Week 2: Introduction to Topics

We have seen that a decision problem can be formulated as a linear programming formulation which can be readily solved to provide the optimal decision and the corresponding optimal profit/cost, and also identify critical resources or bottlenecks. This is based on the data available for the input parameters required for formulating the constraints and the objective function.

Sensitivity Analysis:

Sensitivity analysis in Linear Programming problem (LPP) is a powerful tool that helps evaluate the impact of changes in the coefficients of the objective function or the right-hand side values and other inputs, on the optimal solution of the linear programming problem. This analysis helps decision-makers understand how sensitive the solution to variations in input data is. It also provides important information on the bottlenecks and the impact of releasing them.

Sensitivity analysis in an LPP addresses the following issues:
1. Changes in the values of the RHS of a constraint.
2. Changes in the values of the objective function co-efficient.
3. Addition of a new variable.
4. Addition of new constraint

The key components of sensitivity analysis for prescriptive linear programming:

1. Objective Function Coefficient Sensitivity:
    - Measures how much the optimal solution changes when objective function coefficients are changed.
    - Provides information on:
        - Potential shifts in optimal decisions based on cost or profit changes.
        - Robustness of the optimal solution
        - Relative importance of the decision variables
2. Right-Hand Side Sensitivity:
    - Assesses the impact of changes in constraint right-hand sides (resource availability) on the optimal solution.
    - Provides information on:
        - Resource utilization
        - Identifying critical resources
        - Assessing the effect of resource changes on optimal decisions
        - The extent to which changes in resources impact the optimal solution.
3. Shadow Prices of constraints:
    - Represent the marginal value of a unit increase in a constraint's right-hand side.
    - Indicate the potential benefit of acquiring more resources.
4. Range of Optimality:
    - Specifies the range within which an objective function coefficient can change without affecting the optimal solution.

    - Helps understand the extent to which solutions can accommodate uncertainty.

Benefits of Sensitivity Analysis:

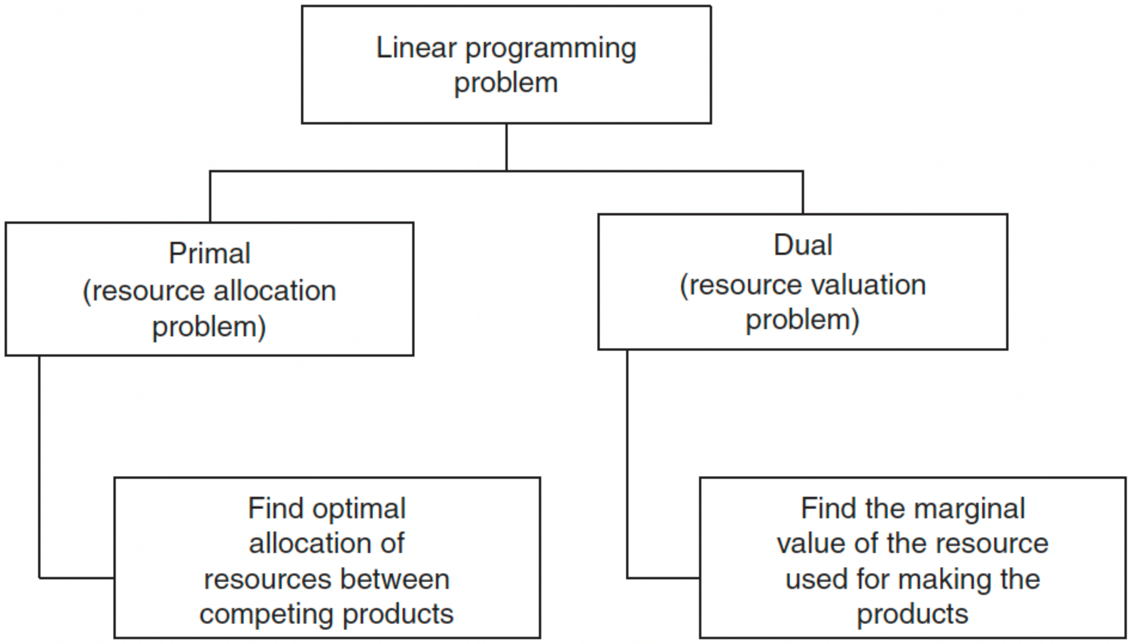
- Robust Decision-Making: Evaluates solution stability under uncertainty, leading to more reliable decisions.
- Resource Allocation Insights: Identifies critical resources and guides prioritization.
- Risk Assessment: Quantifies potential impacts of changes, enabling proactive risk management.
- Trade-Off Analysis: Facilitates exploration of different scenarios and trade-offs.

