2D Geometric Transformations

Basic Transformations

1) Translation

→ applied to an object by repositioning it along a straight line path from one coordinate location to another.

→ done by adding translation distance tx and ty to the original coordinate position (x, y) to move the point to new position (x', y')

$$x' = x + t_x \qquad y' = y + t_y$$

Translation distance pair (t_x, t_y) is called translation vector or shift vector.

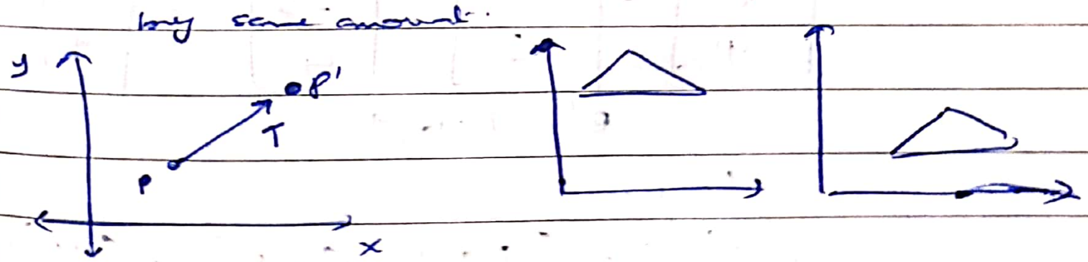
$$P = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad P' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} \qquad T = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

$$P' = P + T$$

I. case matrix transformations are used,

$$P = [x \quad y] \qquad T = [t_x \quad t_y]$$

→ Translation is rigid body transformation that moves objects without deformation ie every point on the object is translated by same amount.



Q1. Translate a polygon with coordinates A(2,5) B(7,10) C(10,2) by 3 units in x direction and 4 units in y direction

Ans. $t_x = 3 \quad t_y = 4$

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$$A' = A + T$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$A' = (5, 9)$$

$$B' = B + T$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \end{bmatrix}$$

$$B' = (10, 14)$$

$$C' = C + T$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 13 \\ 6 \end{bmatrix}$$

$$C' = (13, 6)$$

Q2) Translate a polygon with coordinates A(9,2), B(15,5), C(20,2) and translation vector, T = (-5,4)

Ans- $A' = A + T$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \end{bmatrix} + \begin{bmatrix} -5 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$$A' = (4, 6)$$

$$B' = B + T$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \end{bmatrix} + \begin{bmatrix} -5 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 9 \end{bmatrix}$$

$$B' = (10, 9)$$

$$C' = C + T$$

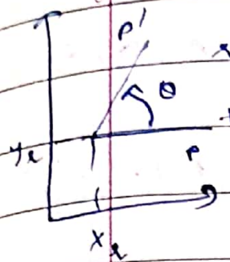
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 20 \\ 2 \end{bmatrix} + \begin{bmatrix} -5 \\ 4 \end{bmatrix} = \begin{bmatrix} 15 \\ 6 \end{bmatrix}$$

$$C' = (15, 6)$$

2) Rotation

→ applied to an object by repositioning it along a circular path in the xy plane.

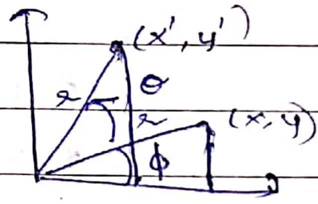
→ To generate a rotation, we specify a rotation angle θ and the position (x_r, y_r) of the rotation point (or pivot point) about which the object is to be rotated.



+ve ⇒ counterclockwise

−ve ⇒ clockwise

Considering a pivot point P at coordinate origin. r is constant distance of point from origin, angle ϕ is original angular position of point from horizontal and θ is rotational angle.



Using standard trigonometric identities,

$$x' = r\cos(\phi+\theta) = r\cos\phi\cos\theta - r\sin\phi\sin\theta$$

$$y' = r\sin(\phi+\theta) = r\cos\phi\sin\theta + r\sin\phi\cos\theta$$

However, $x = r\cos\phi$, $y = r\sin\phi$

$$\therefore\ x' = x\cos\theta - y\sin\theta$$

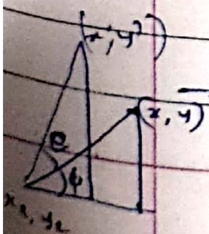
$$y' = x\sin\theta + y\cos\theta$$

$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$P'^T = P^T R^T$$

$$\Rightarrow P' = R \cdot P$$

where $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$



→ About an arbitrary pivot point, rotation is as

$$x' = x_r + (x - x_r)\cos\theta - (y - y_r)\sin\theta$$

$$y' = y_r + (y_r - x_r)\sin\theta + (y - y_r)\cos\theta$$

→ Rotations are rigid body transformations that move objects without deformation.
ie every point on object is rotated through the same angle.

Q. Perform a 45° rotation of a triangle $A(0,0)$ $B(1,1)$ $C(5,2)$ about origin

Ans. About origin:

$A(0,0)$

$B' = [x' \; y'] = [0 \quad 0] \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix}$

$\Rightarrow A' = [0 \quad 0]$

$B(1,1)$

$B' = (x' y') = [1 \quad 1] \begin{bmatrix} \cos 45^\circ & -\sin 45 \\ \sin 45 & \cos 45^\circ \end{bmatrix}$

$= [1 \quad 1] \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$= \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}$

$\Rightarrow B' = (0 \quad \sqrt{2})$

$C(5,2)$

$C' = (x', y') = [5 \quad 2] \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$= [3\sqrt{2}/2, \; 7\sqrt{2}/2]$

$\Rightarrow C = [3\sqrt{2}/2, \; 7\sqrt{2}/2]$

New coordinates are

$A'(0,0)$

$B'(0, \sqrt{2})$

$C'(3\sqrt{2}/2, \; 7\sqrt{2}/2)$

3) Scaling

→ alters the size of the object.

→ carried out for polygons by multiplying the coordinate values (x, y) of each vertex by scaling factors (s_x and s_y) to produce transformed coordinates (x', y')

$$x' = x \cdot s_x \qquad y' = y \cdot s_y$$

↳ scales in x dir.

scales in y. dir.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$P' = S \cdot P$$

where S is 2 x 2 scaling matrix.

$S < 1$ ⇒ decrease in size

$S > 1$ ⇒ increase

$S = 1$ ⇒ same size.

→ Uniform scaling : $s_x = s_y$

maintains relative object proportions.

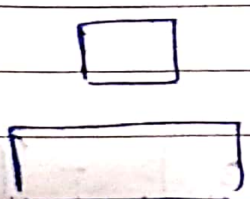
→ Differential scaling : $s_x \neq s_y$

used in design applications where pictures are constructed from a few basic shapes that can be adjusted by scaling and positioning transformations.

