This data can conveniently be grouped and shown in a tabular form as follows :

Class	*Frequency*	*Cumulative frequency*
50—54	1	1
55—59	2	3
60—64	9	12
65—69	7	19
70—74	8	27
75—79	17	44
80—84	6	50
85—89	8	58
90—94	3	61
95—99	3	64
	Total = 64	

It would be seen from the above table that there is one student getting marks between 50—54, two students getting marks between 55—59, nine students getting marks between 60—64 and so on. Thus the 64 figure have been put into only 10 groups, called the **classes**. The width of the class is called the **class interval** and the number in that interval is called the **frequency.** The mid-point or the mid-value of the class is called the **class mark.** The above table showing the classes and the corresponding frequencies is called a *frequency table. Thus a set of raw data summarised by distributing it into a number of classes alongwith their frequencies is known as a* **frequency distribution**.

While forming a frequency distribution, the number of classes should not ordinarily exceed 20, and should not, in general, be less than 10. As far as possible, the class intervals should be of equal width.

(3) **Cumulative frequency.** In some investigations, we require the number of items less than a certain value. We add up the frequencies of the classes upto that value and call this number as the *cumulative frequency*. In the above table, the third column shows the cumulative frequencies, *i.e.*, the number of students, getting less than 54 marks, less than 59 marks and so on.

25.3 GRAPHICAL REPRESENTATION

A convenient way of representing a sample frequency distribution is by means of graphs. It gives to the eyes the general run of the observations and at the same time makes the raw data readily intelligible. We give below the important types of graphs in use :

(1) **Histogram.** A histogram is drawn by erecting rectangles over the class intervals, such that the areas of the rectangles are proportional to the class frequencies. If the class intervals are of equal size, the height of the rectangles will be proportional to the class frequencies themselves (Fig. 25.1).

(2) **Frequency polygon.** A frequency polygon for an ungrouped data can be obtained by joining points plotted with the variable values as the abscissae and the frequencies as the ordinates. For a grouped distribution, the abscissae of the points will be the mid-values of the class intervals. In case the intervals are equal, the frequency polygon can be obtained by joining the middle points of the upper sides of the rectangles of the histogram by straight lines (shown by dotted lines in Fig. 25.1). If the class intervals become very very small, the frequency polygon takes the form of a smooth curve called the *frequency curve*.

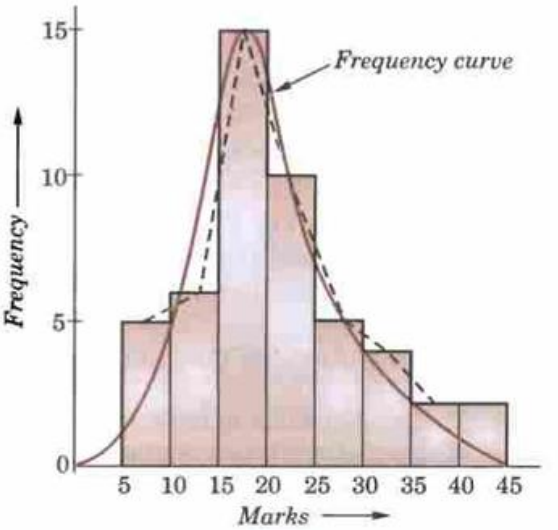


Fig. 25.1

(3) **Cumulative frequency curve-Ogive.** Very often, it is desired to show in a diagrammatic form, not the relative frequencies in the various intervals, but the cumulative frequencies above or below a given value. For example, we may wish to read off from a diagram the number or proportions of people whose income is not less than any given amount, or proportion of people whose height does not exceed any stated value. Diagrams of

this type are known as *cumulative frequency curves* or *ogives*. These are of two kinds '*more than*' or '*less than*' and typically they look somewhat like a long drawn S (Fig. 25.2).

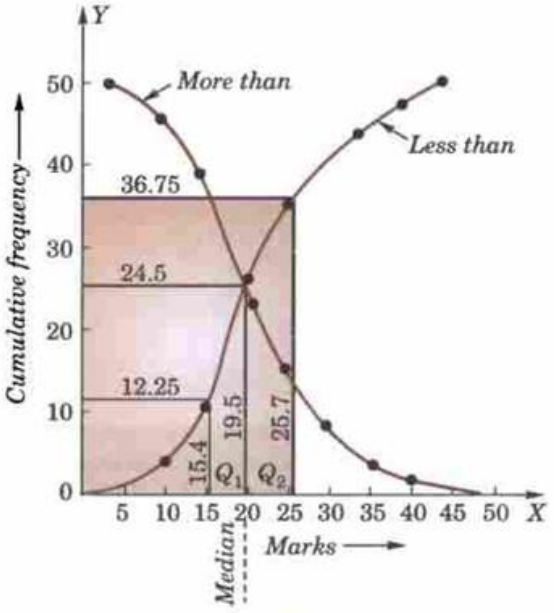


Fig. 25.2

Example 25.1. *Draw the histogram, frequency polygon, frequency curve and the ogive 'less than' and 'more than' from the following distribution of marks obtained by 49 students :*

Class (Marks group)	Frequency (No. of students)	Cumulative frequency	
		(Less than)	(More than)
5—10	5	5	49
10—15	6	11	44
15—20	15	26	38
20—25	10	36	23
25—30	5	41	13
30—35	4	45	8
35—40	2	47	4
40—45	2	49	2

Solution. In Fig. 25.1, the rectangles show the *histogram*; the dotted polygon represents the *frequency polygon* and the smooth curve is the *frequency curve*.

The *ogives* 'less than' and 'more than' are shown in Fig. 25.2.

25.4 COMPARISON OF FREQUENCY DISTRIBUTIONS

The condensation of data in the form of a frequency distribution is very useful as far as it brings a long series of observations into a compact form. But in practice, we are generally interested in comparing two or more series. The inherent inability of the human mind to grasp in its entirety even the data in the form of a frequency distribution compels us to seek for certain constants which could concisely give an insight into the important characteristics of the series. The chief constants which summarise the fundamental characteristics of the frequency distributions are (*i*) *Measures of central tendency*, (*ii*) *Measures of dispersion and* (*ii*) *Measures of skewness*.

25.5 MEASURES OF CENTRAL TENDENCY

A frequency distribution in general, shows clustering of the data around some central value. Finding of this central value or the average is of importance, as it gives a most representative value of the whole group.

Different methods give different averages which are known as the *measures of central tendency*. The commonly used measures of central value are *Mean, Median, Mode, Geometric mean and Harmonic mean.*

(1) Mean. If $x_1, x_2, x_3, ..., x_n$ are a set of n values of a variate, then the *arithmetic mean* (or simply *mean*) is given by

$$\bar{\mathbf{x}} = \frac{x_1 + x_2 + x_3 + ... + x_n}{n}, \text{ i.e. } \frac{\Sigma \mathbf{x}_i}{\mathbf{n}} \qquad ...(1)$$

In a *frequency distribution*, if $x_1, x_2, ..., x_n$ be the mid-values of the class-intervals having frequencies $f_1, f_2, ..., f_n$ respectively, we have

$$\bar{\mathbf{x}} = \frac{f_1x_1 + f_2x_2 + ... + f_nx_n}{f_1 + f_2 + ... + f_n} = \frac{\Sigma \mathbf{f}_i\mathbf{x}_i}{\Sigma \mathbf{f}_i} \qquad ...(2)$$

Calculation of mean. Direct method of computing especially when applied to grouped data involves heavy calculations and in order to avoid these, the following formulae are generally used :

I. Short-cut method $$\bar{\mathbf{x}} = \mathbf{A} + \frac{\Sigma \mathbf{f}_i d_i}{\Sigma \mathbf{f}_i} \qquad ...(3)$$

II. Step-deviation method $$\bar{\mathbf{x}} = \mathbf{A} + \mathbf{h}\frac{\Sigma \mathbf{f}_i \mathbf{u}_i}{\Sigma \mathbf{f}_i} \qquad ...(4)$$

where $d = x - A$ and $u = (x - A)/h$, A being an arbitrary origin and h the equal class interval.

Proof. If $x_1, x_2, ..., x_n$ are the mid-values of the classes with frequencies $f_1, f_2, ..., f_n$, we have

$$\Sigma f_i x_i = \Sigma f_i (A + d_i) = A\Sigma f_i + \Sigma f_i d_i$$

$$\therefore \qquad \bar{x} = \frac{\Sigma f_i x_i}{\Sigma f_i} = A + \frac{\Sigma f_i d_i}{\Sigma f_i}$$

Further $u_i = d_i/h$ or $d_i = hu_i$. Substituting this value in (3), we get (4).

Obs. *The algebraic sum of the deviations of all the variables from their mean is zero, for*

$$\Sigma f_i (x_i - \bar{x}) = \Sigma f_i x_i - \bar{x}\,\Sigma f_i = \Sigma f_i x_i - \frac{\Sigma f_i x_i}{\Sigma f_i} \cdot \Sigma f_i = 0.$$

Cor. *If $\bar{x}_1, \bar{x}_2$ be the means of two samples of size n_1 and n_2, then the mean $\bar{x}$ of the combined sample of size $n_1 + n_2$ is given by*

$$\bar{\mathbf{x}} = \frac{\mathbf{n_1}\bar{\mathbf{x}}_1 + \mathbf{n_2}\bar{\mathbf{x}}_2}{\mathbf{n_1} + \mathbf{n_2}}$$

For $n_1\bar{x}_1$ = sum of all observations of the first sample,

and $n_2\bar{x}_2$ = sum of all observations of the second sample.

$\therefore$ sum of the observations of the combined sample $= n_1\bar{x}_1 + n_2\bar{x}_2$.

Also number of the observations in the combined sample $= n_1 + n_2$.

$\therefore$ mean of the combined sample $= \dfrac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$.

Example 25.2. *The following is the frequency distribution of a random sample of weekly earnings of 509 employees :*

Weekly earnings :	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40
No. of employees :	3	6	10	15	24	42	75	90	79	55	36	26	19	13	9	7

Calculate the average weekly earnings.

Solution. The calculations are arranged in the following table. The arbitrary origin is generally taken as the value corresponding to the maximum frequency.

By direct method, we have

$$\text{Mean } \bar{x} = \frac{\Sigma fx}{\Sigma f} = \frac{13,315}{509} = 26.16$$

By step-deviation method, we have

$$\bar{x} = A + h\frac{\Sigma fu}{\Sigma f} = 25 + 2 \times \frac{295}{509}$$

$$= 25 + 1.16 = 26.16, \text{ which is same as found above.}$$

Weekly earnings	Mid value x	No. of employees f	Step deviations $f \times x$	$u = (x - 25)/2$	$f \times u$
10—12	11	3	33	−7	−21
12—14	13	6	78	−6	−36
14—16	15	10	150	−5	−50
16—18	17	15	255	−4	−60
18—20	19	24	456	−3	−72
20—22	21	42	882	−2	−84
22—24	23	75	1725	−1	−75
24—26	25	90	2250	0	−398
26—28	27	79	2133	1	79
28—30	29	55	1595	2	110
30—32	31	36	1116	3	108
32—34	33	26	858	4	104
34—36	35	19	665	5	95
36—38	37	13	481	6	78
38—40	39	9	351	7	63
40—42	41	7	287	8	56
					+ 693
		$\Sigma f = 509$	$\Sigma fx = 13{,}315$		$\Sigma fu = 295$

(2) Median. *If the values of a variable are arranged in the ascending order of magnitude, the median is the middle item if the number is odd and is the mean of the two middle items if the number is even.* Thus the median is equal to the mid-value, *i.e.*, the value which divides the total frequency into two equal parts.

For the grouped data,

$$\textbf{Median} = \mathbf{L} + \frac{(\frac{1}{2}\mathbf{N} - \mathbf{C})}{\mathbf{f}} \times \mathbf{h}$$

where L = lower limit of the median class, N = total frequency,

f = frequency of the median class, h = width of the median class,

and C = cumulative frequency upto the class preceding the median class.

Quartiles. Quartiles are those values which divide the frequency into four equal parts, when the values are arranged in the ascending order of magnitude. The **lower quartile** (Q_1) is mid-way between the lower extreme and the median. The **upper quartile** (Q_3) is midway between the median the upper extreme.

For the grouped data, these are calculated by the formulae :

$$\mathbf{Q_1} = \mathbf{L} + \frac{(\frac{1}{4}\mathbf{N} - \mathbf{C})}{\mathbf{f}} \times \mathbf{h}$$

and

$$\mathbf{Q_3} = \mathbf{L} + \frac{(\frac{3}{4}\mathbf{N} - \mathbf{C})}{\mathbf{f}} \times \mathbf{h}$$

where L = lower limit of the class in which Q_1 or Q_3 lies, f = frequency of this class, h = width of the class

and C = cumulative frequency upto the class preceding the class in which Q_1 or Q_3 lies.

The difference between the upper and lower quartiles, i.e., Q_3—Q_1 is called the **inter-quartile range.**

Obs. The ogives give a ready method of marking on the curve the values of the median and the quartiles. The two ogives 'less than' and 'more than' cut each other at the median (Fig. 25.2).

(3) Mode. *The mode is defined as that value of the variable which occurs most frequently, i.e., the value of the maximum frequency.*

For a grouped distribution, it is given by the formula

$$\textbf{Mode} = \mathbf{L} + \frac{\Delta_1}{\Delta_1 + \Delta_2}\mathbf{h}$$

where L = lower limit of the class containing the mode,

Δ_1 = excess of modal frequency over frequency of preceding class,

Δ_2 = excess of modal frequency over following class,

and h = size of modal class.

For a frequency curve (Fig. 25.1), the abscissa of the highest ordinate determines the value of the mode. There may be one or more modes in a frequency curve. Curves having a single mode are termed as *unimodal*, those having two modes as *bi-modal* and those having more than two modes as *multi-modal*.

Obs. In a symmetrical distribution, the mean, median and mode coincide. For other distributions, however, they are different and are known to be connected by the empirical relationship :

Mean – Mode = 3(Mean – Median).

Example 25.3. *Calculate median and the lower and upper quartiles from the distribution of marks obtained by 49 students of example 25.1. Find also the semi-interquartile range and the mode.*

Solution. *Median* (or 49/2) falls in the class (15—20) and is given by

$$15 + \frac{(49/2) - 11}{15} \times 5 = 15 + \frac{13.5}{3} = 19.5 \text{ marks.}$$

Lower quartile Q_1 (or 49/4) = 12.25) also falls in the class 15—20.

$$\therefore \qquad Q_1 = 15 + \frac{(49/4) - 11}{15} \times 5 = 15 + \frac{12.5}{3} = 15.4 \text{ marks}$$

Upper quartile (or $\frac{3}{4} \times 49 = 36.75$) falls in the class 25—30.

$$\therefore \qquad Q_3 = 25 + \frac{36.75 - 36}{5} \times 5 = 25.75 \text{ marks.}$$

Semi-interquartile range $= \frac{1}{2}(Q_3 - Q_1) = \dfrac{25.75 - 15.4}{2} = \dfrac{10.35}{2} = 5.175.$

Mode. It is seen that the mode value falls in the class 15—20. Employing the formula for the grouped distribution, we have

$$Mode = 15 + \frac{15 - 6}{(15 - 6) + (15 - 10)} \times 5 = 18.2 \text{ marks.}$$

Obs. In Fig. 25.2, the ogives meet at a point whose abscissa is 19.5 which is the *median* of the distribution. The values for the lower and upper quartiles are similarly seen to be 15.4 (for frequency 12.25) and 25.7 (for frequency 36.75).

Example 25.4. *Given below are the marks obtained by a batch of 20 students in a certain class test in Physics and Chemistry.*

Roll No. of students	*Marks in Physics*	*Marks in Chemistry*	*Roll No. of students*	*Marks in Physics*	*Marks in Chemistry*
1	53	58	11	25	10
2	54	55	12	42	42
3	52	25	13	33	15
4	32	32	14	48	46
5	30	26	15	72	50
6	60	85	16	51	64
7	47	44	17	45	39
8	46	80	18	33	38
9	35	33	19	65	30
10	28	72	20	29	36

In which subject is the level of knowledge of the students higher ?

Solution. The subject for which the value of the median is higher will be the subject in which the level of knowledge of the students is higher. To find the median in each case, we arrange the marks in ascending order of magnitude :

Sr. No.	Marks in Physics	Marks in Chemistry	Sr. No.	Marks in Physics	Marks in Chemistry
1	25	10	11	46	42
2	28	15	12	47	44
3	29	25	13	48	46
4	30	26	14	51	50
5	32	30	15	52	55
6	33	32	16	53	58
7	33	33	17	54	64
8	35	36	18	60	72
9	42	38	19	65	80
10	45	39	20	72	85

Median marks in Physics = A.M. of marks of 10th and 11th terms

$$= \frac{45+46}{2} = 45.5$$

Median marks in Chemistry = A.M. of marks of 10th and 11th items.

$$= \frac{39+42}{2} = 40.5$$

Since the median marks in Physics is greater than the median marks in Chemistry; the level of knowledge in Physics is higher.

Example 25.5. *An incomplete frequency distribution is given as below :*

Variable :	*10—20*	*20—30*	*30—40*	*40—50*	*50—60*	*60—70*	*70—80*
Frequency :	*12*	*30*	*?*	*65*	*?*	*25*	*18*

Given that the total frequency is 229 and median is 46, find the missing frequencies.

Solution. Let f_1, f_2 be the missing frequencies of the classes 30—40 and 50—60 respectively.

Since the median lies in the class 40—50,

$$\therefore \qquad 46 = 40 + \frac{229/2 - (12 + 30 + f_1)}{65} \times 10$$

which gives $f_1 = 33.5$ which can be taken as 34.

$$\therefore \qquad f_2 = 229 - (12 + 30 + 34 + 65 + 25 + 18) = 45.$$

(4) Geometric mean. If $x_1, x_2, \ldots, x_n$ are a set of n observations, then the *geometric mean* is given by

$$\text{G.M.} = (x_1 x_2 \ldots x_n)^{1/n}$$

or

$$\log \text{G.M.} = \frac{1}{n}(\log x_1 + \log x_2 + \ldots + \log x_n) \qquad \ldots(1)$$

In a frequency distribution, let $x_1, x_2, \ldots, x_n$ be the central values with corresponding frequencies $f_1, f_2, \ldots, f_n$, we have

$$\text{G.M.} = \left[(x_1)^{f_1} \cdot (x_2)^{f_2} \ldots (x_n)^{f_n}\right]^{1/n} \quad \text{where } n = \Sigma f_i.$$

or

$$\log \text{G.M.} = \frac{1}{n}[f_1 \log x_1 + f_2 \log x_2 + \ldots + f_n \log x_n] \qquad \ldots(2)$$

Hence (1) and (2) show that logarithm of G.M. = A.M. of logarithms of the values.

(5) Harmonic mean. If $x_1, x_2, \ldots x_n$ be a set of n observations, then the *harmonic mean is defined as the reciprocal of the (arithmetic) mean of the reciprocals of the quantities.* Thus

$$\text{H.M.} = \frac{1}{\frac{1}{n}\left(\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}\right)}$$

In a frequency distribution, $\text{H.M.} = \dfrac{1}{\frac{1}{n}\left(\frac{f_1}{x_1} + \frac{f_2}{x_2} + \ldots + \frac{f_n}{x_n}\right)}$ where $n = \Sigma f_i$.

Example 25.6. *Three cities A, B, C are equidistant from each other. A motorist travels from A to B at 30 km/hr, from B to C at 40 km/hr, from C to A at 50 km/hr. Determine the average speed.*

Solution. Let $AB = BC = CA = s$ km

Time taken to travel from A to $B = s/30$

Time taken to travel from B to $C = s/40$

Time taken to travel from C to $A = s/50$

$\therefore$ average time taken $= \frac{1}{3}\left(\frac{s}{30} + \frac{s}{40} + \frac{s}{50}\right)$

Thus the average speed $= \dfrac{s}{\frac{1}{3}\left(\frac{s}{30} + \frac{s}{40} + \frac{s}{50}\right)}$

In other words, the average speed is the harmonic mean of 30, 40, 50 km/hr.

Hence the average speed $= \dfrac{1}{\frac{1}{3}\left(\frac{1}{30} + \frac{1}{40} + \frac{1}{50}\right)} = 38.3$ km/hr.

Obs. Of the various measures of central tendency, *the mean* is the most important for it can be computed easily. *The median*, though more easily calculable, cannot be applied with ease to theoretical analysis. Median is of advantage when there are exceptionally large and small values at the ends of the distribution.

The mode, though most easily calculated, has the least significance. It is particularly misleading in distributions which are small in numbers or highly unsymmetrical.

The geometrical mean though difficult to compute, finds application in cases like populations where we are concerned with a quantity whose changes tend to be directly proportional to the quantity itself.

The harmonic mean is useful in limited situations where time and rate or prices are involved.

PROBLEMS 25.1

1. Draw the histogram and frequency polygon for the following distribution. Also calculate the arithmetic mean :

Class interval :	0—99	100—199	200—299	300—399	400—499	500—599	600—699	700—799
Frequency :	10	54	184	264	246	40	1	1

2. The following marks were given to a batch of candidates :

66	62	45	79	32	51	56	60	51	49
25	42	54	54	58	70	43	58	50	52
38	67	50	59	48	65	71	30	46	55
82	51	63	45	53	40	35	56	70	52
67	55	57	30	63	42	74	58	44	55

Draw a cumulative frequency curve.

Hence find the proportion of candidates securing more than 50 marks. Also mark off the median, the first and third quartiles.

3. Find the mean, median and mode for the following :

Mid Value :	15	20	25	30	35	40	45	50	55
Frequency :	2	22	19	14	3	4	6	1	1

(*Kerala, 1990*)

4. Calculate mean, median and mode of the following data relating to weight of 120 articles :

Weight (in gm) :	0—10	10—20	20—30	30—40	40—50	50—60
No. of articles :	14	17	22	26	23	18

5. The population of a country was 300 million in 1971. It became 520 million in 1989. Calculate the percentage compound rate of growth per annum.

[**Hint.** Use $P_n = P_0(1 + r)^n$, r being the growth rate.]

6. The number of divorces per 1000 marriages in the United States increased from 84 in 1970 to 108 in 1990. Find the annual increase of the divorce rate for the period 1970 to 1990.

7. An aeroplane flies along the four sides of a square at speeds of 100, 200, 300 and 400 km/hr. respectively. What is the average speed of the plane in its flight around the square.

8. A man having to drive 90 km. wishes to achieve an average speed of 30 km/hr. For the first half of the journey, he averages only 20 km/hr. What must be his average speed for the second half of the journey if his overall average is to be 30 km/hr.

9. Following table gives the cumulative frequency of the age of a group of 199 teachers. Find the mean and median age of the group.

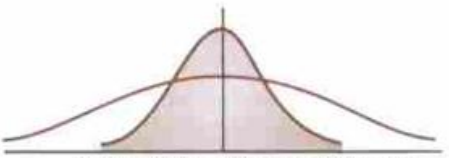
Age in years :	20—25	25—30	30—35	35—40	40—45	45—50	50—55	55—60	60—65	65—70
Cum. frequ. :	21	40	90	130	146	166	176	186	195	199

10. Recast the following cumulative table in the form of an ordinary frequency distribution and determine the median and the mode :

No. of days absent	*No. of students*	*No. of days absent*	*No. of students*
Less than 5	29	Less than 30	644
Less than 10	224	Less than 35	650
Less than 15	465	Less than 40	653
Less than 20	582	Less than 45	655
Less than 25	634		

25.6 MEASURES OF DISPERSION

Although measures of central tendency do exhibit one of the important characteristics of a distribution, yet they fail to give any idea as to how the individual values differ from the central value, *i.e.*, whether they are closely packed around the central value or widely scattered away from it. Two distributions may have the same mean and the same total frequency, yet they may differ in the extent to which the individual values may be spread about the average (*See* Fig. 25.3). The magnitude of such a variation is called *dispersion*. The important measures of dispersion are given below :



Same mean different dispersion

Fig. 25.3

(1) Range. *This is the simplest measure of dispersion and is given by the difference between the greatest and the least values in the distribution.* If the weekly wages of a group of labourers are

₹ 21 23 28 25 35 42 39 48

then *range* = Max. value – Min. value = 48 – 21 = ₹ 27.

(2) Quartile deviation or semi-interquartile range. One half of the interquartile range is called *quartile deviation,* or *semi-interquartile range.* If Q_1 and Q_3 are the first and third quartiles, the semi-interquartile range

$$Q = \tfrac{1}{2}(Q_3 - Q_1).$$

(3) Mean deviation. The mean deviation is the mean of the absolute differences of the values from the mean, median or mode. Thus *mean deviation* (*M.D.*)

$$= \frac{1}{n}\Sigma f_i\ |x_i - A|$$

where A is either the mean or the median or the mode. As the positive and negative differences have equal effects, only the absolute value of differences is taken into account.

(4) Standard deviation. The most important and the most powerful measure of dispersion is the *standard deviation* (*S.D.*) : generally denoted by σ. It is computed as the square root of the mean of the squares of the differences of the *variate* values from their mean.

Thus *standard deviation* (*S.D.*)

$$\sigma = \sqrt{\left[\frac{\Sigma \mathbf{f}_i(\mathbf{x}_i - \bar{\mathbf{x}})^2}{\mathbf{N}}\right]} \qquad ...(1)$$

where N is the total frequency Σf_i.

If however, the deviations are measured from any other value, say A, instead of $\bar{x}$, it is called the *root-mean-square deviation*.

The square of the standard deviation is known as the **variance.**

Calculation of S.D. The change of origin and the change of scale considerably reduces the labour in the calculation of standard deviation. The formulae for the computation of σ are as follows :

I. Short-cut method

$$\sigma = \sqrt{\left[\frac{\Sigma f_i d_i^2}{\Sigma f_i} - \left(\frac{\Sigma f_i d_i}{\Sigma f_i}\right)^2\right]} \quad ...(2)$$

II. Step-deviation method

$$\sigma = h\sqrt{\left\{\frac{\Sigma f_i d_i'^2}{\Sigma f_i} - \left(\frac{\Sigma f_i d_i'}{\Sigma f_i}\right)^2\right\}} \quad ...(3)$$

where $d_i = x_i - A$ and $d'_i = (x_i - A)/h$, being the assumed mean and h the equal class interval.

Proof. We know that $x_i - \bar{x} = (x_i - A) - (\bar{x} - A)$

$$\therefore \quad \Sigma f_i(x_i - \bar{x})^2 = \Sigma f_i[d_i - (\bar{x} - A)]^2 = \Sigma f_i d_i^2 + (\bar{x} - A)^2 \Sigma f_i - 2(\bar{x} - A)\Sigma f_i d_i$$

$$= \Sigma f_i d_i^2 - \frac{(\Sigma f_i d_i)^2}{\Sigma f_i} \qquad \left[\because \ \bar{x} = A + \frac{\Sigma f_i d_i}{\Sigma f_i}\right]$$

Hence $$\sigma^2 = \frac{\Sigma f_i (x - \bar{x})^2}{\Sigma f_i} = \frac{\Sigma f_i d_i^2}{\Sigma f_i} - \left(\frac{\Sigma f_i d_i}{\Sigma f_i}\right)^2$$

Further $d'_i = (x_i - A)/h = d_i/h$ or $d_i = hd'_i$, then substituting this value in (2), we get (3).

Obs. *The root mean square deviation is least when measured from the mean.*
The root mean square deviation is given by

$$s^2 = \frac{\Sigma f_i d_i^2}{\Sigma f_i} \quad \text{and} \quad \frac{\Sigma f_i d_i}{\Sigma f_i} = \left[A + \frac{\Sigma f_i d_i}{\Sigma f_i}\right] - A = \bar{x} - A$$

$\therefore$ from (2), we have $$s^2 = \sigma^2 + (\bar{x} - A)^2 \quad ...(4)$$

This shows that s^2 is always $> \sigma^2$ and the least value of $s^2 = \sigma^2$. This occurs when $A = \bar{x}$.

25.7 (1) COEFFICIENT OF VARIATION

The ratio of the standard deviation to the mean, is known as the *coefficient of variation*. As this is a ratio having no dimension, it is used for comparing the variations between the two groups with different means. It is often expressed as a percentage.

$$\therefore \quad \textit{Coefficient of variation} = \frac{\sigma}{\bar{x}} \times 100$$

(2) Relations between measures of dispersion

(*i*) *Quartile deviation = 2/3 (standard deviation)*

(*ii*) *Mean deviation = 4/5 (standard deviation)*

25.8 STANDARD DEVIATION OF THE COMBINATION OF TWO GROUPS

If m_1, σ_1 be the mean and S.D. of a sample of size n_1 and m_2, σ_2 be those for a sample of size n_2, then the S.D. σ of the combined sample of size $n_1 + n_2$ is given by

$$(n_1 + n_2)\sigma^2 = n_1\sigma_1^2 + n_2\sigma_2^2 + n_1 D_1^2 + n_2 D_2^2$$

where $D_i = m_i - m$, m being the mean of combined sample.

From (4), we have $ns^2 = n\sigma^2 + n(\bar{x} - A)^2$ where n is the size of the sample.

i.e. sum of the squares of the deviations from $A = n\sigma^2 + n(\bar{x} - A)^2$.

Now let us apply this result to the first given sample taking A at m. Then, sum of the squares of the deviations of n_1 items from $m = n_1\sigma_1^2 + n_1(m_1 - m)^2$ ...(5)

Similarly for the second given sample taking A at m, sum of the squares of the deviations of n_2 items from $m = n_2\sigma_2^2 + n_2(m_2 - m)^2$ ...(6)

Adding (5) and (6), sum of the squares of the deviations of $n_1 + n_2$ items from m

$$= n_1\sigma_1^2 + n_2\sigma_2^2 + n_1(m_1 - m)^2 + n_2(m_2 - m)^2$$

$$\therefore \quad (n_1 + n_2)\sigma^2 = n_1\sigma_1^2 + n_2\sigma_2^2 + n_1 D_1^2 + n_2 D_2^2$$

This result can be extended to the combination of any number of samples, giving a result of the form

$$(\Sigma n_i)\sigma^2 = \Sigma(n_i\sigma_i^2) + \Sigma(n_i D_i^2).$$

Example. 25.7. *Calculate the mean and standard deviation for the following :*

Size of item :	6	7	8	9	10	11	12
Frequency :	3	6	9	13	8	5	4

(V.T.U., 2001)

Solution. The calculations are arranged as follows :

Size of item x	*Frequency f*	*Deviation d = x − 9*	*f × d*	*f × d²*
6	3	−3	−9	27
7	6	−2	−12	24
8	9	−1	−9	9
9	13	0	0	0
10	8	1	8	8
11	5	2	10	20
12	4	3	12	36
	$\Sigma f = 48$		$\Sigma fd = 0$	$\Sigma fd^2 = 124$

$$\therefore \qquad mean = 9 + \frac{\Sigma fd}{\Sigma f} = 9$$

$$Standard\ deviation = \sqrt{\left[\frac{\Sigma f(x-\bar{x})^2}{\Sigma f}\right]} = \sqrt{\left(\frac{\Sigma fd^2}{\Sigma f}\right)} = \sqrt{\left(\frac{124}{48}\right)} = 1.607.$$

Example 24.8. *Calculate the mean and standard deviation of the following frequency distribution :*

Weekly wages is ₹	*No. of men*
4.5—12.5	4
12.5—20.5	24
20.5—28.5	21
28.5—36.5	18
36.5—44.5	5
44.5—52.5	3
52.5—60.5	5
60.5—68.5	8
68.5—76.5	2

Solution. The calculations are arranged in the table below :

Wages class ₹	*Mid value x*	*No. of men f*	*Step deviation* $d' = \frac{x-32.5}{8}$	*fd'*	*fd'²*
4.5—12.5	8.5	4	−3	−12	36
12.5—20.5	16.5	24	−2	−48	96
20.5—28.5	24.5	21	−1	−21	21
28.5—36.5	32.5	18	0	0	0
36.5—44.5	40.5	5	1	5	5
44.5—52.5	48.5	3	2	6	12
52.5—60.5	56.5	5	3	15	45
60.5—68.5	64.5	8	4	32	128
68.5—76.5	72.5	2	5	10	50
		$\Sigma f = 90$		$\Sigma fd' = -13$	$\Sigma fd'^2 = 393$

$$\therefore \qquad mean\ wage = 32.5 + 8 \times \frac{\Sigma fd'}{\Sigma f} = 32.5 + 8\left(\frac{-13}{90}\right) = ₹\ 31.35$$

$$Standard\ deviation = 8\sqrt{\frac{\Sigma fd'^2}{\Sigma f} - \left(\frac{\Sigma fd'}{\Sigma f}\right)^2} = 8\sqrt{\frac{393}{90} - \left(\frac{-13}{90}\right)^2} = ₹\ 16.64.$$

Example 25.9. *The following are scores of two batsmen A and B in a series of innings :*

A :	12	115	6	73	7	19	119	36	84	29
B :	47	12	16	42	4	51	37	48	13	0

Who is the better score getter and who is more consistent ? (V.T.U., 2004)

Solution. Let x denote score of A and y that of B.

Taking 51 as the origin, we prepare the following table :

x	$d(=x-51)$	d^2	y	$\delta(=y-51)$	δ^2
12	−39	1521	47	−4	16
115	64	4096	12	−39	1521
6	−45	2025	16	−35	1225
73	22	484	42	−9	81
7	−44	1936	4	−47	2209
19	−32	1024	51	0	0
119	68	4624	37	−14	196
36	−15	225	48	−3	9
84	33	1089	13	−38	1444
29	−22	484	0	−51	2601
Total	−10	17508		−240	9302

For A, $$\text{A.M. } \bar{x} = 51 + \frac{\Sigma d}{n} = 51 - \frac{10}{10} = 50$$

$$\text{S.D. } \sigma_1 = \sqrt{\left\{\frac{\Sigma d^2}{n} - \left(\frac{\Sigma d}{n}\right)^2\right\}} = \sqrt{\{1750.8 - (-1)^2\}} = 41.8$$

$\therefore$ coefficient of variation $= \frac{\sigma_1}{\bar{x}} \times 100 = \frac{41.8}{50} \times 100 = 83.6\%$

For B, $$\text{A.M. } \bar{y} = 51 + \frac{\Sigma \delta}{n} = 51 - \frac{240}{10} = 27$$

$$\text{S.D. } \sigma_2 = \sqrt{\left\{\frac{\Sigma \delta^2}{n} - \left(\frac{\Sigma \delta}{n}\right)^2\right\}} = \sqrt{\{930.2 - (-24)^2\}} = 18.8$$

$\therefore$ coefficient of variation $= \frac{\sigma^2}{\bar{y}} \times 100 = \frac{18.8}{27} \times 100 = 69.6\%$

Since the A.M. of A > A.M. of B, it follows that A is a better score getter (*i.e.*, more efficient) than B.

Since the coefficient of variation of B < the coefficient of variation of A, it means that B is more consistent than A. Thus even though A is a better player, he is less consistent.

Example 25.10. *The numbers examined, the mean weight and S.D. in each group of examination by three medical examiners are given below. Find the mean weight and S.D. of the entire data when grouped together.*

Med. Exam.	*No. Examined*	*Mean Wt. (lbs.)*	*S.D. (lbs.)*
A	*50*	*113*	*6*
B	*60*	*120*	*7*
C	*90*	*115*	*8*

Solution. We have $n_1 = 50,\ \bar{x}_1 = 113,\ \sigma_1 = 6$

$n_2 = 60,\ \bar{x}_2 = 120,\ \sigma_2 = 7$

$n_3 = 90,\ \bar{x}_3 = 115,\ \sigma_3 = 8.$

If $\bar{x}$ is the mean of the entire data,

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2 + n_3\bar{x}_3}{n_1 + n_2 + n_3} = \frac{50 \times 113 + 60 \times 120 + 90 \times 115}{50 + 60 + 90} = \frac{23200}{200} = 116 \text{ lb.}$$

If σ is the S.D. of the entire data,

$$N\sigma^2 = n_1\sigma_1^2 + n_2\sigma_2^2 + n_3\sigma_3^2 + n_1D_1^2 + n_2D_2^2 + n_3D_3^2$$

where $N = n_1 + n_2 + n_3 = 200$, $D_1 = \bar{x}_1 - \bar{x} = -3$, $D_2 = \bar{x}_2 - \bar{x} = 4$ and $D_3 = \bar{x}_3 - \bar{x} = -1$.

$$\therefore \quad 200\sigma^2 = 50 \times 36 + 60 \times 49 + 90 \times 64 + 50 \times 9 + 60 \times 16 + 90 \times 1$$

$$= 1800 + 2940 + 5760 + 450 + 960 + 90$$

$$\sigma^2 = \frac{12000}{200} = 60. \quad \text{Hence} \quad \sigma = \sqrt{60} = 7.746 \text{ lb.}$$

PROBLEMS 25.2

1. The crushing strength of 8 cement concrete experimental blocks, in metric tonnes per sq. cm., was 4.8, 4.2, 5.1, 3.8, 4.4, 4.7, 4.1 and 4.5. Find the mean crushing strength and the standard deviation.
2. Show that the variance of the first n positive integers is $\frac{1}{12}(n^2 - 1)$. *(V.T.U., 2003)*
3. The mean of five items of an observation is 4 and the variance is 5.2. If three of the items are 1, 2 and 6, then find the other two. *(V.T.U., 2002)*
4. For the distribution

x:	5	6	7	8	9	10	11	12	13	14	15
f:	18	15	34	47	68	90	80	62	35	27	11

find the mean, median and lower and upper quartiles, variance and the standard deviation.

5. The following table shows the marks obtained by 100 candidates in an examination. Calculate the mean, median and standard deviation :

Marks obtained :	1—10	11—20	21—30	31—40	41—50	51—60
No. of candidates :	3	16	26	31	16	8

(Osmania, 2003 S ; V.T.U., 2003 S)

6. Compute the quartile deviation and standard deviation for the following :

x:	100—109	110—119	120—129	130—139	140—149	150—159	160—169	170—179
f:	15	44	133	150	125	82	35	16

7. Calculate (*i*) mean deviation about the mean, (*ii*) mean deviation about the median for the following distribution :

Class :	3—4.9	5—6.9	7—8.9	9—10.9	11—12.9	13—14.9	15—16.9
f :	5	8	30	82	45	24	6

(Madras, 2002)

8. Two observers bring the following two sets of data which represent measurements of the same quantity :

I.	105.1	103.4	104.2	104.7	104.8	105.0	104.9
II.	105.3	105.1	104.8	105.2	106.7	102.9	103.1

Calculate the standard deviation in each case. Which set of data is more reliable ? Can the same conclusion be reached by calculating the mean deviation ?

Obs. The smaller the coefficient of variation, the greater is the *reliability* or *consistency* in the data.

9. The heights and weights of the 10 armymen are given below. In which characteristics are they more variable ?

Height in cm.	170	172	168	177	179	171	173	178	173	179
Weight in kg.	75	74	75	76	77	73	76	75	74	75

10. The index number of prices of two articles A and B for six consecutive weeks are given below :

A :	314	326	336	368	404	412
B :	330	331	320	318	321	330

Find which has a more variable price ?

11. The scores of two golfers A and B in 12 rounds are given below. Who is the better player and who is the more consistent player ?

A :	74	75	78	72	78	77	79	81	79	76	72	71
B :	87	84	80	88	89	85	86	82	82	79	86	80

12. The scores obtained by two batsmen A and B in 10 matches are given below :

A :	30	44	66	62	60	34	80	46	20	38
B :	34	46	70	38	55	48	60	34	45	30

Calculating mean, S.D. and coefficient of variation for each batsman, determine who is more efficient and who is more consistent.

13. Find the mean and standard deviation of the following two samples put together :

Sample No.	*Size*	*Mean*	*S.D.*
1	50	158	5.1
2	60	164	4.6

14. A distribution consists of three components with frequencies 200, 250 and 300 having means 25, 10 and 15 and S.Ds. 3, 4 and 5 respectively. Show that the mean of the combined distribution is 16 and its S.D. is 7.2 approximately.

25.9 (1) MOMENTS

The rth moment about the mean $\bar{x}$ of a distribution is denoted by μ_r and is given by

$$\mu_r = \frac{1}{N}\Sigma f_i(x_i - \bar{x})^r \qquad ...(1)$$

The corresponding moment about any point a is defined as

$$\mu_r' = \frac{1}{N}\Sigma f_i(x_i - a)^r \qquad ...(2)$$

In particular, we have $\mu_0 = \mu_0' = 1$ ...(3)

$$\mu_1 = \frac{1}{N}\Sigma f_i(x_i - \bar{x}) = 0;\ \mu_1' = \frac{1}{N}\Sigma f_i(x_i - a) = \bar{x} - a = d, \text{ say} \qquad ...(4)$$

$$\mu_2 = \frac{1}{N}\Sigma f_i(x_i - \bar{x})^2 = \sigma^2. \qquad ...(5)$$

(2) Moments about the mean in terms of moments about any point.

We have

$$\mu_r = \frac{1}{N}\Sigma f_i(x_i - \bar{x})^r = \frac{1}{N}\Sigma f[(x_i - a) - (\bar{x} - a)]^r$$

$$= \frac{1}{N}\Sigma f_i(X_i - d)^r \quad \text{where} \quad X_i = x_i - a,\ d = \bar{x} - a.$$

$$= \frac{1}{N}[\Sigma f_i X_i^r - {}^rC_1 d\Sigma f_i X_i^{r-1} + {}^rC_2 d^2 \Sigma f_i X_i^{r-2} - ...]$$

$$= \mu_r' - {}^rC_1\, d\mu_{r-1}' + {}^rC_2 d^2 \mu_{r-2}'' - ..... \qquad ...(6)$$

In particular,

$$\mu_2 = \mu'_2 - \mu'^2_1 \qquad ...(7)$$

$$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu'^3_1 \qquad ...(8)$$

$$\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2\mu'^2_1 - 3\mu'^4_1 \qquad ...(9)$$

These three results should be committed to memory. It should be noted that in each of these relations, the sum of the coefficients of the various terms on the right side is zero. Also each term on the right side is of the same dimension as the term on the left.

25.10 SKEWNESS

Skewness measures the degree of asymmetry or the departure from symmetry. If the frequency curve has a longer 'tail' to the right, *i.e.*, the mean is to the right of the mode [as in Fig. 25.4 (*a*)], then the distribution is said to have *positive skewness*. If the curve is more elongated to the left, then it is said to have *negative skewness* [Fig. 25.4 (*b*)].

The following three measures of skewness deserve mention :

(*i*) *Pearson's* coefficient of skewness* $= \dfrac{\text{mean} - \text{mode}}{\sigma}$

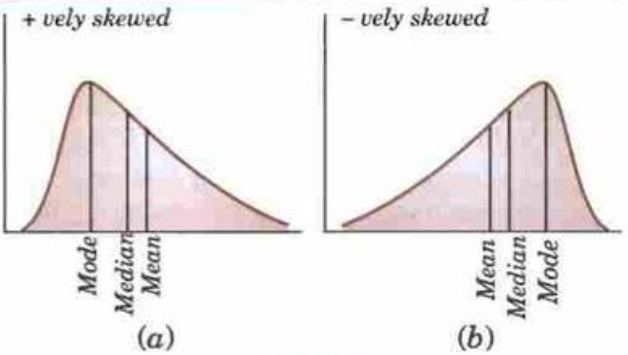


Fig. 25.4

* After the English statistician and biologist *Karl Pearson* (1857–1936) who did pioneering work and found the English school of statistics.

(*ii*) *Quartile coefficient of skewness* $= \dfrac{Q_3 + Q_1 - 2Q_2}{Q_3 - Q_1}$

Its value always lies between – 1 and + 1.

(*iii*) *Coefficient of skewness based on third moment* $\gamma_1 = \sqrt{\beta_1}$.

where $\beta_1 = \mu_3^2/\mu_2^3$

Thus $\gamma_1 = \sqrt{\beta_1}$ gives the simplest measure of *skewness.*

25.11 KURTOSIS

Kurtosis measures the degree of peakedness of a distribution and is given by $\beta_2 = \mu_4/\mu_2^2$.

(*a*) *Leptokurtic* (*b*) *Platykurtic* (*c*) *Mesokurtic*

Fig. 25.5

$\gamma_2 = \beta_2 - 3$ gives the excess of *Kurtosis*. The curves with $\beta_2 > 3$ are called *Leptokurtic* and those with $\beta_2 < 3$ as *Platykurtic*. The normal curve for which $\beta_2 = 3$, is called *Mesokurtic* [Fig. 25.5].

Example 25.11. *The first four moments about the working mean 28.5 of a distribution are 0.294, 7.144, 42.409 and 454.98. Calculate the moments about the mean. Also evaluate β_1, β_2 and comment upon the skewness and kurtosis of the distribution.* (V.T.U., 2[illegible])

Solution. The first four moments about the arbitrary origin 28.5 are $\mu'_1 = 0.294$, $\mu'_2 = 7.144$, $\mu'_3 = 42.409$, $\mu'_4 = 454.98$.

$$\therefore \quad \mu'_1 = \frac{1}{N}\sum f_i(x_i - 28.5) = \frac{1}{N}\sum f_i x_i - 28.5 = \bar{x} - 28.5 = 0.294 \text{ or } \bar{x} = 28.794$$

$$\mu_2 = \mu'_2 - \mu'^2_1 = 7.144 - (0.294)^2 = 7.058$$

$$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu'^3_1 = 42.409 - 3(7.144)(0.294) + 2(0.294)^3 = 36.151.$$

$$\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2\mu'^2_1 - 3\mu'^4_1$$

$$= 454.98 - 4(42.409) \times (0.294) + 6(7.144)(0.294)^2 - 3(0.294)^4 = 408.738$$

Now $\beta_1 = \mu_3^2/\mu_2^3 = (36.151)^2/(7.058)^3 = 3.717$

$\beta_2 = \mu_4/\mu_2^2 = 408.738/(7.058)^2 = 8.205.$

$\therefore$ $\gamma_1 = \sqrt{\beta_1} = 1.928$, which indicates considerable skewness of the distribution.

$\gamma_2 = \beta_2 - 3 = 5.205$ which shows that the distribution is *leptokurtic.*

Example 25.12. *Calculate the median, quartiles and the quartile coefficient of skewness from the following data :*

Weight (lbs) :	*70–80*	*80–90*	*90–100*	*100–110*	*110–120*	*120–130*	*130–140*	*140–150*
No. of persons :	*12*	*18*	*35*	*42*	*50*	*45*	*20*	*8*

Solution. Here total frequency $N = \Sigma f_i = 230$.

The cummulative frequency table is

Weight (lbs) :	*70–80*	*80–90*	*90–100*	*100–110*	*110–120*	*120–130*	*130–140*	*140–150*
f :	12	18	35	42	50	45	20	8
cum. f. :	12	30	65	107	157	202	222	230

Now $N/2 = 230/2 = 115$th item which lies in 110–120 group.

$$\therefore \quad \text{median or } Q_2 = L + \frac{N/2 - C}{f} \times h = 110 + \frac{115 - 107}{50} \times 10 = 111.6$$

Also $N/4 = 230/4 = 57.5$ *i.e.* Q_1 is 57.5th or 58th item which lies in 90–100 group.

$$\therefore \qquad Q_1 = L + \frac{N/4 - C}{f} \times h = 90 + \frac{57.5 - 30}{35} \times 10 = 97.85$$

Similarly, $3N/4 = 172.5$ *i.e.* Q_3 is 173rd item which lies in 120–130 group.

$$\therefore \qquad Q_3 = L + \frac{3N/4 - C}{f} \times h = 120 + \frac{172.5 - 157}{45} \times 10 = 123.44$$

Hence quartile coefficient of skewness $= \dfrac{Q_1 + Q_3 - 2Q_2}{Q_3 - Q_1}$

$$= \frac{97.85 + 123.44 - 2 \times 111.6}{123.44 - 97.85} = -0.07 \text{ (approx.)}.$$

PROBLEMS 25.3

1. Calculate the first four moments of the following distribution about the mean :

x:	0	1	2	3	4	5	6	7	8
f:	1	8	28	56	70	56	28	8	1

Also evaluate β_1 and β_2. *(V.T.U., 2004 ; Madras, 2003)*

2. The following table gives the monthly wages of 72 workers in a factory. Compute the standard deviation, quartile deviation, coefficients of variation and skewness. *(V.T.U., 2001)*

Monthly wages (in ₹)	*No. of workers*	*Monthly wages (in ₹)*	*No. of workers*
12.5—17.5	2	37.5—42.5	4
17.5—22.5	22	42.5—47.5	6
22.5—27.5	19	47.5—52.5	1
27.5—32.5	14	52.5—57.5	1
32.5—37.5	3		

3. Find Pearson's coefficient of skewness for the following data :

Class :	10–19	20–29	30–39	40–49	50–59	60–69	70–79	80–89
Frequency :	5	9	14	20	25	15	8	4

(V.T.U., 2000 S)

4. Compute the quartile coefficient of skewness for the following distribution :

x:	3–7	8–12	13–17	18–22	23–27	28–32	33–37	38–42
f:	2	108	580	175	80	32	18	5

(Madras, 2002 ; V.T.U., 2000)

Also compute the measure of skewness based on the third moment.

5. The first three moments of a distribution about the value 2 of the variable are 1, 16 and – 40. Show that the mean = 3, the variance = 15 and $\mu_3 = -86$. *(V.T.U., 2003 S)*

6. Compute skewness and kurtosis, if the first four moments of a frequency distribution $f(x)$ about the value $x = 4$ are respectively 1, 4, 10 and 45. *(Coimbatore, 1999)*

7. In a certain distribution, the first four moments about a point are – 1.5, 17, – 30 and 108. Calculate the moments about the mean, β_1 and β_2 ; and state whether the distribution is leptokurtic or platykurtic ?

25.12 CORRELATION

So far we have confined our attention to the analysis of observations on a single variable. There are, however, many phenomenae where the changes in one variable are related to the changes in the other variable. For instance, the yield of a crop varies with the amount of rainfall, the price of a commodity increases with the reduction in its supply and so on. Such a simultaneous variation, *i.e.* when the changes in one variable are associated or followed by changes in the other, is called *correlation*. Such a data connecting two variables is called *bivariate population*.

If an increase (or decrease) in the values of one variable corresponds to an increase (or decrease) in the other, the correlation is said to be *positive*. If the increase (or decrease) in one corresponds to the decrease (or increase) in the other, the correlation is said to be *negative*. If there is no relationship indicated between the variables, they are said to be *independent or uncorrelated*.

To obtain a measure of relationship between the two variables, we plot their corresponding values on the graph, taking one of the variables along the x-axis and the other along the y-axis. (Fig. 25.6).

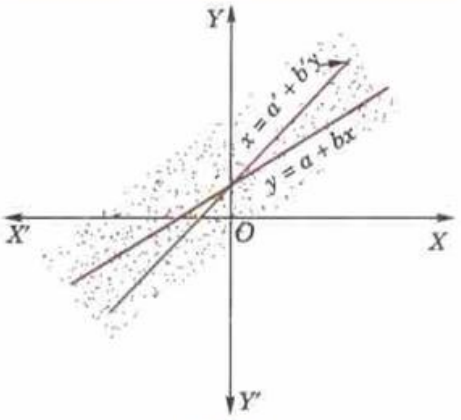


Fig. 25.6

Let the origin be shifted to $(\bar{x}, \bar{y})$, where $\bar{x}, \bar{y}$ are the means of x's and y's that the new co-ordinates are given by

$$X = x - \bar{x}, \qquad Y = y - \bar{y}.$$

Now the points (X, Y) are so distributed over the four quadrants of XY-plane that the product XY is positive in the first and third quadrants but negative in the second and fourth quadrants. The algebraic sum of the products can be taken as describing the trend of the dots in all the quadrants.

$\therefore$ (*i*) If ΣXY is positive, the trend of the dots is through the first and third quadrants,

(*ii*) if ΣXY is negative the trend of the dots is in the second and fourth quadrants, and

(*iii*) if ΣXY is zero, the points indicate no trend *i.e.* the points are evenly distributed over the four quadrants.

The ΣXY or better still $\frac{1}{n}\Sigma XY$, *i.e.*, the average of n products may be taken as a measure of correlation. If we put X and Y in their units, *i.e.*, taking σ_x as the unit for x and σ_y for y, then

$$\frac{1}{n}\Sigma\frac{X}{\sigma_x}.\frac{Y}{\sigma_y}, \text{ i.e., } \frac{\Sigma XY}{n\sigma_x\sigma_y}$$

is the *measure of correlation*.

25.13 COEFFICIENT OF CORRELATION

The numerical measure of correlation is called the *coefficient of correlation* and is defined by the relation

$$r = \frac{\Sigma XY}{n\sigma_x\sigma_y}$$

where X = deviation from the mean $\bar{x} = x - \bar{x}$, Y = deviation from the mean $\bar{y} = y - \bar{y}$,

σ_x = S.D. of x-series, σ_y = S.D. of y-series and n = number of values of the two variables.

Methods of calculation :

(*a*) *Direct method.* Substituting the value of σ_x and σ_y in the above formula, we get

$$r = \frac{\Sigma XY}{\sqrt{(\Sigma X^2 \Sigma Y^2)}} \qquad ...(1)$$

Another form of the formula (1) which is quite handy for calculation is

$$r = \frac{n\Sigma xy - \Sigma x \Sigma y}{\sqrt{[\{n\Sigma x^2 - (\Sigma x)^2\} \times \{n\Sigma y^2 - (\Sigma y)^2\}]}} \qquad ...(2)$$

(*b*) *Step-deviation method.* The direct method becomes very lengthy and tedious if the means of the two series are not integers. In such cases, use is made of assumed means. If d_x and d_y are step-deviations from the assumed means, then

$$r = \frac{n\Sigma d_x d_y - \Sigma d_x \Sigma d_y}{\sqrt{[\{n\Sigma d_x^2 - (\Sigma d_x)^2\} \times \{n\Sigma d_y^2 - (\Sigma d_y)^2\}]}} \qquad ...(3)$$

where $d_x = (x - a)/h$ and $d_y = (y - b)/k$.

Obs. The change of origin and units do not alter the value of the correlation coefficient since r is a pure number.

(*c*) *Co-efficient of correlation for grouped data.* When x and y series are both given as frequency distributions, these can be represented by a two-way table known as the *correlation-table*. It is double-entry table with one series along the horizontal and the other along the vertical as shown on page 848. The co-efficient of correlation for such a *bivariate frequency distribution* is calculated by the formula.

$$r = \frac{n(\Sigma fd_x d_y) - (\Sigma fd_x)(\Sigma fd_y)}{\sqrt{[\{n\Sigma fd_x^2 - (\Sigma fd_x)^2\} \times \{n\Sigma fd_y^2 - (\Sigma fd_y)^2\}]}} \quad ...(4)$$

where d_x = deviation of the central values from the assumed mean of x-series,
d_y = deviation of the central values from the assumed mean of y-series,
f is the frequency corresponding to the pair (x, y)
and $n(= \Sigma f)$ is the total number of frequencies.

Example 25.13. *Psychological tests of intelligence and of engineering ability were applied to 10 students. Here is a record of ungrouped data showing intelligence ratio (I.R.) and engineering ratio (E.R.). Calculate the co-efficient of correlation.*

Student	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*
I.R.	*105*	*104*	*102*	*101*	*100*	*99*	*98*	*96*	*93*	*92*
E.R.	*101*	*103*	*100*	*98*	*95*	*96*	*104*	*92*	*97*	*94*

(Andhra, 2000)

Solution. We construct the following table :

Student	*Intelligence ratio*		*Engineering ratio*				
	x	$x - \bar{x} = X$	y	$y - \bar{y} = Y$	X^2	Y^2	XY
A	105	6	101	3	36	9	18
B	104	5	103	5	25	25	25
C	102	3	100	2	9	4	6
D	101	2	98	0	4	0	0
E	100	1	95	−3	1	9	−3
F	99	0	96	−2	0	4	0
G	98	−1	104	6	1	36	−6
H	96	−3	92	−6	9	36	18
I	93	−6	97	−1	36	1	6
J	92	−7	94	−4	49	16	28
Total	990	0	980	0	170	140	92

From this table, mean of x, *i.e.*, $\bar{x} = 990/10 = 99$ and mean of y, *i.e.* $\bar{y} = 980/10 = 98$.

$$\Sigma X^2 = 170, \Sigma Y^2 = 140 \text{ and } \Sigma XY = 92.$$

Substituting these values in the formula (1) p. 744, we have

$$r = \frac{\Sigma XY}{\sqrt{(\Sigma X^2 \Sigma Y^2)}} = \frac{92}{\sqrt{(170 \times 140)}} = 92/154.3 = 0.59.$$

Example 25.14. *The correlation table given below shows that the ages of husband and wife of 53 married couples living together on the census night of 1991. Calculate the coefficient of correlation between the age of the husband and that of the wife.* *(J.N.T.U., 2003)*

Age of husband	*Age of wife*						*Total*
	15–25	*25–35*	*35–45*	*45–55*	*55–65*	*65–75*	
15–25	*1*	*1*	–	–	–	–	*2*
25–35	*2*	*12*	*1*	–	–	–	*15*
35–45	–	*4*	*10*	*1*	–	–	*15*
45–55	–	–	*3*	*6*	*1*	–	*10*
55–65	–	–	–	*2*	*4*	*2*	*8*
65–75	–	–	–	–	*1*	*2*	*3*
Total	*3*	*17*	*14*	*9*	*6*	*4*	*53*

Solution.

Age of husband				Age of wife x-series							Suppose $d_x = \frac{x-40}{10}$, $d_y = \frac{y-40}{10}$		
				15–25	25–35	35–45	45–55	55–65	65–75				
Years			Mid pt. x	20	30	40	50	60	70	Total f			
Age group	Mid pt. y		d_y \ d_x	−20	−10	0	10	20	30		fd_y	fd_y^2	fd_xd_y
				−2	−1	0	1	2	3				
15–25	20	−20	−2	1 **4**	1 **2**					2	−4	8	**6**
25–35	30	−10	−1	2 **4**	12 **12**	1 **0**				15	−15	15	**16**
35–45	40	0	0		4 **0**	10 **0**	1 **0**			15	0	0	**0**
45–55	50	10	1			3 **0**	6 **6**	1 **2**		10	10	10	**8**
55–65	60	20	2				2 **4**	4 **16**	2 **12**	8	16	32	**32**
65–75	70	30	3					1 **6**	2 **18**	3	9	27	**24**
Total f				3	17	14	9	6	4	53 = n	16	92	**86**
fd_x				−6	−17	0	9	12	12	10	Thick figures in small sqs. stand for fd_xd_y **Check:** $\Sigma fd_xd_y = 86$ from both sides		
fd_x^2				12	17	0	9	24	36	98			
fd_xd_y				**8**	**14**	**0**	**10**	**24**	**30**	**86**			

With the help of the above correlation table, we have

$$r = \frac{n(\Sigma fd_xd_y) - (\Sigma fd_x)(\Sigma fd_y)}{\sqrt{[n\Sigma fd_x^2 - (\Sigma fd_x)^2] \times [n\Sigma fd_y^2 - (\Sigma fd_y)^2]}}$$

$$= \frac{53 \times 86 - 10 \times 16}{\sqrt{[(53 \times 98 - 100) \times (53 \times 92 - 256)]}} = \frac{4398}{\sqrt{(5094 \times 4620)}} = \frac{4398}{4850} = 0.91 \text{ (approx.)}.$$

25.14 LINES OF REGRESSION

It frequently happens that the dots of the scatter diagram generally, tend to cluster along a well defined direction which suggests a linear relationship between the variables x and y. Such a line of best-fit for the given distribution of dots is called the *line of regression* (Fig. 25.6). In fact there are two such lines, one giving the best possible mean values of y for each specified value of x and the other giving the best possible mean values of x for given values of y. The former is known as the *line of regression of y on x* and the latter as the *line of regression of x on y*.

Consider first the line of regression of y on x. Let the straight line satisfying the general trend of n dots in a scatter diagram be

$$y = a + bx \qquad ...(1)$$

We have to determine the constants a and b so that (1) gives for each value of x, the best estimate for the average value of y in accordance with the *principle of least squares* (page 816), therefore, the normal equations for a and b are

$$\Sigma y = na + b\Sigma x \qquad ...(2)$$

and

$$\Sigma xy = a\Sigma x + b\Sigma x^2 \qquad ...(3)$$

(2) gives $\quad \frac{1}{n}\Sigma y = a + b \cdot \frac{1}{n}\Sigma x$ *i.e.*, $\bar{y} = a + b\bar{x}$.

This shows that $(\bar{x}, \bar{y})$, *i.e.*, the means of x and y, lie on (1).

Shifting the origin to $(\bar{x}, \bar{y})$, (3) takes the form

$$\Sigma(x-\bar{x})(y-\bar{y}) = a\Sigma(x-\bar{x}) + b\Sigma(x-\bar{x})^2, \text{ but } a\Sigma(x-\bar{x}) = 0,$$

$$\therefore \quad b = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(x-\bar{x})^2} = \frac{\Sigma XY}{\Sigma X^2} = \frac{\Sigma XY}{n\sigma_x^2} = r\frac{\sigma_y}{\sigma_x} \qquad \left[\because r = \frac{\Sigma XY}{n\sigma_x\sigma_y}\right]$$

Thus the line of best fit becomes $y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$ ...(4)

which is the equation of *the line of regression of y on x*. Its slope is called the *regression coefficient of y on x*.

Interchanging x and y, we find that the line of regression of x on y is

$$x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y}) \qquad ...(5)$$

Thus the *regression coefficient of y on x* $= r\sigma_y/\sigma_x$ ...(6)

and the *regression coefficient of x on y* $= r\sigma_x/\sigma_y$ ...(7)

Cor. *The correlation coefficient r is the geometric mean between the two regression co-efficients.*

For $\quad r\frac{\sigma_y}{\sigma_x} \times r\frac{\sigma_x}{\sigma_y} = r^2$.

Example 25.15. *The two regression equations of the variables x and y are x = 19.13 – 0.87y and y = 11.64 – 0.50x. Find (i) mean of x's, (ii) mean of y's and (iii) the correlation coefficient between x and y.*

(V.T.U., 2004 ; Anna, 2003 ; Burdwan, 2003)

Solution. Since the mean of x's and the mean of y's lie on the two regression lines, we have

$$\bar{x} = 19.13 - 0.87\bar{y} \qquad ...(i)$$

$$\bar{y} = 11.64 - 0.50\bar{x} \qquad ...(ii)$$

Multiplying (*ii*) by 0.87 and subtracting from (*i*), we have

$$[1 - (0.87)(0.50)]\ \bar{x} = 19.13 - (11.64)(0.87) \text{ or } 0.57\ \bar{x} = 9.00 \text{ or } \bar{x} = 15.79$$

$$\therefore \quad \bar{y} = 11.64 - (0.50)(15.79) = 3.74$$

$\therefore$ regression coefficient of y on x is – 0.50 and that of x on y is – 0.87.

Now since the coefficient of correlation is the geometric mean between the two regression coefficients.

$$\therefore \quad r = \sqrt{[(-0.50)(-0.87)]} = \sqrt{(0.43)} = -0.66.$$

[– ve sign is taken since both the regression coefficients are – ve]

Example 25.16. *In the following table are recorded data showing the test scores made by salesmen on an intelligence test and their weekly sales :*

Salesmen	1	2	3	4	5	6	7	8	9	10
Test scores	40	70	50	60	80	50	90	40	60	60
Sales (000)	2.5	6.0	4.5	5.0	4.5	2.0	5.5	3.0	4.5	3.0

Calculate the regression line of sales on test scores and estimate the most probable weekly sales volume if a salesman makes a score of 70.

Solution. With the help of the table below, we have

$$\bar{x} = \text{mean of } x \text{ (test scores)} = 60 + 0/10 = 60$$

$$\bar{y} = \text{mean of } y \text{ (sales)} = 4.5 + (-4.5)/10 = 4.05.$$

Regression line of sales (y) on scores (x) is given by

$$y - \bar{y} = r(\sigma_y / \sigma_x)(x - \bar{x})$$

where

$$r\frac{\sigma_y}{\sigma_x} = \frac{\Sigma XY}{\sigma_x \sigma_y} \times \frac{\sigma_y}{\sigma_x} = \frac{\Sigma XY}{(\sigma_x)^2} = \left[\Sigma d_x d_y - \frac{\Sigma d_x \Sigma d_y}{n}\right] \Big/ \left[\Sigma d_x^2 - (\Sigma d_x)^2 / n\right]$$

$$= \frac{140 - \dfrac{0 \times (-4.5)}{10}}{2400 - 0^2/10} = \frac{140}{2400} = 0.06$$

$\therefore$ the required regression line is

$$y - 4.05 = 0.06(x - 60) \quad \text{or} \quad y = 0.06x + 0.45.$$

For $x = 70, y = 0.06 \times 70 + 0.45 = 4.65.$

Thus the most probable weekly sales volume for a score of 70 is 4.65.

Test scores x	*Sales* y	*Deviation of x from assumed mean (= 60)* d_x	*Deviation of y from assumed average (= 4.5)* d_y	$d_x \times d_y$	d_x^2	d_y^2
40	2.5	− 20	− 2	40	400	4
70	6.0	10	1.5	15	100	2.25
50	4.5	− 10	0	0	100	0
60	5.0	0	0.5	0	0	2.25
80	4.5	20	0	0	400	0
50	2.0	− 10	− 2.5	25	100	6.25
90	5.5	30	1	30	900	1.00
40	3.0	− 20	− 1.5	30	400	2.25
60	4.5	0	0	0	0	0
60	3.0	0	− 1.5	0	0	2.25
		$\Sigma d_x = 0$	$\Sigma d_y = -4.5$	$\Sigma d_x d_y = 140$	$\Sigma d_x^2 = 2400$	$\Sigma d_y^2 = 18.25$

Example 25.17. *If θ is the angle between the two regression lines, show that*

$$\tan\theta = \frac{1 - r^2}{r} \cdot \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}$$

Explain the significance when $r = 0$ and $r = \pm 1$. (U.P.T.U., 2007 ; V.T.U., 2007)

Solution. The equations to the line of regression of y on x and x on y are

$$y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x}) \text{ and } x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y})$$

$\therefore$ their slopes are $m_1 = r\sigma_y/\sigma_x$ and $m_2 = \sigma_y/r\sigma_x$

Thus

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{\sigma_y / r\sigma_x - r\sigma_y / \sigma_x}{1 + \sigma_y^2 / \sigma_x^2} = \frac{1 - r^2}{r} \cdot \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}$$

When $r = 0$, $\tan\theta \to \infty$ or $\theta = \pi/2$ i.e. when the variables are independent, the two lines of regression are perpendicular to each other.

When $r = \pm 1$, $\tan\theta = 0$ i.e., $\theta = 0$ or π. Thus the lines of regression coincide i.e., there is perfect correlation between the two variables.

Example 25.18. *In a partially destroyed laboratory record, only the lines of regression of y on x and x on y are available as $4x - 5y + 33 = 0$ and $20x - 9y = 107$ respectively. Calculate $\bar{x}, \bar{y}$ and the coefficient of co-relation between x and y.* (S.V.T.U., 2009 ; U.P.T.U., 2009 ; V.T.U., 2005)

Solution. Since the regression lines pass through $(\bar{x}, \bar{y})$, therefore,

$$4\bar{x} - 5\bar{y} + 33 = 0, \quad 20\bar{x} - 9\bar{y} = 107.$$

Solving these equations, we get $\bar{x} = 13$, $\bar{y} = 17$.

Rewriting the line of regression of y on x as $y = \frac{4}{5}x + \frac{33}{5}$, we get

$$b_{yx} = r\frac{\sigma_y}{\sigma_x} = \frac{4}{5} \qquad ...(i)$$

Rewriting the line of regression of x on y as $x = \frac{9}{20}y + \frac{107}{9}$, we get

$$b_{xy} = r\frac{\sigma_x}{\sigma_y} = \frac{9}{20} \qquad ...(ii)$$

Multiplying (*i*) and (*ii*), we get

$$r^2 = \frac{4}{5} \times \frac{9}{20} = 0.36 \qquad \therefore \qquad r = 0.6$$

Hence $r = 0.6$, the positive sign being taken as b_{yx} and b_{xy} both are positive.

Example 25.19. *Establish the formula* $r = \dfrac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\sigma_x\sigma_y}$

Hence calculate r from the following data :

x :	21	23	30	54	57	58	72	78	87	90
y :	60	71	72	83	110	84	100	92	113	135

(U.P.T.U., 2002)

Solution. (*a*) Let $z = x - y$ so that $\bar{z} = \bar{x} - \bar{y}$.

$\therefore$ $$z - \bar{z} = (x - \bar{x}) - (y - \bar{y})$$

or $$(z - \bar{z})^2 = (x - \bar{x})^2 + (y - \bar{y})^2 - 2(x - \bar{x})(y - \bar{y})$$

Summing up for *n* terms, we have

$$\Sigma(z - \bar{z})^2 = \Sigma(x - \bar{x})^2 + \Sigma(y - \bar{y})^2 - 2\Sigma(x - \bar{x})(y - \bar{y})$$

or $$\frac{\Sigma(z - \bar{z})^2}{n} = \frac{\Sigma(x - \bar{x})^2}{n} + \frac{\Sigma(y - \bar{y})^2}{n} - 2\frac{\Sigma(x - \bar{x})(y - \bar{y})}{n}$$

i.e., $$\sigma_z^2 = \sigma_x^2 + \sigma_y^2 - 2r\sigma_x\sigma_y \qquad \left[\because r = \frac{\Sigma(x - \bar{x})(y - \bar{y})}{n\sigma_x\sigma_y}\right.$$

which is the required result.

(*b*) To fine *r*, we have to calculate σ_x, σ_y and σ_{x-y}. We make the following table :

x	$X = x - 54$	X^2	y	$Y = y - 100$	Y^2	$y - x$	$(x - y)^2$
21	– 33	1089	60	– 40	1600	39	1521
23	– 31	961	71	– 29	841	48	2304
30	– 24	576	72	– 28	784	42	1764
54	0	0	83	– 17	289	29	841
57	3	9	110	10	100	53	2809
58	4	16	84	– 16	256	26	676
72	18	324	100	0	0	28	784
78	24	576	92	– 8	64	14	196
87	33	1089	113	13	169	26	676
90	36	1296	135	35	1225	45	2025
Total	30	5936		– 80	5328	350	13596

$$\therefore \qquad \sigma_x^2 = \frac{\Sigma X^2}{N} - \left(\frac{\Sigma X}{N}\right)^2 = \frac{5636}{10} - \left(\frac{30}{10}\right)^2 = 593.6 - 9 = 584.6$$

$$\sigma_y^2 = \frac{\Sigma Y^2}{N} - \left(\frac{\Sigma Y}{N}\right)^2 = \frac{5328}{10} - \left(\frac{-80}{10}\right)^2 = 532.8 - 64 = 468.8$$

$$\sigma^2_{x-y} = \frac{\Sigma(x-y)^2}{N} - \left\{\frac{\Sigma(x-y)}{N}\right\}^2 = 1359.6 - 1225 = 134.6$$

From the above formula,

$$r = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\sigma_x\sigma_y} = \frac{584.6 + 468.8 - 134.6}{2 \times 24.18 \times 23.85} = 0.876.$$

Example 25.20. *While calculating correlation coefficient between two variables x and y from 25 pairs of observations, the following results were obtained : $n = 25$, $\Sigma x = 125$, $\Sigma x^2 = 650$, $\Sigma y = 100$, $\Sigma y^2 = 460$, $\Sigma xy = 508$. Later it was discovered at the time of checking that the pairs of values*

x	y
8	12
6	8

were copied down as

x	y
6	14
8	6

Obtain the correct value of correlation coefficient. (V.T.U., 2011 S ; S.V.T.U., 2009)

Solution. To get the correct results, we subtract the incorrect values and add the corresponding correct values.

$\therefore$ The correct results would be

$$\Sigma n = 25, \Sigma x = 125 - 6 - 8 + 8 + 6 = 125, \Sigma x^2 = 650 - 6^2 - 8^2 + 8^2 + 6^2 = 650$$

$$\Sigma y = 100 - 14 - 6 + 12 + 8 = 100, \Sigma y^2 = 460 - 14^2 - 6^2 + 12^2 + 8^2 = 436$$

$$\Sigma xy = 508 - 6 \times 14 - 8 \times 6 + 8 \times 12 + 6 \times 8 = 520$$

$$r = \frac{n\Sigma xy - (\Sigma x)(\Sigma y)}{\sqrt{[\{n\Sigma x^2 - (\Sigma x)^2\}\{n\Sigma y^2 - (\Sigma y^2)\}]}} = \frac{25 \times 520 - 125 \times 100}{\sqrt{[\{25 \times 650 - (125)^2\}\{25 \times 436 - (100)^2\}]}}$$

$$= \frac{20}{\sqrt{(25 \times 36)}} = \frac{2}{3}.$$

25.15 STANDARD ERROR OF ESTIMATE

The sum of the squares of the deviations of the points from the line of regression of y on x is

$$\Sigma(y - a - bx)^2 = \Sigma(Y - bX)^2, \text{ where } X = x - \bar{x}, Y = y - \bar{y}$$

$$= \sum\left(Y - r\frac{\sigma_y}{\sigma_x}X\right)^2 = \Sigma Y^2 - 2r(\sigma_y/\sigma_x)\,\Sigma XY + r^2(\sigma_y^2/\sigma_x^2)\,\Sigma X^2$$

$$= n\sigma_y^2 - 2r(\sigma_y/\sigma_x)\,r\,.\,n\sigma_x\sigma_y + r^2(\sigma_y^2/\sigma_x^2)\,.\,n\sigma_x^2 = n\sigma_y^2(1 - r^2).$$

Denoting this sum of squares by nS_y^2, we have $S_y = \sigma_y\sqrt{(1 - r^2)}$ ...(1)

Since S_y is the root mean square deviation of the points from the regression line of y on x, it is called the *standard error of estimate* of y. Similarly the standard error of estimate of x is given by

$$S_x = \sigma_x\sqrt{(1 - r^2)} \qquad ...(2)$$

Since the sum of the squares of deviations cannot be negative, it follows that

$$r^2 \le 1 \quad \text{or} \quad -1 \le r \le 1.$$

i.e., correlation coefficient lies between – 1 and 1. (J.N.T.U., 2006)

If $r = 1$ or -1, the sum of the squares of deviations from either line of regression is zero. Consequently each deviation is zero and all the points lie on both the lines of regression. These two lines coincide and we say that the correlation between the variables is *perfect*. The nearer r^2 is to unity the closer are the points to the lines of

regression. Thus *the departure of r^2 from unity is a measure of departure from linearity of the relationship between the variables.*

25.16 RANK CORRELATION

A group of n individuals may be arranged in order to merit with respect to some characteristic. The same group would give different orders for different characteristics. Considering the orders corresponding to two characteristics A and B, the correlation between these n pairs of ranks is called the *rank correlation* in the characteristics A and B for that group of individuals.