Duality and the Dual LP:

- Every linear programming problem (primal problem) has a corresponding dual LP that provides an alternative perspective on the optimization problem.
- Formulation:
    - Variables in the dual problem correspond to constraints in the primal problem.
    - Constraints in the dual problem correspond to variables in the primal problem.
    - The objective function of the dual problem is related to the resource values in the primal problem.
    - The RHS of a constraint in the dual problem are related to the objective function coefficient of the corresponding variable in the primal problem.
- Relationship:
    - The optimal solution to the primal problem provides information about the optimal solution to the dual problem, and vice versa.
    - The optimal objective function values of the primal and dual problems are equal (assuming both have feasible and bounded solutions).
    - Shadow prices in the primal problem are equal to the optimal values of the dual variables and vice versa.
    - Reduced costs in the primal problem are the slack/surplus in the dual constraints and vice versa.

Primal and Dual Relationship:

- Primal Problem: This is the original LPP you want to solve, typically a maximization or minimization problem subject to linear constraints.
- Dual Problem: This is an LPP derived from the primal problem, with objective and constraint structures switched. Its objective function minimizes the total cost of resources.
- For every primal linear programming problem, we can formulate a corresponding dual problem.
- For easier understanding, assume that the primal is a resource allocation problem. Then the dual will be resource valuation problem as shown in the Figure below. For every variable in the primal, there will be dual constraint and for every constraint in the primal, there will be a dual variable.



Conversion of a Primal Model to Dual Model

The following steps are used in converting a primal maximization problem to a dual minimization problem:
1. Write the formulation in the standard form. For a maximization problem, in the standard form, all constraints should be less than or equal to constraints (≤). In the case of standard minimization problem, all constraints should be greater than or equal to constraints (≥).
2. For each constraint in the primal problem, define a dual variable. Write the dual objective function Yb, where Y is the dual variables vector and b is the RHS vector in the primal. If the primal is maximization problem, the dual will be a minimization problem and vice versa.
3. For each variable in the primal identify the dual constraint $YA \geq c$, where c is the objective function coefficients vector in the primal.
4. Write the non-negativity constraint $Y \geq 0$.

Microsoft Excel includes a tool called Solver, which can be used to solve optimization problems, including linear and integer programming problems. Here's a step-by-step guide on how to use the Excel Solver for linear programming:

Step 1: Open Excel and Enable Solver

- Open Excel and create a new worksheet or open an existing one.
- Go to the "File" menu and click on "Options."
- In the Excel Options dialog box, select "Add-ins" on the left.
- In the "Manage" box at the bottom, choose "Excel Add-ins" and click "Go."
- In the Add-Ins box, check "Solver Add-in" and click "OK."

Step 2: Define the Objective Function and Constraints

- In your Excel worksheet, set up a table to represent the decision variables, objective function, and constraints. Label each section appropriately.
- Enter the coefficients of the objective function in a row, with each coefficient corresponding to a decision variable.
- Below the objective function, enter the coefficients of the constraints in separate rows.
- In a column to the right of the constraint coefficients, enter the constraint values.
- Make sure to add any necessary constraints such as non-negativity constraints for the decision variables.

Step 3: Set Up Solver

- Go to the "Data" tab and click on "Solver" in the Analysis group.
- In the Solver Parameters dialog box, set the "Set Objective" field to the cell containing the objective function.
- Choose whether you want to maximize or minimize the objective function.
- Set the "By Changing Variable Cells" field to the range containing your decision variables.
- Click on the "Add" button in the Constraints box to add constraints. Enter the range of cells containing the constraint formulas and the relationship (<=, =, >=).
- Click "OK" to close the Add Constraint dialog box.
- Optionally, you can set additional Solver options such as changing the solving method or setting tolerance values.

Step 4: Solve the Problem

- Click on the "Solve" button in the Solver Parameters dialog box.
- Solver will attempt to find a solution that satisfies the constraints and optimizes the objective function.
- If a solution is found, click "OK" to accept the solution.

Solver will adjust the values of the decision variables to optimize the objective function while satisfying the specified constraints.

Solver can handle moderately sized linear programming problems, but for very large and complex problems, it may become computationally intensive and time-consuming.

Use Solver for the Optimal Solution

1. **Open Solver:**
    - Go to "Data" > "Solver" to open the Solver dialog box.
2. **Set Objective:**
    - In the Solver dialog box, set the "Set Objective" field to the cell containing your objective function.
3. **Set Variables to Adjust:**

    - Set the "By Changing Variable Cells" field to the cells containing your decision variables.
4. **Set Constraints:**
    - Add constraints by clicking on "Add" in the Solver dialog box. Input the cells for the constraints and set the relationship (<=, =, or >=).
5. **Solver Options:**
    - In the Solver Options, choose the solving method (Simplex LP is suitable for linear programming problems).
6. **Solve:**
    - Click "Solve" to find the optimal solution.