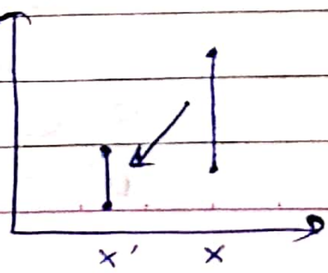
→ Objects in this are both scaled and repositioned

$s_f < 1$ ⇒ move closer to coordinate origin

$s_f > 1$ ⇒ move farther from origin.

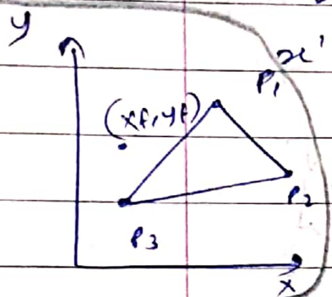


$(s_x, s_y) = (2, 1)$



$s_x = s_y = 0.5$
closer to origin

→ location of scaled object can be controlled by choosing a position called the fixed point (x_f, y_f) so that location of scaled object remains unchanged after scaling transformation.



$x' = x_f + (x - x_f) S_x$ $\quad$ $y' = y_f + (y - y_f) S_y$

can be rewritten as.

$x' = x . S_x + x_f (1 - S_x)$

$y' = y . S_y + y_f (1 - S_y)$

The additive term $x_f(1 - S_x)$ and $y_f(1 - S_y)$ are constant for all points in the object.

Q. Given a polygon A(2,2) B(2,5) C(5,3), D(3,2). Scale it using scaling factors $S_x = 2$, $S_y = 1$

Ans- A (2,2)

$$A' = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 \end{bmatrix}$$

B (2,5)

$$B' = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 5 \end{bmatrix}$$ (vector form)

C (5,3)

$$C' = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 10 \\ 3 \end{bmatrix}$$

D (3,2)

$$D' = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

Matrix representations and homogeneous coordinates

There may be cases wherein we want to perform different operations on a particular object. e.g. suppose we want to translate, then rotate, scale. Instead of calculating transformed coordinates one step at a time, efficient approach would be to combine transformations and directly obtain new coordinates, thus eliminating intermediate steps. For this, 2D transformation equations are expressed as matrix multiplication.

$$P' = M_1 P + M_2$$

M_2 → translational terms

M_1 → multiplicative factors

To represent 2D transformation as matrix multi. we represent each Cartesian position (x, y) with homogeneous coordinate triple (x_h, y_h, h)

where $x = \frac{x_h}{h}$ $\quad y = \frac{y_h}{h}$ and $h \neq 0$

Conveniently $h = 1$.

Coordinates are represented with 3-element column vectors, and transformation operations are written as 3×3 matrices.

Translation:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

or

$$P' = T(t_x, t_y) \cdot P$$

Inverse of $T(t_x, t_y)$ is obtained by replacing the t_x, t_y

with their negatives.

Rotation: (about coordinate origin)

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

or

$$P' = R(\theta) \cdot P$$

Inverse of rotation matrix $R(\theta)$ is found when θ is replaced with $-\theta$.

Scaling: (about coordinate origin)

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$P' = S(s_x, s_y) \cdot P$$

Inverse of S can be found by replacing the factors by $1/s_x$ and $1/s_y$.

Q. Give the transformation matrix for each of the following transitions.

(i) Shift the image to right by 3 units

Ans- $t_y = 0 \quad t_x = 3$

$$T(t_x, t_y) = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(ii) Shift the image up by 2 units

Ans- $t_y = 2 \quad t_x = 0$

$$T(t_x, t_y) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

(iii) Move the image down by 1/2 unit and to right by 1 unit.

Ans $t_y = -0.5 \quad t_x = 1$

$$T(t_x, t_y) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -0.5 \\ 0 & 0 & 1 \end{bmatrix}$$

(iv) Move the image down by 2/3 units and left by 4 units.

Ans $t_x = -4, \quad t_y = -2/3$

$$T(t_x, t_y) = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{bmatrix}$$

Q. Find the transformation of a triangle A (1, 0) B (0, 1) C (1, 1) when it is rotated at 45°.

Ans

$$R(\theta) = \begin{bmatrix} \cos 45 & -\sin 45 & 0 \\ \sin 45 & \cos 45 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} A' \\ B' \\ C' \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 1 & 1 \end{bmatrix}$$

$A' = (1/\sqrt{2}, 1/\sqrt{2}) \quad B' (-1/\sqrt{2}, 1/\sqrt{2})$

$C = (0, 1/\sqrt{2})$

// can do separately.

Composite Transformations

Forming products of transformation matrices is known as concatenation or composition of matrices.

For column matrix representation of coordinate positions, we form composite transformations by multiplying matrices in order from right to left. i.e. each successive transformation matrix premultiplies the product of preceding transformation matrices.

1) Translations

→ perform 2 successive translation operations on an original position

→ If 2 successive translation vectors (t_{x_1}, t_{y_1}) and (t_{x_2}, t_{y_2}) are applied to a coordinate position P, the final transformation location position P' is calculated as:

$$P' = T(t_{x_2}, t_{y_2}) \cdot \{T(t_{x_1}, t_{y_1}) \cdot P\}$$
$$= \{T(t_{x_2}, t_{y_2}) \cdot T(t_{x_1}, t_{y_1})\} P$$

where P and P' are represented as homogeneous coordinate column vectors.

Composite transformation matrix for sequence of translation:

$$\begin{bmatrix} 1 & 0 & t_{x_2} \\ 0 & 1 & t_{y_2} \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & t_{x_1} \\ 0 & 1 & t_{y_1} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_{x_1}+t_{x_2} \\ 0 & 1 & t_{y_1}+t_{y_2} \\ 0 & 0 & 1 \end{bmatrix}$$

or

$$T(t_{x_2}, t_{y_2}) \cdot T(t_{x_1}, t_{y_1}) = T(t_{x_1}+t_{x_2}, t_{y_1}+t_{y_2})$$

⇒ 2 successive translations are additive

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2) Rotation

→ perform 2 successive rotations to point P

$P' = R(\theta_2)\{R(\theta_1).P\}$

$= \{R(\theta_2)\,R(\theta_1)\}P$

By multiplying 2 rotation matrices, we can verify that 2 successive rotations are additive

$R(\theta_2).R(\theta_1) = R(\theta_1+\theta_2)$.

∴ Composite rotation matrix gives final matrix.

$P' = R(\theta_1+\theta_2).P$.

