s}{s^2+1^2}\right\}$ $\quad g(t) = L^{-1}\left\{\frac{s}{s^2+2^2}\right\}$

$f(t) = \cos 1t$, $\quad g(t) = \cos 2t$

$f(t-u) = \cos(t-u)$, $g(u) = \cos 2u$

By Convolution Theorem

$$L^{-1}\{\bar f(s)\bar g(s)\} = \int_0^t f(t-u)\,g(u)\,du$$

$$L^{-1}\left\{\frac{s^2}{(s^2+1)(s^2+4)}\right\} = \int_0^t \cos(t-u)\cos 2u\,du$$

$$= \frac{1}{2}\int_0^t 2\cos 2u\cos(t-u)\,du$$

$$2\cos A\cos B = \cos(A-B)+\cos(A+B)$$

$$= \frac{1}{2}\int_0^t \left[\cos(2u-(t-u)) + \cos(2u+(t-u))\right]du$$

$$= \frac{1}{2}\int_0^t \left[\cos(2u-t+u) + \cos(2u+t-u)\right]du$$

$$= \frac{1}{2}\int_0^t \left[\cos(3u-t) + \cos(u+t)\right]du$$

$$= \frac{1}{2}\left[\left[\frac{\sin(3u-t)}{3} + \frac{\sin(u+t)}{1}\right]\right]_0^t$$

coeff of u

$$= \frac{1}{2}\left[\left\{\frac{\sin(3t-t)}{3} + \sin(t+t)\right\} - \left\{\frac{\sin(3\times 0-t)}{3} + \sin(0+t)\right\}\right]$$

$$= \frac{1}{2}\left[\left\{\frac{\sin 2t}{3} + \sin 2t\right\} - \left\{\frac{\sin(-t)}{3} + \sin t\right\}\right]$$

$$= \frac{1}{2}\left[\frac{\sin 2t}{3} + \sin 2t - \left\{-\frac{\sin t}{3} + \sin t\right\}\right]$$

$$= \frac{1}{2}\left[\frac{\sin 2t}{3} + \sin 2t + \frac{\sin t}{3} - \sin t\right]$$

$$= \frac{1}{2}\left[\frac{\sin 2t + 3\sin 2t + \sin t - 3\sin t}{3}\right]$$

$$= \frac{1}{6}\left[4\sin 2t - 2\sin t\right]$$

$$= \frac{2}{6}\left[2\sin 2t - \sin t\right]$$

$$= \frac{1}{3}\left[2\sin 2t - \sin t\right]$$

$$L^{-1}\left\{\frac{1}{(s+1)(s-2)^2}\right\}$$

$$= L^{-1}\left\{\frac{1}{s+1} \times \frac{1}{(s-2)^2}\right\}$$

$$= L^{-1}\left\{\bar{f}(s) \times \bar{g}(s)\right\}$$

Whn

$$\bar{f}(s) = \frac{1}{s+1} \ \& \ \bar{g}(s) = \frac{1}{(s-2)^2}$$

$$f(t) = L^{-1}\left\{\frac{1}{s+1}\right\} \qquad g(t) = L^{-1}\left\{\frac{1}{(s-2)^2}\right\}$$

$$= e^{-1t} \qquad = e^{2t} L^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= e^{-t} \qquad = e^{2t} \cdot \frac{t^{(2-1)}}{(2-1)!}$$

$$f(t-u) = e^{-(t-u)}$$

$$= e^{-t+u} \qquad g(t) = \frac{e^{2t}\, t^1}{1!} = e^{2t} \cdot t$$

$$f(t-u) = e^{-t} \cdot e^{u} \qquad g(u) = e^{2u} \cdot u$$

By Convolution Th $L^{-1}\left\{\bar{f}(s)\bar{g}(s)\right\} = \int_0^t f(t-u)g(u)\,du$

$$L^{-1}\left\{\frac{1}{(s+1)(s-2)^2}\right\} = \int_0^t e^{-t} \cdot e^{u} \cdot e^{2u} \cdot u\,du = e^{-t} \int_0^t e^{u+2u} \cdot u\,du$$

apply $\int uv$ generalised Rule

$$= e^{-t} \int_0^t u e^{3u}\,du$$

A E (LIATE)

$$= e^{-t}\left[u\frac{e^{3u}}{3} - (1)\frac{e^{3u}}{3 \times 3} + c\right]_0^t$$

$$= e^{-t}\left[\frac{u e^{3u}}{3} - \frac{e^{3u}}{9}\right]_0^t$$

$$= e^{-t}\left[e^{3u}\left(\frac{u}{3} - \frac{1}{9}\right)\right]_0^t$$

$$= e^{-t}\left[\left\{e^{3t}\left(\frac{t}{3} - \frac{1}{9}\right)\right\} - \left\{e^{0}\left(\frac{0}{3} - \frac{1}{9}\right)\right\}\right]$$

$$= e^{-t}\left[e^{3t}\left(\frac{t}{3} - \frac{1}{9}\right) - \left\{1\left(0 - \frac{1}{9}\right)\right\}\right]$$

$$= e^{-t}\left[e^{3t}\left(\frac{t}{3} - \frac{1}{9}\right) + \frac{1}{9}\right]$$