27. If A is any 8 by 8 invertible matrix, then its column space is ____. Why?

28. True or false (with a counterexample if false)?

(a) The vectors b that are not in the column space $C(A)$ form a subspace.

(b) If $C(A)$ contains only the zero vector, then A is the zero matrix.

(c) The column space of $2A$ equals the column space of A.

(d) The column space of $A-I$ equals the column space of A.

29. Construct a 3 by 3 matrix whose column space contains $(1,1,0)$ and $(1,0,1)$ but not $(1,1,1)$. Construct a 3 by 3 matrix whose column space is only a line.

30. If the 9 by 12 system $Ax=b$ is solvable for every b, then $C(A)=$ ____.

31. Why isn't $\mathbf{R}^2$ a subspace of $\mathbf{R}^3$?

2.2 Solving $Ax=0$ and $Ax=b$

Chapter 1 concentrated on square invertible matrices. There was one solution to $Ax=b$ and it was $x=-A^{-1}b$. That solution was found by elimination (not by computing A^{-1}). A rectangular matrix brings new possibilities—U may not have a full set of pivots. This section goes onward from U to a reduced form R—**the simplest matrix that elimination can give**. R reveals all solutions immediately.

For an invertible matrix, the nullspace contains only $x=0$ (multiply $Ax=0$ by A^{-1}). The column space is the whole space ($Ax=b$ has a solution for every b). The new questions appear when the nullspace contains *more than the zero vector* and/or the column space contains *less than all vectors*:

1. Any vector x_n in the nullspace can be added to a particular solution x_p. The solutions to all linear equations have this form, $x=x_p+x_n$:

$$\textbf{Complete solution} \qquad Ax_p=b \quad \textbf{and} \quad Ax_n=0 \qquad \textbf{produce} \quad A(x_p+x_n)=b.$$

2. When the column space doesn't contain every b in $\mathbf{R}^m$, we need the conditions on b that make $Ax=b$ solvable.

A 3 by 4 example will be a good size. We will write down all solutions to $Ax=0$. We will find the conditions for b to lie in the column space (so that $Ax=b$ is solvable). The 1 by 1 system $0x=b$, one equation and one unknown, shows two possibilities:

$0x=b$ has *no solution* unless $b=0$. The column space of the 1 by 1 zero matrix contains only $b=0$.

$0x=0$ has *infinitely many solutions*. The nullspace contains *all* x. A particular solution is $x_p=0$, and the complete solution is $x=x_p+x_n=0+(\text{any } x)$.

Simple, I admit. If you move up to 2 by 2, it's more interesting. The matrix $\left[\begin{smallmatrix}1&1\\2&2\end{smallmatrix}\right]$ is not invertible: $y+z=b_1$ and $2y+2z=b_2$ usually have no solution.

There is ***no solution*** unless $b_2 = 2b_1$. The column space of A contains only those b's, the multiples of $(1,2)$.

When $b_2 = 2b_1$ there are ***infinitely many solutions***. A particular solution to $y+z=2$ and $2y+2z=4$ is $x_p = (1,1)$. The nullspace of A in Figure 2.2 contains $(-1,1)$ and all its multiples $x_n = (-c,c)$:

$$\textbf{Complete solution} \quad \begin{array}{rcrcl} y & + & z & = & 2 \\ 2y & + & 2z & = & 4 \end{array} \quad \text{is solved by} \quad x_p + x_n = \begin{bmatrix}1\\1\end{bmatrix} + c\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}1-c\\1+c\end{bmatrix}.$$