3) Scalings

→ 2 successive scaling operations give

$$\begin{bmatrix} sx_2 & 0 & 0 \\ 0 & sy_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} sx_1 & 0 & 0 \\ 0 & sy_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} sx_1 sx_2 & 0 & 0 \\ 0 & sy_1 sy_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or $S(sx_2, sy_2).S(sx_1, sy_1) = S(sx_1 sx_2, sy_1 sy_2)$

⇒ successive scaling operation is multiplicative

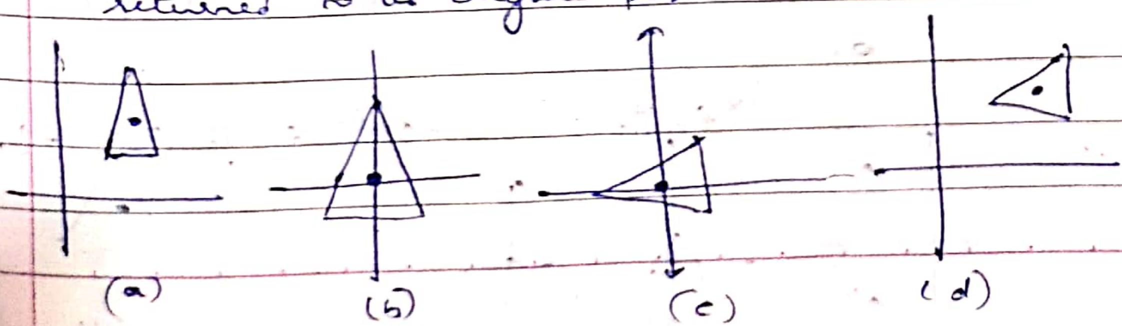
General pivot-point rotation

→ to perform rotation about any selected pivot point (x_r, y_r), perform: translate - rotate - translate

1) Translate object so that the pivot-point position is moved to the coordinate origin

2) Rotate the object about the coordinate origin.

3) Translate the object so that the pivot point is returned to its original position.



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Composite transformation matrix for the sequence is obtained by: Concatenation:

$$\begin{bmatrix} 1 & 0 & x_r \\ 0 & 1 & y_r \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_r \\ 0 & 1 & -y_r \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & x_r(1-\cos\theta)+y_r\sin\theta \\ \sin\theta & \cos\theta & y_r(1-\cos\theta)-x_r\sin\theta \\ 0 & 0 & 1 \end{bmatrix}$$

which can be expressed in the form:

$T(x_r, y_r) \cdot R(\theta) \cdot T(-x_r, -y_r) = R(x_r, y_r, \theta)$

where $T(-x_r, -y_r) = T^{-1}(x_r, y_r)$.

General Fixed point scaling

→ To perform scaling operation on an object with a selected fixed position (x_f, y_f) where scaling function can only scale relative to coordinate origin.

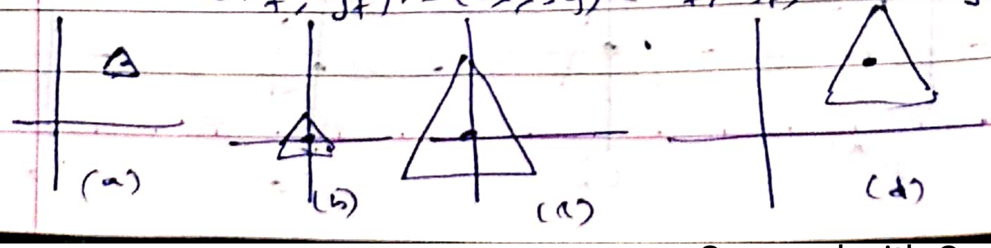
1) Translate object so that fixed point coincides with coordinate origin.
2) Scale the object wrt coordinate origin.
3) Use inverse translation of step 1 to return the object to its original position.

Required scaling matrix using concatenation

$$\begin{bmatrix} 1 & 0 & x_f \\ 0 & 1 & y_f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_f \\ 0 & 1 & -y_f \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & x_f(1-s_x) \\ 0 & s_y & y_f(1-s_y) \\ 0 & 0 & 1 \end{bmatrix}$$

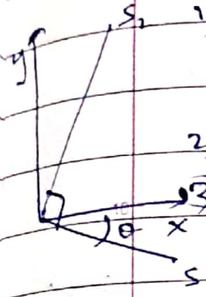
or

$T(x_f, y_f) \cdot S(s_x, s_y) T(-x_f, -y_f) = S(x_f, y_f, s_x, s_y)$



General scaling directions
(Scaling + rotation)

→ To apply scaling factors with values specified by parameters s_1 and s_2 in directions.
→ To accomplish scaling without changing orientation of object,

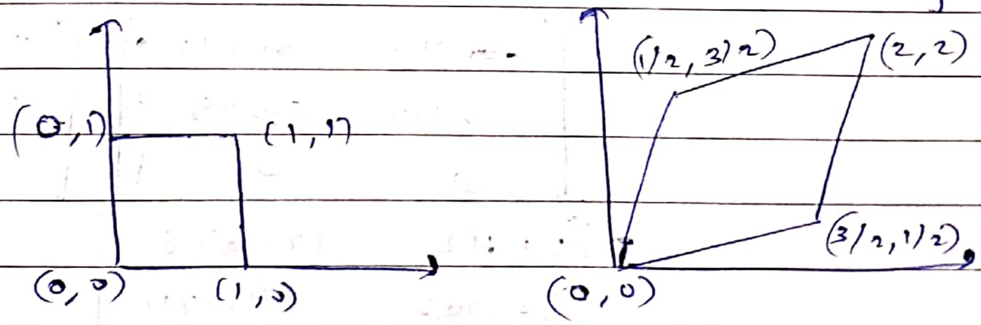


1) Perform rotation so that s_1 and s_2 coincide with x and y axes respectively.
2) Scaling transformation is applied
3) Opposite rotation from 1 to return to original positions.

Composite matrix resulting from product is:

$$R^{-1}(\theta)\, S(s_1, s_2)\, R(\theta)$$

$$= \begin{bmatrix} s_1\cos^2\theta + s_2\sin^2\theta & (s_2 - s_1)\cos\theta\sin\theta & 0 \\ (s_2 - s_1)\cos\theta\sin\theta & s_1\sin^2\theta + s_2\cos^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



for $s_1 = 1$ $s_2 = 2$ $\theta = 45^\circ$

Concatenation properties

→ Matrix multiplication is associative.

$A \cdot B \cdot C = (AB) \cdot C = (AB) \cdot C$

→ Transformation products may not be commutative. However, 2 successive rotations could be performed in either order and final position would be same.