Let x_i, y_i be the ranks of the ith individuals in A and B respectively. Assuming that no two individuals are bracketed equal in either case, each of the variables taking the values 1, 2, 3, ..., n, we have

$$\bar{x} = \bar{y} = \frac{1+2+3+\ldots+n}{n} = \frac{n(n+1)}{2n} = \frac{n+1}{2}$$

If X, Y be the deviations of x, y from their means, then

$$\Sigma X_i^2 = \Sigma(x_i - \bar{x})^2 = \Sigma x_i^2 + n(\bar{x})^2 - 2\bar{x}\,\Sigma x_i = \Sigma n^2 + \frac{n(n+1)^2}{4} - 2\frac{n+1}{2}\,.\,\Sigma n$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)^2}{4} - \frac{n(n+1)^2}{2} = \frac{1}{12}(n^3 - n)$$

Similarly $\quad \Sigma Y_i^2 = \frac{1}{12}(n^3 - n)$

Now let $\quad d_i = x_i - y_i \quad$ so that $\quad d_i = (x_i - \bar{x}) - (y_i - \bar{y}) = X_i - Y_i$

$\therefore \quad \Sigma d_i^2 = \Sigma X_i^2 + \Sigma Y_i^2 - 2\Sigma X_i Y_i$

or $\quad \Sigma X_i Y_i = \frac{1}{2}(\Sigma X_i^2 + \Sigma Y_i^2 - \Sigma d_i^2) = \frac{1}{12}(n^3 - n) - \frac{1}{2}\Sigma d_i^2.$

Hence the correlation coefficient between these variables is

$$r = \frac{\Sigma X_i Y_i}{\sqrt{(\Sigma X_i^2 \Sigma Y_i^2)}} = \frac{\frac{1}{12}(n^3 - n) - \frac{1}{2}\Sigma d_i^2}{\frac{1}{12}(n^3 - n)} = 1 - \frac{6\,\Sigma d_i^2}{n^3 - n}$$

This is called the *rank correlation coefficient* and is denoted by ρ.

Example 25.21. *Ten participants in a contest are ranked by two judges as follows :*

x :	1	6	5	10	3	2	4	9	7	8
y :	6	4	9	8	1	2	3	10	5	7

Calculate the rank correlation coefficient ρ. (V.T.U., 2002)

Solution. If $\quad d_i = x_i - y_i$, then $d_i = -5, 2, -4, 2, 2, 0, 1, -1, 2, 1$

$\therefore \quad \Sigma d_i^2 = 25 + 4 + 16 + 4 + 4 + 0 + 1 + 1 + 4 + 1 = 60$

Hence $\quad \rho = 1 - \frac{6\Sigma d_i^2}{n^3 - n} = 1 - \frac{6 \times 60}{990} = 0.6$ nearly.

Example 25.22. *Three judges, A, B, C, give the following ranks. Find which pair of judges has common approach*

A :	1	6	5	10	3	2	4	9	7	8
B :	3	5	8	4	7	10	2	1	6	9
C :	6	4	9	8	1	2	3	10	5	7

(J.N.T.U., 2003)

Solution. Here $n = 10$.

$A\ (=x)$	Ranks by $B\ (=y)$	$C\ (=z)$	d_1 $x-y$	d_2 $y-z$	d_3 $z-x$	d_1^2	d_2^2	d_3^2
1	3	6	−2	−3	5	4	9	25
6	5	4	1	1	−2	1	1	4
5	8	9	−3	−1	4	9	1	16
10	4	8	6	−4	−2	36	16	4
3	7	1	−4	6	−2	16	36	4
2	10	2	−8	8	0	64	64	0
4	2	3	2	−1	−1	4	1	1
9	1	10	8	−9	1	64	81	1
7	6	5	1	1	−2	1	1	4
8	9	7	−1	2	−1	1	4	1
Total			0	0	0	200	214	60

$$\therefore \qquad \rho(x, y) = 1 - \frac{6\Sigma d_1^2}{n(n^2-1)} = 1 - \frac{6 \times 200}{10 \times 99} = -0.2$$

$$\rho(y, z) = 1 - \frac{6\Sigma d_2^2}{n(n^2-1)} = 1 - \frac{6 \times 214}{10 \times 99} = -0.3$$

$$\rho(z, x) = 1 - \frac{6\Sigma d_3^2}{n(n-1)} = 1 - \frac{6 \times 60}{10 \times 99} = 0.6$$

Since $\rho(z, x)$ is maximum, the pair of judges A and C have the nearest common approach.

PROBLEMS 25.4

1. Find the correlation co-efficient and the regression lines of y and x and x on y for the following data :

x :	1	2	3	4	5
y :	2	5	3	8	7

(V.T.U., 2010)

2. Find the correlation coefficient between x and y from the given data :

x :	78	89	97	69	59	79	68	57
y :	125	137	156	112	107	138	123	108

(J.N.T.U., 2005)

3. Find the co-efficient of correlation between industrial production and export using the following data and comment on the result.

Production (in crore tons) :	55	56	58	59	60	60	62
Exports (in crore tons) :	35	38	38	39	44	43	45

(Madras, 2000)

4. Ten people of various heights as under, were requested to read the letters on a car at 25 yards distance. The number of letters correctly read is given below :

Height (in feet) :	5.1	5.3	5.6	5.7	5.8	5.9	5.10	5.11	6.0	6.1
No. of letters :	11	17	19	14	8	15	20	6	8	12

Is there any correlation between heights and visual power ?

5. Using the formula $r = \dfrac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\sigma_x\sigma_y}$, find r from the following data :

x :	92	89	87	86	83	77	71	63	53	50
y :	86	88	91	77	68	85	52	82	37	57

6. Find the correlation between x (marks in Mathematics) and y (marks in Engineering Drawing) given in the following data :

y \ x	10—40	40—70	70—100	Total
0—30	5	20	—	25
30—60	—	28	2	30
60—90	—	32	13	45
Total	5	80	15	100

7. Find two lines of regression and coefficient of correlation for the data given below :
$n = 18, \Sigma x = 12, \Sigma y = 18, \Sigma x^2 = 60, \Sigma y^2 = 96, \Sigma xy = 48.$ *(U.P.T.U., MCA, 2009)*
8. If the coefficient of correlation between two variables x and y is 0.5 and the acute angle between their lines of regression is $\tan^{-1}(3/8)$, show that $\sigma_x = \frac{1}{2}\sigma_y$. *(V.T.U., 2004)*
9. For two random variables x and y with the same mean, the two regression lines are $y = ax + b$ and $x = \alpha y + \beta$. Show that $\frac{b}{\beta} = \frac{1-a}{1-\alpha}$. Find also the common mean. *(U.P.T.U., 2010)*
10. Two random variables have the regression lines with equations $3x + 2y = 26$ and $6x + y = 31$. Find the mean values and the correlation coefficient between x and y. *(Madras, 2002)*
11. The regression equations of two variables x and y are $x = 0.7y + 5.2$, $y = 0.3x + 2.8$. Find the means of the variables and the coefficient of correlation between them. *(Osmania, 2002)*
12. In a partially destroyed laboratory data, only the equations giving the two lines of regression of y on x and x on y are available and are respectively, $7x - 16y + 9 = 0$, $5y - 4x - 3 = 0$. Calculate the co-efficient of correlation, $\bar{x}$ and $\bar{y}$.
13. The following results were obtained from records of age (x) and blood pressure (y) of a group of 10 men :

	x	y
Mean	53	142
Variance	130	165

and $\Sigma(x - \bar{x})(y - \bar{y}) = 1220.$

Find the appropriate regression equation and use it to estimate the blood pressure of a man whose age is 45.
14. Compute the standard error of estimate S_x for the respective heights of the following 12 couples :

Height x of husband (inches) :	68	66	68	65	69	66	68	65	71	67	68	70
Height y of wife (inches) :	65	63	67	64	68	62	70	66	68	67	69	71

15. Calculate the rank correlation coefficient from the following data showing ranks of 10 students in two subjects :

Maths :	3	8	9	2	7	10	4	6	1	5
Physics :	5	9	10	1	8	7	3	4	2	6

16. Find the rank correlation for the following data :

x :	56	42	72	36	63	47	55	49	38	42	68	60
y :	147	125	160	118	149	128	150	145	115	140	152	155

(S.V.T.U., 2009 ; J.N.T.U., 2003)

25.17 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 25.5

Select the correct answer or fill up the blanks in each of the following questions :

1. The median of the numbers 11, 10, 12, 13, 9 is
(*a*) 12.5 (*b*) 12 (*c*) 10.5 (*d*) 11.
2. The mode of the numbers 7, 7, 7, 9, 10, 11, 11, 11, 12 is
(*a*) 11 (*b*) 12 (*c*) 7 (*d*) 7 and 11.

3. S.D. is defined as

(a) $\sqrt{\frac{\Sigma f(x-\bar{x})^2}{\Sigma f}}$ (b) $\frac{\Sigma f(x-\bar{x})}{\Sigma f}$ (c) $\frac{\Sigma f(x-\bar{x})^2}{\Sigma f}$.

4. Coefficient of variation is

(a) $\frac{\sigma}{\bar{x}} \times 100$ (b) $\frac{\sigma}{x}$ (c) $\sqrt{\frac{\sigma^2}{x}} \times 100$.

5. Average scores of three batsman A, B, C are respectively 40, 45 and 55 and their S.D.s are respectively 9, 11, 16. Which batsman is more consistent ?

(a) A (b) B (c) C.

6. The equations of regression lines are $y = 0.5x + a$ and $x = 0.4y + b$. The correlation coefficient is

(a) $\sqrt{0.2}$ (b) 0.45 (c) $-\sqrt{0.2}$.

7. If the correlation coefficient is 0, the two regression lines are

(a) parallel (b) perpendicular (c) coincident (d) inclined at 45° to each other.

8. If r_1 and r_2 are two regression coefficients, then signs of r_1 and r_2 depend on .....

9. Regression coefficient of y on x is 0.7 and that of x on y is 3.2. Is the correlation coefficient r consistent ?

10. The standard deviation of the numbers 24, 48, 64, 36, 53 is .....

11. If $y = x + 1$ and $x = 3y - 7$ are the two lines of regression then $\bar{x}$ = ..., $\bar{y}$ = ... and r = ...

12. If the two regression lines are perpendicular to each other, then their coefficient of correlation is .....

13. Quartile deviation is defined as .....

14. The minimum value of correlation coefficient is .....

15. Prediction error of Y is defined as .....

16. If X and Y are independent, then the correlation coefficient between X and Y is .....

17. The point of intersection of the two regression lines is .....

18. The smaller the coefficient of variation, the greater is the ..... in the data.

19. The moment coefficient of skewness is given by .....

20. Kurtosis measures the ..... of a distribution.

21. The equation of the line of regression of y on x is .....

22. Coefficient of variation = ...

23. The angle between two regression lines is given by .....

24. A frequency curve is said to be Mesokurtic when β_2 is .....

25. Correlation coefficient is the geometrical mean between .....

26. When the variables are independent, the two lines of regression are .....

27. Arithmetic mean of the coefficients of regression is ..... than the coefficient of correlation.

28. If two regression lines coincide then the coefficient of correlation is .....

29. The rank coefficient is given by .....

30. The ratio of the standard deviation to the mean is known as .....

31. The value of $\Sigma f(x - \bar{x})$ = ...

32. The value of coefficient of correlation lies between ..... and .....

33. If the two regression coefficients are – 0.4 and – 0.9, then the correlation coefficient is .....

34. A distribution with the following constants is positively skew : $Q_1 = 25.8$, median = 49.0, $Q_3 = 64.2$.

(True or False)

35. Quartile coefficient of skewness is $\frac{Q_3 + Q_1 - 2Q_2}{Q_3 - Q_1}$. (True or False)

36. Skewness indicates peakedness of the frequency distribution. (True or False)

CHAPTER 26

Probability and Distributions

1. Introduction, Principle of counting, Permutations and Combinations. 2. Basic terminology, Definition of probability. 3. Probability and Set notations. 4. Addition law of probability. 5. Independent events — Multiplication law of probability. 6. Baye's theorem. 7. Random variable. 8. Discrete probability distribution. 9. Continuous probability distribution. 10. Expectation, Variance, Moments. 11. Moment generating function. 12. Probability generating function. 13. Repeated trials. 14. Binomial distribution. 15. Poisson distribution. 16. Normal distribution. 17. Probable error. 18. Normal approximation to Binomial distribution. 19. Some other distributions. 20. Objective Type of Questions.

26.1 (1) INTRODUCTION

We often hear such statements : 'It is likely to rain today', 'I have a fair chance of getting admission', and 'There is an even chance that in tossing a coin the head may come up'. In each case, we are not certain of the outcome, but we wish to assess the chances of our predictions coming true. The study of probability provides a mathematical framework for such assertions and is essential in every decision making process. Before defining probability, let us explain a few terms :

(2) Principle of counting. If an event can happen in n_1 ways and thereafter for each of these events a second event can happen in n_2 ways, and for each of these first and second events a third event can happen for n_3 ways and so on, then the number of ways these m event can happen is given by the product $n_1 \,.\, n_2 \,.\, n_3 \ldots n_m$.

(3) Permutations. A permutation of a number of objects is their arrangement in some definite order. Given three letters a, b, c, we can permute them two at a time as "bc, cb ; ca, ac; ab, ba" yielding 6 permutations. The combinations or groupings are only 3, *i.e.*, bc, ca, ab. Here the order is immetrial.

The number of permutations of n different thing taken r at a time is

$$n\,(n-1)\,(n-2)\ldots(n-r+1), \text{ which is denoted by } {}^nP_r.$$

Thus $$ {}^nP_r = n\,(n-1)\,(n-2)\ldots(n-r+1) = \frac{n\,!}{(n-r)!}$$

Permutations with repetitions. The number of permutations of n objects of which n_1 are alike, n_2 are alike and n_3 are alike is $\dfrac{n!}{n_1!\,n_2!\,n_3!}$.

(4) Combinations. *The number of combinations of n different objects taken r at a time is denoted by* ${}^nC_r$. If we take any one of the combinations, its r objects can be arranged in $r!$ ways. So the total number of arrangements which can be obtained from all the combinations is ${}^nP_r = {}^nC_r \,.\, r!$.

Thus $$ {}^nC_r = \frac{{}^nP_r}{r!} = \frac{n!}{r!\,(n-r)!}$$

Also $$ {}^nC_{n-r} = {}^nC_r$$

e.g., $$ {}^{25}P_4 = 25 \times 24 \times 23 \times 22;\ {}^{25}C_{21} = {}^{25}C_4 = \frac{25 \times 24 \times 23 \times 22}{4 \times 3 \times 2 \times 1}.$$

Example 26.1. *In how many ways can one make a first, second, third and fourth choice among 12 firms leasing construction equipment.* (*J.N.T.U., 2003*)

Solution. First choice can be made from any of the 12 firms. Thereafter the second choice can be made from among the remaining 11 firms. Then the third choice can be made from the remaining 10 firms and the fourth choice can be made from the 9 firms.

Thus from the *principle of counting*, the number of ways in which first, second, third and fourth choice can be affected = 12 × 11 × 10 × 9 = 11880.

Example 26.2. *Find the number of permutations of all the letters of the word (i) Committee (ii) Engineering.*

Solution. (*i*) $n = 9, n_1 (m, m) = 2, n_2 (t, t) = 2, n_3 (e, e) = 2$

$\therefore$ no. of permutations $= \dfrac{n!}{n_1!\,.\,n_2!\,.\,n_3!} = \dfrac{9!}{2!\,.\,2!\,.\,2!} = 45360.$

(*ii*) $n = 11, n_1(e's) = 3, n_2 (g, g) = 2, n_3 (i, i) = 2, n_4 (n's) = 3$

$\therefore$ no. of permutations $= \dfrac{11!}{3!\,2!\,2!\,3!} = 277200.$

Example 26.3. *From six engineers and five architects a committee is to be formed having three engineers and two architects. How many different committees can be formed if (i) there is no restriction. (ii) two particular engineers must be included. (iii) one particular architect must be excluded.*

Solution. (*i*) Number of committees ${}^6C_3 \times {}^5C_2 = \dfrac{6\,.\,5\,.\,4}{3\,.\,2\,.\,1} \times \dfrac{5\,.\,4}{2\,.\,1} = 200.$

(*ii*) Here we have to choose one engineer from the remaining four engineers.

$\therefore$ no. of committees $= {}^4C_1 \times {}^5C_2 = 4 \times \dfrac{5\,.\,4}{2\,.\,1} = 40$

(*iii*) Here we have to choose two architects from the remaining four architects.

$\therefore$ no. of committees $= {}^6C_3 \times {}^4C_2 = \dfrac{6\,.5\,.\,4}{3\,.2\,.\,1} \times \dfrac{4\,.\,3}{2\,.\,1} = 120.$

PROBLEMS 26.1

1. If a test consists of 12 true-false questions, in how many different ways can a student make the test paper with one answer to each question. (*J.N.T.U., 2003*)
2. How many 4-digit numbers can be formed from the six digits 2, 3, 5, 6, 7 and 9, without repetition ? How many of these are less than 500 ?
3. A student has to answer 9 out of 12 questions. How many choices has he (*i*) if he must answer first two questions (*ii*) if he must answer at least four of the first five questions.
4. How many car number plates can be made if each plate contains two different letters followed by three different digits ? Solve the problem (*a*) with repetitions and (*b*) without repetitions.

26.2 (I) BASIC TERMINOLOGY

(*i*) **Exhaustive events.** A set of events is said to be *exhaustive*, if it includes all the possible events. For example, in tossing a coin there are two exhaustive cases either head or tail and there is no third possibility.

(*ii*) **Mutually exclusive events.** If the occurrence of one of the events procludes the occurrence of all other, then such a set of events is said to be *mutually exclusive*. Just as tossing a coin, either head comes up or the tail and both can't happen at the same time, *i.e.*, these are two mutually exclusive cases.

(*iii*) **Equally likely events.** If one of the events cannot be expected to happen in preference to another then such events are said to be *equally likely*. For instance, in tossing a coin, the coming of the head or the tail is equally likely.

Thus when a *die** is thrown, the turning up of the six different faces of the die are exhaustive, mutually exclusive and equally likely.

(*iv*) **Odds in favour of an event.** If the number of ways favourable to an event A is m and the number of ways not favourable to A is n then *odds in favour of* $A = m/n$ and *odds against* $A = n/m$.

(2) Definition of probability. *If there are n exhaustive, mutually exclusive and equally likely cases of which m are favourable to an event A, then probability (p) of the happening of A is*

$$\mathbf{P(A) = m/n.}$$

As there are $n - m$ cases in which A will not happen (denoted by A'), *the chance of A not happening is q or* $P(A')$ *so that*

$$q = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - p$$

i.e., $P(A') = 1 - P(A)$ so that $\mathbf{P(A) + P(A') = 1,}$

i.e., if an event is certain to happen then its probability is unity, while if it is certain not to happen, its probability is zero.

Obs. This definitions of probability fails when

(*i*) number of outcomes is infinite (not exhaustive) and (*ii*) outcomes are not equally likely.

(3) Statistical (or Empirical) definition of probability. *If in n trials, an event A happens m times, then the probability (p) of happening of A is given by*

$$p = P(A) = \underset{n \to \infty}{Lt} \frac{m}{n}$$

Example 26.4. *Find the chance of throwing (a) four, (b) an even number with an ordinary six faced die.*

Solution. (*a*) There are six possible ways in which the die can fall and of these there is only one way of throwing 4. Thus the required chance $= \frac{1}{6}$.

(*b*) There are six possible ways in which the die can fall. Of these there are only 3 ways of getting 2, 4 or 6. Thus the required chance $= 3/6 = \frac{1}{2}$.

Example 26.5. *What is the chance that a leap year selected at random will contain 53 Sundays ?* (*Madras, 2003*)

Solution. A leap year consists of 366 days, so that there are 52 full weeks (and hence 52 Sundays) and two extra days. These two days can be (*i*) Monday, Tuesday (*ii*) Tuesday, Wednesday, (*iii*) Wednesday, Thursday (*iv*) Thursday, Friday (*v*) Friday, Saturday (*vi*) Saturday, Sunday (*vii*) Sunday, Monday.

Of these 7 cases, the last two are favourable and hence the required probability $= \frac{2}{7}$.

Example 26.6. *A five figure number is formed by the digits 0, 1, 2, 3, 4 without repetition. Find the probability that the number formed is divisible by 4.*

Solution. The five digits can be arranged in 5! ways, out of which 4! will begin with zero.

$\therefore$ total number of 5-figure numbers formed $= 5! - 4! = 96$.

Those numbers formed will be divisible by 4 which will have two extreme right digits divisible by 4, *i.e.*, numbers ending in 04, 12, 20, 24, 32, 40.

Now numbers ending in $04 = 3! = 6$, numbers ending in $12 = 3! - 2! = 4$,
numbers ending in $20 = 3! = 6$, numbers ending in $24 = 3! - 2! = 4$,
numbers ending in $32 = 3! - 2! = 4$, and numbers ending in $40 = 3! = 6$.

[The numbers having 12, 24, 32 in the extreme right are $(3! - 2!)$ since the numbers having zero on the extreme left are to excluded.]

* *Die* is a small cube. Dots 1, 2, 3, 4, 5, 6 are marked on its six faces. The outcome of throwing a die is the number of dots on its upper face.

$\therefore$ total number of favourable ways = 6 + 4 + 6 + 4 + 4 + 6 = 30.

Hence the required probability = $\frac{30}{96} = \frac{5}{16}$.

Example 26.7. *A bag contains 40 tickets numbered 1, 2, 3, ... 40, of which four are drawn at random and arranged in ascending order ($t_1 < t_2 < t_3 < t_4$). Find the probability of t_3 being 25?*

Solution. Here exhaustive number of cases = ${}^{40}C_4$

If $t_3 = 25$, then the tickets t_1 and t_2 must come out of 24 tickets numbered 1 to 24. This can be done in ${}^{24}C_2$ ways.

Then t_4 must come out of the 15 tickets (numbering 25 to 40) which can be done in ${}^{15}C_1$ ways.

$\therefore$ favourable number of cases = ${}^{24}C_2 \times {}^{15}C_1$

Hence the probability of t_3 being 25 = $\frac{{}^{24}C_2 \times {}^{15}C_1}{{}^{40}C_4} = \frac{414}{9139}$.

Example 26.8. *An urn contains 5 red and 10 black balls. Eight of them are placed in another urn. What is the chance that the latter then contains 2 red and 6 black balls ?*

Solution. The number of ways in which 8 balls can be drawn out of 15 is ${}^{15}C_8$.

The number of ways of drawing 2 red balls is ${}^5C_2$ and corresponding to each of these ${}^5C_2$ ways of drawing a red ball, there are ${}^{10}C_6$ ways of drawing 6 black balls.

$\therefore$ the total number of ways in which 2 red and 6 black balls can be drawn is ${}^5C_2 \times {}^{10}C_6$.

$\therefore$ the required probability = $\frac{{}^5C_2 \times {}^{10}C_6}{{}^{15}C_8} = \frac{140}{429}$.

Example 26.9. *A committee consists of 9 students two of which are from 1st year, three from 2nd year and four from 3rd year. Three students are to be removed at random. What is the chance that (i) the three students belong to different classes, (ii) two belong to the same class and third to the different class, (iii) the three belong to the same class ?* (V.T.U., 2002 S)

Solution. (*i*) The total number of ways of choosing 3 students out of 9 is ${}^9C_3$, *i.e.*, 84.

A student can be removed from 1st year students in 2 ways, from 2nd year in 3 ways and from 3rd year in 4 ways, so that the total number of ways of removing three students, one from each group is $2 \times 3 \times 4$.

Hence the required chance = $\frac{2 \times 3 \times 4}{{}^9C_3} = \frac{24}{84} = \frac{2}{7}$.

(*ii*) The number of ways of removing two from 1st year students and one from others

$$= {}^2C_2 \times {}^7C_1.$$

The number of ways of removing two from 2nd year students and one from others

$$= {}^3C_2 \times {}^6C_1.$$

The number of ways of removing 2 from 3rd year students and one from others

$$= {}^4C_2 \times {}^5C_1.$$

$\therefore$ the total number of ways in which two students of the same class and third from the others may be removed

$$= {}^2C_2 \times {}^7C_1 + {}^3C_2 \times {}^6C_1 + {}^4C_2 \times {}^5C_1 = 7 + 18 + 30 = 55.$$

Hence, the required chance = $\frac{55}{84}$.

(*iii*) Three students can be removed from 2nd year group in ${}^3C_3$, *i.e.* 1 way and from 3rd year group in ${}^4C_3$, *i.e.*, 4 ways.

$\therefore$ the total number of ways in which three students belong to the same class = 1 + 4 = 5.

Hence the required chance = $\frac{5}{84}$.

Example 26.10. *A has one share in a lottery in which there is 1 prize and 2 blanks ; B has three shares in a lottery in which there are 3 prizes and 6 blanks ; compare the probability of A's success to that of B's success.*

Solution. A can draw a ticket in ${}^3C_1 = 3$ ways.

The number of cases in which A can get a prize is clearly 1.

$\therefore$ the probability of A's success $= \frac{1}{3}$.

Again B can draw a ticket in ${}^9C_3 = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 84$ ways.

The number of ways in which B gets all blanks $= {}^6C_3 = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20$

$\therefore$ the number of ways of getting a prize $= 84 - 20 = 64$.

Thus the probability of B's success $= 64/84 = 16/21$.

Hence A's probability of success : B's probability of success $= \frac{1}{3} : \frac{16}{21} = 7 : 16$.

26.3 PROBABILITY AND SET NOTATIONS

(1) Random experiment. Experiments which are performed essentially under the same conditions and whose results cannot be predicted are known as *random experiments. e.g.,* Tossing a coin or rolling a die are random experiments.

(2) Sample space. The set of all possible outcomes of a random experiment is called *sample space* for that experiment and is denoted by S.

The elements of the sample space S are called the *sample points.*

e.g., On tossing a coin, the possible outcomes are the head (H) and the tail (T). Thus $S = \{H, T\}$.

(3) Event. The outcome of a random experiment is called an *event*. Thus every subset of a sample space S is an *event*.

The null set ϕ is also an event and is called an *impossible event. Probability of an impossible event* is zero *i.e.,* $P(\phi) = 0$.

(4) Axioms

(*i*) *The numerical value of probability lies between 0 and 1.*

i.e., for any event A of S, $0 \le P(A) \le 1$.

(*ii*) *The sum of probabilities of all sample events is unity i.e.,* $P(S) = 1$.

(*iii*) *Probability of an event made of two or more sample events is the sum of their probabilities.*

(5) Notations

(*i*) *Probability of happening of events* **A** *or* **B** is written as $P(A + B)$ or $P(A \cup B)$.

(*ii*) *Probability of happening of* **both the events A and B** is written as $P(AB)$ or $P(A \cap B)$.

(*iii*) '*Event A implies* ($\Rightarrow$) *event B*' is expressed as $A \subset B$.

(*iv*) '*Events A and B are mutually exclusive*' is expressed as $A \cap B = \phi$.

(6) *For any two events A and B,*

$$\boldsymbol{P(A \cap B') = P(A) - P(A \cap B)}$$

Proof. From Fig. 26.1,

$$(A \cap B') \cup (A \cap B) = A$$

$\therefore$ $P[(A \cap B') \cup (A \cap B)] = P(A)$

or $P(A \cap B') + P(A \cap B) = P(A)$

or $P(A \cap B') = P(A) - P(A \cap B)$

Similarly, $P(A' \cap B) = P(B) - P(A \cap B)$

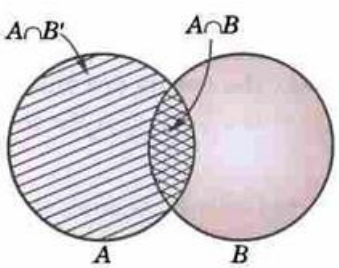


Fig. 26.1

26.4 ADDITION LAW OF PROBABILITY *or* THEOREM OF TOTAL PROBABILITY

(1) *If the probability of an event A happening as a result of a trial is P(A) and the probability of a* **mutually exclusive** *event B happening is P(B), then the probability of* **either** *of the events happening as a result of the trial is* $\mathbf{P(A+B)}$ **or** $\mathbf{P(A \cup B) = P(A) + P(B)}$.

Proof. Let n be the total number of equally likely cases and let m_1 be favourable to the event A and m_2 be favourable to the event B. Then the number of cases favourable to A or B is $m_1 + m_2$. Hence the probability of A or B happening as a result of the trial

$$= \frac{m_1 + m_2}{n} = \frac{m_1}{n} + \frac{m_2}{n} = P(A) + P(B).$$

(2) *If A, B, are any two events* **(not mutually exclusive),** *then*

$$\mathbf{P(A+B) = P(A) + P(B) - P(AB)}$$

or

$$\mathbf{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

If the events A and B are any two events then, there are some outcomes which favour both A and B. If m_3 be their number, then these are included in both m_1 and m_2. Hence the total number of outcomes favouring either A or B or both is

$$m_1 + m_2 - m_3.$$

Thus the probability of occurrence of A or B or both

$$= \frac{m_1 + m_2 - m_3}{n} = \frac{m_1}{n} + \frac{m_2}{n} - \frac{m_3}{n}$$

Hence $\quad P(A+B) = P(A) + P(B) - P(AB)$

or $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Obs. *When A and B are mutually exclusive* $P(AB)$ or $P(A \cap B) = 0$ and we get

$$P(A+B) \text{ or } P(A \cup B) = P(A) + P(B).$$

In general, for a number of **mutually exclusive** events $A_1, A_2, \ldots A_n$, we have

$$P(A_1 + A_2 + \ldots + A_n) \text{ or } P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n).$$

(3) *If A, B, C are any three events, then*

$$\mathbf{P(A+B+C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(CA) + P(ABC).}$$

or

$$\mathbf{P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)}$$

Proof. Using the above result for any two events, we have

$$\begin{aligned}
P(A \cup B \cup C) &= P[(A \cup B) \cup C] \\
&= P(A \cup B) + P(C) - P[(A \cup B) \cap C] \\
&= [P(A) + P(B) - P(A \cap B)] + P(C) - P[(A \cap C) \cup (B \cap C)] && \text{(Distributive Law)} \\
&= P(A) + P(B) + P(C) - P(A \cap B) - \{P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)\} \\
& && [\because (A \cap C) \cap (B \cap C) = A \cap B \cap C] \\
&= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C) && [\because A \cap C = C \cap A.]
\end{aligned}$$

Example 26.11. *In a race, the odds in favour of the four horses H_1, H_2, H_3, H_4 are 1 : 4, 1 : 5, 1 : 6, 1 : 7 respectively. Assuming that a dead heat is not possible, find the chance that one of them wins the race.*

Solution. Since it is not possible for all the horses to cover the same distance in the same time (a dead heat), the events are mutually exclusive.

If p_1, p_2, p_3, p_4 be the probabilities of winning of the horses H_1, H_2, H_3, H_4 respectively, then

$$p_1 = \frac{1}{1+4} = \frac{1}{5} \qquad [\because \text{ Odds in favour of } H_1 \text{ are } 1:4]$$

and

$$p_2 = \frac{1}{6}, p_3 = \frac{1}{7}, p_4 = \frac{1}{8}.$$

Hence the chance that one of them wins $= p_1 + p_2 + p_3 + p_4$

$$= \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} = \frac{533}{840}.$$

Example 26.12. *A bag contains 8 white and 6 red balls. Find the probability of drawing two balls of the same colour.*

Solution. Two balls out of 14 can be drawn in ${}^{14}C_2$ ways which is the total number of outcomes.

Two white balls out of 8 can be drawn in ${}^{8}C_2$ ways. Thus the probability of drawing 2 white balls

$$= \frac{{}^{8}C_2}{{}^{14}C_2} = \frac{28}{91}$$

Similarly 2 red balls out of 6 can be drawn in ${}^{6}C_2$ ways. Thus the probability of drawing 2 red balls

$$= \frac{{}^{6}C_2}{{}^{14}C_2} = \frac{15}{91}.$$

Hence the probability of drawing 2 balls of the same colour (either both white or both red)

$$= \frac{28}{91} + \frac{15}{91} = \frac{43}{91}.$$

Example 26.13. *Find the probability of drawing an ace or a spade or both from a deck of cards* ?*

Solution. The probability of drawing an ace from a deck of 52 cards = 4/52.

Similarly the probability of drawing a card of spades = 13/52, and the probability of drawing an ace of spades = 1/52.

Since the two events (*i.e.*, a card being an ace and a card being of spades) are not mutually exclusive, therefore, the probability of drawing an ace or a spade

$$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{4}{13}.$$

26.5 (1) INDEPENDENT EVENTS

Two events are said to be *independent*, if happening or failure of one does not affect the happening or failure of the other. Otherwise the events are said to be *dependent*.

For two dependent events A and B, the symbol P(B/A) denotes the probability of occurrence of B, when A has already occurred. It is known as the **conditional probability** and is read as a '*probability of B given A*'.

(2) Multiplication law of probability *or* **Theorem of compound probability.** *If the probability of an event A happening as a result of trial is P(A) and after A has happened the probability of an event B happening as a result of another trial (i.e.,* **conditional probability of B given A***) is P(B/A), then the probability of* **both** *the events A and B happening as a result of two trials is* $\mathbf{P(AB)}$ **or** $\mathbf{P(A \cap B) = P(A) \,.\, P(B/A)}$.

Proof. Let n be the total number of outcomes in the first trial and m be favourable to the event A so that $P(A) = m/n$.

Let n_1 be the total number of outcomes in the second trial of which m_1 are favourable to the event B so that $P(B/A) = m_1/n_1$.

Now each of the n outcomes can be associated with each of the n_1 outcomes. So the total number of outcomes in the combined trial is nn_1. Of these mm_1 are favourable to both the events A and B. Hence

$$P(AB) \text{ or } P(A \cap B) = \frac{mm_1}{nn_1} = P(A) \,.\, P(B/A).$$

Similarly, the *conditional probability of A given B is P(A/B).*

$$\therefore \qquad P(AB) \text{ or } P(A \cap B) = P(B) \,.\, P(A/B)$$

Thus $$P(A \cap B) = P(A) \,.\, P(B/A) = P(B) \,.\, P(A/B).$$

(3) *If the events A and B are* **independent,** *i.e.*, if the happening of B does not depend on whether A has happened or not, then $P(B/A) = P(B)$ and $P(A/B) = P(A)$.

$$\therefore \qquad \mathbf{P(AB) \text{ or } P(A \cap B) = P(A) \,.\, P(B).}$$

In general, $P(A_1A_2 \ldots A_n)$ or $P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1) \,.\, P(A_2) \ldots . \, P(A_n)$.

* *Cards* : A pack of cards consists of four suits *i.e.*, Hearts, Diamonds, Spades and Clubs. Each suit has 13 cards : an Ace, a King, a Queen, a Jack and nine cards numbered 2, 3, 4, ..., 10. Hearts and Diamonds are *red* while Spades and Clubs are *black*.

Cor. If p_1, p_2 be the probabilities of happening of two independent events, then

(*i*) the probability that the first event happens and the second fails is $p_1(1-p_2)$.

(*ii*) the probability that both events fail to happen is $(1-p_1)(1-p_2)$.

(*iii*) the probability that at least one of the events happens is

$1-(1-p_1)(1-p_2)$. This is commonly known as their **cumulative probability.**

In general, if $p_1, p_2, p_3, \ldots p_n$ be the chances of happening of n independent events, then their cumulative probability (i.e., the chance that at least one of the events will happen) is

$$1-(1-p_1)(1-p_2)(1-p_3)\ldots(1-p_n).$$

Example 26.14. *Two cards are drawn in succession from a pack of 52 cards. Find the chance that the first is a king and the second a queen if the first card is (i) replaced, (ii) not replaced.*

Solution. (*i*) The probability of drawing a king $= \frac{4}{52} = \frac{1}{13}$.

If the card is replaced, the pack will again have 52 cards so that the probability of drawing a queen is 1/13.

The two events being independent, the probability of drawing both cards in succession $= \frac{1}{13} \times \frac{1}{13} = \frac{1}{169}$.

(*ii*) The probability of drawing a king $= \frac{1}{13}$.

If the card is not replaced, the pack will have 51 cards only so that the chance of drawing a queen is 4/51.

Hence the probability of drawing both cards $= \frac{1}{13} \times \frac{4}{51} = \frac{4}{663}$.

Example 26.15. *A pair of dice is tossed twice. Find the probability of scoring 7 points (a) once, (b) at least once (c) twice.* (*Kurukshetra, 2009 S ; V.T.U., 2004*)

Solution. In a single toss of two dice, the sum 7 can be obtained as (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1) *i.e.*, in 6 ways, so that the probability of getting 7 = 6/36 = 1/6.

Also the probability of not getting 7 = 1 – 1/6 = 5/6.

(*a*) The probability of getting 7 in the first toss and not getting 7 in the second toss = 1/6 × 5/6 = 5/36.

Similarly, the probability of not getting 7 in the first toss and getting 7 in the second toss = 5/6 × 1/6 = 5/36.

Since these are mutually exclusive events, addition law of probability applies.

$\therefore$ required probability $= \frac{5}{36} + \frac{5}{36} = \frac{5}{18}$.

(*b*) The probability of not getting 7 in either toss $= \frac{5}{6} \times \frac{5}{6} = \frac{25}{36}$

$\therefore$ the probability of getting 7 at least once $= 1 - \frac{25}{36} = \frac{11}{36}$.

(*c*) The probability of getting 7 twice $= \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$.

Example 26.16. *There are two groups of objects : one of which consists of 5 science and 3 engineering subjects, and the other consists of 3 science and 5 engineering subjects. An unbiased die is cast. If the number 3 or number 5 turns up, a subject is selected at random from the first group, otherwise the subject is selected at random from the second group. Find the probability that an engineering subject is selected ultimately*

Solution. Prob. of turning up 3 or 5 $= \frac{2}{6} = \frac{1}{3}$.

Prob. of selecting an engg. subject from first group $= \frac{3}{8}$

$\therefore$ Prob of selecting an engg. subject from first group on turning up 3 or 5

$$= \frac{1}{3} \times \frac{3}{8} = \frac{1}{8} \qquad \ldots(i)$$

Now prob. of not turning 3 or 5 $= 1 - \frac{1}{3} = \frac{2}{3}$.

Prob. of selecting an engg. subject from second group $= \frac{5}{8}$

$\therefore$ prob. of selecting an engg. subject from second group on turning up 3 or 5

$$= \frac{2}{3} \times \frac{5}{8} = \frac{5}{12} \qquad ...(ii)$$

Thus the prob. of selecting an engg. subject

$$= \frac{1}{8} + \frac{5}{12} = \frac{13}{24}. \qquad \text{[From } (i) \text{ and } (ii)\text{]}$$

Example 26.17. *A box A contains 2 white and 4 black balls. Another box B contains 5 white and 7 black balls. A ball is transferred from the box A to the box B. Then a ball is drawn from the box B. Find the probability that it is white.* (V.T.U., 2004)

Solution. The probability of drawing a white ball from box B will depend on whether the transferred ball is black or white.

If a black ball is transferred, its probability is 4/6. There are now 5 white and 8 black balls in the box B. Then the probability of drawing white ball from box B is $\frac{5}{13}$.

Thus the probability of drawing a white ball from urn B, if the transferred ball is black

$$= \frac{4}{6} \times \frac{5}{13} = \frac{10}{39}.$$

Similarly the probability of drawing a white ball from urn B, if the transferred ball is white

$$= \frac{2}{6} \times \frac{6}{13} = \frac{2}{13}.$$

Hence required probability $= \frac{10}{39} + \frac{2}{13} = \frac{16}{39}$.

Example 26.18. *(a) A biased coin is tossed till a head appears for the first time. What is the probability that the number of required tosses is odd.* (Mumbai, 2006)

(b) Two persons A and B toss an unbiased coin alternately on the understanding that the first who gets the head wins. If A starts the game, find their respective chances of winning. (Madras, 2000 S)

Solution. (*a*) Let p be the probability of getting a head and q the probability of getting a tail in a single toss, so that $p + q = 1$.

Then probability of getting head on an odd toss

= Probability of getting head in the 1st toss
+ Probability of getting head in the 3rd toss
+ Probability of getting head in the 5th toss + ... ∞

$$= p + qqp + qqqqp + \ldots \infty$$

$$= p\,(1 + q^2 + q^4 + \ldots) = p \cdot \frac{1}{1 - q^2} \quad (q < 1)$$

$$= p \cdot \frac{1}{(1-q)(1+q)} = p \cdot \frac{1}{p\,(1+q)} = \frac{1}{1+q}.$$

(*b*) Probability of getting a head = 1/2. Then A can win in 1st, 3rd, 5th, ... throws.

$$\therefore \text{ the chances of } A\text{'s winning } = \frac{1}{2} + \left(\frac{1}{2}\right)^2 \frac{1}{2} + \left(\frac{1}{2}\right)^4 \frac{1}{2} + \left(\frac{1}{2}\right)^6 \frac{1}{2} + \cdots$$

$$= \frac{1/2}{1 - (1/2)^2} = \frac{1}{2} \cdot \frac{4}{3} = \frac{2}{3}.$$

Hence the chance of B's winning = 1 – 2/3 = 1/3.

Example 26.19. *Two cards are selected at random from 10 cards numbered 1 to 10. Find the probability p that the sum is odd, if*

(i) the two cards are drawn together.

(ii) the two cards are drawn one after the other without replacement.

(iii) the two cards are drawn one after the other with replacement. *(J.N.T.U., 2003)*

Solution. (*i*) Two cards out of 10 can be selected in ${}^{10}C_2 = 45$ ways. The sum is odd if one number is odd and the other number is even. There being 5 odd numbers (1, 3, 5, , , 9) and 5 even numbers (2, 4, 6, 8, 10), an odd and an even number is chosen in $5 \times 5 = 25$ ways.

Thus $$p = \frac{25}{45} = \frac{5}{9}.$$

(*ii*) Two cards out of 10 can be selected one after the other *without replacement* in $10 \times 9 = 90$ ways.

An odd number is selected in $5 \times 5 = 25$ ways and an even number in $5 \times 5 = 25$ ways

Thus $$p = \frac{25+25}{90} = \frac{5}{9}.$$

(*iii*) Two cards can be selected one after the other *with replacement* in $10 \times 10 = 100$ ways.

An odd number in selected in $5 \times 5 = 25$ ways and an even number in $5 \times 5 = 25$ ways.

Thus $$p = \frac{25+25}{100} = \frac{1}{2}.$$

Example 26.20. *Given $P(A) = 1/4$, $P(B) = 1/3$ and $P(A \cup B) = 1/2$, evaluate $P(A/B)$, $P(B/A)$, $P(A \cap B')$ and $P(A/B')$.*

Solution. (*i*) Since $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\therefore \quad \frac{1}{2} = \frac{1}{4} + \frac{1}{3} - P(A \cap B) \text{ or } P(A \cap B) = \frac{1}{12}$$

Thus $$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/12}{1/3} = \frac{1}{4}.$$

(*ii*) $$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1/12}{1/4} = \frac{1}{3}.$$

(*iii*) $$P(A \cap B') = P(A) - P(A \cap B) = \frac{1}{4} - \frac{1}{12} = \frac{1}{6}.$$

(*iv*) $$P(A/B') = \frac{P(A \cap B')}{P(B')} = \frac{1/6}{1 - P(B)} = \frac{1/6}{1 - 1/3} = \frac{1}{4}.$$

Example 26.21. *The odds that a book will be reviewed favourably by three independent critics are 5 to 2, 4 to 3 and 3 to 4. What is the probability that of the three reviews, a majority will be favourable.* *(V.T.U., 2003 S)*

Solution. The probability that the book shall be reviewed favourably by first critic is 5/7, by second 4/7 and by third 3/7.

A majority of the three reviews will be favourable when two or three are favourable.

$\therefore$ prob. that the first two are favourable and the third unfavourable

$$= \frac{5}{7} \times \frac{4}{7} \times \left(1 - \frac{3}{7}\right) = \frac{80}{343}$$

Prob. that the first and third are favourable and second unfavourable

$$= \frac{5}{7} \times \frac{3}{7} \times \left(1 - \frac{4}{7}\right) = \frac{45}{343}$$

Prob. that the second and third are favourable and the first unfavourable

$$= \frac{4}{7} \times \frac{3}{7} \times \left(1 - \frac{5}{7}\right) = \frac{24}{343}$$

Finally, prob. that all the three are favourable $= \frac{5}{7} \times \frac{4}{7} \times \frac{3}{7} = \frac{60}{343}$

Since they are mutually exclusive events, the required prob.

$$= \frac{80}{343} + \frac{45}{343} + \frac{24}{343} + \frac{60}{343} = \frac{209}{343}.$$

Example 26.22. *I can hit a target 3 times in 5 shots, B 2 times in 5 shots and C 3 times in 4 shots. They fire a volley. What is the probability that (i) two shots hit, (ii) atleast two shots hit ?*

(A.M.I.E.T.E., 2003 ; Madras, 2000 S)

Solution. Prob. of A hitting the target = 3/5, prob. of B hitting the target = 2/5

Prob. of C hitting the target = 3/4.

(*i*) In order that two shots may hit the target, the following cases must be considered :

$$p_1 = \text{Chance that } A \text{ and } B \text{ hit and } C \text{ fails to hit} = \frac{3}{5} \times \frac{2}{3} \times \left(1 - \frac{3}{4}\right) = \frac{6}{100}$$

$$p_2 = \text{Chance that } B \text{ and } C \text{ hit and } A \text{ fails to hit} = \frac{2}{5} \times \frac{3}{4} \times \left(1 - \frac{3}{5}\right) = \frac{12}{100}$$

$$p_3 = \text{Chance that } C \text{ and } A \text{ hit and } B \text{ fails to hit} = \frac{3}{4} \times \frac{3}{5} \times \left(1 - \frac{2}{5}\right) = \frac{27}{100}$$

Since these are mutually exclusive events, the probability that any 2 shots hit

$$= p_1 + p_2 + p_3 = \frac{6}{100} + \frac{12}{100} + \frac{27}{100} = 0.45.$$

(*ii*) In order that at least two shots may hit the target, we must also consider the case of all A, B, C hitting the target [in addition to the three cases of (*i*)] for which

$$p_4 = \text{chance that } A, B, C \text{ all hit} = \frac{3}{5} \times \frac{2}{5} \times \frac{3}{4} = \frac{18}{100}$$

Since all these are mutually exclusive events, the probability of *atleast* two shots hit

$$= p_1 + p_2 + p_3 + p_4 = \frac{6}{100} + \frac{12}{100} + \frac{27}{100} + \frac{18}{100} = 0.63.$$

Example 26.23. *A problem in mechanics is given to three students A, B, and C whose chances of solving it are $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$ respectively. What is the probability that the problem will be solved.* *(V.T.U., 2004)*

Solution. The probability that A can solve the problem is 1/2.

The probability that A cannot solve the problem is $1 - \frac{1}{2}$.

Similarly the probabilities that B and C cannot solve the problem are $1 - \frac{1}{3}$ and $1 - \frac{1}{4}$.

$\therefore$ the probability that A, B and C cannot solve the problem is $\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)$.

Hence the probability that the problem will be solved, *i.e.*, at least one student will solve it

$$= 1 - \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right) = \frac{3}{4}.$$

Example 26.24. *The students in a class are selected at random, one after the other, for an examination. Find the probability p that the boys and girls in the class alternate if*

(i) the class consists of 4 boys and 3 girls.

(ii) the class consists of 3 boys and 3 girls. *(J.N.T.U., 2003)*

Solution. (*i*) As there are 7 students in the class, the first examined must be a boy.

$\therefore$ prob. that first is a boy $= \dfrac{4}{7}$

Then the prob. that the second is a girl $= \dfrac{3}{6}$.

$\therefore$ prob. of the next boy $= \dfrac{3}{5}$

Similarly the prob. that the fourth is a girl $= \dfrac{2}{4}$,

the prob. that the fifth is a boy $= \dfrac{2}{3}$,

the prob. that the sixth is a girl $= \dfrac{1}{2}$

and the last is a boy $= \dfrac{1}{1}$.

Thus
$$p = \frac{4}{7}\cdot\frac{3}{6}\cdot\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{1}{1} = \frac{1}{35}.$$

(*ii*) The first student is a boy and the first student is a girl are two mutually exclusive cases. If the first student is a boy, then the probability p_1 that the students alternate is
$$p_1 = \frac{3}{6}\cdot\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{1}{1} = \frac{1}{20}.$$
If the first student is a girl, then the probability p_2 that the students alternate is
$$p_2 = \frac{3}{6}\cdot\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{1}{1} = \frac{1}{20}.$$
Thus the required prob. $p = p_1 + p_2 = \dfrac{1}{20} + \dfrac{1}{20} = \dfrac{1}{10}$.

Example 26.25. (*Huyghen's problem*) *A and B throw alternately with a pair of dice. A wins if he throws 6 before B throws 7 and B wins if he throws 7 before A throws 6. If A begins, find his chance of winning.*
(*Madras, 2006 ; J.N.T.U., 2003*)

Solution. The sum 6 can be obtained as follows : (1, 5), (2, 4), (3, 3), (4, 2), (5, 1), *i.e.*, in 5 ways.

The probability of A's throwing 6 with 2 dice is $\dfrac{5}{36}$.

$\therefore$ the probability of A's not throwing 6 is 31/36.

Similarly the probability of B's throwing 7 is 6/36, *i.e.*, $\dfrac{1}{6}$.

$\therefore$ the probability of B's not throwing 7 is 5/6.

Now A can win if he throws 6 in the first, third, fifth, seventh etc. throws.

$\therefore$ the chance of A's wining
$$= \frac{5}{36} + \frac{31}{36}\times\frac{5}{6}\times\frac{5}{36} + \frac{31}{36}\times\frac{5}{6}\times\frac{31}{36}\times\frac{5}{6}\times\frac{5}{36} + \cdots$$
$$= \frac{5}{36}\left[1 + \left(\frac{31}{36}\times\frac{5}{6}\right) + \left(\frac{31}{36}\times\frac{5}{6}\right)^2 + \left(\frac{31}{36}\times\frac{5}{6}\right)^3 + \cdots\right]$$
$$= \frac{5}{36}\cdot\frac{1}{1-(31/36)\times(5/6)} = \frac{5}{36}\times\frac{36\times 6}{61} = \frac{30}{61}.$$

PROBLEMS 26.2

1. (*i*) Given $P(A) = 1/2$, $P(B) = 1/3$ and $P(AB) = 1/4$, find the value $P(A + B)$. (*Burdwan, 2003*)

   (*ii*) Let A and B be two events with $P(A) = 1/2$, $P(B) = 1/3$ and $P(A \cap B) = 1/4$. Find $P(A/B)$, $P(A \cup B)$, $P(A'/B')$.
   (*Kurukshetra, 2009 ; V.T.U., 2003 S*)

2. In a single throw with two dice, what is the chance of throwing
(*a*) two aces ? (*b*) 7 ? Is this probability the same as that for getting 7 in two throws of a single die ?
3. Compare the chances of throwing 4 with one dice, 8 with two dice and 12 with three dice.
4. Find the probability that a non-leap year should have 53 Saturdays ? (*Madras, 2003*)
5. When a coin is tossed four times, find the probability of getting (*i*) exactly one head, (*ii*) at most three heads and (*iii*) at least two heads ? (*V.T.U., 2000 S*)
6. Ten coins are thrown simultaneously. Find the probability of getting at least seven heads. (*P.T.U., 2003*)
7. If all the letters of word 'ENGINEER' be written at random, what is the probability that all the latters E are found together.
8. A ten digit number is formed using the digits from zero to nine, every digit being used only once. Find the probability that the number is divisible by 4.
9. Four cards are drawn from a pack of 52 cards. What is the chance that
(*i*) no two cards are of equal value ? (*ii*) each belongs to a different suit ?
10. Suppose 5 cards are drawn at random from a pack of 52 cards. If all cards are red what is the probability that all of them are hearts? (*Mumbai, 2005*)
11. Out of 50 rare books, 3 of which are especially valuable, 5 are stolen at random by a thief. What is the probability that
(*a*) none of the 3 is included ? (*b*) 2 of the 3 are included ?
12. Five men in a company of twenty are graduates. If 3 men are picked out of 20 at random, what is the probability that
(*a*) they are all graduates ? (*b*) at least one is graduate ?
13. From 20 tickets marked from 1 to 20, one ticket is drawn at random. Find the probability that it is marked with a multiple of 3 or 5.
14. Five balls are drawn from a bag containing 6 white and 4 black balls. What is the chance that 3 white and 2 black balls are drawn ?
15. The probability of n independent events are $p_1, p_2, p_3, \ldots p_n$. Find the probability that at least one of the events will happen. Use this result to find the chance of getting at least one six in a throw of 4 dice.
16. Find the probability of drawing 4 white balls and 2 black balls without replacement from a bag containing 1 red, 4 black and 6 white balls.
17. A bag contains 10 white and 15 black balls. Two balls are drawn in succession. What is the probability that one of them is black and the other white ?
18. A purse contains 2 silver and 4 copper coins and a second purse contains 4 silver and 4 copper coins. If a coin is selected at random from one of the two purses, what is the probability that it is a silver coin ? (*Osmania 2002*)
19. A box I contains 5 white balls and 6 black balls. Another box II contains 6 white balls and 4 black balls. A box is selected at random and then a ball is drawn from it : (*i*) what is the probability that the ball drawn will be white ? (*ii*) Given that the ball drawn is white, what is the probability that it came from box I. (*Mumbai, 2006*)
20. A party of n persons take their seats at random at a round table ; find the probability that two specified persons do not sit together.
21. A speaks the truth in 75% cases, and B in 80% of the cases. In what percentage of cases, are they likely to contradict each other in stating the same fact ? (*V.T.U., 2002 S*)
22. The probability that Sushil will solve a problem is 1/4 and the probability that Ram will solve it is 2/3. If Sushil and Ram work independently, what is the probability that the problem will be solved by (*a*) both of them, (*b*) at least one of them ?
23. A student takes his examination in four subjects, P, Q, R, S. He estimates his chances of passing in P as 4/5, in Q as 3/4, in R as 5/6 and in S as 2/3. To qualify, he must pass in P and at least two other subjects. What is the probability that he qualifies ? (*Madras, 2000 S*)
24. The probability that a 50 year old man will be alive at 60 is 0.83 and the probability that a 45 year old women will be alive at 55 is 0.87. What is the probability that a man who is 50 and his wife who is 45 will both be alive 10 years hence ?
25. If on an average one birth in 80 is a case of twins, what is the probability that there will be at least one case of twins in a maternity hospital on a day when 20 births occur ?
26. Two persons A and B fire at a target independently and have a probability 0.6 and 0.7 respectively of hitting the target. Find the probability that the target is destroyed.
27. A and B throw alternately with a pair of dice. The one who throws 9 first wins. Show that the chances of their winning are 9 : 8.

26.6 BAYE'S THEOREM

An event A corresponds to a number of exhaustive events $B_1, B_2, \ldots, B_n$. If $P(B_i)$ and $P(A/B_i)$ are given, then

$$P(B_i/A) = \frac{P(B_i)\,P(A/B_i)}{\Sigma P(B_i)P(A/B_i)}.$$

Proof. By the multiplication law of probability,

$$P(AB_i) = P(A)\,P(B_i/A) = P(B_i)\,P(A/B_i) \qquad \ldots(1)$$

$$\therefore \qquad P(B_i/A) = \frac{P(B_i)\,P(A/B_i)}{P(A)} \qquad \ldots(2)$$

Since the event A corresponds to $B_1, B_2, \ldots, B_n$, we have by the addition law of probability,

$$P(A) = P(AB_1) + P(AB_2) + \ldots + P(AB_n) = \Sigma P(AB_i) = \Sigma P(B_i)\,P(A/B_i) \qquad \text{[By (1)]}$$

Hence from (2), we have $\quad P(B_i/A) = \dfrac{P(B_i)\,P(A/B_i)}{\Sigma P(B_i)\,P(A/B_i)}$

which is known as the *theorem of inverse probability.*

Obs. The probabilities $P(B_i)$, $i = 1, 2, \ldots, n$ are called *apriori probabilities* because these exist before we get any information from the experiment.

The probabilities $P(A/B_i)$, $i = 1, 2, \ldots, n$ are called *posteriori probabilities*, because these are found after the experiment results are known.

Example 26.26. *Three machines M_1, M_2 and M_3 produce identical items. Of their respective output 5%, 4% and 3% of items are faulty. On a certain day, M_1 has produced 25% of the total output, M_2 has produced 30% and M_3 the remainder. An item selected at random is found to be faulty. What are the chances that it was produced by the machine with the highest output?*

Solution. Let the event of drawing a faulty item from any of the machines be A, and the event that an item drawn at random was produced by M_i be B_i. We have to find $P(B_i/A)$ for which we proceed as follows :

	M_1	M_2	M_3	*Remarks*
$P(B_i)$	0.25	0.30	0.45	$\because$ sum = 1
$P(A/B_i)$	0.05	0.04	0.03	
$P(B_i)\,P(A/B_i)$	0.0125	0.012	0.0135	sum = 0.38
$P(B_i/A)$	$\frac{0.0125}{0.038}$	$\frac{0.012}{0.038}$	$\frac{0.0135}{0.038}$	by Baye's theorem

The highest output being from M_3, the required probability = 0.0135/0.038 = 0.355.

Example 26.27. *There are three bags : first containing 1 white, 2 red, 3 green balls ; second 2 white, 3 red, 1 green balls and third 3 white, 1 red, 2 green balls. Two balls are drawn from a bag chosen at random. These are found to be one white and one red. Find the probability that the balls so drawn came from the second bag.*

(J.N.T.U., 2003)

Solution. Let B_1, B_2, B_3 pertain to the first, second, third bags chosen and A : the two balls are white and red.

Now $\qquad P(B_1) = P(B_2) = P(B_3) = \dfrac{1}{3}$

$P(A/B_1) = P$ (a white and a red ball are drawn from first bag)

$$= ({}^1C_1 \times {}^2C_1)/{}^6C_2 = \frac{2}{15}$$

Similarly $\qquad P(A/B_2) = ({}^2C_1 \times {}^3C_1)/{}^6C_2 = \dfrac{2}{5}$, $P(A/B_3) = ({}^3C_1 \times {}^1C_1)/{}^6C_2 = \dfrac{1}{5}$

By Baye's theorem, $\quad P(B_2/A) = \dfrac{P(B_2)\,P(A/B_2)}{P(B_1)\,P(A/B_1) + P(B_2)\,P(A/B_2) + P(B_3)\;P(A/B_3)}$

$$= \frac{\frac{1}{3} \times \frac{2}{5}}{\frac{1}{3} \times \frac{2}{15} + \frac{1}{3} \times \frac{2}{5} + \frac{1}{3} \times \frac{1}{5}} = \frac{6}{11}.$$

PROBLEMS 26.3

1. In a certain college, 4% of the boys and 1% of girls are taller than 1.8 m. Further more 60% of the students are girls. If a student is selected at random and is found to be taller than 1.8 m., what is the probability that the student is a girl ?
2. In a bolt factory, machines A, B and C manufacture 25%, 35% and 40% of the total. Of their output 5%, 4% and 2% are defective bolts. A bolt is drawn at random from the product and is found to be defective. What are the probabilities that it was manufactured by machines A, B or C ? *(V.T.U., 2006 ; Rohtak, 2005 ; Madras, 2000 S)*
3. In a bolt factory, there are four machines A, B, C, D manufacturing 20%, 15%, 25% and 40% of the total output respectively. Of their outputs 5%, 4%, 3% and 2% in the same order are defective bolts. A bolt is chosen at random from the factory's production and is found defective. What is the probability that the bolt was manufactured by machine A or machine D ? *(Hissar, 2007 ; J.N.T.U., 2003)*
4. The contents of three urns are : 1 white, 2 red, 3 green balls ; 2 white, 1 red, 1 green balls and 4 white, 5 red, 3 green balls. Two balls are drawn from an urn chosen at random. These are found to be one white and one green. Find the probability that the balls so drawn came from the third urn. *(Kurukshetra, 2007)*

26.7 RANDOM VARIABLE

If a real variable X be associated with the outcome of a random experiment, then since the values which X takes depend on chance, it is called a *random variable* or a *stochastic variable* or simply a *variate*. For instance, if a random experiment E consists of tossing a pair of dice, the sum X of the two numbers which turn up have the value 2, 3, 4, ..., 12 depending on chance. Then X is the random variable. It is a function whose values are real numbers and depend on chance.

If in a random experiment, the event corresponding to a number a occurs, then the corresponding random variable X is said to assume the value a and the probability of the event is denoted by $P(X = a)$. Similarly the probability of the event X assuming any value in the interval $a < X < b$ is denoted by $P(a < X < b)$. The probability of the event $X \le c$ is written as $P(X \le c)$.

If a random variable takes a finite set of values, it is called a *discrete variate*. On the other hand, if it assumes an infinite number of uncountable values, it is called a *continuous variate*.

26.8 (1) DISCRETE PROBABILITY DISTRIBUTION

Suppose a discrete variate X is the outcome of some experiment. If the probability that X takes the values x_i, is p_i, then

$$P(X = x_i) = p_i \text{ or } p(x_i) \text{ for } i = 1, 2, \ldots$$

where (*i*) $p(x_i) \ge 0$ for all values of i, (*ii*) $\Sigma p(x_i) = 1$

The set of values x_i with their probabilities p_i constitute a **discrete probability distribution** *of the discrete variate X.*

For example, the discrete probability distribution for X, the sum of the numbers which turn on tossing a pair of dice is given by the following table :

$X = x_i$	2	3	4	5	6	7	8	9	10	11	12
$p(x_i)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

[$\because$ There are $6 \times 6 = 36$ equally likely outcomes and therefore, each has the probability 1/36. We have $X = 2$ for one outcome, *i.e.* (1, 1) ; $X = 3$ for two outcomes (1, 2) and (2, 1); $X = 4$ for three outcomes (1, 3), (2, 2) and (3, 1) and so on.]

(2) Distribution function. *The distribution function $F(x)$ of the discrete variate X is defined by*

$$F(x) = P(X \le x) = \sum_{i=1}^{x} p(x_i)$$ where x is any integer. The graph of $F(x)$ will be stair step form (Fig. 26.2). The distribution function is also sometimes called *cumulative distribution function*.

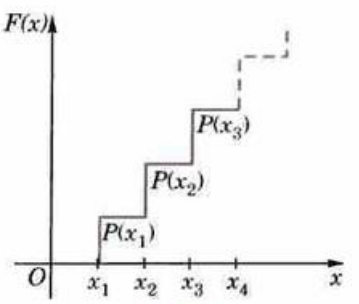


Fig. 26.2

Example 26.28. *A die is tossed thrice. A success is 'getting 1 or 6' on a toss. Find the mean and variance of the number of successes.* (V.T.U., 2011 S ; Rohtak, 2004)

Solution. Probability of a success $= \frac{2}{6} = \frac{1}{3}$, Probability of failures $= 1 - \frac{1}{3} = \frac{2}{3}$.

$\therefore$ prob. of no success = Prob. of all 3 failures $= \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$

Probability of one successes and 2 failures $= 3c_1 \times \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$

Probability of two successes and one failure $= 3c_2 \times \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{2}{9}$

Probability of three successes $= \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$

Now	$x_i = 0$	1	2	3
	$p_i = 8/27$	4/9	2/9	1/27

$\therefore$ mean $\mu = \Sigma p_i x_i = 0 + \frac{4}{9} + \frac{4}{9} + \frac{1}{9} = 1.$

Also $\Sigma p_i x_i^2 = 0 + \frac{4}{9} + \frac{8}{9} + \frac{9}{27} = \frac{5}{3}$

$\therefore$ variance $\sigma^2 = \Sigma p_i x_i^2 - \mu^2 = \frac{5}{3} - 1 = \frac{2}{3}$.

Example 26.29. *The probability density function of a variate X is*

X :	0	1	2	3	4	5	6
$p(X)$:	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

(i) Find $P(X < 4)$, $P(X \geq 5)$, $P(3 < X \leq 6)$. (V.T.U., 2010)

(ii) What will be the minimum value of k so that $P(X \leq 2) > 3$.

Solution. (*i*) If X is a random variable, then

$$\sum_{i=0}^{6} p(x_i) = 1 \text{ i.e., } k + 3k + 5k + 7k + 9k + 11k + 13k = 1 \text{ or } k = 1/49.$$

$\therefore$ $P(X < 4) = k + 3k + 5k + 7k = 16k = 16/49.$

$P(X \geq 5) = 11k + 13k = 24k = 24/49.$

$P(3 < X \leq 6) = 9k + 11k + 13k = 33k = 33/49.$

(*ii*) $P(X \leq 2) = k + 3k + 5k = 9k > 0.3$ or $k > 1/30$

Thus minimum value of $k = 1/30$.

Example 26.30. *A random variable X has the following probability function :*

x :	0	1	2	3	4	5	6	7
$p(x)$:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

(i) Find the value of the k *(ii) Evaluate $P(X < 6)$, $P(X \geq 6)$*

(iii) $P(0 < X < 5)$. (W.B.T.U., 2005 ; J.N.T.U., 2003)

Solution. (*i*) If X is a random variable, then

$$\sum_{i=0}^{7} p(x_i) = 1, \text{ i.e., } 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$

i.e., $7k^2 + 9k - 1 = 0$ *i.e.* $(10 - k)(k + 1) = 0$ *i.e.,* $k = \frac{1}{10}$

(*ii*) $P(X < 6) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5)$

$$= 0 + k + 2k + 2k + 3k + k^2 = 8k + k^2 = \frac{8}{10} + \frac{1}{100} = \frac{81}{100}$$

$$P(X \geq 6) = P(X = 6) + P(X = 7) = 2k^2 + 7k^2 + k = \frac{9}{100} + \frac{1}{10} = \frac{19}{100}$$

(*ii*) $P(0 < X < 5) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$

$$= k + 2k + 2k + 3k = 8k = \frac{8}{10} = \frac{4}{5}.$$

26.9 (1) CONTINUOUS PROBABILITY DISTRIBUTION

When a variate X takes every value in an interval, it gives rise to *continuous distribution* of X. The distributions defined by the variates like heights or weights are continuous distributions.

A major conceptual difference, however, exists between discrete and continuous probabilities. When thinking in discrete terms, the probability associated with an event is meaningful. With continuous events, however, where the number of events is infinitely large, the probability that a specific event will occur is practically zero. For this reason, continuous probability statements must be worded somewhat differently from discrete ones. Instead of finding the probability that x equals some value, we find the probability of x falling in a small interval.

Thus the probability distribution of a continuous variate x is defined by a function $f(x)$ such that the probability of the variate x falling in the small interval $x - \frac{1}{2}dx$ to $x + \frac{1}{2}dx$ is $f(x)\,dx$. Symbolically it can be expressed as $P\left(x - \frac{1}{2}dx \leq x \leq x + \frac{1}{2}dx\right) = f(x)\,dx$. Then $f(x)$ is called the *probability density function* and the continuous curve $y = f(x)$ is called the *probability curve.*

The range of the variable may be finite or infinite. But even when the range is finite, it is convenient to consider it as infinite by supposing the density function to be zero outside the given range. Thus if $f(x) = \phi(x)$ be the density function denoted for the variate x in the interval (a, b), then it can be written as

$$\begin{aligned} f(x) &= 0, & x < a \\ &= \phi(x), & a \leq x \leq b \\ &= 0, & x > b. \end{aligned}$$

The density function $f(x)$ is always positive and $\int_{-\infty}^{\infty} f(x)\,dx = 1$ (*i.e.*, the total area under the probability curve and the x-axis is unity which corresponds to the requirements that the total probability of happening of an event is unity).

(2) Distribution function

If $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx,$

then $F(x)$ is defined as the **cumulative distribution function** *or simply the* **distribution function** *of the continuous variate X.* It is the probability that the value of the variate X will be $\leq x$. The graph of $F(x)$ in this case is as shown in Fig. 26.3(*b*).

The distribution function $F(x)$ has the following properties :

(*i*) $F'(x) = f(x) \geq 0$, so that $F(x)$ is a non-decreasing function.

(*i*) $F(-\infty) = 0$; (*iii*) $F(\infty) = 1$

(*iv*) $P(a \leq x \leq b) = \int_a^b f(x)\,dx = \int_{-\infty}^{b} f(x)\,dx - \int_{-\infty}^{a} f(x)\,dx = F(b) - F(a).$

Example 26.31. (*i*) *Is the function defined as follows a density function?*

$$\begin{aligned} f(x) &= e^{-x}, & x \geq 0 \\ &= 0, & x < 0, \end{aligned}$$

(*ii*) *If so, determine the probability that the variate having this density will fall in the interval (1, 2) ?*

(*iii*) *Also find the cumulative probability function F(2) ?*

Solution. (*i*) $f(x)$ is clearly ≥ 0 for every x in (1, 2) and

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} 0 \cdot dx + \int_{0}^{\infty} e^{-x}\,dx = 1$$

Hence the function $f(x)$ satisfies the requirements for a density function.

(*ii*) Required probability $= P(1 \le x \le 2) = \int_1^2 e^{-x}\,dx = e^{-1} - e^{-2} = 0.368 - 0.135 = 0.233.$

This probability is equal to the shaded area in Fig. 26.3 (*a*).

(*iii*) Cumulative probability function $F(2)$

$$\int_{-\infty}^{2} f(x)\,dx = \int_{-\infty}^{0} 0 \cdot dx + \int_0^2 e^{-x}\,dx = 1 - e^{-2} = 1 - 0.135 = 0.865$$

which is shown in Fig. 26.3 (*b*).

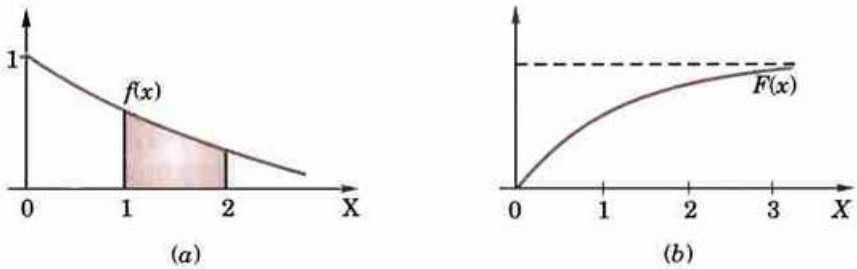


Fig. 26.3

26.10 (1) EXPECTATION

The mean value (μ) *of the probability distribution of a variate X is commonly known as its* **expectation** *and is denoted by* $E(X)$. If $f(x)$ is the probability density function of the variate X, then

$$\sum_i x_i\, f(x_i) \qquad \text{(discrete distribution)}$$

or

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx \qquad \text{(continuous distribution)}$$

In general, expectation of any function $\phi(x)$ is given by

$$E[\phi(x)] = \sum_i \phi(x_i)\, f(x_i) \qquad \text{(discrete distribution)}$$

or

$$E[\phi(x)] = \int_{-\infty}^{\infty} \phi(x)\, f(x)\,dx \qquad \text{(continuous distribution)}$$

(2) **Variance** *of a distribution is given by*

$$\sigma^2 = \sum_i (x_i - \mu)^2\, f(x_i) \qquad \text{(discrete distribution)}$$

or

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2\, f(x)\,dx \qquad \text{(continuous distribution)}$$

where σ is the *standard deviation* of the distribution.