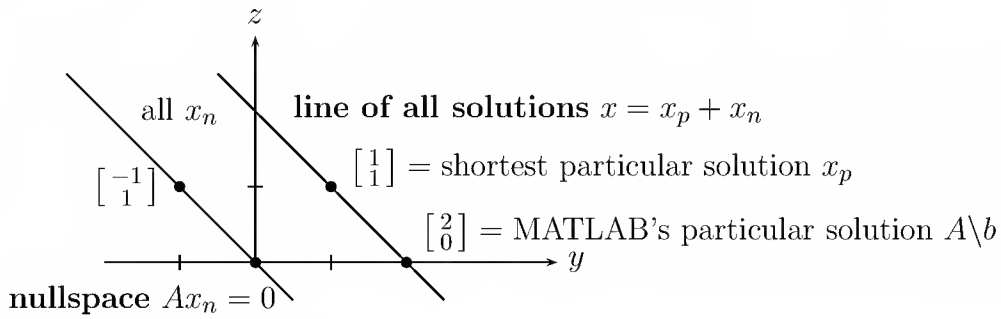

Figure 2.2: The parallel lines of solutions to $Ax_n = 0$ and $\left[\begin{smallmatrix}1&1\\2&2\end{smallmatrix}\right]\left[\begin{smallmatrix}y\\z\end{smallmatrix}\right] = \left[\begin{smallmatrix}2\\4\end{smallmatrix}\right]$.

Echelon Form U and Row Reduced Form R

We start by simplifying this 3 by 4 matrix, first to U and then further to R:

$$\textbf{Basic example} \qquad A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}.$$

The pivot $a_{11} = 1$ is nonzero. The usual elementary operations will produce zeros in the first column below this pivot. The bad news appears in column 2:

$$\textbf{No pivot in column 2} \qquad A \to \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & \mathbf{0} & 3 & 3 \\ 0 & \mathbf{0} & 6 & 6 \end{bmatrix}.$$

The candidate for the second pivot has become zero: *unacceptable*. We look below that zero for a nonzero entry—intending to carry out a row exchange. In this case the *entry below it is also zero*. If A were square, this would signal that the matrix was singular. With a rectangular matrix, we must expect trouble anyway, and there is no reason to stop.

All we can do is to *go on to the next column*, where the pivot entry is 3. Subtracting twice the second row from the third, we arrive at U:

$$\textbf{Echelon matrix } U \qquad U = \begin{bmatrix} \mathbf{1} & 3 & 3 & 2 \\ 0 & 0 & \mathbf{3} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Strictly speaking, we proceed to the fourth column. A zero is in the third pivot position, and nothing can be done. U is upper triangular, but its pivots are not on the main diagonal. The nonzero entries of U have a "staircase pattern," or ***echelon form***. For the 5 by 8 case in Figure 2.3, the starred entries may or may not be zero.

$$U = \begin{bmatrix} \bullet & * & * & * & * & * & * & * \\ 0 & \bullet & * & * & * & * & * & * \\ 0 & 0 & 0 & \bullet & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} \mathbf{1} & \mathbf{0} & * & \mathbf{0} & * & * & * & \mathbf{0} \\ 0 & \mathbf{1} & * & \mathbf{0} & * & * & * & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{1} & * & * & * & \mathbf{0} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Figure 2.3: The entries of a 5 by 8 echelon matrix U and its reduced form R.

We can always reach this echelon form U, with zeros below the pivots:

1. The pivots are the first nonzero entries in their rows.
2. Below each pivot is a column of zeros, obtained by elimination.
3. Each pivot lies to the right of the pivot in the row above. This produces the staircase pattern, and zero rows come last.

Since we started with A and ended with U, the reader is certain to ask: Do we have $A = LU$ as before? There is no reason why not, since the elimination steps have not changed. Each step still subtracts a multiple of one row from a row beneath it. The inverse of each step adds back the multiple that was subtracted. These inverses come in the right order to put the multipliers directly into L:

$$\textbf{Lower triangular} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \quad \text{and} \quad A = LU.$$

Note that L is square. It has the same number of rows as A and U.

The only operation not required by our example, but needed in general, is row exchange by a permutation matrix P. Since we keep going to the next column when no pivots are available, there is no need to assume that A is nonsingular. Here is $PA = LU$ for all matrices:

2B For any m by n matrix A there is a permutation P, a lower triangular L with unit diagonal, and an m by n echelon matrix U, such that $PA = LU$.