Step 4: View Sensitivity Analysis Reports

1. **Sensitivity Report:**
    - After Solver finds the optimal solution, you can save and view the Sensitivity Report.
    - Go to "Data" > "Solver" > "Sensitivity."
2. **Interpret Sensitivity Report:**
    - The Sensitivity Report provides information on the changing cells, objective coefficients, right-hand side changes, etc.
3. **Use Shadow Prices:**
    - Check the shadow prices to understand the impact of a unit increase in the right-hand side values of constraints on the objective function.
4. **Interpret Reduced Costs:**
    - Examine reduced costs to identify variables with non-zero reduced costs.

Components of sensitivity reports from solver:

Sensitivity Report:

The sensitivity report is a key output generated by Solver in Microsoft Excel after solving a linear programming problem. It provides valuable information about how changes in the problem parameters impact the optimal solution and assists decision-makers in understanding the robustness and reliability of the results.

Allowable Increase/Decrease:

- **Variable Cells:**
    - In the sensitivity report, the allowable increase and decrease for variable cells indicate the range within which the coefficients of decision variables can change without affecting the optimality of the solution.
    - Decision-makers can use this information to understand the flexibility in resource allocation and the impact on the objective function.
- **Constraint Cells:**
    - For constraint cells, the allowable increase and decrease indicate the range within which the right-hand side values of constraints can change without altering the current optimal solution.
    - Decision-makers use this information to assess the resilience of the solution to variations in resource availability or other constraints.

Reduced Cost:

- **Definition:**
    - Reduced cost represents the amount by which the objective function coefficient of a non-basic variable would need to improve (decrease for

maximization, increase for minimization) for that variable to enter the optimal solution.

- **Significance:**
    - At optimality reduced costs for all variables <= 0 in a maximization problem and >=0 in a minimization problem.
    - Reduced costs help decision-makers prioritize which variables might contribute to further improvement in the objective function.

Final Value of a Constraint:

- **Definition:**
    - The final value of a constraint represents the value of the LHS of the constraint at the optimal solution.
- **Significance:**
    - For a resource constraint this is the amount of resource consumed.
    - If the resource consumed is equal to the resource available (RHS), the constraint is said to be binding.
    - Decision-makers can assess the impact of variations in constraints on the objective function value, given as the shadow price. The shadow price of a non-binding constraint is always equal to zero.
    - The allowable range for changes in the RHS of a constraint is the range within which the shadow price remains valid.

Objective Function:

- **Definition:**
    - The final value of the objective function in the sensitivity report is the optimal objective function value obtained after Solver converges to a solution.
- **Significance:**
    - Decision-makers use the optimal objective function value to assess the economic impact of the solution.
    - It serves as the basis for comparing different scenarios and making informed decisions.

Simplex Method

The simplex method is a widely used technique for solving linear programming problems. It's an iterative algorithm that moves from one feasible solution to another in order to optimize a linear objective function, while satisfying a set of linear equality and inequality constraints

Steps of the Simplex Method:

1. **Formulate the Linear Programming Problem (LPP):**
    - Define the objective function to be maximized or minimized.
    - Identify the decision variables and write down the constraints.
2. **Convert the Problem to Standard Form:**
    - Express inequalities as equalities by introducing slack or surplus variables.
    - Ensure all variables are non-negative.
3. **Initialize the Simplex Tableau:**
    - Create an initial tableau with the coefficients of the decision variables and slack/surplus variables in the constraints.
    - Include the coefficients of the objective function.

4. **Select a Pivot Column:**
   - Identify the most negative/positive (maximization/minimization problem) coefficient in the bottom row (the objective function row) of the tableau. This column is the pivot column.
5. **Select a Pivot Row:**
   - For each row, calculate the ratio of the right-hand side to the corresponding coefficient in the pivot column. The pivot row is the one with the smallest non-negative ratio.
6. **Update the Pivot Element:**
   - Divide the pivot row by the pivot element (the element at the intersection of the pivot row and pivot column) to make the pivot element equal to 1.
7. **Update Other Rows:**
   - Make all other elements in the pivot column equal to 0 by performing row operations.
8. **Repeat Steps 4-7 Until Optimal Solution is Found:**
   - Continue selecting pivot columns and pivot rows until there are no more negative/positive (maximization/minimization problem) values in the bottom row.
9. **Read the Solution:**
   - The optimal solution is found when there are no negative/positive (maximization/minimization problem) values in the bottom row of the tableau.
   - Read the values of the decision variables from the corresponding columns in the tableau.