2.6. Provide definitions for a global maximizer, a strict global maximizer, a local maximizer, and a strict local maximizer.

2.7. Consider minimizing $f(x)$ for $x \in S$ where S is the set of integers. Prove that every point in S is a local minimizer of f.

2.8. Let $S = \{ x : g_i(x) \ge 0, i = 1, \dots, m \}$ and assume that the functions $\{ g_i \}$ are continuous. Prove that if $g_i(\hat{x}) > 0$ for all i, then $\{ x : \|x - \hat{x}\| < \epsilon \} \subset S$ for some $\epsilon > 0$.

2.9. Let S be the feasible region in Figure 2.1. Show that S can be represented by equality and inequality constraints in such a way that it has no interior points. Thus the interior of a set may depend on the way it is represented.

2.10. Let $S = \{ x : g_i(x) \ge 0, i = 1, \dots, m \}$ and assume that the functions $\{ g_i \}$ are continuous. Assume that there exists a point $\hat{x}$ such that $g_i(\hat{x}) > 0$ for all i. Prove that S has a nonempty interior regardless of how S is represented.

2.3 Convexity

There is one important case where global solutions can be found, the case where the objective function is a convex function and the feasible region is a convex set. Let us first talk about the feasible region.

A set S is *convex* if, for any elements x and y of S,

$$\alpha x + (1 - \alpha) y \in S \quad \text{for all } 0 \le \alpha \le 1.$$

In other words, if x and y are in S, then the line segment connecting x and y is also in S. Examples of convex and nonconvex sets are given in Figure 2.4. More generally, every set defined by a system of linear constraints is a convex set; see the Exercises.

A function f is *convex* on a convex set S if it satisfies

$$f(\alpha x + (1 - \alpha) y) \le \alpha f(x) + (1 - \alpha) f(y)$$

for all $0 \le \alpha \le 1$ and for all $x, y \in S$. This definition says that the line segment connecting the points $(x, f(x))$ and $(y, f(y))$ lies on or above the graph of the function; see Figure 2.5. Intuitively, the graph of the function is bowl shaped.

Analogously, a function is concave on S if it satisfies

$$f(\alpha x + (1 - \alpha) y) \ge \alpha f(x) + (1 - \alpha) f(y)$$

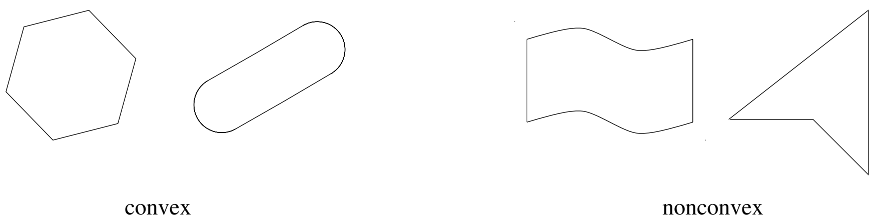

Figure 2.4. *Convex and nonconvex sets.*

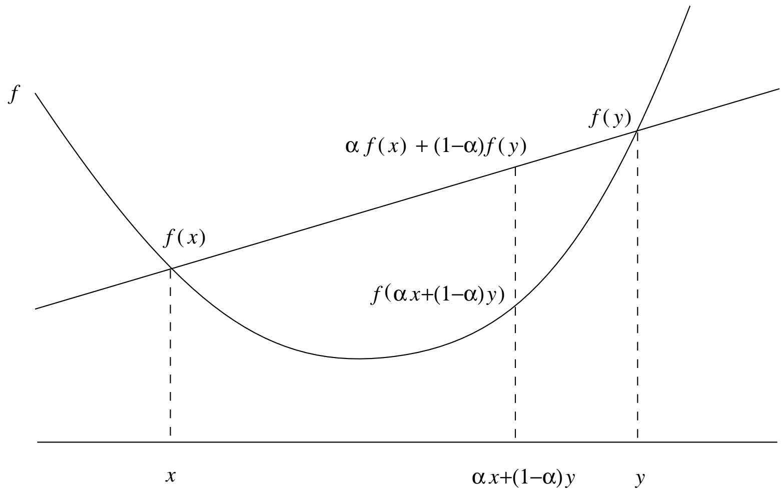

Figure 2.5. *Convex function.*

for all $0 \le \alpha \le 1$ and for all $x, y \in S$. Concave functions are explored in the Exercises below. Linear functions are both convex and concave.