Commutative → successive translation, rotation, ~~uniform~~ scalings, pair (rotation + uniform scaling)

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Q1) Consider a line drawn from (0,0) to (10,5). Find 2D transformation matrix to do the following transformations in sequence and find coordinates of transformed endpoints of given line.

1) Translate using $t_x = 50$, $t_y = 30$.
2) Rotate 45° about origin.
3) Scale line to twice its original size.

Ans- 1)

$$P' = T(t_x, t_y) \cdot P$$

$$= \begin{bmatrix} 1 & 0 & 50 \\ 0 & 1 & 30 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 10 \\ 0 & 5 \\ 1 & 1 \end{bmatrix} \quad 3\times2$$

$$= \begin{bmatrix} 50 & 60 \\ 30 & 35 \\ 1 & 1 \end{bmatrix}$$

2)

$$P' = R(\theta) \cdot P$$

$$= \begin{bmatrix} \cos 45 & -\sin 45 & 0 \\ \sin 45 & \cos 45 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 50 & 60 \\ 30 & 35 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 14.142 & 17.678 \\ 56.568 & 67.174 \\ 1 & 1 \end{bmatrix}$$

3)

$$P' = S(s_x, s_y) \cdot P$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 14.142 & 17.678 \\ 56.568 & 67.174 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 28.284 & 35.356 \\ 113.138 & 134.348 \\ 1 & 1 \end{bmatrix}$$

Q2) Magnify the triangle with vertices $A(0,0)$ $B(2,5)$ $C(7,3)$ to 1.5 times its size while keeping point C fixed.

Ans. $P' = S(x_f, y_f, s_x, s_y) \cdot P$

$$P' = \begin{bmatrix} 1.5 & 0 & x_f(1-s_x) \\ 0 & 1.5 & y_f(1-s_y) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 7 \\ 0 & 5 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$\therefore x_f = 7$
$y_f = 3$

$$= \begin{bmatrix} 1.5 & 0 & -3.5 \\ 0 & 1.5 & -1.5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 7 \\ 0 & 5 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3.5 & 0.5 & 7 \\ -1.5 & 6 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$A' = (-3.5, -1.5)$
$B' = (0.5, 6)$
$C' = (7, 3)$

Q3) Find the transformation matrix that transforms square ABCD whose centre is at $(2,2)$ is reduced to half of its size, with centre still remaining at $(2,2)$. coordinates of square are $A(0,0)$, $B(0,4)$, $C(4,4)$ and $D(4,0)$. Find coordinates of new square.

Ans- // General fixed-point scaling $(T.S.T^{-1})$

$(x_f, y_f) = (2,2)$, $s_x = s_y = 0.5$

Required scaling matrix is found by:

$$\begin{bmatrix} 1 & 0 & x_f \\ 0 & 1 & y_f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_f \\ 0 & 1 & -y_f \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & x_f(1-s_x) \\ 0 & s_y & y_f(1-s_y) \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0 & 2(0.5) \\ 0 & 0.5 & 2(0.5) \\ 0 & 0 & 1 \end{bmatrix}$$

$$S(x_f, y_f, S_x, S_y) = \begin{bmatrix} 0.5 & 0 & 1 \\ 0 & 0.5 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P' = S_{(x_f, y_f, S_x, S_y)} \, P$$

$$= \begin{bmatrix} 0.5 & 0 & 1 \\ 0 & 0.5 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 4 & 4 \\ 0 & 4 & 4 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 3 & 3 \\ 1 & 3 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

New coordinates are:

A (1, 1) B (1, 3) C (3, 3)

D (3, 1)

Q. Compute the composite transformation matrix for the foll. transformations in the given order.

(a) Translate by (-2, 1)

(b) Rotate by 70

(c) Translate by (2, 3)

Ans- (a) $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} \cos 70 & -\sin 70 & 0 \\ \sin 70 & \cos 70 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

Composite transformation matrix is given by

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.342 & -0.939 & 0 \\ 0.939 & 0.342 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.342 & -0.939 & 0.377 \\ 0.939 & 0.342 & 1.464 \\ 0 & 0 & 1 \end{bmatrix}$$

Q Perform following transformations on given polygons
i) rotate triangle MNO, M(1,1), N(2,3), O(4,0) by 60° wrt origin. (5)
ii) Rotate triangle ABC A(1,0) B(3,4) C(0,5) by 90° clockwise about point X(2,2)

Ans- (i) $P = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$R(60°) = \begin{bmatrix} \cos 60 & -\sin 60 & 0 \\ \sin 60 & \cos 60 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0.5 & -0.866 & 0 \\ 0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$P' = R.P$

$= \begin{bmatrix} 0.5 & -0.866 & 0 \\ 0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} -0.366 & -1.598 & 2 \\ 1.366 & 3.232 & 3.464 \\ 0 & 0 & 1 \end{bmatrix}$

coordinates are (-0.366, 1.366), (-1.598, 3.232), (2, 3.464)

(ii) $R(90°, 2, 2) = \begin{bmatrix} \cos(-90) & -\sin 90 & 2(1)+2(-1) \\ \sin(90) & \cos(90) & 2(1)-2(-1) \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

$P' = RP = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 4 & 5 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 5 \\ 3 & 1 & 4 \\ 1 & 1 & 1 \end{bmatrix}$

General composite transformation and computational efficiency

A general 2D transformation, representing a combination of translations, rotations and scalings can be expressed as:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} rs_{xx} & rs_{xy} & trs_x \\ rs_{yx} & rs_{yy} & trs_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

where rs_{ij} are multiplicative rotation-scaling terms in the transformation that involve only rotation angles and scaling factors.

trs_x and trs_y are translational terms containing combinations of translational distances, pivot point and fixed-point coordinates, and θ and s parameters. For eg: centroid (x_c, y_c) then

$$T(t_x, t_y) \cdot R(x_c, y_c, \theta) \cdot S(x_c, y_c, s_x, s_y)$$

$$= \begin{bmatrix} s_x\cos\theta & -s_y\sin\theta & x_c(1-s_x\cos\theta)+y_c s_y\sin\theta+t_x \\ s_x\sin\theta & s_y\cos\theta & y_c(1-s_y\cos\theta)-x_c s_x\sin\theta+t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Although matrix eqn. requires 9 multipli. and 6 additions, explicit calculations for transformed coordinates are:

$$x' = x \cdot rs_{xx} + y \cdot rs_{xy} + trs_x$$

$$y' = x \cdot rs_{yx} + y \cdot rs_{yy} + trs_y$$

$\Rightarrow$ 4 mul + 4 additions.

