Week 3: Introduction to Topics

Assignment Problem

The assignment problem in linear programming (LP) involves the optimal assignment of a set of tasks to a set of agents, minimizing or maximizing a certain objective function while satisfying specific constraints. In LP, the assignment problem is typically formulated using binary decision variables that represent whether a task is assigned to a particular agent. The objective function aims to minimize or maximize the total cost, profit, or some other measure associated with the assignments. Constraints ensure that each task is assigned to exactly one agent, and each agent handles at most one task. The assignment problem finds applications in various fields, including operations research, logistics, resource allocation, and project management, where efficient allocation of resources or tasks is crucial for optimizing performance and minimizing costs. Linear programming techniques provide systematic approaches to solve the assignment problem and derive optimal solutions efficiently.

Here's a breakdown of the key aspects:

1. Problem Formulation:

- Sets: You have two sets: jobs (J) and workers (W).
- Cost Matrix: A matrix C with dimensions |J| x |W|, where C[i,j] represents the cost of assigning job i to worker j.
- Decision Variables: Binary variables x[i,j] that indicate whether job i is assigned to worker j (x[i,j] = 1) or not (x[i,j] = 0).

2. Objective Function:

Minimize the total cost of assignments:

Minimize Z = ΣΣ C[i,j] * x[i,j]

3. Constraints:

  - Each job can be assigned to only one worker:

Σx[i,j] = 1 for all i in J

  - Each worker can be assigned to only one job:

Σx[i,j] = 1 for all j in W

The assignment problem has diverse applications across various fields, including:

- Resource allocation: Assigning tasks to workers, machines, or projects.
- Scheduling: Scheduling flights to gates, students to courses, or patients to doctors.
- Logistics: Assigning delivery routes to drivers or warehouses to customers.

Assignment Problem Using Solver

Here's a guide on how to solve an assignment problem using Excel Solver:

1. Set up the Data:

- Create a table with rows representing jobs and columns representing workers.
- Enter the cost of assigning each job to each worker in the corresponding cells.

2. Access Solver:

- Go to the "Data" tab and click on the "Solver" button in the "Analysis" group. If Solver isn't available, you may need to install it via Excel Options.

3. Set Objective Function:

- In the Solver Parameters dialog box:
  - Set "Set Objective" to the cell containing the formula for the total cost (e.g., SUMPRODUCT(B2:D4,E2:E4)).
  - Choose "Min" to minimize the total cost.

4. Add Constraints:

- Click the "Add" button to add constraints:
  - Each job must be assigned to exactly one worker:
    - Set "Cell Reference" to the range of assignment cells for each job.
    - Choose "=1" as the constraint.
  - Each worker can be assigned to only one job:
    - Set "Cell Reference" to the range of assignment cells for each worker.
    - Choose "=1" as the constraint.

5. Solve:

- Click the "Solve" button.

- If Solver finds a solution, it will display a message with the optimal total cost.
- Click "Keep Solver Solution" to update the assignment cells with the optimal solution.

6. Interpret Results:

a) Assignment Table:

   Look at the cells where you had the decision variables (representing assignments).

   A value of 1 in a cell indicates that the corresponding job is assigned to the corresponding worker.

   A value of 0 implies that there is no assignment between that job and worker in the optimal solution.

b) Optimal Cost:

   The cell you specified as the objective function now reflects the minimized total cost based on the optimal assignment.

c) Constraint Satisfaction:

   Solver ensures that all constraints you defined are met.

   Check the constraint cells (e.g., cells representing "each job assigned once" or "each worker assigned once") to verify they all equal 1.

d) Additional Interpretations:

   Analyse the distribution of assignments: Are certain workers assigned more jobs than others? Does this make sense considering skills or workload?

   Examine the cost breakdown: Which job-worker pairings contribute the most to the total cost? Are there opportunities for cost reduction adjustments?

   Consider the sensitivity of the solution: Use Solver's sensitivity analysis to see how changes in individual cost values might affect the optimal assignment and total cost.

Transportation Problem:

The transportation problem is a classic optimization problem in linear programming (LP) that deals with efficiently allocating goods from multiple suppliers to multiple destinations while minimizing transportation costs or maximizing profits.

Minimizing the total cost of transporting goods from origins to destinations, satisfying supply and demand constraints.

1. Problem Formulation:

a) Sets: You have two sets: origins (O) and destinations (D).
b) Supply: Each origin i has a known supply s_i of goods.
c) Demand: Each destination j has a known demand d_j for goods.
d) Cost Matrix: A matrix C with dimensions |O| x |D|, where C[i,j] represents the cost of transporting one unit of goods from origin i to destination j.
e) Decision Variables: Non-negative variables x[i,j] representing the amount of goods shipped from origin ito destination j.

2. Objective Function:

Minimize the total transportation cost:

Minimize Z = ΣΣ C[i,j] * x[i,j]

3. Constraints:

- Supply constraint: The total amount shipped from each origin cannot exceed its supply:

Σx[i,j] <= s_i for all i in O

- Demand constraint: The total amount received by each destination must meet its demand:

Σx[i,j] >= d_j for all j in D

- Non-negativity: The amount shipped cannot be negative:

x[i,j] >= 0 for all (i,j)

4.Solving Methods:

- North-West Corner Method: A simple manual method to find an initial feasible solution.
- Least Cost Method: A more efficient manual method to find a better initial solution.
- Vogel's Approximation Method: A method to select the initial cell with the largest opportunity cost reduction, often leading to a good initial solution.
- Specialized LP solvers: Can be used to solve the LP formulation directly, especially for large problems.

