Chapter 1

Matrices and Gaussian Elimination

1.1 Introduction

This book begins with the central problem of linear algebra: *solving linear equations*. The most important ease, and the simplest, is when the number of unknowns equals the number of equations. We have n **equations in** n **unknowns**, starting with $n = 2$:

$$\begin{array}{lrcrcr} \textbf{Two equations} & 1x & + & 2y & = & 3 \\ \textbf{Two unknowns} & 4x & + & 5y & = & 6. \end{array} \tag{1}$$

The unknowns are x and y. I want to describe two ways, *elimination* and *determinants*, to solve these equations. Certainly x and y are determined by the numbers 1, 2, 3, 4, 5, 6. The question is how to use those six numbers to solve the system.

1. **Elimination** Subtract 4 times the first equation from the second equation. This eliminates x from the second equation. and it leaves one equation for y:

$$(\textbf{equation 2}) - 4(\textbf{equation 1}) \qquad -3y = -6. \tag{2}$$

   Immediately we know $y = 2$. Then x comes from the first equation $1x + 2y = 3$:

$$\textbf{Back-substitution} \qquad 1x + 2(2) = 3 \quad \text{gives} \quad x = -1. \tag{3}$$

   Proceeding carefully, we cheek that x and y also solve the second equation. This should work and it does: 4 times ($x = -1$) plus 5 times ($y = 2$) equals 6.

2. **Determinants** The solution $y = 2$ depends completely on those six numbers in the equations. There most be a formula for y (and also x) It is a "ratio of determinants" and I hope you will allow me to write it down directly:

$$y = \frac{\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}} = \frac{1 \cdot 6 - 3 \cdot 4}{1 \cdot 5 - 2 \cdot 4} = \frac{-6}{-3} = 2. \tag{4}$$

That could seem a little mysterious, unless you already know about 2 by 2 determinants. They gave the same answer $y = 2$, coming from the same ratio of -6 to -3. If we stay with determinants (which we don't plan to do), there will be a similar formula to compute the other unknown, x:

$$x = \frac{\begin{vmatrix} 3 & 2 \\ 6 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}} = \frac{3 \cdot 5 - 2 \cdot 6}{1 \cdot 5 - 2 \cdot 4} = \frac{3}{-3} = -1. \tag{5}$$

Let me compare those two approaches, looking ahead to real problems when n is much larger ($n = 1000$ is a very moderate size in scientific computing). The truth is that direct use of the determinant formula for 1000 equations would be a total disaster. It would use the million numbers on the left sides correctly, but not efficiently. We will find that formula (Cramer's Rule) in Chapter 4, but we want a good method to solve 1000 equations in Chapter 1.

That good method is *Gaussian Elimination*. This is the algorithm that is constantly used to solve large systems of equations. From the examples in a textbook ($n = 3$ is close to the upper limit on the patience of the author and reader) too might not see much difference. Equations (2) and (4) used essentially the same steps to find $y = 2$. Certainly x came faster by the back-substitution in equation (3) than the ratio in (5). For larger n there is absolutely no question. Elimination wins (and this is even the best way to compute determinants).

The idea of elimination is deceptively simple—you will master it after a few examples. It will become the basis for half of this book, simplifying a matrix so that we can understand it. Together with the mechanics of the algorithm, we want to explain four deeper aspects in this chapter. They are:

1. Linear equations lead to ***geometry of planes***. It is not easy to visualize a nine-dimensional plane in ten-dimensional space. It is harder to see ten of those planes, intersecting at the solution to ten equations—but somehow this is almost possible. Our example has two lines in Figure 1.1, meeting at the point $(x,y) = (-1,2)$. Linear algebra moves that picture into ten dimensions, where the intuition has to imagine the geometry (and gets it right)

2. We move to ***matrix notation***, writing the n unknowns as a vector x and the n equations as $Ax = b$. We multiply A by "elimination matrices" to reach an upper triangular matrix U. Those steps factor A into L times U, where L is lower triangular. I will write down A and its factors for our example, and explain them at the right time:

$$\textbf{Factorization} \qquad A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} = L \textbf{ times } U. \tag{6}$$