$\Rightarrow$ Max no of computations required for the transformation sequence once the individual matrices have been concatenated.

and the elements of the composite matrix evaluated.

For a rigid body transformation matrix (translation + rot only)

$$\begin{bmatrix} r_{xx} & r_{xy} & tr_x \\ r_{yx} & r_{yy} & tr_y \\ 0 & 0 & 1 \end{bmatrix}$$

Each vector has unit length

$$r_{xx}^2 + r_{xy}^2 = r_{yx}^2 + r_{yy}^2 = 1$$

and dot product $= 0 \Rightarrow$ vectors are $\perp r$

$$r_{xx} r_{yx} + r_{xy} r_{yy} = 0$$

$$\Rightarrow \begin{bmatrix} r_{xx} & r_{xy} & 0 \\ r_{yx} & r_{yy} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{xx} \\ r_{xy} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} r_{xx} & r_{xy} & 0 \\ r_{yx} & r_{yy} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{yx} \\ r_{yy} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Each step: 2 multi. and 2 add.

(Code is asked, in reader 219 - 220)

Other transformations

Reflection

→ transformation that produces a mirror image of an object. For a 2-D reflection, generated relative to an axis of reflection by rotating the object 180° about the reflection axis.

→ Reflection axis $\perp r$ plane.

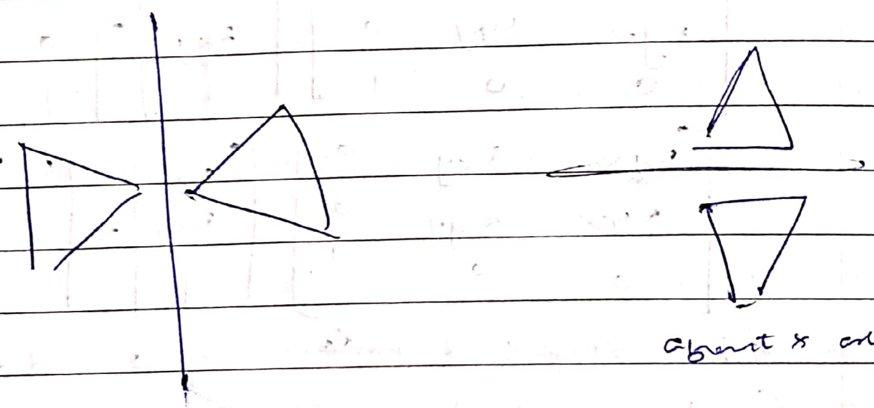
→ Reflection about line $y=0$ is accomplished with transformation matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

keeps x values same, but flips y values of coordinate positions

→ Reflection about y axis flips x coordinates while keeping y unchanged: Trans matrix /about line $x=0$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

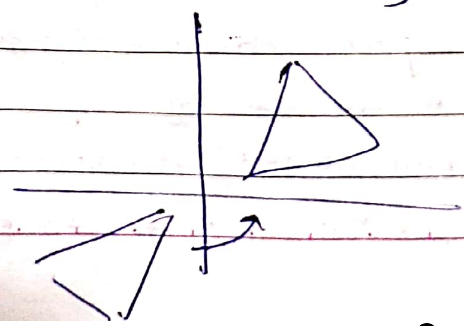


about x axis

about y axis

→ Reflecting rel to xy plane, flip both x and y coordinates and passing through origin

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



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→ Reflection axis : diagonal line $y = x$.
Reflection matrix is.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This sequence is derived by concatenating a sequence of rotation and coordinate axis reflection matrices.

clockwise (45^0) + reflection (x axis) + anticlock (45)

→ Reflection about $y = -x$

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

clockwise (45) + reflec (y axis) + anticlock (45)

→ $y = mx + b$
translate + rotate + reflect

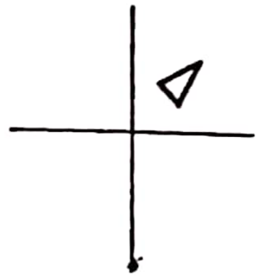
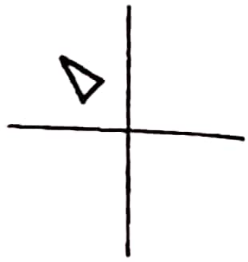
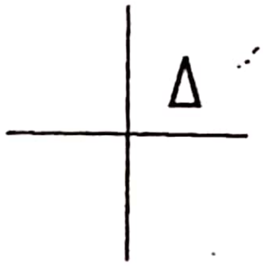
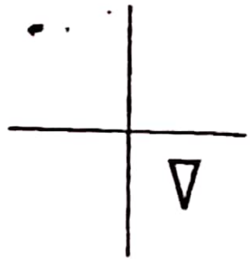
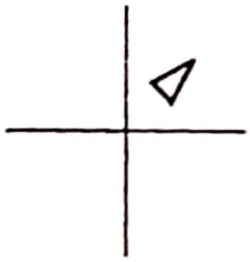
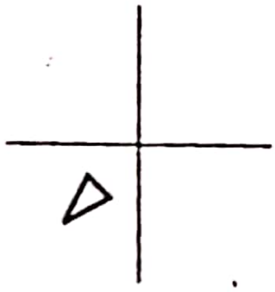
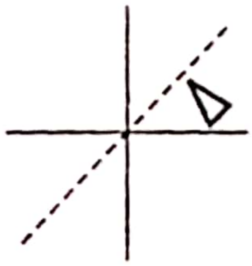
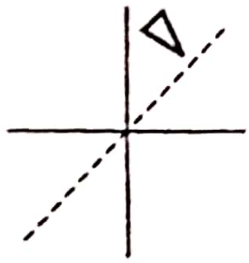
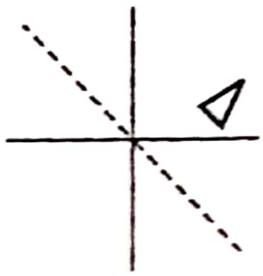
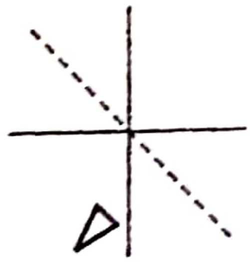
Reflection	Transformation matrix	Original image	Reflected image
Reflection about Y-axis	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$		
Reflection about X axis	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$		
Reflection about origin	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$		
Reflection about line $y = x$	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$		
Reflection about line $y = -x$	$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$		

Table 4.1 Common reflections

Shear

→ transformation that distorts the object such that the transformed shape appears as if the object were composed of internal layers that had been caused to slide over each other.

