Chapter 3

Orthogonality

3.1 Orthogonal Vectors and Subspaces

A basis is a set of independent vectors that span a space. Geometrically, it is a set of coordinate axes. A vector space is defined without those axes, but every time I think of the x-y plane or three-dimensional space or $\mathbf{R}^n$, the axes are there. They are usually perpendicular! *The coordinate axes that the imagination constructs are practically always orthogonal.* In choosing a basis, we tend to choose an orthogonal basis.

The idea of an orthogonal basis is one of the foundations of linear algebra. We need a basis to convert geometric constructions into algebraic calculations, and we need an orthogonal basis to make those calculations simple. A further specialization makes the basis just about optimal: The vectors should have *length* 1. For an ***orthonormal basis*** (orthogonal unit vectors), we will find

1. the length $\|x\|$ of a vector;

2. the test $x^{\mathrm{T}}y = 0$ for perpendicular vectors; and

3. how to create perpendicular vectors from linearly independent vectors.

More than just vectors, *subspaces* can also be perpendicular. We will discover, so beautifully and simply that it will be a delight to see, that ***the fundamental subspaces meet at right angles***. Those four subspaces are perpendicular in pairs, two in $\mathbf{R}^m$ and two in $\mathbf{R}^n$. That will complete the fundamental theorem of linear algebra.

The first step is to find the ***length of a vector***. It is denoted by $\|x\|$, and in two dimensions it comes from the hypotenuse of a right triangle (Figure 3.1a). The square of the length was given a long time ago by Pythagoras: $\|x\|^2 = x_1^2 + x_2^2$.

In three-dimensional space, $x = (x_1, x_2, x_3)$ is the diagonal of a box (Figure 3.1b). Its length comes from *two* applications of the Pythagorean formula. The two-dimensional case takes care of $(x_1, x_2, 0) = (1, 2, 0)$ across the base. This forms a right angle with the vertical side $(0, 0, x_3) = (0, 0, 3)$. The hypotenuse of the bold triangle (Pythagoras again)

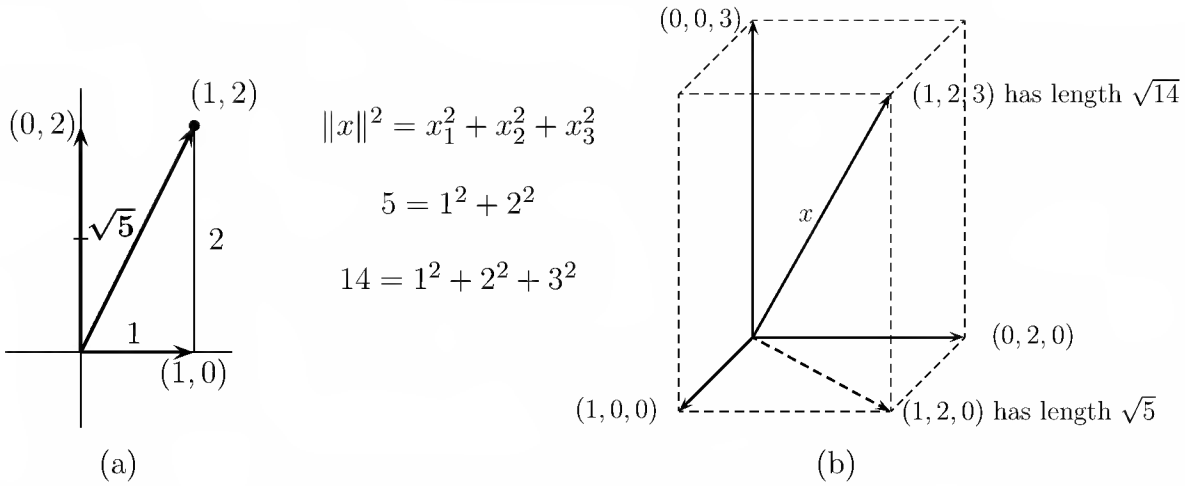

Figure 3.1: The length of vectors (x_1, x_2) and (x_1, x_2, x_3).

is the length $\|x\|$ we want:

$$\textbf{Length in 3D} \qquad \|x\|^2 = 1^2 + 2^2 + 3^2 \quad \text{and} \quad \|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

The extension to $x = (x_1, \ldots, x_n)$ in n dimensions is immediate. By Pythagoras $n-1$ times, ***the length*** $\|x\|$ ***in*** $\mathbf{R}^n$ ***is the positive square root of*** $x^{\mathrm{T}}x$:

$$\textbf{Length squared} \qquad \|x\|^2 = x_1^2 + x_2^2 + \cdots + x_n^2 = x^{\mathrm{T}}x. \tag{1}$$

The sum of squares matches $x^{\mathrm{T}}x$—and the length of $x = (1, 2, -3)$ is $\sqrt{14}$:

$$x^{\mathrm{T}}x = \begin{bmatrix} 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = 1^2 + 2^2 + (-3)^2 = 14.$$