5. Applications:

- Optimizing transportation routes for goods like products, raw materials, or resources.
- Assigning workers to tasks or resources to projects.
- Scheduling flights to airports or deliveries to customers.

Transportation Simplex Method

The transportation simplex method is a special-purpose solution procedure applicable to any network model having the special structure of the transportation problem. It is actually a clever implementation of the general simplex method for linear programming that takes advantage of the special mathematical structure of the transportation problem; but because of the special structure, the transportation simplex method is hundreds of times faster than the general simplex method.

To apply the transportation simplex method, you must have a transportation problem with total supply equal to total demand; thus, for some problems you may need to add a dummy origin or dummy destination to put the problem in this form. The transportation simplex method takes the problem in this form and applies a two-phase solution procedure. In phase I, apply the minimum cost method to find an initial feasible solution. In phase II, begin with the initial feasible solution and iterate until you reach an optimal solution. The steps of the transportation simplex method for a minimization problem are summarized as follows:

Phase I

Find an initial feasible solution using the minimum cost method.

Phase II

Step 1. If the initial feasible solution is degenerate with less than m + n - 1 occupied cells, add an artificially occupied cell or cells so that m+n -1 occupied cells exist in locations that enable use of the MODI method.

Step 2. Use the MODI method to compute row indexes, ui, and column indexes, vj.

Step 3. Compute the net evaluation index eij=cij-ui- vj for each unoccupied cell.

Step 4. If eij >=0 for all unoccupied cells, stop; you have reached the minimum cost solution. Otherwise, proceed to step 5.

Step 5. Identify the unoccupied cell with the smallest (most negative) net evaluation index and select it as the incoming cell.

Step 6. Find the stepping-stone path associated with the incoming cell. Label each cell on the stepping-stone path whose flow will increase with a plus sign and each cell whose flow will decrease with a minus sign.

Step 7. Choose as the outgoing cell the minus-sign cell on the stepping-stone path with the smallest flow. If there is a tie, choose any one of the tied cells. The tied cells that are not chosen will be artificially occupied with a flow of zero at the next iteration.

Step 8. Allocate to the incoming cell the amount of flow currently given to the outgoing cell; make the appropriate adjustments to all cells on the stepping-stone path and continue with step 2.

Problem Variations:

The following problem variations can be handled, with slight adaptations, by the transportation simplex method:

1. Total supply not equal to total demand
2. Maximization objective function
3. Unacceptable routes

The case where the total supply is not equal to the total demand can be handled easily by the transportation simplex method if we first introduce a dummy origin or a dummy destination. If total supply is greater than total demand, we introduce a dummy destination with demand equal to the excess of supply over demand. Similarly, if total demand is greater than total supply, we introduce a dummy origin with supply equal to the excess of demand over supply. In either case, the use of a dummy destination or a dummy origin will equalize total supply and total demand so that we can use the transportation simplex method. When a dummy destination or origin is present, we assign cost coefficients of zero to every arc into a dummy destination and to every arc out of a dummy origin. The reason is that no shipments will actually be made from a dummy origin or to a dummy destination when the solution is implemented and thus a zero cost per unit is appropriate. The transportation simplex method also can be used to solve maximization problems. The only modification necessary involves the selection of an incoming cell. Instead of picking the cell with the smallest or most negative eij value, we pick that cell for which eij is largest. That is, we pick the cell that will cause the largest increase per unit in the objective function. If eij<=0 for all unoccupied cells, we stop; the maximization solution has been reached.

To handle unacceptable routes in a minimization problem, infeasible arcs must carry an extremely high cost, denoted M, to keep them out of the solution. Thus, if we have a route (arc) from an origin to a destination that for some reason cannot be used, we simply assign this arc a cost per unit of M, and it will not enter the solution. Unacceptable arcs would be assigned a profit per unit of -M in a maximization problem.

Multi-Period (Stage) Models

Manufacturing companies face the problem of production-inventory planning for a number of future time periods under given constraints. This problem is known as the Multi-Period Production Scheduling problem. In manufacturing companies, production-inventory demand varies across multiple period horizons. The main goal is to level up the production needs for individual products in this fluctuating demand.
The objective function of such a problem is minimising the total costs associated with production and inventory. We can determine the production schedules for each period using Linear programming in a rolling way.

In many real-life applications, the decisions are taken over several periods or stages. The decision maker has to take decisions during every stage of the planning horizon that optimizes the objective function.

In Multi-Period Production Scheduling problems, the multi-period problem refers to the fact that production planning and inventory management occur over multiple time periods, instead of just a single period. This adds another layer of complexity compared to single-period models.

Here's how the multi-period aspect manifests in a Multi-Period Production Scheduling problem:

Demand:

- Demand for the product is usually not constant and varies across different periods. The model needs to consider these changing demands while planning production.

Inventory:

- Inventory levels become crucial as you need to balance production to meet demand while avoiding excessive inventory holding costs. Decisions in one period affect inventory levels in future periods.

Production Capacity:

- Production capacity might be limited and may vary across periods due to factors like machine availability, workforce constraints, or maintenance schedules.

Cost Considerations:

- Additional costs, like setup costs for changing production levels between periods, need to be factored into the optimization.

Solution Approaches:

- Dynamic Programming: This approach breaks down the problem into smaller sub-problems for each period, optimizing decisions sequentially.
- Linear Programming (LP) formulations: While LP can be used for specific cases, it may not be efficient for large problems due to the increased number of variables and constraints.
- Specialized Heuristics and Metaheuristics: These provide efficient approximation algorithms for complex scenarios.