(3) *The* **rth moment** *about the mean* (denoted by μ_r) *is defined by*

$$\mu_r = \Sigma (x_i - \mu)^r f(x_i) \qquad \text{(discrete distribution)}$$

or

$$\mu_r = \int_{-\infty}^{\infty} (x - \mu)^r\, f(x)\,dx \qquad \text{(continuous distribution)}$$

(4) **Mean deviation from the mean** *is given by*

$$\Sigma\, |x_i - \mu|\, f(x_i) \qquad \text{(discrete distribution)}$$

or by

$$\int_{-\infty}^{\infty} |x - \mu|\, f(x)\,dx \qquad \text{(continuous distribution)}$$

Example 26.32. *In a lottery, m tickets are drawn at a time out of n tickets numbered from 1 to n. Find the expected value of the sum of the numbers on the tickets drawn.*

Solution. Let $x_1, x_2, ..., x_n$ be the variables representing the numbers on the first, second, ..., nth ticket. The probability of drawing a ticket out of n tickets being in each case $1/n$, we have

$$E(x_i) = 1 \cdot \frac{1}{n} + 2 \cdot \frac{1}{n} + 3 \cdot \frac{1}{n} + ... + n \cdot \frac{1}{n} = \frac{1}{2}(n+1)$$

$\therefore$ expected value of the sum of the numbers on the tickets drawn

$$= E(x_1 + x_2 + ... + x_m) = E(x_1) + E(x_2) + ... + E(x_m)$$

$$= mE(x_i) = \frac{1}{2} m (n+1).$$

Example 26.33. *X is a continuous random variable with probability density function given by*

$$f(x) = kx \ (0 \le x < 2)$$
$$= 2k \ (2 \le x < 4)$$
$$= -kx + 6k \ (4 \le x < 6)$$

Find k and mean value of X. *(J.N.T.U., 2003)*

Solution. Since the total probability is unity

$$\therefore \quad \int_0^6 f(x)\, dx = 1$$

i.e.,
$$\int_0^2 kx\, dx + \int_2^4 2k dx + \int_4^6 (-kx + 6k)\, dx = 1$$

or
$$k \left| x^2/2 \right|_0^2 + 2k \left| x \right|_2^4 + \left(-kx^2/2 + 6kx\right)_4^6 = 1$$

or
$$2k + 4k + (-10k + 12k) = 1 \ i.e., \ k = 1/8.$$

$$\text{Mean of } X = \int_0^6 x\, f(x)\, dx$$

$$= \int_0^2 kx^2\, dx + \int_2^4 2kx\, dx + \int_4^6 x(-kx + 6k) dx$$

$$= k \left| x^3/3 \right|_0^2 + 2k \left| x^2/2 \right|_2^4 + \left(-k \left| x^3/3 \right|_4^6 + 6k \left| x^2/2 \right|_4^6\right)$$

$$= k(8/3) + k(12) - k(152/3) + 3k(20) = \frac{1}{8}(24) = 3.$$

Example 26.34. *A variate X has the probability distribution*

x :	-3	6	9
$P(X = x)$:	$1/6$	$1/2$	$1/3$

Find E (X) and E (X²). Hence evaluate E (2X + 1)².

Solution.
$$E(X) = -3 \times \frac{1}{6} + 6 \times \frac{1}{2} + 9 \times \frac{1}{3} = 11/2.$$

$$E(X)^2 = 9 \times \frac{1}{6} + 36 \times \frac{1}{2} + 81 \times \frac{1}{3} = 93/2$$

$$\therefore \quad E(2X+1)^2 = E(4X^2 + 4X + 1) = 4E(X^2) + 4E(X) + 1$$
$$= 4(93/2) + 4(11/2) + 1 = 209.$$

Example 26.35. *The frequency distribution of a measurable characteristic varying between 0 and 2 is as under*

$$f(x) = x^3, \quad 0 \le x \le 1$$
$$= (2-x)^3, \quad 1 \le x \le 2.$$

Calculate the standard deviation and also the mean deviation about the mean.

Solution. Total frequency $N = \int_0^1 x^3\,dx + \int_1^2 (2-x)^3\,dx = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$$\therefore \quad \mu_1'\text{ (about the origin)} = \frac{1}{N}\left[\int_0^1 x \cdot x^3\,dx + \int_1^2 x(2-x)^3\,dx\right]$$

$$= 2\left\{\left|\frac{x^5}{5}\right|_0^1 + \left|-x\cdot\frac{(2-x)^4}{4}\right|_1^2 - \left|\frac{(2-x)^5}{20}\right|_1^2\right\} = 2\left(\frac{1}{5}+\frac{1}{4}+\frac{1}{20}\right) = 1$$

$$\mu_2'\text{ (about the origin)} = \frac{1}{N}\left[\int_0^1 x^2 \cdot x^3 dx + \int_1^2 x^2\,(2-x)^3\,dx\right]$$

$$= 2\left\{\left|\frac{x^6}{6}\right|_0^1 + \left|-x^2\,\frac{(2-x)^4}{4}\right|_1^2 + \frac{1}{2}\int_1^2 x\,(2-x)^4\,dx\right\}$$

$$= 2\left\{\frac{1}{6}+\frac{1}{4}+\frac{1}{2}\left[\frac{1}{5}+\frac{1}{30}\right]\right\} = \frac{16}{15}$$

Hence $\qquad \sigma^2 = \mu_2 = \mu_2' - (\mu_1')^2 = \frac{1}{15}$

i.e., standard deviation $\sigma = \frac{1}{\sqrt{15}}$.

Mean deviation about the mean

$$= \frac{1}{N}\left\{\int_0^1 |x-1|\,x^3\,dx + \int_1^2 |x-1|\,(2-x)^3\,dx\right\}$$

$$= 2\left\{\int_0^1 (1-x)x^3\,dx + \int_1^2 (x-1)(2-x)^3\,dx\right\}$$

$$= 2\left\{\left(\frac{1}{4}-\frac{1}{5}\right)+\left(0+\frac{1}{20}\right)\right\} = \frac{1}{5}.$$

26.11 MOMENT GENERATING FUNCTION

(1) *The moment generating function (m.g.f.) of the discrete probability distribution of the variate X about the value x = a is defined as the expected value of* $e^{t(x-a)}$ and is denoted by $M_a(t)$. Thus

$$M_a(t) = \Sigma p_i\, e^{t(x_i - a)} \qquad \text{...(1)}$$

which is a function of the parameter t only.

Expanding the exponential in (1), we get

$$M_a(t) = \Sigma p_i + t\Sigma p_i\,(x_i - a) + \frac{t^2}{2!}\,\Sigma p_i (x_i - a)^2 + \ldots + \frac{t^r}{r!}\,\Sigma x_i\,(x_i - a)^r + \ldots$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\,\mu'_2 + \ldots + \frac{t^r}{r!}\mu'_r + \ldots \qquad \text{...(2)}$$

where μ'_r is the moment of order r about a. Thus $M_a(t)$ generates moments and that is why it is called the moment generating function. From (2), we find

μ'_r = coefficient of $t^r/r!$ in the expansion of $M_a(t)$.

Otherwise differentiating (2) r times with respect to t and then putting $t = 0$, we get

$$\mu'_r = \left[\frac{d^r}{dt^r}\,M_a(t)\right]_{t=0} \qquad \text{...(3)}$$

Thus the moment about any point $x = a$ can be found from (2) or more conveniently from the formula (3).

Rewriting (1) as

$$M_a(t) = e^{-at}\,\Sigma p_i\, e^{tx_i} \quad \text{or} \quad M_a(t) = e^{-at}\,M_0(t) \qquad \text{...(4)}$$

Thus the m.g.f. about the point $a = e^{-at}$ *(m.g.f. about the origin).*

Obs. *The m.g.f. of the sum of two independent variables is the product of their m.g.fs.* ...(5)

(2) *If $f(x)$ is the density function of a continuous variate X, then the moment generating function of this continuous probability distribution about $x = a$ is given by*

$$M_a(t) = \int_{-\infty}^{\infty} e^{t(x-a)} f(x)\, dx.$$

Example 26.36. *Find the moment generating function of the exponential distribution*

$f(x) = \frac{1}{c} e^{-x/c}$, $0 \le x \le \infty$, $c > 0$. *Hence find its mean and S.D.* *(Kurukshetra, 2009)*

Solution. The moment generating function about the origin is

$$M_0(t) = \int_0^{\infty} e^{tx} \cdot \frac{1}{c} e^{-x/c}\, dx = \frac{1}{c} \int_0^{\infty} e^{(t-1/c)x}\, dx \qquad \left[\because\ |t| < \frac{1}{c}\right]$$

$$= \frac{1}{c} \frac{\left| e^{(t-1/c)x} \right|_0^{\infty}}{|(t-1/c)|} = (1-ct)^{-1} = 1 + ct + c^2t^2 + c^3t^3 + \ldots$$

$$\therefore \qquad \mu'_1 = \left[\frac{d}{dt} M_0(t)\right]_{t=0} = (c + 2c^2t + 3c^3t^2 + \ldots)_{t=0} = c$$

$$\mu'_2 = \left[\frac{d^2}{dt^2} M_0(t)\right]_{t=0} = 2c^2, \text{ and } \mu_2 = \mu'_2 - (\mu'_1)^2 = 2c^2 - c^2 = c^2.$$

Hence the mean is c and S.D. is also c.

26.12 PROBABILITY GENERATING FUNCTION

The probability generating function (p.g.f.) $P_x(t)$ for a random variable x which takes integral values 0, 1, 2, 3, ... only, is defined by

$$P_x(t) = p_0 + p_1 t + p_2 t^2 + \ldots = \sum_{n=0}^{\infty} p_n t^n = E(t^x)$$

The coefficient of t^n in the expansion of $P(t)$ in powers of t gives $P(t)_{x=n}$.

$$\frac{\partial P}{\partial t} = \sum_{n=0}^{\infty} n p_n t^{n-1} \qquad \text{or} \qquad \left(\frac{\partial P}{\partial t}\right)_{t=1} = \Sigma\, n p_n = \mu_1'$$

$$\frac{\partial^2 P}{\partial t^2} = \sum_{n=0}^{\infty} n(n-1)\, p_n t^{n-2} \qquad \text{or} \qquad \left(\frac{\partial^2 P}{\partial t^2}\right)_{t=1} = \Sigma\, n(n-1)\, p_n = \mu_2' - \mu_1'$$

$$= \mu_2 + \mu_1'^2 - \mu_1' \text{ and so on}$$

Also $$\left(\frac{\partial^k P}{\partial t^k}\right)_{t=0} = n!\, p_n,\ k = 1, 2, \ldots n.$$

For integral valued variates, we have

$$P_x(e^t) = E(e^{tx}) = m.g.f. \text{ for } x.$$

Obs. The p.g.f. of the sum of two independent random variables is the product of their p.g.f.'s.

Example 26.37. *If x be a random variable with probability generating function $P_x(t)$, find the probability generating function of*

(i) $x + 2$ *(ii) $2x$.*

Solution. We have $P_x(t) = \sum_{k=0}^{\infty} p_k t^k$

(*i*) Probability generating function of $x + 2 = \sum_{k=0}^{\infty} p_k\, t^{k+2} = t^2 \sum_{k=0}^{\infty} p_k\, t^k = t^2\, P_x(t)$.

(*ii*) Probability generating function of $2x = \sum_{k=0}^{\infty} p_k\, t^{2k} = \sum_{k=0}^{\infty} p_k (t^2)^k = P(t^2)$.

PROBLEMS 26.4

1. A random variable x has the following probability function :

Values of x :	-2	-1	0	1	2	3
$p(x)$:	0.1	k	0.2	$2k$	0.3	k

Find the value of k and calculate mean and variance. (*S.V.T.U., 2007 ; V.T.U., 2004 ; Madras, 2003*)

2. Find the standard deviation for the following discrete distribution :

x :	8	12	16	20	24
$p(x)$:	1/8	1/6	3/8	1/4	1/12

3. Obtain the distribution function of the total number of heads occurring in three tosses of an unbiased coin.
4. Show that for any discrete distribution $\beta_2 \geq 1$.
5. From an urn containing 3 red and 2 white balls, a man is to draw 2 balls at random without replacement, being promised Rs. 20 for each red ball he draws and Rs. 10 for each white one. Find his expectation.
6. Four coins are tossed. What is the expectation of the number of heads ?
7. The diameter of an electric cable is assumed to be a continuous variate with p.d.f. $f(x) = 6x(1-x)$, $0 \leq x \leq 1$. Verify that the above is a p.d.f. Also find the mean and variance.
8. A random variable gives measurements X between 0 and 1 with a probability function

$$f(x) = 12x^3 - 21x^2 + 10x, \quad 0 \leq x \leq 1$$
$$= 0$$

(*i*) Find $P\left(X \leq \frac{1}{2}\right)$ and $P\left(X > \frac{1}{2}\right)$

(*ii*) Find a number k such that $P(X \leq k) = \frac{1}{2}$ (*J.N.T.U., 2003*)

9. The power reflected by an aircraft that is received by a radar can be described by an exponential random variable X.

The probability density of X is given by $f(x) = \begin{cases} \frac{1}{x_0} e^{-x/x_0}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

where x_0 is the average power received by the radar.

(*i*) What is the probability that the radar will receive power larger than the power received on the average ? (*ii*) What is the probability that the radar will receive power less than the power received on the average ?

(*Mumbai, 2006*)

10. A function is defined as follows :

$$f(x) = 0, \qquad x < 2$$
$$= \frac{1}{18}(2x+3), \qquad 2 \leq x \leq 4$$
$$= 0, \qquad x > 4.$$

Show that it is a density function. Find the probability that a variate having this density will fall in the interval $2 \leq x \leq 3$?

11. A continuous distribution of a variable x in the range $(-3, 3)$ is defined as

$$f(x) = \frac{1}{16}(3+x)^2, \qquad -3 \leq x < -1$$
$$= \frac{1}{16}(2-6x^2), \qquad -1 \leq x < 1$$
$$= \frac{1}{16}(3-x)^2, \qquad 1 \leq x \leq 3.$$

Verify that the area under the curve is unity. Show that the mean is zero. *(Kurukshetra, 2005)*

12. The frequency function of a continuous random variable is given by

$$f(x) = y_0 x(2-x),\ 0 \le x \le 2.$$

Find the value of y_0, mean and variance of x. *(Kerala, 2005 ; J.N.T.U., 2003)*

13. The probability density $p(x)$ of a continuous random variable is given by

$$p(x) = y_0 e^{-|x|},\ -\infty < x < \infty.$$

Prove that $y_0 = 1/2$. Find the mean and variance of the distribution.

(S.V.T.U., 2008 ; Kurukshetra, 2007 ; V.T.U., 2004)

14. If $f(x) = \begin{cases} \frac{1}{2}(x+1), & -1 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$

represents the density of a random variable X, find $E(X)$ and Var (X).

15. A function is defined as under :

$$f(x) = 1/k,\ x_1 \le x \le x_2$$
$$= 0, \text{ elsewhere.}$$

Find the cumulative distribution of the variate x when k satisfies the requirements for $f(x)$ to be a density function.

26.13 REPEATED TRIALS

We know that the probability of getting a head or a tail on tossing a coin is $\frac{1}{2}$. If the coin is tossed thrice, the probability of getting one head and two tails can be combined as $H—T—T$, $T—H—T$, $T—T—H$. The probability of each one of these being $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$, *i.e.*, $\left(\frac{1}{2}\right)^3$, their total probability shall be $3(1/2)^3$.

Similarly if a trial is repeated n times and if p is the probability of a success and q that of a failure, then the probability of r successes and $n-r$ failures is given by $p^r q^{n-r}$.

But these r successes and $n-r$ failures can occur in any of the ${}^nC_r$ ways in each of which the probability is same.

Thus the probability of r successes is ${}^nC_r\, p^r q^{n-r}$.

Cor. *The probabilities of at least r successes in n trials*

$$= \text{the sum of the probabilities of } r, r+1, \ldots, n \text{ successes}$$
$$= {}^nC_r\, p^r q^{n-r} + {}^nC_{r+1}\, p^{r+1} q^{n-r-1} + \ldots + {}^nC_n\, p^n.$$

26.14 (1) BINOMIAL DISTRIBUTION*

It is concerned with trials of a repetitive nature in which only the occurrence or non-occurrence, success or failure, acceptance or rejection, yes or no of a particular event is of interest.

If we perform a series of independent trials such that for each trial p is the probability of a success and q that of a failure, then the probability of r successes in a series of n trials is given by ${}^nC_r\, p^r q^{n-r}$, where r takes any integral value from 0 to n. The probabilities of 0, 1, 2, ... r, ..., n successes are, therefore, given by

$$q^n,\ {}^nC_1 pq^{n-1},\ {}^nC_2 p^2 q^{n-2},\ \ldots,\ {}^nC_r\, p^r q^{n-r},\ \ldots,\ p^n.$$

The probability of the number of successes so obtained is called *the* **binomial distribution** for the simple reason that the probabilities are the successive terms in the expansion of the binomial $(q+p)^n$.

$\therefore$ the sum of the probabilities

$$= q^n + {}^nC_1 pq^{n-1} + {}^nC_2 p^2 q^{n-2} + \ldots + p^n = (q+p)^n = 1.$$

(2) Constants of the binomial distribution. The moment generating function about the origin is

$$M_0(t) = E(e^{tx}) = \Sigma\, {}^nC_x\, p^x q^{n-x}\, e^{tx} \qquad \text{[By (1) § 26.11]}$$
$$= \Sigma\, {}^nC_x\, (pe^t)^x q^{n-x} = (q + pe^t)^n$$

* It was discovered by a Swiss mathematician *Jacob Bernoulli* and was published posthumously in 1713.

Differentiating with respect to t and putting $t = 0$ and using (3) § 26.11, we get the mean

$$\mu'_1 = np.$$

Since $M_a(t) = e^{-at} M_0(t)$, the m.g.f. of the binomial distribution about its mean $(m) = np$, is given by

$$M_m(t) = e^{-npt}(q + pe^t)^n = (qe^{-pt} + pe^{qt})^n$$

$$= \left(1 + pq\frac{t^2}{2!} + pq(q^2 - p^2)\frac{t^3}{3!} + pq(q^3 + p^3)\frac{t^4}{4!} + \cdots\right)^n$$

or
$$1 + \mu_1 t + \mu_2 \frac{t^2}{2!} + \mu_3 \frac{t^3}{3!} + \mu_4 \frac{t^4}{4!} + \cdots$$

$$= 1 + npq\frac{t^2}{2!} + npq(q - p)\frac{t^3}{3!} + npq\,[1 + 3(n - 2)pq]\frac{t^4}{4!} + \ldots$$

Equating the coefficients of like powers of t on either side, we have

$$\mu_2 = npq,\ \mu_3 = npq(q - p),\ \mu_4 = npq\,[1 + 3(n - 2)pq].$$

Also
$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(q - p)^2}{npq} = \frac{(1 - 2p)^2}{npq} \quad \text{and} \quad \beta_2 = \frac{\mu_4}{\mu_2^2} = 3 + \frac{1 - 6pq}{npq}$$

Thus *mean* $= np$, *standard deviation* $= \sqrt{(npq)}$.

skewness $= (1 - 2p)/\sqrt{(npq)}$, *kurtosis* $= \beta_2$.

Obs. The skewness is positive for $p < \frac{1}{2}$ and negative for $p > \frac{1}{2}$. When $p = \frac{1}{2}$, the skewness is zero, *i.e.*, the probability curve of the binomial distribution will be symmetrical (bell-shaped).

As n the number of trials increase indefinitely, $\beta_1 \to 0$, and $\beta_2 \to 3$.

(3) Binomial frequency distribution. If n independent trials constitute one experiment and this experiment be repeated N times, then the frequency of r successes is $N\,{}^nC_r\,p^r\,q^{n-r}$. The possible number of successes together with these expected frequencies constitute the *binomial frequency distribution.*

(4) Applications of Binomial distribution. This distribution is applied to problems concerning : (*i*) Number of defectives in a sample from production line,

(*ii*) Estimation of reliability of systems,

(*iii*) Number of rounds fired from a gun hitting a target,

(*iii*) Radar detection.

Example 26.38. *The probability that a pen manufactured by a company will be defective is 1/10. If 12 such pens are manufactured, find the probability that*

(*a*) *exactly two will be defective.* (*b*) *at least two will be defective.*

(*c*) *none will be defective.* (*V.T.U., 2004 ; Burdwan, 2003*)

Solution. The probability of a defective pen is 1/10 = 0.1

$\therefore$ The probability of a non-defective pen is $1 - 0.1 = 0.9$

(*a*) *The probability that exactly two will be defective*

$$= {}^{12}C_2\,(0.1)^2\,(0.9)^{10} = 0.2301$$

(*b*) *The probability that at least two will be defective*

$$= 1 - (\text{prob. that either none or one is non-defective})$$

$$= 1 - [{}^{12}C_0(0.9)^{12} + {}^{12}C_1(0.1)\,(0.9)^{11}] = 0.3412$$

(*c*) *The probability that none will be defective*

$$= {}^{12}C_{12}(0.9)^{12} = 0.2833.$$

Example 26.39. *In 256 sets of 12 tosses of a coin, in how many cases one can expect 8 heads and 4 tails.* (*J.N.T.U., 2003*)

Solution. $P(\text{head}) = \frac{1}{2}$ and $P(\text{tail}) = \frac{1}{2}$

By binomial distribution, probability of 8 heads and 4 tails in 12 trials is

$$P(X=8) = {}^{12}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^4 = \frac{12!}{8!\,4!} \cdot \frac{1}{2^{12}} = \frac{495}{4096}$$

$\therefore$ the expected number of such cases in 256 sets

$$= 256 \times P(X=8) = 256\ \frac{495}{4096} = 30.9 = 31 \text{ (say)}.$$

Example 26.40. *In sampling a large number of parts manufactured by a machine, the mean number of defectives in a sample of 20 is 2. Out of 1000 such samples, how many would be expected to contain at least 3 defective parts.* (V.T.U., 2004)

Solution. Mean number of defectives = 2 = np = $20p$.

$\therefore$ The probability of a defective part is $p = 2/20 = 0.1$.

and the probability of a non-defective part = 0.9

$\therefore$ The probability of at least three defectives in a sample of 20.

$$= 1 - (\text{prob. that either none, or one, or two are non-defective parts})$$
$$= 1 - [{}^{20}C_0(0.9)^{20} + {}^{20}C_1(0.1)(0.9)^{19} + {}^{20}C_2(0.1)^2(0.9)^{18}]$$
$$= 1 - (0.9)^{18} \times 4.51 = 0.323.$$

Thus *the number of samples having at least three defective parts* out of 1000 samples

$$= 1000 \times 0.323 = 323.$$

Example 26.41. *The following data are the number of seeds germinating out of 10 on damp filter paper for 80 sets of seeds. Fit a binomial distribution to these data :*

x :	0	1	2	3	4	5	6	7	8	9	10
f :	6	20	28	12	8	6	0	0	0	0	0

Solution. Here $n = 10$ and $N = \Sigma f_i = 80$

$$\therefore \qquad \text{mean} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{20+56+36+32+30}{80} = \frac{174}{80} = 2.175$$

Now the mean of a binomial distribution = np

i.e., $\qquad np = 10p = 2.175 \qquad \therefore \quad p = 0.2175,\ q = 1 - p = 0.7825$

Hence the binomial distribution to be fitted is

$$N(q+p)^n = 80\,(0.7825 + 0.2175)^{10}$$
$$= 80.{}^{10}C_0\,(0.7825)^{10} + 80.{}^{10}C_1\,(0.7825)^9\,(0.2175)^1 + {}^{10}C_2\,(0.7825)^8\,(0.2175)^2 + \ldots + {}^{80}C_9\,(0.7825)^1\,(0.2175)^9 + {}^{80}C_{10}\,(0.2175)^{10}$$
$$= 6.885 + 19.13 + 23.94 + \ldots + 0.0007 + 0.00002$$

$\therefore$ the successive terms in the expansion give the expected or theoretical frequencies which are

x :	0	1	2	3	4	5	6	7	8	9	10
f :	6.9	19.1	24.0	17.8	8.6	2.9	0.7	0.1	0	0	0

PROBLEMS 26.5

1. Determine the binomial distribution for which mean = 2 (variance) and mean + variance = 3. Also find $P(X \le 3)$. (*Kerala, 2005*)
2. An ordinary six-faced die is thrown four times. What are the probabilities of obtaining 4, 3, 2, 1 and 0 faces ?
3. If the chance that one of the ten telephone lines is busy at an instant is 0.2.
   (*a*) What is the chance that 5 of the lines are busy ?
   (*b*) What is the most probable number of busy lines and what is the probability of this number ?
   (*c*) What is the probability that all the lines are busy ? (*V.T.U., 2002 S*)
4. If the probability that a new-born child is a male is 0.6, find the probability that in a family of 5 children there are exactly 3 boys. (*Kurukshetra, 2005*)

5. If on an average 1 vessel in every 10 is wrecked, find the probability that out of 5 vessels expected to arrive, at least 4 will arrive safely. *(P.T.U., 2005)*
6. The probability that a bomb dropped from a plane will strike the target is 1/5. If six bombs are dropped, find the probability that (*i*) exactly two will strike the target, (*ii*) at least two will strike the target.
7. A sortie of 20 aeroplanes is sent on an operational flight. The chances that an aeroplane fails to return is 5%. Find the probability that (*i*) one plane does not return (*ii*) at the most 5 planes do not return, and (*iii*) what is the most probable number of returns ? *(Hissar, 2007)*
8. The probability that an entering student will graduate is 0.4. Determine the probability that out of 5 students (*a*) none (*b*) one and (*c*) at least one will graduate.
9. Out of 800 families with 5 children each, how many would you expect to have (*a*) 3 boys, (*b*) 5 girls, (*c*) either 2 or 3 boys ? Assume equal probabilities for boys and girls. *(V.T.U., 2004)*
10. If 10 per cent of the rivets produced by a machine are defective, find the probability that out of 5 rivets chosen at random (*i*) none will be defective, (*ii*) one will be defective, and (*iii*) at least two will be defective.
11. In a bombing action there is 50% chance that any bomb will strike the target. Two direct hits are needed to destroy the target completely. How many bombs are required to be dropped to give a 99% chance or better of completely destroying the target. *(V.T.U., 2003 S)*
12. A product is 0.5% defective and is packed in cartons of 100. What percentage contains not more than 3 defectives ?
13. If in a lot of 500 solenoids 25 are defective, find the probability of 0, 1, 2, 3 defective solenoids in a random sample of 20 solenoids.
14. 500 articles were selected at random out of a batch containing 10,000 articles, and 30 were found to be defective. How many defectives articles would you reasonably expect to have in the whole batch ? *(J.N.T.U., 2003)*
15. Fit a binomial distribution for the following data and compare the theoretical frequencies with the actual ones :

x:	0	1	2	3	4	5
f:	2	14	20	34	22	8

(Bhopal, 2006)

16. Fit a binomial distribution to the following frequency distribution :

x:	0	1	2	3	4	5	6
f:	13	25	52	58	32	16	4

(Kurukshetra, 2009 ; S.V.T.U., 2007)

26.15 (1) POISSON DISTRIBUTION*

It is a distribution related to the probabilities of events which are extremely rare, but which have a large number of independent opportunities for occurrence. The number of persons born blind per year in a large city and the number of deaths by horse kick in an army corps are some of the phenomena, in which this law is followed.

This distribution can be derived as a limiting case of the binomial distribution by making n very large and p very small, keeping np fixed (= m, say).

The probability of r successes in a binomial-distribution is

$$P(r) = {}^nC_r\, p^r q^{n-r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}\, p^r q^{n-r}$$

$$= \frac{np(np-p)(np-2p)\cdots(np-\overline{r-1}p)}{r!}\,(1-p)^{n-r}$$

As $n \to \infty$, $p \to 0$ $(np = m)$, we have

$$P(r) = \frac{m^r}{r!} \operatorname*{Lt}_{n\to\infty} \frac{(1-m/n)^n}{(1-m/n)^r} = \frac{m^r}{r!}\, e^{-m}$$

so that the probabilities of 0, 1, 2..., r,... successes in a Poisson distribution are given by

$$e^{-m},\ me^{-m},\ \frac{m^2e^{-m}}{2!}, \cdots, \frac{m^re^{-m}}{r!}, \cdots$$

The sum of these probabilities is unity as it should be.

* It was discovered by a French mathematician S.D. Poisson in 1837.

(2) Constants of the Poisson distribution. These constants can easily be derived from the corresponding constants of the binomial distribution simply by making $n \to \infty, p \to 0, (q \to 1)$ and noting that $np = m$

$$Mean = \text{Lt}\,(np) = m$$

$$\mu_2 = \text{Lt}\,(npq) = m\ \text{Lt}\,(q) = m$$

$\therefore$ $\quad Standard\ deviation = \sqrt{m}$

Also $\mu_3 = m$, $\mu_4 = m + 3m^2$

$\therefore$ $\quad Skewness\ (= \sqrt{\beta_1}\,) = 1/m$, $Kurtosis\ (= \beta_2) = 3 + 1/m$.

Since μ_3 is positive, Poisson distribution is positively skewed and since $\beta_2 > 3$, it is *Leptokurtic*.

(3) Applications of Poisson distribution. This distribution is applied to problems concerning : (*i*) Arrival pattern of 'defective vehicles in a workshop', 'patients in a hospital' or 'telephone calls'.

(*ii*) Demand pattern for certain spare parts.

(*iii*) Number of fragments from a shell hitting a target.

(*iv*) Spatial distribution of bomb hits.

Example 26.42. *If the probability of a bad reaction from a certain injection is 0.001, determine the chance that out of 2,000 individuals more than two will get a bad reaction.* (*V.T.U., 2008 ; Kottayam, 2005*)

Solution. It follows a Poisson distribution as the probability of occurrence is very small.

Mean $m = np = 2000(0.001) = 2$

Probability that more than 2 will get a bad reaction

= 1 – [prob. that no one gets a bad reaction + prob. that one gets a bad reaction + prob. that two get bad reaction]

$$= 1 - \left[e^{-m} + \frac{m^1 e^{-m}}{1!} + \frac{m^2 e^{-m}}{2!}\right] = 1 - \left[\frac{1}{e^2} + \frac{2}{e^2} + \frac{2}{e^2}\right] \qquad [\because\ m = 2]$$

$$= 1 - \frac{5}{e^2} = 0.32. \qquad [\because\ e = 2.718\,]$$

Example 26.43. *In a certain factory turning out razor blades, there is a small chance of 0.002 for any blade to be defective. The blades are supplied in packets of 10, use Poisson distribution to calculate the approximate number of packets containing no defective, one defective and two defective blades respectively in a consignment of 10,000 packets.* (*Kurukshetra, 2009 S; Madras, 2006 ; V.T.U., 2004*)

Solution. We know that $m = np = 10 \times 0.002 = 0.02$

$$e^{-0.02} = 1 - 0.02 + \frac{(0.02)^2}{2!} - \ldots = 0.9802 \text{ approximately}$$

Probability of no defective blade is $e^{-m} = e^{-0.02} = 0.9802$

$\therefore$ no. of packets containing no defective blade is

$$10{,}000 \times 0.9802 = 9802$$

Similarly the number of packets containing one defective blade $= 10{,}000 \times me^{-m}$

$$= 10{,}000 \times (0.02) \times 0.9802 = 196$$

Finally the number of packets containing two defective blades

$$= 10{,}000 \times \frac{m^2 e^{-m}}{2!} = 10{,}000 \times \frac{(0.02)^2}{2!} \times 0.9802 = 2 \text{ approximately.}$$

Example 26.44. *Fit a Poisson distribution to the set of observations :*

x :	0	1	2	3	4
f :	122	60	15	2	1

(*Bhopal, 2007 S ; V.T.U., 2004 ; U.P.T.U., 2003*)

Solution. Mean $= \dfrac{\Sigma f_i x_i}{\Sigma f_i} = \dfrac{60 + 36 + 6 + 4}{200} = 0.5.$

$\therefore$ mean of Poisson distribution *i.e.*, $m = 0.5$.

Hence the theoretical frequency for r successes is

$$\frac{Ne^{-m}(m)^r}{r!} = \frac{200e^{-0.5}(.5)^r}{r!} \text{ where } r = 0, 1, 2, 3, 4$$

$\therefore$ the theoretical frequencies are

x:	0	1	2	3	4
f:	121	61	15	2	0

($\because$ $e^{-.5} = 0.61$)

PROBLEMS 26.6

1. If a random variable has a Poisson distribution such that $P(1) = P(2)$, find (*i*) mean of the distribution. (*ii*) $P(4)$. (*V.T.U., 2003*)
2. X is a Poisson variable and it is found that the probability that $X = 2$ is two-thirds of the probability that $X = 1$. Find the probability that $X = 0$ and the probability that $X = 3$. What is the probability that X exceeds 3 ?
3. For Poisson distribution, prove that $m\,\mu_2\,\gamma_1\,\gamma_2 = 1$, where symbols have their usual meanings. (*S.V.T.U., 2008*)
4. A certain screw making machine produces on average of 2 defective screws out of 100, and packs them in boxes of 500. Find the probability that a box contains 15 defective screws. (*Kurukshetra, 2006*)
5. A manufacturer knows that the condensers he makes contain on the average 1 % defectives. He packs them in boxes of 100. What is the probability that a box picked at random will contain 3 or more faulty condensers ?
6. A car-hire firm has two cars which it hires out day by day. The number of demands for a car on each day is distributed as a Poisson distribution with mean 1.5. Calculate the proportion of days (*i*) on which there is no demand, (*ii*) on which demand is refused. ($e^{-1.5} = 0.2231$). (*Bhopal, 2008 S ; J.N.T.U., 2003*)
7. The incidence of occupational disease in an industry is such that the workmen have a 10% chance of suffering from it. What is probability that in a group of 7, five or more will suffer from it ?
8. The frequency of accidents per shift in a factory is as shown in the following table :

Accidents per shift :	0	1	2	3	4
Frequency :	180	92	24	3	1

Calculate the mean number of accidents per shift and the corresponding Poisson distribution and compare with actual observations.

9. A source of liquid is known to contain bacteria with the mean number of bacteria per cubic centimetre equal to 3. Ten 1 c.c., test-tubes are filled with the liquid. Assuming that Poisson distribution is applicable, calculate the probability that all the test-tubes will show growth *i.e.*, contain atleast 1 bacterium each.
10. Find the expectation of the function $\phi(x) = xe^{-x}$ in a Poisson distribution. (*V.T.U., 2003*)

[**Hint :** If m be the mean of the Poisson distribution, then expectation of

$$\phi(x) = \sum_{x=0}^{\infty} \frac{\phi(x) \cdot m^x e^{-m}}{x!} = m \exp.\, m\,(e^{-1} - m - 1)$$

11. Fit a Poisson distribution to the following :

x:	0	1	2	3	4
f:	46	38	22	9	1

(*Kurukshetra, 2009 ; Bhopal, 2008 ; V.T.U., 2003 S*)

12. Fit a Poisson distribution to the following data given the number of yeast cells per square for 400 squares :

No. of cells per sq. :	0	1	2	3	4	5	6	7	8	9	10
No. of squares :	103	143	98	42	8	4	2	0	0	0	0

(*S.V.T.U., 2007*)

26.16 (1) NORMAL DISTRIBUTION*

Now we consider a continuous distribution of fundamental importance, namely the normal distribution. *Any quantity whose variation depends on random causes is distributed according to the normal law.* Its importance lies in the fact that a large number of distributions approximate to the normal distribution.

* In 1924, *Karl Pearson* found this distribution which *Abraham De Moivre* had discovered as early as 1733. See footnote p. 843 and 647.

Let us define a variate $z = \dfrac{x - np}{\sqrt{(npq)}}$ ...(1)

where x is a binomial variate with mean np and S.D. $\sqrt{(npq)}$ so that z is a variate with mean zero and variance unity. In the limit as n tends to infinity, the distribution of z becomes a continuous distribution extending from $-\infty$ to ∞.

It can be shown that *the limiting form of the binomial distribution (1) for large values of n when neither p nor q is very small, is the normal distribution.* The normal curve is of the form

$$y = \frac{1}{\sigma\sqrt{(2\pi)}} e^{-(x-\mu)^2/2\sigma^2} \qquad ...(2)$$

where μ and σ are the mean and standard deviation respectively.

(2) Properties of the normal distribution

I. The normal curve (2) is bell-shaped and is symmetrical about its mean. It is unimodal with ordinates decreasing rapidly on both sides of the mean (Fig. 26.3). The maximum ordinate is $1/\sigma\sqrt{(2\pi)}$, found by putting $x = \mu$ in (2).

As it is symmetrical, its mean, median and mode are the same. Its points of inflexion (found by putting $d^2y/dx^2 = 0$ and verifying that at these points $d^3y/dx^3 \neq 0$) are given by $x = \mu \pm \sigma$, *i.e.*, these points are equidistant from the mean on either side.

II. Mean deviation from the mean μ

$$= \int_{-\infty}^{\infty} |x - \mu| \cdot \frac{1}{\sigma\sqrt{(2\pi)}} e^{-(x-\mu)^2/2\sigma^2}\, dx \qquad [\text{Put } z = (x-\mu)/\sigma]$$

$$= \frac{\sigma}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} |z| e^{-z^2/2}\, dz$$

$$= \frac{\sigma}{\sqrt{(2\pi)}} \left[\int_{-\infty}^{0} -ze^{-z^2/2} dz + \int_{0}^{\infty} ze^{-z^2/2} dz \right] = \frac{2\sigma}{\sqrt{(2\pi)}} \int_{0}^{\infty} ze^{-z^2/2}\, dz$$

$$= \frac{2\sigma}{\sqrt{(2\pi)}} \left| -e^{-z^2/2} \right|_0^{\infty} = -\sqrt{\left(\frac{2}{\pi}\right)} \sigma(0-1) = 0.7979\,\sigma \approx (4/5)\,\sigma$$

III. Moments about the mean

$$\mu_{2n+1} = \int_{-\infty}^{\infty} (x-\mu)^{2n+1} \frac{1}{\sigma\sqrt{(2\pi)}} e^{-(x-\mu)^2/2\sigma^2}\, dx$$

$$= \frac{\sigma^{2n+1}}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} z^{2n+1} e^{-z^2/2}\, dz \text{ where } z = (x-\mu)/\sigma$$

$= 0$, since the integral is an odd function.

Thus all odd order moments about the mean vanish.

$$\mu_{2n} = \int_{-\infty}^{\infty} (x-\mu)^{2n} \frac{1}{\sigma\sqrt{(2\pi)}} e^{-(x-\mu)^2/2\sigma^2}\, dx$$

$$= \frac{\sigma^{2n}}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} z^{2n-1} e^{-z^2/2} \,.\, z dz \qquad [\text{Integrate by parts}]$$

$$= \frac{\sigma^{2n}}{\sqrt{(2\pi)}} \left[\left| -z^{2n-1} e^{-z^2/2} \right|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} (2n-1) z^{2n-2} e^{-z^2/2}\, dz \right]$$

$$= \frac{\sigma^{2n}}{\sqrt{(2\pi)}} (0-0) + (2n-1)\,\sigma^2\, \mu_{2n-2}$$

Repeated application of this reduction formula, gives

$$\mu_{2n} = (2n-1)(2n-3) \ldots 3 \,.\, 1\sigma^{2n}$$

In particular, $\mu_2 = \sigma^2$, $\mu_4 = 3\sigma^4$.

Hence $\quad \beta_1 = \dfrac{\mu_3^{\,2}}{\mu_2^{\,3}} = 0$ and $\beta_2 = \dfrac{\mu_4}{\mu_2^{\,2}} = 3$

i.e., the coefficient of skewness is zero (*i.e.* the curve is symmetrical) and the Kurtosis is 3. This is the basis for the choice of the value 3 in the definitions of platykurtic and leptokurtic (page 844).

IV. The probability of x lying between x_1 and x_2 is given by the area under the normal curve from x_1 to x_2, *i.e.*, $P(x_1 \le x \le x_2)$

$$= \frac{1}{\sigma\sqrt{(2\pi)}} \int_{x_1}^{x_2} e^{-(x-\mu)^2/2\sigma^2}\, dx$$

$$= \frac{1}{\sqrt{(2\pi)}} \int_{z_1}^{z_2} e^{-z^2/2}\, dz \text{ where } z = (x-\mu)/\sigma,\ dz = dx/\sigma \text{ and } z_1 = (x_1-\mu)/\sigma,\ z_2 = (x_2-\mu)/\sigma.$$

$$= \frac{1}{\sqrt{(2\pi)}} \left\{ \int_0^{z_2} e^{-z^2/2}\, dz - \int_0^{z_1} e^{-z^2/2}\, dz \right\} = P_2(z) - P_1(z)$$

The values of each of the above integrals can be found from the table III–Appendix 2, which gives the values of

$$P(z) = \frac{1}{\sqrt{(2\pi)}} \int_0^z e^{-z^2/2}\, dz$$

for various values of z. This integral is called the *probability integral* or the *error function* due to its use in the theory of sampling and the theory of errors.

Using this table, we see that the area under the normal curve from $z = 0$ to $z = 1$, *i.e.* from $x = \mu$ to $\mu + \sigma$ is 0.3413.

$\therefore$ (*i*) The area under the normal curve between the ordinates $x = \mu - \sigma$ and $x = \mu + \sigma$ is 0.6826, ~ *68% nearly. Thus approximately 2/3 of the values lie within these limits.*

(*ii*) The area under the normal curve between $x = \mu - 2\sigma$ and $x = \mu + 2\sigma$ is 0.9544 ~ 95.5%, which implies that *about* $4\frac{1}{2}$ *% of the values lie outside these limits.*

(*ii*) *99.73% of the values lie between* $x = \mu - 3\sigma$ and $x = \mu + 3\sigma$ *i.e., only a quarter % of the whole lies outside these limits.*

(*iv*) *95% of the values lie between* $x = \mu - 1.96\sigma$ *and* $x = \mu + 1.96\,\sigma$ *i.e., only 5% of the values lie outside these limits.*

(*v*) *99% of the values lie between* $x = \mu - 2.58\sigma$ *and* $x = \mu + 2.58\sigma$ *i.e., only 1% of the values lie outside these limits.*

(*vi*) 99.9% of the values lie between $x = \mu - 3.29\sigma$ and $x = \mu + 3.29\sigma$.

In other words, a value that deviates more than σ from μ occurs about once in 3 trials. A value that deviates more than 2σ or 3σ from μ occurs about once in 20 or 400 trials. Almost all values lie within 3σ of the mean.

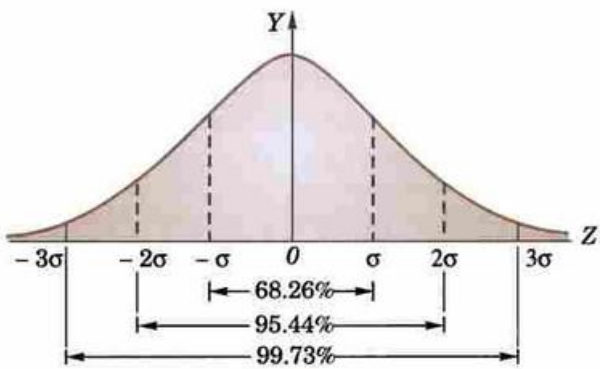


Fig. 26.4

The shape of the standardised normal curve is

$$y = \frac{1}{\sqrt{(2\pi)}} e^{-z^2/2} \text{ where } z = (x-\mu)/\sigma \qquad \text{...(3)}$$

and the respective areas are shown in Fig. 26.4. 'z' is called a *normal variate*.

(3) Normal frequency distribution. We can fit a normal curve to any distribution. If N be the total frequency, μ the mean and σ the standard deviation of the given distribution then the curve

$$y = \frac{N}{\sigma\sqrt{(2\pi)}} e^{-(x-\mu)^2/2\sigma^2} \qquad \text{...(4)}$$

will fit the given distribution as best as the data will permit. The frequency of the variate between x_1 and x_2 as given by the fitted curve, will be the area under (1) from x_1 to x_2.

(4) Applications of normal distribution. This distribution is applied to problems concerning :

(*i*) Calculation of errors made by chance in experimental measurements.

(*ii*) Computation of hit probability of a shot.

(*iii*) Statistical inference in almost every branch of science.

26.17 PROBABLE ERROR

Any lot of articles manufactured to certain specifications is subject to small errors. In fact, measurement of any physical quantity shows slight error. In general, these errors of manufacture or experiment are of random nature and therefore, follow a normal distribution. While quoting a specification of an experimental result, we usually mention the *probable error* (λ). *It is such that the probability of an error falling within the limits* $\mu - \lambda$ *and* $\mu + \lambda$ *is exactly equal to the chance of an error falling outside these limits, i.e.* the chance of an error lying within $\mu - \lambda$ and $\mu + \lambda$ is $\frac{1}{2}$.

$$\therefore \qquad \frac{1}{\sigma\sqrt{(2\pi)}} \int_{\mu-\lambda}^{\mu+\lambda} e^{-(x-\mu)^2/2\sigma^2}\, dx = \frac{1}{2}$$

or
$$\frac{1}{\sqrt{(2\pi)}} \int_0^{\lambda/\sigma} e^{-z^2/2} dz = \frac{1}{4} \qquad \left[z = \frac{x-\mu}{\sigma}\right]$$

The table V, (Appendix 2) gives $\lambda/\sigma = 0.6745$

Hence the probable error $\lambda = 0.6745\sigma \sim \frac{2}{3}\sigma$.

Obs. Quartile deviation $= \frac{1}{2}(Q_3 - Q_1) \sim \frac{2}{3}\sigma$; Mean deviation $\approx \frac{4}{5}\sigma$ [p. 839]

$\therefore$ *Q.D. : M.D. : S.D.* = 10 : 12 : 15. *(Madras, 2003)*

Example 26.45. *X is a normal variate with mean 30 and S.D. 5, find the probabilities that (i) $26 \le X \le 40$, (ii) $X \ge 45$ and (iii) $|X - 30| > 5$.* *(J.N.T.U., 2005)*

Solution. We have $\mu = 30$ and $\sigma = 5$

$$\therefore \qquad z = \frac{X-\mu}{\sigma} = \frac{X-30}{5}$$

(*i*) When $X = 26, z = -0.8$; when $X = 40, z = 2$

$$\begin{aligned}\therefore \qquad P(26 \le X \le 40) &= P(-0.8 \le z \le 2)\\ &= P(-0.8 \le z \le 0) + P(0 \le z \le 2) \qquad \text{[Using Table } III\text{]}\\ &= P(0 \le z \le 0.8) + 0.4772\\ &= 0.2881 + 0.4772 = 0.7653\end{aligned}$$

(*ii*) When $X = 45, z = 3$

$$\begin{aligned}\therefore \qquad P(X \ge 45) &= P(z \ge 3) = 0.5 - P(0 \le z \le 3)\\ &= 0.5 - 0.4986 = 0.0014\end{aligned}$$

(*iii*)
$$\begin{aligned}P[|X - 30| \le 5] &= P[25 \le X \le 35]\\ &= P(-1 \le z \le 1) = 2P(0 \le z \le 1)\\ &= 2 \times 0.3413 = 0.6826\end{aligned}$$

$$\begin{aligned}\therefore \qquad P[|X - 30| > 5] &= 1 - P[|X - 30| \le 5]\\ &= 1 - 0.6826 = 0.3174.\end{aligned}$$

Example 26.46. *A certain number of articles manufactured in one batch were classified into three categories according to a particular characteristic, being less than 50, between 50 and 60 and greater than 60. If this characteristic is known to be normally distributed, determine the mean and standard deviation for this batch if 60%, 35% and 5% were found in these categories.*

Solution. Let μ be the mean (at $z = 0$) and σ the standard deviation of the normal curve (Fig. 26.5).

Now 60% of the articles have the characteristic below 50, 35% between 50 and 60 and only 5% greater than 60.

Let the area to the left of the ordinate PQ be 60% and that between the ordinates PQ and ST be 35% so that the areas to the left of PQ ($z = z_1$) and ST ($z = z_2$) are 0.6 and 0.95 respectively, *i.e.*, the area $OPQR = 0.6 - 0.5 = 0.1$ and the area $OSTR = 0.45$.

$\therefore$ area corresponding to $z_1 \left(= \dfrac{50-\mu}{\sigma}\right) = 0.1$

and that corresponding to $z_2 \left(= \dfrac{60-\mu}{\sigma}\right) = 0.45$

From the table III, we have

$(50-\mu)/\sigma = 0.2533$ and $(60-\mu)/\sigma = 1.645$

whence $\sigma = 7.543$ and $\mu = 48.092$.

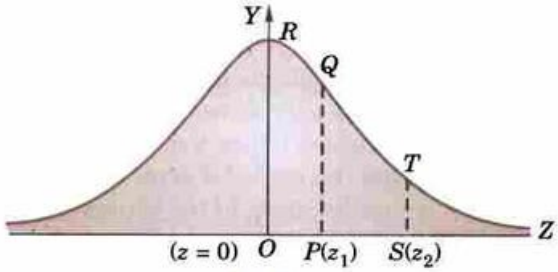


Fig. 26.5

Example 26.47. *In a normal distribution, 31% of the items are under 45 and 8% are over 64. Find the mean and S.D. of the distribution.* (V.T.U., 2009 ; S.V.T.U., 2008 ; Kurukshetra, 2007 S)

Solution. Let $\bar{x}$ be the mean and σ the S.D. 31% of the items are under 45 means area to the left of the ordinate $x = 45$. (Fig. 26.6)

When $x = 45$, let $z = z_1$ so that $z_1 = \dfrac{45-\bar{x}}{\sigma}$ ...(*i*)

$\therefore$ $\displaystyle\int_{-\infty}^{z_1} \phi(z)\,dz = 0.31$ or $\displaystyle\int_{-\infty}^{0} \phi(z)\,dz - \int_{z_1}^{0} \phi(z)\,dz = 0.31$

Hence $\displaystyle\int_{z_1}^{0} \phi(z)\,dz = \int_{-\infty}^{0} \phi(z)\,dz - 0.31 = 0.5 - 0.31 = 0.19$

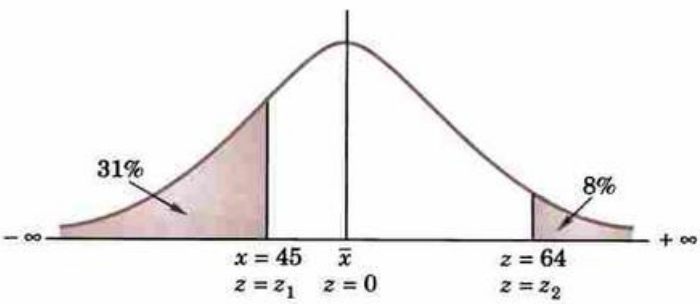


Fig. 26.6

From table III, $z_1 = -0.5$ ...(*ii*)

When $x = 64$, let $z = z_2$ so that $z_2 = (64 - \bar{x})/\sigma$ ...(*iii*)

$\therefore$ $\displaystyle\int_{z_2}^{\infty} \phi(z)\,dz = 0.08$ or $\displaystyle\int_{0}^{\infty} \phi(z)\,dz - \int_{0}^{z_2} \phi(z)\,dz = 0.08$

Hence $\displaystyle\int_{0}^{z_2} \phi(z)\,dz = \int_{0}^{\infty} \phi(z)\,dz - 0.08 = 0.5 - 0.08 = 0.42$

From table III, $z_2 = 1.4$ ...(*iv*)

From (*i*) and (*ii*), $45 - \bar{x} = -0.5\sigma$

From (*iii*) and (*iv*), $64 - \bar{x} = 1.4\sigma$

Solving these equations, we get $\bar{x} = 50$ and $\sigma = 10$.

Example 26.48. *In a test on 2000 electric bulbs, it was found that the life of a particular make, was normally distributed with an average life of 2040 hours and S.D. of 60 hours. Estimate the number of bulbs likely to burn for*

(a) more than 2150 hours, *(b) less than 1950 hours and*

(c) more than 1920 hours and but less than 2160 hours. (Bhopal, 2008 S ; U.P.T.U., 2008)

Solution. Here $\mu = 2040$ hours and $\sigma = 60$ hours.

(*a*) For $x = 2150$, $z = \dfrac{x-\mu}{\sigma} = 1.833$.

$\therefore$ area against $z = 1.83$ in the table III = 0.4664.

We, however, require the area to the right of the ordinate at $z = 1.83$. This area = 0.5 – 0.4664 = 0.0336.

Thus the number of bulbs expected to burn for more than 2150 hours

$$= 0.0336 \times 2000 = 67 \text{ approximately.}$$

(*b*) For $x = 1950$, $z = \dfrac{x-\mu}{\sigma} = -1.5$

The area required in this case is to the left of $z = -1.33$

i.e., $= 0.5 - 0.4082$ (table value for $z = 1.33$)

$= 0.0918.$

$\therefore$ the number of bulbs expected to burn for less than 1950 hours

$$= 0.0918 \times 2000 = 184 \text{ approximately.}$$

(*c*) When $x = 1920$, $\quad z = \dfrac{1920-2040}{60} = -2$

When $x = 2160$, $\quad z = \dfrac{2160-2040}{60} = 2.$

The number of bulbs expected to burn for more than 1920 hours but less than 2160 hours will be represented by the area between $z = -2$ and $z = 2$. This is twice the area from the table for $z = 2$, *i.e.*, $= 2 \times 0.4772 = 0.9544$.

Thus the required number of bulbs = 0.9544 × 2000 = 1909 nearly.

Example 26.49. *If the probability of committing an error of magnitude x is given by*

$$y = \frac{h}{\sqrt{\pi}} e^{-h^2x^2};$$

compute the probable error from the following data :

$m_1 = 1.305$; $m_2 = 1.301$; $m_3 = 1.295$; $m_4 = 1.286$;
$m_5 = 1.318$; $m_6 = 1.321$; $m_7 = 1.283$; $m_8 = 1.289$;
$m_9 = 1.300$; $m_{10} = 1.286$. *(Kurukshetra, 2005)*

Solution. From the given data which is normally distributed, we have

$$\text{mean} = \frac{1}{10}\Sigma m_i = \frac{12.984}{10} = 1.2984$$

and
$$\sigma^2 = \frac{1}{10}\Sigma(m_i - \text{mean})^2$$

$$= \frac{1}{10}[(0.007)^2 + (0.003)^2 + (0.003)^2 + (0.012)^2 + (0.02)^2 + (0.023)^2 + (0.015)^2 + (0.009)^2 + (0.002)^2 + (0.012)^2]$$

$$= 0.0001594 \text{ whence } \sigma = 0.0126.$$

$\therefore$ *probable error* $= \dfrac{2}{3}\sigma = 0.0084$ approx.

Example 26.50. *Fit a normal curve to the following distribution.*

x:	2	4	6	8	10
f:	1	4	6	4	1

(V.T.U., 2001)

Solution.
$$\text{Mean} = \frac{\Sigma fx}{\Sigma f} = \frac{2+16+36+32+10}{16} = 6$$

$$\text{S.D.} = \sqrt{\left[\frac{\Sigma fx^2}{\Sigma f} - \left(\frac{\Sigma fx}{\Sigma f}\right)^2\right]} = \sqrt{(40-36)} = 2$$

Taking $\mu = 6$, $\sigma = 2$ and $N = 16$, the equation of the normal curve is

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \text{ or } y = \frac{1}{2\sqrt{2\pi}} e^{-(x-6)^2/8} \qquad ...(i)$$

Area under (i) in (x_1, x_2) or (z_1, z_2)

$$= \frac{1}{\sqrt{2\pi}} \int_0^{z_2} e^{-z^2/2} dz - \frac{1}{\sqrt{2\pi}} \int_0^{z_1} e^{-z^2/2} dz \qquad \text{where } z = \frac{x-6}{2}$$

To evaluate these integrals, we refer to table III.

Calculations :

Mid x	(x_1, x_2)	(z_1, z_2)	*Area under (i) in* (z_1, z_2)	*Expected frequency*
2	(1, 3)	(– 2.5, – 1.5)	0.4938 – 0.4332	16 × 0.606 = 0.97
4	(3, 5)	(– 1.5, – 0.5)	0.4332 – 0.1915	16 × 0.2417 = 3.9
6	(5, 7)	(– 0.5, 0.5)	0.1915 + 0.1915	16 × 0.383 = 6.1
8	(7, 9)	(0.5, 1.5)	0.4332 – 0.1915	16 × 0.2417 = 3.9
10	(9, 11)	(1.5, 2.5)	0.4938 – 0.4332	16 × 0.606 = 0.97

Hence the expected (theoretical) frequencies corrected to nearest integer are 1, 4, 6, 4, 1 which agree with the observed frequencies. This shows that the normal curve (i) is a proper fit to the given distribution.

PROBLEMS 26.7

1. Show that the standard deviation for a normal distribution is approximately 25% more than the mean deviation.
2. For a normally distributed variate with mean 1 and S.D. 3, find the probabilities that
   (i) $3.43 \le x \le 6.19$ $\qquad$ (ii) $-1.43 \le x \le 6.19$.
3. If z is normally distributed with mean 0 and variance 1, find
   (i) $P_r\{z \le -1.64\}$; $\qquad$ (ii) z_1 if $P_r\{z \ge z_1\} = 0.84$.
4. In a certain examination, the percentage of candidates passing and getting distinctions were 45 and 9 respectively. Estimate the average marks obtained by the candidates, the minimum pass and distinction marks being 40 and 75 respectively. (Assume the distribution of marks to be normal). *(Kottayam, 2005)*
5. A manufacturer of air-mail envelopes knows from experience that the weight of the envelopes is normally distributed with mean 1.95 gm and standard deviation 0.05 gm. About how many envelopes weighing (i) 2 gm or more ; (ii) 2.05 gm or more can be expected in a given packet of 100 envelopes.
6. The mean height of 500 students is 151 cm. and the standard deviation is 15 cm. Assuming that the heights are normally distributed, find how many students' heights lie between 120 and 155 cm. *(Burdwan, 2003)*
7. The mean and standard deviation of the marks obtained by 1000 students in an examination are respectively 34.4 and 16.5. Assuming the normality of the distribution, find the approximate number of students expected to obtain marks between 30 and 60.
8. In an examination taken by 500 candidates, the average and the standard deviation of marks obtained (normally distributed) are 40% and 10%. Find approximately
   (i) how many will pass, if 50% is fixed as a minimum ?
   (ii) what should be the minimum if 350 candidates are to pass ?
   (iii) how many have scored marks above 60% ?
9. The mean inside diameter of a sample of 200 washers produced by a machine is 5.02 mm and the standard deviation is 0.05 mm. The purpose for which these washers are intended allows a maximum tolerance in the diameter of 4.96 to 5.08 mm, otherwise the washers are considered defective. Determine the percentage of defective washers produced by the machine, assuming the diameters are normally distributed.
   [**Hint.** 4.96 in standard units = (4.96 – 5.02)/0.05 = – 1.2
   5.08 in standard units = (5.08 – 5.02)/0.05 = 1.2
   Proportion of non-defective washers = 2 (area between $z = 0$ and $z = 1.2$)
   = 0.7698 or 77% nearly.
   ∴ percentage of defective washers = 100 – 77 = 23%.]
10. Assuming that the diameters of 1000 brass plugs taken consecutively from a machine, form a normal distribution with mean 0.7515 cm. and standard deviation 0.0020 cm., how many of the plugs are likely to be rejected if the approved diameter is 0.752 ± 0.004 cm. ? *(Bhopal, 2002)*

11. It is given that the age of thermostats of a particular make follow the normal law with mean 5 years and S.D. 2 years. 1000 units are sold out every month. How many of them will have to be replaced at the end of the second year.

12. The income of a group of 10,000 persons was found to be normally distributed with mean Rs. 750 p.m., and standard deviation of Rs. 50. Show that, of this group, about 95% had income exceeding Rs. 668 and only 5% had income exceeding Rs. 832. Also find the lowest income among the richest 100. *(U.P.T.U., 2004 S)*

13. Find the equation of the best fitting normal curve to the following distribution :

x :	0	1	2	3	4	5
y :	13	23	34	15	11	4

14. Obtain the equation of the normal probability curve that may be fitted to the following data :

Variable :	4	6	8	10	12	14	16	18	20	22	24
Frequency :	1	7	15	22	35	43	38	20	13	5	1

15. A factory turns out an article by mass production and it is found that 10% of the product is rejected. Find the S.D. of the number of rejects and the equation to the normal curve to represent the number of rejects.

[**Hint.** $p = 0.1, q = 0.9, n = 100$.

$\therefore$ binomial distribution of rejects gives mean $= np = 10$, S.D. $= \sqrt{(npq)} = 3$

If this binomial distribution is approximated by a normal distribution, then the equation to the normal curve is

$$y = \frac{100}{\sigma\sqrt{(2\pi)}} e^{-(x-\mu)^2/2\sigma^2} \quad \text{where } \mu = 10, \sigma = 3.]$$

16. Given that the probability of committing an error of magnitude x is

$$y = \frac{h}{\sqrt{\pi}} e^{-h^2x^2}, \text{ show that the probable error is } 0.4769/h.$$

26.18 NORMAL APPROXIMATION TO BINOMIAL DISTRIBUTION

If the number of successes in a Binomial distribution range from x_1 to x_2, then the probability of getting these successes

$$= \sum_{r=x_1}^{x_2} {}^nC_r \, p^r \, q^{n-r}$$

As the number of trials increases, the Binomial distribution becomes approximated to the Normal distribution. The mean np and the variance npq of the binomial distribution will be quite close to the mean and standard deviation of the approximated normal distribution. Thus *for n sufficiently large (≥ 30), the binomial distribution with probability of success p, is approximated by the normal distribution with* $\mu = np, \sigma = \sqrt{npq}$.

We must however, be careful to get the correct values of z. For any success x, real class interval is $(x - 1/2, x + 1/2)$. Hence

$$z_1 = \frac{x_1 - \frac{1}{2} - \mu}{\sigma} = \frac{x_1 - \frac{1}{2} - np}{\sqrt{npq}}; \; z_2 = \frac{x_2 + \frac{1}{2} - np}{\sqrt{npq}}$$

so that $P(x_1 < x < x_2) = P(z_1 < z < z_2) = \int_{z_1}^{z_2} \phi(z)\, dz$ which can be calculated by using table III–Appendix 2.

Example 26.51. *In a referendum 60% of voters voted in favour. A random sample of 200 voters was selected. What is the probability that in the sample*

(a) more than 130 voted in favour ?

(b) between 105 and 130 inclusive voted in favour ?

(c) 120 voted in favour ?

Solution. Here $n = 200, p = 0.6, q = 0.4$

$\therefore \qquad \mu = np = 200 \times 0.6 = 120; \; \sigma = \sqrt{npq} = \sqrt{48} = 6.928$

(*a*) $P(x > 130) = P(x > 130.5) = P\left(x > \frac{130.5 - 120}{\sqrt{48}}\right) = P(z > 1.516) = 0.0648$

(b) $P(105 < x < 130) = P(105.5 < x < 129.5)$

$$= P\left(\frac{105.5-120}{\sqrt{48}} < z < \frac{129.5-120}{\sqrt{48}}\right) = P(-2.09 < z < 1.37) = 0.8964$$

(c) $P(x = 120) = P(119.5 < x < 120.5)$

$$= P(-0.072 < z < 0.072) = 0.0575.$$

PROBLEMS 26.8

1. A pair of unbiased dice are rolled 180 times and their score recorded. Find
   (a) $P(x \le 20)$, (b) $P(20 \le x \le 40)$, (c) $P(20 < x \le 30)$.
2. A marksman has a probability of 0.9 of hitting a target on a single shot. If the marksman has 40 shots, what is the probability that he hits the target (a) at least 35 times; (b) between 34 and 36 times; (c) 37 times.
3. A certain drug is effective in 72% of cases. Given 2000 people are treated with the drug, what is the probability that it will be effective for (a) at least 1400 patients, (b) less than 1390 patients, (c) 1420 patients.

26.19 SOME OTHER DISTRIBUTIONS

Discrete distributions

(1) Geometric distribution. If p be the probability of success and k be the numbers of failures preceding the first success then this distribution is

$$P(k) = q^k p, \qquad k = 0, 1, 2, \ldots, q = 1 - p.$$

Obviously $\displaystyle\sum_{k=0}^{\infty} P(k) = p \sum_{k=0}^{\infty} q^k = p \cdot \frac{1}{1-q} = 1.$

It can easily be shown that mean $= q/p$, and variance $= q/p^2$.

(2) Negative binomial distribution. This distribution gives the probability that the event occurs for the kth time on the rth trial ($r \ge k$). If p be the probability of occurrence of an event then

$$P(k, r) = {}^{r-1}C_{k-1}\, p^k q^{r-k}.$$

It contains two parameters p and k. If $k = 1$, the Negative binomial distribution reduces to the geometric distribution.

(3) Hypergeometric distribution. Suppose a bag contains m white and n black balls. If r balls are drawn one at a time (*with replacement*), then the probability that k of them will be white is

$$P(k) = {}^mC_k\, {}^nC_{r-k} / {}^{m+n}C_r,\ k = 0, 1, \ldots r, r \le m, r \le n.$$

This distribution is known as *Hypergeometric distribution.*

For $\displaystyle\sum_{k=0}^{r} P(k) = 1$, since $\displaystyle\sum_{k=0}^{r} {}^mC_k\, {}^nC_{r-k} = {}^{m+n}C_r$

This can be proved by equating the coefficient of t^r in

$$(1+t)^m (t+1)^n = (1+t)^{m+n}$$

Continuous distributions

(4) Uniform (or Rectangular) distribution. A random variable X is said to be uniformly distributed over the interval $-\infty < a < b < \infty$, if its density is given by

$$f(x) = \frac{1}{b-a}, a < x < b \qquad \ldots(i)$$

The distribution given by (i) is called a *uniform distribution*. In this distribution, X takes the values with the same probability.

Its *mean* $\mu = \displaystyle\int_a^b x \cdot f(x)\, dx = \frac{1}{b-a}\left|\frac{x^2}{2}\right|_a^b = \frac{a+b}{2}.$

and *variance* $\sigma^2 = \mu_2' - (\mu)^2 = \displaystyle\int_a^b x^2 \cdot \frac{1}{b-a}\, dx - \left(\frac{a+b}{2}\right)^2 = \frac{1}{12}(b-a)^2.$

(5) Gamma distribution. This continuous distribution is given by $f(x) = \frac{\lambda}{\Gamma(r)} (\lambda x)^{r-1} e^{-\lambda x}$ for all $x \geq, 0$, where r and λ (both > 0) are called the parameters of the *gamma distribution*. Its mean $= r/\lambda$ and variance $= r/\lambda^2$.

Gamma distribution tends to normal distribution as the parameter r tends to infinity.

(6) Exponential distribution. This distribution is a special case of gamma distribution when $r = 1$ so that $f(x) = \lambda e^{-\lambda x}$ for $x > 0$, where λ is a parameter.

It can be seen that mean $= 1/\lambda$, standard deviation $= 1/\lambda$.

This distribution plays an important role in the reliability and queuing theory.

(7) Weibull distribution*. This distribution is given by

$$f(x) = \frac{\alpha}{c} x^{\alpha-1} e^{-x^{\alpha}/c}, x > 0, c > 0$$

where c is a scale parameter and α a shape parameter.

Initially this distribution was used to describe experimentally observed variation in the fatigue resistance of steel and its elastic limits. But it has also been employed to study the variation of length of service of radio service equipment.

Example 26.52. *A die is cast until 6 appears. What is the probability that it must be cast more than 5 times ?*

Solution. Here probability of getting 6 is $p = \frac{1}{6}$. Then $q = \frac{5}{6}$.

If X is the number of tosses required for the first success, then

$$P(X = x) = q^{x-1} p \text{ for } x = 1, 2, 3, \ldots$$

$\therefore$ required probability $= P(X > 5) = 1 - P(X \leq 5)$

$$= 1 - \sum_{x=1}^{5} \left(\frac{5}{6}\right)^{x-1} \cdot \left(\frac{1}{6}\right) = 1 - \frac{1}{6}\left\{1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^3 + \left(\frac{5}{6}\right)^4\right\} = \left(\frac{5}{6}\right)^5.$$

Example 26.53. *A random variable X has a uniform distribution over (– 3, 3), find k for which*

$$P(X > k) = \frac{1}{3}.$$

Also evaluate $P(X < 2)$ and $P[|X - 2| < 2]$.

Solution. (*i*) Density of $X = f(x) = \frac{1}{b-a} = \frac{1}{3-(-3)} = \frac{1}{6}$

$\therefore$ $$P(X > k) = 1 - P(X \leq k) = 1 - \int_{-3}^{k} f(x)\, dx$$

$$= 1 - \frac{1}{6}\int_{-3}^{k} dx = 1 - \frac{1}{6}(k+3) = \frac{1}{3} \qquad \text{(given)}$$

This gives $k = 1$.

(*ii*) $$P(X < 2) = \int_{-3}^{2} f(x)\, dx = \frac{1}{6}\int_{-3}^{2} dx = \frac{5}{6}.$$

(*iii*) $$P[|X - 2| < 2] = P[2 - 2 < X < 2 + 2] = P[0 < x < 4] = \int_0^3 f(x)\, dx = \frac{1}{6}\int_0^3 dx = \frac{1}{2}.$$

PROBLEMS 26.9

1. Show that the mode of the *geometric distribution* $P(x) = \left(\frac{1}{2}\right)^x$, $x = 1, 2, 3 \ldots$, is unity.
2. Show that for the *rectangular distribution* $f(x) = 1, 0 \leq x \leq 1$,

$$\text{mean} = \frac{1}{2}, \text{ variance} = \frac{1}{12} \text{ and mean deviation} = \frac{1}{4}.$$

* It was first used by Swedish scientist *Weibull* in 1951.

3. Find the mean and variance of the *uniform distribution* given by $f(x) = 1/n$, $x = 1, 2, ..., n$.
4. Show that for the *exponential distribution*
$dP = y_0 e^{-x/\sigma}$, $0 \le x \le \infty$,
the mean and S.D. are both equal to σ.
5. Find the mean and variance of the *exponential distribution* $f(x) = \frac{1}{b} e^{-(x-a)/b}$ $x > a$. (*Mumbai, 2005*)
6. Find the moment generating function for the *triangular distribution* given by
$$f(x) = x, \quad 0 \le x \le 1$$
$$= 2 - x, \quad 1 \le x \le 2.$$
7. Show that for the *Gamma distribution* $f(x) = \frac{e^{-x} x^{l-1}}{\Gamma(l)}$, $0 < x < \infty$, the mean and variance are both equal to l.
8. Find the moment generating function of the *Gamma distribution* $f(x) = \frac{1}{\Gamma\left(\frac{1}{4}\right)} e^{-x} x^{-3/4}$, $x \ge 0$, at the origin.
(*J.N.T.U., 2006 ; Madras, 2000 S*)

[**Chebyshev's inequality***. If x is a continuous random variable with mean μ and variance σ^2, then for any positive real parameter t,
$$P(|x - 2| \ge t) \le \sigma^2/t^2 \text{ or } P(|x - \mu| \le t) \ge 1 - \sigma^2/t^2.$$
This result is known as *Chebyshev's inequality*. It gives limits to the probability that the value of the variate chosen at random will differ from mean by more than t.]

9. For the points on a symmetrical die, prove that *Chebyshev's inequality* gives
$$P(|x - \bar{x}| > 2.5) < 0.478,$$
while the actual probability is zero.
10. For the *Geometrical distribution* $P(x) = 2^{-x}$, $x = 1, 2, 3, ...$, prove that *Chebyshev's inequality* gives
$$P(|x - 2| < 2) > \frac{1}{2},$$
while the actual probability is 15/16.

26.20 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 26.10

Select the correct answer or fill up the blanks in each of the following problems :

1. The probability that A happens is 1/3. The odds against happening of A are
(*a*) 2 : 1 (*b*) 2 : 3 (*c*) 3 : 2 (*d*) 5 : 2.
2. The odds in favour of an event A are 5 to 4. The probability of success of A is
(*a*) 4/5 (*b*) 5/9 (*c*) 4/9.
3. The probability that A passes a test is 2/3 and the probability that B passes the same test is 3/5. The probability that only one of them passes is
(*a*) 2/5 (*b*) 4/15 (*c*) 2/15 (*d*) 7/15.
4. A buys a lottery ticket in which the chance of winning is 1/10 ; B has a ticket in which his chance of winning is 1/20. The chance that atleast one of them wins is
(*a*) 1/200 (*b*) 29/200 (*c*) 30/200 (*d*) 170/200.
5. The probability that a non-leap year should have 53 Tuesdays is ...
6. The probability of getting 2 or 3 or 4 from a throw of single dice is ...
7. The mean of the Binomial distribution with n observations and probability of success p, is
(*a*) pq (*b*) np (*c*) $\sqrt{np}$ (*d*) $\sqrt{pq}$.
8. If the mean of a Poisson distribution is m, then S.D. of this distribution is
(*a*) m^2 (*b*) $\sqrt{m}$ (*c*) m (*d*) none of these.

* See footnote on page 571.

9. The S.D. of the Binomial distribution is

(a) $\sqrt{npq}$ (b) $\sqrt{np}$ (c) npq (d) pq.

10. In a Poisson distribution if $2P(x = 1) = P(x = 2)$, then the variance is

(a) 0 (b) -1 (c) 4 (d) 2.