→ x - direction shear relative to x axis is produced with transformation matrix

$$\begin{bmatrix} 1 & sh_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which transforms coordinate positions as:

$$x' = x + sh_x y \qquad y' = y.$$

Any real no can be assigned to shear parameter sh_x. A coordinate position (x, y) is then shifted horizontally by an amount proportional to its distance (y-value) from the x axis $(y=0)$.

→ x-direction shears relative to other ref line

$$\begin{bmatrix} 1 & sh_x & -sh_x \cdot y_{ref} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$x' = x + sh_x(y - y_{ref})$, $y' = y$

→ y direction shear relative to line $x = x_{ref}$

$$\begin{bmatrix} 1 & 0 & 0 \\ sh_y & 1 & sh_y \cdot x_{ref} \\ 0 & 0 & 1 \end{bmatrix}$$

$x' = x$, $y' = sh_y(x - x_{ref}) + y$.

Q1) $sh_x = \frac{1}{2}$, $y_{ref} = -1$. Find coordinates of ~~parallelogram~~ square $A(0,0)$, $B(0,1)$, $C(1,0)$, $D(1,1)$ after shear has been applied along x-direction.

Ans- Transformation matrix:

$$\begin{bmatrix} 1 & sh_x & -sh_x y_{ref} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$x' = x + sh_x(y - y_{ref})$ $y' = y$.

At A $(0,0)$

$x' = 0 + \frac{1}{2}(0+1) = 1/2$ $y' = 0$

At B $(0, 1)$

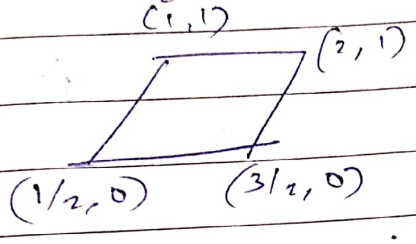
$x' = 0 + \frac{1}{2}(1+1) = 1$ $\qquad y' = 1$

At C $(1, 0)$

$x' = 1 + \frac{1}{2}(0+1) = \frac{3}{2}$ $\qquad y' = 0$

At D $(1, 1)$

$x' = 1 + \frac{1}{2}(1+1) = 2$ $\qquad y' = 1$



Q2) Apply shear $sh_x = 2$ to unit square positioned at $A(0,0)$, $B(0,1)$, $C(1,0)$, $D(1,1)$ applied along x - direction ~~...~~

Ans:- $x' = x + sh\,(y - y_{ref})$ $\qquad y' = y$

$A(0,0)$

$x' = 0 + 2(0) = 0$ $\qquad y' = 0$

$B(0,1)$

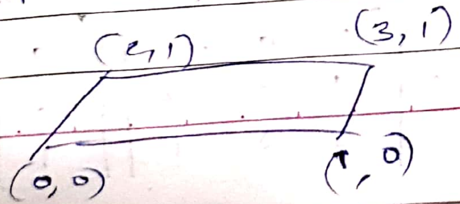
$x' = 0 + 2(1) = 2$ $\qquad y' = 1$

$C(1,0)$

$x' = 1 + 2(0) = 1$ $\qquad y' = 0$

$D(1,1)$

$x' = 1 + 2(1) = 3$ $\qquad y' = 1$



Q3) Apply shear $sh_y = 1/2$ & $x_{ref} = -1$ in the y-direction on unit square D(0,0), A(0,1), B(1,1), C(1,0)

A(0,1)

Ans

$x' = x$ $\qquad$ $y' = sh_y(x - x_{ref}) + y$

$x' = 0$ $\qquad$ $y' = \frac{1}{2}(0+1) + 1$

$\Rightarrow 3/2$

B(1,1)

$x' = 1$ $\qquad$ $y' = \frac{1}{2}(1+1) + 1$

C = (1,0)

$x' = 1$ $\qquad$ $y' = \frac{1}{2}(0+1) + \ldots$

$= 1$

D = (0,0)

$x' = 0$ $\qquad$ $y' = \frac{1}{2}(0+1) + 0$

$= 1/2$

Q.4) Magnify triangle with vertices A(0,0), B(1,... and C(5,2) to twice its size keeping C(5,2) fixed

Ans $\quad s_x = 2 = s_y \qquad x_f = 5 \qquad y_f = 2$

$T(x_f, y_f) \cdot S(s_x, s_y) \cdot T(-x_f, -y_f) = S(x_f, y_f, \ldots)$

$$S = \begin{bmatrix} s_x & 0 & x_f(1-s_x) \\ 0 & s_y & y_f(1-s_y) \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 5(1-2) \\ 0 & 2 & 2(1-2) \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -5 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

for point A(0,0) $\quad A' = SA$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -5 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$$

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B (1, 1), $B' = S \cdot B$.

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -5 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

C (5, 2) $C' = S \cdot C$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -5 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$$

New coordinates are A' (−5, −2), B' (−3, 0), C' (5, 2)

Q5) Perform a 45° rotation of a triangle A (0, 0), B (1, 1), C (5, 2) :

(i) about origin

(ii) about P (−1, −1)

Ans- $P = \begin{bmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

(a) About origin :

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$P' = R(\theta) \cdot P$

$$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 3\sqrt{2}/2 \\ 0 & \sqrt{2} & 7\sqrt{2}/2 \\ 1 & 1 & 1 \end{bmatrix}$$

(b) About P (−1, −1)

$$T_1 \cdot R(\theta) \cdot T_2 = \begin{bmatrix} \cos\theta & -\sin\theta & -x_r\cos\theta + y_r\sin\theta + x_1 \\ \sin\theta & \cos\theta & -x_r\sin\theta - y_r\cos\theta + y_1 \\ 0 & 0 & 1 \end{bmatrix}$$

Given $\theta = 45^\circ$ and $(x_1, y_1) = (-1, -1)$

$$\therefore T_1 R(\theta) \cdot T_2 = \begin{bmatrix} 0.707 & -0.707 & -1 \\ 0.707 & 0.707 & 0.414 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} 0.707 & -0.707 & 1 \\ 0.707 & 0.707 & 0.414 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 & 1.121 \\ 0.414 & 1.828 & 5.363 \\ 1 & 1 & 1 \end{bmatrix}$$

New coordinates are

$A'(-1, 0.414)$, $B'(-1, 1.828)$, $C(1.121, 5.363)$

Q6) A line is denoted by its endpoint $(0,0)$ and $(3,5)$ in a 2-D graphics system. Express the line in matrix notation and perform the following transformation on the line.