Now comes R. We can go further than U, to make the matrix even simpler. Divide the second row by its pivot 3, so that ***all pivots are*** 1. Then use the pivot row to produce ***zero above the pivot***. This time we subtract a row from a *higher row*. The final result (the best form we can get) is the ***reduced row echelon form*** R:

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & \mathbf{1} & \mathbf{1} \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{0} & \mathbf{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} = R.$$

This matrix R is the final result of elimination on A. MATLAB would use the command R = rref(A). Of course rref(R) would give R again!

What is the row reduced form of a square invertible matrix? In that case R is the *identity matrix*. There is a full set of pivots, all equal to 1, with zeros above and below. So rref(A) = I, when A is invertible.

For a 5 by 8 matrix with four pivots, Figure 2.3 shows the reduced form R. **It still contains an identity matrix, in the four pivot rows and four pivot columns**. From R we will quickly find the nullspace of A. $Rx = 0$ has the same solutions as $Ux = 0$ and $Ax = 0$.

Pivot Variables and Free Variables

Our goal is to read off all the solutions to $Rx = 0$. The pivots are crucial:

$$\begin{array}{c}\textbf{Nullspace of } R \\ \textbf{(pivot columns} \\ \textbf{in boldface)}\end{array} \qquad Rx = \begin{bmatrix} \mathbf{1} & 3 & \mathbf{0} & -1 \\ \mathbf{0} & 0 & \mathbf{1} & 1 \\ \mathbf{0} & 0 & \mathbf{0} & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The unknowns u, v, w, y go into two groups. One group contains the ***pivot variables***, those that correspond to ***columns with pivots***. The first and third columns contain the pivots, so u and w are the pivot variables. The other group is made up of the ***free variables***, corresponding to ***columns without pivots***. These are the second and fourth columns, so v and y are free variables.

To find the most general solution to $Rx = 0$ (or, equivalently, to $Ax = 0$) we may assign arbitrary values to the free variables. Suppose we call these values simply v and y. The pivot variables are completely determined in terms of v and y:

$$Rx = 0 \qquad \begin{array}{rcl} u + 3v - y = 0 & \text{yields} & u = -3v + y \\ w + y = 0 & \text{yields} & w = \quad -y \end{array} \tag{1}$$

There is a "double infinity" of solutions, with v and y free and independent. The complete solution is a combination of two **special solutions**:

Nullspace contains all combinations of special solutions

$$x = \begin{bmatrix} -3v+y \\ v \\ -y \\ y \end{bmatrix} = v \begin{bmatrix} \mathbf{-3} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + y \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{-1} \\ \mathbf{1} \end{bmatrix}. \tag{2}$$

Please look again at this complete solution to $Rx = 0$ and $Ax = 0$. The special solution $(-3,1,0,0)$ has free variables $v = 1$, $y = 0$. The other special solution $(1,0,-1,1)$ has $v = 0$ and $y = 1$. *All solutions are linear combinations of these two.* The best way to find all solutions to $Ax = 0$ is from the special solutions:

1. After reaching $Rx = 0$, identify the pivot variables and free variables.
2. Give one free variable the value 1, set the other free variables to 0, and solve $Rx = 0$ for the pivot variables. This x is a special solution.
3. Every free variable produces its own "special solution" by step 2. The combinations of special solutions form the nullspace—all solutions to $Ax = 0$.

Within the four-dimensional space of all possible vectors x, the solutions to $Ax = 0$ form a ***two-dimensional subspace***—the nullspace of A, In the example, $\boldsymbol{N}(A)$ is generated by the special vectors $(-3,1,0,0)$ and $(1,0,-1,1)$. The combinations of these two vectors produce the whole nullspace.

Here is a little trick. The special solutions are especially easy from R. The numbers 3 and 0 and -1 and 1 lie in the "nonpivot columns" of R. **Reverse their signs to find the pivot variables** (not free) **in the special solutions**. I will put the two special solutions from equation (2) into a nullspace matrix N, so you see this neat pattern:

Nullspace matrix (columns are special solutions)

$$N = \begin{bmatrix} \mathbf{-3} & \mathbf{1} \\ 1 & 0 \\ \mathbf{0} & \mathbf{-1} \\ 0 & 1 \end{bmatrix} \begin{array}{l} \text{not free} \\ \text{free} \\ \text{not free} \\ \text{free} \end{array}$$

The free variables have values 1 and 0. When the free columns moved to the right-hand side of equation (2), their coefficients 3 and 0 and -1 and 1 switched sign. That determined the pivot variables in the special solutions (the columns of N).