11. If the probability of hitting a target by one shot be $p = 0.8$, then the probability that out of ten shots, seven will hit the target is ...
12. For a Poisson variate $x : P(x = 1) = P(x = 2)$, then the mean of x is ...
13. If $P(A) = 0.35$, $P(B) = 0.73$ and $P(A \cap B) = 0.14$, then $P(A \cap B') = ...$
14. If A and B are independent, $P(B) = 0.14$ and $P(A/B) = 0.24$, then $P(A) = ...$
15. The probability distribution of the number of heads, when two coins are tossed, is ...
16. The multiplication law of probability states that ...
17. The area under the standard normal curve which lies between $z = 0.90$ and $z = -1.85$ is ...
[Given $P(0 < z < 1.85) = 0.4678$, $P(0 < z < 0.9) = 0.3159$]
18. The mean, median and mode of a normal distribution are ...
19. The mean and variance of a Poisson distribution are ...
20. If A and B are two mutually exclusive events, then $P(A \cup B) = ...$
21. For a normal distribution $\beta_1 = ...$ and $\beta_2 = ...$
22. The number of ways in which five people can be lined up to get on a bus are ...
23. A shipment of 10 television sets contains 3 defective sets. The number of ways in which one can purchase 4 of these sets and receive 2 defective sets are ...
24. The probability of getting a total of 5 when a pair of dice is tossed is ...
25. If $P(B) = 0.81$ and $P(A \cap B) = 0.18$, then $P(A/B) = ...$
26. If two unbiased dice are thrown simultaneously, the probability that the sum of the numbers on them is at least 10, is ..... .
27. If X is a Poisson variate such that $P(X = 2) = P(X = 3)$, then $P(X = 0) = ...$
28. An unbiased die is tossed twice, then the probability of obtaining the sum 6, is ...
29. The variance of Poisson distribution with parameter $\lambda = 2$ is ...
30. The distribution in which mean, median, mode are equal is ...
31. For the Poisson variate, probability of getting at least one success is ...
32. Total number of events in rolling of an ideal die is ...
33. If X be normal with mean 10 and variance 4, then $P(X < 11) = ...$
34. If X is a binomial variate with parameters n and p, then its m.g.f. about the origin is ...
35. In a normal distribution, mean deviation : standard deviation = ...
36. If A and B are independent and $P(A) = 1/2$, $P(B) = 1/3$ then $P(A \cap B) = ...$
37. If X is the random variable representing the outcome of the roll of an ideal die, then $E(X) = ...$
38. If X is a binomial variate with $p = 1/5$ for the experiment of 50 trials, then the standard deviation is ...
39. The area under the whole normal curve is ...
40. Given $X = B(n, p)$, then the conditions under which X tends to a Poisson distribution, are ...
41. If A and B are mutually exclusive events then $P(A \cup B) = ...$
42. The probability of selecting x white balls from a bag containing y white and z red balls is ...
43. The mean of the binomial distribution is ...
44. If A and B are mutually exclusive events, $P(A) = 0.29$, $P(B) = 0.43$, then $P(A \cup B) = ...$ and $P(A \cap B') = ...$
45. If the mean and variance of a binomial variate are 12 and 4, then the distribution is ...
46. If x is a Poisson variable such that $P(x = 2) = 9P(x = 4) + 90P(x = 6)$, then the mean = ...
47. μ_r' the rth moment about the origin in terms of the m.g.f. is ...
48. The chance of throwing 7 in a single throw with two dice is ...
49. If A and B are any two events with $P(A) = 1/2$, $P(B) = 1/3$ and $P(A \cap B) = 1/4$, then $P(A/B) = ...$
50. In the roll of an ideal die, the probability of getting a prime number is ...
51. If A and B are mutually exclusive events, $P(A \cup B) = 0.6$, $P(B) = 0.4$, then $P(A) = ...$
52. The probability that a leap year should have 53 Sundays is ...
53. The probability density function of a binomial distribution is ...
54. The probable error is ... times S.D. approximately.
55. To fit a normal distribution, the parameters required are ...

56. A card is drawn from a well-shuffled pack of 52 cards, then the probability of this card being a red coloured ace is ...
57. If $P(1) = P(2)$, then the mean of the Poisson distribution is ...
58. Baye's theorem states that ...
59. If x is a Poisson variate such that $P(x = 1) = 0.3$ and $P(x = 2) = 0.2$, then $P(x = 0) = ...$
60. If A and B are events such that $P(A \cup B) = 3/4$, $P(A \cap B) = 1/4$, and $P(A) = 2/3$, then $P(A) = ...$
61. The chance of throwing 7 in a single throw with two dice is the same as that of getting 7 in two throws of a single die. (True or False)
62. If the mean of a Poisson distribution is 5, then its variance is 10. (True or False)
63. If X is normal with mean 3 and variance 1, then $X - 3$ is a variate with mean 0 and variance 1. (True or False)
64. If X is a binomial variable with parameters $n = 10$, $p = 1/4$, then its standard deviation is 2.25. (True or False)
65. The mean of a binomial distribution is 5 and S.D. is 3. (True or False)
66. The mean and variance of Poisson distribution are equal. (True or False)
67. The graph of the normal distribution is symmetric with respect to the line $y = x$. (True or False)
68. The standard deviation of a binomial distribution is np. (True or False)
69. $f(x) = kx$ in $0 < x < 1$ is a valid probability density function, if $k = ...$
70. If $V(x) = 2$, then $V(2x + 3) = ...$
71. The p.d.f. of an exponential distribution is ...
72. If X is uniformly distributed in $(-2, 3)$, then its variance is ...
73. The variance of Poisson distribution with parameter $\lambda = 2$ is ...
74. In Gamma distribution with parameter l, the variance is ...
75. If $f(x) = kx^3$, $0 < x < 1$ and 0 elsewhere, is a p.d.f., then $k = ...$
76. The m.g.f. of a random variable X is $(1 - 2t)^{-4}$, then $E(x)$ is ...
77. A random variable X has F-distribution with (m, n) degrees of freedom, then $1/X$ has the same distribution with ... degrees of freedom.
78. If X is a continuous random variable having the p.d.f. $f(x)$, then the m.g.f. about the origin is given by ...
79. If $f(X) = X + 2/k$, $X = 1, 2, 3, 4, 5$ is the probability distribution of a discrete random variable, then $k = ...$
80. The p.d.f. of Gamma variate is ...
81. If X is uniformly distributed in $[a, b]$, then $E(X) = ...$
82. The marks obtained by students were found normally distributed with mean 75 and variance 100. The percentage of students who scored more than 75 marks is ...
83. When four unbiased coins are tossed, the probability of getting two heads is ...
84. If $f(x) = \begin{cases} kxe^{-x}, & x > 0 \\ 0, & \text{elsewhere, is the p.d.f. of } x, \text{ then } k = ... \end{cases}$
85. If x is a uniform distribution defined in the interval $(4, 7)$, then its variance is ...
86. The p.d.f. of a continuous random variable is $f(x) = k/x^3$, $5 \le x \le 10$; 0 elsewhere, then the value of k is
(a) 1 (b) 50 (c) 200/3 (d) 200.
87. The relation between probability density function and cumulative density function of a random variable is ...
88. If X has Poisson distribution with parameter λ, then $P(X \text{ is even}) = ...$
89. Range of t- distribution is ...
90. If the p.d.f. of x is $f(x) = kx(1 - x)$, $0 < x < 1$, then $k = ...$
91. The function $f(x) = \begin{cases} kx^2, & 0 < x < 3 \\ 0, & \text{otherwise, is a probability function, then } k = ... \end{cases}$
92. If the random variable x is uniformly distributed in $[0, 3]$, then its p.d.f. is $f(x) = 3$, $0 < x < 3$; 0, elsewhere. (True or False)
93. Exponential distribution $f(x)$ is defined by $f(x) = ae^{-2x}$, $0 < x < \infty$, then $a = ...$
94. The p.d.f. of Beta distribution with $\alpha = 1$, $\beta = 4$ is $f(x) = ...$
95. For a standard normal variate z, $P(-0.72 \le z \le 0) = ...$

CHAPTER 27

Sampling & Inference

1. Introduction. 2. Sampling distribution ; Standard error. 3. Testing of Hypothesis ; Errors. 4. Level of significance ; Tests of significance. 5. Confidence limits. 6. Simple sampling of attributes. 7. Test of significance for large samples. 8. Comparison of large samples. 9. Sampling of variables. 10. Central limit theorem. 11. Confidence limits for unknown means. 12. Test of significance for means of two large samples. 13. Sampling of variables—small samples. 14. Student's *t*-distribution. 15. Significance test of a sample mean. 16. Significance test of difference between sample means. 17. Chi-square test. 18. Goodness of fit. 19. F-distribution. 20. Fisher's *z*-distribution. 21. Objective Type of Questions.

27.1 (1) INTRODUCTION

We know that a small section selected from the population is called a *sample* and the process of drawing a sample is called *sampling*. It is essential that a sample must be a *random* selection so that each member of the population has the same chance of being included in the sample. Thus the fundamental assumption underlying theory of sampling is *Random sampling*.

A special case of random sampling in which each event has the same probability p of success and the chance of success of different events are independent whether previous trials have been made or not, is known as *simple sampling*.

The statistical constants of the population such as mean (μ), standard deviation (σ) etc. are called the *parameters*. Similarly, constants for the *sample* drawn from the given population *i.e.*, mean ($\bar{x}$), standard deviation (S) etc. are called the *statistic*. The population parameters are in general, not known and their estimates given by the corresponding sample statistic are used. We use the Greek letters to denote the population parameters and Roman letters for sample statistic.

(2) Objectives of sampling. Sampling aims at gathering the maximum information about the population with the minimum effort, cost and time. The object of sampling studies is to obtain the best possible values of the parameters under specific conditions. Sampling determines the reliability of these estimates. The logic of the sampling theory is the logic of induction in which we pass from a particular (sample) to general (population). Such a generalisation from sample to population is called **Statistical Inference.**

27.2 SAMPLING DISTRIBUTION

Consider all possible samples of size n which can be drawn from a given population at random. For each sample, we can compute the mean. The means of the samples will not be identical. If we group these different means according to their frequencies, the frequency distribution so formed is known as *sampling distribution of the mean*. Similarly we can have *sampling distribution of the standard deviation* etc.

While drawing each sample, we put back the previous sample so that the parent population remains the same. This is called *sampling with replacement* and all the subsequent formulae will pertain to sampling with replacement.

(2) Standard error. The standard deviation of the sampling distribution is called the *standard error* (*S.E.*). Thus the standard error of the sampling distribution of means is called standard error of means. The standard error is used to assess the difference between the expected and observed values. The reciprocal of the standard error is called *precision*.

If $n \geq 30$, a sample is called *large* otherwise *small*. The sampling distribution of large samples is assumed to be normal.

27.3 (1) TESTING A HYPOTHESIS*

To reach decisions about populations on the basis of sample information, we make certain assumptions about the populations involved. Such assumptions, which may or may not be true, are called *statistical hypothesis*. By testing a hypothesis is meant a process for deciding whether to accept or reject the hypothesis. The method consists in assuming the hypothesis as correct and then computing the probability of getting the observed sample. If this probability is less than a certain preassigned value the hypothesis is rejected.

(2) Errors. If a hypothesis is rejected while it should have been accepted, we say that a *Type I error* has been committed. On the other hand, if a hypothesis is accepted while it should have been rejected, we say that a *Type II error* has been made. The statistical testing of hypothesis aims at limiting the Type I error to a preassigned value (say : 5% or 1%) and to minimize the Type II error. The only way to reduce both types of errors is to increase the sample size, if possible.

(3) Null hypothesis. The hypothesis formulated for the sake of rejecting it, under the assumption that it is true, is called the *null hypothesis* and is denoted by H_0. To test whether one procedure is better than another, we assume that there is no difference between the procedures. Similarly to test whether there is a relationship between two variates, we take H_0 that there is no relationship. By accepting a null hypothesis, we mean that on the basis of the statistic calculated from the sample, we do not reject the hypothesis. It however, does not imply that the hypothesis is proved to be true. Nor its rejection implies that it is disproved.

27.4 (1) LEVEL OF SIGNIFICANCE

The probability level below which we reject the hypothesis is known as the *level of significance*. The region in which a sample value falling is rejected, is known as the *critical region*. We generally take two critical regions which cover 5% and 1% areas of the normal curve. The shaded portion in the figure corresponds to 5% level of significance. Thus the *probability of the value of the variate falling in the critical region is the level of significance*.

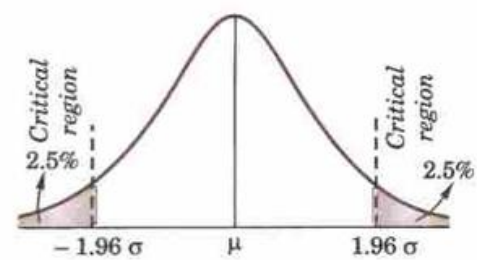


Fig. 27.1

Depending on the nature of the problem, we use a *single-tail test* or *double-tail test* to estimate the significance of a result. In a double-tail test, the areas of both the tails of the curve representing the sampling distribution are taken into account whereas in the single tail test, only the area on the right of an ordinate is taken into consideration. For instance, to test whether a coin is biased or not, double-tail test should be used, since a biased coin gives either more number of heads than tails (which corresponds to right tail), or more number of tails than heads (which corresponds to left tail only).

(2) Tests of significance. The procedure which enables us to decide whether to accept or reject the hypothesis is called the *test of significance*. Here we test whether the differences between the sample values and the population values (or the values given by two samples) are so large that they signify evidence against the hypothesis or these differences are so small as to account for fluctuations of sampling.

27.5 CONFIDENCE LIMITS**

Suppose that the sampling distribution of a statistic S is normal with mean μ and standard deviation σ. As in the Fig. 27.1 the sample statistic S can be expected to lie in the interval $(\mu - 1.96\sigma, \mu + 1.96\sigma)$ for 95% times *i.e.*, we can be confident of finding μ in the interval $(S - 1.96\sigma, S + 1.96\sigma)$ in 95% cases. Because of this, we call

*The American statistician *J. Neyman* (1894—1981) and the English statistician *E.S. Pearson* (1895—1980)–son of *Karl Pearson* (See footnote p. 843), developed a systematic theory of tests around 1930.

***J. Neyman* developed the modern theory and terminology of confidence limits.

$(S - 1.96\sigma, S + 1.96\sigma)$ the 95% confidence interval for estimation of μ. The ends of this interval (*i.e.* $S \pm 1.96\sigma$) are called 95% *confidence limits* (or *fiducial limits*) for S. Similarly $S \pm 2.58\sigma$ are 99% confidence limits. The numbers 1.96, 2.58 etc. are called *confidence coefficients*. The values of confidence coefficients corresponding to various levels of significance can be found from the normal curve area table VI – Appendix 2.

27.6 SIMPLE SAMPLING OF ATTRIBUTES

The sampling of attributes may be regarded as the selection of samples from a population whose members possess the attribute K or not K. The presence of K may be called a success and its absence a failure.

Suppose we draw a simple sample of n items. Clearly it is same as a series of n independent trials with the same probability p of success. The probabilities of 0, 1, 2, ..., n successes are the terms in the binomial expansion of $(q + p)^n$ where $q = 1 - p$.

We know that the mean of this distribution is np and standard deviation is $\sqrt{(npq)}$ *i.e.*, the expected value of success in a sample of size n is np and the standard error is $\sqrt{(npq)}$.

If we consider the proportion of successes, then

(*i*) mean proportion of successes $= np/n = p$.

(*ii*) standard error of the proportion of successes

$$= \sqrt{\left(n \cdot \frac{p}{n} \cdot \frac{q}{n}\right)} = \sqrt{\left(\frac{pq}{n}\right)}$$

and (*iii*) precision of the proportion of successes $= \sqrt{(n/pq)}$, which varies as $\sqrt{n}$, since p and q are constants.

27.7 TEST OF SIGNIFICANCE FOR LARGE SAMPLES

We know that the binomial distribution tends to normal for large n. Suppose we wish to test the hypothesis that the probability of success in such trial is p. Assuming it to be true, the mean μ and the standard deviation σ of the sampling distribution of number of successes are np and $\sqrt{(npq)}$ respectively.

For a normal distribution, only 5% of the members lie outside $\mu \pm 1.96\sigma$ while only 1% of the members lie outside $\mu \pm 2.58\sigma$.

If x be the observed number of successes in the sample and z is the standard normal variate then $z = (x - \mu)/\sigma$.

Thus we have the following test of significance :

(*i*) *If* $|z| < 1.96$, *difference between the observed and expected number of successes is not significant.*

(*ii*) *If* $|z| > 1.96$, *difference is significant at 5% level of significance.*

(*iii*) *If* $|z| > 2.58$, *difference is significant at 1% level of significance.*

Example 27.1. *A coin was tossed 400 times and the head turned up 216 times. Test the hypothesis that the coin is unbiased at 5% level of significance.* (V.T.U., 2007)

Solution. Suppose the coin is unbiased.

Then the probability of getting the head in a toss $= \frac{1}{2}$

$\therefore$ expected number of successes $= \frac{1}{2} \times 400 = 200$

and the observed value of successes = 216

Thus the excess of observed value over expected value = 216 – 200 = 16

Also S.D. of simple sampling $= \sqrt{npq} = \sqrt{\left(400 \times \frac{1}{2} \times \frac{1}{2}\right)} = 10$

Hence $z = \dfrac{x - np}{\sqrt{(npq)}} = \dfrac{16}{10} = 1.6$

As $z < 1.96$, the hypothesis is accepted at 5% level of significance *i.e.*, we conclude that the coin is unbiased at 5% level of significance.

Example 27.2. *A die was thrown 9000 times and a throw of 5 or 6 was obtained 3240 times. On the assumption of random throwing, do the data indicate an unbiased die ?* (V.T.U., 2010)

Solution. Suppose the die is unbiased.

Then the probability of throwing 5 or 6 with one die $= \frac{1}{3}$

The expected number of successes $= \frac{1}{3} \times 9000 = 3000$

and the observed value of successes = 3240

Thus the excess of observed value over expected value 3240 – 3000 = 240

Also S.D. of simple sampling $= \sqrt{npq} = \sqrt{\left(9000 \times \frac{1}{3} \times \frac{2}{3}\right)} = 44.72$

Hence $z = \frac{x - np}{\sqrt{(npq)}} = \frac{240}{44.72} = 5.4$ nearly.

As $z > 2.58$, the hypothesis has to be rejected at 1% level of significance and we conclude that the die is biased.

Example 27.3. *In a locality containing 18000 families, a sample of 840 families was selected at random. Of these 840 families, 206 families were found to have a monthly income of ₹ 250 or less. It is desired to estimate how many out of 18,000 families have a monthly income of ₹ 250 or less. Within what limits would you place your estimate ?*

Solution. Here $p = \frac{206}{840} = \frac{103}{420}$ and $q = \frac{317}{420}$

$\therefore$ standard error of the population of families having a monthly income of ₹ 250 or less

$$= \sqrt{\left(\frac{pq}{n}\right)} = \sqrt{\left(\frac{103}{420} \times \frac{317}{420} \times \frac{1}{840}\right)} = .015 = 1.5\%$$

Hence taking $\frac{103}{420}$ (or 24.5%) to be the estimate of families having a monthly income of ₹ 250 or less in the locality, the limits are $(24.5 \pm 3 \times 1.5)\%$ *i.e.*, 20% and 29% approximately.

27.8 COMPARISON OF LARGE SAMPLES

Two large samples of sizes n_1, n_2 are taken from two populations giving proportions of attributes A's as p_1, p_2 respectively.

(*a*) On the hypothesis that the populations are similar as regards the attribute A, we combine the two samples to find an estimate of the common value of proportion of A's in the populations which is given by

$$p = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2}$$

If e_1, e_2 be the standard errors in the two samples then

$$e_1^2 = \frac{pq}{n_1} \text{ and } e_2^2 = \frac{pq}{n_2}$$

If e be the standard error of the difference between p_1 and p_2, then

$$e^2 = e_1^2 + e_2^2 = pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right) \quad \therefore \quad z = \frac{p_1 \sim p_2}{e}$$

If $z > 3$, the difference between p_1 and p_2 is real one.

If $z < 2$, the difference may be due to fluctuations of simple sampling.

But if z lies between 2 and 3, then the difference is significant at 5% level of significance.

(*b*) If the proportions of A's are not the same in the two populations from which the samples are drawn, but p_1 and p_2 are the true values of proportions then S.E. e of the difference $p_1 \sim p_2$ is given by

$$e^2 = \frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}$$

If $z = \dfrac{p_1 \sim p_2}{e} < 3$, the difference could have arisen due to fluctuations of simple sampling.

Example 27.4. *In a city A 20% of a random sample of 900 school boys had a certain slight physical defect. In another city B, 18.5% of a random sample of 1600 school boys had the same defect. Is the difference between the proportions significant ?* *(V.T.U., 2003 S)*

Solution. We have $n_1 = 900, n_2 = 1600$

and $p_1 = \dfrac{20}{100} = \dfrac{1}{5}, p_2 = \dfrac{18.5}{100}$

$\therefore$ $$p = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \frac{180 + 296}{900 + 1600} = 0.19$$

and $q = 1 - 0.19 = 0.81$

Thus $$e^2 = pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right) = 0.19 \times 0.81\left(\frac{1}{900} + \frac{1}{1600}\right) = 0.0017$$

giving $e = 0.04$ nearly.

Also $$p_1 - p_2 = \frac{1.5}{100} = 0.015 \quad \therefore \quad z = \frac{p_1 - p_2}{e} = \frac{.015}{.04} = 0.37$$

As $z < 1$, the difference between the proportions is not significant.

Example 27.5. *In two large populations there are 30% and 25% respectively of fair haired people. Is this difference likely to be hidden in samples of 1200 and 900 respectively from the two populations ?* *(Coimbatore, 2001)*

Solution. Here $p_1 = 0.3, p_2 = 0.25$ so that $p_1 - p_2 = 0.05$.

$\therefore$ $$e^2 = \frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2} = \frac{0.3 \times 0.7}{1200} + \frac{0.25 \times 0.75}{900}$$

so that $e = 0.0195$

$\therefore$ $$z = \frac{p_1 - p_2}{e} = \frac{0.05}{0.0195} = 2.5 \text{ nearly}$$

Hence it is unlikely that the real difference will be hidden.

PROBLEMS 27.1

1. A die is tossed 960 times and it falls with 5 upwards 184 times. Is the die biased ? *(V.T.U., 2006)*
2. 12 dice are thrown 3086 times and a throw of 2, 3, 4 is reckoned as a success. Suppose that 19142 throws of 2, 3, 4 have been made out. Do you think that this observed value deviates from the expected value ? If so, can the deviation from the expected value be due to fluctuations of simple sampling ?
3. Balls are drawn from a bag containing equal number of black and white balls, each ball being replaced before drawing another. In 2250 drawings 1018 black and 1232 white balls have been drawn. Do you suspect some bias on the part of the drawer ?
4. A sample of 1000 days is taken from meteorological records of a certain district and 120 of them are found to be foggy. What are the probable limits to the percentage of foggy days in the district ?
5. In a group of 50 first cousines there were found to be 27 males and 23 females. Ascertain if the observed proportions are inconsistent with the hypothesis that the sexes should be in equal proportion.
6. A random sample of 500 apples was taken from a large consignment and 65 were found to be bad. Estimate the proportion of the bad apples in the consignment as well as the standard error of the estimate. Deduce that the percentage of bad apples in the consignment almost certainly lies between 8.5 and 17.5.
7. 400 children are chosen in an industrial town and 150 are found to be under weight. Assuming the conditions of simple sampling, estimate the percentage of children who are under weight in the industrial town and assign limits within which the percentage probably lies ?

8. A machine produces 16 imperfect articles in a sample of 500. After machine is overhauled, it produces 3 imperfect articles in a batch of 100. Has the machine been improved ? *(Rohtak, 2005 ; Madras, 2003)*
9. One type of aircraft is found to develop engine trouble in 5 flights out of a total of 100 and another type in 7 flights out of a total of 200 flights. Is there a significant difference in the two types of aircrafts so far as engine defects are concerned ?
10. In a sample of 600 men from a certain city, 450 are found smokers. In another sample of 900 men from another city, 450 are smokers. Do the data indicate that the cities are significantly different with respect to the habit of smoking among men ? *(J.N.T.U., 2003)*
11. In a sample of 500 people from a state 280 take tea and rest take coffee. Can we assume that tea and coffee are equally popular in the state at 5% level of significance ?

27.9 (1) SAMPLING OF VARIABLES

We now consider sampling of a variable such as weight, height, etc. Each member of the population gives a value of the variable and the population is a frequency distribution of the variable. Thus a random sample of size n from the population is same as selecting n values of the variables from those of the distribution.

(2) Sampling distribution of the mean. *If a population is distributed normally with mean μ and standard deviation σ, then the means of all positive random samples of size n, are also distributed normally with mean μ and standard error $\sigma/\sqrt{n}$.* This result shows how the precision of a sample mean increases as the sample size increases.

27.10 CENTRAL LIMIT THEOREM

This is a very important theorem regarding the distribution of the mean of a sample *if the parent population is non-normal and the sample size is large.*

If the variable X has a non-normal distribution with mean μ and standard deviation σ, then the limiting distribution of

$z = \dfrac{\bar{x} - \mu}{\sigma\sqrt{n}}$ as $n \to \infty$, *is the standard normal distribution* (*i.e.*, with mean 0 and unit S.D.)

There is no restriction upon the distribution of X except that it has a finite mean and variance. This theorem holds well for a sample of 25 or more which is regarded as large.

Thus *if the population is normal, the sampling distribution of the mean is also normal with mean μ and S.E. $\sigma/\sqrt{n}$, while for large samples the same result holds even if the distribution of the population is non-normal.* This property is of universal use and throws light on the importance of normal distribution in statistical theory.

27.11 CONFIDENCE LIMITS FOR UNKNOWN MEAN

Let the population from which a random sample of size n is drawn, have mean μ and S.D. σ. If μ is not known, there will be a range of values of μ for which observed mean $\bar{x}$ of the sample is not significant at any assigned level of probability. The relative deviation of $\bar{x}$ from μ is $(\bar{x} - \mu)/\sqrt{\sigma}$.

If $\bar{x}$ is not significant at 5% level of probability, then

$$|(\bar{x} - \mu)\sqrt{n}/\sigma| < 1.96 \quad i.e. \quad \bar{x} - 1.96\sigma/\sqrt{n} < \mu < \bar{x} + 1.96\sigma/\sqrt{n}$$

Thus 95% confidence or fiducial limits for the mean of the population corresponding to given sample are $\bar{x} \pm 1.96\sigma/\sqrt{n}$.

Similarly *99% confidence limits for μ are $\bar{x} \pm 2.58\sigma/\sqrt{n}$.*

Example 27.6. *A sample of 900 members is found to have a mean of 3.4 cm. Can it be reasonably regarded as a truly random sample from a large population with mean 3.25 cm and S.D. 1.61 cm.*

Solution. Here $\bar{x} = 3.4$ cm, $n = 900$, $\mu = 3.25$ and $\sigma = 1.61$ cm

$$\therefore \qquad z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{3.4 - 3.25}{1.61/30} = 2.8$$

As $z > 1.96$, the deviation of the sample mean from the mean of the population is significant at 5% level of significance. Hence it cannot be regarded as a random sample.

Example 27.7. *The mean of a certain normal population is equal to the standard error of the mean of the samples of 100 from that distribution. Find the probability that the mean of the sample of 25 from the distribution will be negative ?*

Solution. If μ be the mean and σ the S.D. of the distribution, then

$$\mu = \text{S.E. of the sample means} = \frac{\sigma}{\sqrt{100}} = \frac{\sigma}{10}$$

Also for a sample of size 25, we have $z = \dfrac{\bar{x} - \mu}{\sigma/\sqrt{25}} = \dfrac{\bar{x} - \sigma/10}{6/5} = \dfrac{5\bar{x}}{\sigma} - \dfrac{1}{2}$

Since $\bar{x}$ is negative, $z < -\frac{1}{2}$.

$\therefore$ the probability that a normal variate $z < -\frac{1}{2}$

$$= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{1}{2}} e^{-\frac{1}{2}z^2}\, dz = \frac{1}{\sqrt{2\pi}} \int_{1/2}^{\infty} e^{-\frac{1}{2}z^2}\, dz$$

$= 0.5 - 0.915 = 0.3085$, from the tables.

Example 27.8. *An unbiased coin is thrown n times. It is desired that the relative frequency of the appearance of heads should lie between 0.49 and 0.51. Find the smallest value of n that will ensure this result with 90% confidence.*

Solution. S.E. of the proportion of heads $= \sqrt{\dfrac{pq}{n}} = \sqrt{\dfrac{1}{2} \cdot \dfrac{1}{2} \cdot \dfrac{1}{n}} = \dfrac{1}{2\sqrt{n}}$

90% of confidence = 45% or .45 of the total area under the normal curve on each side of the mean.

$\therefore$ the corresponding value of $z = 1.645$, from the tables.

Thus $\quad p \mp 1.645\sigma = 0.49$ or 0.51.

i.e., $\quad 0.5 - 1.645 \cdot \dfrac{1}{2\sqrt{n}} = 0.49$ and $0.5 + 1.645 \cdot \dfrac{1}{2\sqrt{n}} = 0.51$

whence $\quad \dfrac{1.645}{2\sqrt{n}} = 0.01$ or $\sqrt{n} = \dfrac{329}{4}$ or $n = 6765$ approximately.

27.12 TEST OF SIGNIFICANCE FOR MEANS OF TWO LARGE SAMPLES

(*a*) Suppose two random samples of sizes n_1 and n_2 have been *drawn from the same population* with S.D. σ. We wish to test whether the difference between the sample means $\bar{x}_1$ and $\bar{x}_2$ is significant or is merely due to fluctuations of sampling.

If the samples are independent, then the standard error e of the difference of their means is given by

$$e^2 = e_1^2 + e_2^2$$

where $e_1 = \sigma/\sqrt{n_1}$, $e_2 = \sigma/\sqrt{n_2}$ are the S.E.s of the means of the two samples.

$$\therefore \qquad e = \sigma\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}. \quad \text{Hence} \quad z = \frac{\bar{x}_1 - \bar{x}_2}{\sigma\sqrt{(1/n_1 + 1/n_2)}}$$

is normally distributed with mean zero and S.D. 1.

Test of significance (n_1, n_2 being large) :

If $z > 1.96$, then the difference is significant at 5% level of significance.

If $z > 3$, it is highly probable that either the samples have not been drawn from the same population or the sampling is not simple.

(*b*) *If the samples are known to be drawn from different populations* with means μ_1, μ_2 and standard deviations σ_1 and σ_2. Then the standard error e of their means is given by

$$e = \sqrt{\left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)}$$

Assuming that the two populations have the same mean (*i.e.*, $\mu_1 = \mu_2$), the difference of the means of the samples will be normally distributed with mean zero and S.D. e. Now the same procedure of test of significance is applied.

Example 27.9. *The means of simple samples of sizes 1000 and 2000 are 67.5 and 68.0 cm respectively. Can the samples be regarded as drawn from the same population of S.D. 2.5 cm.* (*Madras, 2002*)

Solution. We have $\bar{x}_1 = 67.5,\ \bar{x}_2 = 68.0$

$n_1 = 1000,\ n_2 = 2000.$

On the hypothesis, that the samples are drawn from the same population of S.D. $\sigma = 2.5$, we get

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sigma\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{67.5 - 68.0}{2.5\sqrt{\left(\frac{1}{1000} + \frac{1}{2000}\right)}}$$

$$= \frac{0.5}{2.5 \times 0.0387} = \frac{0.5}{0.09675} = 5.1$$

Hence the difference between the sample means *i.e.*, 5.1 is very much greater than 1.96 and is therefore significant. Thus, the samples cannot be regarded as drawn from the same population.

Example 27.10. *A sample of height of 6400 soldiers has a mean of 67.85 inches and a standard deviation of 2.56 inches while a simple sample of heights of 1600 sailors has a mean of 68.55 inches and a standard deviation of 2.52 inches. Do the data indicate that the sailors are on the average taller than soldiers?*

Solution. Here $\bar{x}_1 = 67.85,\ \sigma_1 = 2.56,\ n_1 = 6400$

$\bar{x}_2 = 68.55,\ \sigma_2 = 2.52,\ n_2 = 1600.$

$\therefore$ S.E. of the difference of the mean heights is

$$e = \sqrt{\left[\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right]} = \sqrt{\left[\frac{(2.56)^2}{6400} + \frac{(2.52)^2}{1600}\right]}$$

$$= \sqrt{[.001024 + .003969]} = 0.005 \text{ nearly.}$$

Also difference between the means $= \bar{x}_2 - \bar{x}_1 = 0.7$, which $> 10e$. This is highly significant. Hence the data indicates that the sailors are on the average taller than the soldiers.

PROBLEMS 27.2

1. A sample of 400 items is taken from a normal population whose mean is 4 and variance 4. If the sample mean is 4.45, can the samples be regarded as a simple sample ?
2. To know the mean weights of all 10-year old boys in Delhi, a sample of 225 is taken. The mean weight of the sample is found to be 67 pounds with a S.D. of 12 pounds. Can you draw any inference from it about the mean weight of the population ?
3. A normal population has a mean 0.1 and a S.D. of 2.1. Find the probability that the mean of simple sample of 900 members will be negative.
4. If the mean breaking strength of copper wire is 575 lbs. with a standard deviation of 8.3 lbs., how large a sample must be used in order that there be one chance in 100 that the mean breaking strength of the sample is less than 572 lbs. ?

   [**Hint.** $|z| = \left|\frac{\bar{x} - \mu}{\sigma}\sqrt{n}\right| = \frac{3}{8.3}\sqrt{n}$

   Also from table IV, $z = 2.33$. Hence $n = 42$ nearly.]

5. A research worker wishes to estimate mean of a population by using sufficiently large sample. The probability is 95% that sample mean will not differ from the true mean by more than 25% of the S.D. How large a sample should be taken ?
6. The density function of a random variable x is $f(x) = ke^{-2x^2+10x}$. Find the upper 5% point of the distribution of the means of the random sample of size 25 from the above population.
7. The means of two large samples of 1000 and 2000 members are 168.75 cms. and 170 cms. respectively. Can the samples be regarded as drawn from the same population of standard deviation 6.25 cms.
8. If 60 new entrants in a given university are found to have a mean height of 68.60 inches and 50 seniors a mean height of 69.51 inches ; is the evidence conclusive that the mean height of the seniors is greater than that of the new entrants ? Assume the standard deviation of height to be 2.48 inches.
9. A sample of 100 electric bulbs produced by manufacturer A showed a mean life time of 1190 hours and a standard deviation of 90 hours. A sample of 75 bulbs produced by manufacturer B showed a mean life time of 1230 hours, with a standard deviation of 120 hours. Is there a difference between the mean life time of two brands at a significance level of (*i*) 0.05 (*ii*) 0.01.
10. A random sample of 1000 men from North India shows that their mean wage is ₹ 5 per day with a S.D. of ₹ 1.50. A sample of 1500 men from South India gives a mean wage of ₹ 4.50 per day with a standard deviation of ₹ 2. Does the mean rate of wages varies as between the two regions ?

27.13 SAMPLING OF VARIABLES—SMALL SAMPLES

In case of large samples, sampling distribution approaches a normal distribution and values of sample statistic are considered best estimates of the parameters in a population. It will no longer be possible to assume that statistics computed from small samples are normally distributed. As such, *a new technique has been devised for small samples which involves the concept of 'degrees of freedom' which we explain below.*

Number of degrees of freedom *is the number of values in a set which may be assigned arbitrarily.* For instance, if $x_1 + x_2 + x_3 = 15$ and we assign any values of two of the variables (say : x_1, x_2), then the values of x_3 will be known. The two variables are therefore, free and independent choices for finding the third. Hence these are the degrees of freedom. *If there are n observations, the degrees of freedom (d.f.) are* $(n - 1)$. In other words, while finding the mean of a small sample, one degree of freedom is used up and $(n - 1)$ *d.f.* are left to estimate the population variance.

27.14 (1) STUDENT'S t-DISTRIBUTION

Consider a small sample of size n, drawn from a normal population with mean μ and s.d. σ. If $\bar{x}$ and σ_s be the sample mean and s.d., then the statistic, 't' is defined as

$$t = \frac{\bar{x} - \mu}{\sigma}\sqrt{n} \quad \text{or} \quad t = \frac{\bar{x} - \mu}{\sigma_s}\sqrt{(n-1)},$$

where $\nu = n - 1$ denotes the *df.* of t. If we calculate t for each sample, we obtain the sampling distribution for t. This distribution known as *Student's t-distribution**, is given by

$$y = \frac{y_0}{(1 + t^2/\nu)^{(\nu+1)/2}} \qquad ...(1)$$

where y_0 is constant such that the area under the curve is unity.

(2) Properties of t-distribution.

I. This curve is symmetrical about the line $t = 0$, like the normal curve, since only even powers of t appear in (1). But it is more peaked than the normal curve with the same S.D. The t-curve approaches the horizontal axis less rapidly than the normal curve. Also t-curve attains its maximum value at $t = 0$ so that its mode coincides with the mean. (Fig. 27.2)

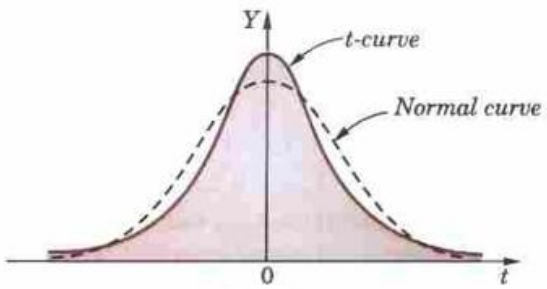


Fig. 27.2

*This distribution was first found by the English statistician *W.S. Gosset* in 1908 who wrote under the pen-name of '*Student*'. *R.A. Fisher* defined t correctly and found its distribution in 1926.

II. The limiting form of t-distribution when $\nu \to \infty$ is given by $y = y_0 e^{-\frac{1}{2}t^2}$ which is a normal curve. This shows that *t is normally distributed for large samples.*

III. The probability P that the value of t will exceed t_0 is given by

$$P = \int_{t_0}^{\infty} y\, dx$$

The values of t_0 have been tabulated for various values of P for various values of ν from 1 to 30 (Table IV – Appendix 2).

IV. Moments about the mean

All the moments of odd order about the mean are zero, due to its symmetry about the line $t = 0$.

Even order moments about the mean are

$$\mu_2 = \frac{\nu}{\nu - 2},\ \mu_4 = \frac{3\nu^2}{(\nu - 2)(\nu - 4)}, \ldots$$

The t-distribution is often used in tests of hypothesis about the mean when the population standard deviation σ is unknown.

27.15 SIGNIFICANCE TEST OF A SAMPLE MEAN

Given a random small sample $x_1, x_2, x_3 \ldots x_n$ from a normal population, we have to test the hypothesis that mean of the population is μ. For this, we first calculate $t = (\bar{x} - \mu)\sqrt{n}/\sigma_s$

where
$$\bar{x} = \frac{1}{n}\sum_1^n x_i,\ \sigma_s^2 = \frac{1}{n-1}\sum_1^n (x_i - \bar{x})^2$$

Then find the value of P for the given df from the table.

If the calculated value of $t > t_{0.05}$, the difference between $\bar{x}$ and μ is said to be significant at 5% level of significance.

If $t > t_{0.01}$, the difference is said to be significant at 1% level of significance.

If $t < t_{0.05}$, the data is said to be consistent with the hypothesis that μ is the mean of the population.

Example 27.11. *A certain stimulus administered to each of 12 patients resulted in the following increases of blood pressure : 5, 2, 8, – 1, 3, 0, – 2, 1, 5, 0, 4, 6. Can it be concluded that the stimulus will in general be accompanied by an increase in blood pressure.* (V.T.U., 2007)

Solution. Let us assume that the stimulus administered to all the 12 patients will increase the B.P. Taking the population to be normal with mean $\mu = 0$ and S.D. σ,

$$\bar{d} = \frac{5 + 2 + 8 - 1 + 3 + 0 - 2 + 1 + 5 + 0 + 4 + 6}{12} = 2.583$$

$$\sigma^2 = \frac{\Sigma d^2}{n} - \bar{d}^2 = \frac{1}{12}\ [5^2 + 2^2 + 8^2 + (-1^2) + 3^2 + 0^2 + (-2)^2 + 1^2 + 5^2 + 0^2 + 4^2 + 6^2] - (2.583)^2$$

$$= 8.744. \quad \therefore \quad \sigma = 2.9571$$

Now
$$t = \frac{\bar{d} - \mu}{\sigma_s}\sqrt{(n-1)} = \frac{2.583 - 0}{2.9571}\sqrt{(12 - 1)} = 2.897$$

Here $d.f.$ $\gamma = 12 - 1 = 11$.

For $\gamma = 11$, $t_{0.05} = 2.2$, from table IV.

Since the $|t| > t_{0.05}$, our assumption is rejected *i.e.*, the stimulus does not increase the B.P.

Example 27.12. *The nine items of a sample have the following values : 45, 47, 50, 52, 48, 47, 49, 53, 51. Does the mean of these differ significantly from the assumed mean of 47.5 ?* (V.T.U., 2010)

Solution. We find the mean and standard deviation of the sample as follows :

x	$d = x - 48$	d^2
45	−3	9
47	−1	1
50	2	4
52	2	4
48	0	0
47	−1	1
49	1	1
53	5	25
51	3	9
Total	10	66

$$\therefore \qquad \bar{x} = \text{mean} = 48 + \frac{\Sigma d}{9} = 48 + \frac{10}{9} = 49.1$$

$$\sigma_s^{\,2} = \frac{\Sigma d^2}{9} - \left(\frac{\Sigma d}{9}\right)^2 = \frac{66}{9} - \frac{100}{81} = \frac{494}{81}$$

$$\therefore \qquad \sigma_s = 2.47$$

Hence $$t = \frac{\bar{x} - \mu}{\sigma_s}\sqrt{(n-1)} = \frac{49.1 - 47.5}{2047}\sqrt{8} = 1.83$$

Here $\qquad d.f.\ \nu = 9 - 1 = 8$

For $\nu = 8$, we get from table IV, $t_{0.05} = 2.31$.

As calculated value of $t < t_{0.05}$, the value of t is not significant at 5% level of significance which implies that there is no significant difference between $\bar{x}$ and μ. Thus the test provides no evidence against the population mean being 47.5.

Example 27.13. *A machinist is making engine parts with axle diameter of 0.7 inch. A random sample of 10 parts shows mean diameter 0.742 inch with a standard deviation of 0.04 inch. On the basis of this sample, would you say that the work is inferior ?* *(V.T.U., 2009)*

Solution. Here we have $\mu = 0.700$, $\bar{x} = 0.742$, $\sigma_x = 0.040$, $n = 10$.

Taking the hypothesis that the product is not inferior *i.e.*, there is no significant difference between $\bar{x}$ and μ.

$$\therefore \qquad t = \frac{\bar{x} - \mu}{\sigma_x}\sqrt{(n-1)} = \frac{0.742 - 0.700}{0.040}\sqrt{(10-1)} = \frac{0.126}{0.040} = 3.16$$

Degrees of freedom $\rho = 10 - 1 = 9$.

For $\rho = 9$, we get from table IV, $t_{0.05} = 2.262$.

As the calculated value of $t > t_{0.05}$, the value of t is significant at 5% level of significance. This implies that $\bar{x}$ differs significantly from μ and the hypothesis is rejected. Hence the work is inferior. In fact, the work is inferior even at 2% level of significance.

Example 27.14. *Show that 95% confidence limits for the mean μ of the population are $\bar{x} \pm \frac{\sigma_s}{\sqrt{n}} t_{0.05}$. Deduce that for a random sample of 16 values with mean 41.5 inches and the sum of the squares of the deviations from the mean 135 inches^2 and drawn from a normal population, 95% confidence limits for the mean of the population are 39.9 and 43.1 inches.*

Solution. (*a*) The probability P that $t \le t_{0.05}$ is 0.95. Hence the 95% confidence limits for μ are given by

$$\left| \frac{\bar{x} - \mu}{\sigma_s}\sqrt{n} \right| \le t_{0.05}$$

or $$|\bar{x} - \mu| \le \frac{\sigma_s}{\sqrt{n}} t_{0.05} \quad \text{or} \quad \bar{x} - \frac{\sigma_s}{\sqrt{n}} t_{0.05} \le \mu \le \bar{x} + \frac{\sigma_s}{\sqrt{n}} t_{0.05}$$

We can, therefore, say with a confidence coefficient 0.95 that the confidence interval $\bar{x} \pm \frac{\sigma_s}{\sqrt{n}} t_{0.05}$ contains the population mean μ.

(*b*) Here, $n = 16$, $\nu = n - 1 = 15$, $\sigma_s = \sqrt{\frac{135}{15}} = 3$.

Also from table IV, $t_{0.05}$ (for $\nu = 15$) = 2.13

$$\therefore \qquad \frac{\sigma_s}{\sqrt{n}} t_{0.05} = \frac{3}{4} \times 2.13 = 1.6 \text{ approx.}$$

Hence the required confidence limits are 41.5 ± 1.6 *i.e.*, 39.9 and 43.1 inches.

27.16 SIGNIFICANCE TEST OF DIFFERENCE BETWEEN SAMPLE MEANS

Given two independent samples $x_1, x_2, x_3, \ldots, x_{n_1}$ and $y_1, y_2, \ldots, y_{n_2}$ with means $\bar{x}$ and $\bar{y}$ and standard deviations σ_x and σ_y from a normal population with the same variance, we have to test the hypothesis that the population means μ_1 and μ_2 are the same.

For this, we calculate
$$t = \frac{\bar{x} \sim \bar{y}}{\sigma\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \qquad \ldots(1)$$

where
$$\bar{x} = \frac{1}{n_1}\sum_1^{n_1} x_i, \; \bar{y} = \frac{1}{n_2}\sum_1^{n_2} y_i$$

and
$$\sigma_s^2 = \frac{1}{n_1 + n_2 - 2}\left[(n_1 - 1)\sigma_x^2 + (n_2 - 1)\sigma_y^2\right] = \frac{1}{n_1 + n_2 - 2}\left\{\sum_1^{n_1}(x_i - \bar{x})^2 + \sum_1^{n_2}(y_j - \bar{y})^2\right\}$$

It can be shown that the variate t defined by (1) follows the t-distribution with $n_1 + n_2 - 2$ degrees of freedom.

If the calculated value of $t > t_{0.05}$, the difference between the sample means is said to be significant at 5% level of significance.

If $t > t_{0.01}$, the difference is said to be significant at 1% level of significance.

If $t < t_{0.05}$, the data is said to be consistent with the hypothesis, that $\mu_1 = \mu_2$.

Cor. If the two samples are of the same size and the data are paired, then t is defined by

$$t = \frac{\bar{d} - 0}{(\sigma/\sqrt{n})} \quad \text{where} \quad \sigma^2 = \frac{1}{n-1}\sum_1^n (d_i - \bar{d})^2$$

d_i = difference of the ith members of the samples ;

$\bar{d}$ = mean of the differences = $\Sigma d_i/n$; and the number of *d.f.* = $n - 1$.

Example 27.15. *Eleven students were given a test in statistics. They were given a month's further tuition and a second test of equal difficulty was held at the end of it. Do the marks give evidence that the students have benefitted by extra coaching ?*

Boys :	1	2	3	4	5	6	7	8	9	10	11
Marks I test :	23	20	19	21	18	20	18	17	23	16	19
Marks II test :	24	19	22	18	20	22	20	20	23	20	17

(*V.T.U., 2011 S*)

Solution. We compute the mean and the S.D. of the difference between the marks of the two tests as under :

$$\bar{d} = \text{mean of } d\text{'s} = \frac{11}{11} = 1 \; ; \; \sigma_s^{-2} = \frac{\Sigma(d - \bar{d})^2}{n-1} = \frac{50}{10} = 5 \quad i.e., \quad \sigma_s = 2.24$$

Assuming that the students have not been benefited by extra coaching, it implies that the mean of the difference between the marks of the two tests is zero *i.e.*, $\mu = 0$.

Then $$t = \frac{\bar{d} - \mu}{\sigma_s}\sqrt{n} = \frac{1-0}{2.24}\sqrt{11} = 1.48 \text{ nearly and } df\ \nu = 11 - 1 = 10.$$

Students	x_1	x_2	$d = x_2 - x_1$	$d - \bar{d}$	$(d - \bar{d})^2$
1	23	24	1	0	0
2	20	19	−1	−2	4
3	19	22	3	2	4
4	21	18	−3	−4	16
5	18	20	2	1	1
6	20	22	2	1	1
7	18	20	2	1	1
8	17	20	3	2	4
9	23	23	—	−1	1
10	16	20	4	3	9
11	19	17	−2	−3	9
			$\Sigma d = 11$		$\Sigma(d - \bar{d})^2 = 50$

From table IV, we find that $t_{0.05}$ (for $\nu = 10$) = 2.228. As the calculated value of $t < t_{0.05}$, the value of t is not significant at 5% level of significance *i.e.*, the test provides no evidence that the students have benefited by extra coaching.

Example 27.16. *From a random sample of 10 pigs fed on diet A, the increases in weight in a certain period were 10, 6, 16, 17, 13, 12, 8, 14, 15, 9 lbs. For another random sample of 12 pigs fed on diet B, the increases in the same period were 7, 13, 22, 15, 12, 14, 18, 8, 21, 23, 10, 17 lbs. Test whether diets A and B differ significantly as regards their effect on increases in weight ?*

Solution. We calculate the means and standard deviations of the samples as follows :

	Diet A			*Diet B*	
x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$	y_i	$y_i - \bar{y}$	$(y_i - \bar{y})^2$
10	−2	4	7	−8	64
6	−6	36	13	−2	4
16	4	16	22	7	49
17	5	25	15	0	0
13	1	1	12	−3	9
12	0	0	14	−1	1
8	−4	16	18	3	9
14	2	4	8	−7	49
15	3	9	21	6	36
9	−3	9	23	8	64
			10	−5	25
			17	2	4
120	0	120	180	0	314

$$\bar{x} = \frac{120}{10} = 12 \text{ lbs.}, \ \bar{y} = \frac{180}{12} = 15 \text{ lbs.}$$

$$\sigma_s^2 = [\Sigma(x_i - \bar{x})^2 + \Sigma(y_i - \bar{y})^2]/(n_1 + n_2 - 2)$$

$$= (120 + 314)/(10 + 12 - 2) = (434/20) = 21.1$$

$$\therefore \quad \sigma_s = 4.65$$

Assuming that the samples do not differ in weight so far as the two diets are concerned *i.e.*, $\mu_1 - \mu_2 = 0$.

Hence $$t = \frac{(\bar{y} - \bar{x}) - 0}{\sigma_s\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{15 - 12}{4.65\sqrt{\left(\frac{1}{10} + \frac{1}{12}\right)}} = \frac{3}{4.65}\sqrt{\frac{120}{22}} = 1.6 \text{ nearly}$$

Here $\qquad d.f.\ \nu = n_1 + n_2 - 2 = 10 + 12 - 2 = 20.$

For $\nu = 20$, we find $t_{0.05} = 2.09$ [From table IV

$\therefore$ the calculated value of $t < t_{0.05}$.

Hence the difference between the sample means is not significant *i.e.*, the two diets do not differ significantly as regards their effect on increase in weight.

PROBLEMS 27.3

1. Find the student's t for the following variable values in a sample of eight : $-4, -2, -2, 0, 2, 2, 3, 3$; taking the mean of the universe to be zero.
2. A random sample of 10 boys had the following I.Q. :

   70, 120, 110, 101, 88, 83, 95, 98, 107, 100.

   Do these data support the assumption of a population mean I.Q. of 100 (at 5% level of significance) ? *(V.T.U., 2006 ; Coimbatore, 2001)*
3. A sample of 10 measurements of the diameter of a sphere gave a mean of 12 cm and a standard deviation 0.15 cm. Find 95% confidence limits for the actual diameter.
4. A random sample of size 25 from a normal population has the mean $\bar{x} = 47.5$ and s.d. $S = 8.4$. Does this information refute the claim that the mean of the population is $\mu = 42.1$. *(J.N.T.U., 2003)*
5. A process for making certain bearings is under control if the diameter of the bearings have the mean 0.5 cm. What can we say about this process if a sample of 10 of these bearings has a mean diameter of 0.506 cm. and s.d. of 0.004 cm ?
6. A machine is supposed to produce washers of mean thickness 0.12 cm. A sample of 10 washers was found to have a mean thickness of 0.128 cm and standard deviation 0.008. Test whether the machine is working in proper order at 5% level of significance.
7. Find out the reliability of the sample mean of the following data : *Breaking strength of 10 specimens of 1.04 cms diameter hard-drawn copper wire :*

Specimen :	1	2	3	4	5	6	7	8	9	10
Breaking Strength (kgs) :	578	572	570	568	572	570	570	572	526	584

8. Test runs with 6 models of an experimental engine showed that they operated for 24, 28, 21, 23, 32 and 22 minutes with a gallon of fuel. If the probability of a Type I error is at the most 0.01, is this evidence against a hypothesis that on the average this kind of engine will operate for atleast 29 minutes per gallon of the same fuel. Assume normality. *(J.N.T.U., 2003)*
9. Two horses A and B were tested according to the time (in seconds) to run a particular race with the following results :

Horse A :	28	30	32	33	33	29	and 34
Horse B :	29	30	30	24	27	and 29	

   Test whether you can discriminate between two horses ? *(Rohtak, 2005 ; Coimbatore, 2001)*
10. A group of 10 rats fed on a diet A and another group of 8 rats fed on a different diet B, recorded the following increase in weights :

Diet A :	5	6	8	1	12	4	3	9	6	10	gm
Diet B :	2	3	6	8	10	1	2	8	gm.		

   Does it show that superiority of diet A over that of B ? *(Madras, 2003)*
11. A group of boys and girls were given an intelligence test. The mean score, S.D.s and numbers in each group are as follows :

	Boys	*Girls*
Mean	124	121
S.D.	12	10
n	18	14

   Is the mean score of boys significantly different from that of girls ?
12. The means of two random samples of sizes 9 and 7 are 196.42 and 198.82 respectively. The sum of the squares of the deviations from the mean are 26.94 and 18.73 respectively. Can the sample be considered to have been drawn from the same normal population ? *(Mumbai, 2004)*

27.17 (1) CHI-SQUARE (χ^2) TEST

When a fair coin is tossed 80 times, we expect from theoretical considerations that heads will appear 40 times and tail 40 times. But this never happens in practice *i.e.*, the results obtained in an experiment do not agree exactly with the theoretical results. *The magnitude of discrepancy between observation and theory is given by the quantity* χ^2 (pronounced as *chi-square*). If $\chi^2 = 0$, the observed and theoretical frequencies completely agree. As the value of χ^2 increases, the discrepancy between the observed and theoretical frequencies increases.

(1) Definition. *If $O_1, O_2, ..., O_n$ be a set of observed (experimental) frequencies and $E_1, E_2, ..., E_n$ be the corresponding set of expected (theoretical) frequencies, then χ^2 is defined by the relation*

$$\chi^2 = \frac{(O_1 - E_1)^2}{E_1} + \frac{(O_2 - E_2)^2}{E_2} + ... + \frac{(O_n - E_n)^2}{E_n}$$

$$= \sum \frac{(O_i - E_i)^2}{E_i} \qquad ...(1)$$

with $n - 1$ degrees of freedom. [$\Sigma O_i = \Sigma E_i = n$ the total frequency]

(2) Chi-square distribution*

If $x_1, x_2, ..., x_n$ be n independent normal variates with mean zero and s.d. unity, then it can be shown that $x_1^2 + x_2^2 + ... + x_n^2$, is a random variate having χ^2-distribution with *ndf*.

The equation of the χ^2-curve is

$$y = y_0 e^{-\chi^2/2} (\chi^2)^{(\nu - 1)/2} \qquad ...(2)$$

where $\nu = n - 1$ (Fig. 27.3).

(3) Properties of χ^2-distribution

I. If $\nu = 1$, the χ^2-curve (2) reduces to $y = y_0 e^{-\chi^2/2}$, which is the exponential distribution.

II. If $\nu > 1$, this curve is tangential to x-axis at the origin and is positively skewed as the mean is at ν and mode at $\nu - 2$.

III. The probability P that the value of χ^2 from a random sample will exceed χ_0^2 is given by

$$P = \int_{\chi_0^2}^{\infty} y\, dx.$$

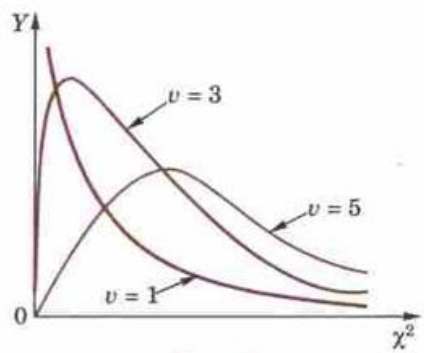


Fig. 27.3

The values of χ_0^2 have been tabulated for various values of P and for values of ν from 1 to 30. (Table-V-Appendix 2)

For $\nu > 30$, the χ^2-curve approximates to the normal curve and we should refer to normal distribution tables for significant values of χ^2.

IV. Since the equation of χ^2-curve does not involve any parameters of the population, *this distribution does not depend on the form of the population* and is therefore, very useful in a large number of problems.

V. Mean = γ and *variance* = 2γ.

27.18 GOODNESS OF FIT

The value of χ^2 is used to test whether the deviations of the observed frequencies from the expected frequencies are significant or not. It is also used to test how will a set of observations fit a given distribution, χ^2 therefore, provides a test of goodness of fit and may be used to examine the validity of some hypothesis about an observed frequency distribution. As a test of goodness of fit, it can be used to study the correspondence between theory and fact.

This is a non-parametric distribution-free test since in this we make no assumption about the distribution of the parent population.

Hamlet discovered this distribution in 1875. *Karl Pearson* rediscovered it independently in 1900 and applied it to test 'goodness of fit'.

Procedure to test significance and goodness of fit.

(*i*) Set up a 'null hypothesis' and calculate

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i}$$

(*ii*) Find the *df* and read the corresponding values of χ^2 at a prescribed significance level from Table V.

(*iii*) From χ^2-table, we can also find the probability P corresponding to the calculated values of χ^2 for the given *d.f.*

(*iv*) If $P < 0.05$, the observed value of χ^2 is significant at 5% level of significance.

If $P < 0.01$, the value is significant at 1% level.

If $P > 0.05$, it is a good fit and the value is not significant.

Example 27.17. *In experiments on pea breeding, the following frequencies of seeds were obtained :*

Round and yellow	*Wrinkled and yellow*	*Round and green*	*Wrinkled and green*	*Total*
315	*101*	*108*	*32*	*556*

Theory predicts that the frequencies should be in proportions 9 : 3 : 3 : 1. Examine the correspondence between theory and experiment.

Solution. The corresponding frequencies are

$$\frac{9}{16}\times 556, \frac{3}{16}\times 556, \frac{3}{16}\times 556, \frac{1}{16}\times 556 \text{ } i.e., 313, 104, 104, 35.$$

Hence
$$\chi^2 = \frac{(315-313)^2}{313} + \frac{(101-104)^2}{104} + \frac{(108-104)^2}{104} + \frac{(32-35)^2}{35}$$

$$= \frac{4}{313} + \frac{9}{104} + \frac{16}{104} + \frac{9}{35} = 0.51 \quad \text{and} \quad df\, \nu = 4 - 1 = 3.$$

For $\nu = 3$, we have $\chi^2_{0.05} = 7.815$ [From Table V

Since the calculated value of χ^2 is much less than $\chi^2_{0.05}$, there is a very high degree of agreement between theory and experiment.

Example 27.18. *A set of five similar coins is tossed 320 times and the result is*

No. of heads :	*0*	*1*	*2*	*3*	*4*	*5*
Frequency :	*6*	*27*	*72*	*112*	*71*	*32*

Test the hypothesis that the data follow a binomial distribution.

(Kottayam, 2005 ; P.T.U., 2005 ; V.T.U., 2004)

Solution. For $\nu = 5$, we have $\chi^2_{0.05} = 11.07$.

p, probability of getting a head $= \frac{1}{2}$; q, probability of getting a tail $= \frac{1}{2}$.

Hence the theoretical frequencies of getting 0, 1, 2, 3, 4, 5 heads are the successive terms of the binomial expansion $320\,(p+q)^5$

$$= 320\,[p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5]$$

$$= 320\left[\frac{1}{32} + \frac{5}{32} + \frac{10}{32} + \frac{10}{32} + \frac{5}{32} + \frac{1}{32}\right] = 10 + 50 + 100 + 100 + 50 + 10$$

Thus the theoretical frequencies are 10, 50, 100, 100, 50, 10.

Hence

$$\chi^2 = \frac{(6-10)^2}{10} + \frac{(27-50)^2}{50} + \frac{(72-100)^2}{100} + \frac{(112-100)^2}{100} + \frac{(71-50)^2}{50} + \frac{(32-10)^2}{10}$$

$$= \frac{1}{100}(160 + 1058 + 784 + 144 + 882 + 4840) = \frac{7868}{100} = 78.68$$

and $\quad df\, \nu = 6 - 1 = 5.$

Since the calculated value of χ^2 is much greater than $\chi^2_{0.05}$, the hypothesis that the data follow the binomial law is rejected.

Example 27.19. *Fit a Poisson distribution to the following data and test for its goodness of fit at level of significance 0.05.*

x:	0	1	2	3	4
f:	419	352	154	56	19

(V.T.U., 2008)

Solution. Mean $m = \dfrac{\Sigma f x_i}{\Sigma f_i} = \dfrac{904}{1000} = 0.904$. $\therefore$ $e^{-m} = e^{-0.904} = 0.4049$.

Hence the theoretical frequencies are $\dfrac{1000 \times e^{-m}(m)^r}{r!}$, $r = 0, 1, 2, 3, 4$

i.e.,

x:	0	1	2	3	4	Total
f:	404.9	366	165.4	49.8	11.3	997.4
	(406.2)				(12.6)	

In order that the total observed and expected frequencies may agree, we take the first and last theoretical frequencies as 406.2 and 12.6 instead of 404.9 and 11.3 as shown in brackets. (In case, the expected frequencies are less than 10, we group together such classes. Here of course, none of the frequencies < 10).

Hence

$$\chi^2 = \frac{(419-406.2)^2}{406.2} + \frac{(352-366)^2}{366} + \frac{(154-165.4)^2}{165.4} + \frac{(56-49.8)^2}{49.8} + \frac{(19-12.6)^2}{12.6}$$
$$= 0.403 + 0.536 + 0.786 + 0.772 + 3.251 = 5.748$$

Since the mean of the theoretical distribution has been estimated from the given data and the totals have been made to agree, there are two constraints so that the number of degrees of freedom $\nu = 5 - 2 = 3$.

For $\nu = 3$, we have $\chi^2_{0.05} = 7.82$. [From Table V]

Since the calculated value of $\chi^2 < \chi^2_{0.05}$, the agreement between the fact and theory is good and hence the Poisson distribution can be fitted to the data.

Example 27.20. *Fit a normal distribution to the following data of weights of 100 students of Delhi University and test the goodness of fit.*

Weights (kg) :	60-62	63-65	66-68	69-71	72-74
Frequency :	5	18	42	27	8

Solution. Here $N = 100$, mean $m = 67.45$ and S.D. $\sigma = 2.92$. The calculations are arranged as follows* :

Class boundary (x)	$z = (x - m)/\sigma$	*Area under normal curve from 0 to z*	*Area for each class (A)**	*Expected frequency* ($f_e = N \times A$)
59.5	– 2.72	0.4967		
			0.0413	4.13
62.5	– 1.70	0.4554		
			0.2068	20.68
65.5	– 0.67	0.2486		
			0.3892	38.92
68.5	0.36	0.1406		
			0.2771	27.71
71.5	1.39	0.4177		
			0.0743	7.43
74.5	2.41	0.4920		

* A is obtained by subtracting the successive areas in the third column when the corresponding values of z have the same sign and adding them when the z values have opposite signs (which occurs only once).

$$\therefore \quad \chi^2 = \frac{(5-4.13)^2}{4.13} + \frac{(18-20.68)^2}{20.68} + \frac{(42-38.92)^2}{38.92} + \frac{(27-27.71)^2}{27.71} + \frac{(8-7.43)^2}{7.43}$$

$$= 0.1833 + 0.3473 + 0.2437 + 0.0182 + 0.0437 = 0.8362$$

As regards the number of degrees of freedom (γ), there are three constraints (*i*) discrepancy between total observed and total estimated frequencies (*ii*) and (*iii*) mean (m) and standard deviation (σ) have been estimated from the sample data. $\therefore \quad r = 5 - 3 = 2$.

For $\gamma = 2$, $\chi^2_{0.05} = 0.103$ from table V.

Since $\chi^2 = 0.8362 > 0.103$. Hence the fit is not good.

PROBLEMS 27.4

1. Five dice were thrown 96 times and the number of times 4, 5 or 6 were thrown were :

No. of dice showing 4, 5 or 6 :	5	4	3	2	1	0
Frequency :	8	18	35	24	10	1

Find the probability of getting this result by chance ?

2. Genetic theory states that children having one parent of blood type M and other blood type N will always be one of the three types M, MN, N and that the proportions of these types will on average be 1 : 2 : 1. A report states that out of 300 children having one M parent and one N parent, 30% were found to be of type M, 45% of type MN and remainder of type N. Test the hypothesis by χ^2 test.

3. A die was thrown 60 times and the following frequency distribution was observed :

Faces :	1	2	3	4	5	6
f_0 :	15	6	4	7	11	17

Test whether the die is unbiased ?

4. The following table gives the number of aircraft accidents that occurred during the various days of the week. Find whether the accidents are uniformly distributed over the week ?

Days :	Sun	Mon	Tue	Wed	Thu	Fri	Sat	Total
No. of accidents :	14	16	8	12	11	9	14	84

(*Hissar, 2005*)

5. Fit a binomial distribution to the data :

x :	0	1	2	3	4	5
f :	38	144	342	287	164	25

and test for goodness of fit, at the level of significance 0.05.

6. In 1000 extensive sets of trials for an event of small probability, the frequencies f_0 of the number x of successes proved to be :

x :	0	1	2	3	4	5	6	7
f_0 :	305	366	210	80	28	9	2	1

Fit a Poisson distribution to the data and test the goodness of fit.

7. The frequencies of localities according to the number of deaths due to cholera during eight years in 1000 localities is as follows :

No. of deaths :	0	1	2	3	4	5	6	7
No. of localities :	314	355	204	86	29	9	3	0

Fit a suitable distribution to the data and test the goodness of fit.

8. Obtain the equation of the normal curve that may be fitted to the data and test the goodness of fit.

x :	4	6	8	10	12	14	16	18	20	22	24	Total
$f(x)$:	1	7	15	22	35	43	38	20	13	5	1	200

27.19 (1) F-DISTRIBUTION*

Let $x_1, x_2, \ldots x_{n_1}$ and $y_1, y_2, \ldots y_{n_2}$ be the values of two independent random samples drawn from the normal populations σ^2 having equal variances.

* This distribution was introduced by the English statistician *Prof. R.A. Fisher* (1890–1962) who had greatly influenced the development of modern statistics.

Let $\bar{x}_1$ and $\bar{x}_2$ be the sample means and $s_1^2 = \frac{1}{n_1 - 1}\sum_1^{n_1}(x_i - \bar{x})^2$, $s_2^2 = \frac{1}{n_2 - 1}\sum_1^{n_2}(y_i - \bar{y})^2$ be the sample variances.

Then we define F by the relation

$$F = \frac{s_1^2}{s_2^2} \qquad (s_1^2 > s_2^2)$$

This gives F-distribution (also known as variance ratio distribution) with $\gamma_1 = n_1 - 1$ and $\gamma_2 = n - 2$ degrees of freedom. *The larger of the variances is placed in the numerator.*

(2) Properties. I. The F-distribution curve lies entirely in the first quadrant and is *unimodal*.

II. The F-distribution is independent of the population variance σ^2 and depends on γ_1 and γ_2 only.

III. $F_\alpha(\gamma_1, \gamma_2)$ is the value of F for γ_1 and γ_2 of such that the area to the right of F_α is α.

IV. It can be shown that the mode of F-distribution is less than unity.

(3) Significance test. Snedecor's F-tables give 5% and 1% points of significance for F. (Table VI – Appendix 2). 5% points of F mean that area under the F-curve to the right of the ordinate at a value of F, is 0.05. Clearly value of F at 5% significance is lower than that at 1%. F-distribution is very useful for testing the equality of population means by comparing sample variances. As such it forms the basis of *analysis of variance*.

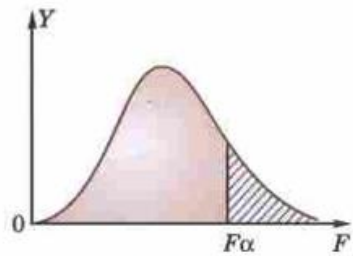


Fig. 27.4

Example 27.21. *Two samples of sizes 9 and 8 give the sum of squares of deviations from their respective means equal to 160 inches² and 91 inches² respectively. Can these be regarded as drawn from the same normal population ?* (V.T.U., 2002)

Solution. We have $\Sigma(x - \bar{x})^2 = 160$ and $\Sigma(y - \bar{y})^2 = 91$

$$\therefore \qquad s_1^2 = \frac{160}{8} = 20$$

and
$$s_2^2 = \frac{91}{7} = 13.$$

Hence
$$F = \frac{s_1^2}{s_2^2} = \frac{20}{13} = 1.54 \text{ nearly.}$$

For $\gamma_1 = 8$, $\gamma_2 = 7$, we have $F_{0.05} = 3.73$. [From Table VI]

Since the calculated value of $F < F_{0.05}$, the population variances are not significantly different. Thus the two samples can be regarded as drawn from two normal populations with the same variance. If the two populations are to be same, their means should also be the same which can be verified by applying t-test provided the sample means are known.

Example 27.22. *Measurements on the length of a copper wire were taken in 2 experiments A and B as under :*

A's measurements (mm) :	12.29,	12.25,	11.86,	12.13,	12.44,	12.78,	12.77,	11.90,	12.47.
B's measurements (mm) :	12.39,	12.46,	12.34,	12.22,	11.98,	12.46,	12.23,	12.06.	

Test whether B's measurements are more accurate than A's. (The readings taken in both cases being unbiased)

Solution. Readings in both cases being unbiased, B's measurements will be taken more accurate if its population variance is less than that of A's measurements.

Under the hypothesis that the two populations have the same variance (*i.e.* $\sigma_1^2 = \sigma_2^2$), we have

$$F = \frac{s_1^2}{s_2^2}$$

with $\gamma_1 = n_1 - 1 = 8$ and $\gamma_2 = n_2 - 1 = 7$.

We calculate the s.d.'s of the two series as follows :

A's measurements			B's measurements		
x	$u = 100(x - 12)$	u^2	y	$v = 100(y - 12)$	v^2
12.29	29	841	12.39	39	1521
12.25	25	625	12.46	46	2116
11.86	− 14	196	12.34	34	1156
12.13	13	169	12.22	22	484
12.44	44	1936	11.98	− 2	4
12.78	78	6084	12.46	46	2116
12.77	77	5929	12.23	23	529
11.90	− 10	100	12.06	6	36
12.47	47	2209			
	289	18089		214	7962

$$\therefore \qquad s_1^2 = \frac{1}{n_1 - 1}\left[18089 - \frac{(289)^2}{n_1}\right] = \frac{1}{8}(18089 - 9280) = 1101.1$$

$$s_2^2 = \frac{1}{n_2 - 1}\left[7962 - \frac{(214)^2}{n_2}\right] = \frac{1}{7}(7962 - 5724) = 319.7$$

$$\therefore \qquad F = \frac{s_1^2}{s_2^2} = \frac{1101.1}{319.7} = 3.44$$

For $\gamma_1 = 8$ and $\gamma_2 = 7$, from table VI, $F_{0.05} = 3.73$ and $F_{0.01} = 6.84$.

Since the calculated value of F is less than both $F_{0.05}$ and $F_{0.01}$, the result is insignificant at both 5% and 1% level.

Hence there is no reason to say that B's measurements are more accurate than those of A's.

27.20 (1) FISHER'S z-DISTRIBUTION

Changing the variable F to z by the substitution $z = \frac{1}{2}\log_e F$ or $F = e^{2z}$ in the F-distribution, we get the *Fisher's z-distribution.*

It is more nearly symmetrical than F-distribution. Table showing the values of z that will be exceeded in simple sampling with probabilities 0.05 and 0.01 have been prepared for various values of v_1 and v_2.

(2) Significance test. As z-table give only critical values corresponding to right hand tail areas, therefore 5% (or 1%) points of z imply that the area to the right of the ordinate at z is 0.05 (or 0.01). In other words, *5% and 1% points of z correspond to 10% and 2% levels of significance respectively.*

Example 27.23. *Two gauge operations are tested for precision in making measurements. One operator completes a set of 26 readings with a standard deviations of 1.34 and the other does 34 readings with a standard deviations of 0.98. What is the level of significance of this difference.*

(Given that for $\nu_1 = 25$ and $\nu_2 = 33$, $z_{0.05} = 0.305$, $z_{0.01} = 0.432$)

Solution. We have $n_1 = 26$, $\sigma_x = 1.34$; $n_2 = 34$, $\sigma_y = 0.98$

$$\therefore \qquad s_1^2 = \frac{n_1}{n_1 - 1}\cdot\sigma_x^2 = \frac{26}{25}(1.34)^2 \approx (1.34)^2 \quad \text{and} \quad s_2^2 = \frac{n_2}{n_2 - 1}\cdot\sigma_y^2 = \frac{34}{33}(0.98)^2 \approx (0.98)^2$$

$$\text{Hence} \qquad F = \left(\frac{1.34}{0.98}\right)^2 = 1.8696 \quad \text{and} \quad z = \frac{1}{2}\log_e F = 1.1513 \log_{10} 1.8696 = 0.3129$$

Since the calculated value of z is just greater than $z_{0.05}$ and less than $z_{0.01}$, the difference between the standard deviation is just significant at 5% level and insignificant at 1% level.