(i) Scale the line by a factor of 3.0 in x direction and 2.0 in y direction wrt $(3,4)$

(ii) Translate the original line by 2 units in x direction and 3 units in negative y direction

(iii) Rotate the line by 90° about the origin

Ans - (i) $P = \begin{bmatrix} 0 & 3 \\ 0 & 5 \\ 1 & 1 \end{bmatrix}$

$$S = \begin{bmatrix} S_x & 0 & x_f(1-S_x) \\ 0 & S_y & y_f(1-S_y) \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3(1-3) \\ 0 & 2 & 4(1-2) \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & -6 \\ 0 & 2 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

$P' = S.P$

$$= \begin{bmatrix} 3 & 0 & -6 \\ 0 & 2 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 0 & 5 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 3 \\ -4 & 6 \\ 1 & 1 \end{bmatrix}$$

New coordinates are $(-6,-4)$ and $(3,6)$

(ii) $$T = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$P' = TP$

$$= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 0 & 5 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -3 & 2 \\ 1 & 1 \end{bmatrix}$$

New coordinates are : $(2,5)$ and $(-3,2)$

(iii) $$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.866 & 0 \\ 0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} 0.5 & -0.866 & 0 \\ 0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 0 & 5 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -2.83 \\ 0 & 5.098 \\ 1 & 1 \end{bmatrix}$$

$\Rightarrow$ New coordinates are $(0,0)$ and $(-2.83, 5.098)$

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Q7) Consider a line drawn from (0,0) to (10,5). Find the 2-D transformation matrix to do the following transformation in sequence and find the coordinates of the transformed endpoints of the given line.

(a) Translate using tx = 50 and ty = 30

(b) Rotate 45° about origin of coordinate system.

(c) Scale line to twice its original size.

Ans- Composite transformation matrix for above is given by:

$$M = T(50,30) \cdot R(45°) \cdot S(2,2)$$

$$\therefore M = S(2,2) \times R(45°) \times T(50,30)$$

$$M = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 50 \\ 0 & 1 & 30 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.414 & -1.414 & 28.280 \\ 1.414 & 1.414 & 113.120 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P' = MP$$

$$\begin{bmatrix} 1.414 & -1.414 & 28.280 \\ 1.414 & 1.414 & 113.120 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 10 \\ 0 & 5 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 28.280 & 35.350 \\ 113.120 & 134.330 \\ 1 & 1 \end{bmatrix}$$

New coordinates are ~~(...)~~ A(28.280, 113.120) and B(35.350, 134.330)

Q8) Suppose that the dimensions of a triangle are represented as follows:

$$\begin{bmatrix} 15 & 35 & 1 \\ 10 & 25 & 1 \\ 20 & 25 & 1 \end{bmatrix}$$

where each row represents a vertex of a triangle and is in the form $\{x, y, 1\}$. It is required to rotate the triangle through clockwise angle of 90° about the point $(5, 20)$ and then scale the triangle using scale factor $S_x = S_y = 3$. After doing the above transformations, it is required to translate the triangle with $t_x = 20$, $t_y = 10$. Find transformation matrix and dimensions of resulting triangle.

Ans

$$P = \begin{bmatrix} 15 & 10 & 20 \\ 35 & 25 & 25 \\ 1 & 1 & 1 \end{bmatrix}$$

$$M = T(20, 10) \cdot S(3, 3) \cdot R(90^\circ)_{(5,20)}$$

$$= \begin{bmatrix} 1 & 0 & 20 \\ 0 & 1 & 10 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & -15 \\ -1 & 0 & 25 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 3 & -25 \\ -3 & 0 & 85 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P' = MP$$

$$= \begin{bmatrix} 0 & 3 & -25 \\ -3 & 0 & 85 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 15 & 10 & 20 \\ 35 & 25 & 25 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 80 & 50 & 50 \\ 40 & 55 & 25 \\ 1 & 1 & 1 \end{bmatrix}$$

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New coordinates are (80, 40), (50, 55) and (50, 25)

Q 9) For a quadrilateral ABCD with vertex coordinates as A (25, 20), B (−20, 30), C (−20, −10), D(15, 10) apply the following transformations individually.

(i) Translate with $t_x = 10$, $t_y = 15$
(ii) Perform scaling with $S_x = 1.2$, $S_y = 0.5$
(iii) Reflection about axis i.e. to xy plane or passing through coordinate axis.

Ans − $P = \begin{bmatrix} 25 & -20 & -20 & 15 \\ 20 & 30 & -10 & 10 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

(i) $T = \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & 15 \\ 0 & 0 & 1 \end{bmatrix}$

$P' = T P$

$= \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & 15 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 25 & -20 & -20 & 15 \\ 20 & 30 & -10 & 10 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 35 & -10 & -10 & 25 \\ 35 & 45 & 5 & 25 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

(ii) $S = \begin{bmatrix} 1.2 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$P' = S P$

$= \begin{bmatrix} 1.2 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 25 & -20 & -20 & 15 \\ 20 & 30 & -10 & 10 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 30 & -24 & -24 & 18 \\ 10 & 15 & -5 & 5 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$ 5, 15)

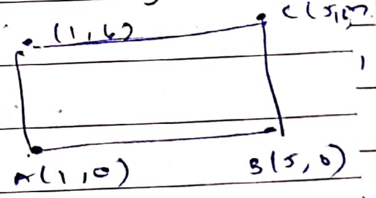
(ii) $R_c = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$P' = (R_c)P$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 25 & -20 & -20 & 15 \\ 20 & 30 & -10 & 10 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -25 & 20 & 20 & 15 \\ -20 & -30 & 10 & -10 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Q10) Perform shear transformation on rectangle ABCD A(1,0), B(5,0), C(5,6), D(1,6) keeping base AB fixed and shear by 2 units right



Ans $Sh = 2$

AB is x axis:

relative to x' axis

$$T = \begin{bmatrix} 1 & Sh_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ $x' = x + Sh_x y$

$y' = y$

A (1,0) : $y' = 0$ $x' = 1 + 0 = 1$ $A' = (1, 0)$

B (5,0) : $y' = 0$ $x' = 5 + 0 = 5$ $B' = (5, 0)$

C (5,6) $y' = 6$ $x' = 5 + 2(6) = 17$ $C'(17, 6)$

D (1,6) $y' = 6$ $x' = 1 + 2(6) = 13$ $D' = (13, 6)$.

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Q11) Prove that transformation matrix for reflection about the line $y = x$ is equivalent to a reflection relative to x axis followed by a counter clockwise rotation of 90°.

Ans - To prove: $R_{f@y=x} = R_{f@x} + R_{90°}$

LHS = Reflection at line $y = x$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

RHS = Reflection at x axis × Rotation

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

= LHS

=> proved

Q12) Rotate a triangle ABC by an angle 30° in the clockwise direction where the triangle has coordinates A (0, 5), B(10, 2) and C (7, 4). Give resultant matrix.