This is the place to recognize one extremely important theorem. Suppose a matrix has more columns than rows, $n > m$. Since m rows can hold at most m pivots, ***there must be at least*** $n - m$ ***free variables***. There will be even more free variables if some rows of R reduce to zero; but no matter what, at least one variable must be free. This free variable can be assigned any value, leading to the following conclusion:

2C If $Ax = 0$ has more unknowns than equations ($n > m$), it has at least one special solution: There are more solutions than the trivial $x = 0$.

There must be infinitely many solutions, since any multiple cx will also satisfy $A(cx) = 0$. The nullspace contains the line through x. And if there are additional free variables, the nullspace becomes more than just a line in n-dimensional space. *The nullspace has the same "dimension" as the number of free variables and special solutions.*

This central idea—the ***dimension*** of a subspace—is made precise in the next section. We count the free variables for the nullspace. We count the pivot variables for the column space!

Solving $Ax = b$, $Ux = c$, and $Rx = d$

The case $b \neq 0$ is quite different from $b = 0$. The row operations on A must act also on the right-hand side (on b). We begin with letters (b_1, b_2, b_3) to find the solvability condition—for b to lie in the column space. Then we choose $b = (1,5,5)$ and find all solutions x.

For the original example $Ax = b = (b_1, b_2, b_3)$, apply to both sides the operations that led from A to U. The result is an upper triangular system $Ux = c$:

$$Ux = c \qquad \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 - 2b_2 + 5b_1 \end{bmatrix}. \tag{3}$$

The vector c on the right-hand side, which appeared after the forward elimination steps, is just $L^{-1}b$ as in the previous chapter. Start now with $Ux = c$.

It is not clear that these equations have a solution. The third equation is very much in doubt, because its left-hand side is zero. ***The equations are inconsistent unless*** $b_3 - 2b_2 + 5b_1 = 0$. Even though there are more unknowns than equations, there may be no solution. We know another way of answering the same question: $Ax = b$ can be solved if and only if b lies in the column space of A. This subspace comes from the four columns of A (not of U!):

$$\textbf{Columns of } A \textbf{ "span" the column space} \qquad \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}.$$

Even though there are four vectors, their combinations only fill out a plane in three-dimensional space. Column 2 is three times column 1. The fourth column equals the third minus the first. *These dependent columns, the second and fourth, are exactly the ones without pivots.*

The column space $C(A)$ can be described in two different ways. On the one hand, it is *the plane generated by columns 1 and 3.* The other columns lie in that plane, and contribute nothing new. Equivalently, it is the plane of all vectors b that satisfy $b_3 - 2b_2 + 5b_1 = 0$; this is the constraint if the system is to be solvable. ***Every column***

satisfies this constraint, so it is forced on b***!*** Geometrically, we shall see that the vector $(5,-2,1)$ is perpendicular to each column.

If b belongs to the column space, the solutions of $Ax=b$ are easy to find. The last equation in $Ux=c$ is $0=0$. To the free variables v and y, we may assign any values, as before. The pivot variables u and w are still determined by back-substitution. For a specific example with $b_3-2b_2+5b_1=0$, choose $b=(1,5,5)$:

$$Ax=b \qquad \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}.$$

Forward elimination produces U on the left and c on the right:

$$Ux=c \qquad \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}.$$

The last equation is $0=0$, as expected. Back-substitution gives

$$\begin{aligned} 3w+3y&=3 \qquad \text{or} \qquad w=1-y \\ u+3v+3w+2y&=1 \qquad \text{or} \qquad u=-2-3v+y. \end{aligned}$$

Again there is a double infinity of solutions: v and y are free, u and w are not:

$$\textbf{Complete solution} \quad x=x_p+x_n \qquad x=\begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}. \tag{4}$$