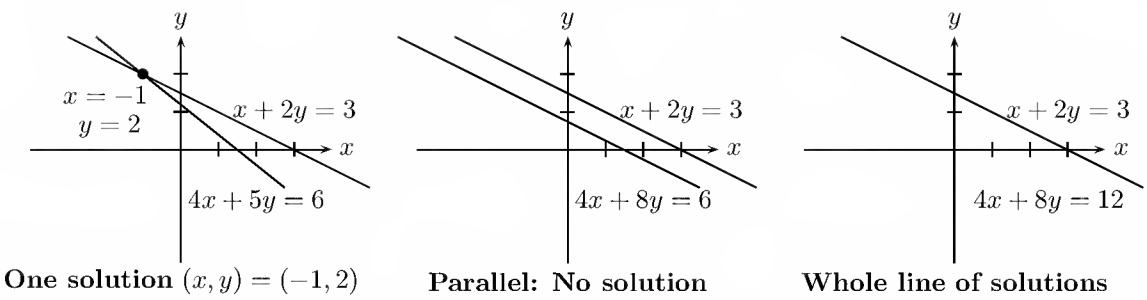


Figure 1.1: The example has one solution. Singular cases have none or too many.

First we have to introduce matrices and vectors and the rules for multiplication. Every matrix has a ***transpose*** A^{T}. This matrix has an ***inverse*** A^{-1}.

3. In most cases elimination goes forward without difficulties. The matrix has an inverse and the system $Ax = b$ has one solution. In exceptional cases the method will *break down*—either the equations were written in the wrong order, which is easily fixed by exchanging them, or the equations don't have a unique solution.

   That ***singular case*** will appear if 8 replaces 5 in our example:

$$\begin{array}{lrcrcr} \textbf{Singular case} & 1x & + & 2y & = & 3 \\ \textbf{Two parallel lines} & 4x & + & 8y & = & 6. \end{array} \tag{7}$$

   Elimination still innocently subtracts 4 times the first equation from the second. But look at the result!

$$(\textbf{equation 2}) - 4(\textbf{equation 1}) \qquad 0 = -6.$$

   This singular case has ***no solution***. Other singular cases have ***infinitely many solutions***. (Change 6 to 12 in the example, and elimination will lead to $0 = 0$. Now y can have *any value*,) When elimination breaks down, we want to find every possible solution.

4. We need a rough count of the ***number of elimination steps*** required to solve a system of size n. The computing cost often determines the accuracy in the model. A hundred equations require a third of a million steps (multiplications and subtractions). The computer can do those quickly, but not many trillions. And already after a million steps, roundoff error could be significant. (Some problems are sensitive; others are not.) Without trying for full detail, we want to see large systems that arise in practice, and how they are actually solved.

The final result of this chapter will be an elimination algorithm that is about as efficient as possible. It is essentially the algorithm that is in constant use in a tremendous variety of applications. And at the same time, understanding it in terms of *matrices*—the coefficient matrix A, the matrices E for elimination and P for row exchanges, and the

final factors L and U—is an essential foundation for the theory. I hope you will enjoy this book and this course.

1.2 The Geometry of Linear Equations

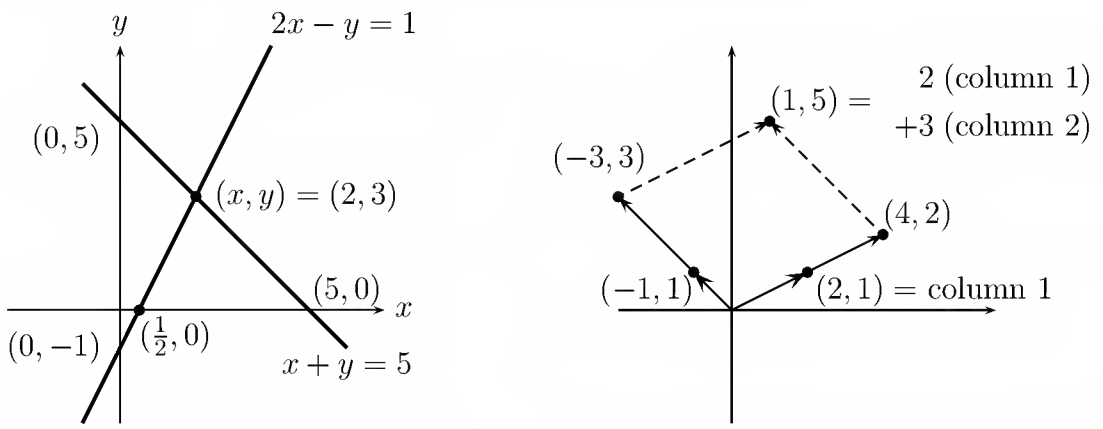
The way to understand this subject is by example. We begin with two extremely humble equations, recognizing that you could solve them without a course in linear algebra. Nevertheless I hope you will give Gauss a chance:

$$\begin{aligned} 2x - y &= 1 \\ x + y &= 5. \end{aligned}$$

We can look at that system *by rows* or *by columns*. We want to see them both.