PROBLEMS 27.5

1. Two samples of 9 and 7 individuals have variances 4.8 and 9.6 respectively. Is the variance 9.6 significantly greater than the variance 4.6 ?
2. Test for breaking strength were carried out on two lots of 5 and 9 steel wires. The variance of one lot was 230 and that of other was 492. Is there a significant difference in their variability ?
3. Show how you would use Fisher's z-test to decide whether the two sets of observations 17, 27, 18, 25, 27, 29, 27, 23, 17 and 16, 16, 20, 16, 20, 17, 15, 21, indicate samples from the same universe.
4. In two groups of ten children each, the increase in weight due to different diets during the same period, were in pounds

   3, 7, 5, 6, 5, 4, 4, 5, 3, 6

   8, 5, 7, 8, 3, 2, 7, 6, 5, 7.

   Is there a significant difference in their variability ?
5. The mean diameter of rivets produced by two firms A and B are practically the same but their standard deviations are different. For 16 rivets manufactured by firm A, the S.D. is 3.8 mm while for 22 rivets manufactured by firm B is 2.9 mm. Do you think products of firm B are of better quality than those of firm A.
6. The I.Q.'s of 25 students from one college showed a variance of 16 and those of an equal number from the other college had a variance of 8. Discuss whether there is any significant difference in variability of intelligence.
7. Two random samples from two normal populations are given below :

Sample I :	16	26	27	23	24	22
Sample II :	33	42	35	32	28	31

   Do the estimates of population variances differ significantly ?

Degrees of freedom :	(5, 5)	(5, 6)	(6, 5)
5% value of F :	5.05	4.39	4.95.

8. Two independent samples of sizes 7 and 6 have the following values :

Sample A :	28	30	32	33	33	29	34
Sample B :	29	30	30	24	27	29	

   Examine whether the samples have been drawn from normal populations having the same variance ?

   [Given that the values of F at 5% level for (6, 5) d.f. is 4.95 and for (5, 6) d.f. is 4.39].

27.21 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 27.6

Select the correct answer or fill up the blanks in each of the following questions :

1. The 'null hypothesis' implies that .....
2. The uses of t-distribution are .....
3. Type I and type II errors are such that .....
4. A single-tailed test is used when .....
5. Control limit theorem states that .....
6. A hypothesis is true, but is rejected. Then this is an error of type .....
7. If the standard deviation of a χ^2 distribution is 10, then its degree of freedom is .....
8. Range of F-distribution is .....
9. A hypothesis is false but accepted, then there is an error of type .....
10. The mean and variance of a χ^2 distribution with 8 degrees of freedom are ..... and ..... respectively.
11. In a t-distribution of sample size n, the degrees of freedom are .....
12. The test statistic $F = \frac{s_1^2}{s_2^2}$ is used when

    (*i*) $s_2^2 > s_1^2$ (*ii*) $s_2^2 < s_1^2$ (*iii*) $s_1^2 = s_2^2$ (*iv*) none of these.
13. The t-test is applicable to samples for which n is .....
14. The two main uses of χ^2-test are .....
15. Range of t-distribution is .....
16. If two samples are taken from two populations of unequal variances, we can apply t-test to test the difference of means. (True or False)
17. The Chi-square distribution is continuous. (True or False).

CHAPTER 28

Numerical Solution of Equations

1. Introduction. 2. Solution of algebraic and transcendental equations—Bisection method, Method of false position, Newton's method. 3. Useful deductions from the Newton-Raphson Formula. 4. Approximate solution of equations—Horner's method. 5. Solution of linear simultaneous equations. 6. Direct methods of solution—Gauss elimination method, Gauss-Jordan method, Factorization method. 7. Iterative methods of solution—Jacobi's method, Gauss-Seidal method, Relaxation method. 8. Solution of non-linear simultaneous equations—Newton-Raphson method. 9. Determination of eigen values by iteration. 10. Objective Type of Questions.

28.1 INTRODUCTION

The limitations of analytical methods have led the engineers and scientists to evolve graphical and numerical methods. As seen in § 1.8, the graphical methods, though simple, give results to a low degree of accuracy. Numerical methods can, however, be derived which are more accurate. With the advent of high speed digital computers and increasing demand for numerical answers to various problems, numerical techniques have become indispensible tool in the hands of engineers.

Numerical methods are often, of a repetitive nature. These consist in repeated execution of the same process where at each step the result of the preceding step is used. This is known as *iteration process* and is repeated till the result is obtained to a desired degree of accuracy.

In this chapter, we shall discuss some numerical methods for the solution of algebraic and transcendental equations and simultaneous linear and non-linear equations. We shall close the chapter by describing an iterative method for the solution of eigen-value problem. For a detailed study of these topics, the reader should refer to author's book '*Numerical Methods in Engineering & Science*'.

28.2 SOLUTION OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS

To find the roots of an equation $f(x) = 0$, we start with a known approximate solution and apply any of the following methods :

(1) Bisection method. This method consists in locating the root of the equation $f(x) = 0$ between a and b. If $f(x)$ is continuous between a and b, and $f(a)$ and $f(b)$ are of opposite signs then there is a root between a and b. For definiteness, let $f(a)$ be negative and $f(b)$ be positive. Then the first approximation to the root is $x_1 = \frac{1}{2}(a + b)$.

If $f(x_1) = 0$, then x_1 is a root of $f(x) = 0$. Otherwise, the root lies between a and x_1 or x_1 and b according as $f(x_1)$ is positive or negative. Then we bisect the interval as before and continue the process until the root is found to desired accuracy.

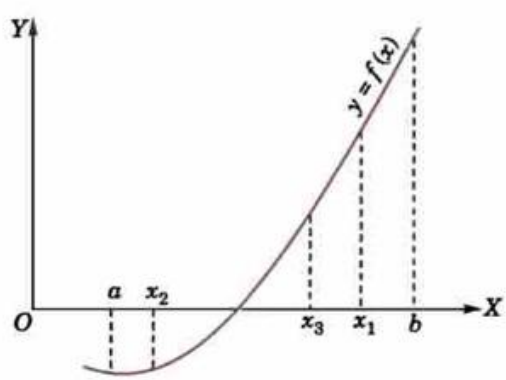


Fig. 28.1

In the Fig. 28.1, $f(x_1)$ is +ve, so that the root lies between a and x_1. Then the second approximation to the root is $x_2 = \frac{1}{2}(a + x_1)$. If $f(x_2)$ is – ve, the root lies between x_1 and x_2. Then the third approximation to the root is $x_3 = \frac{1}{2}(x_1 + x_2)$ and so on.

Example 28.1. *(a) Find a root of the equation $x^3 - 4x - 9 = 0$, using the bisection method correct to three decimal places.* (Mumbai, 2003)

(b) Using bisection method, find the negative root of the equation $x^2 - 4x + 9 = 0$. (J.N.T.U., 2009)

Solution. (*a*) Let $f(x) = x^3 - 4x - 9$

Since $f(2)$ is –ve and $f(3)$ is +ve, a root lies between 2 and 3

$\therefore$ first approximate to the root is

$$x_1 = \frac{1}{2}(2 + 3) = 2.5$$

Thus $\quad f(x_1) = (2.5)^3 - 4(2.5) - 9 = -3.375 \quad$ *i.e.*, – ve

$\therefore$ the root lies between x_1 and 3. Thus the second approximation to the root is

$$x_2 = \frac{1}{2}(x_1 + 3) = 2.75$$

Then $\quad f(x_2) = (2.75)^3 - 4(2.75) - 9 = 0.7969 \quad$ *i.e.*, + ve

$\therefore$ the root lies between x_1 and x_2. Thus the third approximation to the root is

$$x_3 = \frac{1}{2}(x_1 + x_2) = 2.625$$

Then $\quad f(x_3) = (2.625)^3 - 4(2.625) - 9 = -1.4121 \quad$ *i.e.*, – ve

$\therefore$ the root lies between x_2 and x_3. Thus the fourth approximation to the root is

$$x_4 = \frac{1}{2}(x_2 + x_3) = 2.6875$$

Repeating this process, the successive approximations are

$x_5 = 2.71875, \quad x_6 = 2.70313, \quad x_7 = 2.71094$

$x_8 = 2.70703, \quad x_9 = 2.70508, \quad x_{10} = 2.70605$

$x_{11} = 2.70654, \quad x_{12} = 2.70642$

Hence the root is 2.7064

(*b*) If α, β, γ are the roots of the given equation, then $-\alpha, -\beta, -\gamma$ are the roots of $(-x)^3 - 4(-x) + 9 = 0$

$\therefore$ the negative root of the given equation is the positive root of $x^3 - 4x - 9 = 0$ which we have found above to be 2.7064.

Hence the negative root for the given equation is – 2.7064.

Example 28.2. *By using the bisection method, find an approximate root of the equation $\sin x = 1/x$, that lies between $x = 1$ and $x = 1.5$ (measured in radians). Carry out computations upto the 7th stage.* (V.T.U., 2003 S)

Solution. Let $f(x) = x \sin x - 1$. We know that $1^r = 57.3°$.

Since $\quad f(1) = 1 \times \sin(1) - 1 = \sin(57.3°) - 1 = -0.15849$

and $\quad f(1.5) = 1.5 \times \sin(1.5)^r - 1 = 1.5 \times \sin(85.95)° - 1 = 0.49625$;

a root lies between 1 and 1.5.

$\therefore$ first approximation to the root is $x_1 = \frac{1}{2}(1 + 1.5) = 1.25$.

Then $f(x_1) = (1.25)\sin(1.25) - 1 = 1.25 \sin(71.625°) - 1 = 0.18627$ and $f(1) < 0$.

$\therefore$ a root lies between 1 and $x_1 = 1.25$.

Thus the second approximation to the root is $x_2 = \frac{1}{2}(1 + 1.25) = 1.125$.

Then $f(x_2) = 1.125 \sin(1.125) - 1 = 1.125 \sin(64.46)° - 1 = 0.01509$ and $f(1) < 0$.

$\therefore$ a root lies between 1 and $x_2 = 1.125$.

Thus the third approximation to the root is $x_3 = \frac{1}{2}(1 + 1.125) = 1.0625$

Then $f(x_3) = 1.0625 \sin(1.0625) - 1 = 1.0625 \sin(60.88) - 1 = -0.0718 < 0$

and $f(x_2) > 0$, *i.e.* now the root lies between $x_3 = 1.0625$ and $x_2 = 1.125$.

$\therefore$ fourth approximation to the root is $x_4 = \frac{1}{2}(1.0625 + 1.125) = 1.09375$

Then $f(x_4) = -0.02836 < 0$ and $f(x_2) > 0$,

i.e., the root lies between $x_4 = 1.09375$ and $x_2 = 1.125$.

$\therefore$ fifth approximation to the root is $x_5 = \frac{1}{2}(1.09375 + 1.125) = 1.10937$

Then $f(x_5) = -0.00664 < 0$ and $f(x_2) > 0$.

$\therefore$ the root lies between $x_5 = 1.10937$ and $x_2 = 1.125$.

Thus the sixth approximation to the root is

$$x_6 = \frac{1}{2}(1.10937 + 1.125) = 1.11719$$

Then $f(x_6) = 0.00421 > 0$. But $f(x_5) < 0$.

$\therefore$ the root lies between $x_5 = 1.10937$ and $x_6 = 1.11719$.

Thus the seventh approximation to the root is $x_7 = \frac{1}{2}(1.10937 + 1.11719) = 1.11328$

Hence the desired approximation to the root is 1.11328.

(2) Method of false position *or* **Regula-falsi method.** This is the oldest method of finding the real root of an equation $f(x) = 0$ and closely resembles the bisection method. Here we choose two points x_0 and x_1 such that $f(x_0)$ and $f(x_1)$ are of opposite signs *i.e.*, the graph of $y = f(x)$ crosses the x-axis between these points (Fig. 28.2). This indicates that a root lies between x_0 and x_1 consequently $f(x_0) f(x_1) < 0$.

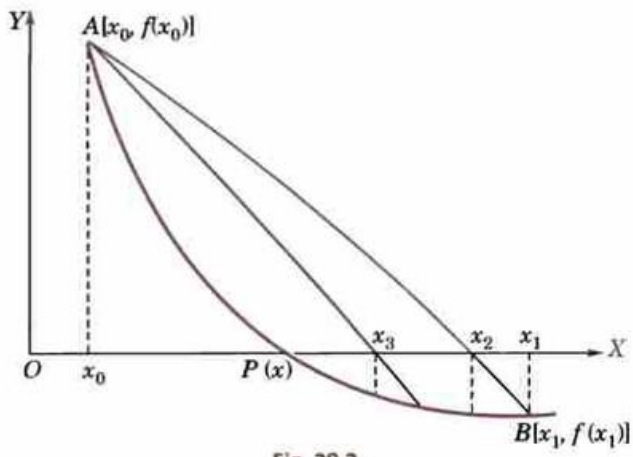


Fig. 28.2

Equation of the chord joining the points $A[x_0, f(x_0)]$ and $B[x_1, f(x_1)]$ is

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) \qquad ...(1)$$

The method consists in replacing the curve AB by means of the chord AB and taking the point of intersection of the chord with the x-axis as an approximation to the root. So the abscissa of the point where the chord cuts the x-axis ($y = 0$) is given by

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \qquad ...(2)$$

which is an approximation to the root.

If now $f(x_0)$ and $f(x_2)$ are of opposite signs, then the root lies between x_0 and x_2. So replacing x_1 by x_2 in (2), we obtain the next approximation x_3. (The root could as well lie between x_1 and x_2 and we would obtain x_3 accordingly). This procedure is repeated till the root is found to desired accuracy. The iteration process based on (1) is known as the *method of false position.*

Example 28.3. *Find a real root of the equation $x^3 - 2x - 5 = 0$ by the method of false position correct to three decimal places.* (*Manipal, 2005*)

Solution. Let $f(x) = x^3 - 2x - 5$

so that $f(2) = -1$ and $f(3) = 16$ *i.e.*, A root lies between 2 and 3.

$\therefore$ taking $x_0 = 2, x_1 = 3, f(x_0) = -1, f(x_1) = 16$, in the method of false position, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 2 + \frac{1}{17} = 2.0588 \qquad ...(i)$$

Now $f(x_2) = f(2.0588) = -0.3908$ *i.e.*, the root lies between 2.0588 and 3.

$\therefore$ taking $x_0 = 2.0588, x_1 = 3, f(x_0) = -0.3908, f(x_1) = 16$, in (i), we get

$$x_3 = 2.0588 - \frac{0.9412}{16.3908}(-0.3908) = 2.0813$$

Repeating this process, the successive approximations are

$$x_4 = 2.0862, x_5 = 2.0915, x_6 = 2.0934, x_7 = 2.0941, x_8 = 2.0943 \text{ etc.}$$

Hence the root is 2.094 correct to 3 decimal places.

Example 28.4. *Find the root of the equation $\cos x = xe^x$ using the regula-falsi method correct to four decimal places.* (Bhopal, 2009)

Solution. Let $f(x) = \cos x - xe^x = 0$

So that $f(0) = 1, f(1) = \cos 1 - e = -2.17798$

i.e., the root lies between 0 and 1.

$\therefore$ taking $x_0 = 0, x_1 = 1, f(x_0) = 1$ and $f(x_1) = -2.17798$ in the regula-falsi method, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 0 + \frac{1}{3.17798} \times 1 = 0.31467 \qquad ...(i)$$

Now $f(0.31467) = 0.51987$

i.e., the root lies between 0.31467 and 1.

$\therefore$ taking $x_0 = 0.31467, x_1 = 1, f(x_0) = 0.51987, f(x_1) = -2.17798$ in (i), we get

$$x_3 = 0.31467 + \frac{0.68533}{2.69785} \times 0.51987 = 0.44673$$

Now $f(0.44673) = 0.20356$

i.e., the root lies between 0.44673 and 1.

$\therefore$ taking $x_0 = 0.44673, x_1 = 1, f(x_0) = 0.20356, f(x_1) = -2.17798$ in (i), we get

$$x_4 = 0.44673 + \frac{0.55327}{2.38154} \times 0.20356 = 0.49402$$

Repeating this process, the successive approximations are

$$x_5 = 0.50995, \quad x_6 = 0.51520, \quad x_7 = 0.51692$$
$$x_8 = 0.51748, \quad x_9 = 0.51767, \quad x_{10} = 0.51775 \text{ etc.}$$

Hence the root is 0.5177 correct to 4 decimal places.

Example 28.5. *Find a real root of the equation $x \log_{10} x = 1.2$ by regula-falsi method correct to four decimal places.* (V.T.U., 2010 ; J.N.T.U., 2008 ; Kottayam, 2005)

Solution. Let $f(x) = x \log_{10} x - 1.2$

so that $f(1) = -$ ve, $f(2) = -$ ve and $f(3) = +$ ve.

$\therefore$ a root lies between 2 and 3.

Taking $x_0 = 2$ and $x_1 = 3, f(x_0) = -0.59794$ and $f(x_1) = 0.23136$, in the method of false position, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 2.72102 \qquad ...(i)$$

Now $f(x_2) = f(2.72102) = -0.01709$

i.e., the root lies between 2.72102 and 3.

$\therefore$ taking $x_0 = 2.72102, x_1 = 3, f(x_0) = -0.01709$

and $f(x_1) = 0.23136$ in (i), we get

$$x_3 = 2.72102 + \frac{0.27898}{0.23136 + 0.01709} \times 0.01709 = 2.74021$$

Repeating this process, the successive approximations are

$$x_4 = 2.74024, x_5 = 2.74063 \text{ etc.}$$

Hence the root is 2.7406 correct to 4 decimal places.

Example 28.6. *Use the method of false position, to find the fourth root of 32 correct to three decimal places.*

Solution. Let $x = (32)^{1/4}$ so that $x^4 - 32 = 0$

Take $f(x) = x^4 - 32$. Then $f(2) = -16$ and $f(3) = 49$, *i.e.*, a root lies between 2 and 3.

$\therefore$ taking $x_0 = 2, x_1 = 3, f(x_0) = -16, f(x_1) = 49$ in the method of false position, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 2 + \frac{16}{65} = 2.2462 \qquad ...(i)$$

Now $f(x_2) = f(2.2462) = -6.5438$ *i.e.* the root lies between 2.2462 and 3.

$\therefore$ taking $x_0 = 2.2462, x_1 = 3, f(x_0) = -6.5438, f(x_1) = 49$

in (*i*), we get
$$x_3 = 2.2462 - \frac{3 - 2.2462}{49 + 6.5438}(-6.5438) = 2.335$$

Now $f(x_3) = f(2.335) = -2.2732$ *i.e.* the root lies between 2.335 and 3.

$\therefore$ taking $x_0 = 2.335$ and $x_1 = 3, f(x_0) = -2.2732$ and $f(x_1) = 49$ in (*i*), we obtain

$$x_4 = 2.335 - \frac{3 - 2.335}{49 + 2.2732}(-2.2732) = 2.3645$$

Repeating this process, the successive approximations are $x_5 = 2.3770, x_6 = 2.3779$ etc.

Since $x_5 = x_6$ upto 3 decimal places, we take $(32)^{1/4} = 2.378$.

(3) Newton-Raphson method*. Let x_0 be an approximate root of the equation $f(x) = 0$. If $x_1 = x_0 + h$ be the exact root, then $f(x_1) = 0$.

$\therefore$ expanding $f(x_0 + h)$ by Taylor's series

$$f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + ... = 0$$

Since h is small, neglecting h^2 and higher powers of h, we get

$$f(x_0) + hf'(x_0) = 0 \quad \text{or} \quad h = -\frac{f(x_0)}{f'(x_0)} \qquad ...(1)$$

$\therefore$ a closer approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Similarly, starting with x_1, a still better approximation x_2 is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

In general,
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad ...(2)$$

which is known as the *Newton-Raphson formula* or *Newton's iteration formula.*

Obs. 1. *Newton's method is useful in cases of large values of $f'(x)$ i.e.* when the graph of $f(x)$ while crossing the x-axis is nearly vertical.

Obs. 2. Newton's method has a second order of quadratic convergence. Suppose x_n differs from the root α by a small quantity ε_n so that $x_0 = \alpha + \varepsilon_n$ and $x_{n+1} = \alpha + \varepsilon_{n+1}$.

Then (2) becomes $\alpha + \varepsilon_{n+1} = \alpha + \varepsilon_n - \dfrac{f(\alpha + \varepsilon_n)}{f'(\alpha + \varepsilon_n)}$

i.e.,
$$\varepsilon_{n+1} = \varepsilon_n - \frac{f(\alpha + \varepsilon_n)}{f'(\alpha + \varepsilon_n)} = \varepsilon_n - \frac{f(\alpha) + \varepsilon_n f'(\alpha) + \frac{1}{2!}\varepsilon_n^2 f''(\alpha) + ...}{f'(\alpha) + \varepsilon_n f''(\alpha) + ...} \qquad \text{[By Taylor's expansion.]}$$

$$= \varepsilon_n - \frac{\varepsilon_n f'(\alpha) + \frac{1}{2}\varepsilon_n^2 f''(\alpha) + ...}{f'(\alpha) + \varepsilon_n f''(\alpha) + ...} \qquad [\because f(\alpha) = 0]$$

$$= \frac{\varepsilon_n^2 f''(\alpha)}{2[f'(\alpha) + \varepsilon_n f''(\alpha)]} = \frac{\varepsilon_n^2}{2} \cdot \frac{f''(\alpha)}{f'(\alpha)}. \qquad \left[\begin{array}{l}\text{neglecting third and}\\ \text{higher powers of } \varepsilon_n.\end{array}\right]$$

This shows that the subsequent error at each step, is proportional to the square of the previous error and as such the convergence is quadratic. *(P.T.U., 2005)*

Obs. 3. Geometrical interpretation. Let x_0 be a point near the root α of the equation $f(x) = 0$ (Fig. 28.3). Then the equation of the tangent at $A_0\,[x_0, f(x_0)]$ is $y - f(x_0) = f'(x_0)(x - x_0)$.

*See footnote p. 466. Named after the English mathematician *Joseph Raphson* (1648–1715) who suggested a method similar to Newton's method.

It cuts the x-axis at $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$ which is a first approximation to the root α. If A_1 is the point corresponding to x_1 on the curve, then the tangent at A_1 will cut the x-axis of x_2 which is nearer to α and is, therefore, a second approximation to the root. Repeating this process, we approach to the root α quite rapidly. Hence *the method consists in replacing the part of the curve between the point A_0 and the x-axis by means of the tangent to the curve at A_0.*]

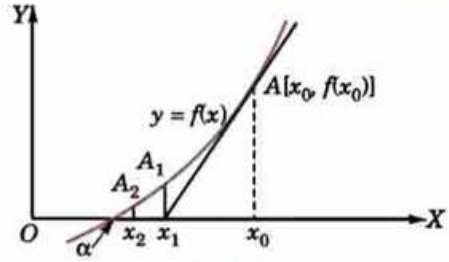


Fig. 28.3

Example 28.7. *Find the positive root of $x^4 - x = 10$ correct to three decimal places, using Newton-Raphson method.* (J.N.T.U., 2008 ; Madras, 2006)

Solution. Let $\quad f(x) = x^4 - x - 10$

So that $\quad f(1) = -10 = -\text{ve}, f(2) = 16 - 2 - 10 = 4 = +\text{ve}$

$\therefore$ a root of $f(x) = 0$ lies between 1 and 2. Let us take $x_0 = 2$

Also $\quad f'(x) = 4x^3 - 1$

Newton-Raphson's formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad ...(i)$$

Putting $n = 0$, the first approximation x_1 is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{4}{4 \times 2^3 - 1} = 2 - \frac{4}{31} = 1.871$$

Putting $n = 1$ in (i), the second approximation is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.871 - \frac{f(1.871)}{f'(1.871)}$$

$$= 1.871 - \frac{(1.871)^4 - (1.871) - 10}{4(1.871)^3 - 1} = 1.871 - \frac{0.3835}{25.199} = 1.856$$

Putting $n = 2$ in (i), the third approximation is

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.856 - \frac{(1.856)^4 - (1.856) - 10}{4(1.856)^3 - 1}$$

$$= 1.856 - \frac{0.010}{24.574} - 1.856$$

Here $x_2 = x_3$. Hence the desired is 1.856 correct to three decimal places.

Example 28.8. *Find the Newton's method, the real root of the equation $3x = \cos x + 1$.* (V.T.U., 2009 ; S.V.T.U., 2007)

Solution. Let $\quad f(x) = 3x - \cos x - 1$

$f(0) = -2 = -\text{ve}, f(1) = 3 - 0.5403 - 1 = 1.4597 = +\text{ve}.$

So a root of $f(x) = 0$ lies between 0 and 1. It is nearer to 1. Let us take $x_0 = 0.6$.

Also $\quad f'(x) = 3 + \sin x$

$\therefore$ Newton's iteration formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n}$$

$$= \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n} \qquad ...(i)$$

Putting $n = 0$, the first approximation x_1 is given by

$$x_1 = \frac{x_0 \sin x_0 + \cos x_0 + 1}{3 + \sin x_0} = \frac{(0.6)\sin(0.6) + \cos(0.6) + 1}{3\sin(0.6)}$$

$$= \frac{0.6 \times 0.5729 + 0.82533 + 1}{3 + 0.5729} = 0.6071$$

Putting $n = 1$ in (i), the second approximation is

$$x_2 = \frac{x_1 \sin x_1 + \cos x_1 + 1}{3 + \sin x_1} = \frac{0.6071 \sin (0.6071) + \cos (0.6071) + 1}{3 + \sin (0.6071)}$$

$$= \frac{0.6071 \times 0.57049 + 0.8213 + 1}{3 + 0.57049} = 0.6071 \quad \text{Clearly, } x_1 = x_2.$$

Hence the desired root is 0.6071 correct to four decimal places.

Example 28.9. *Using Newton's iterative method, find the real root of $x \log_{10} x = 1.2$ correct to five decimal places.* *(V.T.U., 2005 ; Mumbai, 2004 ; Burdwan, 2003)*

Solution. Let $f(x) = x \log_{10} x - 1.2$

$f(1) = -1.2 = -\text{ve}, f(2) = 2 \log_{10} 2 - 1.2 = 0.59794 = -\text{ve}$

and $f(3) = 3 \log_{10} 3 - 1.2 = 1.4314 - 1.2 = 0.23136 = +\text{ve}$

So a root of $f(x) = 0$ lies between 2 and 3. Let us take $x_0 = 2$

Also $f'(x) = \log_{10} x + x \cdot \frac{1}{x} \log_{10} e = \log_{10} x + 0.43429$

$\therefore$ Newton's iteration formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \frac{0.43429\, x_n + 1.2}{\log_{10} x_n + 0.43429} \qquad ...(i)$$

Putting $n = 0$, the first approximation is

$$x_1 = \frac{0.43429 \times x_0 + 1.2}{\log_{10} x_0 + 0.43429} = \frac{0.43429 \times 2 + 1.2}{\log_{10} 2 + 0.43429} = \frac{0.86858 + 1.2}{0.30103 + 0.43429} = 2.81$$

Similarly putting $n = 1, 2, 3, 4$ in (i), we get

$$x_2 = \frac{0.43429 \times 2.81 + 1.2}{\log_{10} 2.81 + 0.43429} = 2.741$$

$$x_3 = \frac{0.43429 \times 2.741 + 1.2}{\log_{10} 2.81 + 0.43429} = 2.74064$$

$$x_4 = \frac{0.43429 \times 2.74064 + 1.2}{\log_{10} 2.74064 + 0.43429} = 2.74065$$

$$x_5 = \frac{0.43429 \times 2.74065 + 1.2}{\log_{10} 2.74065 + 0.43429} = 2.74065$$

Clearly $x_4 = x_5$.

Hence the required root is 2.74065 correct to five decimal places.

28.3 USEFUL DEDUCTIONS FROM THE NEWTON-RAPHSON FORMULA

(1) *Iterative formula to find $1/N$ is* $\quad \boldsymbol{x_{n+1} = x_n (2 - Nx_n)}$

(2) *Iterative formula to find $\sqrt{N}$ is* $\quad \boldsymbol{x_{n+1} = \frac{1}{2}(x_n + N/x_n)}$

(3) *Iterative formula to find $1/\sqrt{N}$ is* $\quad \boldsymbol{x_{n+1} = \frac{1}{2}(x_n + 1/Nx_n)}$

(4) *Iterative formula to find $\sqrt[k]{N}$ is* $\quad \boldsymbol{x_{n+1} = \frac{1}{k}[(k-1)x_n + N/x_n^{k-1})]}$

Proofs. (1) Let $x = 1/N$ or $1/x - N = 0$

Taking $f(x) = 1/x - N$, we have $f'(x) = -x^{-2}$

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(1/x_n - N)}{-x_n^{-2}} = x_n + \left(\frac{1}{x^n} - N\right) x_n^2 = x_n + x_n - Nx_n^2 = x_n(2 - Nx_n)$$

(2) Let $x = \sqrt{N}$ or $x^2 - N = 0$

Taking $f(x) = x^2 - N$, we have $f'(x) = 2x$

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - N}{2x_n} = \frac{1}{2}(x_n + N/x_n) \qquad \textit{(Madras, 2006)}$$

(3) Let $x = \dfrac{1}{\sqrt{N}}$ or $x^2 - \dfrac{1}{N} = 0$

Taking $f(x) = x^2 - 1/N$, we have $f'(x) = 2x$

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 1/N}{2x_n} = \frac{1}{2}\left(x_n + \frac{1}{Nx_n}\right)$$

(4) Let $x = \sqrt[k]{N}$ or $x^k - N = 0$

Taking $f(x) = x^k - N$, we have $f'(x) = kx^{k-1}$

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^k - N}{kx_n^{k-1}} = \frac{1}{k}\left[(k-1)x_n + \frac{N}{x_n^{k-1}}\right].$$

Example 28.10. *Evaluate the following (correct to four decimal places) by Newton's iteration method :*

(i) $1/31$ *(ii)* $\sqrt{5}$ *(Anna, 2007)*

(iii) $1/\sqrt{14}$ *(iv)* $\sqrt[3]{24}$ *(Madras, 2003)* *(v)* $(30)^{-1/5}$.

Solution. (*i*) Taking $N = 31$, the above formula (1) becomes

$$x_{n+1} = x_n(2 - 31x_n)$$

Since an approximate value of $1/31 = 0.03$, we take $x_0 = 0.03$

Then
$$x_1 = x_0(2 - 31x_0) = 0.03\,(2 - 31 \times 0.03) = 0.0321$$
$$x_2 = x_1(2 - 31x_1) = 0.0321(2 - 31 \times 0.0321) = 0.032257$$
$$x_3 = x_2\,(2 - 31x_2) = 0.032257\,(2 - 31 \times 0.032257) = 0.03226$$

Since $x_2 = x_3$ upto 4 decimal places, we have $1/31 = 0.0323$.

(*ii*) Taking $N = 5$, the above formula (2), becomes $x_{n+1} = \dfrac{1}{2}(x_n + 5/x_n)$

Since an approximate value of $\sqrt{5} = 2$, we take $x_0 = 2$

Then
$$x_1 = \frac{1}{2}(x_0 + 5/x_0) = \frac{1}{2}(2 + 5/2) = 2.25$$
$$x_2 = \frac{1}{2}(x_1 + 5/x_1) = 2.2361$$
$$x_3 = \frac{1}{2}(x_2 + 5/x_2) = 2.2361$$

Since $x_2 = x_3$ upto 4 decimal places, we have $\sqrt{5} = 2.2361$.

(*iii*) Taking $N = 14$, the above formula (3), becomes $x_{n+1} = \dfrac{1}{2}[x_n + 1/(14x_n)]$

Since an approximate value of $1/\sqrt{14} = 1/\sqrt{16} = \dfrac{1}{4} = 0.25$, we take $x_0 = 0.25$

Then
$$x_1 = \frac{1}{2}[x_0 + (14x_0)^{-1}] = \frac{1}{2}[0.25 + (14 \times 0.25)^{-1}] = 0.26785$$
$$x_2 = \frac{1}{2}[x_1 + (14x_1)^{-1}] = \frac{1}{2}[0.26785 + (14 \times 0.26785)^{-1}] = 0.2672618$$
$$x_3 = \frac{1}{2}[x_2 + (14x_2)^{-1}] = \frac{1}{2}[0.2672618 + (14 \times 0.2672618)^{-1}] = 0.2672612$$

Since $x_2 = x_3$ upto 4 decimal places, we take $1/\sqrt{14} = 0.2673$.

(*iv*) Taking $N = 24$ and $k = 3$, the above formula (4) becomes $x_{n+1} = \dfrac{1}{3}[2x_n + 24/x_n^2]$

Since an approximate value of $(24)^{1/3} = (27)^{1/3} = 3$, we take $x_0 = 3$.

Then $$x_1 = \frac{1}{3}(2x_0 + 24/x_0^2) = \frac{1}{3}(6 + 24/9) = 2.88889$$

$$x_2 = \frac{1}{3}(2x_1 + 24/x_1^2) = \frac{1}{3}[(2 \times 2.88889) + 24/(2.88889)^2] = 2.88451$$

$$x_3 = \frac{1}{3}(2x_2 + 24/x_2^2) = \frac{1}{3}[2 \times 2.88451 + 24/(2.88451)^2] = 2.8845$$

Since $x_2 = x_3$ upto 4 decimal places, we take $(24)^{1/3} = 2.8845$

(*v*) Taking $N = 30$ and $k = -5$, the above formula (4) becomes

$$x_{n+1} = \frac{1}{-5}(-6x_n + 30/x_n^{-6}) = \frac{x_n}{5}(6 - 30x_n^5)$$

Since an approximate value of $(30)^{-1/5} = (32)^{-1/5} = 1/2$, we take $x_0 = 1/2$

Then $$x_1 = \frac{x_0}{5}(6 - 30x_0^5) = \frac{1}{10}(6 - 30/2^5) = 0.50625$$

$$x_2 = \frac{x_1}{5}(6 - 30x_1^5) = \frac{0.50625}{5}[6 - 30(0.50625)^5] = 0.506495$$

$$x_3 = \frac{x_2}{5}(6 - 30x_2^5) = \frac{0.506495}{5}[6 - 30(0.506495)^5] = 0.506496.$$

Since $x_2 = x_3$ upto 4 decimal places, we take $(30)^{-1/5} = 0.5065$.

PROBLEMS 28.1

1. Find a root of the following equations, using the bisection method correct to three decimal places :
   (*i*) $x^3 - 2x - 5 = 0$ (*P.T.U., 2005*) (*ii*) $x^3 - x^2 - 1 = 0$ (*J.N.T.U., 2009*)
   (*iii*) $x^3 - x - 11 = 0$ which lies between 2 and 3 (*iv*) $2x^3 + x^2 - 20x + 12 = 0$.
2. Using the bisection method, find a real root of the following equations correct to three decimal places :
   (*i*) $\cos x = xe^x$ (*Mumbai, 2004*) (*ii*) $x \log_{10} x = 1.2$ lying between 2 and 3
   (*iii*) $e^x - x = 2$ lying between 1 and 1.4 (*iv*) $e^x = 4 \sin x$.
3. Find a real root of the following equations correct to three decimal places by the method of false position :
   (*i*) $x^3 + x - 1 = 0$ (*ii*) $x^3 - 4x - 9 = 0$ (*V.T.U., 2007*)
   (*iii*) $x^3 + x - 1 = 0$ near $x = 1$ (*iv*) $x^6 - x^4 - x^3 - 1 = 0$. (*Nagarjuna, 2001*)
4. Using regula-falsi method, compute the real root of the following equations correct to three decimal places :
   (*i*) $xe^x = 2$ (*S.V.T.U., 2007*) (*ii*) $\cos x = 3x - 1$ (*iii*) $x \tan x - 1 = 0$
   (*iv*) $2x - \log x = 7$ (*J.N.T.U., 2006*) (*v*) $xe^x = \sin x$. (*P.T.U., 2005*)
5. Find the fourth root of 12 correct to three decimal places using the method of false position.
6. Find by Newton's method, a root of the following equations correct to 3 decimal places :
   (*i*) $x^3 - 3x + 1 = 0$ (*Bhopal, 2009*) (*ii*) $x^3 - 2x - 5 = 0$ (*P.T.U., 2005*)
   (*iii*) $x^3 - 5x + 3 = 0$ (*Mumbai, 2004*)
   (*iv*) $3x^3 - 9x^2 + 8 = 0$ lying between 1 and 2. (*Madras, 2003*)
7. Find a root of the following equations correct to three significant figures using Newton's iterative method :
   (*i*) $x^4 + x^3 - 7x^2 - x + 5 = 0$ lying between 2 and 3 (*Madras, 2003*)
   (*ii*) $x^5 - 5x^2 + 3 = 0$.
8. Find the negative root of the equation $x^3 - 21x + 3500 = 0$ correct to two decimal places by Newton's method.
9. Using Newton-Raphson method, find a root of the following equations correct to the three decimal places :
   (*i*) $xe^x - 2 = 0$ (*V.T.U., 2005*) (*ii*) $x^2 + 4 \sin x = 0$ (*Hazaribagh, 2009*)
   (*iii*) $x \tan x + 1 = 0$ which is near $x = \pi$ (*J.N.T.U., 2006 ; V.T.U., 2006*)
   (*iv*) $e^x = x^2 + \cos 25x$ which is near $x = 4.5$. (*V.T.U., 2007*)
10. Find by Newton's method, the root of the equations :
    (*i*) $\cos x = xe^x$ (*J.N.T.U., 2009 ; V.T.U., 2003*) (*ii*) $x \log_{10} x = 12.34$ (*Anna, 2004*)
    (*iii*) $10^x + x - 4 = 0$ (*iv*) $x + \log_{10} x = 3.375$ (*Rohtak, 2003*)
11. Develop a recurrence formula for finding $\sqrt{N}$, using Newton-Raphson method and hence compute to three decimal places
    (*i*) $\sqrt{13}$ (*U.P.T.U., 2008*) (*ii*) $\sqrt{10}$ (*J.N.T.U., 2008*)

12. Find the cube root of 41, using Newton-Raphson method. *(Madras, 2003)*
13. Develop an algorithm using N-R method, to find the fourth root of a positive number N and hence find $(32)^{1/4}$. *(W.B.T.U., 2005)*
14. Evaluate the following (correct to 3 decimal places) by using the Newton-Raphson method :
    (*i*) 1/18 *J.N.T.U., 2004)* (*ii*) $1/\sqrt{15}$ (*iii*) $(28)^{-1/4}$.

28.4 APPROXIMATE SOLUTION OF EQUATIONS—HORNER'S METHOD

This is the best method of finding approximate values of both rational and irrational roots of a numerical equation. Horner's method consists in diminution of the root of an equation by successive digits occurring in the roots.

If the root of an equation lies between a and $a + 1$, then the value of this root will be $a \,.\, bcd \ldots$, where $b, c, d \ldots$ are digits in its decimal part. To obtain these, we proceed as follows :

(*i*) Diminish the roots of the given equation by a so that the root of the new equation is $0 \,.\, bcd \ldots$

(*ii*) Then multiply the roots of the transformed equation by 10 so that the root of the new equation is $b \,.\, cd \ldots$

(*iii*) Now diminish the root by b and multiply the roots of the resulting equation by 10 so that the root is $c \,.\, d \ldots$

(*iv*) Next diminish the root by c and so on. By continuing this process, the root may be evaluated to any desired degree of accuracy digit by digit. The method will be clear from the following example.

Example 28.11. *Find by Horner's method, the positive root of the equation $x^3 + x^2 + x - 100 = 0$ correct to three decimal places.*

Solution. *Step I.* Let $f(x) = x^3 + x^2 + x - 100$

By Descartes' rule of signs, there is only one positive root. Also $f(4) = -$ ve and $f(5) = +$ ve, therefore, the root lies between 4 and 5.

Step II. Diminish the roots of given equation by 4 so that the transformed equation is

$$x^3 + 13x^2 + 57x - 16 = 0 \qquad \ldots(i)$$

Its root lies between 0 and 1. (We draw a zig-zag line above the set of figures 13, 57, – 16 which are the coefficients of the terms in (*i*) as shown below. Now multiply the roots of (*i*) by 10 for which multiply the second term by 10, the third term by 100 and the fourth term by 1000 (*i.e.* attach one zero to the second term, two zeros to the third term and three zeros to the fourth term). Then we get the equation

$$f_1(x) = x^3 + 130x^2 + 5700x - 16000 = 0 \qquad \ldots(ii)$$

1	1	1	– 100	(4.264
	4	20	84	
	5	21	– 16000	
	4	36	11928	
	9	5700	– 4072000	
	4	264	3788376	
	130	5964	– 283624000	
	2	268		
	132	623200		
	2	8196		
	134	631396		
	2	8232		
	1360	63962800		
	6			
	1366			
	6			
	1372			
	6			
	13780			

Its root lies between 0 and 10.

Clearly $f_1(2) = -\text{ve}, f_1(3) = +\text{ve}$

$\therefore$ the root of (*ii*) lies between 2 and 3 *i.e.*, first figure after decimal is 2.

Step III. Diminish the roots of $f_1(x) = 0$ by 2 so that the next transformed equation is

$$x^3 + 136x^2 + 6232x - 4072 = 0 \qquad ...(iii)$$

Its root lies between 0 and 1. (We draw the second zig-zag line above the set of figures 136, 6232, – 4072). Multiply the roots of (*iii*), by 10, *i.e.* attach one zero to second term, two zeros to third term and three zeros to the fourth term. Then the new equation is

$$f_2(x) = x^3 + 1360x^2 + 623200x - 4072000 = 0$$

Its root lies between 0 and 10, which is nearly $= \dfrac{4072000}{623200} = 6$

Hence second figure after decimal place is 6.

Step IV. Diminish the roots of $f_2(x) = 0$ by 6, so that the transformed equation is

$$x^3 + 1378x^2 + 639628x - 283624 = 0.$$

Its root lies between 0 and 1. (We draw the third zig-zag line above the set of figures 1378, 639628, – 283624.) As before multiply its roots by 10, *i.e.*attach one zero to the second term, two zeros to the third term and three zeros to the fourth term. Then the equation becomes

$$f_3(x) = x^3 + 13780x^2 + 63962800x - 283624000 = 0$$

Its root lies between 0 and 10, which is nearly $= \dfrac{283624000}{63962800} = 4$. Thus the roots of $f_3(x) = 0$ are to be diminished by 4 *i.e.* the third figure after decimal place is 4. But there is no need to proceed further as the root is required correct to three decimal places only. Hence the root is 4.264.

Obs. 1. *After two steps of diminishing, we apply the* **principle of trial divisor** *in which we divide the last coefficient by last but one coefficient to get the next integer by which the roots are to be diminished. These last two coefficients should have opposite signs.*

Obs. 2. *At any stage if the trial divisor suggests the next integer to be zero, then we should again multiply the roots by 10 and write zero in decimal place of the root.*

Example 28.12. *Find the cube root of 30 correct to 3 decimal places, using Horner's method.*

Solution. *Step I.* Let $x = \sqrt[3]{30}$ *i.e.* $f(x) = x^3 - 30 = 0$

Now $f(3) = -3\ (-\text{ve}),\quad f(4) = 34\ (+\text{ve})$

$\therefore$ the root lies between 3 and 4.

Step II. Diminish the roots of the given equation by 3 so that the transformed equation is

$$x^3 + 9x^2 + 27x - 3 = 0 \qquad ...(i)$$

Its roots lies between 0 and 1. (We draw a zig-zag line above the set of numbers 9, 27, – 3 which are the coefficients of the terms in (*i*)). Now multiply the roots of (*i*) by 10 for which attach one zero to the second term, two zeros to the third term and three zeros to the fourth term. Then we get the equation

$$f_1(x) = x^3 + 90x^2 + 2700x - 3000 = 0 \qquad ...(ii)$$

Its roots lies between 0 and 10.

Clearly $f_1(1) = -\text{ve},\quad f_2(2) = +\text{ve}$

$\therefore$ the root of (*ii*) lies between 1 and 2 *i.e.*, first figure after decimal place is 1.

Step III. Diminish the roots of $f_1(x) = 0$ by 1, so that the next transformed equation is

$$x^3 + 93x^2 + 2883x - 209 = 0 \qquad ...(iii)$$

Its root lies between 0 and 1. (We draw a second zig-zag line above the set of figures 93, 2883, – 209). Multiply the roots of (*iii*) by 10 *i.e.*, attach one zero to second term, two zeros to third term and three zeros to the fourth term. Then the new equation is

$$f_2(x) = x^3 + 930x^2 + 288300x - 209000 = 0$$

Its root lies between 0 and 10, which is nearly

$$= 209000/288300 = 0.724 > 0 \text{ and } < 1.$$

Hence second figure after decimal place is 0.

1	0	0	−30 (3.107
	3	9	27
	3	9	−30000
	3	18	2791
	6	2700	−209000000
	3	91	
	90	2791	
	1	92	
	91	28830000	
	1		
	92		
	1		
	9300		

Step IV. Diminish the root of $f_2(x) = 0$ by 0 and then multiply its roots by 10 so that

$f_3(x) = x^3 + 9300x^2 + 28830000x - 209000000 = 0.$

Its root lies between 0 and 10, which is nearly = 209000000/28830000 = 7.2 > 7 and < 8. Thus the roots of $f_3(x) = 0$ are to be diminished by 7 *i.e.*, the third figure after decimal is 7. Hence the required root is 3.107.

PROBLEMS 28.2

1. Find by Horner's method, the root (correct to three decimal places) of the equation
   (*i*) $x^3 - 3x + 1 = 0$ which lies between 1 and 2
   (*ii*) $x^3 + x - 1 = 0$ (*Coimbatore, 1997*) (*iii*) $x^3 - 6x - 13 = 0$
   (*iv*) $x^3 - 3x^2 + 2.5 = 0$ which lies between 1 and 2. (*Madras, 2000 S*)
2. Using Horner's method, find the largest real root of $x^3 - 4x + 2 = 0$ correct to three decimal places.
3. Show that the root of the equation $x^4 + x^3 - 4x^2 - 16 = 0$ lies between 2 and 3. Find its value correct to two decimal places by Horner's method.
4. Find the negative root of the equation $x^3 - 9x^2 + 18 = 0$ correct to two decimal places by Horner's method.
5. Find the cube root of 25 by Horner's method correct to 3 decimal places.

28.5 SOLUTION OF LINEAR SIMULTANEOUS EQUATIONS

Simultaneous linear equations occur in various engineering problems. The student knows that a given system of linear equations can be solved by Cramer's rule or by Matrix method (§ 2.10). But these methods become tedious for large systems. However, there exist other numerical methods of solution which are well-suited for computing machines. We now explain some direct and iterative methods of solution.

28.6 DIRECT METHODS OF SOLUTION

(1) Gauss elimination method*. In this method, the unknowns are eliminated successively and the system is reduced to an upper triangular system from which the unknowns are found by back substitution. The method is quite general and is well-adapted for computer operations. Here we shall explain it by considering a system of three equations for the sake of clarity.

Consider the equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \qquad \ldots(1)$$

Step I. To eliminate x from second and third equations.

Assuming $a_1 \neq 0$, we eliminate x from the second equation by subtracting (a_2/a_1) times the first equation from the second equation. Similarly we eliminate x from the third equation by eliminating (a_3/a_1) times the first equation from the third equation. We thus, get the new system

*See footnote p. 37.

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ b_2'y + c_2'z &= d_2' \\ b_3'y + c_3'z &= d_3' \end{aligned}\right\} \qquad ...(2)$$

Here the first equation is called the *pivotal equation* and a_1 is called the *first pivot*.

Step II. To eliminate y from third equation in (2).

Assuming $b_2' \neq 0$, we eliminate y from the third equation of (2), by subtracting (b_3'/b_2') times the second equation from the third equation. We thus, get the new system

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ b_2'y + c_2'z &= d_2' \\ c_3''z &= d_3'' \end{aligned}\right\} \qquad ...(3)$$

Here the second equation is the *pivotal equation* and b_2' is the *new pivot*.

Step III. To evaluate the unknowns.

The values of x, y, z are found from the reduced system (3) by back substitution.

Obs. 1. On writing the given equations as $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ *i.e.*, $AX = D$, this method consists in ***transforming the coefficient matrix A to upper triangular matrix*** by elementary row transformations only.

Obs. 2. Clearly the method will fail if any one of the pivots a_1, b_2' or c_3'' becomes zero. In such cases, we rewrite the equations in a different order so that the pivots are non-zero.

Obs. 3. *Partial and complete pivoting.* In the first step, the numerically largest coefficient of x is chosen from all the equations and brought as the first pivot by interchanging the first equation with the equation having the largest coefficient of x. In the second step, the numerically largest coefficient of y is chosen from the remaining equations (leaving the first equation) and brought as the *second pivot* by interchanging the second equation with the equation having the largest coefficient of y'. This process is continued till we arrive at the equation with the single variable. This modified procedure is called *partial pivoting*.

If we are not taken about the elimination of x, y, z in a specified order, then we choose at each stage the numerically largest coefficient of the entire matrix of coefficients. This requires not only an interchange of equations but also an interchange of the position of the variables. This method of elimination is called *complete pivoting*. It is more complicated and does not appreciably improve the accuracy.

Example 28.13. *Apply Gauss elimination method to solve the equations $x + 4y - z = -5$; $x + y - 6z = -12$; $3x - y - z = 4$.* *(Mumbai, 2009)*

Solution. We have

		Check sum	
	$x + 4y - z = -5$	-1	...(*i*)
	$x + y - 6z = -12$	-16	...(*ii*)
	$3x - y - z = 4$	5	...(*iii*)

Step I. Operate (*ii*) – (*i*) and (*iii*) – 3(*i*) to eliminate x :

	Check sum	
$-3y - 5z = -7$	-15	...(*iv*)
$-13y + 2z = 19$	8	...(*v*)

Step II. Operate (*v*) – $\frac{13}{3}$ (*iv*) to eliminate y :

	Check sum	
$\frac{71}{3}z = \frac{148}{3}$	73	...(*vi*)

Step III. By back-substitution, we get

From (*vi*) : $\quad z = \frac{148}{71} = 2.0845$

From (*iv*) : $\quad y = \frac{7}{3} - \frac{5}{3}\left(\frac{148}{71}\right) = -\frac{81}{71} = -1.1408$

From (i): $$x = -5 - 4\left(-\frac{81}{71}\right) + \frac{148}{71} = \frac{117}{71} = 1.6479$$

Hence $x = 1.6479, y = -1.1408, z = 2.0845$

Note. A useful check is provided by noting the sum of the coefficients and terms on the right, operating on those numbers as on the equations and checking that the derived equations have the correct sum.

Otherwise : We have $\begin{bmatrix} 1 & 4 & -1 \\ 1 & 1 & -6 \\ 3 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -12 \\ 4 \end{bmatrix}$

Operate $R_2 - R_1$ and $R_3 - 3R_1$, $\begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & -5 \\ 0 & -13 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ 19 \end{bmatrix}$

Operate $R_3 - \frac{13}{3}R_2$, $\begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & -5 \\ 0 & 0 & 71/3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ 148/3 \end{bmatrix}$

Thus, we have $z = 148/71 = 2.0845$,

$$3y = 7 - 5z = 7 - 10.4225 = -3.4225 \quad i.e., \quad y = -1.1408$$

and $$x = -5 - 4y + z = -5 + 4\,(1.1408) + 2.0845 = 1.6479$$

Hence $x = 1.6479, y = -1.1408, z = 2.0845$.

Example 28.14. *Solve $10x - 7y + 3z + 5u = 6$, $-6x + 8y - z - 4u = 5$, $3x + y + 4z + 11u = 2$, $5x - 9y - 2z + 4u = 7$ by Gauss elimination method.* (S.V.T.U., 2007)

Solution. We have

	Check sum	
$10x - 7y + 3z + 5u = 6$	17	...(i)
$-6x + 8y - z - 4u = 5$	2	...(ii)
$3x + y + 4z + 11u = 2$	21	...(iii)
$5x - 9y - 2z + 4u = 7$	5	...(iv)

Step I. To eliminate x, operate $\left[(ii) - \left(\frac{-6}{10}\right)(i)\right], \left[(iii) - \frac{3}{10}(i)\right], \left[(iv) - \frac{5}{10}(i)\right]$:

	Check sum	
$3.8y + 0.8z - u = 8.6$	12.2	...(v)
$3.1y + 3.1z + 9.5u = 0.2$	15.9	...(vi)
$-5.5y - 3.5z + 1.5u = 4$	-3.5	...(vii)

Step II. To eliminate y, operate $\left[(vi) - \frac{3.1}{3.8}(v)\right], \left[(vii) - \left(\frac{-5.5}{3.8}\right)(v)\right]$:

$$2.4473684z + 10.315789u = -6.8157895 \quad ...(viii)$$
$$-2.3421053z + 0.0526315u = 16.447368 \quad ...(ix)$$

Step III. To eliminate z, operate $\left[(ix) - \left(\frac{-2.3421053}{2.4473684}\right)(viii)\right]$:

$$9.9249319u = 9.9245977$$

Step IV. By back-substitution, we get

$$u = 1, z = -7, y = 4 \text{ and } x = 5.$$

(2) Gauss-Jordan method*. This is a modification of the Gauss elimination method. In this method, elimination of unknowns is performed not in the equations below but in the equations above also, ultimately **reducing the system to a diagonal matrix form *i.e.,*** *each equation involving only one unknown. From these equations the unknowns x, y, z can be obtained readily.*

Thus in this method, the labour of back-substitution for finding the unknowns is saved at the cost of additional calculations.

*See footnote p. 37.

Example 28.15. *Apply Gauss-Jordan method to solve the equations*
$x + y + z = 9$; $2x - 3y + 4z = 13$; $3x + 4y + 5z = 40$. (V.T.U., 2009 ; P.T.U., 2005)

Solution. We have

$$x + y + z = 9 \quad ...(i)$$
$$2x - 3y + 4z = 13 \quad ...(ii)$$
$$3x + 4y + 5z = 40 \quad ...(iii)$$

Step I. Operate $(ii) - 2(i)$ and $(iii) - 3(i)$ to eliminate x from (ii) and (iii).

$$x + y + z = 9 \quad ...(iv)$$
$$-5y + 2z = -5 \quad ...(v)$$
$$y + 2z = 13 \quad ...(vi)$$

Step II. Operate $(iv) + \frac{1}{5}(v)$ and $(vi) + \frac{1}{5}(v)$ to eliminate y from (iv) and (vi) :

$$x + \frac{7}{5}z = 8 \quad ...(vii)$$
$$-5y + 2z = -5 \quad ...(viii)$$
$$\frac{12}{5}z = 12 \quad ...(ix)$$

Step III. Operate $(vii) - \frac{7}{12}(ix)$ and $(viii) - \frac{5}{6}(ix)$ to eliminate z from (vii) and $(viii)$:

$$x = 1$$
$$-5y = -15$$
$$\frac{12}{5}z = 12$$

Hence the solution is $x = 1, y = 3, z = 5$.

Otherwise : Rewriting the equations as
$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 40 \end{bmatrix}$$

Operate $R_2 - 2R_1, R_3 - 3R_1$,
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & 2 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \\ 13 \end{bmatrix}$$

Operate $R_3 + \frac{1}{5}R_2$,
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & 2 \\ 0 & 0 & 12/5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \\ 12 \end{bmatrix}$$

Operate $-R_2, 5R_3$
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & -2 \\ 0 & 0 & 12 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 60 \end{bmatrix}$$

Operate $R_2 + \frac{1}{6}R_3, \frac{1}{12}R_3$
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 15 \\ 5 \end{bmatrix}$$

Operate $\frac{1}{5}R_2$
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ 5 \end{bmatrix}$$

Operate $R_1 - R_2 - R_3$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

Hence, $x = 1, y = 3, z = 5$.

Obs. Here the process of elimination of variables amounts to *reducing the given coefficient metric to a* **diagonal matrix** *by elementary row transformations only.*

Example 28.16. *Solve the equations of example 28.14, by Gauss-Jordan method.*

Solution. We have

$$10x - 7y + 3z + 5u = 6 \quad ...(i)$$
$$-6x + 8y - z - 4u = 5 \quad ...(ii)$$
$$3x + y + 4z + 11u = 2 \quad ...(iii)$$
$$5x - 9y - 2z + 4u = 7 \quad ...(iv)$$

Step I. To eliminate x, operate $\left[(ii) - \left(\frac{-6}{10}\right)(i)\right], \left[(iii) - \left(\frac{3}{10}\right)(i)\right], \left[(iv) - \left(\frac{5}{10}\right)(i)\right]$:

$$10x - 7y + 3z + 5u = 6 \quad ...(v)$$
$$3.8y + 0.8z - u = 8.6 \quad ...(vi)$$
$$3.1y + 3.1z + 9.5u = 0.2 \quad ...(vii)$$
$$-5.5y - 3.5z + 1.5u = 4 \quad ...(viii)$$

Step II. To eliminate y, operate $\left[(v) - \left(\frac{-7}{3.8}\right)(vi)\right], \left[(vii) - \left(\frac{3.1}{3.8}\right)(vi)\right], \left[(viii) - \left(\frac{-5.5}{3.8}\right)(vi)\right]$:

$$10x + 4.4736842z + 3.1578947u = 21.842105 \quad ...(ix)$$
$$3.8y + 0.8z - u = 8.6 \quad ...(x)$$
$$2.4473684z + 10.315789u = -6.8157895 \quad ...(xi)$$
$$-2.3421053x + 0.0526315u = 16.447368 \quad ...(xii)$$

Step III. To eliminate z, operate $\left[(ix) - \left(\frac{4.473684}{2.4473684}\right)(xi)\right]$,

$$\left[(x) - \left(\frac{0.8}{2.4473684}\right)(xi)\right], \left[(xii) - \left(\frac{-2.3421053}{2.4473684}\right)(xi)\right] :$$

$$10x - 15.698923u = 34.301075$$
$$3.8y - 4.3720429u = 10.827957$$
$$2.4473684z + 10.315789u = -6.8157895$$
$$9.9247309u = 9.9245975$$

Step IV. From the last equation $u = 1$ nearly.

Substitution of $u = 1$ in the above three equations gives $x = 5, y = 4, z = -7$.

(3) Factorization method*. This method is based on the fact that every matrix A can be expressed as the product of a lower triangular matrix and an upper triangular matrix, provided all the principal minors of A are non-singular, *i.e.*, if $A = [a_{ij}]$, then

$$a_{11} \neq 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0, \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0, \text{ etc.}$$

Also such a factorization if it exists, is unique.

Now consider the equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

which can be written as

$$AX = B \quad ...(1)$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Let

$$A = LU, \quad ...(2)$$

where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

*Another name given to this discomposition is **Doolittle's method.**

Then (1) becomes $LUX = B$ ...(3)

Writing $UX = V$, ...(4)

(3) becomes $LV = B$ which is equivalent to the equations

$$v_1 = b_1 \,;\, l_{21}v_1 + v_2 = b_2 \,;\, l_{31}v_1 + l_{31}v_2 + v_3 = b_3$$

Solving these for v_1, v_2, v_3, we know V. Then, (4) becomes

$$u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = v_1 \,;\, u_{22}x_2 + u_{23}x_3 = v_2 \,;\, u_{33}x_3 = v_3,$$

from which x_3, x_2 and x_1 can be found by *back-substitution.*

To compute the matrices L and U, we write (2) as

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 0 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Multiplying the matrices on the left and equating corresponding elements from both sides, we obtain

(*i*) $u_{11} = a_{11}$, $\quad u_{12} = a_{12}$, $\quad u_{13} = a_{13}$

(*ii*) $l_{21}u_{11} = a_{21}$ or $l_{21} = a_{21}/a_{11}$

$l_{31}u_{11} = a_{31}$ or $l_{31} = a_{31}/a_{11}$

(*iii*) $l_{21}u_{12} + u_{22} = a_{22}$ or $u_{22} = a_{22} - \dfrac{a_{21}}{a_{11}}\, a_{12}$

$l_{21}u_{13} + u_{23} = a_{23}$ or $u_{23} = a_{23} - \dfrac{a_{21}}{a_{11}}\, a_{13}$

(*iv*) $l_{31}u_{12} + l_{33}u_{22} = a_{32}$ or $l_{32} = \dfrac{1}{u_{22}}\left[a_{32} - \dfrac{a_{31}}{a_{11}}a_{12}\right]$

(*v*) $l_{31}u_{13} + l_{33}u_{23} + u_{33} = a_{33}$ which gives u_{33}.

Thus we compute the elements of L and U in the following set order :

(*i*) First row of U, (*ii*) First column of L,

(*iii*) Second row of U, (*iv*) Second column of L, (*v*) Third row of U.

This procedure can easily be generalised.

Obs. This method is superior to Gauss elimination method and is often used for the solution of linear systems and **for finding the inverse of a matrix.** Among the direct methods, Factorization method is also preferred as the software for computers.

Example 28.17. *Apply factorization method to solve the equations :*

$$3x + 2y + 7z = 4 \,;\, 2x + 3y + z = 5 \,;\, 3x + 4y + z = 7.$$ *(Madras, 2000 S)*

Solution. Let $\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix}$ (*i.e.* A),

so that

(*i*) $u_{11} = 3, u_{12} = 2, \quad u_{13} = 7.$

(*ii*) $l_{21}u_{11} = 2, \quad \therefore\ l_{21} = 2/3$

$l_{31}u_{11} = 3, \quad \therefore\ l_{31} = 1.$

(*iii*) $l_{21}u_{12} + u_{22} = 3, \quad \therefore\ u_{22} = 5/3,$

$l_{21}u_{13} + u_{23} = 1, \quad \therefore\ u_{23} = -11/3.$

(*iv*) $l_{31}u_{12} + l_{32}u_{22} = 4, \quad \therefore\ l_{32} = 6/5.$

(*v*) $l_{31}u_{13} + l_{32}u_{23} + u_{33} = 1$

$\therefore \quad u_{33} = -8/5$

Thus $$A = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1 & 6/5 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 & 7 \\ 0 & 5/3 & -11/3 \\ 0 & 0 & -8/5 \end{bmatrix}$$

Writing $UX = V$, the given system becomes $\begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1 & 6/5 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$

Solving this system, we have $v_1 = 4$,

$$\frac{2}{3}v_1 + v_2 = 5 \qquad \text{or} \qquad v_2 = \frac{7}{3}$$

$$v_1 + \frac{6}{5}v_2 + v_3 = 7 \qquad \text{or} \qquad v_3 = \frac{1}{5}$$

Hence the original system becomes

$$\begin{bmatrix} 3 & 2 & 7 \\ 0 & 5/3 & -11/3 \\ 0 & 0 & -8/5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 7/3 \\ 1/5 \end{bmatrix}$$

i.e.,
$$3x + 2y + 7z = 4\,;\ \frac{5}{3}y - \frac{11}{3}z = \frac{7}{3}\,;\ -\frac{8}{5}z = \frac{1}{5}$$

By back-substitution, we have $z = -1/8$, $y = 9/8$ and $x = 7/8$.

Example 28.18. *Solve the equations of Example 28.14 by factorization method.*

Solution. Let
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{13} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix} = \begin{bmatrix} 10 & -7 & 3 & 5 \\ -6 & 8 & -1 & -4 \\ 3 & 1 & 4 & 11 \\ 5 & -9 & -2 & 4 \end{bmatrix} \quad (i.e., A)$$

so that

(*i*) R_1 of U : $u_{11} = 10$, $u_{12} = -7$, $u_{13} = 3$, $u_{14} = 5$
(*ii*) C_1 of L : $l_{21} = -0.6$, $l_{31} = 0.3$, $l_{41} = 0.5$
(*iii*) R_2 of U : $u_{22} = 3.8$, $u_{23} = 0.8$, $u_{24} = -1$
(*iv*) C_2 of L : $l_{32} = 0.81579$, $l_{42} = -1.44737$
(*v*) R_3 of U : $u_{33} = 2.44737$, $u_{34} = 10.31579$
(*vi*) C_3 of L : $l_{43} = -0.95699$
(*vii*) R_4 of U : $u_{44} = 9.92474$

Thus

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -0.6 & 1 & 0 & 0 \\ 0.3 & 0.81579 & 1 & 0 \\ 0.5 & -1.44737 & -0.95699 & 1 \end{bmatrix}\begin{bmatrix} 10 & -7 & 3 & 5 \\ 0 & 3.8 & 0.8 & -1 \\ 0 & 0 & 2.44737 & 10.31579 \\ 0 & 0 & 0 & 9.92474 \end{bmatrix}$$

Writing $UX = V$, the given system becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -0.6 & 1 & 0 & 0 \\ 0.3 & 0.81579 & 1 & 0 \\ 0.5 & -1.44737 & -0.95699 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 2 \\ 7 \end{bmatrix}$$

Solving this system, we get

$$v_1 = 6,\ v_2 = 8.6,\ v_3 = -6.81579,\ v_4 = 9.92474.$$

Hence the original system becomes

$$\begin{bmatrix} 10 & -7 & 3 & 5 \\ 0 & 3.8 & 0.8 & -1 \\ 0 & 0 & 2.44737 & 10.31579 \\ 0 & 0 & 0 & 9.92474 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 6 \\ 8.6 \\ -6.81579 \\ 9.92474 \end{bmatrix}$$

i.e.,
$$10x - 7y + 3z + 5u = 6, \quad 3.8y + 0.8z - u = 8.6,$$
$$2.44737z + 10.31579u = -6.81579,\ u = 1.$$

By back-substitution, we get $u = 1$, $z = -7$, $y = 4$, $x = 5$.

PROBLEMS 28.3

Solve the following equations by Gauss elimination method :

1. $2x + y + z = 10$; $3x + 2y + 3z = 18$; $x + 4y + 9z = 16$. *(P.T.U., 2005)*
2. $2x + 2y + z = 12$; $3x + 2y + 2z = 8$; $5x + 10y - 8z = 10$. *(W.B.T.U., 2004)*
3. $2x - y + 3z = 9$; $x + y + z = 6$; $x - y + z = 2$. *(Bhopal, 2009)*
4. $2x_1 + 4x_2 + x_3 = 3$; $3x_1 + 2x_2 - 2x_3 = -2$; $x_1 - x_2 + x_3 = 6$. *(Marathwada, 2008)*
5. $5x_1 + x_2 + x_3 + x_4 = 4$; $x_1 + 7x_2 + x_3 + x_4 = 12$;
   $x_1 + x_2 + 6x_3 + x_4 = -5$; $x_1 + x_2 + x_3 + 4x_4 = -6$.

Solve the following equations by Gauss-Jordan method :

6. $2x + 5y + 7z = 52$; $2x + y - z = 0$; $x + y + z = 9$. *(V.T.U., 2010)*
7. $2x - 3y + z = -1$; $x + 4y + 5z = 25$; $3x - 4y + z = 2$. *(Kerala, 2003)*
8. $x + 3y + 3z = 16$; $z + 4y + 3z = 18$; $x + 3y + 4z = 19$. *(Anna, 2005)*
9. $2x + y + z = 10$; $3x + 2y + 3z = 18$; $x + 4y + 9z = 16$. *(V.T.U., 2008)*
10. $2x_1 + x_2 + 5x_3 + x_4 = 5$; $x_1 + x_2 - 3x_3 + 4x_4 = -1$;
    $3x_1 + 6x_2 - 2x_3 + x_4 = 8$; $2x_1 + 2x_2 + 2x_3 - 3x_4 = 2$.

Solve the following equations by factorization method :

11. $10x + y + z = 12$; $2x + 10y + z = 13$; $2x + 2y + 10z = 14$. *(Andhra, 2004 ; P.T.U., 2003)*
12. $x + 2y + 3z = 14$; $2x + 3y + 4z = 20$; $3x + 4y + z = 14$.
13. $2x + 3y + z = 9$; $x + 2y + 3z = 6$; $3x + y + 2z = 8$.
14. $2x_1 - x_2 + x_3 = -1$; $2x_2 - x_3 + x_4 = 1$; $x_1 + 2x_3 - x_4 = -1$; $x_1 + x_2 + 2x_4 = 5$.
15. Find the inverse of the matrix $\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & 4 \\ 1 & 2 & 2 \end{bmatrix}$ by Crout's method.

28.7 ITERATIVE METHODS OF SOLUTION

The preceding methods of solving simultaneous linear equations are known as *direct methods* as they yield exact solutions. On the other hand, an iterative method is that in which we start from an approximation to the true solution and obtain better and better approximations from a computation cycle repeated as often as may be necessary for achieving a desired accuracy.

Simple iteration methods can be devised for systems in which the coefficients of the leading diagonal are large compared to others. We now explain three such methods :

(1) Jacobi's iteration method*. Consider the equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \qquad \text{...(1)}$$

If a_1, b_2, c_3 are large as compared to other coefficients, then solving these for x, y, z respectively, the system can be written in the form

$$\left.\begin{aligned} x &= k_1 - l_1y - m_1z \\ y &= k_2 - l_2x - m_2z \\ z &= k_3 - l_3x - m_3y \end{aligned}\right\} \qquad \text{...(2)}$$

Let us start with the initial approximations x_0, y_0, z_0 (each = 0) for the values of x, y, z. Substituting these on the right, we get the first approximations $x_1 = k_1, y_1 = k_2, z_1 = k_3$.

Substituting these on the right-hand sides of (2), the second approximations are given by

$$\begin{aligned} x_2 &= k_1 - l_1y_1 - m_1z_1 \\ y_2 &= k_2 - l_2x_1 - m_2z_1 \\ z_2 &= k_3 - l_3x_1 - m_3y_1 \end{aligned}$$

This process is repeated till the difference between two consecutive approximations is negligible.

*See footnote p. 215.

Example 28.19. *Solve by Jacobi's iteration method, the equations $10x + y - z = 11.19$, $x + 10y + z = 28.08$, $-x + y + 10z = 35.61$, correct to two decimal places.* *(Anna, 2007)*

Solution. Rewriting the given equations as

$$x = \frac{1}{10}(11.19 - y + z),\ y = \frac{1}{10}(28.08 - x - z),\ z = \frac{1}{10}(35.61 + x - y)$$

We start from an approximation, $x_0 = y_0 = z_0 = 0$.

First iteration $\quad x_1 = \frac{11.19}{10} = 1.119,\ y_1 = \frac{28.08}{10} = 2.808,\ z_1 = \frac{35.61}{10} = 3.561$

Second iteration

$$x_2 = \frac{1}{10}(11.19 - y_1 + z_1) = 1.19$$

$$y_2 = \frac{1}{10}(28.08 - x_1 - z_1) = 2.24$$

$$z_2 = \frac{1}{10}(35.61 + x_1 - y_1) = 3.39$$

Third iteration

$$x_3 = \frac{1}{10}(11.19 - y_2 + z_2) = 1.22$$

$$y_3 = \frac{1}{10}(28.03 - x_2 - z_2) = 2.35$$

$$z_3 = \frac{1}{10}(35.61 + x_2 - y_2) = 3.45$$

Fourth iteration

$$x_4 = \frac{1}{10}(11.19 - y_3 + z_3) = 1.23$$

$$y_4 = \frac{1}{10}(28.03 - x_3 - z_3) = 2.34$$

$$z_4 = \frac{1}{10}(35.61 + x_3 - y_3) = 3.45$$

Fifth iteration

$$x_5 = \frac{1}{10}(11.19 - y_4 + z_4) = 1.23$$

$$y_5 = \frac{1}{10}(28.08 - x_4 - z_4) = 2.34$$

$$z_5 = \frac{1}{10}(35.61 + x_4 - y_4) = 3.45$$

Hence $x = 1.23$, $y = 2.34$, $z = 3.45$.

Example 28.20. *Solve, by Jacobi's iteration method, the equations*

$$20x + y - 2z = 17\ ;\ 3x + 20y - z = -18\ ;\ 2x - 3y + 20z = 25.$$ *(Bhopal, 2009)*

Solution. We write the given equations in the form

$$\left.\begin{aligned} x &= \frac{1}{20}(17 - y + 2z) \\ y &= \frac{1}{20}(-18 - 3x + z) \\ z &= \frac{1}{20}(25 - 2x + 3y) \end{aligned}\right\} \quad \ldots(i)$$

We start from an approximation $x_0 = y_0 = z_0 = 0$.

Substituting these on the right sides of the equations (i), we get

$$x_1 = \frac{17}{20} = 0.85\ ;\ y_1 = -\frac{18}{20} = -0.9\ ;\ z_1 = \frac{25}{20} = 1.25$$

Putting these values on the right of the equations (i), we obtain

$$x_2 = \frac{1}{20}(17 - y_1 + 2z_1) = 1.02$$

$$y_2 = \frac{1}{20}(-18 - 3x_1 + z_1) = -0.965$$

$$z_2 = \frac{1}{20}(25 - 2x_1 + 3y_1) = 1.1515$$

Substituting these values in the right sides of the equations (*i*), we have

$$x_3 = \frac{1}{20}(17 - y_2 + 2z_2) = 1.0134$$

$$y_3 = \frac{1}{20}(-18 - 3x_2 + z_2) = -0.9954$$

$$z_3 = \frac{1}{20}(25 - 2x_2 + 3y_2) = 1.0032$$

Substituting these values, we get

$$x_4 = \frac{1}{20}(17 - y_3 + 2z_3) = 1.0009$$

$$y_4 = \frac{1}{20}(-18 - 3x_3 + z_3) = -1.0018$$

$$z_4 = \frac{1}{20}(25 - 2x_3 + 3y_3) = 0.9993$$

Putting these values, we have

$$x_5 = \frac{1}{20}(17 - y_4 + 2z_4) = 1.0000$$

$$y_5 = \frac{1}{20}(-18 - 3x_4 + z_4) = -1.0002$$

$$z_5 = \frac{1}{20}(25 - 2x_4 + 3y_4) = 0.9996$$

Again substituting these values, we get

$$x_6 = \frac{1}{20}(17 - y_5 + 2z_5) = 1.0000$$

$$y_6 = \frac{1}{20}(-18 - 3x_5 + z_5) = -1.0000$$

$$z_6 = \frac{1}{20}(25 - 2x_5 + 3y_5) = 1.0000$$

The values in the 5th and 6th iterations being practically the same, we can stop.