This has all solutions to $Ax=0$, plus the new $x_p=(-2,0,1,0)$. That x_p is ***a particular solution*** to $Ax=b$. The last two terms with v and y yield more solutions (because they satisfy $Ax=0$). ***Every solution to*** $Ax=b$ ***is the sum of one particular solution and a solution to*** $Ax=0$:

$$x_{\textbf{complete}} = x_{\textbf{particular}} + x_{\textbf{nullspace}}$$

The particular solution in equation (4) comes from solving the equation *with all free variables set to zero*. That is the only new part, since the nullspace is already computed. When you multiply the highlighted equation by A, you get $Ax_{\text{complete}}=b+0$.

Geometrically, the solutions again fill a two-dimensional surface—but it is not a subspace. It does not contain $x=0$. It is *parallel* to the nullspace we had before, shifted by the particular solution x_p as in Figure 2.2. Equation (4) is a good way to write the answer:

1. Reduce $Ax=b$ to $Ux=c$.

2. With free variables $= 0$, find a particular solution to $Ax_p = b$ and $Ux_p = c$.

3. Find the special solutions to $Ax = 0$ (or $Ux = 0$ or $Rx = 0$). Each free variable, in turn, is 1. Then $x = x_p +$ (any combination x_n of special solutions).

When the equation was $Ax = 0$, the particular solution was the zero vector! It fits the pattern, but $x_{\text{particular}} = 0$ was not written in equation (2). Now x_p is added to the nullspace solutions, as in equation (4).

Question: How does the reduced form R make this solution even clearer? You will see it in our example. Subtract equation 2 from equation 1, and then divide equation 2 by its pivot. On the left-hand side, this produces R, as before. On the right-hand side, these operations change $c = (1,3,0)$ to a new vector $d = (-2,1,0)$:

$$\textbf{Reduced equation} \quad Rx = d \qquad \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}. \tag{5}$$

Our particular solution x_p, (one choice out of many) has free variables $v = y = 0$. Columns 2 and 4 can be ignored. Then we immediately have $u = -2$ and $w = 1$, exactly as in equation (4). **The entries of d go directly into x_p.** This is because the identity matrix is sitting in the pivot columns of R!

Let me summarize this section, before working a new example. Elimination reveals the pivot variables and free variables. ***If there are r pivots, there are r pivot variables and $n-r$ free variables***. That important number r will be given a name—it is the ***rank of the matrix***.

2D Suppose elimination reduces $Ax = b$ to $Ux = c$ and $Rx = d$, with r pivot rows and r pivot columns. **The rank of those matrices is r.** The last $m-r$ rows of U and R are zero, so there is a solution only if the last $m-r$ entries of c and d are also zero.

The complete solution is $x = x_p + x_n$. One particular solution x_p has all free variables zero. Its pivot variables are the first r entries of d, so $Rx_p = d$.

The nullspace solutions x_n are combinations of $n-r$ special solutions, with one free variable equal to 1. The pivot variables in that special solution can be found in the corresponding column of R (with sign reversed).

You see how the rank r is crucial. It counts the pivot rows in the "row space" and the pivot columns in the column space. There are $n-r$ special solutions in the nullspace. There are $m-r$ solvability conditions on b or c or d.

Another Worked Example

The full picture uses elimination and pivot columns to find the column space, nullspace, and rank. The 3 by 4 matrix A has rank 2:

$$Ax = b \quad \text{is} \quad \begin{array}{rcrcrcrcl} \mathbf{1}x_1 & + & \mathbf{2}x_2 & + & \mathbf{3}x_3 & + & \mathbf{5}x_4 & = & b_1 \\ \mathbf{2}x_1 & + & \mathbf{4}x_2 & + & \mathbf{8}x_3 & + & \mathbf{12}x_4 & = & b_2 \\ \mathbf{3}x_1 & + & \mathbf{6}x_2 & + & \mathbf{7}x_3 & + & \mathbf{13}x_4 & = & b_3 \end{array} \tag{6}$$

1. Reduce $[A \;\; b]$ to $[U \;\; c]$, to reach a triangular system $Ux = c$.
2. Find the condition on b_1, b_2, b_3 to have a solution.
3. Describe the column space of A: Which plane in $\mathbf{R}^3$?
4. Describe the nullspace of A: Which special solutions in $\mathbf{R}^4$?
5. Find a particular solution to $Ax = (0, 6, -6)$ and the complete $x_p + x_n$.
6. Reduce $[U \;\; c]$ to $[R \;\; d]$: Special solutions from R and x_p from d.