The first approach concentrates on the separate equations (the ***rows***). That is the most familiar, and in two dimensions we can do it quickly. The equation $2x - y = 1$ is represented by a *straight line* in the x-y plane. The line goes through the points $x = 1$, $y = 1$ and $x = \frac{1}{2}$, $y = 0$ (and also through $(2,3)$ and all intermediate points). The second equation $x + y = 5$ produces a second line (Figure 1.2a). Its slope is $dy/dx = -1$ and it crosses the first line at the solution.

The point of intersection lies on both lines. It is the only solution to both equations. That point $x = 2$ and $y = 3$ will soon be found by "elimination."



(a) Lines meet at $x = 2$, $y = 3$ (b) Columns combine with 2 and 3

Figure 1.2: Row picture (two lines) and column picture (combine columns).

The second approach looks at the ***columns*** of the linear system. The two separate equations are really ***one vector equation***:

$$\textbf{Column form} \qquad x\begin{bmatrix} 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

The problem is ***to find the combination of the column vectors on the left side that produces the vector on the right side***. Those vectors $(2,1)$ and $(-1,1)$ are represented by the bold lines in Figure 1.2b. The unknowns are the numbers x and y that multiply the column vectors. The whole idea can be seen in that figure, where 2 times column 1 is added to 3 times column 2. Geometrically this produces a famous parallelogram. Algebraically it produces the correct vector $(1,5)$, on the right side of our equations. The column picture confirms that $x = 2$ and $y = 3$.

More time could be spent on that example, but I would rather move forward to $n = 3$. Three equations are still manageable, and they have much more variety:

$$\textbf{Three planes} \qquad \begin{array}{rcrcrcr} 2u & + & v & + & w & = & 5 \\ 4u & - & 6v & & & = & -2 \\ -2u & + & 7v & + & 2w & = & 9. \end{array} \tag{1}$$

Again we can study the rows or the columns, and we start with the rows. Each equation describes a ***plane*** in three dimensions. The first plane is $2u+v+w=5$, and it is sketched in Figure 1.3. It contains the points $(\frac{5}{2},0,0)$ and $(0,5,0)$ and $(0,0,5)$. It is determined by any three of its points—provided they do not lie on a line.

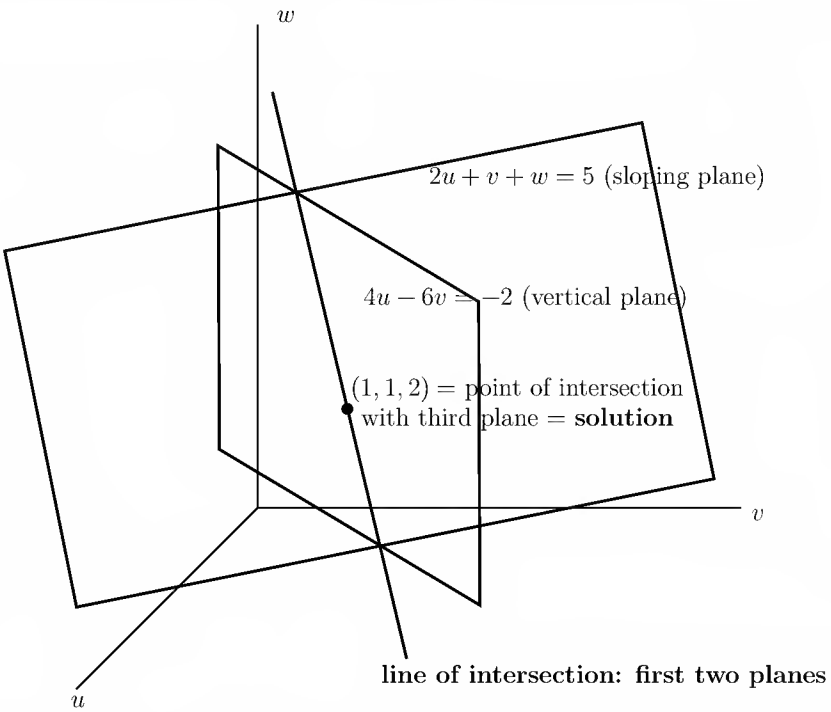


Figure 1.3: The row picture: three intersecting planes from three linear equations.