We say that a function is *strictly convex* if

$$f(\alpha x + (1-\alpha)y) < \alpha f(x) + (1-\alpha) f(y)$$

for all $x \neq y$ and $0 < \alpha < 1$ where $x, y \in S$.

Let us now return to the discussion of local and global solutions. We define a *convex optimization problem* to be a problem of the form

$$\underset{x \in S}{\text{minimize}} \;\; f(x),$$

where S is a convex set and f is a convex function on S. A problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \ge 0, \; i = 1, \ldots, m, \end{array}$$

is a convex optimization problem if f is convex and the functions $\{ g_i \}$ are concave; see the Exercises.

The following theorem shows that any local solution of such a problem is also a global solution. This result is important to linear programming, since every linear program is a convex optimization problem.

Theorem 2.1 (Global Solutions of Convex Optimization Problems)**.** *Let x_* be a local minimizer of a convex optimization problem. Then x_* is also a global minimizer. If the objective function is strictly convex, then x_* is the unique global minimizer.*

Proof. The proof is by contradiction. Let x_* be a local minimizer and suppose, by contradiction, that it is not a global minimizer. Then there exists some point $y \in S$ satisfying

$f(y) < f(x_*)$. If $0 < \alpha < 1$, then

$$\begin{aligned} f(\alpha x_* + (1-\alpha)y) &\le \alpha f(x_*) + (1-\alpha)f(y) \\ &< \alpha f(x_*) + (1-\alpha)f(x_*) = f(x_*). \end{aligned}$$

This shows that there are points arbitrarily close to x_* (i.e., when α is arbitrarily close to 1) whose function values are strictly less than $f(x_*)$. These points are in S because S is convex. This contradicts the definition of a local minimizer. Hence a point such as y cannot exist, and x_* must be a global minimizer.

If the objective function is strictly convex, then a similar argument can be used to show that x_* is the unique global minimizer; see the Exercises. □

For general problems it may be as difficult to determine if the function f and the region S are convex as it is to find a global solution, so this result is not always useful. However, there are important practical problems, such as linear programs, where convexity can be guaranteed.

We conclude this section by defining a *convex combination* (weighted average) of a finite set of points. A convex combination is a linear combination whose coefficients are nonnegative and sum to one. Algebraically, the point y is a convex combination of the points $\{x_i\}_{i=1}^k$ if

$$y = \sum_{i=1}^{k} \alpha_i x_i,$$

where

$$\sum_{i=1}^{k} \alpha_i = 1 \quad \text{and} \quad \alpha_i \ge 0, \quad i = 1, \ldots, k.$$

There will normally be many ways in which y can be expressed as a convex combination of $\{x_i\}$.

As an example, consider the points $x_1 = (0,0)^T$, $x_2 = (1,0)^T$, $x_3 = (0,1)^T$, and $x_4 = (1,1)^T$. If $y = (\frac{1}{2}, \frac{1}{2})^T$, then y can be expressed as a convex combination of $\{x_i\}$ in the following ways:

$$\begin{aligned} y &= 0x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{2}x_3 + 0x_4 \\ &= \tfrac{1}{2}x_1 + 0x_2 + 0x_3 + \tfrac{1}{2}x_4 \\ &= \tfrac{1}{4}x_1 + \tfrac{1}{4}x_2 + \tfrac{1}{4}x_3 + \tfrac{1}{4}x_4, \end{aligned}$$

and so forth.

2.3.1 Derivatives and Convexity

If a one-dimensional function f has two continuous derivatives, then an alternative definition of convexity can be given that is often easier to check. Such a function is convex if and only if

$$f''(x) \ge 0 \quad \text{for all } x \in S;$$

see the Exercises in Section 2.6. For example, the function $f(x) = x^4$ is convex on the entire real line because $f''(x) = 12x^2 \geq 0$ for all x. The function $f(x) = \sin x$ is neither convex nor concave on the real line because $f''(x) = -\sin x$ can be both positive and negative.