Hence the solution is $x = 1, y = -1, z = 1$.

(2) Gauss-Seidel iteration method*. This is a modification of the Jacobi's iteration method. As before, we start with initial approximations x_0, y_0, z_0 (each = 0) for x, y, z respectively. Substituting $y = y_0, z = z_0$ in the first of the equations (2) on page 837, we get

$$x_1 = k_1$$

Then putting $x = x_1, z = z_0$ in the second of the equations (2) on page 837, we have

$$y_1 = k_2 - l_2x_1 - m_2z_0$$

Next substituting $x = x_1, y = y_1$ in the third of the equations (2) on page 837, we obtain

$$z_1 = k_3 - l_3x_1 - m_3y_1$$

and so on, *i.e.*, as soon as new approximation for an unknown is found, it is immediately used in the next step.

This process of iteration is continued till convergency to the desired degree of accuracy is obtained.

Obs 1. Since the most recent approximation of the unknowns are used while proceeding to the next step, the convergence in the Gauss-Seidel method is faster than in Jacobi's method.

Obs 2. Gauss-Sedial method converges if in each equation, the absolute value of the largest coefficient is greater than the sum of the absolute values of the remaining coefficients.

*See footnote p. 37. *After Philipp Ludwig Von Seidel* (1821–1896) who also suggested a similar method.

Example 28.21. *Apply Gauss-Seidel iteration method to solve the equations of Ex. 28.20.*
(V.T.U., 2011 ; Rohtak, 2005 ; Madras, 2003)

Solution. We write the given equation in the form

$$x = \frac{1}{20}(17 - y + 2z)\,;\; y = \frac{1}{20}(-18 - 3x + z)\,;\; z = \frac{1}{20}(25 - 2x + 3y) \qquad ...(i)$$

We start from the approximation $x_0 = y_0 = z_0 = 0$. Substituting $y = y_0$, $z = z_0$ in the right side of the first of equations (i), we get

$$x_1 = \frac{1}{20}(17 - y_0 + 2z_0) = 0.8500$$

Putting $x = x_1$, $z = z_0$ in the second of the equations (i), we have

$$y_1 = \frac{1}{20}(-18 - 3x_1 + z_0) = -1.0275$$

Putting $x = x_1$, $y = y_1$ in the last of the equations (i), we obtain

$$z_1 = \frac{1}{20}(25 - 2x_1 + 3y_1) = 1.0109$$

For the second iteration, we have

$$x_2 = \frac{1}{20}(17 - y_1 + 2z_1) = 1.0025$$

$$y_2 = \frac{1}{20}(-18 - 3x_2 + z_1) = -0.9998$$

$$z_2 = \frac{1}{20}(25 - 2x_2 + 3y_2) = 0.9998$$

For the third iteration, we get

$$x_3 = \frac{1}{20}(17 - y_2 + 2z_2) = 1.0000$$

$$y_3 = \frac{1}{20}(-18 - 3x_3 + z_2) = -1.0000$$

$$z_3 = \frac{1}{20}(25 - 3x_3 + 2y_3) = 1.0000$$

The values in the 2nd and 3rd iterations being practically the same, we can stop.
Hence the solution is $x = 1, y = -1, z = 1$.

Example 28.22. *Solve the equations :*

$$10x_1 - 2x_2 - x_3 - x_4 = 3$$
$$-2x_1 + 10x_2 - x_3 - x_4 = 15$$
$$-x_1 - x_2 + 10x_3 - 2x_4 = 27$$
$$-x_1 - x_2 - 2x_3 + 10x_4 = -9$$

by Gauss-Seidal iteration method. *(Bhopal, 2009 ; J.N.T.U., 2004)*

Solution. Rewriting the given equations as

$$x_1 = 0.3 + 0.2x_2 + 0.1x_3 + 0.1x_4 \qquad ...(i)$$
$$x_2 = 1.5 + 0.2x_1 + 0.1x_3 + 0.1x_4 \qquad ...(ii)$$
$$x_3 = 2.7 + 0.1x_1 + 0.1x_2 + 0.2x_4 \qquad ...(iii)$$
$$x_4 = -0.9 + 0.1x_1 + 0.1x_2 + 0.2x_3 \qquad ...(iv)$$

First iteration

Putting $x_2 = 0, x_3 = 0, x_4 = 0$ in (i), we get $x_1 = 0.3$
Putting $x_1 = 0.3, x_3 = 0, x_4 = 0$ in (ii), we obtain $x_2 = 1.56$
Putting $x_1 = 0.3, x_2 = 1.56, x_4 = 0$ in (iii), we obtain $x_3 = 2.886$
Putting $x_1 = 0.3, x_2 = 1.56, x_3 = 2.886$ in (iv), we get $x_4 = -0.1368$

Second iteration

Putting $x_2 = 1.56, x_3 = 2.886, x_4 = -0.1368$ in (i), we obtain

$x_1 = 0.8869$

Putting $x_1 = 0.8869, x_3 = 2.886, x_4 = -0.1368$ in (ii), we obtain

$x_2 = 1.9523$

Putting $x_1 = 0.8869, x_2 = 1.9523, x_4 = -0.1368$ in (iii), we have

$x_3 = 2.9566$

Putting $x_1 = 0.8869, x_2 = 1.9523, x_3 = 2.9566$ in (iv), we get

$x_4 = -0.0248.$

Third iteration

Putting $x_2 = 1.9523, x_3 = 2.9566, x_4 = -0.0248$ in (i), we obtain

$x_1 = 0.9836$

Putting $x_1 = 0.9836, x_3 = 2.9566, x_4 = -0.0248$ in (ii), we obtain

$x_2 = 1.9899$

Putting $x_1 = 0.9836, x_2 = 1.9899, x_4 = -0.0248$ in (iii), we get

$x_3 = 2.9924$

Putting $x_1 = 0.9836, x_1 = 1.9899, x_3 = 2.9924$ in (iv), we get

$x_4 = -0.0042.$

Fourth iteration. Proceeding as above

$x_1 = 0.9968, x_2 = 1.9982, x_3 = 2.9987, x_4 = -0.0008.$

Fifth iteration is

$x_1 = 0.9994, x_2 = 1.9997, x_3 = 2.9997, x_4 = -0.0001.$

Sixth iteration is

$x_1 = 0.9999, x_2 = 1.9999, x_3 = 2.9999, x_4 = -0.0001.$

Hence the solution is $x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 0.$

(3) Relaxation method*. Consider the equations

$$a_1x + b_1y + c_1z = d_1\,;\; a_2x + b_2y + c_2z = d_2\,;\; a_3x + b_3y + c_3z = d_3$$

We define the residuals R_x, R_y, R_z by the relations

$$R_x = d_1 - a_1x - b_1y - c_1z\,;\; R_y = d_2 - a_2x - b_2y - c_2z\,;\; R_z = d_3 - a_3x - b_3y - c_3z \qquad ...(1)$$

To start with we assume $x = y = z = 0$ and calculate the initial residuals. Then the residuals are reduced step by step by giving increments to the variables. For this purpose, we construct the following *operation table :*

	δR_x	δR_y	δR_z
$\delta x = 1$	$-a_1$	$-a_2$	$-a_3$
$\delta y = 1$	$-b_1$	$-b_2$	$-b_3$
$\delta z = 1$	$-c_1$	$-c_2$	$-c_3$

We note from the equations (1) that if x is increased by 1 (keeping y and z constant), R_x, R_y and R_z decrease by a_1, a_2, a_3 respectively. This is shown in the above table alongwith the effects on the residuals when y and z are given unit increments. (The table is the transpose of the coefficient matrix).

At each step, the numerically largest residual is reduced to almost zero. To reduce a particular residual, the value of the corresponding variable is changed ; *e.g.*, to reduce R_x by p, x should be increased by p/a_1.

When all the residuals have been reduced to almost zero, the increments in x, y, z are added separately to give the desired solution.

Obs. As a check, the computed values of x, y, z are substituted in (1) and the residuals are calculated. If these residuals are not all negligible, then there is some mistake and the entire process should be rechecked.

Example 28.23. *Solve, by Relaxation method, the equations :*

$$9x - 2y + z = 50,\; x + 5y - 3z = 18,\; -2x + 2y + 7z = 19. \qquad \textit{(Madras, 2000 S)}$$

*This method was originally developed by *R.V. Southwell* in 1935, for application to structural engineering problems.

Solution. The residuals are given by

$R_x = 50 - 9x + 2y - z$; $R_y = 18 - x - 5y + 3z$; $R_z = 19 + 2x - 2y - 7z$

The operations table is

	δR_x	δR_y	δR_z
$\delta x = 1$	-9	-1	2
$\delta y = 1$	2	-5	-2
$\delta z = 1$	-1	3	-7

The relaxation table is

	R_x	R_y	R_z	
$x = y = z = 0$	50	18	19	...(*i*)
$\delta x = 5$	5	13	29	...(*ii*)
$\delta z = 4$	1	25	1	...(*iii*)
$\delta y = 5$	11	0	-9	...(*iv*)
$\delta x = 1$	2	-1	-7	...(*v*)
$\delta z = -1$	3	-4	0	...(*vi*)
$\delta y = -0.8$	1.4	0	1.6	...(*vii*)
$\delta z = 0.23$	1.17	0.69	$-.09$	...(*viii*)
$\delta x = 0.13$	0	0.56	0.17	...(*ix*)
$\delta y = 0.112$	0.224	0	-0.054	...(*x*)

$\Sigma\delta x = 6.13, \Sigma\delta y = 4.31, \Sigma\delta z = 3.23$

Thus $x = 6.13, y = 4.31, z = 3.23.$

[**Explanation.** In (*i*), the largest residual is 50. To reduce it, we give an increment $dx = 5$ and the resulting residuals are shown in (*ii*). Of these $R_x = 29$ is the largest and we given an increment $\delta z = 4$ to get the results in (*iii*). In (*vi*), $R_y = -4$ is the (numerically) largest and we give an increment $\delta y = -4/5 = -0.8$ to obtain the results in (*vii*). Similarly the other steps have been carried out.]

Example 28.24. *Solve by Relaxation method, the equations :*

$10x - 2y - 3z = 205$; $-2x + 10y - 2z = 154$; $-2x - y + 10z = 120$. *(V.T.U., 2011 S ; Rohtak, 2005)*

Solution. The residuals are given by

$R_x = 205 - 10x + 2y + 3z$; $R_y = 154 + 2x - 10y + 2z$; $R_z = 120 + 2x + y - 10z$.

The operations table is

	δR_x	δR_y	δR_z
$\delta x = 1$	-10	2	2
$\delta y = 1$	2	-10	-1
$\delta z = 1$	3	2	-10

The relaxation table is :

	R_x	R_y	R_z
$x = y = z = 0$	205	154	120
$\delta x = 20$	5	194	160
$\delta y = 19$	43	4	179
$\delta z = 18$	97	40	-1
$\delta x = 10$	-3	60	19
$\delta y = 6$	9	0	25
$\delta z = 2$	15	4	5
$\delta x = 2$	-5	8	9
$\delta z = 1$	-2	10	-1
$\delta y = 1$	0	0	0

$\Sigma\delta x = 32, \Sigma y = 26, \Sigma z = 21.$

Hence $x = 32, y = 26, z = 21.$

PROBLEMS 28.4

1. Solve by Jacobi's method, the equations : $5x - y + z = 10$; $2x + 4y = 12$; $x + y + 5z = -1$. Start with the solution (2, 3, 0).
2. Solve the equations $27x + 6y - z = 85$; $x + y + 54z = 110$; $6x + 15y + 2z = 72$.
   by (*a*) Jacobi's method (*b*) Gauss-Seidel method. *(Anna, 2006)*

   Solve the following equations by Gauss-Seidel method :
3. $2x + y + 6z = 9$; $8x + 3y + 2z = 13$; $x + 5y + z = 7$.
4. $28x + 4y - z = 32$; $x + 3y + 10z = 24$; $2x + 17y + 4z = 35$. *(Mumbai, 2009)*
5. $10x + y + z = 12$; $2x + 10y + z = 13$; $2x + 2y + 10z = 104$. *(V.T.U., MCA, 2007)*
6. $83x + 11y - 4z = 95$; $7x + 52y + 13z = 104$; $3x + 8y + 29z = 71$. *(Hazaribagh, 2009)*
7. $3x_1 - 0.1x_2 - 0.2x_3 = 7.85$; $0.1x_1 + 7x_2 - 0.3x_3 = -19.3$; $0.3x_1 - 0.2x_2 + 10x_3 = 71.4$ *(Mumbai, 2004)*
8. $1.2x + 2.1y + 4.2z = 9.9$; $5.3x + 6.1y + 4.7z = 21.6$; $9.2x + 8.3y + z = 15.2$.
9. $\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \end{bmatrix}$

   Solve by Relaxation method, the following sets of equations :
10. $3x + 9y - 2z = 11$; $4x + 2y + 13z = 24$; $4x - 4y + 3z = -8$. *(Bhopal, 2002)*
11. $10x - 2y - 2z = 6$; $-x + 10y - 2z = 7$; $-x - y + 10z = 8$.
12. $-9x + 3y + 4z + 100 = 0$; $x - 7y + 3z + 80 = 0$; $2x + 3y - 5z + 60 = 0$.
13. $54x + y + z = 110$; $2x + 15y + 6z = 72$; $-x + 6y + 27z = 85$. *(Bhopal, 2003)*

28.8 SOLUTION OF NON-LINEAR SIMULTANEOUS EQUATIONS—NEWTON-RAPHSON METHOD

Consider the equations

$$f(x, y) = 0, g(x, y) = 0 \qquad ...(1)$$

If an initial approximation (x_0, y_0) to a solution has been found by graphical method or otherwise, then a better approximation (x_1, y_1) can be obtained as follows :

Let $x_1 = x_0 + h, y_1 = y_0 + k$, so that $f(x_0 + h, y_0 + k) = 0, g(x_0 + h, y_0 + k) = 0$ ...(2)

Expanding each of the functions in (2) by Taylor's series to first degree terms, we get approximately

$$\left.\begin{aligned} f_0 + h\frac{\partial f}{\partial x_0} + k\frac{\partial f}{\partial y_0} = 0 \\ g_0 + h\frac{\partial g}{\partial x_0} + k\frac{\partial g}{\partial y_0} = 0 \end{aligned}\right\} \qquad ...(3)$$

where $f_0 = f(x_0, y_0)$, $\dfrac{\partial f}{\partial x_0} = \left(\dfrac{\partial f}{\partial x}\right)_{x_0, y_0}$ etc.

Solving the equations (3) for h and k, we get a new approximation to the root as

$$x_1 = x_0 + h, y_1 = y_0 + k$$

This process is repeated till we get the values to the desired accuracy.

Example 28.25. *Solve the system of non-linear equations :*

$$x^2 + y = 11, y^2 + x = 7. \qquad \textit{(Pune, 2000)}$$

Solution. An initial approximation to the solution is obtained from a rough graph of the given equations, as $x_0 = 3.5$ and $y_0 = -1.8$.

We have $f = x^2 + y - 11$ and $g = y^2 + x - 7$ so that

$$\frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial y} = 1 \quad \text{and} \quad \frac{\partial g}{\partial x} = 1, \frac{\partial g}{\partial y} = 2y.$$

Then Newton-Raphson's equations (3) above will be

$$7h + k = 0.55, h - 3.6k = 0.26$$

Solving these, we get $h = 0.0855$, $k = -0.0485$

$\therefore$ the better approximation to the root is

$$x_1 = x_0 + h = 3.5855, y_1 = y_0 + k = -1.8485$$

Repeating the above process, replacing (x_0, y_0) by (x_1, y_1), we obtain $x_2 = 3.5844, y_2 = -1.8482$.

PROBLEMS 28.5

1. Solve the equations $x^2 + y = 5$; $y^2 + x = 3$.
2. Solve the non-linear equations $x = 2(y + 1)$, $y^2 = 3xy - 7$ correct to three decimals.
3. Use Newton-Raphson method to solve the equations $x = x^2 + y^2$, $y = x^2 - y^2$ correct to two decimals, starting with the approximation (0.8, 0.4).
4. Solve the non-linear equations $x^2 - y^2 = 4$, $x^2 + y^2 = 16$ numerically with $x_0 = y_0 = 2.828$ using N.R. method. Carry out two iterations. *(V.T.U., MCA, 2007)*
5. Solve the equations $2x^2 + 3xy + y^2 = 3$; $4x^2 + 2xy + y^2 = 30$. Correct to three decimal places, using Newton-Raphson method, given that $x_0 = -3$, and $y_0 = 2$.

28.9 DETERMINATION OF EIGEN VALUES BY ITERATION

In § 2.14, we came across equations of the type

$$\left.\begin{aligned} (a_{11} - \lambda)x_1 + a_{12}x_2 + a_{13}x_3 &= 0 \\ a_{21}x_1 + (a_{22} - \lambda)x_2 + a_{23}x_3 &= 0 \\ a_{31}x_1 + a_{32}x_2 + (a_{33} - \lambda)x_3 &= 0 \end{aligned}\right\} \quad \text{...(1)}$$

which in matrix form, may be written as $[A - \lambda I]\, X = 0$ or $AX = \lambda X$ ...(2)

where $A = [a_{ij}]$ and X is the column matrix $[x_i]$.

Equation (1) will have a non-trivial solution if the coefficient matrix vanishes *e.g.*,

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} = 0$$

This gives a cubic in λ whose roots are *eigen values* of (2) and corresponding to each *eigen value*, we have a non-zero solution $X = [x_1, x_2, x_3]$ which is called an *eigen vector*. Such an equation can ordinarily be solved easily.

In some applications, it is required to compute the numerically largest *eigen value* and the corresponding *eigen vector*. In such cases, the following iterative method is more convenient which is also well-suited for computing machines.

If X_1, X_2, X_3 be the eigen vectors corresponding to the eigen values $\lambda_1, \lambda_2, \lambda_3$, then an arbitrary column vector can be written as $X = k_1X_1 + k_2X_2 + k_3X_3$

Then $\quad AX = k_1AX_1 + k_2AX_2 + k_3AX_3 = k_1\lambda_1X_1 + k_2\lambda_2X_2 + k_3\lambda_3X_3$

Similarly $\quad A^2X = k_1\lambda_1^2X_1 + k_2\lambda_2^2X_2 + k_3\lambda_3^2X_3$

and $\quad A^rX = k_1\lambda_1^r X_1 + k_2\lambda_2^rX_2 + k_3\lambda_3^rX_3$

If $|\lambda_1| > |\lambda_2| > |\lambda_3|$, then the contribution of the term $k_1\lambda_1^rX_1$ to the sum on the right increases with r and therefore, every time we multiply a column vector by A, it becomes nearer to the eigen vector X_1. Then we make the largest component of the resulting column vector unity to avoid the factor k_1.

Thus we start with a column vector X which is as near the solution as possible and evaluate AX which is written as $\lambda^{(1)} X^{(1)}$ after normalisation. This gives the first approximation $\lambda^{(1)}$ to the eigen value and $X^{(1)}$ to eigen vector. Similarly we evaluate $AX^{(1)} = \lambda^{(2)} X^{(2)}$ which gives the second approximation. We repeat this process till $[X^{(r)} - X^{(r-1)}]$ becomes negligible. Then $\lambda^{(r)}$ will be the largest eigen value of (1) and $X^{(r)}$, the corresponding eigen vector.

*This iterative procedure for finding the dominant eigen value of a matrix is known as Rayleigh's power method.**

*After the English mathematician and physicist John William Strut known as *Lord Rayleigh* (1842–1919) who made important contributions to the theory of waves, elasticity and hydrodynamics. He was professor at Cambridge and London.

Example 28.26. *Determine the largest eigen value and the corresponding eigen vector of the matrices using the power method :*

(*i*) $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ (*ii*) $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ (*V.T.U., 2007*)

Solution. (*i*) Let the initial approximation to the eigen vector corresponding to the largest eigen value of A be $X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Then $$AX = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0.2 \end{bmatrix} = \lambda^{(1)}X^{(1)}$$

So the first approximation to the eigen value is $\lambda^{(1)} = 5$ and the corresponding eigen vector is $X^{(1)} = \begin{bmatrix} 1 \\ 0.2 \end{bmatrix}$.

Now $$AX^{(1)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 5.8 \\ 1.4 \end{bmatrix} = 5.8\begin{bmatrix} 1 \\ 0.241 \end{bmatrix} = \lambda^{(2)}X^{(2)}$$

Thus the second approximation to the eigen-value is $\lambda^{(2)} = 5.8$ and the corresponding eigen-vector is $X^{(2)} = \begin{bmatrix} 1 \\ 0.241 \end{bmatrix}$, repeating the above process, we get

Now $$AX^{(2)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.241 \end{bmatrix} = 5.966\begin{bmatrix} 1 \\ 0.248 \end{bmatrix} = \lambda^{(3)}X^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.249 \end{bmatrix} = 5.994\begin{bmatrix} 1 \\ 0.250 \end{bmatrix} = \lambda^{(4)}X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.250 \end{bmatrix} = 5.999\begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = \lambda^{(5)}X^{(5)}$$

$$AX^{(5)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = \lambda^{(6)}X^{(6)}$$

Clearly $\lambda^{(5)} = \lambda^{(6)}$ and $X^{(5)} = X^{(6)}$ upto 3 decimal places. Hence the largest eigen-value is 6 and the corresponding eigen vector is $\begin{bmatrix} 1 \\ 0.25 \end{bmatrix}$.

(*ii*) Let the initial approximation to the required eigen vector be $X = [1, 0, 0]'$.

Then $$AX = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \lambda^{(1)}X^{(1)}.$$

So the first approximation to the eigen value is $\lambda^{(1)} = 2$ and the corresponding eigen vector

$$X^{(1)} = [1, -0.5, 0]'.$$

Hence $$AX^{(1)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -2 \\ 0.5 \end{bmatrix} = 2.5\begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix} = \lambda^{(2)}X^{(2)}.$$

Repeating the above process, we get

$$AX^{(2)} = 2.8\begin{bmatrix} 1 \\ -1 \\ 0.43 \end{bmatrix} = \lambda^{(3)}X^{(3)}\,;\; AX^{(3)} = 3.43\begin{bmatrix} 0.87 \\ -1 \\ 0.54 \end{bmatrix} = \lambda^{(4)}X^{(4)}$$

$$AX^{(4)} = 3.41\begin{bmatrix} 0.80 \\ -1 \\ 0.61 \end{bmatrix} = \lambda^{(5)}X^{(5)}\,;\; AX^{(5)} = 3.41\begin{bmatrix} 0.76 \\ -1 \\ 0.65 \end{bmatrix} = \lambda^{(6)}X^{(6)}\,;\; AX^{(6)} = 3.41\begin{bmatrix} 0.74 \\ -1 \\ 0.67 \end{bmatrix} = \lambda^{(7)}X^{(7)}$$

Clearly $\lambda^{(6)} = \lambda^{(7)}$ and $X^{(6)} = X^{(7)}$ approximately.

Hence the largest eigen value is 3.41 and the corresponding eigen vector is $[0.74, -1, 0.67]'$.

PROBLEMS 28.6

1. Find by power method, the larger eigen-value of the matrices :

   (*a*) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (*Anna, 2005*) (*b*) $\begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$.

2. Obtain the largest eigen-value and the corresponding eigen-vector for the equations
   $(2-\lambda)x_1 - x_2 = 0 \; ; -x_1 + (2-\lambda)x_2 - x_3 = 0 \; ; -x_2 + (2-\lambda)x_3 = 0$
   by Rayleigh Quotient method.

3. Find the dominant eigen value and the corresponding eigen vector of the following matrices using the power method :

   (*a*) $\begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}$ (*V.T.U., 2011*) (*b*) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ (*V.T.U., 2011 S*)

4. Find the largest eigen-value and the corresponding eigen-vector of the matrices :

   (*a*) $\begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$ (*Anna, 2005*) (*b*) $\begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$ (*V.T.U., 2008*)

   (*c*) $\begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}$ with initial approximation $[1, 1, 0]^T$. (*Madras, 2006*)

28.10 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 28.7

Fill up the blanks or select the correct answer to each of the following problems :

1. Out of Regula-falsi method and Newton-Rephson method, the rate of convergence is faster for ..... .
2. If x_n is the nth iterate, then the Newton-Raphson formula is ..... .
3. In the Regula-falsi method of finding the real root of an equation, the curve AB is replaced by ..... .
4. Newton's iterative formula to find the value of $\sqrt{N}$ is ..... .
5. Newton-Raphson formula converges when ..... .
6. In solving simultaneous equations by Gauss-Jordan method, the coefficient matrix is reduced to ..... matrix.
7. In the case of bisection method, the convergence is
   (*a*) linear (*b*) quadratic (*c*) very slow.
8. The order of convergence in Newton-Raphson method is
   (*a*) 2 (*b*) 3 (*c*) 0 (*d*) none.
9. The Newton-Raphson algorithm for finding the cube root of N is ..... .
10. The bisection method for finding the root of an equation $f(x) = 0$ is ..... .
11. In Regula-falsi method, the first approximation is given by ..... .
12. The order of convergence in Newton-Raphson method is
    (*a*) 2 (*b*) 3 (*c*) 0 (*d*) none.
13. The iterative formula for finding the reciprocal of N is $x_{n+1} =$ ..... .
14. As soon as a new value of a variable is found by iteration, it is used immediately in the following equations, this method is called
    (*a*) Gauss-Jordan method (*b*) Gauss-Seidal method
    (*c*) Jacobi's method (*d*) Relaxation method.
15. Out of Regula-falsi method and Newton-Raphson method, the rate of convergence is faster for ..... .
16. The difference between direct and iterative methods of solving simultaneous linear equations is ..... .
17. To which form the coefficient matrix is transformed when $AX = B$ is solved by Gauss elimination method.
18. Jacobi's iteration method can be used to solve a system of non-linear equations. (True or False)
19. The convergence in the Gauss-Seidal method is thrice as fast as in Jacobi's method. (True or False)
20. By Gauss elimination method, solve $x + y = 2$ and $2x + 3y = 5$. (*Anna, 2007*)

CHAPTER 29

Finite Differences and Interpolation

1. Finite differences. 2. Differences of a polynomial. 3. Factorial notation 4. Relations between the operators. 5. To find one or more missing terms. 6. Newton's interpolation formulae. 7. Central difference interpolation formulae—Gauss's interpolation formulae ; Stirling's formula ; Bessel's formula ; Everett's formula. 8. Choice of an interpolation formula. 9. Interpolation with unequal intervals. 10. Lagrange's formula. 11. Divided differences. 12. Newton's divided difference formula. 13. Inverse interpolation. 14. Objective Type of Questions.

29.1 FINITE DIFFERENCES

Suppose we are given the following values of $y = f(x)$ for a set of values of x :

x :	x_0	x_1	$x_2 \ldots x_n$.
y :	y_0	y_1	$y_2 \ldots y_n$.

Then the process of finding the values of y corresponding to any value of $x = x_i$ between x_0 and x_n is called *interpolation*. Thus *interpolation is the technique of estimating the value of a function for any intermediate value of the independent variable* while the process of computing the value of the function outside the given range is called *extrapolation*. The study of the interpolation is based on the concept of differences of a function which we proceed to discuss. For a detailed study, the reader should refer to author's book *'Numerical Methods in Engineering and Science'*.

Suppose that the function $y = f(x)$ is tabulated for the equally spaced values $x = x_0, x_0 + h, x_0 + 2h, \ldots, x_0 + nh$ giving $y = y_0, y_1, y_2, \ldots, y_n$. To determine the values of $f(x)$ or $f'(x)$ for some intermediate values of x, the following three types of differences are found useful :

(1) Forward differences. The differences $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ when denoted by $\Delta y_0, \Delta y_1, \ldots, \Delta y_{n-1}$ respectively are called the *first forward differences* where Δ is the *forward difference operator*. Thus the first forward differences are $\Delta y_r = y_{r+1} - y_r$.

Similarly, the second forward differences are defined by

$$\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r$$

In general,

$$\Delta^p y_r = \Delta^{p-1} y_{r+1} - \Delta^{p-1} y_r$$

defines the *pth forward differences.*

These differences are systematically set out as follows in what is called a *Forward Difference Table.*

In a difference table, x is called the *argument* and y the *function* or the *entry* y_0, the first entry is called the *leading term* and $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$ etc. are called the *leading differences.*

Obs. Any higher order forward difference can be expressed in terms of the entries.

We have $\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0) = y_3 - 3y_2 + 3y_1 - y_0$$

$$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0 = (y_4 - 3y_3 + 3y_2 - y_1) - (y_3 - 3y_2 + 3y_1 - y_0) = y_4 - 4y_3 + 6y_2 - 4y_1 + y_0$$

The coefficients occurring on the right hand side being the binomial coefficient, we have in general,

$$\Delta^n y_0 = y_n - {}^nc_1 y_{n-1} + {}^nc_2 y_{n-2} - \ldots + (-1)^n y_0$$

Forward Difference Table

Value of x	*Value of y*	*1st. diff.*	*2nd diff.*	*3rd diff.*	*4th diff.*	*5th diff.*
x_0	y_0					
		Δy_0				
$x_0 + h$	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
$x_0 + 2h$	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
$x_0 + 3h$	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
$x_0 + 4h$	y_4		$\Delta^2 y_3$			
		Δy_4				
$x_0 + 5h$	y_5					

(2) Backward differences. The differences $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ when denoted by $\nabla y_1, \nabla y_2, \ldots, \nabla y_n$ respectively, are called the *first backward differences* where ∇ is the *backward difference operator*. Similarly we define higher order backward differences. Thus we have

$$\nabla y_r = y_r - y_{r-1},\ \nabla^2 y_r = \nabla y_r - \nabla y_{r-1},$$
$$\nabla^3 y_r = \nabla^2 y_r - \nabla^2 y_{r-1} \text{ etc.}$$

The differences are exhibited in the following :

Backward Difference Table

Value of x	*Value of y*	*1st. diff.*	*2nd diff.*	*3rd diff.*	*4th diff.*	*5th diff.*
x_0	y_0					
		∇y_1				
$x_0 + h$	y_1		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_3$		
$x_0 + 2h$	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$
$x_0 + 3h$	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$	
		∇y_4		$\nabla^3 y_5$		
$x_0 + 4h$	y_4		$\nabla^2 y_5$			
		∇y_5				
$x_0 + 5h$	y_5					

(3) Central differences. Sometimes it is convenient to employ another system of differences known as *central differences*. In this system, the *central difference operator* δ is defined by the relations :

$$y_1 - y_0 = \delta y_{1/2}, y_2 - y_1 = \delta y_{3/2}, \ldots, y_n - y_{n-1} = \delta y_{n-1/2}$$

Similarly, higher order central differences are defined as

$$\delta y_{3/2} - \delta y_{1/2} = \delta^2 y_1,\ \delta y_{5/2} - \delta y_{3/2} = \delta^2 y_2, \ldots,$$
$$\delta^2 y_2 - \delta^2 y_1 = \delta^3 y_{3/2} \text{ and so on.}$$

These differences are shown in the following :

Central Difference Table

Value of x	*Value of y*	*1st. diff.*	*2nd diff.*	*3rd diff.*	*4th diff.*	*5th diff.*
x_0	y_0					
		$\delta y_{1/2}$				
$x_0 + h$	y_1		$\delta^2 y_1$			
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$		
$x_0 + 2h$	y_2		$\delta^2 y_2$		$\delta^4 y_2$	
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$
$x_0 + 3h$	y_3		$\delta^2 y_3$		$\delta^4 y_3$	
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$		
$x_0 + 4h$	y_4		$\delta^2 y_4$			
		$\delta y_{9/2}$				
$x_0 + 5h$	y_5					

We see from this table that the central differences on the same horizontal line have the same suffix. Also the differences of odd order are known only for half values of the suffix and those of even order for only integral values of the suffix.

It is often required to find the mean of adjacent values in the same column of differences. We denote this mean by μ. Thus

$$\mu\delta y_1 = \tfrac{1}{2}(\delta y_{1/2} + \delta y_{3/2}),\ \mu\delta^2 y_{3/2} = \tfrac{1}{2}(\delta^2 y_1 + \delta^2 y_2) \text{ etc.}$$

Obs. The reader should note that it is only the notation which changes and not the differences.

$$y_1 - y_0 = \Delta y_0 = \nabla y_1 = \delta y_{1/2}.$$

Of all the interpolation formulae, those involving central differences are most useful in practice as the coefficients in such formulae decrease much more rapidly.

Example 29.1. *Evaluate (i) $\Delta \tan^{-1} x$ (ii) $\Delta(e^x \log 2x)$ (iii) $\Delta(x^2/\cos 2x)$ (iv) $\Delta^2 \cos 2x$.* *(P.T.U., 2001)*

Solution. (*i*) $\Delta \tan^{-1} x = \tan^{-1}(x+h) - \tan^{-1} x$

$$= \tan^{-1}\left\{\frac{x+h-x}{1+(x+h)x}\right\} = \tan^{-1}\left\{\frac{h}{1+hx+x^2}\right\}$$

(*ii*)
$$\Delta(e^x \log 2x) = e^{x+h}\log 2(x+h) - e^x \log 2x$$
$$= e^{x+h}\log 2(x+h) - e^{x+h}\log 2x + e^{x+h}\log 2x - e^x \log 2x$$
$$= e^{x+h}\log\frac{x+h}{x} + (e^{x+h} - e^x)\log 2x$$
$$= e^x\left[e^h \log\left(1+\frac{h}{x}\right) + (e^h - 1)\log 2x\right]$$

(*iii*)
$$\Delta\left(\frac{x^2}{\cos 2x}\right) = \frac{(x+h)^2}{\cos 2(x+h)} - \frac{x^2}{\cos 2x} = \frac{(x+h)^2\cos 2x - x^2\cos 2(x+h)}{\cos 2(x+h)\cos 2x}$$
$$= \frac{[(x+h)^2 - x^2]\cos 2x + x^2[\cos 2x - \cos 2(x+h)]}{\cos 2(x+h)\cos 2x}$$
$$= \frac{(2hx+h^2)\cos 2x + 2x^2\sin(h)\sin(2x+h)}{\cos 2(x+h)\cos 2x}$$

(*iv*)
$$\Delta^2\cos 2x = \Delta\{\cos 2(x+h) - \cos 2x\}$$
$$= \Delta\cos 2(x+h) - \Delta\cos 2x$$
$$= [\cos 2(x+2h) - \cos 2(x+h)] - [\cos 2(x+h) - \cos 2x]$$
$$= -2\sin(2x+3h)\sin h + 2\sin(2x+h)\sin h$$
$$= -2\sin h\,[\sin(2x+3h) - \sin(2x+h)]$$
$$= -2\sin h\,[2\cos(2x+2h)\sin h] = -4\sin^2 h\cos(2x+2h).$$

Example 29.2. *Evaluate (i) $\Delta^2\left(\dfrac{5x+12}{x^2+5x+16}\right)$ (Mumbai, 2003) (ii) $\Delta^2(ab^x)$ (iii) $\Delta^n(e^x)$ interval of differencing being unity.* *(Rohtak, 2003)*

Solution. (*i*) $\Delta^2\left(\dfrac{5x+12}{x^2+5x+6}\right) = \Delta^2\left\{\dfrac{5x+12}{(x+2)(x+3)}\right\} = \Delta^2\left\{\dfrac{2}{x+2} + \dfrac{3}{x+3}\right\}$

$$= \Delta\left\{\Delta\left(\frac{2}{x+2}\right) + \Delta\left(\frac{3}{x+3}\right)\right\} = \Delta\left\{2\left(\frac{1}{x+3} - \frac{1}{x+2}\right) + 3\left(\frac{1}{x+4} - \frac{1}{x+3}\right)\right\}$$
$$= -2\Delta\left\{\frac{1}{(x+2)(x+3)}\right\} - 3\Delta\left\{\frac{1}{(x+3)(x+4)}\right\}$$
$$= -2\left\{\frac{1}{(x+3)(x+4)} - \frac{1}{(x+2)(x+3)}\right\} - 3\left\{\frac{1}{(x+4)(x+5)} - \frac{1}{(x+3)(x+4)}\right\}$$

$$= \frac{4}{(x+2)(x+3)(x+4)} + \frac{6}{(x+3)(x+4)(x+5)} = \frac{2(5x+16)}{(x+2)(x+3)(x+4)(x+5)}$$

(*ii*) $\Delta(ab^x) = a\,\Delta(b^x) = a(b^{x+1} - b^x) = ab^x(b-1)$

$\Delta^2(ab^x) = \Delta[\Delta ab^x] = a(b-1)\,\Delta(b^x)$

$= a(b-1)(b^{x+1} - b^x) = a(b-1)^2 - b^x.$

(*iii*) $\Delta e^x = e^{x+1} - e^x = (e-1)e^x$

$\Delta^2 e^x = \Delta(\Delta e^x) = \Delta[(e-1)\,e^x]$

$= (e-1)\,\Delta e^x = (e-1)(e-1)\,e^x = (e-1)^2\,e^x$

Similarly $\Delta^3 e^x = (e-1)^3 e^x$, $\Delta^4 e^x = (e-1)^4 e^x$, ... and $\Delta^n e^x = (e-1)^n e^x$.

29.2 DIFFERENCES OF A POLYNOMIAL

The nth differences of a polynomial of the nth degree are constant and all higher order differences are zero.

Let the polynomial of the nth degree in x, be

$$f(x) = ax^n + bx^{n-1} + cx^{n-2} + \ldots + k(x+h) + l$$

$\therefore$ $\Delta f(x) = f(x+h) - f(x)$

$= a[(x+h)^n - x^n] + b[(x+h)^{n-1} - x^{n-1}] + \ldots + kh$

$= anhx^{n-1} + b'x^{n-2} + c'x^{n-3} + \ldots + k'x + l'$ ...(1)

where $b', c', \ldots, l'$ are new constant coefficients.

Thus the first differences of a polynomial of the nth degree is a polynomial of degree $(n-1)$.

Similarly $\Delta^2 f(x) = \Delta[f(x+h) - f(x)] = \Delta f(x+h) - \Delta f(x)$

$= anh[(x+h)^{n-1} - x^{n-1}] + b'\,[(x+h)^{n-2} - x^{n-2}] + \ldots + k'h$

$= an(n-1)\,h^2x^{n-2} + b''x^{n-3} + c''\,x^{n-4} + \ldots + k''$, [by (1)]

$\therefore$ the second differences represent a polynomial of degree $(n-2)$.

Continuing this process, for the nth differences we get a polynomial of degree zero *i.e.*

$\Delta^n f(x) = an(n-1)(n-2)\ldots 1\,.\,h^n = an\,!\,h^n$ ...(2)

which is a constant. Hence the $(n+1)$th and higher differences of a polynomial of nth degree will be zero.

Obs. The converse of this theorem is also true *i.e. if the nth differences of a function tabulated at equally spaced intervels are constant, the function is a polynomial of degree n.* This fact is important in numerical analysis as it enables us to approximate a function by a polynomial of nth degree, if its nth order differences become nearly constant.

Example 29.3. *Evaluate* $\Delta^{10}[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)]$.

Solution. $\Delta^{10}[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)] = \Delta^{10}[abcd\,x^{10} + (\ )\,x^9 + (\ )\,x^8 + \ldots + 1]$

$= abcd\,\Delta^{10}(x^{10})$ [$\because$ $\Delta^{10}(x^n) = 0$ for $n < 10$]

$= abcd\,(10\,!)$. [by (2) above]

29.3 (1) FACTORIAL NOTATION

A product of the form $x(x-1)(x-2)\ldots(x-r+1)$ is denoted by $[x]^r$ and is called a **factorial.**

In particular $[x] = x$, $[x]^2 = x(x-1)$

$[x]^3 = x(x-1)(x-2)$, etc.

In general $[x]^n = x(x-1)(x-2)\ldots(x-n+1)$

In case, the interval of differencing is h, then

$$[x]^n = x(x-h)(x-2h)\ldots(x-\overline{n-1}h)$$

which is called a *Factorial polynomial or function.*

The factorial notation is of special utility in the theory of finite differences. It helps in finding the successive differences of a polynomial directly by simple rule of differentiation.

The result of differencing $[x]^r$ is analogous to that of differentiating x^r.

(2) To express a polynomial in the factorial notation

(i) arrange the coefficients of the powers of x in descending order, replacing missing powers by zeros ;

(ii) using detached coefficients divide by x, x – 1, x – 2, etc. successively.

Obs. Every polynomial of degree n can be expressed as a factorial polynomial of the same degree and *vice versa*.

Example 29.4. *Express $y = 2x^3 - 3x^2 + 3x - 10$ in a factorial notation and hence show that $\Delta^3 y = 12$.* *(Bhopal, 2007 ; P.T.U., 2005)*

Solution. First method : Let $y = A[x]^3 + B[x]^2 + C[x] + D$.

Then

	x^3	x^2	x	
1	2	– 3	3	$-10 = D$
	—	2	– 1	
2	2	– 1	$2 = C$	
	—	4		
3	2	$3 = B$		
	—			
	$2 = A$			

Hence $\quad y = 2[x]^3 + 3[x]^2 + 2[x] - 10$

$\therefore \quad \Delta y = 2 \times 3[x]^2 + 3 \times 2[x] + 2$

$\Delta^2 y = 6 \times 2[x] + 6$

$\Delta^3 y = 12$, which shows that the third differences of y are constant, as they should be.

Obs. *The coefficient of the highest power of x remains unchanged while transforming a polynomial to factorial notation.*

Second method (*Direct method*) :

Let $\quad y = 2x^3 - 3x^2 + 3x - 10$

$= 2x(x-1)(x-2) + Bx(x-1) + Cx + D$

Putting $\quad x = 0, -10 = D$

Putting $\quad x = 1, 2 - 3 + 3 - 10 = C + D$

$\therefore \quad C = -8 - D = -8 + 10 = 2$

Putting $\quad x = 2, 16 - 12 + 6 - 10 = 2B + 2C + D$

$\therefore \quad B = \frac{1}{2}(-2C - D) = \frac{1}{2}(-4 + 10) = 3.$

Hence $\quad y = 2x(x-1)(x-2) + 3x(x-1) + 2x - 10 = 2[x]^3 + 3[x]^2 + 2[x] - 10$

$\therefore \quad \Delta y = 2 \times 3[x]^2 + 3 \times 2[x] + 2, \Delta^2 y = 6 \times 2[x] + 6, \Delta^3 y = 12.$

Example 29.5. *Find the missing values in the following table :*

x :	*45*	*50*	*55*	*60*	*65*
y :	*3.0*	—	*2.0*	—	*– 2.4*

(Bhopal, 2007 ; V.T.U., 2001)

Solution. The difference table is as follows :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
45	$y_0 = 3$			
		$y_1 - 3$		
50	y_1		$5 - 2y_1$	
		$2 - y_1$		$3y_1 + y_3 - 9$
55	$y_2 = 2$		$y_1 + y_3 - 4$	
		$y_3 - 2$		$3.6 - y_1 - 3y_3$
60	y_3	$-2.4 - y_3$	$-0.4 - 2y_3$	
65	$y_4 = -2.4$			

As only three entries y_0, y_2, y_4 are given, the function y can be represented by a second degree polynomial.

$\therefore$ $\Delta^3 y_0 = 0$ and $\Delta^3 y_1 = 0$

i.e., $3y_1 + y_3 = 9$; $y_1 + 3y_3 = 3.6$

Solving these, we get $y_1 = 2.925, y_3 = 0.225$.

Otherwise : As only three entries $y_0 = 3, y_2 = 2, y_4 = -2.4$ are given, the function y can be represented by a second degree polynomial.

$\therefore$ $\Delta^3 y_0 = 0$ and $\Delta^3 y_1 = 0$

i.e., $(E-1)^3 y_0 = 0$ and $(E-1)^3 y_1 = 0$

i.e., $(E^3 - 3E^2 + 3E - 1) y_0 = 0$ and $(E^3 - 3E^2 + 3E - 1) y_1 = 0$

i.e., $y_3 - 3y_2 + 3y_1 - y_0 = 0$

$y_4 - 3y_3 + 3y_2 - y_1 = 0$

i.e., $y_3 + 3y_1 = 9$; $3y_3 + y_1 = 3.6$

Solving these, we get $y_1 = 2.925, y_3 = 0.225$.

Example 29.6. *Assuming that the following values of y belong to a polynomial of degree 4, compute the next three values :*

x :	0	1	2	3	4	5	6	7
y :	1	−1	1	−1	1	—	—	—

Solution. We construct the following difference table from the given data :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	$y_0 = 1$				
		-2			
1	$y_1 = -1$		4		
		2		-8	
2	$y_2 = 1$		-4		16
		-2		8	
3	$y_3 = -1$		4		16
		2		$\Delta^3 y_2$,	
4	$y_4 = 1$		$\Delta^2 y_3$		16
		Δy_4		$\Delta^2 y_3$,	
5	y_5		$\Delta^2 y_4$		16
		Δy_5		$\Delta^3 y_4$,	
6	y_6		$\Delta^2 y_5$		
		Δy_6			
7	y_7				

Since the values of y belong to a polynomial of degree 4, the fourth differences must be constant. But $\Delta^4 y = 16$.

$\therefore$ The other fourth order differences must also be 16. Thus

$$\Delta^4 y_1 = 16 = \Delta^3 y_2 - \Delta^3 y_1$$

i.e.,
$$\Delta^3 y_2 = \Delta^3 y_1 + \Delta^4 y_1 = 8 + 16 = 24$$
$$\Delta^2 y_3 = \Delta^2 y_2 + \Delta^3 y_2 = 4 + 24 = 28$$
$$\Delta y_4 = \Delta y_3 + \Delta^2 y_3 = 2 + 28 = 30$$

and
$$y_5 = y_4 + \Delta y_4 = 1 + 30 = 31$$

Similarly starting with $\Delta^4 y_2 = 16$, we get

$$\Delta^3 y_3 = 40, \Delta^2 y_4 = 68, \Delta y_5 = 98, y_6 = 129.$$

Starting with $\Delta^4 y_3 = 16$, we obtain

$$\Delta^3 y_4 = 56, \Delta^2 y_5 = 124, \Delta y_6 = 222, y_7 = 351.$$

PROBLEMS 29.1

1. Construct the table of differences for the data below :

x :	0	1	2	3	4
$f(x)$:	1.0	1.5	2.2	3.1	4.6

Evaluate $\Delta^3 f(2)$.

2. If $u_0 = 3$, $u_1 = 12$, $u_2 = 18$, $u_3 = 2000$, $u_4 = 100$, calculate Δu_0.
3. Show that $\Delta^3 y_i = y_{i+3} - 3y_{i+2} + 3y_{i+1} - y_i$.
4. Form the table of backward differences of the function
$$f(x) = x^3 - 3x^2 - 5x - 7 \text{ for } x = -1, 0, 1, 2, 3, 4, 5.$$
5. Form a table of differences for the function
$$f(x) = x^3 + 5x - 7 \text{ for } x = -1, 0, 1, 2, 3, 4, 5$$
Continue the table to obtain $f(6)$.
6. Extend the following table to two more terms on either side by constructing the difference table :

x :	$-.2$	0.0	0.2	0.4	0.6	0.8	1.0
y :	2.6	3.0	3.4	4.28	7.08	14.2	29.0

7. Show that

(*i*) $\Delta\left[\dfrac{1}{f(x)}\right] = \dfrac{-\Delta f(x)}{f(x)\,f(x+1)}$; *(Raipur, 2005)* (*ii*) $\Delta \log f(x) = \log\left\{1 + \dfrac{\Delta f(x)}{f(x)}\right\}$.

8. Evaluate :

(*i*) $\Delta(x + \cos x)$ (*ii*) $\Delta \tan^{-1}\left(\dfrac{n-1}{n}\right)$ (*iii*) $\Delta\left\{\dfrac{1}{x(x+4)(x+6)}\right\}$ *(Madras, 2001)*

(*iv*) $\Delta^2\left(\dfrac{1}{x^2+5x+6}\right)$ *(P.T.U., 2001)*

9. Evaluate :

(*i*) $\Delta(e^{3x}\log 2x)$ (*ii*) $\Delta(2^x/x!)$ (*iii*) $\Delta^n(a^x)$ *(Burdwan, 2003)* (*iv*) $\Delta^n\left(\dfrac{1}{x}\right)$.

10. If $f(x) = e^{ax+b}$, show that its leading differences form a geometric progression. *(Mumbai, 2003)*
11. Prove that

(*i*) $y_3 = y_2 + \Delta y_1 + \Delta^2 y_0 + \Delta^3 y_0$ (*ii*) $\nabla^2 y_8 = y_8 - 2y_7 + y_6$; $\delta^2 y_5 = y_6 - 2y_5 + y_4$.

12. Evaluate :

(*i*) $\Delta^3[(1-x)(1-2x)(1-3x)]$

(*ii*) $\Delta^{10}[(1-x)(1-2x^2)(1-3x^3)(1-4x^4)]$, if the interval of differencing is 2.

13. Express $x^3 - 2x^2 + x - 1$ into factorial polynomial. Hence show that $\Delta^4 f(x) = 0$ *(P.T.U., 2001)*
14. Express $u = x^4 - 12x^3 + 24x^2 - 30x + 9$ and its successive differences in factorial notation. Hence show that $\Delta^5 u = 0$.
15. Find the first and second differences of $x^4 - 6x^3 + 11x^2 - 5x + 8$ with $h = 1$. Show that the fourth difference is constant.
16. Obtain the function whose first difference is $2x^3 + 3x^2 - 5x + 4$.
17. Find the first term of the series whose second and subsequent terms are 8, 3, 0, -1, 0.
18. If $u(x)$ and $v(x)$ be two functions of x, prove that

(*i*) $\Delta[u(x)\,v(x)] = u(x)\,\Delta v(x) + v(x+1)\,\Delta u(x)$. (*ii*) $\Delta\left\{\dfrac{u(x)}{v(x)}\right\} = \dfrac{v(x)\,\Delta u(x) - u(x)\,\Delta v(x)}{v(x)\,v(x+1)}$.

29.4 (1) OTHER DIFFERENCE OPERATORS

We have already introduced the operators Δ, ∇ and δ. Besides these, there are the operators E and μ, which we define below :

(*i*) **Shift operator E** *is the operation of increasing the argument x by h* so that
$$Ef(x) = f(x+h),\ E^2 f(x) = f(x+2h),\ E^3 f(x) = f(x+3h) \text{ etc.}$$
The inverse operator E^{-1} is defined by $E^{-1} f(x) = f(x-h)$

If y_x is the function $f(x)$, then $Ey_x = y_{x+h}$, $E^{-1}y_x = y_{x-h}$, $E^n y_x = y_{x+nh}$, where n may be any real number.

(*ii*) **Averaging operator** μ is defined by the equation $\mu y_x = \frac{1}{2}(y_{x+h/2} + y_{x-h/2})$

Obs. *In the difference calculus, Δ and E are regarded as the fundamental operators and ∇, δ, μ can be expressed in terms of these.*

(2) Relations between the operators. We shall now establish the following identities :

(*i*) $\Delta = \mathbf{E - 1}$ (*ii*) $\nabla = \mathbf{1 - E^{-1}}$

(*iii*) $\delta = \mathbf{E^{1/2} - E^{-1/2}}$ (*iv*) $\mu = \frac{1}{2}(\mathbf{E^{1/2} + E^{-1/2}})$

(*v*) $\Delta = \mathbf{E\nabla = \nabla E = \delta E^{1/2}}$ (*vi*) $\mathbf{E = e^{hD}}$.

Proofs. (*i*) $\Delta y_x = y_{x+h} - y_x = Ey_x - y_x = (E-1)\,y_x$.

This shows that the operators Δ and E are connected by the symbolic relation

$$\Delta = E - 1 \quad \text{or} \quad E = 1 + \Delta$$

(*ii*) $$\nabla y_x = y_x - y_{x-h} = y_x - E^{-1}y_x = (1 - E^{-1})y_x$$

$\therefore$ $$\nabla = 1 - E^{-1} \quad \text{or} \quad E = (1 - \nabla)^{-1}$$

(*iii*) $$\delta y_x = y_{x+h/2} - y_{x-h/2} = E^{1/2}y_x - E^{-1/2}y_x = (E^{1/2} - E^{-1/2})y_x$$

$\therefore$ $$\delta = E^{1/2} - E^{-1/2}.$$

(*iv*) $$\mu y_x = \frac{1}{2}(y_{x+h/2} + y_{x-y/2}) = \frac{1}{2}(E^{1/2}y_x + E^{-1/2}y_x) = \frac{1}{2}(E^{1/2} + E^{-1/2})\,y_x$$

$\therefore$ $$\mu = \frac{1}{2}(E^{1/2} + E^{-1/2}).$$

(*v*) $$E\nabla y_x = E(y_x - y_{x-h}) = Ey_x - Ey_{x-h} = y_{x+h} - y_x = \Delta y_x \quad \therefore \quad E\nabla = \Delta$$

Also $$\nabla E y_x = \nabla y_{x+h} = y_{x+h} - y_x = \Delta y_x \quad \therefore \quad \nabla E = \Delta$$

$$\delta E^{1/2} y_x = \delta y_{x+h/2} = y_{x+h/2+h/2} - y_{x+h/2-h/2} = y_{x+h} - y_x = \Delta y_x$$

$\therefore$ $$\delta E^{1/2} = \Delta$$

Hence $$\Delta = E\nabla = \nabla E = \delta E^{1/2}.$$

(*vi*) $$Ef(x) = f(x+h)$$

$$= f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \ldots \qquad \text{[By Taylor's series]}$$

$$= f(x) + hD\,f(x) + \frac{h^2}{2!}D^2 f(x) + \ldots = \left(1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \ldots\ldots\right) f(x) = e^{hD} f(x)$$

$\therefore$ $$E = e^{hD}$$

Cor. 1. $$E = 1 + \Delta = e^{hD}.$$

2. $$D = \frac{1}{h}\log(1+\Delta) = \frac{1}{h}\left(\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \ldots\right)$$ *(Burdwan, 2003)*

Note. *A table showing the symbolic relations between the various operators is given below for ready reference. To prove such relations between the operators, always express each operator in terms of the fundamental operator E.*

(3) Relations between the various operators

In terms of	E	Δ	∇	δ	hD
E		$\Delta + 1$	$(1+\nabla)^{-1}$	$1 + \frac{1}{2}\delta^2 + \delta\sqrt{(1+\delta^2/4)}$	e^{hD}
Δ	$E - 1$	—	$(1-\nabla)^{-1} - 1$	$\frac{1}{2}\delta^2 + \delta\sqrt{(1+\delta^2/4)}$	$e^{hD} - 1$
∇	$1 - E^{-1}$	$1 - (1+)^{-1} - 1$	—	$-\frac{1}{2}\delta^2 + \delta\sqrt{(1+\delta^2/4)}$	$1 - e^{-hD}$
δ	$E^{1/2} - E^{-1}/2$	$\Delta(1+\Delta)^{-1/2}$	$\nabla(1-\nabla)^{-1/2}$	—	$2\sinh(hD/2)$
μ	$\frac{1}{2}(E^{1/2} + E^{-1/2})$	$(1+\Delta/2)(1+\Delta)^{-1/2}$	$(1+\nabla/2)(1+\nabla)^{-1/2}$	$\sqrt{(1+\delta^2/4)}$	$\cosh(hD/2)$
hD	$\log E$	$\log(1+\Delta)$	$\log(1-)^{-1}$	$2\sinh^{-1}(\delta/2)$	

Example 29.7. *Prove that*

$$e^x = \left(\frac{\Delta^2}{E}\right)e^x \cdot \frac{Ee^x}{\Delta^2 e^x}, \text{ the interval of differencing being } h.$$ *(Bhopal, 2009)*

Solution. Since $\left(\frac{\Delta^2}{E}\right)e^x = \Delta^2 . E^{-1}e^x = \Delta^2 e^{x-h} = \Delta^2 e^x . e^{-h} = e^{-h}\Delta^2 e^x$

$\therefore$ $$\text{R.H.S.} = e^{-h}\Delta^2 e^x . \frac{Ee^x}{\Delta^2 e^x} = e^{-h} Ee^x = e^{-h} . e^{x+h} = e^x.$$

Example 29.8. *Prove with the usual notations, that*

(i) $hD = \log(1+\Delta) = -\log(1-\nabla) = \sinh^{-1}(\mu\delta)$ *(Rohtak, 2005)*

(ii) $(E^{1/2} + E^{-1/2})(1+\Delta)^{1/2} = 2 + \Delta$ *(Bhopal, 2009 ; U.P.T.U., 2009)*

(iii) $\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{(1+\delta^2/4)}$ *(iv)* $\Delta^3 y_2 = \nabla^3 y_5$

Solution. *(i)* We know that $e^{hD} = E = 1 + \Delta \quad \therefore \quad hD = \log(1+\Delta)$

Also $\quad hD = \log E = -\log(E^{-1}) = -\log(1-\nabla) \quad [\because \; E^{-1} = 1 - \nabla]$

We have proved that $\quad \mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$ and $\delta = E^{1/2} - E^{-1/2}$

$\therefore$ $$\mu\delta = \tfrac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2}) = \tfrac{1}{2}(E - E^{-1}) = \tfrac{1}{2}(e^{hD} - e^{-hD}) = \sinh(hD)$$

i.e. $\quad hD = \sinh^{-1}(\mu\delta).$

Hence $\quad hD = \log(1+\Delta) = -\log(1-\nabla) = \sinh^{-1}(\mu\delta)$

(ii) $(E^{1/2} + E^{-1/2})(1+\Delta)^{1/2} = (E^{1/2} + E^{-1/2})E^{1/2} = E + 1 = 1 + \Delta + 1 = 2 + \Delta.$

(iii) $\frac{1}{2}\delta^2 + \delta\sqrt{(1+\delta^2/4)}$

$$= \tfrac{1}{2}(E^{1/2} - E^{-1/2})^2 + (E^{1/2} - E^{-1/2})\sqrt{[1 + (E^{1/2} - E^{-1/2})^2/4]}$$

$$= \tfrac{1}{2}(E + E^{-1} - 2) + (E^{1/2} - E^{-1/2})\sqrt{[(E + E^{-1} + 2)/4]}$$

$$= \tfrac{1}{2}(E + E^{-1} - 2) + \tfrac{1}{2}(E^{1/2} - E^{-1/2})(E^{1/2} + E^{-1/2})$$

$$= \tfrac{1}{2}[(E + E^{-1} - 2) + (E - E^{-1})] = \tfrac{1}{2}(2E - 2) = E - 1 = \Delta.$$

(iv) $\quad \Delta^3 y_2 = (E-1)^3 y_2 \quad [\because \; \Delta = E - 1]$

$$= (E^3 - 3E^2 + 3E - 1)y_2 = y_5 - 3y_4 + 3y_3 - y_2 \quad ...(1)$$

$$\nabla^3 y_5 = (1 - E^{-1})^3 y_5 \quad [\because \; \Delta = 1 - E^{-1}]$$

$$= (1 - 3E^{-1} + 3E^{-2} - E^{-3})y_5 = y_5 - 3y_4 + 3y_3 - y_2 \quad ...(2)$$

From (1) and (2), $\quad \Delta^3 y_2 = \nabla^3 y_5.$

29.5 TO FIND ONE OR MORE MISSING TERMS

When one or more values of $y = f(x)$ corresponding to the equidistant values of x are missing, we can find these using any of the following two methods :

First method : We assume the missing term or terms as a, b etc. and form the difference table. Assuming the last difference as zero, we solve these equations for a, b. These give the missing term/terms.

Second method : If n entries of y are given, $f(x)$ can be represented by $a(n-1)$th degree polynomial *i.e.*, $\Delta^n y = 0$. Since $\Delta = E - 1$, therefore $(E-1)^n y = 0$. Now expanding $(E-1)^n$ and substituting the given values, we obtain the missing term/terms.

Example 29.9. *Find the missing term in the table :*

x :	2	3	4	5	6
y :	45.0	49.2	54.1	...	67.4

(U.P.T.U., 2008)

Solution. Let the missing term be a. Then the difference table is as follows :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
2	$45.0\ (=y_0)$				
		4.2			
3	$49.2\ (=y_1)$		0.7		
		49.9		$a-59.7$	
4	$54.1\ (=y_2)$		$a-59.0$		$240.2-4a$
		$a-54.1$		$180.5-3a$	
5	$a\ (=y_3)$		$121.5-a$		
		$67.4-a$			
6	$67.4\ (=y_4)$				

We know that $\Delta^4 y = 0$ *i.e.*, $240.2 - 4a = 0$.

Hence $a = 60.05$.

Otherwise: As only four entries y_0, y_1, y_2, y_3 are given, therefore $y = f(x)$ can be represented by a third degree polynomial.

$\therefore$ $\Delta^3 y = \text{constant}$ or $\Delta^4 y = 0$ *i.e.*, $(E-1)^4 = 0$

i.e., $(E^4 - 4E^3 + 6E^2 - 4E + 1) = 0$ or $y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$

Let the missing entry y_3 be a so that

$$67.4 - 4a + 6(54.1) - 4(49.2) + 45 = 0 \text{ or } -4a = -240.2$$

Hence $a = 60.05$.

Example 29.10. *Find the missing values in the following data :*

x :	45	50	55	60	65
y :	3.0	...	2.0	...	-2.4

(Bhopal, 2007)

Solution. Let the missing value be a, b. Then the difference table is as follows :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
45	$3(=y_0)$			
		$a-3$		
50	$a(=y_1)$		$5-2a$	
		$2-a$		$3a+b-9$
55	$2(=y_2)$		$b+a-4$	
		$b-2$		$3.6-a-36$
60	$b(=y_3)$		$-0.4-2b$	
		$-2.4-b$		
65	$-2.4\ (=y_4)$			

As only three entries y_0, y_2, y_4 are given, y can be represented by a second degree polynomial having third differences as zero.

$\therefore$ $\Delta^3 y_0 = 0$ and $\Delta^3 y_1 = 0$

i.e., $3a + b = 9,\ a + 3b = 3.6$

Solving these, we get $a = 2.925$, $b + 0.0225$.

Otherwise. As only three entries $y_0 = 3$, $y_2 = 2$, $y_4 = -2.4$ are given, y can be represented by a second degree polynomial having third differences as zero.

$\therefore$ $\Delta^3 y_0 = 0$ and $\Delta^3 y_1 = 0$

i.e., $(E-1)^3 y_0 = 0$ and $(E-1)^3 y_1 = 0$

i.e., $(E^3 - 3E^3 + 3E - 1)\, y_0 = 0\ ;\ (E^3 - 3E^2 + 3E - 1)\,.\, y_1 = 0$

or $y_3 - 3y_2 + 3y_1 - y_0 = 0\ ;\ y_4 - 3y_3 + 3y_2 - y_1 = 0$

or $y_3 + 3y_1 = 9\ ;\ 3y_3 + y_1 = 3.6$

Solving three, we get $y_1 = 2.925$, $y_2 = 0.225$.

Example 29.11. *If $y_{10} = 3$, $y_{11} = 6$, $y_{12} = 11$, $y_{13} = 18$, $y_{14} = 27$, find y_4.* *(Mumbai, 2005)*

Solution. Taking y_{14} as u_0, we are required to find y_4 *i.e.*, u_{-10}. Then the difference table is

x	u	Δu	$\Delta^2 u$	
x_{-4}	$y_{10} = u_{-4} = 3$			
		3		
x_{-3}	$y_{11} = u_{-3} = 6$		2	
		5		0
x_{-2}	$y_{12} = u_{-2} = 11$		2	
		7		0
x_{-1}	$y_{13} = u_{-1} = 18$		2	
		9		
x_0	$y_{14} = u_0 = 27$			

Then
$$y_4 = u_{-10} = (E^{-1})^{10} u_0 = (1 - \nabla)^{10} u_8$$
$$= \left(1 - 10\nabla + \frac{10.9}{2}\nabla^2 - \frac{10.9.8}{1.2.3}\nabla^3 + \ldots\right) u_0$$
$$= u_0 - 10\nabla u_0 + 45\nabla^2 u_0 - 120\nabla^3 u_0$$
$$= 27 - 10 \times 9 + 45 \times 2 - 120 \times 0 = 27.$$

Example 29.12. *If y_x is a polynomial for which fifth difference is constant and $y_1 + y_7 = -7845$, $y_2 + y_6 = 686$, $y_3 + y_5 = 1088$, find y_4.* *(Mumbai, 2004)*

Solution. Starting with y_1 instead of y_0, we note that $\Delta^6 y_1 = 0$ [$\because$ $\Delta^5 y_1$ is constant.

i.e.,
$$(E - 1)^6 y_1 = (E^6 - 6E^5 + 15E^4 - 20E^3 + 15E^2 - 6E + 1) y_1 = 0$$
$$\therefore \quad y_7 - 6y_6 + 15y_5 - 20y_4 + 15y_3 - 6y_2 + y_1 = 0$$
or
$$(y_7 + y_1) - 6(y_6 + y_2) + 15(y_5 + y_3) - 20y_4 = 0$$
i.e.
$$y_4 = \frac{1}{20}[(y_1 + y_7) - 6(y_2 + y_6) + 15(y_3 + y_5)]$$
$$= \frac{1}{20}[-784 - 6(686) + 15(1088)] = 571.$$

Example 29.13. *Prove the following identities :*

(*i*) $u_1 x + u_2 x^2 + u_3 x^3 + \ldots = \dfrac{x}{1-x} u_1 + \left(\dfrac{x}{1-x}\right)^2 \Delta u_1 + \left(\dfrac{x}{1-x}\right)^3 \Delta^2 u_1 + \ldots$

(*ii*) $u_0 + \dfrac{u_1 x}{1!} + \dfrac{u_2 x^2}{2!} + \dfrac{u_3 x^3}{3!} + \ldots = e^x \left(u_0 + x\Delta u_0 + \dfrac{x^2}{2!}\Delta^2 u_0 + \dfrac{x^3}{3!}\Delta^3 u_0 + \ldots\right)$.

Solution. (*i*) L.H.S. $= xu_1 + x^2 E u_1 + x^3 E^2 u_1 + \ldots = x(1 + xE + x^2E^2 + \ldots) u_1$ [$\because$ $u_{x+h} = E^h u_x$]
$$= x \cdot \frac{1}{1 - xE} u_1, \text{ taking sum of infinite G.P.}$$
$$= x \left[\frac{1}{1 - x(1 + \Delta)}\right] u_1 \qquad [\because \ E = 1 + \Delta]$$
$$= x \left(\frac{1}{1 - x - x\Delta}\right) u_1 = \frac{x}{1-x}\left(1 - \frac{x\Delta}{1-x}\right)^{-1} u_1 = \frac{x}{1-x}\left(1 + \frac{x\Delta}{1-x} + \frac{x^2\Delta^2}{(1-x^2)} + \ldots\ldots\right) u_1$$
$$= \frac{x}{1-x} u_1 + \frac{x^2}{(1-x)^2}\Delta u_1 + \frac{x^2}{(1-x)^3}\Delta^2 u_1 + \ldots = \text{R.H.S.}$$

(*ii*) $$\text{L.H.S.} = u_0 + \frac{x}{1!}Eu_0 + \frac{x^2}{2!}E^2u_0 + \frac{x^3}{3!}E^3u_0 + \ldots$$

$$= \left(1 + \frac{xE}{1!} + \frac{x^2E^2}{2!} + \frac{x^3E^3}{3!} + \ldots\right)u_0 = e^{xE}\,u_0 = e^{x(1+\Delta)}\,u_0$$

$$= e^x \,.\, e^{x\Delta}\,u_0 = e^x\left(1 + \frac{x\Delta}{1!} + \frac{x^2\Delta^2}{2!} + \frac{x^3\Delta^3}{3!} + \ldots\right)u_0$$

$$= e^x\left(u_0 + \frac{x}{1!}\Delta u_0 + \frac{x^2}{2!}\Delta^2 u_0 + \frac{x^3}{3!}\Delta^3 u_0 + \ldots\right) = \text{R.H.S.}$$

PROBLEMS 29.2

1. Explain the difference between $\left(\frac{\Delta^2}{E}\right)u_x$ and $\frac{\Delta^2 u_x}{Eu_x}$. (*Madras, 2003*)

2. Evaluate taking h as the interval of differencing :

   (*i*) $\frac{\Delta^2}{E}\sin x$ (*ii*) $\left(\frac{\Delta^2}{E}\right)x^4, (h = 1)$ (*W.B.T.U., 2005*)

   (*iii*) $\left(\frac{\Delta^2}{E}\right)\sin(x+h) + \frac{\Delta^2 \sin(x+h)}{E\sin(x+h)}$ (*iv*) $(\Delta + \nabla)^2(x^2 + x), (h = 1)$.

3. With the usual notations, show that

   (*i*) $\nabla = 1 - e^{-hD}$ (*ii*) $D = \frac{2}{h}\sinh^{-1}\left(\frac{\delta}{2}\right)$

   (*iii*) $(1 + \Delta)(1 - \nabla) = 1$. (*iv*) $\Delta - \nabla = \nabla\Delta = \delta^2$. (*Mumbai, 2005*)

4. Prove that

   (*i*) $\delta = \Delta(1+\Delta)^{-1/2} = \nabla(1-\nabla)^{-1/2}$ (*ii*) $\mu^2 = 1 + \frac{\delta^2}{4}$ (*U.P.T.U, 2009*)

   (*iii*) $\delta(E^{1/2} + E^{-1/2}) = \Delta E^{-1} + \Delta$ (*iv*) $\nabla = \Delta E^{-1} = E^{-1}\Delta = 1 - E^{-1}$

5. Show that (*i*) $\mu\delta = \frac{1}{2}(\Delta + \nabla)$ (*ii*) $1 + \delta^2/2 = \sqrt{(1 + \delta^2\mu^2)}$ (*U.P.T.U., MCA, 2008*)

   (*iii*) $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$ (*U.P.T.U., 2009*) (*iv*) $\nabla^2 = h^2D^2 - h^3D^3 + \frac{7}{12}h^4D^4 - \ldots\ldots$

6. Prove that

   (*i*) $\nabla^r f_k = \Delta^r f_{k-r}$ (*ii*) $\Delta f_k^2 = (f_k + f_{k+1})\,\Delta f_k$ (*J.N.T.U., MCA, 2006*)

   (*iii*) $\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{\left(1 + \frac{\delta^2}{4}\right)}$ (*iv*) $E^{1/2} = (1 + \delta^2/4)^{1/2} + \delta/2$.

7. Prove that $\nabla y_{n+1} = h\left(1 + \frac{1}{2}\nabla + \frac{5}{12}\nabla^2 + \ldots\right)y'_n$.

8. The following table gives the values of y which is a polynomial of degree five. It is known that $f(3)$ is in error. Correct the error.

x :	0	1	2	3	4	5	6
y :	1	2	33	254	1025	3126	7777

(*Mumbai, 2004*)

9. Estimate the missing term in the following table :

x :	0	1	2	3	4
$f(x)$:	1	3	9	—	81

(*S.V.T.U., 2007*)

10. Find the missing terms of the following data :

x :	1	1.5	2	2.5	3	3.5	4
$f(x)$:	6	?	10	20	?	1.5	5

(*U.P.T.U., 2010*)

11. Find the missing values in the following table :

x :	0	1	2	3	4	5	6
y :	5	11	22	40	...	140	...

(V.T.U., 2006)

12. If $u_{13} = 1$, $u_{14} = -3$, $u_{15} = -1$, $u_{16} = 13$ find u_8. (Mumbai, 2004)

13. Evaluate y_4 from the following data (stating the assumptions you make) :

$$y_0 + y_8 = 1.9243, y_1 + y_7 = 1.9590, y_2 + y_6 = 1.9823, y_3 + y_5 = 1.9956.$$ (Mumbai, 2003)

14. Using the method of separation of symbols, prove that

(i) $u_0 + u_1 + u_2 + \ldots\ldots + u_n = {}^{n+1}C_1 u_0 + {}^{n+1}C_3 \Delta^2 u_0 + \ldots\ldots + {}^{n+1}C_{n+1} \Delta^n u_0$.

(ii) $y_x = y_n - {}^{n-x}C_1 \Delta y_{n-1} + {}^{n-x}C_2 \Delta^2 y_{n-2} - \ldots\ldots + (-1)^{n-x} \Delta^{n-x} y_{n-(n-x)}$.

15. Using the method of finite differences, sum the following series :

(i) $2.5 + 5.8 + 8.11 + 11.14 + \ldots$ to n terms.

(ii) $1.2.3 + 2.3.4 + 3.4.5 + \ldots$ to n terms.

16. Prove that $u_0 + u_1 x + u_2 x^2 + \ldots\ldots \infty = \dfrac{u_0}{1-x} + \dfrac{x \Delta u_0}{(1-x)^2} + \dfrac{x^2 \Delta^2 u_0}{(1-x)^3} + \ldots\ldots \infty$

Hence sum the series $1.2 + 2.3x + 3.4x^2 + \ldots\ldots \infty$.

29.6 NEWTON'S INTERPOLATION FORMULAE*

We now derive two important interpolation formulae by means of the forward and backward differences of a function. These formulae are often employed in engineering and scientific problems.