Changing 5 *to* 10, *the plane* $2u+v+w=10$ *would be parallel to this one*. It contains $(5,0,0)$ and $(0,10,0)$ and $(0,0,10)$, twice as far from the origin—which is the center point $u=0$, $v=0$, $w=0$. Changing the right side moves the plane parallel to itself, and the plane $2u+v+w=0$ goes through the origin.

The second plane is $4u - 6v = -2$. It is drawn vertically, because w can take any value. The coefficient of w is zero, but this remains a plane in 3-space. (The equation $4u = 3$, or even the extreme case $u = 0$, would still describe a plane.) The figure shows the intersection of the second plane with the first. That intersection is a line. *In three dimensions a line requires two equations*; in n dimensions it will require $n-1$.

Finally the third plane intersects this line in a point. The plane (not drawn) represents the third equation $-2u+7v+2w=9$, and it crosses the line at $u=1$, $v=1$, $w=2$. That triple intersection point $(1,1,2)$ solves the linear system.

How does this row picture extend into n dimensions? The n equations will contain n unknowns. The first equation still determines a "plane." It is no longer a two-dimensional plane in 3-space; somehow it has "dimension" $n-1$. It must be flat and extremely thin within n-dimensional space, although it would look solid to us.

If time is the fourth dimension, then the plane $t = 0$ cuts through four-dimensional space and produces the three-dimensional universe we live in (or rather, the universe as it was at $t = 0$). Another plane is $z = 0$, which is also three-dimensional; it is the ordinary x-y plane taken over all time. Those three-dimensional planes will intersect! They share the ordinary x-y plane at $t = 0$. We are down to two dimensions, and the next plane leaves a line. Finally a fourth plane leaves a single point. It is the intersection point of 4 planes in 4 dimensions, and it solves the 4 underlying equations.

I will be in trouble if that example from relativity goes any further. The point is that linear algebra can operate with any number of equations. The first equation produces an $(n-1)$-dimensional plane in n dimensions, The second plane intersects it (we hope) in a smaller set of "dimension $n-2$." Assuming all goes well, every new plane (every new equation) reduces the dimension by one. At the end, when all n planes are accounted for, the intersection has dimension zero. It is a *point*, it lies on all the planes, and its coordinates satisfy all n equations. It is the solution!

Column Vectors and Linear Combinations

We turn to the columns. This time the vector equation (the same equation as (1)) is

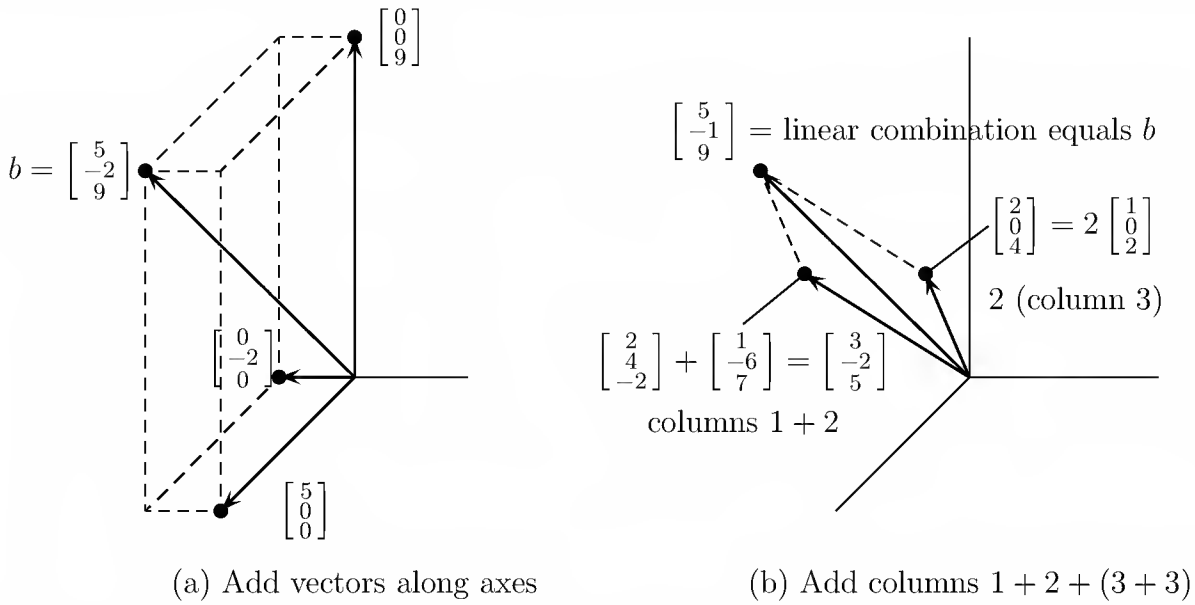
$$\textbf{Column form} \qquad u\begin{bmatrix}2\\4\\-2\end{bmatrix} + v\begin{bmatrix}1\\-6\\7\end{bmatrix} + w\begin{bmatrix}1\\0\\2\end{bmatrix} = \begin{bmatrix}5\\-2\\9\end{bmatrix} = b. \tag{2}$$

