In QDA $\rightarrow \hat{\Sigma}_0 \neq \hat{\Sigma}_1 \neq \Sigma_p$

But, we are assuming that both $\hat{\Sigma}_0$ and $\hat{\Sigma}_1$ is non singular i.e. Its inverse will be -

$$\hat{\Sigma}_0 = \begin{bmatrix} \hat{\sigma}_0 x_1^2 & 0 \\ 0 & \hat{\sigma}_0 x_2^2 \end{bmatrix} \quad \hat{\Sigma}_0^{-1} = \begin{bmatrix} \frac{1}{\hat{\sigma}_0 x_1^2} & 0 \\ 0 & \frac{1}{\hat{\sigma}_0 x_2^2} \end{bmatrix}$$

Similarly -

$$\hat{\Sigma}_1 = \begin{bmatrix} \hat{\sigma}_{1x_1}^2 & 0 \\ 0 & \hat{\sigma}_{1x_2}^2 \end{bmatrix}, \quad \hat{\Sigma}_1^{-1} = \begin{bmatrix} \frac{1}{\hat{\sigma}_{1x_1}^2} & 0 \\ 0 & \frac{1}{\hat{\sigma}_{1x_2}^2} \end{bmatrix}$$

Hold on guys, it's going to be a long one —

$$\frac{P\left(C=1 \mid x=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)}{P\left(C=0 \mid x=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)} = \frac{\cancel{\frac{1}{(\sqrt{2\pi})^2}} \frac{1}{\sqrt{|\hat{\Sigma}_1|}} e^{-\frac{1}{2}\begin{bmatrix} x_1 - \hat{\mu}_1^{x_1} \\ x_2 - \hat{\mu}_1^{x_2} \end{bmatrix}^T \hat{\Sigma}_1^{-1} \begin{bmatrix} x_1 - \hat{\mu}_1^{x_1} \\ x_2 - \hat{\mu}_1^{x_2} \end{bmatrix}}}{\cancel{\frac{1}{(\sqrt{2\pi})^2}} \frac{1}{\sqrt{|\hat{\Sigma}_0|}} e^{-\frac{1}{2}\begin{bmatrix} x_1 - \hat{\mu}_0^{x_1} \\ x_2 - \hat{\mu}_0^{x_2} \end{bmatrix}^T \hat{\Sigma}_0^{-1} \begin{bmatrix} x_1 - \hat{\mu}_1^{x_1} \\ x_2 - \hat{\mu}_1^{x_2} \end{bmatrix}}}$$

$$= \frac{\sqrt{|\hat{\Sigma}_0|}}{\sqrt{|\hat{\Sigma}_1|}} e^{\frac{1}{2}\begin{bmatrix} x_1 - \hat{\mu}_0^{x_1} \\ x_2 - \hat{\mu}_0^{x_2} \end{bmatrix}^T \hat{\Sigma}_0^{-1} \begin{bmatrix} x_1 - \hat{\mu}_0^{x_1} \\ x_2 - \hat{\mu}_1^{x_2} \end{bmatrix} - \frac{1}{2}\begin{bmatrix} x_1 - \hat{\mu}_1^{x_1} \\ x_2 - \hat{\mu}_1^{x_2} \end{bmatrix}^T \hat{\Sigma}_1^{-1} \begin{bmatrix} x_1 - \hat{\mu}_1^{x_1} \\ x_2 - \hat{\mu}_1^{x_2} \end{bmatrix}}$$

$$= \frac{\sqrt{|\hat{\Sigma}_0|}}{\sqrt{|\hat{\Sigma}_1|}} e^{\frac{1}{2}\begin{bmatrix} x_1 - \hat{\mu}_0^{x_1} \\ x_2 - \hat{\mu}_0^{x_2} \end{bmatrix}^T \begin{bmatrix} \frac{1}{\hat{\sigma}_0^2 x_1} & 0 \\ 0 & \frac{1}{\hat{\sigma}_0^2 x_2} \end{bmatrix} \begin{bmatrix} x_1 - \hat{\mu}_0^{x_1} \\ x_2 - \hat{\mu}_0^{x_2} \end{bmatrix} - \frac{1}{2}\begin{bmatrix} x_1 - \hat{\mu}_1^{x_1} \\ x_2 - \hat{\mu}_1^{x_2} \end{bmatrix}^T \begin{bmatrix} \frac{1}{\hat{\sigma}_1^2 x_1} & 0 \\ 0 & \frac{1}{\hat{\sigma}^2 x_2} \end{bmatrix} \begin{bmatrix} x_1 - \hat{\mu}_1^{x_1} \\ x_2 - \hat{\mu}_1^{x_2} \end{bmatrix}}$$

$$= \frac{\sqrt{|\hat{\Sigma}_0|}}{\sqrt{|\hat{\Sigma}_1|}} e^{\frac{1}{2}\left[\frac{(x_1-\hat{\mu}_0^{x_1})^2}{\hat{\sigma}_0^{x_1 2}} + \frac{(x_2-\hat{\mu}_0^{x_2})^2}{\hat{\sigma}_0^{x_2 2}} - \frac{(x_1-\hat{\mu}_1^{x_1})^2}{\hat{\sigma}_1^{x_1 2}} - \frac{(x_2-\hat{\mu}_1^{x_2})^2}{\hat{\sigma}_1^{x_2 2}}\right]}$$

