Week 1 – Introduction to Topics

Introduction to Prescriptive Analytics:

Prescriptive Analytics makes use of Optimization models which are mathematical methods for arriving at optimal or near optimal decisions for given managerial objectives under various constraints.
Applications of such methods is ubiquitous as all decision makers are faced with constraints under which they need to find the most optimal decision(s) that meet the stated objective. It is a powerful tool in the hands of decision-makers seeking to navigate complex scenarios and make data-driven choices and helps organizations make informed choices, optimize processes, and allocate resources efficiently.

The objective of the course is to familiarize the student with construction of optimization models as decision making tools and be able to use Excel based Solver to obtain solutions and interpret them. The emphasis is on understanding the concepts and interpretation of the solution rather than on theoretical details and proofs or derivations of results.

Introduction to Linear Programming:

Linear programming (LP) is one such mathematical modelling technique used for finding optimal decisions in the presence of constraints and an objective, specifically in situations where resources are limited. It involves maximizing or minimizing a linear objective function while satisfying a set of linear equality and inequality constraints.

Key Components of Linear Programming:

1. **Decision Variables:**
   - These represent the quantities to be determined, often denoted by $x_1, x_2, \dots, x_n$. They are the decision-makers' choices that need to be optimized.
2. **Objective Function:**
   - The objective function is a linear equation that needs to be maximized or minimized. It is typically represented as $Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$, where $c_1, c_2, \dots, c_n$ are coefficients and Z is the objective function.
3. **Constraints:**
   - Linear programming involves constraints that restrict the values of the decision variables. These constraints are represented by linear inequalities or equations, such as $a_1x_1 + a_2x_2 + \dots + a_nx_n \leq b$

Steps in Linear Programming:

1. **Define the Decision Variables:**
   - Identify the variables whose values are to be determined.
2. **Formulate the Objective Function:**

- Create an equation representing the objective to be maximized or minimized.

3. **Establish Constraints:**
    - Identify and formulate constraints that limit the decision choices.
4. **Solve the Linear Programming Problem:**
    - Use solution methods or algorithms to find the optimal solution that maximizes or minimizes the objective function within the given constraints.
5. **Interpret Results:**
    - Analyse the results to make informed decisions. The optimal solution provides insights into the best allocation of resources.

Applications of Linear Programming: Linear programming finds applications in various fields, including:

- Supply chain management
- Transportation and logistics
- Finance and investment planning
- Manufacturing process optimization
- Project management

In summary, linear programming is a valuable tool for decision-makers facing resource allocation challenges. By optimizing objective functions subject to constraints, it enables efficient and effective decision-making in complex scenarios.

Linear programming (LP) Formulation

Linear programming (LP) formulation is the process of translating real-world problems into a mathematical model that can be solved using linear programming techniques. The key components of a linear programming formulation include decision variables, an objective function, and constraints.

1. **Decision Variables:**

Decision variables represent the quantities to be determined or optimized. These variables are denoted by symbols such *as* $x1,x2,\ldots,xn$, and they are the unknowns in the optimization problem.

2. **Objective Function:**

The objective function defines the goal of the optimization problem. It is a linear equation **representing the quantity** to be maximized or minimized. The general form $Z = c_1x_1+c_2x_2+\ldots+c_nx_n$, where $c_1,c_2,\ldots,c_n$ are known values or objective function coefficients of $x1,x2,\ldots,xn$, and Z is the objective function value.

3. **Constraints:**

Constraints are conditions or limitations that restrict the values of the decision variables. They are expressed as linear inequalities or equations. The general form is $a_1x_1+a_2x_2+\ldots+a_nx_n\leq b$ where $a_1,a_2,\ldots,a_n$ are known values or coefficients of the unknowns $x1,x2,\ldots,xn$, and b is a constant.

4. **Non-Negativity Constraints:**
   In linear programming problems, decision variables are taken to be non-negative, i.e. ($x_i \geq 0$), which is usually true by default as negative values may not have practical interpretations in most cases.

5. **Formulating the Problem:**
   The linear programming problem is formulated by defining the decision variables, constructing the objective function, and specifying the constraints. The goal is to create a mathematical representation of the real-world problem.

6. **Mathematical Representation:**
   The complete mathematical representation of a linear programming problem is often summarized as:

$$\text{Maximize (or Minimize)} \quad Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

$$\text{Subject to} \quad \begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2 \\ \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m \\ x_1 \geq 0,\ x_2 \geq 0, \ldots, x_n \geq 0 \end{cases}$$

7. **Solving the Linear Programming Problem:**
   Once formulated, the linear programming problem can be solved using various methods, such as the simplex method or graphical methods, to find the optimal values of the decision variables that maximize or minimize the objective function.

   Linear programming formulation is a crucial step in leveraging the power of mathematical optimization to solve real-world problems efficiently and make informed decisions.