(1) Newton's forward interpolation formula. Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \ldots$ corresponding to the values $x_0, x_0 + h, x_0 + 2h, \ldots$ of x. Suppose it is required to evaluate $f(x)$ for $x = x_0 + ph$, where p is any real number.

For any real number p, we have defined E such that

$$E^p f(x) = f(x + ph)$$

$$\therefore \quad y_p = f(x_0 + ph) = E^p f(x_0) = (1 + \Delta)^{-p} y_0 \qquad [\because \ E = 1 + \Delta]$$

$$= \left\{1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \ldots\right\} y_0 \qquad \text{[Using Binomial theorem]}$$

i.e.,

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots \qquad \ldots(1)$$

It is called **Newton's forward interpolation formula** as (1) contains y_0 and the forward differences of y_0.

Obs. *This formula is used for interpolating the values of y near the beginning of a set of tabulated values and extrapolating values of y a little backward (i.e. to the left) of y_0.*

(2) Newton's backward interpolation formula. Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \ldots$ corresponding to the values $x_0, x_0 + h, x_0 + 2h, \ldots$ of x. Suppose it is required to evaluate $f(x)$ for $x = x_0 + ph$, where p is any real number. Then we have

$$y_p = f(x_n + ph) = E^p f(x_n) = (1 - \nabla)^{-p} y_n \qquad [\because \ E^{-1} = 1 - \nabla]$$

$$= \left[1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^2 + \ldots\right] y_n \qquad \text{[Using Binomial theorem]}$$

i.e.,

$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots \qquad \ldots(2)$$

It is called *Newton's backward interpolation formula as* (2) contains y_n and backward differences of y_n.

Obs. *This formula is used for interpolating the values of y near the end of a set of tabulated values and also for extrapolating values of y a little ahead* (to the right) *of y_n.*

Example 29.14. *The table gives the distances in nautical miles of the visible horizon for the given heights in feet above the earth's surface :*

x = height :	100	150	200	250	300	350	400
y = distance :	10.63	13.03	15.04	16.81	18.42	19.90	21.27

Find the values of y when (i) x = 218 ft (Madras, 2003 S) *(ii) 410 ft.* (V.T.U., 2002)

*See foot note p.466.

Solution. The difference table is as under :

x	y	Δ	Δ^2	Δ^3	Δ^4
100	10.63				
		2.40			
150	13.03		– 0.39		
		2.01		0.15	
200	**15.04**		– 0.24		– 0.07
		1.77		0.08	
250	16.81		**– 0.16**		– 0.05
		1.61		**0.03**	
300	18.42		– 0.13		**– 0.01**
		1.48		**0.02**	
350	19.90		**– 0.11**		
		1.37			
400	**21.27**				

(*i*) If we take $x_0 = 200$, then $y_0 = 15.04$, $\Delta y_0 = 1.77$, $\Delta^2 y_0 = -0.16$, $\Delta^3 y_0 = 0.03$ etc.

Since $x = 218$ and $h = 50$, $\therefore \quad p = \dfrac{x - x_0}{h} = \dfrac{18}{50} = 0.36$

$\therefore$ Using Newton's forward interpolation formula, we get

$$y_{218} = y_0 + p\Delta y_0 + \frac{p(p-1)}{1.2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{1.2.3}\Delta^3 y_0 + \ldots$$

$$f(218) = 15.04 + 0.36(1.77) + \frac{0.36(-0.64)}{2}(-0.16) + \frac{0.36(-0.64)(-1.64)}{6}(0.03) + \ldots$$

$$= 15.04 + 0.637 + 0.018 + 0.001 + \ldots = 15.696 \quad \textit{i.e.}, 15.7 \text{ nautical miles}$$

(*ii*) Since $x = 410$ is near the end of the table, we use Newton's backward interpolation formula.

$\therefore$ taking $x_n = 400, p = \dfrac{x - x_n}{h} = \dfrac{10}{50} = 0.2$

Using the line of backward differences

$$y_n = 21.27, \nabla y_n = 1.37, \nabla^2 y_n = -0.11, \nabla^3 y_n = 0.02 \text{ etc.}$$

$\therefore$ newton's backward formula gives

$$y_{410} = y_{400} + p\nabla y_{400} + \frac{p(p+1)}{2}\nabla^2 y_{400} + \frac{p(p+1)(p+2)}{1.2.3.}\nabla^3 y_{400} + \ldots$$

$$= 21.27 + 0.2(1.37) + \frac{0.2(1.2)}{2}(-0.11) + \ldots = 21.53 \text{ nautical miles.}$$

Example 29.15. *From the following table, estimate the number of students who obtained marks between 40 and 45 :*

Marks :	*30—40*	*40—50*	*50—60*	*60—70*	*70—80*
No. of Students :	*31*	*42*	*51*	*35*	*31*

(V.T.U., 2011 S ; S.V.T.U., 2007 ; Madras, 2006)

Solution. First we prepare the cumulative frequency table, as follows :

Marks less than (x) :	40	50	60	70	80
No. of Students (y_x) :	31	73	124	159	190

Now the difference table is

x	y	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta^4 y_x$
40	**31**				
		42			
50	73		**9**		
		51		**– 25**	
60	124		– 16		37
		35		12	
70	159		– 4		
		31			
80	190				

We shall find y_{45} *i.e.* number of students with marks less than 45.

Taking $x_0 = 40, x = 45$, we have $p = \dfrac{x - x_0}{h} = \dfrac{5}{10} = 0.5$ $[\because \ h = 10]$

$\therefore$ using Newton's forward interpolation formula, we get

$$y_{45} = y_{40} + p\Delta y_{40} + \frac{p(p-1)}{2}\Delta^2 y_{40} + \frac{p(p-1)(p-2)}{1.2.3}\Delta^3 y_{40} + \ldots$$

$$= 31 + 0.5 \times 42 + \frac{0.5(-0.5)}{2} \times 9 + \frac{0.5(0.5)(-1.5)}{6} \times (-25) + \frac{0.5(-0.5)(-1.5)(-2.5)}{24} \times 37$$

$= 47.87$, on simplification.

$\therefore$ the number of students with marks less than 45 is 47.87 *i.e.*, 48.

But the number of students with marks less than 40 is 31.

Hence the number of students getting marks between 40 and 45 = 48 – 31 = 17.

Example 29.16. *Find the cubic polynomial which takes the following values :*

x :	0	1	2	3
$f(x)$:	1	2	1	10

Hence or otherwise evaluate $f(4)$. *(Bhopal, 2009 ; Rohtak, 2005 ; W.B.T.U., 2005)*

Solution. The difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	**1**			
		1		
1	2		**– 2**	
		– 1		**12**
2	1		10	
		9		
3	10			

We take $x_0 = 0$ and $p = \dfrac{x - 0}{h} = x$ $[\because \ h = 1]$

$\therefore$ using Newton's forward interpolation formula, we get

$$f(x) = f(0) + \frac{x}{1}\Delta f(0) + \frac{x(x-1)}{1.2}\Delta^2 f(0) + \frac{x(x-1)(x-2)}{1.2.3}\Delta^3 f(0)$$

$$= 1 + x(1) + \frac{x(x-1)}{2}(-2) + \frac{x(x-1)(x-2)}{6}(12)$$

$= 2x^3 - 7x^2 + 6x + 1$, which is the required polynomial.

To compute $f(4)$, we take $x_n = 3, x = 4$ so that $p = \dfrac{x - x_n}{h} = 1$ $[\because \ h = 1]$

Using Newton's backward interpolation formula, we get

$$f(4) = f(3) + p\nabla f(3) + \frac{p(p+1)}{1.2}\nabla^2 f(3) + \frac{p(p+1)(p+2)}{1.2.3}\nabla^3 f(3)$$

$= 10 + 9 + 10 + 12 + 41.$

which is the same value as that obtained by substituting $x = 4$ in the cubic polynomial above.

Obs. The above example shows that *if a tabulated function is a polynomial, then interpolation and extrapolation give the same values.*

Example 29.17. *In the table below, the values of y are consecutive terms of a series of which 23.6 is the 6th term. Find the first and tenth terms of the series :*

x :	3	4	5	6	7	8	9
y :	4.8	8.4	14.5	23.6	36.2	52.8	73.9

(Anna, 2007)

Solution. The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
3	4.8				
		3.6			
4	8.4		2.5		
		6.1		0.5	
5	14.5		3.0		0
		9.1		0.5	
6	23.6		3.5		0
		12.6		0.5	
7	36.2		4.0		0
		16.6		0.5	
8	52.8		4.5		
		21.1			
9	73.9				

To find the first term, use Newton's forward interpolation formula with $x_0 = 3, x = 1, h = 1$ and $p = -2$. We have

$$y(1) = 4.8 + \frac{(-2)}{1} \times 3.6 + \frac{(-2)(-3)}{1.2} \times 2.5 + \frac{(-2)(-3)(-4)}{1.2.3} \times 0.5 = 3.1$$

To obtain the tenth term, use Newton's backward interpolation formula with $x_n = 9, x = 10, h = 1$ and $p = 1$. This gives

$$y(10) = 73.9 + \frac{1}{1} \times 21.1 + \frac{1(2)}{1.2} \times 4.5 + \frac{1(2)(3)}{1.2.3} \times 0.5 = 100.$$

PROBLEMS 29.3

1. Using Newton's forward formula, find the value of $f(1.6)$, if

x :	1	1.4	1.8	2.2
$f(x)$:	3.49	4.82	5.96	6.5

(J.N.T.U., 2006)

2. State Newton's interpolation formula and use it to calculate the value of exp (1.85), given the following table :

x :	1.7	1.8	1.9	2.0	2.1	2.2	2.3
$f(x)$:	5.474	6.050	6.686	7.389	8.166	9.025	9.974

(Kottayam, 2005)

3. If $f(1.15) = 1.0723$, $f(1.20) = 1.0954$, $f(1.25) = 1.1180$ and $f(1.30) = 1.1401$, find $f(1.28)$.
4. Given $\sin 45° = 0.7071$, $\sin 50° = 0.7660$, $\sin 55° = 0.8192$, $\sin 60° = 0.8660$, find $\sin 52°$, using Newton's forward formula.
5. From the following table of half-yearly premium for policies maturing at different ages, estimate the premium for policies maturing at age of 46 :

Age :	45	50	55	60	65
Premium (in rupees) :	114.84	96.16	83.32	74.48	68.48

(U.P.T.U., 2010)

6. The area A of a circle of diameter d is given for the following values :

d :	80	85	90	95	100
A :	5026	5674	6362	7088	7854

(V.T.U., 2010)

Calculate the area of a circle of diameter 105.

7. Estimate the value of $f(22)$ and $f(42)$ from the following available data :

x :	20	25	30	35	40	45
$f(x)$:	354	332	291	260	231	204

(J.N.T.U., 2007)

8. From the following table :

$x°$:	10	20	30	40	50	60	70	80
$\cos x$:	0.9848	0.9397	0.8660	0.7660	0.6428	0.5000	0.3420	0.1737

Calculate cos 25° and cos 73° using Gregory Newton formulae. *(U.P.T.U., 2006)*

9. Find the number of men getting wages below Rs. 15 from the following data :

Wages in Rs. :	0—10	10—20	20—30	30—40
Frequency :	9	30	35	42

(Nagarjuna, 2001)

10. Find the polynomial interpolating the data :

x :	0	1	2
$f(x)$:	0	5	2

(U.P.T.U., 2008)

11. Construct Newton's forward interpolation polynomial for the following data :

x :	4	6	8	10
y :	1	3	8	16

(Madras, 2006)

Hence evaluate y for $x = 5$.

12. Construct the difference table for the following data :

x :	0.1	0.3	0.5	0.7	0.9	1.1	1.3
$f(x)$:	0.003	0.067	0.148	0.248	0.370	0.518	0.697

Evaluate $f(0.6)$ *(J.N.T.U., 2007)*

13. Estimate from following table $f(3.8)$ to three significant figures using Gregory Newton backward interpolation formula:

x :	0	1	2	3	4
$f(x)$:	1	1.5	2.2	3.1	4.6

(U.P.T.U., 2009)

14. The following table gives the population of a town during the last six censuses. Estimate the increase in the population during the period from 1976 to 1978 :

Year :	1941	1951	1961	1971	1981	1991
Population (in thousands) :	12	15	20	27	39	52

(U.P.T.U., 2009)

15. In the following table, the values of y are consecutive terms of a series of which 12.5 is the 5th term. Find the first and tenth terms of the series.

x :	3	4	5	6	7	8	9
y :	2.7	6.4	12.5	21.6	34.3	51.2	72.9

(P.T.U., 2001)

16. Given $u_1 = 40$, $u_3 = 45$, $u_5 = 54$, find u_2 and u_4. *(Nagarjuna, 2003 S)*

17. If $u_{-1} = 10$, $u_1 = 8$, $u_2 = 10$, $u_4 = 50$, find u_0 and u_3.

18. Given $y_0 = 3$, $y_1 = 12$, $y_2 = 81$, $y_3 = 200$, $y_4 = 100$, $y_5 = 8$, without forming the difference table, find $\Delta^5 y_0$.

29.7 CENTRAL DIFFERENCE INTERPOLATION FORMULAE

In the preceding section, we derived Newton's forward and backward interpolation formulae which are applicable for interpolation near the beginning and end of tabulated values. Now we shall develop central difference formulae which are best suited for interpolation near the middle of the table.

If x takes the values $x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h$ and the corresponding values of $y = f(x)$ are $y_{-2}, y_{-1}, y_0, y_1, y_2$, then we can write the difference table in the two notations as follows :

x	y	*1st diff.*	*2nd diff.*	*3rd diff.*	*4th diff.*
$x_0 - 2h$	y_{-2}				
		$\Delta y_{-2} (= \delta y_{-3/2})$			
$x_0 - h$	y_{-1}		$\Delta^2 y_{-2} (= \delta^2 y_{-1})$		
		$\Delta y_{-1} (= \delta y_{-1/2})$		$\Delta^3 y_{-2} (= \delta^3 y_{-1/2})$	
x_0	y_0		$\Delta^2 y_{-1} (\delta^2 y_0)$		$\Delta^4 y_{-2} (= \delta^4 y_0)$
		$\Delta y_0 (= \delta y_{1/2})$		$\Delta^3 y_{-1} (= \delta^3 y_{1/2})$	
$x_0 + h$	y_1		$\Delta^2 y_0 (= \delta^2 y_1)$		
		$\Delta y_1 (= \delta y_{3/2})$			
$x_0 + 2h$	y_2				

(1) Gauss's forward interpolation formula. The Newton's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{1.2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{1.2.3}\Delta^3 y_0 + \ldots \qquad \ldots(1)$$

We have $\Delta^2 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1}$

i.e., $\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$ ...(2)

Similarly $\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$ ...(3)

$\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1}$ etc. ...(4)

Also $\Delta^3 y_{-1} - \Delta^3 y_{-2} = \Delta^4 y_{-2}$

i.e., $\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$

Similarly $\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$ etc. ...(5)

Substituting for $\Delta^2 y_0, \Delta^3 y_0, \Delta^4 y_0$ ... from (2), (3), (4) ... in (1), we get

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{1.2}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{p(p-1)(p-2)}{1.2.3}(\Delta^3 y_{-1} + \Delta^4 y_{-1})$$
$$+ \frac{p(p-1)(p-2)(p-3)}{1.2.3.4}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \ldots$$

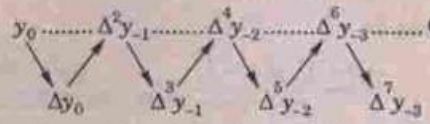
Hence $$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)\,p(p-1)}{3!}\Delta^3 y_{-1}$$
$$+ \frac{(p+1)\,p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \ldots \text{ [Using (5)]}$$

which is called *Gauss's forward interpolation formula.*

Cor. *In the central differences notation, this formula will be*

$$y_p = y_0 + p\delta y_{1/2} + \frac{p(p-1)}{2!}\delta^2 y_0 + \frac{(p+1)\,p(p-1)}{3!}\delta^3 y_{1/2} + \frac{(p+1)\,p(p-1)(p-2)}{4!}\delta^4 y_0 + \ldots$$

Obs. 1. *It employs odd differences just below the central line and even difference on the central line as shown below:*



Obs. 2. This formula is used to interpolate the values of y for p $(0 < p < 1)$ measured forwardly from the origin.

(2) Gauss's backward interpolation formula. The Newton's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{1.2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{1.2.3}\Delta^3 y_0 + \ldots \quad \ldots(1)$$

We have $\Delta y_0 - \Delta y_{-1} = \Delta^2 y_{-1}$

i.e., $\Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1}$ ...(2)

Similarly $\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$ ...(3)

$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$ etc. ...(4)

Also $\Delta^3 y_{-1} - \Delta^3 y_{-2} = \Delta^4 y_{-2}$

i.e., $\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$ ...(5)

Similarly $\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$ etc. ...(6)

Substituting for $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$, ... from (2), (3), (4) in (1), we get

$$y_p = y_0 + p(\Delta y_{-1} + \Delta^2 y_{-1}) + \frac{p(p-1)}{1.2}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{p(p-1)(p-2)}{1.2.3}(\Delta^3 y_{-1} + \Delta^4 y_{-1})$$
$$+ \frac{p(p-1)(p-2)(p-3)}{1.2.3.4}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \ldots$$
$$= y_0 + p\Delta y_{-1} + \frac{(p+1)p}{1.2}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{1.2.3}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{1.2.3.4}\Delta^4 y_{-1}$$
$$+ \frac{p(p-1)(p-2)(p-3)}{1.2.3.4}\Delta^5 y_{-1} + \ldots$$

$$= y_0 + p\Delta y_{-1} + \frac{(p+1)p}{1.2}\Delta^2 y_{-1} + \frac{(p+1)\,p\,(p-1)}{1.2.3}(\Delta^3 y_{-2} + \Delta^4 y_{-2})$$

$$+ \frac{(p+1)\,p\,(p-1)(p-2)}{1.2.3.4}(\Delta^4 y_{-2} + \Delta^5 y_{-2}) + \ldots \quad \text{[Using (5) and (6)]}$$

Hence $$y_p + y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)\,p\,(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p+1)\,p\,(p-1)}{4!}\Delta^4 y_{-2} + \ldots$$

which is called *Gauss's backward interpolation formula.*

Cor. *In the central differences notation, this formula will be*

$$y_p = y_0 + p\delta y_{-1/2} + \frac{(p+1)p}{2!}\delta^2 y_0 + \frac{(p+1)\,p(p-1)}{3!}\delta^3 y_{-1/2} + \frac{(p+2)(p+1)\,p\,(p-1)}{4!}\delta^4 y_0 + \ldots$$

Obs. 1. This formula contains odd differences above the central line and even differences on the central line as shown below :

$$\begin{array}{ccccccccc} & \Delta y_{0-1} & & \Delta^3 y_{-2} & & \Delta^5 y_{-3} & & & \\ \nearrow & & \searrow & & \nearrow & & \searrow & & \\ y_0 \ldots\ldots & & \Delta^2 y_{-1} \ldots\ldots & & \Delta^4 y_{-2} \ldots\ldots & & \Delta^6 y_{-3} \ldots\ldots & \text{Central line} & \end{array}$$

Obs. 2. It is used to interpolate the values of y for a negative value of p lying between -1 and 0.

Obs. 3. Gauss's forward and backward formulae are not of much practical use. However, these serve as intermediate steps for obtaining the important formulae of the following sections.

(3) **Stirling's formula.*** Gauss's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p\,(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)\,p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)\,p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \ldots \quad \ldots(1)$$

Gauss's backward interpolation formula is

$$y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)\,p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p+1)\,p\,(p-1)}{4!}\Delta^4 y_{-2} + \ldots \quad \ldots(2)$$

Taking the mean of (1) and (2), we obtain

$$y_p = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p\,(p^2-1)}{3!} \times \left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2\,(p^2-1)}{4!}\Delta^4 y_{-2} + \ldots \quad \ldots(3)$$

which is called *Stirling's formula.*

Cor. *In the central differences notation, (3) takes the form*

$$y_p = y_0 + p\,\mu\delta y_0 + \frac{p^2}{2!}\delta^2 y_0 + \frac{p\,(p^2+1^2)}{3!}\mu\delta^3 y_0 + \frac{p^2\,(p^2-1^2)}{4!}\delta^4 y_0 + \ldots \quad \ldots(4)$$

for $$\frac{1}{2}(\Delta y_0 + \Delta y_{-1}) = \frac{1}{2}(\delta y_{1/2} + \delta y_{-1/2}) = \mu\delta y_0$$

$$\frac{1}{2}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) = \frac{1}{2}(\delta^3 y_{1/2} + \delta^3 y_{-1/2}) = \mu\delta^3 y_0 \text{ etc.}$$

Obs. *This formula involves means of the odd differences just above and below the central line and even differences on this line* as shown below :

$$\ldots y_0 \ldots \begin{pmatrix}\Delta y_{-1}\\ \Delta y_0\end{pmatrix} \ldots \Delta^2 y_{-1} \ldots \begin{pmatrix}\Delta^3 y_{-2}\\ \Delta^3 y_{-1}\end{pmatrix} \ldots \Delta^4 y_{-2} \ldots \begin{pmatrix}\Delta^5 y_{-3}\\ \Delta^5 y_{-2}\end{pmatrix} \ldots \Delta^6 y_{-3} \ldots \text{ Central line}$$

(4) **Bessel's formula.**** Gauss's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p\,(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)\,p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)\,p(p-1)(p-2)}{4!}\Delta^4 y_{-2} \ldots \quad \ldots(1)$$

*Named after the Scotish mathematicians *James Stirling* (1692-1770).

**See footnote p. 550.

We have $\Delta^2 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1}$

i.e., $\Delta^2 y_{-1} = \Delta^2 y_0 - \Delta^3 y_{-1}$ ...(2)

Similarly $\Delta^4 y_{-2} = \Delta^4 y_{-1} - \Delta^5 y_{-2}$ etc. ...(3)

Now (1) can be written as

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\left(\frac{1}{2}\Delta^2 y_{-1} + \frac{1}{2}\Delta^2 y_{-1}\right) + \frac{p(p^2-1)}{3!}\Delta^3 y_{-1}$$
$$+ \frac{p(p^2-1)(p-2)}{4!}\left(\frac{1}{2}\Delta^4 y_{-2} + \frac{1}{2}\Delta^4 y_{-2}\right) + \dots$$

$$= y_0 + p\Delta y_0 + \frac{1}{2}\frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{1}{2}\frac{p(p-1)}{2!}(\Delta^2 y_0 - \Delta^3 y_{-1}) + \frac{p(p^2-1)}{3!}\Delta^3 y_{-1}$$
$$+ \frac{1}{2}\frac{p(p^2-1)(p-2)}{4!}\Delta^4 y_{-2} + \frac{1}{2}\frac{p(p^2-1)(p-2)}{4!} \times (\Delta^4 y_{-1} - \Delta^5 y_{-2}) + \dots \quad \text{[Using (2), (3) etc.]}$$

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\cdot\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{p(p-1)}{2!}\times\left(\frac{p+1}{3} - \frac{1}{2}\right)\Delta^3 y_{-1}$$
$$+ \frac{p(p^2-1)(p-2)}{4!}\cdot\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \dots$$

Hence $$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\cdot\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{(p-1/2)\,p(p-1)}{3!}\Delta^3 y_{-1}$$
$$+ \frac{(p+1)\,p(p-1)(p-2)}{4!}\cdot\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \dots \quad \dots(4)$$

which is known as the *Bessel's formula.*

Cor. *In the central differences notation, (4) becomes*

$$y_p = y_0 + p\delta y_{1/2} + \frac{p(p-1)}{2!}\mu\delta^2 y_{1/2} + \frac{(p-1/2)\,p(p-1)}{3!}\delta^3 y_{1/2} + \frac{(p+1)\,p(p-1)(p-2)}{4!}\mu\delta^4 y_{1/2} + \dots \quad \dots(5)$$

for $\frac{1}{2}(\Delta^2 y_{-1} + \Delta^2 y_0) = y\delta^2 y_{1/2}$, $\frac{1}{2}(\Delta^4 y_{-2} + \Delta^4 y_{-1}) = \mu\delta^4 y_{1/2}$ etc.

Obs. *This is a very useful formula for practical purposes. It involves odd differences below the central line and means of even differences of and below his line as shown below :*

$$y_0 \text{———} \Delta y_0 \left\{\begin{matrix}\Delta^2 y_{-1}\\ \Delta^2 y_0\end{matrix}\right\} \text{———} \Delta^3 y_{-1} \left\{\begin{matrix}\Delta^4 y_{-2}\\ \Delta^4 y_{-1}\end{matrix}\right\} \text{———} \Delta^5 y_{-2} \left\{\begin{matrix}\Delta^6 y_{-3}\\ \Delta^6 y_{-2}\end{matrix}\right\} \text{——— Central line}$$

(5) Everett's formula. Gauss's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)\,p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)\,p(p-1)(p-2)}{4!}\Delta^4 y_{-2}$$
$$+ \frac{(p+2)(p+1)\,p(p-1)(p-2)}{5!}\Delta^5 y_{-2} + \dots \quad \dots(1)$$

We eliminate the odd difference in (1) by using the relations

$$\Delta y_0 = y_1 - y_0,\ \Delta^3 y_{-1} = \Delta^2 y_0 - \Delta^2 y_{-1},\ \Delta^5 y_{-2} = \Delta^4 y_{-1} - \Delta^4 y_{-2} \text{ etc.}$$

Then (1) becomes

$$y_p = y_0 + p(y_1 - y_0) + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)\,p(p-1)}{3!}(\Delta^2 y_0 - \Delta^2 y_{-1})$$
$$+ \frac{(p+1)\,p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \frac{(p+2)(p+1)\,p(p-1)(p-2)}{5!} \times (\Delta^4 y_{-1} - \Delta^4 y_{-2}) + \dots$$

$$= (1-p)\,y_0 + py_1 - \frac{p(p-1)(p-2)}{3!}\Delta^2 y_{-1} + \frac{(p+1)\,p(p-1)}{3!}\Delta^2 y_0$$

$$-\frac{(p+1)\,p\,(p-1)\,(p-2)\,(p-3)}{5\,!}\,\Delta^4 y_{-2}+\frac{(p+2)\,(p+1)\,p\,(p-1)\,(p-2)}{5\,!}\,\Delta^4 y_{-1}-\ldots$$

To change the terms with negative sign, putting $p = 1 - q$, we obtain

$$y_p = qy_0 + \frac{q(q^2-1^2)}{3\,!}\Delta^2 y_{-1} + \frac{q(q^2-1^2)(q^2-2^2)}{5\,!}\Delta^4 y_{-2} + \ldots\ldots + py_1 + \frac{p(p^2-1^2)}{3\,!}\Delta^2 y_0$$

$$+\frac{p\,(p^2-1^2)\,(p^2-2^2)}{5\,!}\Delta^4 y_{-1}+\ldots$$

This is known as *Everett's formula.*

Obs. *This formula is extensively used and involves only even differences on and below the central line* as shown below :

$$y_0 \;\text{—}\; \Delta^2 y_{-1} \;\text{—}\; \Delta^4 y_{-2} \;\text{—}\; \Delta^6 y_{-3} \;\text{—}\; \text{Central line}$$

$$y_1 \quad \Delta^2 y_0 \quad \Delta^4 y_{-1} \quad \Delta^6 y_{-2}$$

29.8 CHOICE OF AN INTERPOLATION FORMULA

The coefficients in the central difference formulae are smaller and converge faster than those in Newton's formulae. After a few terms, the coefficients in the Stirling's formula decrease more rapidly than those of the Bessel's formula and the coefficients of Bessel's formula decrease more rapidly than those of Newton's formula. As much, whenever possible, *central difference formulae should be used in preference to Newton's formulae.*

The right choice of an interpolation formula however, depends on the position of the interpolated value in the given data.

The following rules will be found useful :

1. *To find a tabulated value near the beginning of the table, use Newton's forward formula.*
2. *To find a value near the end of the table, use Newton's backward formula.*
3. *To find an interpolated value near the centre of the table, use either Stirling's or Bessel's or Everett's formula.*

If interpolation is required for p lying between $-1/4$ and $1/4$, prefer Stirling's formula.

If interpolation is desired for p lying between $1/4$ and $3/4$, use Bessel's or Everett's formula.

Example 29.18. *Find f(22) from the Gauss forward formula :*

x :	20	25	30	35	40	45
$f(x)$:	354	332	291	260	231	204

(J.N.T.U., 2007)

Solution. Taking $x_0 = 25$, $h = 5$, we have to find the value of $f(x)$ for $x = 22$.

i.e., for $\quad p = \dfrac{x - x_0}{h} = \dfrac{22-25}{5} = -0.6$

The difference table is as follows :

x	p	y_p	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$	$\Delta^5 y_p$
20	-1	354 $(= y_{-1})$					
			-22				
25	0	332 $(= y_0)$		-19			
			-41		29		
30	1	291 $(= y_1)$		10		-37	
			-31		-8		45
35	2	260 $(= y_2)$		2		8	
			-29		0		
40	3	231 $(= y_3)$		2			
			-27				
45	4	204 $(= y_4)$					

Gauss forward formula is

$$y^p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta y_{-1} + \frac{(p+1)\,p\,(p-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(p+1)\,p\,(p-1)\,(p-2)}{4!}\Delta^4 y_{-2} + (p+1)\,(p-1)\,(p-2)\,(p+2)\,\Delta^5 y_{-2}$$

$$\therefore \qquad f(22) = 332 + (0.6)\,(-41) + \frac{(-0.6)\,(-0.6-1)}{2!}(-19) + \frac{(-0.6+1)\,(-0.6)\,(-0.6-1)}{3!}(-8)$$

$$+ \frac{(-0.6-1)\,(-0.6)\,(-0.6-1)\,(-0.6-2)}{4!}(-37)$$

$$+ \frac{(-0.6+1)\,(-0.6)\,(-0.6-1)\,(-0.6-2)\,(-0.6+2)}{5!}(45)$$

$$= 332 + 24.6 - 9.12 + 1.5392 - 0.5241$$

Hence $\qquad f(22) = 347.983.$

Example 29.19. *Interpolate by means of Gauss's backward formula, the population of a town for the year 1974, given that :*

Year :	*1939*	*1949*	*1959*	*1969*	*1979*	*1989*
Population (in thousands) :	*12*	*15*	*20*	*27*	*39*	*52*

(Kottayam, 2005 ; Madras, 2003)

Solution. Taking $x_0 = 1969$, $h = 10$, the population of the town is to be found for $p = \dfrac{1974 - 1969}{10} = 0.5$.

The central difference table is

x	p	y_p	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$	$\Delta^5 y_p$
1939	−3	12					
			3				
1949	−2	15		2			
			5		0		
1959	−1	20		2		3	
			7		3		−10
1969	0	27		5		−7	
			12		−4		
1979	1	39		1			
			13				
1989	2	52					

Gauss's backward formula is

$$y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)\,p}{2!}\Delta^2 y_{-1} + \frac{(p+1)\,p\,(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+1)\,(p+1)\,p\,(p-1)}{4!}\Delta^4 y_{-2}$$

$$+ \frac{(p+2)\,(p+1)\,p\,(p-1)\,(p-2)}{5!}\Delta^5 y_3 + \ldots$$

i.e.,

$$y_{.5} = 27 + (0.5)(7) + \frac{(1.5)(.5)}{2}(5) + \frac{(1.5)\,(.5)\,(-.5)}{6}(3) + \frac{(2.5)\,(1.5)\,(-.5)}{24}(-7)$$

$$+ \frac{(2.5)\,(1.5)\,(.5)\,(-.5)\,(-1.5)}{120}(-10)$$

$$= 27 + 3.5 + 1.875 - 0.1875 + 0.2743 - 0.1172 = 32.345 \text{ thousands approx.}$$

Example 29.20. *Given*

$\theta°$:	0	5	10	15	20	25	30
$\tan\theta$:	0	0.0875	0.1763	0.2679	0.3640	0.4663	0.5774

Using Stirling's formula, estimate the value of tan 16°. (*Anna, 2005*)

Solution. Taking the origin at $\theta° = 15°$, $h = 5°$ and $p = \dfrac{\theta - 15}{5}$, we have the following central difference table :

p						
−3	0.0000					
		0.08575				
−2	0.0875		0.0013			
		0.0888				
−1	0.1763		0.0028			
		0.0916		0.0015		
0	0.2679		0.0045		0.0002	
		0.0961		0.0017		−0.0002
1	0.3640		0.0062		0.0000	
		0.1023		0.0017		0.0009
2	0.4663		0.0088		0.0009	
		0.1111		0.0026		
3	0.5774					

At $\theta = 16°$, $\quad p = \dfrac{16 - 15}{5} = 0.2$

Stirling's formula is

$$y_p = y_0 + \frac{p}{1} \cdot \frac{\Delta y_{-1} + \Delta y_0}{2} + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!} \cdot \frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2} + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \ldots$$

$$\therefore \quad y_{0.2} = 0.2679 + (0.2)\left(\frac{0.0916 + 0.0961}{2}\right) + \frac{(0.2)^2}{2!}(0.0045) + \ldots$$

$$= 0.2679 + 0.01877 + 0.00009 + \ldots = 0.28676$$

Hence $\quad \tan 16° = 0.28676.$

Example 29.21. *Employ Stirling's formula to compute $y_{12.2}$ from the following table ($y_x = 1 + \log_{10} \sin x$) :*

$x°$:	10	11	12	13	14
$10^5 y_x$:	23,967	28,060	31,788	35,209	38,368

(*V.T.U., 2004*)

Solution. Taking the origin at $x_0 = 12°$, $h = 1$ and $p = x - 12$, we have the following central table :

p	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta^4 y_x$
−2	0.23967				
		0.04093			
−1	0.28060		−0.00365		
		0.03728		0.00058	
0	0.31788		−0.00307		−0.00013
		0.034121		−0.00045	
1	0.35209		−0.00062		
		0.03159			
2	0.38368				

At $x = 12.2$, $p = 0.2$. (As p lies between $-1/4$ and $1/4$, the use of Stirling's formula will be quite suitable.)

Stirling's formula is

$$y_p = y_0 + \frac{p}{1} \cdot \frac{\Delta y_{-1} + \Delta y_{-0}}{2} + \frac{p^2}{2!} \Delta^2 y_{-1} + \frac{p(p^2-1)}{3!} \cdot \frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2} + \frac{p^2(p^2-1)}{4!} \Delta^4 y_{-2} + \ldots$$

When $p = 0.2$, we have

$$\therefore \quad y_{0.2} = 0.31788 + 0.2\left(\frac{0.03728 + 0.03421}{2}\right) + \frac{(0.2)^2}{2}(-0.00307)$$

$$+ \frac{(0.2)[(0.2)^2 - 1]}{6}\left(\frac{0.00058 - 0.00045}{2}\right) + \frac{(0.2)^2[(0.2)^2 - 1]}{24}(-0.00013)$$

$$= 0.31788 + 0.00715 - 0.00006 - 0.000002 + 0.0000002 = 0.32497.$$

Example 29.22. *Apply Bessel's formula to obtain y_{25}, given $y_{20} = 2854$, $y_{24} = 3162$, $y_{28} = 3544$, $y_{32} = 3992$.* (S.V.T.U., 2007 ; V.T.U., 2000 S)

Solution. Taking the origin at $x_0 = 24$, $h = 4$, we have $p = \frac{1}{4}(x - 24)$.

$\therefore$ The central difference table is

p	y	Δy	$\Delta^2 y$	$\Delta^3 y$
-1	2854			
		308		
0	3162		74	
		382		-8
1	3544		66	
		448		
2	3992			

At $x = 25$, $p = (25 - 24)/4 = 1/4$. (As p lies between 1/4 and 3/4, the use of Bessel's formula will yield accurate esult.)

Bessel's formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{(p - 1/2)\, p\,(p-1)}{3!} \Delta^3 y_{-1} + \ldots \qquad \ldots(1)$$

When $p = 0.25$, we have

$$y_p = 3162 + 0.25 \times 382 + \frac{0.25(-0.75)}{2}\left(\frac{74 + 66}{2}\right) + \frac{(-0.25)\,0.25\,(-0.75)}{6}(-8)$$

$$= 3162 + 95.5 - 6 - 5625 - 0.0625 = 3250.875 \text{ approx.}$$

Example 29.23. *Apply Bessel's formula to find the value of f(27.5) from the table :*

x :	25	26	27	28	29	30
$f(x)$:	4.000	3.846	3.704	3.571	3.448	3.333

(U.P.T.U., 2009)

Solution. Taking the origin at $x_0 = 27$, $h = 1$, we have $p = x - 27$

The central difference table is

x	p	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
25	-2	4.000				
			-0.154			
26	-1	3.846		0.012		
			-0.142		-0.003	
27	0	3.704		0.009		0.004
			-0.133		-0.001	
28	1	3.571		0.010		-0.001
			-0.123		-0.002	
29	2	3.448		0.008		
			-0.115			
30	3	3.333				

At $x = 27.5$, $p = 0.5$ (As p lies between 1/4 and 3/4, the use of Bessel's formula will yield accurate result)
Bessel's formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{\left(p - \frac{1}{2}\right)p(p-1)}{3!}\Delta^3 y_{-1}$$
$$+ \frac{(p+1)\,p(p-1)\,(p-2)}{4!}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \ldots$$

When $p = 0.5$, we have

$$y_p = 3.704 - \frac{(0.5)\,(0.5-1)}{2}\left(\frac{0.009 + 0.010}{2}\right) + 0$$
$$+ \frac{(0.5+1)\,(0.5)\,(0.5-1)\,(0.5-2)}{24}\left(\frac{-0.001 - 0.004}{2}\right)$$
$$= 3.704 - 0.11875 - 0.00006 = 3.585$$

Hence $f(27.5) = 3.585$.

Example 29.24. *Given the table*

x :	310	320	330	340	350	360
$\log x$:	2.49136	2.50515	2.51851	2.53148	2.54407	2.55630

find the value of log 337.5 by Everett's formula.

Solution. Taking the origin at $x_0 = 330$ and $h = 10$, we have $p = \dfrac{x - 330}{10}$

$\therefore$ The central difference table is

p	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
−2	2.49136					
		0.01379				
−1	2.50515		−0.00043			
		0.01336		0.00004		
0	2.51881		−0.00039		−0.00003	
		0.01297		0.00001		0.00004
1	2.53148		−0.00038		0.00001	
		0.01259		0.00002		
2	2.54407		−0.00036			
		0.01223				
3	2.55630					

To evaluate log 337.5 *i.e.* for $x = 337.5$, $p = \dfrac{337.5 - 330}{10} = 0.75$

(As $p > 0.5$ and $= 0.75$, Everett's formula will be quite suitable)
Everett's formula is

$$y_p = qy_0 + \frac{q(q^2-1^2)}{3!}\Delta^2 y_{-1} + \frac{q(q^2-1^2)(q^2-2^2)}{5!}\Delta^4 y_{-2} + \ldots + py_1 + \frac{p(p^2-1^2)}{3!}\Delta^2 y_0$$
$$+ \frac{p(p^2-1^2)(p^2-2^2)}{5!}\Delta^4 y_{-1} + \ldots$$
$$= 0.25 \times 2.51851 + \frac{0.25\,(0.0625-1)}{6} \times (-0.00039) + \frac{0.25\,(0.0625-1)\,(0.0625-4)}{120}$$
$$\times (-0.00003) + 0.75 \times 2.53148 + \frac{0.75\,(0.5625-1)}{6} \times (-0.00038)$$
$$+ \frac{0.75\,(0.5625-1)\,(0.5625-4)}{120} \times (0.00001)$$
$$= 0.62963 + 0.00002 - 0.0000002 + 1.89861 + 0.00002 + 0.0000001 = 2.52828 \text{ nearly.}$$

PROBLEMS 29.4

1. Using Gauss's forward formula, evaluate $f(3.75)$ from the table :

x :	2.5	3.0	3.5	4.0	4.5	5.0
y :	24.145	22.043	20.225	18.644	17.262	16.047

(Bhopal, 2002 ; Madras, 2000)

2. Using Gauss's backward difference formula, find $y(8)$ from the following table :

x :	0	5	10	15	20	25
y :	7	11	14	18	24	32

(J.N.T.U., 2007)

3. Using Gauss's backward formula, estimate the number of persons earning wages between Rs. 60 and Rs. 70 from the following data :

Wages (₹) :	Below 40	40–60	60–80	80–100	100–120
No. of persons : (in thousands)	250	120	100	70	50

(Madras, 2000)

4. From the following table :

x :	1.00	1.05	1.10	1.15	1.20	1.25	1.30
e^x :	2.7183	2.8577	3.0042	3.1582	3.3201	3.4903	3.6693

(U.P.T.U., 2006)

Find $e^{1.17}$, using Gauss forward formula.

5. The pressure p of wind corresponding to velocity v is given by the following data. Estimate p when $y = 25$.

v :	10	20	30	40
p :	1.1	2	4.4	7.9

6. Using Stirling's formula find y_{35}, given $y_{20} = 512, y_{30} = 439, y_{40} = 346, y_{50} = 243$, where y_x represents the number of persons at age x years in a life table. *(Nagarjuna, 2003 S)*

7. Employ Bessel's formula to find the value of F at $x = 1.95$, given that

x :	1.7	1.8	1.9	2.0	2.1	2.2	2.3
F :	2.979	3.144	3.283	3.391	3.463	3.997	4.491

Which other interpolation formula can be used here ? Which is more appropriate ? Give reasons.

8. Calculate the value of $f(1.5)$ using Bessel's interpolation formula, from the following table :

x :	0	1	2	3
$f(x)$:	3	6	12	15

(U.P.T.U., 2008)

9. Apply Everett's formula to obtain u_{25}, given $u_{20} = 854, u_{24} = 3162, u_{28} = 3544, u_{32} = 3992$. *(S.V.T.U., 2007)*

10. Using Everett's formula, evaluate $f(30)$, if $f(20) = 2854, f(28) = 3162, f(36) = 7088, f(44) = 7984$ *(U.P.T.U., 2006)*

11. Given the table :

x :	310	320	330	340	350	360
$\log x$:	2.4914	2.5052	2.5185	2.5315	2.5441	2.5563

Find the value of log 337.5 by Gauss's, Stirling's and Bessel's formulae.

29.9 INTERPOLATION WITH UNEQUAL INTERVALS

The various interpolation formulae derived so far possess the disadvantages of being applicable only to equally spaced values of the argument. It is, therefore, desirable to develop interpolation formulae for unequally spaced values of x. Now we shall study two such formulae :

(*i*) Lagrange's interpolation formula

(*ii*) Newton's general interpolation formula with divided differences.

29.10 LAGRANGE'S INTERPOLATION FORMULA

If $y = f(x)$ takes the value $y_0, y_1, \ldots, y_n$ corresponding to $x = x_0, x_1, \ldots, x_n$, then

$$f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}\,y_0 + \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}\,y_1 + \ldots + \frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}\,y_n \qquad \ldots(1)$$

This is known as *Lagrange's interpolation formula for unequal intervals.*

Proof. Let $y = f(x)$ be a function which takes the values $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$. Since there are $n + 1$ pairs of values of x and y, we can represent $f(x)$ by a polynomial in x of degree n. Let this polynomial be of the form

$$y = f(x) = a_0(x - x_1)(x - x_2) \ldots (x - x_n) + a_1(x - x_0)(x - x_2) \ldots (x - x_n)$$
$$+ a_2(x - x_0)(x - x_1)(x - x_3) \ldots (x - x_n) + \ldots + a_n(x - x_0)(x - x_1) \ldots (x - x_{n-1}) \quad \ldots(2)$$

Putting $x = x_0$, $y = y_0$, in (2), we get

$$y_0 = a_0(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)$$
$$a_0 = y_0/[(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)]$$

Similarly putting $x = x_1$, $y = y_1$ in (2), we have $a_1 = y_1/[(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)]$

Proceeding the same way, we find $a_2, a_3 \ldots a_n$

Substituting the values of $a_0, a_1, \ldots, a_n$ in (2), we get (1).

Obs. Lagranges interpolation formula (1) for n points is a polynomial of degree $(n - 1)$ which is known as *Lagrangian polynomial* and is very simple to implement on a computer.

This formula can also be used to split the given function into partial fractions.

For on dividing both sides of (1) by $(x - x_0)(x - x_1) \ldots (x - x_n)$, we get

$$\frac{f(x)}{(x_0 - x_0)(x_0 - x_1)\ldots(x_0 - x_n)} = \frac{y_0}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)} \cdot \frac{1}{x - x_0}$$
$$+ \frac{y_0}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)} \cdot \frac{1}{x - x_1} + \ldots + \frac{y_n}{(x_n - x_0)(x_n - x_1)\ldots(x_n - x_{n-1})} \cdot \frac{1}{x - x_n}.$$

Example 29.25. *Given the values*

x :	5	7	11	13	17
$f(x)$:	150	392	1492	2366	5202

evaluate $f(9)$, using (i) Lagrange's formula. *(Anna, 2006)*

Solution. (*i*) Here $x_0 = 5, x_1 = 7, x_2 = 11, x_3 = 13, x_4 = 17$

and $y_0 = 150, y_1 = 392, y_2 = 1452, y_3 = 2366, y_4 = 5202.$

Putting $x = 9$ and substituting the above values in Lagrange's formula, we get

$$f(9) = \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)} \times 150 + \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)} \times 392$$
$$+ \frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)} \times 1452 + \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)} \times 2366$$
$$+ \frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)} \times 5202 = -\frac{50}{3} + \frac{3136}{15} + \frac{3872}{3} - \frac{2366}{3} + \frac{578}{5} = 810.$$

Example 29.26. *Find the polynomial $f(x)$ by using Lagrange's formula and hence find $f(3)$ for*

x :	0	1	2	5
$f(x)$:	2	3	12	147

(Anna, 2005)

Solution. Here $x_0 = 0, x_1, x_2 = 2, x_3 = 5$

and $y_0 = 2, y_1 = 3, y_2 = 12, y_3 = 147$

Lagrange's formula is

$$y = \frac{(x - x_1)(x - x_2)\ldots(x - x_3)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_3)} y_0 + \frac{(x - x_0)(x - x_2)\ldots(x - x_3)}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_3)} y_1$$
$$+ \frac{(x - x_0)(x - x_1)\ldots(x - x_3)}{(x_2 - x_0)(x_2 - x_1)\ldots(x_2 - x_3)} y_2 + \frac{(x - x_0)(x - x_2)\ldots(x - x_5)}{(x_3 - x_0)(x_3 - x_2)\ldots(x_3 - x_2)} y_3$$
$$= \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}(2) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(3)$$
$$+ \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(12) + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(147)$$

Hence $\quad f(x) = x^3 + x^2 - x + 2$

$\therefore \quad f(3) = 27 + 9 - 3 + 2 = 35.$

Example 29.27. *A curve passes through the point (0, 18), (1, 10), (3, – 18) and (6, 90). Find the slope of the curve at $x = 2$.* (J.N.T.U., 2009)

Solution. Here $x_0 = 0, x_1 = 1, x_2 = 3, x_3 = 6$ and $y_0 = 18, y_1 = 10, y_2 = -18, y_3 = 90$

Since the values of x are unequally spaced, we use the Lagrange's formula :

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$= \frac{(x-1)(x-3)(x-6)}{(0-1)(0-2)(0-6)}(18) + \frac{(x-0)(x-3)(x-6)}{(1-0)(1-3)(1-6)}(10)$$

$$+ \frac{(x-0)(x-1)(x-6)}{(3-0)(3-1)(3-6)}(-18) + \frac{(x-0)(x-1)(x-3)}{(6-0)(6-1)(6-3)}(90)$$

$$= (-x^3 + 10x^2 - 27x + 18) + (x^3 - 9x^2 + 18x) + (x^3 - 7x^2 + 6x) + (x^3 - 4x^2 + 3x)$$

i.e., $\quad y = 2x^3 - 10x^2 + 18$

Thus the slope of the curve at $(x = 2) = \left(\frac{dy}{dx}\right)_{x=2}$

$$= (6x^2 - 20x)_{x=2} = -16.$$

Example 29.28. *Using Lagrange's formula, express the function* $\frac{3x^2 + x + 1}{(x-1)(x-2)(x-3)}$ *as a sum of partial fractions.*

Solution. Let us evaluate $y = 3x^2 + x + 1$ for $x = 1, x = 2$ and $x = 3$

These values are

x :	$x_0 = 1$	$x_1 = 2$	$x_2 = 3$
y :	$y_0 = 5$	$y_1 = 15$	$y_2 = 31$

Lagrange's formula is

$$y = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2$$

$$= \frac{(x-1)(x-2)}{(0-1)(0-2)}(5) + \frac{(x-1)(x-3)}{(2-1)(2-3)}(15) + \frac{(x-1)(x-2)}{(3-1)(3-1)}(31)$$

Substituting the above values, we get

$$= \frac{(x-2)(x-3)}{(1-2)(1-3)}(5) + \frac{(x-1)(x-3)}{(2-1)(2-3)}(15) + \frac{(x-1)(x-2)}{(3-1)(3-2)}(31)$$

$$= 2.5(x-2)(x-3) - 15(x-1)(x-3) + 15.5(x-1)(x-2)$$

Thus $\quad \frac{3x^2 + x + 1}{(x-1)(x-2)(x-3)} = \frac{2.5(x-2)(x-3) - 15(x-1)(x-3) + 15.5(x-1)(x-2)}{(x-1)(x-2)(x-3)}$

$$= \frac{2.5}{x-1} - \frac{15}{x-2} + \frac{15.5}{x-3}.$$

Example 29.29. *Find the distance moved by a particle and its acceleration at the end of 4 seconds, if the time verses velocity data is as follows :*

t :	0	1	3	4
v :	21	15	12	10

Solution. Since the values of t are not equispaced, we use Lagrange's formula :

$$v = \frac{(t-t_1)(t-t_2)(t-t_3)}{(t_0-t_1)(t_0-t_2)(t_0-t_3)}v_0 + \frac{(t-t_0)(t-t_2)(t-t_3)}{(t_1-t_0)(t_1-t_2)(t_1-t_3)}v_1$$

$$+ \frac{(t-t_0)(t-t_1)(t-t_3)}{(t_2-t_0)(t_2-t_1)(t_2-t_3)}v_2 + \frac{(t-t_0)(t-t_2)(t-t_5)}{(t_3-t_0)(t_3-t_2)(t_3-t_2)}v_3$$

$$v = \frac{(t-1)(t-3)(t-4)}{(-1)(-2)(-4)}(21) + \frac{t(t-3)(t-4)}{(1)(-2)(-3)}(15) + \frac{t(t-1)(t-4)}{(3)(2)(-1)}(12) + \frac{t(t-1)(t-3)}{(4)(3)(1)}(10)$$

i.e., $$v = \frac{1}{12}(-5t^3 + 38t^2 - 105t + 252)$$

$\therefore$ Distance moved $s = \int_0^4 v\,dt = \frac{1}{12}\int_0^4 (-5t^3 + 38t^2 - 105t + 252)\,dt$ $\left[\because\ v = \frac{ds}{dt}\right]$

$$= \frac{1}{12}\left(-\frac{5t^4}{4} + \frac{38t^3}{3} - \frac{105t^2}{2} + 252t\right)_0^4$$

$$= \frac{1}{12}\left(-320 + \frac{2432}{3} - 840 + 1008\right) = 54.9$$

Also acceleration $= \frac{dv}{dt} = \frac{1}{2}(-15t^2 + 76t - 105 + 0)$

Hence acceleration at $(t = 4) = \frac{1}{12}(-15(16) + 76(4) - 105) = -3.4$.

PROBLEMS 29.5

1. Use Lagrange's interpolation formula to find the value of y when $x = 10$, if the following values of x and y are given :

x :	5	6	9	11
y :	12	13	14	16

(*U.P.T.U., 2009 ; J.N.T.U., 2008*)

2. Given $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$, $\log_{10} 661 = 2.8202$, find by using Lagrange's formula, the value of $\log_{10} 656$. (*Hazaribagh, 2009*)

3. The following are the measurements T made on a curve recorded by oscilograph representing a change of current I due to a change in the conditions of an electric current.

T :	1.2	2.0	2.5	3.0
I :	1.36	0.58	0.34	0.20

Using Lagrange's formula, find I at $T = 16$. (*J.N.T.U., 2009*)

4. Using Lagrange's interpolation, calculate the profit in the year 2000 from the following data :

Year :	1997	1999	2001	2002
Profit in Lakhs of ₹ :	43	65	159	248

(*Anna, 2004*)

5. Use Lagrange's formula to find the form of $f(x)$, given

x :	0	2	3	6
$f(x)$:	648	704	729	792

(*Madras, 2003 S*)

6. If $y(1) = -3$, $y(3) = 9$, $y(4) = 30$, $y(6) = 132$, find the Lagrange's interpolation polynomial that takes the same values as y at the given points. (*V.T.U., 2006*)

7. Given $f(0) = -18$, $f(1) = 0$, $f(3) = 0$, $f(5) = -248$, $f(6) = 0$, $f(9) = 13104$, find $f(x)$. (*Nagarjuna, 2003*)

8. Find the missing term in the following table using interpolation

x :	1	2	4	5	6
y :	14	15	5	...	9

9. Using Lagrange's formula, express the function $\frac{x^2 + x - 3}{x^3 - 2x^2 - x + 2}$ as sum of partial fractions.

29.11 DIVIDED DIFFERENCES

The Lagrange's formula has the drawback that if another interpolation value were inserted, then the interpolation coefficients are required to be recalculated. The labour of recomputing the interpolation

coefficients is saved by using Newton's general interpolation formula which employs what are called 'divided differences'. Before deriving this formula, we shall first define these differences.

If $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots$ be given points, then the *first divided difference* for the arguments, x_0, x_1 is defined by the relation $[x_0, x_1] = \dfrac{y_1 - y_0}{x_1 - x_0}$.

Similarly $[x_1, x_2] = \dfrac{y_2 - y_1}{x_2 - x_1}$ and $[x_2, x_3] = \dfrac{y_3 - y_2}{x_3 - x_2}$ etc.

The *second divided difference* for x_0, x_1, x_2 is defined as $[x_0, x_1, x_2] = \dfrac{[x_1 - x_2] - [x_0, x_1]}{x_2 - x_0}$

The *third divided difference* for x_0, x_1, x_2, x_3 is defined as

$[x_0, x_1, x_2, x_3] = \dfrac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0}$ and so on.

Obs. 1. *The divided differences are symmetrical in their arguments i.e. independent of the order of the arguments.*

For it is easy to write $[x_0, x_1] = \dfrac{y_0}{x_0 - x_1} + \dfrac{y_1}{x_1 - x_0} = [x_1, x_0]$ $[x_0, x_1, x_2]$

$$= \frac{y_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} \cdot \frac{y_2}{(x_2 - x_0)(x_2 - x_1)}$$

$= [x_1, x_2, x_0]$ or $[x_2, x_0, x_1]$ and so on.

Obs. 2. *The nth divided differences of a polynomial of the nth degree are constant.*

Let the arguments be equally spaced so that, $x_1 - x_0 = x_2 - x_1 = \ldots = x_n - x_{n-1} = h$. Then

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h}$$

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0} = \frac{1}{2h}\left\{\frac{\Delta y_1}{h} - \frac{\Delta y_0}{h}\right\}$$

$= \dfrac{1}{2!\,h^2}\,\Delta^2 y_0$ and in general, $[x_0, x_1, x_2, \ldots x_n] = \dfrac{1}{n!\,h^n}\,\Delta^n y_0$.

If the tabulated function is a nth degree polynomial, then $\Delta^n y_0$ will be constant. Hence the nth divided differences will also be constant.

29.12 NEWTON'S DIVIDED DIFFERENCE FORMULA

Let $y_0, y_1, \ldots, y_n$ be the values of $y = f(x)$ corresponding to the arguments $x_0, x_1, \ldots, x_n$. Then from the definition of divided differences, we have

$$[x, x_0] = \frac{y - y_0}{x - x_0}$$

so that
$$y = y_0 + (x - x_0)\,[x, x_0] \qquad \ldots(1)$$

Again
$$[x, x_0, x_1] = \frac{[x, x_0] - [x_0, x_1]}{x - x_1}$$

which gives
$$[x, x_0] = [x_0, x_1] + (x - x_1) + (x - x_1)\,[x, x_0, x_1]$$

Substituting this value of $[x, x_0]$ in (1), we get

$$y = y_0 + (x - x_0)\,[x_0, x_1] + (x - x_0)(x - x_1)\,[x, x_0, x_1] \qquad \ldots(2)$$

Also
$$[x, x_0, x_1, x_2] = \frac{[x, x_0, x_1] - [x_0, x_1, x_2]}{x - x_2}$$

which gives $[x, x_0, x_1,] = [x_0, x_1, x_2] + (x, x_2)\,[x, x_0, x_1, x_2]$

Substituting this value of $[x, x_0, x_1]$ in (2), we obtain

$$y = y_0 + (x - x_0)\,[x_0, x_1] + (x - x_0)(x - x_1)\,[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)\,[x, x_0, x_1, x_2]$$

Proceeding in this manner, we get

$$y = f(x) = y_0 + (x - x_0)\,[x_0, x_1] + (x - x_0)(x - x_1)\,[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)\,[x_0, x_1, x_2, x_3] + \ldots + (x - x_0)(x - x_1)\ldots(x - x_n)\,[x, x_0, x_1, \ldots, x_n] \quad \ldots(3)$$

which is called ***Newton's general interpolation formula with divided differences.***

Example 29.30. *Given the values*

x :	5	7	11	13	17
$f(x)$:	150	392	1452	2366	5202,

evaluate $f(9)$, using Newton's divided difference formula. (V.T.U., 2010 ; P.T.U., 2005)

Solution. **The divided difference table is**

x	y	*1st divided differences*	*2nd divided differences*	*3rd divided differences*
5	150			
		$\frac{392-150}{7-5} = 121$		
7	392		$\frac{265-121}{11-5} = 24$	
		$\frac{1452-392}{11-7} = 265$		$\frac{32-24}{13-5} = 1$
11	1452		$\frac{457-265}{13-7} = 32$	
		$\frac{2366-1452}{13-11} = 457$		$\frac{42-32}{17-7} = 1$
13	2366		$\frac{709-457}{17-11} = 42$	
		$\frac{5202-2366}{17-13} = 709$		
17	5202			

Taking $x = 9$ in the Newton's divided difference formula, we obtain

$$f(9) = 150 + (9-5) \times 121 + (9-5)(9-7) \times 24 + (9-5)(9-7)(9-11) \times 1$$
$$= 150 + 484 + 192 - 16 = 810.$$

Example 29.31. *Determine $f(x)$ as a polynomial in x for the following data :*

x :	-4	-1	0	2	5
$f(x)$:	1245	33	5	9	1335

(V.T.U., 2007)

Solution. **The divided differences table is**

x	$f(x)$	*1st divided differences*	*2nd divided differences*	*3rd divided differences*	*4th divided differences*
-4	1245				
		-404			
-1	33		94		
		-28		-14	
0	5		10		3
		2		13	
2	9		88		
		442			
5	1335				

Applying Newton's divided difference formula

$$f(x) = f(x_0) + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + \ldots$$
$$= 1245 + (x + 4)(-404) + (x + 4)(x + 1)(94)$$
$$+ (x + 4)(x + 1)(x - 0)(-14) + (x + 4)(x + 1)x(x - 2)(3)$$
$$= 3x^4 - 5x^3 + 6x^2 - 14x + 5.$$

PROBLEMS 29.6

1. Find the third divided difference with arguments 2, 4, 9, 10 of the function $f(x) = x^3 - 2x$. (U.P.T.U., 2005)
2. Use Newton's divided difference method to compute $f(5.5)$ from the following data :

x :	0	1	4	5	6
$f(x)$:	1	14	15	6	3

(U.P.T.U., 2010)

3. Using Newton's divided difference formula, evaluate $f(8)$ and $f(15)$ given :

x :	4	5	7	10	11	13
$f(x)$:	48	100	294	900	1210	2028

(U.P.T.U., MCA, 2009, V.T.U., 2008)

4. Obtain the Newton's divided difference interpolation polynomial and hence find $f(6)$:

x :	3	7	9	10
$f(x)$:	168	120	72	63

(U.P.T.U., 2007)

5. Using Newton's divided difference interpolation, find the polynomial of the given data :

x :	-1	0	1	3
$f(x)$:	2	1	0	-1

(Anna, 2007)

6. For the following table, find $f(x)$ as a polynomial in x using Newton's divided difference formula:

x :	5	6	9	11
$f(x)$:	12	13	14	16

7. Using the following table, find $f(x)$ as a polynomial in

x :	-1	0	3	6	7
$f(x)$:	3	-6	39	822	1611

(U.P.T.U., 2009)

8. Find the missing term in the following table using Newton's divided difference formula

x :	0	1	2	3	4
y :	1	3	9	...	81

29.13 INVERSE INTERPOLATION

So far, given a set of values of x and y, we have been finding the values of y corresponding to a certain value of x. On the other hand, the process of estimating the value of x for a value of y (which is not in the table) is called the *inverse interpolation*.

Lagrange's formula is merely a relation between two variables either of which may be taken as the independent variable. Therefore, on inter-changing x and y in the Lagrange's formula, we obtain

$$x = \frac{(y - y_1)(y - y_2)\ldots(y - y_n)}{(y_0 - y_1)(y_0 - y_2)\ldots(y_0 - y_n)} x_0 + \frac{(y - y_0)(y - y_2)\ldots(y - y_n)}{(y_1 - y_0)(y_1 - y_2)\ldots(y_1 - y_n)} x_1$$
$$+ \ldots + \frac{(y - y_0)(y - y_1)\ldots(y - y_{n-1})}{(y_n - y_0)(y_n - y_1)\ldots(y_n - y_{n-1})} x_n \quad \ldots(1)$$

which is used for inverse interpolation.

Example 29.32. *The following table gives the values of x and y :*

x :	1.2	2.1	2.8	4.1	4.9	6.2
y :	4.2	6.8	9.8	13.4	15.5	19.6

Find the value of x corresponding to $y = 12$, using Lagrange's technique. (V.T.U., 2009)

Solution. Here $x_0 = 1.2, x_1 = 2.1, x_2 = 2.8, x_3 = 4.1, x_4 = 4.9, x_5 = 6.2$

and $y_0 = 4.2, y_1 = 6.8, y_2 = 9.8, y_3 = 13.4, y_4 = 15.5, y_5 = 19.6$

Taking $y = 12$, the above formula (1) gives

$$x = \frac{(12-6.8)(12-9.8)(12-13.4)(12-15.5)(12-19.6)}{(4.2-6.8)(4.2-9.8)(4.2-13.4)(4.2-15.5)(4.2-19.6)} \times 1.2$$

$$+ \frac{(12-4.2)(12-9.8)(12-13.4)(12-15.5)(12-19.6)}{(6.8-4.2)(6.8-9.8)(6.8-13.4)(6.8-15.5)(6.8-19.6)} \times 2.1$$

$$+ \frac{(12-4.2)(12-6.8)(12-13.4)(12-15.5)(12-19.6)}{(9.8-4.2)(9.8-6.8)(9.8-13.4)(9.8-15.5)(9.8-19.6)} \times 2.8$$

$$+ \frac{(12-4.2)(12-6.8)(12-9.8)(12-15.5)(12-19.6)}{(13.4-4.2)(13.4-6.8)(13.4-9.8)(13.4-15.5)(13.4-19.6)} \times 4.1$$

$$+ \frac{(12-4.2)(12-6.8)(12-9.8)(12-13.4)(12-19.6)}{(15.5-4.2)(15.5-6.8)(15.5-9.8)(15.5-13.4)(15.5-19.6)} \times 4.9$$

$$+ \frac{(12-4.2)(12-6.8)(12-9.8)(12-13.4)(12-15.5)}{(19.6-4.2)(19.6-6.8)(19.6-9.8)(19.6-13.4)(19.6-15.5)} \times 6.2$$

$$= 0.022 - 0.234 + 1.252 + 3.419 - 0.964 + 0.055 = 3.55.$$

Example 29.33. *Apply Lagrange's formula inversely to obtain a root of the equation $f(x) = 0$, given that $f(30) = -30$, $f(34) = -13$, $f(38) = 3$, and $f(42) = 18$.* (V.T.U., 2009 S)

Solution. Here $x_0 = 30, x_1 = 34, x_2 = 38, x_3 = 42$

and $y_0 = -30, y_1 = -13, y_2 = 3, y_3 = 18$

It is required to find x corresponding to $y = f(x) = 0$.

Taking $y = 0$, the Lagrange's formula gives,

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)} x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y-y_2)(y_1-y_3)} x_1$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)} x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)} x_3$$

$$= \frac{13(-3)(-18)}{(-17)(-33)(-48)} \times 30 + \frac{30(-3)(-18)}{17(-16)(-31)} \times 34 + \frac{30(13)(-18)}{33(16)(-15)} \times 38 + \frac{30(13)(-3)}{48(31)(15)} \times 42$$

$$= -0.782 + 6.532 + 33.682 - 2.202 = 37.23$$

Hence the desired root of $f(x) = 0$ is 37.23.

PROBLEMS 29.7

1. Apply Lagrange's method to find the value of x when $f(x) = 15$ from the given data :

x :	5	6	9	11
$f(x)$:	12	13	14	16

(*Madras, 2000*)

2. Obtain the value of t when $A = 85$ from the following table, using Lagrange's method :

t :	2	5	8	14
A :	94.8	87.9	81.3	68.7

29.14 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 29.8

Select the correct answer or fill up the blanks in the following problems :

1. Newton's backward interpolation formula is .....
2. Bessel's formula is most appropriate when p lies between

(*a*) – 0.25 and 0.25 (*b*) 0.25 and 0.75 (*c*) 0.75 and 1.00.

3. From the divided difference table for the following data :

x :	5	15	22
y :	7	36	160

4. Interpolation is the technique of estimating the value of a function for any .....
5. Bessel's formula for interpolation is .....
6. The 4th divided differences for x_0, x_1, x_2, x_3, x_4 = ...
7. Stirling's formula is best suited for p lying between ..... .
8. Newton's divided differences formula is ..... .
9. Given $(x_0, y_0), (x_1, y_1), (x_2, y_2)$, Lagrange's interpolation formula is .....
10. If $f(0) = 1$, $f(2) = 5$, $f(3) = 10$ and $f(x) = 14$, then $x = 0$ ..... .
11. Gauss forward interpolation formula involves
    (*a*) even differences above the central line and odd differences on the central line
    (*b*) even differences below the central line and odd differences on the central line
    (*c*) odd differences below the central line and even differences on the central line
    (*d*) odd differences above the central line and even differences on the central line.
12. If $y(1) = 4$, $y(3) = 12$, $y(4) = 19$ and $y(x) = 7$ find x using Lagrange's formula.
13. Extrapolation is defined as .....
14. The second divided difference of $f(x) = 1/x$, with arguments, a, b, c, is..... . *(Anna, 2007)*
15. Gauss-forward interpolation formula is used to interpolate values of y for
    (*a*) $0 < p < 1$ (*b*) $-1 < p < 0$
    (*c*) $0 < p < -\alpha$ (*d*) $-\alpha < p < 0$.
16. Given

x :	0	1	3	4
y :	– 12	0	6	12

    Using Lagrange's formula, a polynomial that can be fitted to the data is .....
17. The nth divided difference of a polynomial of degree n is
    (*a*) zero (*b*) a constant
    (*c*) a variable (*d*) none of these.
18. If h is the interval of differencing, $\Delta^2 x^3$ = ...

CHAPTER 30

Numerical Differentiation & Integration

1. Numerical differentiation. 2. Formulae for derivatives. 3. Maxima and minima of a tabulated function. 4. Numerical integration. 5. Newton-Cotes quadrature formula. 6. Trapezoidal rule. 7. Simpson's 1/3rd rule. 8. Simpson's 3/8th rule. 9. Boole's rule. 10. Weddle's rule. 11. Objective Type of Questions.

30.1 NUMERICAL DIFFERENTIATION

It is the process of calculating the value of the derivative of a function at some assigned value of x from the given set of values (x_i, y_i). To compute dy/dx, we first replace the latter as many times as we desire. The choice of the interpolation formula to be used, will depend on the assigned value of x at which dy/dx, is desired.

If the values of x are equi-spaced and dy/dx, is required near the beginning of the table, we employ Newton's forward formula. If it is required near the end of the table, we use Newton's backward formula. For values near the middle of the table, dy/dx, is calculated by means of Stirling's or Bessel's formula. If the values of x are not equi-spaced, we use Newton's divided difference formula to represent the function.

30.2 FORMULAE FOR DERIVATIVES

Consider the function $y = f(x)$ which is tabulated for the values $x_i\ (= x_0 + ih)$, $i = 0, 1, 2, \ldots n$.

(1) Derivatives using forward difference formula. Newton's forward interpolation formula (p. 958) is

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots$$

Differentiating both sides w.r.t. p, we have

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \ldots$$

Since $p = \dfrac{(x-x_0)}{h}$, therefore $\dfrac{dp}{dx} = \dfrac{1}{h}$.

Now
$$\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \frac{4p^3-18p^2+22p-6}{4i}\Delta^4 y_0 + \cdots\right] \quad \ldots(1)$$

At $x = x_0, p = 0$. Hence putting $p = 0$,

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + \cdots\right] \quad \ldots(2)$$

Again differentiating (1) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dp}\right)\frac{dp}{dx}$$

$$= \frac{1}{h}\left[\frac{2}{2!}\Delta^2 y_0 + \frac{6p-6}{3!}\Delta^3 y_0 + \frac{12p^2-36p+22}{4!}\Delta^4 y_0 + \cdots\right]\frac{1}{h}$$

Putting $p = 0$, we obtain

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 + \cdots\right] \quad ...(3)$$

Similarly $$\left(\frac{d^3y}{dx^3}\right)_{x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \cdots\right] \quad ...(4)$$

Otherwise : We know that $1 + \Delta = E = e^{hD}$

$\therefore$ $$hD = \log(1+\Delta) = \Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \ldots$$

or $$D = \frac{1}{h}\left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \cdots\right]$$

and $$D^2 = \frac{1}{h^2}\left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \cdots\right]^2 = \frac{1}{h^2}\left[\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 + \cdots\right]$$

and $$D^3 = \frac{1}{h^2}\left[\Delta^3 - \frac{3}{2}\Delta^4 + \cdots\right]$$

Now applying the above identities to y_0, we get

Dy_0, *i.e.*, $$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 - \cdots\right]$$

and $$\left(\frac{d^3y}{dx^3}\right)_{x_0} = \frac{1}{h^3}\left[\Delta^2 y_0 - \frac{3}{2}\Delta^4 y_0 + \cdots\right]$$

which are the same as (2), (3) and (4) respectively.

(2) Derivatives using backward difference formula. Newton's backward interpolation formula (p. 958) is

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots$$

Differentiating both sides w.r.t. p, we get

$$\frac{dy}{dp} = \nabla y_n + \frac{2p+1}{2!}\nabla^2 y_n + \frac{3p^2+6p+2}{3!}\nabla^3 y_n + \cdots$$

Since $p = \dfrac{x - x_n}{h}$, therefore $\dfrac{dp}{dx} = \dfrac{1}{h}$.

Now $$\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx} = \frac{1}{h}\left[\nabla y_n \frac{2p+1}{2!}\nabla^2 y_n + \frac{3p^3+6p+2}{3!}\nabla^3 y_n + \cdots\right] \quad ...(5)$$

At $x = x_n$, $p = 0$. Hence putting $p = 0$, we get

$$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \frac{1}{6}\nabla^6 y_n + \cdots\right] \quad ...(6)$$

Again differentiating (5) w.r.t. x, we have

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dx}\right)\frac{dp}{dx} = \frac{1}{h^2}\left[\nabla^2 y_n + \frac{6p+6}{3!}\nabla^3 y_n + \frac{6p^2+18p+11}{12}\nabla^4 y_n + \cdots\right]$$

Putting $p = 0$, we obtain

$$\left(\frac{d^2y}{dx^2}\right)_{x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \cdots\right] \quad ...(7)$$

Similarly, $$\left(\frac{d^3y}{dx^3}\right)_{x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \cdots\right] \quad ...(8)$$

Otherwise : We know that $1 - \nabla = E^{-1} = e^{-hD}$

$$\therefore \quad -hD = \log(1-\nabla) = -\left[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{4}\nabla^4 + \cdots\right]$$

or $$D = \frac{1}{h}\left[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{4}\nabla^4 + \cdots\right]$$

$$\therefore \quad D^2 = \frac{1}{h^2}\left[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \cdots\right]^2 = \frac{1}{h^2}\left[\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \cdots\right]$$

Similarly, $$D^3 = \frac{1}{h^3}\left[\nabla^3 + \frac{3}{2}\nabla^4 + \cdots\right]$$

Applying these identities to y_n, we get

Dy_n, *i.e.*, $$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y^n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \frac{1}{6}\nabla^6 y_n + \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x_n} = \frac{1}{h^2}\left(\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \cdots\right)$$

and $$\left(\frac{d^3y}{dx^3}\right)_{x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \cdots\right]$$

which are the same as (6), (7) and (8).

(3) Derivatives using central difference formulae. Stirling's formula (p. 964) is

$$y_p = y_0 + \frac{p}{1!}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1^2)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2(p^2-1^2)}{4!}\Delta^4 y_{-2} + \ldots$$

Differentiating both sides w.r.t. p., we get

$$\frac{dy}{dp} = \left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{2p}{2!}\Delta^2 y_{-1} + \frac{3p^2-1}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{4p^3-2p}{4!}\Delta^4 y_{-2} + \ldots$$

Since $p = \dfrac{x - x_0}{h}$, $\therefore \quad \dfrac{dp}{dx} = \dfrac{1}{h}$.