Same is the case for

But these two terms, x_1^2 term can't be cancelled because denominator is different

$$= \frac{\sqrt{|\hat{\Sigma}_0|}}{\sqrt{|\hat{\Sigma}_1|}} \, e^{\left[\hat{\theta}_3 \boxed{x_1^2} + \hat{\theta}_1 \boxed{x_1} + \hat{\theta}_0 + \hat{\theta}_2 \boxed{x_2} + \hat{\theta}_4 \boxed{x_2^2} \right]}$$

$$\hat{\theta}_3 = \frac{(\sigma_1^{x_1^2} - \sigma_0^{x_1^2})}{2\sigma_1^{x_1^2} \cdot \sigma_0^{x_1^2}}$$

$$\hat{\theta}_1 = \frac{(2\hat{\mu}_1^{x_1} \cdot \hat{\sigma}_0^{x_1^2} - 2\hat{\mu}_0^{x_1} \cdot \hat{\sigma}_1^{x_1^2})}{2\hat{\sigma}_1^{x_1^2} \cdot \hat{\sigma}_0^{x_1^2}}$$

$$\hat{\theta}_0 = \frac{(\hat{\mu}_0^{x_1^2} \cdot \hat{\sigma}_1^{x_1^2} - \hat{\mu}_1^{x_1^2} \cdot \hat{\sigma}_0^{x_1^2})}{2\hat{\sigma}_1^{x_1^2} \cdot \hat{\sigma}_0^{x_1^2}} + \frac{(\hat{\mu}_0^{x_2^2} \cdot \hat{\sigma}_1^{x_2^2} - \hat{\mu}_1^{x_2^2} \cdot \hat{\sigma}_0^{x_2^2})}{2\hat{\sigma}_1^{x_2^2} \cdot \hat{\sigma}_0^{x_2^2}}$$

$$\hat{\theta}_2 = \frac{(2\hat{\mu}_1^{x_2} \cdot \hat{\sigma}_0^{x_2^2} - 2\hat{\mu}_0^{x_2} \cdot \hat{\sigma}_1^{x_2^2})}{2\hat{\sigma}_1^{x_2^2} \cdot \hat{\sigma}_0^{x_2^2}}$$

$$\hat{\theta}_4 = \frac{(\sigma_1^{x_2^2} - \sigma_0^{x_2^2})}{2\sigma_1^{x_2^2} \cdot \sigma_0^{x_2^2}}$$

$$\frac{P\left(C=1 \,\middle|\, X=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)}{P\left(C=0 \,\middle|\, X=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)} = \frac{\sqrt{|\hat{\Sigma}_0|}}{\sqrt{|\hat{\Sigma}_1|}} e^{\left(\hat{\theta}_0 + \hat{\theta}_1 x_1 + \hat{\theta}_2 x_2 + \hat{\theta}_3 x_1^2 + \hat{\theta}_4 x_2^2\right)}$$

$$\frac{P\left(C=1 \mid X=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)}{P\left(C=0 \mid X=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)} = \frac{\sqrt{|\hat{\Sigma}_0|}}{\sqrt{|\hat{\Sigma}_1|}} e^{\left(\hat{\theta}_0 + \hat{\theta}_1 x_1 + \hat{\theta}_2 x_2 + \hat{\theta}_3 x_1^2 + \hat{\theta}_4 x_2^2\right)}$$

Applying log on both sides

Quadratic discriminant-function

$$\underbrace{\log_e \frac{P\left(C=1 \mid X=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)}{P\left(C=0 \mid X=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)}}_{y = f(x_1, x_2) = f(X)} = \left(\hat{\theta}_0 + \hat{\theta}_1 x_1 + \hat{\theta}_2 x_2 + \hat{\theta}_3 x_1^2 + \hat{\theta}_4 x_2^2\right)$$

$$y = f(x_1, x_2) = f(X) = \hat{\theta}_0 + \hat{\theta}_1 x_1 + \hat{\theta}_2 x_2 + \underbrace{\hat{\theta}_3 x_1^2 + \hat{\theta}_4 x_2^2}_{\text{Quadratic terms}}$$

$$\frac{P\left(C=1 \,\middle|\, X=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)}{P\left(C=0 \,\middle|\, X=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)} = e^{\left(\hat{\theta}_0 + \hat{\theta}_1 x_1 + \hat{\theta}_2 x_2 + \hat{\theta}_3 x_1^2 + \hat{\theta}_4 x_2^2\right)}$$

When we solve this we get —

$$\boxed{P\left(C=1 \,\middle|\, X=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \frac{1}{1+e^{-\left(\hat{\theta}_0 + \hat{\theta}_1 x_1 + \hat{\theta}_2 x_2 + \hat{\theta}_3 x_1^2 + \hat{\theta}_4 x_2^2\right)}}}$$

where $\hat{\theta} = \begin{bmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \\ \hat{\theta}_3 \\ \hat{\theta}_4 \end{bmatrix}$ and -

$$X' = \begin{bmatrix} x_1 \\ x_2 \\ x_1^2 \\ x_2^2 \end{bmatrix}$$