Orthogonal Vectors

How can we decide whether two vectors x and y are perpendicular? What is the test for orthogonality in Figure 3.2? In the plane spanned by x and y, those vectors are orthogonal provided they form a *right triangle*. We go back to $a^2 + b^2 = c^2$:

$$\textbf{Sides of a right triangle} \qquad \|x\|^2 + \|y\|^2 = \|x - y\|^2. \tag{2}$$

Applying the length formula (1), this test for orthogonality in $\mathbf{R}^n$ becomes

$$\left(x_1^2 + \cdots + x_n^2\right) + \left(y_1^2 + \cdots + y_n^2\right) = (x_1 - y_1)^2 + \cdots + (x_n - y_n)^2.$$

The right-hand side has an extra $-2x_iy_i$ from each $(x_i - y_i)^2$:

$$\text{right-hand side} = \left(x_1^2 + \cdots + x_n^2\right) - 2(x_1y_1 + \cdots + x_ny_n) + \left(y_1^2 + \cdots + y_n^2\right).$$

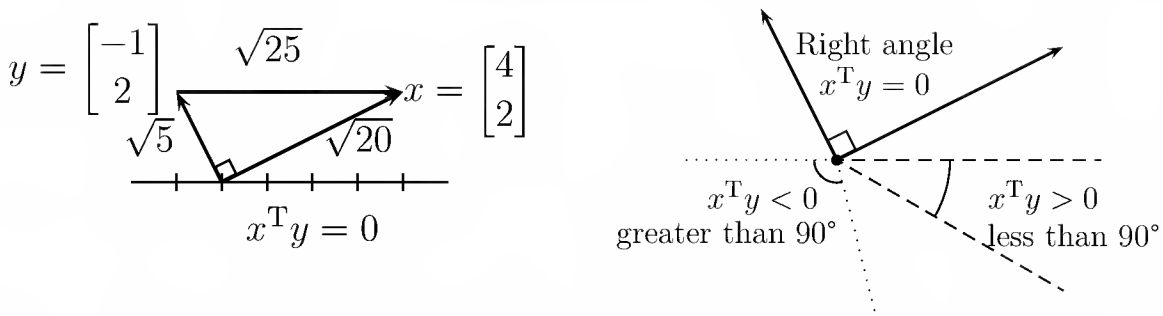

Figure 3.2: A right triangle with $5+20=25$. Dotted angle 100°, dashed angle 30°.

We have a right triangle when that sum of cross-product terms $x_i y_i$ is zero:

$$\textbf{Orthogonal vectors} \qquad x^{\mathrm{T}}y = x_1y_1 + \cdots + x_ny_n = 0. \tag{3}$$

This sum is $x^{\mathrm{T}}y = \sum x_iy_i = y^{\mathrm{T}}x$, the row vector x^{T} times the column vector y:

$$\textbf{Inner product} \qquad x^{\mathrm{T}}y = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x_1y_1 + \cdots + x_ny_n. \tag{4}$$

This number is sometimes called the scalar product or dot product, and denoted by (x,y) or $x \cdot y$. We will use the name ***inner product*** and keep the notation $x^{\mathrm{T}}y$.

3A The inner product $x^{\mathrm{T}}y$ is zero if and only if x and y are orthogonal vectors. If $x^{\mathrm{T}}y > 0$, their angle is less than 90°. If $x^{\mathrm{T}}y < 0$, their angle is greater than 90°.

The length squared is the inner product of x with itself: $x^{\mathrm{T}}x = x_1^2 + \cdots + x_n^2 = \|x\|^2$. The only vector with length zero—the only vector orthogonal to itself—is the zero vector. This vector $x = 0$ is orthogonal to every vector in $\mathbf{R}^n$.

Example 1. $(2,2,-1)$ is orthogonal to $(-1,2,2)$. Both have length $\sqrt{4+4+1} = 3$.

Useful fact: **If nonzero vectors $v_1, \ldots, v_k$ are mutually orthogonal** (every vector is perpendicular to every other), **then those vectors are linearly independent**.

Proof. Suppose $c_1v_1 + \cdots + c_kv_k = 0$. To show that c_1 must be zero, take the inner product of both sides with v_1. Orthogonality of the v's leaves only one term:

$$v_1^{\mathrm{T}}(c_1v_1 + \cdots + c_kv_k) = c_1v_1^{\mathrm{T}}v_1 = 0. \tag{5}$$

The vectors are nonzero, so $v_1^{\mathrm{T}}v_1 \neq 0$ and therefore $c_1 = 0$. The same is true of every c_i. The only combination of the v's producing zero has all $c_i = 0$: *independence*! □

The coordinate vectors $e_1, \ldots, e_n$ in $\mathbf{R}^n$ are the most important orthogonal vectors. Those are the columns of the identity matrix. They form the simplest basis for $\mathbf{R}^n$, and