Solution. (Notice how the right-hand side is included as an extra column!)

1. The multipliers in elimination are 2 and 3 and -1, taking $[A \;\; b]$ to $[U \;\; c]$.

$$\left[\begin{array}{ccccc} 1 & 2 & 3 & 5 & b_1 \\ 2 & 4 & 8 & 12 & b_2 \\ 3 & 6 & 7 & 13 & b_3 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2b_1 \\ 0 & 0 & -2 & -2 & b_3 - 3b_1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 5b_1 \end{array}\right].$$

2. The last equation shows the solvability condition $b_3 + b_2 - 5b_1 = 0$. Then $0 = 0$.
3. The column space of A is the plane containing all combinations of the pivot columns $(1, 2, 3)$ and $(3, 8, 7)$.

   Second description: The column space contains all vectors with $b_3 + b_2 - 5b_1 = 0$. That makes $Ax = b$ solvable, so b is in the column space. *All columns of A pass this test $b_3 + b_2 - 5b_1 = 0$. This is the equation for the plane (in the first description of the column space).*

4. The special solutions in N have free variables $x_2 = 1$, $x_4 = 0$ and $x_2 = 0$, $x_4 = 1$:

$$\begin{array}{l} \textbf{Nullspace matrix} \\ \textbf{Special solutions to } Ax = 0 \\ \textbf{Back-substitution in } Ux = 0 \\ \textbf{Just switch signs in } Rx = 0 \end{array} \qquad N = \begin{bmatrix} -2 & -2 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}.$$

5. Choose $b = (0,6,-6)$, which has $b_3 + b_2 - 5b_1 = 0$. Elimination takes $Ax = b$ to $Ux = c = (0,6,0)$. Back-substitute with free variables $= 0$:

$$\textbf{Particular solution to } Ax_p = (0,6,-6) \qquad x_p = \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix} \begin{matrix} \\ \text{free} \\ \\ \text{free} \end{matrix}$$

The complete solution to $Ax = (0,6,-6)$ is (this x_p) + (all x_n).

6. In the reduced R, the third column changes from $(3,2,0)$ to $(0,1,0)$. The right-hand side $c = (0,6,0)$ becomes $d = (-9,3,0)$. Then -9 and 3 go into x_p:

$$\begin{bmatrix} U & c \end{bmatrix} = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & 0 \\ 0 & 0 & 2 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{bmatrix} R & d \end{bmatrix} = \left[\begin{array}{cccc|c} \mathbf{1} & 2 & 0 & 2 & \mathbf{-9} \\ 0 & 0 & \mathbf{1} & 1 & \mathbf{3} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

That final matrix $[R \;\; d]$ **is** $\mathsf{rref}([A \;\; b]) = \mathsf{rref}([U \;\; c])$. The numbers 2 and 0 and 2 and 1 in the free columns of R have opposite sign in the special solutions (the nullspace matrix N). Everything is revealed by $Rx = d$.

Problem Set 2.2

1. Construct a system with more unknowns than equations, but no solution. Change the right-hand side to zero and find all solutions x_n.

2. Reduce A and B to echelon form, to find their ranks. Which variables are free?

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Find the special solutions to $Ax = 0$ and $Bx = 0$. Find all solutions.

3. Find the echelon form U, the free variables, and the special solutions:

$$A = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 2 & 0 & 6 \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

$Ax = b$ is consistent (has a solution) when b satisfies $b_2 =$ ____. Find the complete solution in the same form as equation (4).