Those are *three-dimensional column vectors. The vector b is identified with the point whose coordinates are* 5, -2, 9. Every point in three-dimensional space is matched to a vector, and vice versa. That was the idea of Descartes, who turned geometry into algebra by working with the coordinates of the point. We can write the vector in a column, or we can list its components as $b = (5,-2,9)$, or we can represent it geometrically by an arrow from the origin. You can choose *the arrow*, or *the point*, or *the three numbers*. In six dimensions it is probably easiest to choose the six numbers.

We use parentheses and commas when the components are listed horizontally, and square brackets (with no commas) when a column vector is printed vertically. What really matters is ***addition of vectors*** and ***multiplication by a scalar*** (a number). In Figure 1.4a you see a vector addition, component by component:

$$\textbf{Vector addition} \qquad \begin{bmatrix}5\\0\\0\end{bmatrix}+\begin{bmatrix}0\\-2\\0\end{bmatrix}+\begin{bmatrix}0\\0\\9\end{bmatrix}=\begin{bmatrix}5\\-2\\9\end{bmatrix}.$$

In the right-hand figure there is a multiplication by 2 (and if it had been -2 the vector



(a) Add vectors along axes (b) Add columns $1+2+(3+3)$

Figure 1.4: The column picture: linear combination of columns equals b.

would have gone in the reverse direction):

$$\textbf{Multiplication by scalars} \qquad 2\begin{bmatrix}1\\0\\2\end{bmatrix}=\begin{bmatrix}2\\0\\4\end{bmatrix}, \quad -2\begin{bmatrix}1\\0\\2\end{bmatrix}=\begin{bmatrix}-2\\0\\-4\end{bmatrix}.$$

Also in the right-hand figure is one of the central ideas of linear algebra. It uses *both* of the basic operations; vectors are *multiplied by numbers and then added*. The result is called a ***linear combination***, and this combination solves our equation:

$$\textbf{Linear combination} \qquad \mathbf{1}\begin{bmatrix}2\\4\\-2\end{bmatrix}+\mathbf{1}\begin{bmatrix}1\\-6\\7\end{bmatrix}+\mathbf{2}\begin{bmatrix}1\\0\\2\end{bmatrix}=\begin{bmatrix}5\\-2\\9\end{bmatrix}.$$

Equation (2) asked for multipliers u, v, w that produce the right side b. Those numbers are $u=\mathbf{1}$, $v=\mathbf{1}$, $w=\mathbf{2}$. They give the correct combination of the columns. They also gave the point $(1,1,2)$ in the row picture (where the three planes intersect).

Our true goal is to look beyond two or three dimensions into n dimensions. With n equations in n unknowns, there are n planes in the row picture. There are n vectors in the column picture, plus a vector b on the right side. The equations ask for a ***linear combination of the n columns that equals b***. For certain equations that will be impossible. Paradoxically, the way to understand the good case is to study the bad one. Therefore we look at the geometry exactly when it breaks down, in the **singular case**.