Ans $P' = R(-30°) P$

$$= \begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 \\ -1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 10 & 7 \\ 0 & 2 & 4 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 5\sqrt{3}+1 & \frac{7\sqrt{3}+2}{2} \\ 0 & \sqrt{3}-5 & -7/2+2\sqrt{3} \\ 1 & 1 & 1 \end{bmatrix}$$

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Q13) Magnify triangle with vertices A(10, 20), B(15, 25) C(20, 20) to 2.5 times its original size by keeping C as a fixed point.

Ans- $P' = S(x_f, y_f, s_x, s_y) \cdot P$

$$= \begin{bmatrix} 2.5 & 0 & 20(1-2.5) \\ 0 & 2.5 & 20(1-2.5) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 & 15 & 20 \\ 20 & 25 & 20 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2.5 & 0 & -30 \\ 0 & 2.5 & -30 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 & 15 & 20 \\ 20 & 25 & 20 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 7.5 & 20 \\ 20 & 32.5 & 20 \\ 1 & 1 & 1 \end{bmatrix}$$

Q14) Give 3×3 homogeneous transformation matrix for each of the following transformation sequence
(a) Rotate clockwise about origin 45° and then scale in x-direction by 1/2 as large.
(b) scale in y-direction by twice as tall, shift down by 1 unit and then rotate clockwise by 30°.

Ans- (a) Rotate + scale

$$TM = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2\sqrt{2} & 1/2\sqrt{2} & 0 \\ -1/2\sqrt{2} & 1/2\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) scale + Translate + Rotate

PTO.

$$= \begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 \\ -1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 \\ -1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & 1 & 0 \\ -1/2 & \sqrt{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Q15) Create appropriate 2-D transformation which reflects a figure in point (0.5, 0.5).

Ans - Translating given point to origin, $T = \begin{bmatrix} 1 & 0 & -0.5 \\ 0 & 1 & -0.5 \\ 0 & 0 & 1 \end{bmatrix}$

Reflecting about origin, $M = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Translating back, $T^{-1} = \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}$

Transformation, $R_t = T M T^{-1}$

$$= \begin{bmatrix} -1 & \cancel{-1} & -0.5 \\ 0 & -1 & -0.5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 0.5 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Q16) Give homogeneous coordinate matrix rep. for the following transformation.

(a) Reflection about the origin

$$R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Shear in both x and y direction

$$S = \begin{bmatrix} \\ \\ \end{bmatrix}$$

(c) counter clockwise by 180°

$$R = \begin{bmatrix} \cos 180 & -\sin 180 & 0 \\ \sin 180 & \cos 180 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(d) clockwise by 90°

$$R = \begin{bmatrix} \cos(-90) & \sin 90 & 0 \\ -\sin 90 & \cos(-90) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Q17) Prove that transformation matrix for a reflection about the line $y=-x$ is equivalent to a reflection relative to the y-axis followed by a counter-clockwise rotation by 90°.

Ans: To prove: $R_f @ y=-x = R_f @ y + R_{90°}$

$LHS = R_f @ y=-x$

$$= \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$RHS = R_f @ y + R_{90°}$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$= LHS$

$\Rightarrow$ proved

Q18) A mirror is placed vertically such that it passes through the points (10,0) and (0,10).

Find the reflected view of the triangle ABC with coordinates A(5, 50), B(20, 40) and C(10, 70)

Ans – $y - y_1 = m(x - x_1)$ (0, 10)

$y - 10 = \frac{0 - 10}{10 - 0}(x - 0)$ (10, 0)

$y - 10 = -x$

$y = -x + 10$ $m = -1$, $c = 10$ $\theta = \tan^{-1}(-1) = -45°$ (clockwise)

For line translate + rotate + reflect about x-axis

Composite transformation matrix given by:

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -10 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.7071 & 0.7071 & -7.071 \\ 0.7071 & -0.7071 & 7.071 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence $= \begin{bmatrix} 0.707 & 0.707 & -7.071 \\ 0.707 & -0.707 & 7.071 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 20 & 10 \\ 50 & 40 & 70 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 31.819 & 35.35 & 49.497 \\ -24.74 & -7.071 & -35.35 \\ 1 & 1 & 1 \end{bmatrix}$

Q19) Scale a triangle with vertices (10, 10), (16, 20) and (20, 10) from a pivot point (16, 16) to half its size

Ans – $P' = SP$

$= \begin{bmatrix} 1/2 & 0 & 16(1/2) \\ 0 & 1/2 & 16(1/2) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 & 16 & 20 \\ 10 & 20 & 10 \\ 1 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1/2 & 0 & 8 \\ 0 & 1/2 & 8 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 & 16 & 20 \\ 10 & 20 & 10 \\ 1 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 13 & 16 & 18 \\ 13 & 18 & 13 \\ 1 & 1 & 1 \end{bmatrix}$

⇒ new coordinates are (13, 13), (16, 18), (18, 13)

Q20) Explain transformation of a ΔABC with vertices (4,1), (5,2) and (4,3) in foll. ways individually.

(a) Reflect about x-axis.

(b) About line $y = -x$.

(c) Rotate triangle about origin by angle 270°.

Ans (a) $R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 5 & 4 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 4 & 5 & 4 \\ -1 & -2 & -3 \\ 1 & 1 & 1 \end{bmatrix}$

(b) $R_2 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 5 & 4 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} -1 & -2 & -3 \\ -4 & -5 & -4 \\ 1 & 1 & 1 \end{bmatrix}$

(c) $R_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 5 & 4 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 & 3 \\ -4 & -5 & -4 \\ 1 & 1 & 1 \end{bmatrix}$

Q 21) Prove that reflection of square ABCD [(2,2), (4,2), (4,4), (2,4)] about x-axis and then rotation of the resulting square about 60° will not be same if order of transformation is changed.

Ans case (i): Reflect + Rotate (60°)

$$T_1 = \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$LHS = P' = T_1 P$

$$= \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 4 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{3}+1 & 2+\sqrt{3} & 2+2\sqrt{3} & 1+2\sqrt{3} \\ \sqrt{3}-1 & 2\sqrt{3}-1 & 2\sqrt{3}-2 & \sqrt{3}-2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Case (2):

$$T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$RHS = P' = T_2 P$

$$= \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 4 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\sqrt{3} & 2-\sqrt{3} & \ldots & 1-2\sqrt{3} \\ -\sqrt{3}-1 & -2\sqrt{3} & \ldots & -\sqrt{3}-2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$\neq RHS$

$\Rightarrow$ proved.