1.3 An Example of Gaussian Elimination

The way to understand elimination is by example. We begin in three dimensions:

$$\textbf{Original system} \qquad \begin{array}{rcrcrcr} 2u & + & v & + & w & = & 5 \\ 4u & - & 6v & & & = & -2 \\ -2u & + & 7v & + & 2w & = & 9. \end{array} \tag{1}$$

The problem is to find the unknown values of u, v, and w, and we shall apply Gaussian elimination. (Gauss is recognized as the greatest of all mathematicians, but certainly not because of this invention, which probably took him ten minutes. Ironically, it is the most frequently used of all the ideas that bear his name.) The method starts by ***subtracting multiples of the first equation from the other equations***. The goal is to eliminate u from the last two equations. This requires that we

(a) subtract 2 times the first equation from the second

(b) subtract -1 times the first equation from the third.

$$\textbf{Equivalent system} \qquad \begin{array}{rcrcrcr} \mathbf{2}u & + & v & + & w & = & 5 \\ & - & \mathbf{8}v & - & 2w & = & -12 \\ & & 8v & + & 3w & = & 14. \end{array} \tag{2}$$

The coefficient 2 is the ***first pivot***. Elimination is constantly dividing the pivot into the numbers underneath it, to find out the right multipliers.

The pivot for the **second stage of elimination** is -8. We now ignore the first equation. A multiple of the second equation will be subtracted from the remaining equations (in this case there is only the third one) so as to eliminate v. We add the second equation to the third or, in other words, we

(c) subtract -1 times the second equation from the third.

The elimination process is now complete, at least in the "forward" direction:

$$\textbf{Triangular system} \qquad \begin{array}{rcrcrcr} \mathbf{2}u & + & v & + & w & = & 5 \\ & - & \mathbf{8}v & - & 2w & = & -12 \\ & & & & \mathbf{1}w & = & 2. \end{array} \tag{3}$$

This system is solved backward, bottom to top. The last equation gives $w = 2$. Substituting into the second equation, we find $v = 1$. Then the first equation gives $u = 1$. This process is called ***back-substitution***.

To repeat: Forward elimination produced the pivots 2, -8, 1. It subtracted multiples of each row from the rows beneath, It reached the "triangular" system (3), which is solved in reverse order: Substitute each newly computed value into the equations that are waiting.

Remark. One good way to write down the forward elimination steps is to include the right-hand side as an extra column. There is no need to copy u and v and w and $=$ at every step, so we are left with the bare minimum:

$$\begin{bmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

At the end is the triangular system, ready for back-substitution. You may prefer this arrangement, which guarantees that operations on the left-hand side of the equations are also done on the right-hand side—because *both sides are there together.*

In a larger problem, forward elimination takes most of the effort. We use multiples of the first equation to produce zeros below the first pivot. Then the second column is cleared out below the second pivot. The forward step is finished when the system is triangular; equation n contains only the last unknown multiplied by the last pivot. Back-substitution yields the complete solution in the opposite order—beginning with the last unknown, then solving for the next to last, and eventually for the first.

By definition, ***pivots cannot be zero***. We need to divide by them.

The Breakdown of Elimination

Under what circumstances could the process break down? Something must go wrong in the singular case, and something might go wrong in the nonsingular case. This may seem a little premature—after all, we have barely got the algorithm working. But the possibility of breakdown sheds light on the method itself.

The answer is: With a full set of n pivots, there is only one solution. The system is non singular, and it is solved by forward elimination and back-substitution. But ***if a zero appears*** in a pivot position, elimination has to stop—either temporarily or permanently. The system might or might not be singular.

If the first coefficient is zero, in the upper left corner, the elimination of u from the other equations will be impossible. The same is true at every intermediate stage. Notice that a zero can appear in a pivot position, even if the original coefficient in that place was not zero. Roughly speaking, ***we do not know whether a zero will appear until we try***, by actually going through the elimination process.

In many cases this problem can be cured, and elimination can proceed. Such a system still counts as nonsingular; it is only the algorithm that needs repair. In other cases a breakdown is unavoidable. Those incurable systems are singular, they have no solution or else infinitely many, and a full set of pivots cannot be found.

Example 1. Nonsingular (cured by exchanging equations 2 and 3)

$$\begin{array}{rcrcrcl} u & + & v & + & w & = & __ \\ 2u & + & 2v & + & 5w & = & __ \\ 4u & + & 6v & + & 8w & = & __ \end{array} \rightarrow \begin{array}{rcrcrcl} u & + & v & + & w & = & __ \\ & & & & 3w & = & __ \\ & & 2v & + & 4w & = & __ \end{array} \rightarrow \begin{array}{rcrcrcl} u & + & v & + & w & = & __ \\ & & 2v & + & 4w & = & __ \\ & & & & 3w & = & __ \end{array}$$

The system is now triangular, and back-substitution will solve it.

Example 2. Singular (incurable)

$$\begin{array}{rcrcrcl} u & + & v & + & w & = & ____ \\ 2u & + & 2v & + & 5w & = & ____ \\ 4u & + & 4v & + & 8w & = & ____ \end{array} \longrightarrow \begin{array}{rcrcrcl} u & + & v & + & w & = & ____ \\ & & & & 3w & = & ____ \\ & & & & 4w & = & ____ \end{array}$$

There is no exchange of equations that can avoid zero in the second pivot position. The equations themselves may be solvable or unsolvable. If the last two equations are $3w = 6$ and $4w = 7$, there is no solution. If those two equations happen to be consistent—as in $3w = 6$ and $4w = 8$—then this singular case has an infinity of solutions. We know that $w = 2$, but the first equation cannot decide both u and v.

Section 1.5 will discuss row exchanges when the system is not singular. Then the exchanges produce a full set of pivots. Chapter 2 admits the singular case, and limps forward with elimination. The $3w$ can still eliminate the $4w$, and we will call 3 the second pivot. (There won't be a third pivot.) For the present we trust all n pivot entries to be nonzero, without changing the order of the equations. That is the best case, with which we continue.

The Cost of Elimination

Our other question is very practical. *How many separate arithmetical operations does elimination require, for n equations in n unknowns?* If n is large, a computer is going to take our place in carrying out the elimination. Since all the steps are known, we should be able to predict the number of operations.

For the moment, ignore the right-hand sides of the equations, and count only the operations on the left. These operations are of two kinds. We divide by the pivot to find out what multiple (say ℓ) of the pivot equation is to be subtracted. When we do this subtraction, we continually meet a "multiply-subtract" combination; the terms in the pivot equation are multiplied by ℓ, and then subtracted from another equation.

Suppose we call each division, and each multiplication-subtraction, one operation. In column 1, *it takes n operations for every zero we achieve*—one to find the multiple ℓ, and the other to find the new entries along the row. There are $n-1$ rows underneath the first one, so the first stage of elimination needs $n(n-1) = n^2 - n$ operations. (Another approach to $n^2 - n$ is this: *All n^2 entries need to be changed, except the n in the first row.*) Later stages are faster because the equations are shorter.