<u>LINEAR PROGRAMMING PROBLEM (LPP) TERMINOLOGIES</u>

The following terminologies are used in LPP:

1. **Feasible region**: Feasible region is the set of all solutions that satisfies all the constraints for the decision problem. Such solutions are called feasible solutions. The feasible region is always convex and may be bounded or unbounded.
2. Binding constraints: Binding constraints are constraints for which the left-hand side (LHS) and right-hand side (RHS) values of the constraint are equal at the optimal solution. A binding constraint is, therefore, a bottleneck where the resource has been completely exhausted. Hence, changing the RHS value of a binding constraint can change the values of decision variables in the optimal solution and have an impact on the objective function value. The change in the objective function value per unit change in the RHS (within a range) of a binding constraint is the marginal worth of the constraint and is called shadow price.
3. **Non-binding constraint***:* Non-binding constraints are constraints for which the values of LHS and RHS of the constraint are not equal at the optimal solution. For a non-binding constraint, changing the value of the RHS (within a range) will not change the optimal solution or impact the objective function value.
4. **Slack variable:** In a less than or equal to constraint (in the current example, all constraints are less than or equal to constraints), slack variable is a variable which when added to LHS will make it an equality constraint. The value of a slack variable in a constraint is the amount of unused resource (note that resource is a generic term that we are using, we must interpret the slack based on the context).
5. **Surplus variable**: In a greater than or equal to constraint, surplus variable is the excess resource used. In general, surplus variable is a variable which when subtracted from LHS of a greater than or equal to constraint will make the constraint an equality constraint
6. **Basic variable:** A decision variable with a non-zero value in the optimal solution is called a basic variable, assuming that the variables have a lower bound of zero and have no upper bound. The set of basic variables is called the basis (term borrowed from matrix algebra).
7. . **Non-basic variable**: In general, a decision variable which has a value of zero in the optimal solution is called a non-basic variable.
8. **Shadow price**: Shadow price is the marginal value of a resource, that is, it is the value by which the objective function value changes for unit change in the RHS of a constraint.
9. **Reduced cost**: In the optimal solution for an LPP, it is possible that some decision variables are at a value of zero or are non-basic variables. Reduced cost is the value by which the objective function value will deteriorate when a non-basic variable is forced to become a basic variable (that is, when the value of the non-basic variable is changed from 0 to 1).

ASSUMPTIONS OF LINEAR PROGRAMMING

LPP models are built under the following assumptions:

- **Additivity**: The value of the objective function is assumed to be sum of contribution of each decision variable value. That is, we assume that the contribution of each decision variable is independent. This assumption may not be satisfied in many cases. For example, assume that a marketing team is trying to find an optimal advertisement mix that will have maximum reach at minimum cost. The various channels for advertisement are: television, newspapers, radio, social media, etc. However, all these channels are not independent since there will be an overlap between channels. That is, if 5 million customers can be reached using television and 3 customers using radio, we cannot conclude that if we use both channels then 8 million customers can be reached since there will be common customers between television and radio. This assumption, however, does not pose a problem in general. For instance, in this example the dependent part can be addressed in the data being used a the given input.
- **Proportionality:** In LPP we assume that a change in a variable will result in proportionate change in the objective function value. This may not be true in all cases. For example, a manufacturer may provide discount based on the quantity of purchase (quantity discounts) and thus the contribution may follow a step function, or there might be economies of scale. There are ways of addressing violation of this assumption by introducing integer variables to handle discounts which gives rise to integer programming problems, approximation of the non-linearity with piece-wise linear functions or using non-linear programming methods instead.
- **Divisibility:** An important assumption in LPP is that a decision variable can be allowed to take non-integer values mathematically and be rounded off to an integer value for practical purposes.
- **Certainty:** We assume that all the model parameters (values of the coefficients in the objective function and constraints, RHS values of constraints) are known with certainty.

Linear Programming Using Graphical Method:

The graphical method is a visual approach to solving linear programming problems (LPPs) with two decision variables. It involves plotting the constraints of the problem on a graph and then identifying the feasible region/area, which is the set of solutions that satisfies all the constraints. The optimal solution of the LPP is then found by locating the point within the feasible region that optimizes the objective function value, depending on whether it is a maximization or minimization problem.

The following steps are used in graphical method:

Step 1: Formulate the Objective Function and Constraints

Write down the objective function and constraints.

Step 2: Plot the Constraints

Create a coordinate system and plot each constraint as a straight line on the coordinate system. To do this, either find two points on the line and join them which will uniquely determine the line or rewrite each constraint in the slope-intercept form and pot the line.

For example, the constraint:

$2x+3y \leq 12$, can be rewritten as $y \leq 4 - 2x/3$.

Step 3: Identify the Feasible Region

The feasible region represents the area or set of solutions that satisfy all the constraints simultaneously.

Step 4: Identify the Corner Points

The corner points of the feasible region are the points where the constraint lines intersect. These points are potential candidates for optimality for the linear programming problem as the optimal solution to an LPP always lies at a corner point of the feasible region. This result is easy to see graphically and can be proven mathematically.

Step 5: Evaluate the Objective Function at Corner Points

Calculate the objective function's value at each corner point. The point that either maximizes or minimizes the objective function is the optimal solution. This can be inefficient when the number of corner points is large, and an for solving LPPs algorithm will only look at a subset of the corner points to determine the optimal one. Alternatively, to find the optimal solution graphically the objective function line can be plotted and moved in an improving direction till it cannot be moved anymore without leaving the feasible region.

Step 6: Check for Unbounded Solutions

Feasible regions can be unbounded. In such cases either an optimal solution exists or the solution is unbounded, i.e. the values of the variables can be increased or decreased without any bound. In the latter case there is usually something that has been missed and need to be corrected.

Step 7: Interpret the Solution

Interpret the solution in the context of the original problem.

Let's go through a simple example:

Maximize $Z = 3x+2y$

Subject to the constraints:

$2x+y \leq 20$

$4x-5y \geq -10$

$X \geq 0, y \geq 0$ (non-negativity)

Plot these constraints, find the feasible region, and then check the corner points for the optimal solution.