Row picture: Intersection of planes	***Column picture***: Combination of columns

The Singular Case

Suppose we are again in three dimensions, and the three planes in the row picture *do not intersect*. What can go wrong? One possibility is that two planes may be parallel. The equations $2u+v+w=5$ and $4u+2v+2w=11$ are inconsistent—and parallel planes give no solution (Figure 1.5a shows an end view). In two dimensions, parallel lines are the only possibility for breakdown. But three planes in three dimensions can be in trouble without being parallel.

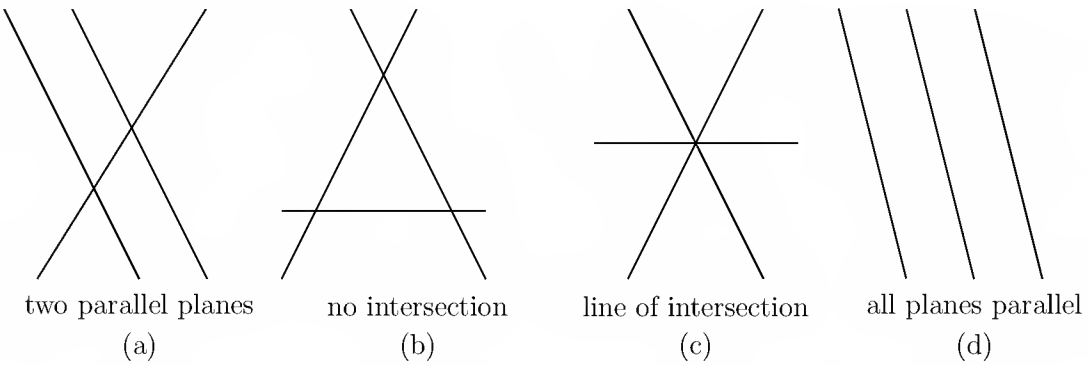


Figure 1.5: Singular cases: no solution for (a), (b), or (d), an infinity of solutions for (c).

The most common difficulty is shown in Figure 1.5b. From the end view the planes form a triangle. Every pair of planes intersects in a line, and those lines are parallel. The third plane is not parallel to the other planes, but it is parallel to their line of intersection. This corresponds to a singular system with $b=(2,5,6)$:

$$\textbf{No solution, as in Figure 1.5b} \qquad \begin{array}{rcrcrcr} u & + & v & + & w & = & 2 \\ 2u & & & + & 3w & = & 5 \\ 3u & + & v & + & 4w & = & 6. \end{array} \tag{3}$$

The first two left sides add up to the third. On the right side that fails: $2+5\neq 6$. Equation 1 plus equation 2 minus equation 3 is the impossible statement $0=1$. Thus the equations are ***inconsistent***, as Gaussian elimination will systematically discover.

Another singular system, close to this one, has an **infinity of solutions**. When the 6 in the last equation becomes 7, the three equations combine to give $0 = 0$. Now the third equation is the sum of the first two. In that case the three planes have a whole *line in common* (Figure 1.5c). Changing the right sides will move the planes in Figure 1.5b parallel to themselves, and for $b = (2,5,7)$ the figure is suddenly different. The lowest plane moved up to meet the others, and there is a line of solutions. Problem 1.5c is still singular, but now it suffers from ***too many solutions*** instead of too few.

The extreme case is three parallel planes. For most right sides there is no solution (Figure 1.5d). For special right sides (like $b = (0,0,0)$!) there is a whole plane of solutions—because the three parallel planes move over to become the same.

What happens to the ***column picture*** when the system is singular? it has to go wrong; the question is how, There are still three columns on the left side of the equations, and we try to combine them to produce b. Stay with equation (3):

Singular case: Column picture
Three columns in the same plane
Solvable only for b in that plane

$$u\begin{bmatrix}1\\2\\3\end{bmatrix} + v\begin{bmatrix}1\\0\\1\end{bmatrix} + w\begin{bmatrix}1\\3\\4\end{bmatrix} = b. \qquad (4)$$

For $b = (2,5,7)$ this was possible; for $b = (2,5,6)$ it was not. The reason is that ***those three columns lie in a plane***. Then every combination is also in the plane (which goes through the origin). If the vector b is not in that plane, no solution is possible (Figure 1.6). That is by far the most likely event; a singular system generally has no solution. But there is a chance that b *does* lie in the plane of the columns. In that case there are too many solutions; the three columns can be combined in *infinitely many ways* to produce b. That column picture in Figure 1.6b corresponds to the row picture in Figure 1.5c.

