$A^{\mathrm{T}}A$ has the same nullspace as A.

Certainly if $Ax = 0$ then $A^{\mathrm{T}}Ax = 0$. Vectors x in the nullspace of A are also in the nullspace of $A^{\mathrm{T}}A$. To go in the other direction, start by supposing that $A^{\mathrm{T}}Ax = 0$, and take the inner product with x to show that $Ax = 0$:

$$x^{\mathrm{T}}A^{\mathrm{T}}Ax = 0, \quad \text{or} \quad \|Ax\|^2 = 0, \quad \text{or} \quad Ax = 0.$$

The two nullspaces are identical. In particular, if A has independent columns (and only $x = 0$ is in its nullspace), then the same is true for $A^{\mathrm{T}}A$:

3M If A has independent columns, then $A^{\mathrm{T}}A$ is *square, symmetric*, and *invertible*.

We show later that $A^{\mathrm{T}}A$ is also positive definite (all pivots and eigenvalues are positive).

This case is by far the most common and most important. Independence is not so hard in m-dimensional space if $m > n$. We assume it in what follows.

Projection Matrices

We have shown that the closest point to b is $p = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}b$. *This formula expresses in matrix terms the construction of a perpendicular line from b to the column space of* A. The matrix that gives p is a projection matrix, denoted by P:

$$\textbf{Projection matrix} \qquad P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}. \tag{4}$$

This matrix projects any vector b onto the column space of A.[1] In other words, $p = Pb$ is the component of b in the column space, and the error $e = b - Pb$ is the component in the orthogonal complement. ($I - P$ is also a projection matrix! It projects b onto the orthogonal complement, and the projection is $b - Pb$.)

In short, we have a matrix formula for splitting any b into two perpendicular components. Pb is in the column space $\boldsymbol{C}(A)$, and the other component $(I - P)b$ is in the left nullspace $\boldsymbol{N}(A^{\mathrm{T}})$—which is orthogonal to the column space.

These projection matrices can be understood geometrically and algebraically.

3N The projection matrix $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ has two basic properties:

(i) It equals its square: $P^2 = P$.

(ii) It equals its transpose: $P^{\mathrm{T}} = P$.

Conversely, any symmetric matrix with $P^2 = P$ represents a projection.

[1] There may be a risk of confusion with permutation matrices, also denoted by P, but the risk should be small, and we try never to let both appear on the same page.