In the multidimensional case the Hessian matrix of second derivatives must be positive semidefinite; that is, at every point $x \in S$

$$y^T \nabla^2 f(x) y \geq 0 \quad \text{for all } y;$$

see the Exercises in Section 2.6. (The Hessian matrix is defined in Appendix B.4.) Notice that the vector y is not restricted to lie in the set S. The quadratic function

$$f(x_1, x_2) = 4x_1^2 + 12x_1x_2 + 9x_2^2$$

is convex over any subset of $\Re^2$ since

$$\begin{aligned} y^T \nabla^2 f(x) y &= \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} 8 & 12 \\ 12 & 18 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ &= 8y_1^2 + 24y_1y_2 + 18y_2^2 \\ &= 2(2y_1 + 3y_2)^2 \geq 0. \end{aligned}$$

Alternatively, it would have been possible to show that the eigenvalues of the Hessian matrix were all greater than or equal to zero.

In the one-dimensional case, if a function satisfies

$$f''(x) > 0 \quad \text{for all } x \in S,$$

then it is strictly convex on S. In the multidimensional case, if the Hessian matrix $\nabla^2 f(x)$ is positive definite for all $x \in S$, then the function is strictly convex on S. This is not an "if and only if" condition, since the Hessian of a strictly convex function need not be positive definite everywhere (see the Exercises).

Now we consider another characterization of convexity that can be applied to functions that have one continuous derivative. In this case a function f is convex over a convex set S if and only if it satisfies

$$f(y) \geq f(x) + \nabla f(x)^T (y - x)$$

for all $x, y \in S$. This property states that the function is on or above any of its tangents. (See Figure 2.6.)

To prove this property, note that if f is convex, then for any x and y in S and for any $0 < \alpha \leq 1$,

$$f(\alpha y + (1 - \alpha)x) \leq \alpha f(y) + (1 - \alpha) f(x),$$

so that

$$\frac{f(x + \alpha(y - x)) - f(x)}{\alpha} \leq f(y) - f(x).$$

If we let α approach 0 from above, we can conclude that $f(y) \geq f(x) + \nabla f(x)^T (y - x)$.

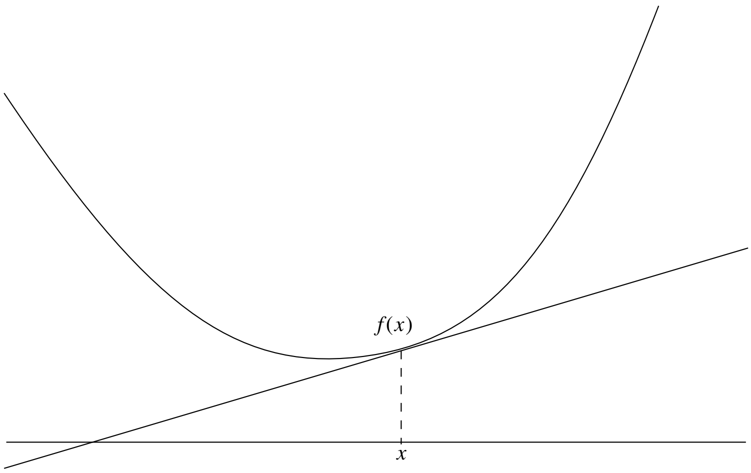

Figure 2.6. *Convex function with continuous first derivative.*

Conversely, suppose that the function f satisfies $f(y) \geq f(x) + \nabla f(x)^T(y - x)$ for all x and y in S. Let $t = \alpha x + (1 - \alpha)y$. Then t is also in the set S, so

$$f(x) \geq f(t) + \nabla f(t)^T(x - t)$$

and

$$f(y) \geq f(t) + \nabla f(t)^T(y - t).$$

Multiplying the two inequalities by α and $1 - \alpha$, respectively, and then adding yields the desired result. See the Exercises for details.

Exercises

3.1. Prove that the intersection of a finite number of convex sets is also a convex set.

3.2. Let $S_1 = \{ x : x_1 + x_2 \leq 1, x_1 \geq 0 \}$ and $S_2 = \{ x : x_1 - x_2 \geq 0, x_1 \leq 1 \}$, and let $S = S_1 \cup S_2$. Prove that S_1 and S_2 are both convex sets but S is not a convex set. This shows that the union of convex sets is not necessarily convex.

3.3. Consider a feasible region S defined by a set of linear constraints

$$S = \{ x : Ax \leq b \}.$$

Prove that S is convex.

3.4. Prove that a function f is concave if and only if $-f$ is convex.

3.5. Let $f(x)$ be a function on $\Re^n$. Prove that f is both convex and concave if and only if $f(x) = c^T x$ for some constant vector c.

3.6. Prove that a convex combination of convex functions all defined on the same convex set S is also a convex function on S.