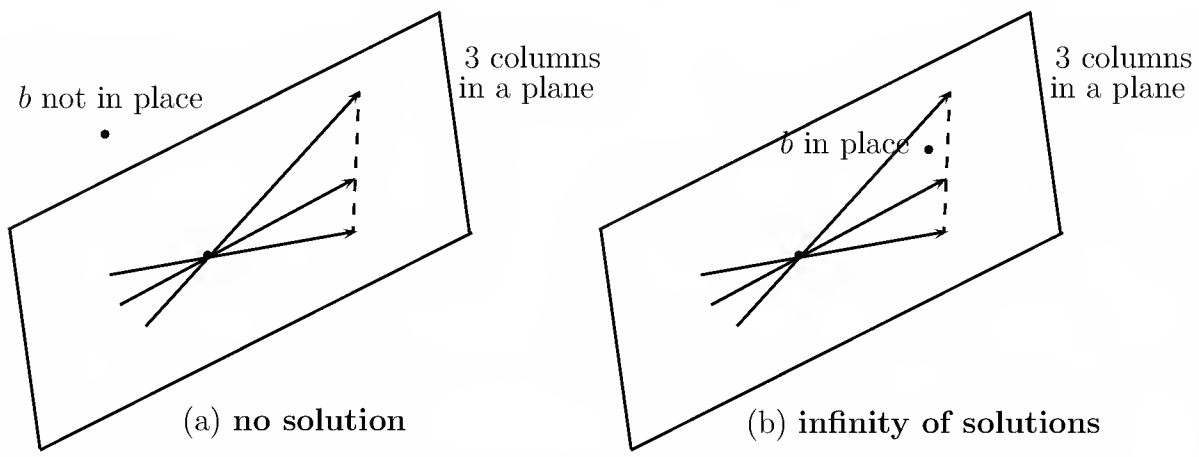


Figure 1.6: Singular cases: b outside or inside the plane with all three columns.

How do we know that the three columns lie in the same plane? One answer is to find a combination of the columns that adds to zero. After some calculation, it is $u = 3$, $v = 1$, $w = -2$. Three times column 1 equals column 2 plus twice column 3. Column 1 is in

the plane of columns 2 and 3. Only two columns are independent.

The vector $b = (2,5,7)$ is in that plane of the columns—it is column 1 plus column 3—so (1, 0, 1) is a solution. *We can add an multiple of the combination* $(3,-1,-2)$ *that gives* $b = 0$. So there is a whole line of solutions—as we know from the row picture.

The truth is that we *knew* the columns would combine to give zero, because the rows did. That is a fact of mathematics, not of computation—and it remains true in dimension n. ***If the n planes have no point in common, or infinitely many points, then the n columns lie in the same plane.***

If the row picture breaks down, so does the column picture. That brings out the difference between Chapter 1 and Chapter 2. This chapter studies the most important problem—the *nonsingular* case—where there is one solution and it has to be found. Chapter 2 studies the general case, where there may be many solutions or none. In both cases we cannot continue without a decent notation (*matrix notation*) and a decent algorithm (*elimination*). After the exercises, we start with elimination.

Problem Set 1.2

1. For the equations $x+y=4$, $2x-2y=4$, draw the row picture (two intersecting lines) and the column picture (combination of two columns equal to the column vector $(4,4)$ on the right side).

2. Solve to find a combination of the columns that equals b:

$$\textbf{Triangular system} \qquad \begin{array}{rcrcrcl} u & - & v & - & w & = & b_1 \\ & & v & + & w & = & b_2 \\ & & & & w & = & b_3. \end{array}$$

3. (Recommended) Describe the intersection of the three planes $u+v+w+z=6$ and $u+w+z=4$ and $u+w=2$ (all in four-dimensional space). Is it a line or a point or an empty set? What is the intersection if the fourth plane $u=-1$ is included? Find a fourth equation that leaves us with no solution.

4. Sketch these three lines and decide if the equations are solvable:

$$\textbf{3 by 2 system} \qquad \begin{array}{rcrcl} x & + & 2y & = & 2 \\ x & - & y & = & 2 \\ & & y & = & 1. \end{array}$$

What happens if all right-hand sides are zero? Is there any nonzero choice of right-hand sides that allows the three lines to intersect at the same point?

5. Find two points on the line of intersection of the three planes $t=0$ and $z=0$ and $x+y+z+t=1$ in four-dimensional space.