Now $$\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx} = \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + p\Delta^2 y_{-1} + \frac{3p^2-1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{2p^3-p}{12}\Delta^4 y_{-2} + \cdots\right]$$

At $x = x_0$, $p = 0$. Hence putting $p = 0$, we get

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{1}{6}\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} + \frac{1}{30}\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2} + \cdots\right] \quad ...(9)$$

Similarly $$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} - \cdots\right] \quad ...(10)$$

Obs. We can similarly use any other interpolation formula for computing the derivatives.

Example 30.1. *Given that*

x :	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y :	7.989	8.403	8.781	9.129	9.451	9.750	10.031

find $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ *at (a)* $x = 1.1$ *(V.T.V., 2006; Madras, 2003 S)*

(b) $x = 1.6$. *(Rohtak, 2006; J.N.T.U., 2004 S)*

Solution. (*a*) The difference table is :

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
1.0	7.989						
		0.414					
1.1	8.403		– 0.036				
		0.378		0.006			
1.2	8.781		– 0.030		– 0.002		
		0.348		0.004		0.001	
1.3	9.129		– 0.026		– 0.001		0.002
		0.322		0.003		0.003	
1.4	9.451		– 0.023		0.002		
		0.299		0.005			
1.5	9.750		– 0.018				
		0.281					
1.6	10.031						

We have

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^2 y_0 - \frac{1}{6}\Delta^6 y_0 + \cdots\right] \quad ...(i)$$

and

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 - \cdots\right] \quad ...(ii)$$

Here $h = 0.1$, $x_0 = 1.1$, $\Delta y_0 = 0.378$, $\Delta^2 y_0 = -0.03$ etc.

Substituting these values in (*i*) and (*ii*), we get

$$\left(\frac{dy}{dx}\right)_{1.1} = \frac{1}{0.1}\left[0.378 - \frac{1}{2}(-0.03) + \frac{1}{3}(0.004) - \frac{1}{4}(-0.001) + \frac{1}{5}(0.003)\right] = 3.952$$

$$\left(\frac{d^2y}{dx^2}\right)_{1.1} = \frac{1}{(0.1)^2}\left[-0.03 - (0.004) + \frac{11}{12}(-0.001) - \frac{5}{6}(0.003)\right] = -3.74$$

(*b*) We use the above difference table and the backward difference operator ∇ instead of Δ.

$$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^2 y_n + \frac{1}{6}\nabla^6 y_n + \cdots\right] \quad ...(i)$$

and

$$\left(\frac{d^2y}{dx^2}\right)_{x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \cdots\right] \quad ...(ii)$$

Here $h = 0.1$, $x_n = 1.6$, $\nabla y_n = 0.281$, $\nabla^2 y_n = -0.018$ etc.

Putting these values in (*i*) and (*ii*), we get

$$\left[\frac{dy}{dx}\right]_{1.6} = \frac{1}{0.1}\left[0.281 + \frac{1}{2}(-0.018) + \frac{1}{3}(0.005) + \frac{1}{4}(0.002) + \frac{1}{5}(0.003) + \frac{1}{6}(0.002)\right] = 2.75$$

$$\left(\frac{d^2y}{dx^2}\right)_{1.6} = \frac{1}{(0.1)^2}\left[-0.018 + 0.005 + \frac{11}{12}(0.002) + \frac{5}{6}(0.003) + \frac{137}{180}(0.002)\right]$$

$$= -0.715.$$

Example 30.2. *The following data gives the velocity of a particle for 20 seconds at an interval of 5 seconds. Find the initial acceleration using the entire data :*

Time t (sec) :	0	5	10	15	20	
Velocity v (m/sec) :	0	3	14	69	228	(Anna, 2004)

Solution. The difference table is :

t	v	Δv	$\Delta^2 v$	$\Delta^3 v$	$\Delta^4 v$
0	0				
		3			
5	3		8		
		11		36	
10	14		44		24
		55		60	
15	69		104		
		159			
20	228				

An initial acceleration $\left(i.e. \dfrac{dv}{dt}\right)$ at $t = 0$ is required, we use Newton's forward formula :

$$\left(\frac{dv}{dt}\right)_{t=0} = \frac{1}{h}\left(\Delta v_0 - \frac{1}{2}\Delta^2 v_0 + \frac{1}{3}\Delta^3 v_0 - \frac{1}{4}\Delta^4 v_0 + \cdots\right)$$

$$\therefore \quad \left(\frac{dv}{dt}\right)_{t=0} = \frac{1}{5}\left[3 - \frac{1}{2}(8) + \frac{1}{3}(36) - \frac{1}{4}(24)\right] = \frac{1}{5}\ (3 - 4 + 12 - 6) = 1$$

Hence the initial acceleration is 1 m/sec^2.

Example 30.3. *A slider in a machine moves along a fixed straight rod. Its distance x cm. along the rod is given below for various values of the time t seconds. Find the velocity of the slider and its acceleration when t = 0.3 seconds.*

$t =$	0	0.1	0.2	0.3	0.4	0.5	0.6	
$x =$	30.13	31.62	32.87	33.64	33.95	33.81	33.24	(V.T.U, 2009)

Solution. The difference table is :

t	x	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
0	30.13						
		1.49					
0.1	31.62		– 0.24				
		1.25		– 0.24			
0.2	32.87		– 0.48		0.26		
		0.77		0.02		– 0.27	
0.3	33.64		– 0.46		– 0.01		0.29
		0.31		0.01		0.02	
0.4	33.95		– 0.45		0.01		
		– 0.14		0.02			
0.5	33.81		– 0.43				
		– 0.57					
0.6	33.24						

As the derivatives are required near the middle of the table, we use Stirling's formulae :

$$\left(\frac{dx}{dt}\right)_{t_0} = \frac{1}{h}\left(\frac{\Delta x_0 + \Delta x_{-1}}{2}\right) - \frac{1}{6}\left(\frac{\Delta^3 x_{-1} + \Delta^3 x_{-2}}{2}\right) + \frac{1}{30}\left(\frac{\Delta^5 x_{-2} + \Delta^5 x_{-3}}{2}\right) + \ldots \qquad \ldots(i)$$

$$\left(\frac{d^2x}{dt^2}\right)_{t_0} = \frac{1}{h^2}\left[\Delta^2 x_{-1} - \frac{1}{12}\Delta^4 x_{-2} + \frac{1}{90}\Delta^6 x_{-3} \cdots\right] \qquad ...(ii)$$

Here $h = 0.1$, $t_0 = 0.3$, $\Delta x_0 = 0.31$, $\Delta x_{-1} = 0.77$, $\Delta^2 x_{-1} = -0.46$ etc.

Putting these values in (*i*) and (*ii*), we get

$$\left(\frac{dx}{dt}\right)_{0.3} = \frac{1}{0.1}\left[\frac{0.31+0.77}{2} - \frac{1}{6}\left(\frac{0.01+0.02}{2}\right) + \frac{1}{30}\left(\frac{0.02-0.27}{2}\right) - \cdots\right] = 5.33$$

$$\left(\frac{d^2x}{dt^2}\right)_{0.3} = \frac{1}{(0.1)^2}\left[-0.46 - \frac{1}{12}(-0.01) + \frac{1}{90}(0.29) - \cdots\right] = -45.6$$

Hence the required velocity is 5.33 cm/sec and acceleration is – 45.6 cm/sec^2.

Example 30.4. *Using Bessel's formula, find f′ (7.5) from the following table :*

x :	7.47	7.48	7.49	7.50	7.51	7.52	7.53
f(x) :	0.193	0.195	0.198	0.201	0.203	0.206	0.208

(J.N.T.U., 2006)

Solution. Taking $x_0 = 7.50$, $h = 0.1$, we have $p = \dfrac{x - x_0}{h} = \dfrac{x - 7.50}{0.01}$

The difference table is :

x	p	y_p	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
7.47	– 3	0.193						
			0.002					
7.48	– 2	0.195		0.001				
			0.003		– 0.001			
7.49	– 1	0.198		0.000		0.000		
			0.003		– 0.001		0.003	
7.50	0	0.201		– 0.001		0.003		– 0.01
			0.002		0.002		– 0.007	
7.51	1	0.203		0.001		– 0.004		
			0.003		– 0.002			
7.52	2	0.206		– 0.001				
			0.002					
7.53	3	0.208						

Bessel's formula (p. 550) is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\cdot\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{\left(p - \frac{1}{2}\right)p(p-1)}{3!}\cdot\Delta^3 y_{-1}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\cdot\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \frac{\left(p - \frac{1}{2}\right)(p+1)p(p-1)(p-2)}{5!}\cdot\Delta^5 y_{-2}$$

$$+ \frac{(p+2)p(p+1)p(p-1)(p-2)(p-3)}{6!}\cdot\frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2} + \cdots \qquad ...(i)$$

Since $\quad p = \dfrac{x - x_0}{h}, \quad \therefore \quad \dfrac{dp}{dx} = \dfrac{1}{h}$ and $\dfrac{dy}{dx} = \dfrac{dy}{dp}\cdot\dfrac{dp}{dx} = \dfrac{1}{h}\dfrac{dy}{dp}$

Differentiating (*i*) w.r.t. p and putting $p = 0$, we get

$$\left(\frac{dy}{dx}\right)_{7.5} = \frac{1}{h}\left(\frac{dy}{dp}\right)_{p=0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{h}(\Delta^2 y_{-1} + \Delta^2 y_0) + \frac{1}{12}\Delta^3 y_{-1} + \frac{1}{24}(\Delta^4 y_{-2} + \Delta^4 y_{-1})\right.$$

$$\left. - \frac{1}{120}\Delta^5 y_{-2} - \frac{1}{240}(\Delta^6 y_{-3} + \Delta^6 y_{-2})\right]$$

$$\left(\frac{dy}{dx}\right)_{7.5} = \frac{1}{0.01}\left[0.002 - \frac{1}{4}(-0.001 + 0.001) + \frac{1}{12}(0.002)^1 + \frac{1}{24}(-0.004 + 0.003) - \frac{1}{120}(-0.007)\frac{-1}{240}(0.010 + 0)\right]$$

$[\because \quad \Delta^6 y_{-2} = 0]$

$$= 0.2 + 0 + 0.01666 - 0.00583 + 0.00416 = 0.223.$$

Example 30.5. *Find $f'(0)$ from the following data :*

x :	3	5	11	27	34
$f(x)$:	−13	23	899	17315	35606

Solution. As the values of x are not equi-spaced, we shall use Newton's divided difference formula. The divided difference table is

x	$f(x)$	1st div. diff.	2nd div. diff.	3rd div. diff.	4th div. diff.
3	−13				
		18			
5	23		16		
		146		0.998	
11	899		39.96		0.0002
		1025		1.003	
27	17315		69.04		
		2613			
34	35606				

Fifth difference being zero, Newton's divided difference formula is

$$f(x) = f(x_0) + (x - x_0) f(x_0 - x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + (x - x_0)(x - x_1) \times (x - x_2)(x - x_3) f(x_0, x_1, x_2 x_3, x_4)$$

Differentiating it w.r.t. x, we get

$$f'(x) = f(x_0, x_1) + (2x - x_0 - x_1) f(x_0, x_1, x_2) + [3x^2 - 2x(x_0 + x_1 + x_2) + (x_0x_1 + x_1x_2 + x_2x_0)] \times f(x_0, x_1, x_2, x_3) + [4x^3 - 3x^2(x_0 + x_1 + x_2 + x_3) + 2x(x_0x_1 + x_1x_2 + x_2x_3 + x_3x_0 + x_1x_3 + x_0x_2) - x_0x_1x_2 + x_1x_2x_3 + x_2x_3x_0 + x_0x_1x_3] f(x_0, x_1, x_2, x_3, x_4)$$

Putting $x_0 = 3, x_1 = 5, x_2 = 11, x_3 = 27$ and $x = 10$, we obtain

$$f'(x) = 18 + 12 \times 16 + 23 \times 0.998 - 426 \times 0.0002 = 232.869.$$

30.3 MAXIMA AND MINIMA OF A TABULATED FUNCTION

Newton's forward interpolation formula is

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots$$

Differentiating it w.r.t. p, we get

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2 - 6p + 2}{6}\Delta^3 y_0 + \ldots \qquad \ldots(1)$$

For maxima or minima, $dy/dp = 0$. Hence equating the right hand side of (1) to zero and retaining only upto third differences, we obtain

$$\Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2 - 6p + 2}{6}\Delta^3 y_0 = 0$$

i.e.,

$$\left(\frac{1}{2}\Delta^3 y_0\right) p^2 + (\Delta^2 y_0 - \Delta^3 y_0) p + \left(\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0\right) = 0$$

Substituting the values of $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$ from the difference table, we solve this quadratic for p. Then the corresponding values of $x = x_0 + ph$ at which y is maximum or minimum.

Example 30.6. *Find the maximum and minimum value of y from the following data :*

x :	-2	-1	0	1	2	3	4
y :	2	-0.25	0	-0.25	2	15.75	56

(Anna, 2004)

Solution. The difference table is :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
-2	2					
		-2.25				
-1	-0.25		2.5			
		0.25		-3		
0	0		-0.5		6	
		-0.25		3		0
1	-0.25		2.5		6	
		0.25		9		0
2	2		11.5		6	
		13.75		15		
3	15.75		26.5			
		40.25				
4	56					

Taking $x_0 = 0$, we have $y_0 = 0$, $\Delta y_0 = -0.25$, $\Delta^2 y_0 = 2.5$, $\Delta^3 y_0 = 9$, $\Delta^4 y_0 = 6$.

Newton's forward difference formula for the first derivative gives

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 - \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 - \frac{4p^3-18p^2+22p-6}{4!}\Delta^4 y_0 - \cdots\right]$$

$$= \frac{1}{1}\left[-0.25 + \frac{2x-1}{2}(2.5) + \frac{1}{6}(3x^2-6x+2)(9) + \frac{1}{24}(4x^2-18x^2+22x-6)(6)\right]$$

$$= \frac{1}{1}\,[-0.25 + 2.5x - 1.25 + 4.5x^2 - 9x + 3 + x^3 - 4.5x^2 + 5.5x - 1.5] = x^3 - x$$

For y to be maximum or minimum, $\dfrac{dy}{dx} = 0$ *i.e.*, $x^3 - x = 0$

i.e., $$x = 0, 1, -1$$

Now $$\frac{d^2y}{dx^2} = 3x^2 - 1 = -\text{ve for } x = 0$$
$$= +\text{ve for } x = 1$$
$$= +\text{ve for } x = -1$$

Since $$y = y_0 + x\Delta y_0 + \frac{x(x-1)}{2!}\Delta^2 y_0 + \ldots, y(0) = 0$$

Thus y is maximum for $x = 0$, and maximum value $= y(0) = 0$.

Also y is minimum for $x = 1$ and minimum value $= y(0) = -0.25$.

PROBLEMS 30.1

1. Find $y'(0)$ and y'' from the following table :

x :	0	1	2	3	4	5
y :	4	8	15	7	6	2

2. Find the first and second derivatives of $f(x)$ at $x = 1.5$ if

x :	1.5	2.0	2.5	3.0	3.5	4.0
$f(x)$:	3.375	7.000	13.625	24.000	38.875	59.000

(S.V.T.U., 2007)

3. Find the first and second derivatives of the function tabulated below, at the point $x = 1.1$:

x :	1.0	1.2	1.4	1.6	1.8	2.0
y :	0	0.128	0.544	1.296	2.432	4.000

(U.P.T.U., 2010; Bhopal, 2009)

4. Given the following table of values of x and y

x :	1.00	1.05	1.10	1.15	1.20	1.25	1.30
y :	1.000	1.025	1.049	1.072	1.095	1.118	1.140

find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at (a) $x = 1.05$ (b) $x = 1.25$ (c) $x = 1.15$. (V.T.U., 2008)

5. For the following values of x and y, find the first derivative at $x = 4$.

x :	1	2	4	8	10
y :	0	1	5	21	27

(J.N.T.U., 2009)

6. From the following table, find the values of dy/dx and d^2y/dx^2 at $x = 2.03$.

x :	1.96	1.98	2.00	2.02	2.04
y :	0.7825	0.7739	0.7651	0.7563	0.7473

(Anna, 2005)

7. Find the value of cos 1.74 from the following table :

x :	1.7	1.74	1.78	1.82	1.86
$\sin x$:	0.9916	0.9857	0.9781	0.9691	0.9584

(J.N.T.U., 2009)

8. The distance covered by an athlete for the 50 metre is given in the following table :

Time (sec) :	0	1	2	3	4	5	6
Distance (metre) :	0	2.5	8.5	15.5	24.5	36.5	50

Determine the speed of the athlete at $t = 5$ sec. correct to two decimals. (U.P.T.U., 2009)

9. The following data gives corresponding values of pressure and specific volume of a superheated stream.

v :	2	4	6	8	10
p :	105	42.7	25.3	16.7	13

Find the rate of change of

(*i*) pressure with respect to volume when $v = 2$,

(*ii*) volume with respect to pressure when $p = 105$.

10. The table below reveals the velocity v of a body during the specific time t, find its acceleration at $t = 1.1$?

t :	1.0	1.1	1.2	1.3	1.4
v :	43.1	47.7	52.1	56.4	60.8

(J.N.T.U., 2009)

11. The elevation above a datum line of 7 points of a road is given below :

x :	0	300	600	900	1200	1500	1800
y :	135	149	157	183	201	205	193

Find the gradient of the road at the middle point.

12. A rod is rotating in a plane. The following table gives the angle θ (radians) through which the rod has turned for various values of the time t second.

t :	0	0.2	0.4	0.6	0.8	1.0	1.2
θ :	0	0.12	0.49	1.12	2.02	3.20	4.67

Calculate the angular velocity and the angular acceleration of the rod, when $t = 0.6$ second. (V.T.U., 2004)

13. Find the value of $f'(x)$ at $x = 0.4$ from the following table using Bessel's formula

x :	0.01	0.02	0.03	0.04	0.05	0.06
$f(x)$:	0.1023	0.1047	0.1071	0.1096	0.1122	0.1148

14. If $y = f(x)$ and y_n denotes $f(x_0 + nh)$, prove that, if powers of h above h^6 be neglected.

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{3}{4h}\left[(y_1 - y_{-1}) - \frac{1}{5}(y_2 - y_{-2}) + \frac{1}{45}(y_3 - y_{-3})\right]$$ (U.P.T.U., 2006)

[**Hint:** Differentiate Stirling's formula w.r.t. x, and put $x = 0$]

15. Find the value of $f'(8)$ from the table given below :

x :	6	7	9	12
$f(x)$:	1.556	1.690	1.908	2.158

(Anna, 2007)

16. Find the $f'(6)$ from the following data :

x :	0	2	3	4	7	8
$f(x)$:	4	26	58	112	466	922

(J.N.T.U., 2009 ; U.P.T.U., 2008)

17. Find the maximum and minimum values of y from the following table :

x :	0	1	2	3	4	5
$f(x)$:	0	0.25	0	2.25	16	56.25

18. Find the value of x for which $f(x)$ is minimum, using the table

x :	9	10	11	12	13	14
$f(x)$:	1330	1340	1320	1250	1120	930

Also find the maximum value of $f(x)$?

30.4 NUMERICAL INTEGRATION

The process of evaluating a definite integral from a set of tabulated values of the integrand $f(x)$ is called *numerical integration*. This process when applied to a function of a single variable, is known as *quadrature*.

The problem of numerical integration, like that of numerical differentiation, is solved by representing $f(x)$ by an interpolation formula and then integrating it between the given limits. In this way, we can derive quadrature formula for approximate integration of a function defined by a set of numerical values only.

30.5 NEWTON-COTES QUADRATURE FORMULA

Let $$I = \int_a^b f(x)\,dx$$

where $f(x)$ takes the values $y_0, y_1, y_2, \ldots y_n$ for $x = x_0, x_1, x_2, \ldots x_n$. (Fig. 30.1)

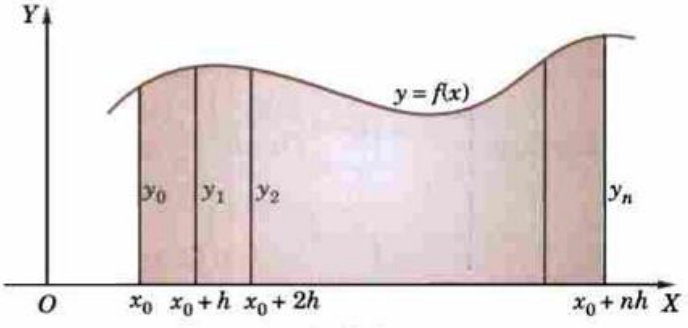


Fig. 30.1

Let us divide the interval (a, b) into n sub-intervals of width h so that $x_0 = a$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h, \ldots x_n = x_0 + nh = b$. Then

$$I = \int_{x_0}^{x_0+nh} f(x)\,dx = h \int_0^n f(x_0 + rh)\,dr,$$

putting $x = x_0 + rh$, $dx = hdr$

$$= h \int_0^n \left[y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0 + \frac{r(r-1)(r-2)(r-3)(r-4)}{5!}\Delta^5 y_0 + \frac{r(r-1)(r-2)(r-3)(r-4)(r-5)}{6!}\Delta^6 y_0 + \cdots \right] dr$$

[By Newton's forward interpolation formula]

Integrating term by, we obtain

$$\int_{x_0}^{x_0+nh} f(x)\,dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 \right] + \left(\frac{n^4}{5} - \frac{3n^3}{2} + \frac{11n^2}{3} - 3n \right)\frac{\Delta^4 y_0}{4!} + \left(\frac{n^5}{6} - 2n^4 + \frac{35n^3}{4} - \frac{50n^2}{3} + 12n \right)\frac{\Delta^5 y_0}{5!} + \left(\frac{n^6}{7} - \frac{15n^5}{6} + 17n^4 - \frac{225n^3}{4} + \frac{274n^2}{3} - 60n \right)\frac{\Delta^6 y_0}{6!} + \cdots \Bigg] \quad \ldots(A)$$

This is known as *Newton-Cotes quadrature formula*. From this general formula, we deduce the following important quadrature rules by taking $n = 1, 2, 3 \ldots$

30.6 TRAPEZOIDAL RULE

Putting $n = 1$ in (A) § 30.5 and taking the curve through (x_0, y_0) and (x_1, y_1) as a straight line *i.e.* a polynomial of first order so that differences of order higher than first become zero, we get

$$\int_{x_0}^{x_0+h} f(x)\,dx = h\left(y_0 + \frac{1}{2}\Delta y_0\right) = \frac{h}{2}(y_0 + y_1)$$

Similarly $$\int_{x_0}^{x_0+2h} f(x)\,dx = h\left(y_1 + \frac{1}{2}\Delta y_1\right) = \frac{h}{2}(y_1 + y_2)$$

.....

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}(y_{(n-1)} + y_n)$$

Adding these n integrals, we obtain

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})]$$

This is known as the ***trapezium rule.***

Obs. The area of each strip (trapezium) is found separately. Then the area under the curve and the *ordinates* at x_0 and $x_0 + nh$ is approximately equal to the areas of the trapeziums.

30.7 SIMPSON'S ONE-THIRD RULE

Putting $n = 2$ in (A) above and taking the curve through (x_0, y_0), (x_1, y_1) and (x_2, y_2) as a parabola *i.e.*, a polynomial of second order so that differences of order higher than second vanish, we get

$$\int_{x_0}^{x_0+2h} f(x)\,dx = 2h\,(y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0)\frac{h}{3}\,(y_0 + 4y_1 + y_2)$$

Similarly, $$\int_{x_0+2h}^{x_0+nh} f(x)\,dx = \frac{h}{3}\,(y_2 + 4y_3 + y_4) \quad \text{when}$$

.....

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n),\ n \text{ being even.}$$

Adding all these integrals, we have (when n is even)

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3) + \dots + y_{n-1}) + 2(y_2 + y_4 + \dots + y_{n-2})]$$

This is known as the *Simpson's one-third rule* or simply *Simpson's rule* and is most commonly used.

Obs. While applying *Simpson's 1/3rd rule, the given interval must be divided into even number of equal subintervals*, since we find the area of two strips at a time.

30.8 SIMPSON'S THREE-EIGHTH RULE

Putting $n = 3$ in (A) above and taking the curve through (x_i, y_i): $i = 0, 1, 2, 3$ as a polynomial of third order so that differences above the third order vanish, we get

$$\int_{x_0}^{x_0+3h} f(x)\,dx = 3h\left(y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right)$$

$$= \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3)$$

Similarly,

$$\int_{x_0+3h}^{x_0+nh} f(x)\,dx = \frac{3h}{8}(y_3 + 3y_4 + 3y_5 + y_6) \text{ and so on.}$$

Adding all such expressions from x_0 to $x_0 + nh$, where n is a multiple of 3, we obtain

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \dots + y_{n-1}) + 2(y_3 + y_6 + \dots + y_{n-3}]$$

which is known as *Simpson's three-eighth rule.*

Obs. While applying *Simpson's 3/8th rule, the number of sub-intervals should be taken as multiple of 3.*

30.9 BOOLE'S RULE

Putting $n = 4$ in (A) above and neglecting all differences above the fourth, we obtain

$$\int_{x_0}^{x_0+4h} f(x)\,dx = 4h\left(y_0 + 2\Delta y_0 \frac{5}{3}\Delta^2 y_0 + \frac{2}{3}\Delta^3 y_0 + \frac{7}{90}\Delta^4 y_0\right)$$

$$= \frac{2h}{45}(7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_8)$$

Similarly

$$\int_{x_0+4h}^{x_0+8h} f(x)\,dx = \frac{2h}{45}(7y_4 + 32y_5 + 12y_6 + 32y_7 + 7y_8) \text{ and so on.}$$

Adding all these integrals from x_0 to $x_0 + nh$, where n is a multiple of 4, we get

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{2h}{45}(7y_0 + 32y_1 + 12y_2 + 32y_3 + 14y_4 + 32y_5 + 12y_6 + 32y_7 + 14y_8 + \ldots$$

This is known as *Boole's rule*.

Obs. While applying *Boole's rule*, the number of sub-intervals should be taken as a multiple of 4.

30.10 WEDDLE'S RULE

Putting $n = 6$ in (A) above and neglecting all differences above the sixth, we obtain

$$\int_{x_0}^{x_0+6h} f(x)\,dx = \left(y_0 + 3\Delta y_0 + \frac{9}{2}\Delta^2 y_0 + 4\Delta^3 y_0 + \frac{123}{60}\Delta^4 y_0 + \frac{11}{20}\Delta^5 x_0 + \frac{1}{6}\cdot\frac{41}{140}\Delta^6 y_0\right)$$

If we replace $\frac{41}{140}\Delta^6 y_0$ by $\frac{3}{10}\Delta^6 y_0$, the error made will be negligible.

$$\therefore \quad \int_{x_0}^{x_0+6h} f(x)\,dx = \frac{3h}{10}(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6)$$

Similarly

$$\int_{x_0+6h}^{x_0+12h} f(x)\,dx = \frac{3h}{10}(y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}) \text{ and so on.}$$

Adding all these integrals from x_0 to $x_0 + nh$, where n is a multiple of 6, we get

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{10}(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + \ldots)$$

This is known as *Weddle's rule*.

Obs. While applying *Weddle's rule the number of sub-intervals should be taken as a multiple of 6*. Weddle's rule is generally more accurate than any of the others. Of the two Simpson rules, the 1/3 rule is better.

Example 30.7. *Evaluate* $\int_0^6 \frac{dx}{1+x^2}$ *by using (i) Trapezoidal rule,*

(i) Simpson's 1/3 rule, *(Mumbai, 2005)*
(ii) Simpson's 3/8 rule, *(J.N.T.U., 2008)*
(iii) Weddle's rule and compare the results with its actual value. *(V.T.U., 2008)*

Solution. Divide the interval (0, 6) into six parts each of width $h = 1$. The values of $f(x) = \frac{1}{1+x^2}$ are given below :

x	0	1	2	3	4	5	6
$f(x)$	1	0.5	0.2	0.1	0.05884	0.0385	0.027
$= y$	y_0	y_1	y_2	y_3	y_4	y_5	y_6

(*i*) By Trapezoidal rule,

$$\int_0^6 \frac{1}{1+x^2} = \frac{h}{2}[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{1}{2}[(1 + 0.027) + 2(0.5 + 0.2 + 0.1 + 0.0588 + 0.0385)] = 1.4108.$$

(*ii*) By Simpson's 1/3 rule,

$$\int_0^6 \frac{1}{1+x^2} = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{1}{3}[(1 + 0.027) + 4(0.5 + 0.1 + 0.0385) + 2(0.2 + 0.0588)] = 1.3662.$$

(*iii*) By Simpson's 3/8 rule,

$$\int_0^6 \frac{1}{1+x^2} = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3}{8}[(1 + 0.027) + 3(0.5 + 0.2 + 0.0588 + 0.0385) + 2(0.1)] = 1.3571.$$

(*iv*) By Weddle's rule,

$$\int_0^6 \frac{1}{1+x^2} = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4\ 5y_5 + y_6]$$

$$= 0.3[1 + 5(0.5) + 0.2 + 6(0.1) + 0.0588 + 5(0.0385) + 0.027] = 1.3735.$$

Also, $\displaystyle\int_0^6 \frac{dx}{1+x^2} = \left|\tan^{-1} x\right|_0^6 = 1.4056$

Obs. This shows that the value of the integral found by Weddle's rule is the nearest to the actual value followed by its value given by Simpson's 1/3rd.

Example 30.8. *Use the Trapezoidal rule to estimate the integral* $\int_0^2 e^{x^2}\,dx$ *taking 10 intervals.*

(U.P.T.U., 2008)

Solution. Let $y = e^{x^2}$, $h = 0.2$ and $n = 10$.

The values of x and y are as follows :

x :	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
y :	1	1.0408	1.1735	1.4333	1.8964	2.1782	4.2206	7.0993	12.9358	25.5337	54.5981
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

By Trapezoidal rule, we have

$$\int_0^1 e^{x^2}\,dx = \frac{h}{2}[(y_0 + y_{10}) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9)]$$

$$= \frac{0.2}{2}[(1 + 54.5981) + 2(1.0408 + 1.1735 + 1.4333 + 1.8964$$
$$+ 2.178 + 4.2206 + 7.0993 + 12.9358 + 25.5337)]$$

Hence $\displaystyle\int_0^2 e^{x^2}\,dx = 17.0621.$

Example 30.9. *Use Simpson's 1/3rd rule to find* $\int_0^{0.6} e^{-x^2}\,dx$ *by taking seven ordinates.*

(V.T.U., 2011 ; Bhopal, 2009)

Solution. Divide the interval (0, 0.6) into six parts each of width $h = 0.1$. The values of $y = f(x) = e^{-x^2}$ are given below :

x	0	0.1	0.2	0.3	0.4	0.5	0.6
x^2	0	0.01	0.04	0.09	0.16	0.25	0.36
y	1	0.9900	0.9608	0.9139	0.8521	0.7788	0.6977
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 1/3rd rule, we have

$$\int_0^{0.6} e^{-x^2}\,dx = \frac{h}{3}\,[(y_0 + y_6) + 4(y_1 + y_3 + y_5 + 2(y_2 + y_4)]$$

$$= \frac{0.1}{3}\,[(1 + 0.6977) + 4(0.99 + 0.9139 + 0.7788 + 2(0.9608 + 0.8521)]$$

$$= \frac{0.1}{3}\,[1.6977 + 10.7308 + 3.6258] = \frac{0.1}{3}\,(16.0543) = 0.5351.$$

Example 30.10. *Compute the value of* $\int_{0.2}^{1.4} (\sin x - \log x + e^x)\,dx$ *using Simpson's* $\frac{3}{8}$*th rule.*

(Mumbai, 2005)

Solution. Let $y = \sin x - \log_e x + e^x$ and $h = 0.2$, $n = 6$.

The values of y are as given below :

x :	0.2	0.4	0.6	0.8	1.0	1.2	1.4
y :	3.0295	2.7975	2.8976	3.1660	3.5597	4.0658	4.4042
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's $\frac{3}{8}$ th rule, we have

$$\int_{0.2}^{1.4} y\,dx = \frac{3h}{8}\,[(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)]$$

$$= \frac{3}{8}\,(0.2)\,[7.7336 + 2(3.1660) + 3\,(13.3247)] = 4.053$$

Hence $\int_{0.2}^{1.4} (\sin x - \log_e x + e^x)\,dx = 4.053.$

Obs. Applications of Simpson's rule. If the various ordinates in §30.5 represent equispaced cross-sectional areas, then Simpson's rule gives the volume of the solid. As such, Simpson's rule is very useful to civil engineers for calculating the amount of earth that must be moved to fill a depression or make a dam. Similar if the ordinates denote velocities at equal intervals of time, the Simpson's rule gives the distance travelled. The following examples illustrate these applications.

Example 30.11. *The velocity v (km / min) of a moped which starts from rests, is given at fixed intervals of time t (min) as follows :*

t :	2	4	6	8	10	12	14	16	18	20
y :	10	18	25	29	32	20	11	5	2	0

Estimate approximately the distance covered in 20 minutes.

Solution. If s km be the distance covered in t (min), then $\frac{ds}{dt} = v$

$$\therefore \qquad \Big|\, s \,\Big|_{t=0}^{20} = \int_0^{20} v\,dt = \frac{h}{3}\,[X + 4.O + 2E], \text{ by Simpson's rule}$$

Hence $h = 2$, $v_0 = 0$, $v_1 = 10$, $v_2 = 18$, $v_3 = 25$ etc.

$$\therefore \qquad X = v_0 + v_{10} = 0 + 0 = 0$$

$$O = v_1 + v_3 + v_5 + v_7 + v_9 = 10 + 25 + 32 + 11 + 2 = 80$$
$$E = v_2 + v_4 + v_6 + v_8 = 18 + 29 + 20 + 5 = 72$$

Hence the required distance $= \left| s \right|_{t=0}^{20} = \frac{2}{3}(0 + 4 \times 80 + 2 \times 72)$

$= 309.33$ km.

Example 30.12. *The velocity v of a particle at distance s from a point on its linear path is given by the following table :*

s (m) :	0	2.5	5.0	7.5	10.0	12.5	15.0	17.5	20.0
v (m/sec) :	16	19	21	22	20	17	13	11	9

Estimate the time taken by the particle to traverse the distance of 20 metres, using Boole's rule. (U.P.T.U. 2007)

Solution. If t sec be the time taken to traverse a distance s (m) then $\frac{ds}{dt} = v$

or $$\frac{dt}{ds} = \frac{1}{v} = y \text{ (say)},$$

$\therefore$ then $$\left| t \right|_{s=0}^{s=20} = \int_0^{20} y\,ds$$

Here $h = 2.5$ and $n = 8$

Also $$y_0 = \frac{1}{16}, y_1 = \frac{1}{19}, y_2 = \frac{1}{4}, y_3 = \frac{1}{22}, y_4 = \frac{1}{20}, y_5 = \frac{1}{17}, y_6 = \frac{1}{13}, y_7 = \frac{1}{11}, y_8 = \frac{1}{9}$$

$\therefore$ by Boole's Rules, we have

$$\left| t \right|_{s=0}^{s=20} = \int_0^{20} y\,ds = \frac{2h}{45}\,[7y_0 + 32y_1 + 12y_2 + 32y_3 + 14y_4 + 32y_5 + 12y_6 + 32y_7 + 14y_8]$$

$$= \frac{2(2.5)}{45}\left[7\left(\frac{1}{16}\right) + 32\left(\frac{1}{19}\right) + 12\left(\frac{1}{21}\right) + 32\left(\frac{1}{22}\right) + 14\left(\frac{1}{20}\right) + 32\left(\frac{1}{17}\right) + 12\left(\frac{1}{13}\right) + 32\left(\frac{1}{11}\right) + 14\left(\frac{1}{9}\right)\right]$$

$$= \frac{1}{9}(12.11776) = 1.35$$

Hence the required time = 1.35 sec.

Example 30.13. *A solid of revolution is formed by rotating about the x-axis, the area between the x-ax the lines $x = 0$ and $x = 1$ and a curve through the points with the following co-ordinates*

x :	0.00	0.25	0.50	0.75	1.00
y :	1.0000	0.9896	0.9589	0.9089	0.8415

Estimate the volume of the solid formed using Simpson's rule. (Raipur, 200

Solution. Here $h = 0.25, y_0 = 1, y_1 = 0.9896, y_2 = 0.9589$, etc.

$\therefore$ Required volume of the solid generated

$$= \int_0^1 \pi y^2\,dx = \pi \cdot \frac{h}{3}\,[(y_0^2 + y_4^2) + 4(y_1^2 + y_3^2) + 2y_2^2]$$

$$= 0.25\,\frac{\pi}{3}\,[\{1 + (0.8415)^2\} + 4\{(0.9896)^2 + (0.9089)^2\} + 2(0.0589)^2]$$

$$= \frac{0.25 \times 3.1416}{3}\,[1.7081 + 7.2216 + 1.839] = 0.2618\,(10.7687) = 2.8192.$$

PROBLEMS 30.2

1. Evaluate $\int_0^1 \frac{dx}{1+x}$ applying

   (*i*) Trapezoidal rule (*J.N.T.U., 2009*) (*ii*) Simpson's 1/3rd rule

   (*iii*) Simpson's 3/8th rule. (*Mumbai, 2004*)

2. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using (*i*) Trapezoidal rule taking $h = 1/4$

   (*ii*) Simpson's 1/3rd rule taking $h = 1/4$. (*J.N.T.U., 2008*)

   (*iii*) Simpson's 3/8th rule taking $h = 1/6$. (*U.P.T.U, 2010 ; V.T.U., 2007*)

   (*iv*) Weddle's rule taking $h = 1/6$. (*Bhopal, 2009*)

   Hence compute an approximate value of π in each case.

3. Find an approximate value of $\log_e 5$ by calculating to 4 decimal places, by Simpson's 1/3 rule, $\int_0^5 \frac{dx}{4x+5}$, dividing the range into 10 equal parts. (*Anna., 2005*)

4. Evaluate $\int_0^6 x \sec x \, dx$ using eight intervals by Trapezoidal rule. (*U.P.T.U., 2009*)

5. Evaluate using Simpson's $\frac{1}{3}$rd rule (*i*) $\int_0^6 \frac{e^x}{1+x} dx$ (*U.P.T.U., 2006*)

   (*ii*) $\int_0^2 e^{-x^2} dx$ (Take $h = 0.25$). (*J.N.T.U., 2007*)

6. Evaluate using Simpson's 1/3rd rule $\int_0^1 \frac{dx}{x^3+x+1}$, choose step length 0.25. (*U.P.T.U., 2009*)

7. Evaluate using Simpson's 1/3rd rule, (*i*) $\int_0^\pi \sin x \, dx$ using 11 ordinates.

   (*ii*) $\int_0^{\pi/2} \sqrt{\cos\theta} \, d\theta$ taking 9 ordinates. (*V.T.U., 2009*)

8. Evaluate correct to 4 decimal places, by Simpson's $\frac{3}{8}$th rule

   (*i*) $\int_0^9 \frac{dx}{1+x^3}$ (*U.P.T.U., M. Tech., 2010*) (*ii*) $\int_0^{\pi/2} e^{\sin x} dx$ (*U.P.T.U., 2007*)

9. Given that

x :	4.0	4.2	4.4	4.6	4.8	5.0	5.2
$\log x$:	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487

   evaluate $\int_4^{5.2} \log x \, dx$ by

   (*a*) Trapezoidal r[illegible] (*b*) Simpson's 1/3rd rule, (*Kerala, 2003*)

   (*c*) Simp[illegible] 3/8 [illegible] T.U., 2006) (*d*) Weddle's rule (*V.T.U., 2008*)

10. Use Boo[illegible] compute $\int_0^{\pi/2} \sqrt{\sin x} \, dx$. (*U.P.T.U., 2008*)

11. The [illegible] (*t*) as a function of time :

t	5	6	7
$f(t)$	78	70	60

   Using Sim[illegible] (*J.N.T.U., 2007*)

12. A curve i[illegible] the following table :

x :	3.5	4
y :	2.6	[illegible]

   Estimate the area bounded by the cu[illegible] = 4. (*Bhopal, 2007*)

13. A river is 80 ft wide. The depth d in feet at a distance x ft. from one bank is given by the following table :

x :	0	10	20	30	40	50	60	70	80
y :	0	4	7	9	12	15	14	8	3

Find approximately the area of the cross-section. *(Rohtak, 2005)*

14. A curve is drawn to pass through the points given by following table :

x :	1	1.5	2	2.5	3	3.5	4
y :	2	2.4	2.7	2.8	3	2.6	2.1

Using Weddle's rule, estimate the area bounded by the curve, the x-axis and the lines $x = 1, x = 4$. *(V.T.U., 2011 S)*

15. A body is in the form of a solid of revolution. The diameter D in cms of its sections at distances x cm. from the one end are given below. Estimate the volume of the solid.

x :	0	2.5	5.0	7.5	10.0	12.5	15.0
D :	5	5.5	6.0	6.75	6.25	5.5	4.0

16. The velocity v of a particle at distances s from a point on its path is given by the table :

s ft :	0	10	20	30	40	50	60
v ft/sec :	47	58	64	65	61	52	38

Estimate the time taken to travel 60 ft. by using Simpson's 1/3 rule. *(U.P.T.U., 2007)*

Compare the result with Simpson's 3/8 rule. *(Madras, 2003)*

17. The following table gives the velocity v of a particle at time t :

t (second) :	0	2	4	6	8	10	12
v (m/sec) :	4	6	16	34	60	94	136

Find the distance moved by the particle in 12 seconds and also the acceleration at $t = 2$ sec. *(S.V.T.U., 2007)*

18. A rocket is launched from the ground. Its acceleration is registered during the first 80 seconds and is given in the table below. Using Simpson's $\frac{1}{3}$rd rule, find the velocity of the rocket at $t = 80$ seconds.

t sec :	0	10	20	30	40	50	60	70	80
f (cm/sec^2) :	30	31.63	33.34	35.47	37.75	40.33	43.25	46.69	50.67

(Mumbai, 2004)

19. A reservoir discharging water through sluices at a depth h below the water surface has a surface area A for various values of h as given below :

h (ft.) :	10	11	12	13	14
A (sq.ft.) :	950	1070	1200	1350	1530

If t denotes time in minutes, the rate of fall of the surface is given by $dh/dt = -48\sqrt{h}/A$.

Estimate the time taken for the water level to fall from 14 to 10 ft. above the sluices.

30.12 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 30.3

Select the correct answer or fill up the blanks in the following questions :

1. The value of $\int_0^1 \frac{dx}{1+x}$ by Simpson's rule is

(*a*) 0.96315 (*b*) 0.63915 (*c*) 0.69315 (*d*) 0.69..

2. Using forward differences, the formula for $f'(a) = \ldots$

3. In application of Simpson's 1/3 rule, the interval h for closer approximation should be ...

4. $f(x)$ is given by

x :	0	0.5	1
$f(x)$:	1	0.8	0.5,

then using Trapezoidal rule, the value of $\int_0^1 f(x)\, dx$ is ...

5. If

x :	0	0.5	1	1.5	2
$f(x)$:	0	0.25	1	2.25	4

then the value of $\int_0^2 f(x)\,dx$ by Simpson's 1/3rd rule is ...

6. Simpson's 3/8th rule states that ...

7. For the data :

t :	3	6	9	12
$y(t)$:	–1	1	2	3,

the value of $\int_3^{12} y(t)\,dt$ when computed by Simpson's $\frac{1}{3}$rd rule is

(*a*) 15 (*b*) 10 (*c*) 0 (*d*) 5.

8. While evaluating a definite integral by Trapezoidal rule, the accuracy can be increased by taking ...

9. The value of $\int_0^1 \frac{dx}{1+x^2}$ by Simpson's 1/3rd rule (taking $n = 1/4$) is ...

10. For the data:

x :	2	4	6	8
$f(x)$:	3	5	6	7,

$\int_2^8 f(x)\,dx$ when found by the Trapezoidal rule is

(*a*) 18 (*b*) 25 (*c*) 16 (*d*) 32.

11. The expression for $\left(\frac{dy}{dx}\right)_{x=x_0}$ using backward differences is ...

12. The number of strips required in Weddle's rule is ...

13. The number of strips required in Simpson's 3/8th rule is a multiple of

(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 6.

14. If $y_0 = 1, y_1 = \frac{16}{17}, y_2 = \frac{4}{5}, y_3 = \frac{16}{25}, y_4 = \frac{1}{2}$ and $h = \frac{1}{4}$, then using Trapezoidal rule, $\int_0^4 y\,dx = ...$

15. Using Simpson's $\frac{1}{3}$rd rule, $\int_0^1 \frac{dx}{x} = ...$ (taking $n = 4$).

16. If $y_0 = 1, y_1 = 0.5, y_2 = 0.2, y_3 = 0.1, y_4 = 0.06, y_5 = 0.04$ and $y_6 = 0.03$, then $\int_0^4 y\,dx$ by Simpson's $\frac{3}{8}$th, rule is = ...

17. If $f(0) = 1, f(1) = 2.7, f(2) = 7.4, f(3) = 20.1, f(4) = 54.6$ and $h = 1$, then $\int_0^4 f(x)\,dx$ by Simpson's $\frac{1}{3}$rd rule = ...

18. Simpson's 1/3rd rule and direct integration give the same result if ...

19. To evaluate $\int_{x_0}^{x_n} y\,dx$ by Simpson's 1/3rd rule as well as Simpson's 3/8th rule, the number of intervals should be ..... and ..... respectively.

20. Whenever Trapezoidal rule is applicable, Simpson's 1/3rd rule can also be applied. (True or False)

CHAPTER 31

Difference Equations

1. Introduction. 2. Definition. 3. Formation of difference equations. 4. Linear difference equations. 5. Rules for finding complementary function. 6. Rules for finding particular integral. 7. Simultaneous difference equations with constant coefficients. 8. Application to deflection of a loaded string. 9. Objective Type of Questions.

31.1 INTRODUCTION

Difference calculus also forms the basis of Difference equations. These equations arise in all situations in which sequential relation exists at various discrete values of the independent variable. The need to work with discrete functions arises because there are physical phenomena which are inherently of a discrete nature. In control engineering, it often happens that the input is in the form of discrete pulses of short duration. The radar tracking devices receive such discrete pulses from the target which is being tracked. As such differences equations arise in the study of electrical networks, in the theory of probability, in statistical problems and many other fields.

Just as the subject of Differential equations grew out Differential calculus to become one of the most powerful instruments in the hands of a practical mathematician when dealing with continuous processes in nature, so the subject of Difference equations is forcing its way to the fore for the treatment of discrete processes. Thus the difference equations may be thought of as the discrete counterparts of the differential equations.

31.2 DEFINITION

(1) A difference equation *is a relation between the differences of an unknown function at one or more general values of the argument.*

Thus $\Delta y_{(n+1)} + y_{(n)} = 2$ ...(1) and $\Delta y_{(n+1)} + \Delta^2 y_{(n-1)} = 1$ ...(2)

are difference equations.

An alternative way of writing a difference equation is as under :

Since $\Delta y_{(n+1)} = y_{(n+2)} - y_{(n+1)}$, therefore (1) may be written as

$$y_{(n+2)} - y_{(n+1)} + y_{(n)} = 2 \qquad ...(3)$$

Also since, $\Delta^2 y_{(n-1)} = y_{(n+1)} - 2y_{(n)} + y_{(n-1)}$, therefore (2) takes the form :

$$y_{(n+2)} - 2y_{(n)} + y_{(n-1)} = 1 \qquad ...(4)$$

Quite often, difference equations are met under the name of *recurrence relations.*

(2) Order of a difference equation *is the difference between the largest and the smallest arguments occuring in the difference equation divided by the unit of increment.*

Thus (3) above is the *second order*, for

$$\frac{\text{largest argument} - \text{smallest argument}}{\text{unit of increment}} = \frac{(n+2)-n}{1} = 2,$$

and (4) is of the *third order*, for $\dfrac{(n+2)-(n-1)}{1} = 3.$

Obs. *While finding the order of a difference equation, it must always be expressed in a form free of Δs,* for the highest power of Δ does not give order of the difference equation.

(3) Solution *of a difference equation is an expression for $y_{(n)}$ which satisfies the given difference equation.*

The general solution *of a difference equation is that in which the number of arbitrary constants is equal to the order of the difference equation.*

A particular solution or **particular integral** *is that solution which is obtained from the general solution by giving particular values to the constants.*

31.3 FORMATION OF DIFFERENCE EQUATIONS

The following examples illustrate the way in which difference equations arise and are formed.

Example 31.1. *Form the difference equation corresponding to the family of curves*

$$y = ax + bx^2 \qquad ...(i)$$

Solution. We have $\Delta y = a\Delta(x) + b\Delta(x^2) = a\,(x + 1 - x) + b[(x + 1)^2 - x^2]$

$$= a + b(2x + 1) \qquad ...(ii)$$

and $$\Delta^2 y = 2b[(x + 1) - x] = 2b \qquad ...(iii)$$

To eliminate a and b, we have from (iii), $b = \frac{1}{2}\Delta^2 y$

and form (ii), $a = \Delta y - b\,(2x + 1) = \Delta y - \frac{1}{2}\Delta^2 y\,(2x + 1)$

Substituting these values of a and b in (i), we get

$$y = \left[\Delta y - \frac{1}{2}\Delta^2 y\,(2x + 1)\right]x + \frac{1}{2}\Delta^2 y\,.\,x^2$$

or $$(x^2 + x)\,\Delta^2 y - 2x\,\Delta y + 2y = 0$$

This is the desired difference equation which may equally well be written in terms of E as

or $$(x^2 + x)\,y_{x+2} - (2x^2 + 4x)y_{x+1} + (x^2 + 3x + 2)y_x = 0.$$

Example 31.2. *From $y_n = A2^n + B(-3)^n$, derive a difference equation by eliminating the constants.*

Solution. We have $y_n = A.2^n + B(-3)^n$, $y_{n+1} = 2A.2^n - 3B(-3)^n$

and $y_{n+2} = 4A.2^n + 9B(-3)^n$.

Eliminating A and B, we get

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 2 & -3 \\ y_{n+2} & 4 & 9 \end{vmatrix} = 0 \quad \text{or} \quad y_{n+2} + y_{n+1} - 6y_n = 0$$

which is the desired difference equation.

PROBLEMS 31.1

1. Write the difference equation $\Delta^3 y_x + \Delta^2 y_x + \Delta y_x + y_x = 0$ in the subscript notation.
2. Assuming $\frac{\log(1 - z)}{1 + z} = y_0 + y_1 z + y_2 z^2 + \ldots\ldots + y_n z^n, \ldots\ldots$, find the difference equations satisfied by y_n.
3. Form a difference equation by eliminating arbitrary constant from $u_n = a2^{n+1}$. *(Anna, 2008)*
4. Find the difference equation satisfied by
   (i) $y = a/x + b$ *(Tiruchirapalli, 2001)* (ii) $y = ax^2 - bx$.
5. Derive the difference equations in each of the following cases :
   (i) $y_n = A.3^n + B.5^n$ (ii) $y_n = (A + Bx)\,2^x$. *(Madras, 2001)*
6. Form the difference equations generated by
   (i) $y_n = ax + b2^x$ (ii) $y_n = a2^n + b(-2)^n$ (iii) $y_x = a2^x + b3^x + c$.

31.4 LINEAR DIFFERENCE EQUATIONS

(1) Def. *A **linear difference equation** is that in which y_{n+1}, y_{n+2}, etc. occur to the first degree only and are not multiplied together.*

*A **linear difference equation with constant coefficient** is of the form*

$$y_{n+r} + a_1 y_{n+r-1} + a_2 y_{n+r-2} + \dots + a_r y_n = f(n) \qquad \dots(1)$$

where $a_1, a_2, \dots a_r$ are constants.

Now we shall deal with linear difference equations with constant coefficients only. Their properties are analogous to those of linear differential equations with constant co-efficients.

(2) Elementary properties. If $u_1(n), u_2(n), \dots, u_r(n)$ be r independent solution of the equation

$$y_{n+r} + a_1 y_{n+r-1} + \dots + a_r y_n = 0 \qquad \dots(2)$$

then its complete solution is $U_n = c_1 u_1(n) + \dots + c_r u_r(n)$

where $c_1, c_2, \dots, c_r$ are arbitrary constants.

If V_n is a particular solution of (1), then the complete solution of (1) is $y_n = U_n + V_n$. *The part U_n is called the **complementary function (C.F.)** and the part V_n is called the **particular integral (P.I.)** of (1).*

*Thus the **complete solution (C.S.)** of (1) is* $\mathbf{y_n = C.F. + P.I.}$

31.5 RULES FOR FINDING THE COMPLEMENTARY FUNCTION

(*i.e., rules to solve a linear difference equation with constant coefficients having right hand side zero*).

(1) To begin with, consider the first order linear equation $y_{n+1} - \lambda y_n = 0$, where λ is a constant.

Rewriting it as $\dfrac{y_{n+1}}{\lambda^{n+1}} - \dfrac{y_n}{\lambda^n} = 0$, we have $\Delta\left(\dfrac{y_n}{\lambda^n}\right) = 0$, which gives $y_n/\lambda^n = c$, a constant.

Thus the solution of $(E - \lambda)\, y_n = 0$ is $\mathbf{y_n = c.\lambda^n}$.

(2) Now consider the *second order linear equation* $y_{n+2} + a y_{n+1} + b y_n = 0$ which in *symbolic form is*

$$(E^2 + aE + b) y_n = 0 \qquad \dots(1)$$

Its symbolic co-efficient equated to zero *i.e.*, $E^2 + aE + b = 0$

is called the *auxiliary equation.* Let its roots be λ_1, λ_2.

Case I. *If these roots are real and distinct,* then (1) is to equivalent to

$$(E - \lambda_1)(E - \lambda_2)\, y_n = 0 \qquad \dots(2)$$

$$(E - \lambda_2)(E - \lambda_1)\, y_n = 0 \qquad \dots(3)$$

If y_n satisfies the subsidiary equation $(E - \lambda_1) y_n = 0$, then it will also satisfy (3).

Similarly, if y_n satisfies the subsidiary equation $(E - \lambda_2) y_n = 0$, then it will also satisfy (2).

$\therefore$ it follows that we can derive two independent solutions of (1), by solving the two subsidiary equations

$$(E - \lambda_1)\, y_n = 0 \quad \text{and} \quad (E - \lambda_2)\, y_n = 0$$

Their solutions are respectively, $y_n = c_1(\lambda_1)^n$ and $y_n = c_2(\lambda_2)^n$

where c_1 and c_2 are arbitrary constants.

Thus the general solution of (1) is $y_n = c_1(\lambda_1)^n + c_2(\lambda_2)^n$

Case II. *If the roots are real and equal* (*i.e.*, $\lambda_1 = \lambda_2$), then (2) becomes

$$(E - \lambda_1)^2 y_n = 0 \qquad \dots(4)$$

Let $\qquad y_n = (\lambda_1)^n z_n$

where z_n is a new dependent variable. Then (4) takes the form

$$(\lambda_1)^{n+2} z_{n+2} - 2\lambda_1(\lambda_1)^{n+1} z_{n+1} + \lambda_1^2 . (\lambda_1)^n z_n = 0$$

or $\qquad z_{n+2} - 2z_{n+1} + z_n = 0 \quad i.e., \quad \Delta^2 z_n = 0$

$\therefore \qquad z_n = c_1 + c_2 n$, where c_1, c_2 are arbitrary constants.

Thus the solution of (1) becomes $y_n = (c_1 + c_2 n)(\lambda_1)^n$.

Case III. *If the roots are imaginary,* (*i.e.* $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$) then the solution of (1) is

$$y_n = c_1(\alpha + i\beta)^n + c_2(\alpha - i\beta)^n \qquad [\text{Put } \alpha = r\cos\theta \text{ and } \beta = r\sin\theta]$$

$$= r^n\,[c_1(\cos n\theta + i\sin n\theta) + c_2(\cos n\theta - i\sin n\theta)]$$

$$= r^n [A_1 \cos n\theta + A_2 \sin n\theta]$$

where A_1, A_2 are arbitrary constants are $r = \sqrt{(\alpha^2 + \beta^2)}$, $\theta = \tan^{-1}(\beta/\alpha)$.

(3) *In general, to solve the equation* $y_{n+r} + a_1 y_{n+r-2} + \ldots + a_r y_n = 0$ where a's are constants :

(*i*) Write the equation in the *symbolic form* $(E^r + a_1 E^{r-1} + \ldots + a_r) y_n = 0$.

(*ii*) Write down the *auxiliary equation i.e.*, $E^r + a_1 E^{r-1} \ldots + a_r = 0$ and solve it for E.

(*iii*) Write the solution as follows :

Roots of A.E.	*Solution, i.e. C.F.*
1. $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots$ (real and distinct roots)	$c_1(\lambda_1)^n + c_2(\lambda_2)^n + c_3(\lambda_3)^n + \ldots\ldots$
2. $\lambda_1, \lambda_1, \lambda_3, \ldots\ldots$ (2 real and equal roots)	$(c_1 + c_2 n)(\lambda_1)^n + c_3(\lambda_3)^n + \ldots\ldots$
3. $\lambda_1, \lambda_1, \lambda_1, \ldots\ldots$ (3 real and equal roots)	$(c_1 + c_2 n + c_3 n^2)(y_1)^n + \ldots\ldots$
4. $\alpha + i\beta, \alpha - i\beta, \ldots\ldots$ (a pair of imaginary roots)	$r^n (c_1 \cos\theta + c_2 \sin n\theta)$ where $r = \sqrt{(\alpha^2 + \beta^2)}$ and $\theta = \tan^{-1}(\beta/\alpha)$

Example 31.3. *Solve the difference equation* $u_{n+3} - 2u_{n+2} - 5u_{n+1} + 6u_n = 0$.

Solution. Given equation in *symbolic form* is $(E^3 - 2E^2 - 5E + 6)u_n = 0$

$\therefore$ its auxiliary equation is $E^3 - 2E^2 - 5E + 6 = 0$

or $$(E-1)(E+2)(E-3) = 0. \qquad \therefore \quad E = 1, -2, 3$$

Thus the complete solution is $u_n = c_1(1)^n + c_2(-2)^n + c_3(3)^n$.

Example 31.4. *Solve* $u_{n+2} - 2u_{n+1} + u_n = 0$.

Solution. Given difference equation in *symbolic form* is $(E^2 - 2E + 1)\, u_n = 0$.

$\therefore$ its auxiliary equation is $E^2 - 2E + 1 = 0$

or $$(E-1)^2 = 0. \qquad \therefore \quad E = 1, 1$$

Thus the required solution is $u_n = (c_1 + c_2 n)(1)^n$, *i.e.*, $u_n = c_1 + c_2 n$.

Example 31.5. *Solve* $y_{n+1} - 2y_n \cos\alpha + y_{n-1} = 0$.

Solution. This is a second order difference equation in y_{n-1} ; which in *symbolic form* is

$$(E^2 - 2E\cos\alpha + 1)\, y_n = 0$$

The *auxiliary equation* is $E^2 - 2E\cos\alpha + 1 = 0$

$$E = \frac{2\cos\alpha \pm \sqrt{(4\cos^2\alpha - 4)}}{4} = \cos\alpha \pm i \sin\alpha$$

Thus the solution is $y_{n-1} = (1)^{n-1} [c_1 \cos(n-1)\alpha + c_2 \sin(n-1)\alpha]$

or $$y_n = c_1 \cos n\alpha + c_2 \sin n\alpha.$$

Example 31.6. *The integers 0, 1, 1, 2, 3, 5, 8, 13, 21, ... are said to form a Fibonacci sequence. Form the Fibonacci difference equation and solve it.*

Solution. In this sequence, each number beyond the second, is the sum of its two previous number. If y_n be the nth number then $y_n = y_{n-1} + y_{n-2}$ for $n > 2$.

or $$y_{n+2} - y_{n+1} - y_n = 0 \text{ (for } n > 0)$$

or $(E^2 - E - 1)y_n = 0$ is the difference equation.

Its A.E. is $E^2 - E - 1 = 0$ which gives $E = \frac{1}{2}(1 \pm \sqrt{5})$.

Thus the solution is $$y_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^2 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n, \text{ for } n > 0$$

When $n = 1, y = 0$

$$\therefore \qquad c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) = 0 \qquad ...(i)$$

When $n = 2, y_2 = 0$

$$\therefore \qquad c_1\left(\frac{1+\sqrt{5}}{2}\right)^2 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^2 = 0 \qquad ...(ii)$$

Solving (i) and (ii), we get

$$c_1 = \frac{5-\sqrt{5}}{10} \text{ and } c_2 = \frac{5+\sqrt{5}}{10}$$

Hence the complete solution is

$$y_n = \frac{5-\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{5+\sqrt{5}}{2}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

PROBLEMS 31.2

Solve the following difference equations :

1. $u_{x+2} - 6u_{x+1} + 9u_x = 0.$
2. $y_{n+2} + y_{n+1} + 2y_n = 0.$
3. $\Delta^2 u_n + 2\Delta u_n + u_n = 0.$
4. $(\Delta^2 - 3\Delta + 2)y_n = 0.$
5. $4y_n - y_{n+2} = 0$ given that $y_0 = 0, y_1 = 2.$
6. $u_{k+3} - 3u_{k+2} + 4u_k = 0.$
7. $f(x+3) - 3f(x+1) - 2f(x) = 0.$
8. $u_{n+3} - 3u_{n+1} + 2u_n = 0$, given $u_1 = 0, u_2 = 8$ and $u_3 = -2.$
9. $(E^3 - 5E^2 + 8E - 4)y_n = 0$, given that $y_0 = 3, y_1 = 2, y_3 = 22.$
10. $u_{n+1} - 2u_n + 2u_{n-1} = 0.$
11. $y_{m+3} + 16y_{m-1} = 0.$
    [**Hint.** $E^4 = -16 = 16\,[\cos(2n+1)\pi + i\sin(2n+1)\pi]$; use De Moivre's theorem.]
12. Show that the difference equation $I_{m+1} - (2 + r_o/r)\,I_m + I_{m-1} = 0$ has the solution.
    $$I_m = I_o \sinh(n-m)\,\alpha/\sinh(n-1)\,\alpha, \text{ if } I = I_0 \text{ and } I_n = 0, \alpha \text{ being} = 2\sinh^{-1}\tfrac{1}{2}\,(r_0/r)^{1/2}.$$
13. A series of values of y_n satisfy the relation, $y_{n+2} + ay_{n+1} + by_n$.
    Given that $y_0 = 0, y_1 = 1, y_2 = y_3 = 2$. Show that $y_n = 2^{n/2}\sin n\pi/4$.
14. A plant is such that each of its seeds when one year old produces 8-fold and produces 18-fold when two years old or more. A seed is planted and as soon as a new seed is produced it is planted. Taking y_n to be the number of seeds produced at the end of the nth year, show that $y_{n+1} = 8y_n + 18(y_1 + y_2 + ... + y_{n-1})$.
    Hence show that $y_{n+2} - 9y_{n+1} - 10y_n = 0$ and find y_n.

31.6 RULES FOR FINDING THE PARTICULAR INTEGRAL

Consider the equation $y_{n+r} + a_1 y_{n+r-1} + ... + a_r y_n = f(n)$

which in *symbolic form* is $\phi(E)y_n = f(n)$ ...(1)

where $\qquad \phi(E) = E^r + a_1 E^{r-1} + ... + a_r$

Then the particular integral is given by P.I. $= \dfrac{1}{\phi(E)}\,f(n)$.

Case I. When $f(n) = a^n$

$$\text{P.I.} = \frac{1}{\phi(E)}\,a_n, \text{ put } E = a$$

$$= \frac{1}{\phi(a)}\,a^n, \text{ provided } \phi(a) \neq 0$$

If $\phi(a) = 0$, *then for the equation*

(i) $(E - a)y_n = a^n, \qquad \text{P.I.} = \dfrac{1}{E-a}\,a^n = na^{n-1}$

(ii) $(E-a)^2 y_n = a^n$, $\quad$ P.I. $= \dfrac{1}{(E-a)^2} a^n = \dfrac{n(n-1)}{2!} a^{n-2}$

(iii) $(E-a)^3 y^n = a^n$, $\quad$ P.I. $= \dfrac{1}{(E-a)^3} a^n = \dfrac{n(n-1)(n-2)}{3!} a^{n-3}$

and so on.

Example 31.7. *Solve* $y_{n+2} - 4y_{n+1} + 3y_n = 5^n$.

Solution. Given equation in *symbolic form* is $(E^2 - 4E + 3)y_n = 5^n$

$\therefore$ The auxiliary equation is $E^2 - 4E + 3 = 0$

or $\quad (E-1)(E-3) = 0. \quad \therefore \quad E = 1, 3$

$\therefore$ $\quad$ C.F. $= c_1(1)^n + c_2(3)^n = c_1 + c_2 . 3^n$

and $\quad$ P.I. $= \dfrac{1}{E^2 - 4E + 3} 5^n$ $\quad$ [Put $E = 5$]

$$= \frac{1}{25 - 4.5 + 3} 5^n = \frac{1}{8} . 5^n$$

Thus the complete solution is $\quad y_n = c_1 + c_2 . 3^n + 5^n/8.$

Example 31.8. *Solve* $u_{n+2} - 4u_{n+1} + 4u_n = 2^n$.

Solution. Given equation in *symbolic form* is $(E^2 - 4E + 4)u_n = 2^n$.

The auxiliary equation is $E^2 - 4E + 4 = 0. \quad \therefore \quad E = 2, 2.$

$$\text{C.F.} = (c_1 + c_2 n)2^n$$

$$\text{P.I.} = \frac{1}{(E-2)^2} . 2^n = \frac{n(n-1)}{2!} . 2^{n-2} = n(n-1)\, 2^{n-3}$$

Hence the complete solution is $u_n = (c_1 + c_2 n)\, 2^n + n(n-1)\, 2^{n-3}$.

Case II. *When* $f(n) = \sin kn$.

$$\text{P.I.} = \frac{1}{\phi(E)} \sin kn = \frac{1}{\phi(E)} \left(\frac{e^{ikn} - e^{-ikn}}{2i} \right) = \frac{1}{2i} \left[\frac{1}{\phi(E)} a^n - \frac{1}{\phi(E)} b^n \right]$$

where $a = e^{ik}$ and $b = e^{-ik}$.

Now proceed as in case I.

(2) *When* $f(n) = \cos kn$ $\quad$ P.I. $= \dfrac{1}{\phi(E)} \cos kn = \dfrac{1}{\phi(E)} \left(\dfrac{e^{ikn} + e^{-ikn}}{2} \right)$

$$= \frac{1}{2} \left[\frac{1}{\phi(E)} a^n + \frac{1}{\phi(E)} b^n \right] \text{ as before}$$

Now proceed as in case I.

Example 31.9. *Solve* $y_{n+2} - 2 \cos \alpha . y_{n+1} + y_n = \cos \alpha n$. *(Nagpur, 2008)*

Solution. Given equation in *symbolic form* is $(E^2 - 2 \cos \alpha . E + 1)y_n = \cos \alpha n$.

The *auxiliary equation is* $E^2 - 2 \cos \alpha . E + 1 = 0$.

$\therefore$ $$E = \frac{2\cos \alpha \pm \sqrt{(4\cos^2 \alpha - 4)}}{2} = \cos \alpha \pm i \sin \alpha$$

$\therefore$ $\quad$ C.F. $= (1)^n [c_1 \cos \alpha n + c_2 \sin \alpha n]$ *i.e.*, $c_1 \cos \alpha n + c_2 \sin \alpha n$

$$\text{P.I.} = \frac{1}{E^2 - 2E \cos \alpha + 1} \cos \alpha n$$

$$= \frac{1}{E^2 - E(e^{i\alpha} + e^{-i\alpha}) + 1} \left(\frac{e^{ian} + e^{-ian}}{2} \right)$$

$$= \frac{1}{2}\left[\frac{1}{(E-e^{i\alpha})(E-e^{-i\alpha})}e^{i\alpha n} + \frac{1}{(E-e^{i\alpha})(E-e^{-i\alpha})}e^{-i\alpha n}\right]$$

$$[\text{Put } E = e^{i\alpha}] \qquad [\text{Put } E = e^{-i\alpha}]$$

$$= \frac{1}{2}\left[\frac{1}{(E-e^{i\alpha})}\cdot\frac{1}{e^{i\alpha}-e^{-i\alpha}}e^{i\alpha n} + \frac{1}{E-e^{-i\alpha}}\cdot\frac{1}{e^{-i\alpha}-e^{i\alpha}}e^{-i\alpha n}\right]$$

$$= \frac{1}{4i\sin\alpha}\left[\frac{1}{E-e^{i\alpha}}e^{i\alpha n} - \frac{1}{E-e^{-i\alpha}}e^{-i\alpha n}\right] = \frac{1}{4i\sin\alpha}\left[n\,.\,e^{i\alpha(n-1)} - n\,e^{-i\alpha(n-1)}\right]$$

$$= \frac{n}{2\sin\alpha}\left[\frac{e^{i\alpha(n-1)} - e^{-i\alpha(n-1)}}{2i}\right] = \frac{n\sin(n-1)\alpha}{2\sin\alpha}$$

Hence the complete solution is

$$y_n = c_1\cos\alpha n + c_2\sin\alpha n + \frac{n\sin(n-1)\alpha}{2\sin\alpha}.$$

Case III. *When* $f(n) = n^p$. $\qquad$ P.I. $= \dfrac{1}{\phi(E)}n^p = \dfrac{1}{\phi(1+\Delta)}n^p$

(1) Expand $[\phi(1+\Delta)]^{-1}$ in ascending powers of Δ by the Binomial theorem as far as the term in Δ^p.

(2) Expand n^p in the factorial form (p. 950) and operate on it with each term of the expansion.

Example 31.10. *Solve* $y_{n+2} - 4y_n = n^2 + n - 1$. *(Madras, 1999)*

Solution. Given equation is $(E^2 - 4)y_n = n^2 + n - 1$.

The *auxiliary equation is* $E^2 - 4 = 0$, $\therefore$ $E = \pm 2$.

$\therefore$ $\qquad$ C.F. $= c_1(2)^n + c_2(-2)^n$.

$\therefore$ $\qquad$ P.I. $= \dfrac{1}{E^2-4}(n^2+n-1) = \dfrac{1}{(1+\Delta)^2-4}[n(n-1)+2n-1]$.

$$= \frac{1}{\Delta^2+2\Delta-3}([n]^2+2[n]-1) = -\frac{1}{3}\left[1-\left(\frac{2}{3}\Delta+\frac{\Delta^2}{3}\right)\right]^{-1}\{[n]^2+2[n]-1\}$$

$$= -\frac{1}{3}\left[1+\left(\frac{2}{3}\Delta+\frac{\Delta^2}{3}\right)+\left(\frac{2}{3}\Delta+\frac{\Delta^2}{3}\right)+\ldots\right]\{[n]^2+2[n]-1\}$$

$$= -\frac{1}{3}\left\{1+\frac{2}{3}\Delta+\frac{7}{9}\Delta^2+\ldots\right\}\{[n]^2+2[n]-1\} = -\frac{1}{3}\left\{[n]^2+2[n]-1+\frac{2}{3}(2[n]+2)+\frac{7}{9}\times 2\right\}$$

$$= -\frac{1}{3}\left\{[n]^2+\frac{10}{3}[n]+\frac{17}{9}\right\} = -\frac{n^2}{3}-\frac{7}{9}n-\frac{17}{27}.$$

Hence the *complete solution* is $y_n = c_1 2^n + c_2(-2)^n - \dfrac{n^2}{3} - \dfrac{7}{9}n - \dfrac{17}{27}$.

Case IV. *When* $f(n) = a^n F(n)$, $F(n)$, being a polynomial of finite degree in n.

$$\text{P.I.} = \frac{1}{\phi(E)}a^n F(n) = a^n\frac{1}{\phi(aE)}F(a)$$

Now $F(n)$ being a polynomial in n, proceed as in case III.

Example 31.11. *Solve* $y_{n+2} - 2y_{n+1} + y_n = n^2 . 2^n$. *(Nagpur, 2008)*

Solution. Given equation is $(E^2 - 2E + 1)y_n = n^2 . 2^n$.

Its $\qquad$ C.F. $= c_1 + c_2 n$

and $\qquad$ P.I. $= \dfrac{1}{(E-1)^2}2^n . n^2 = 2^n\dfrac{1}{(2E-1)^2}n^2 = 2^n\dfrac{1}{(1+2\Delta)^2}n^2$

$$= 2^n (1 + 2\Delta)^{-2} n(n-1) + n = 2^n (1 - 4\Delta + 12\Delta^2 - \ldots)([n]^2 + [n])$$
$$= 2^n \{[n]^2 + [n] - 4(2[n] + 1) + 12 \times 2\}$$
$$= 2^n ([n]^2 - 7[n] + 20) = 2^n (n^2 - 8n + 20)$$

Hence the *complete solution* is $y_n = c_1 + c_2 n + 2^n (n^2 - 8n + 20)$.

PROBLEMS 31.3

Solve the following difference equations :

1. $y_{n+2} - 5y_{n+1} - 6y_n = 4^n, y_0 = 0, y_1 = 1.$ *(Madras, 2003)*
2. $y_{n+2} + 6y_{n+1} + 9y_n = 2^n, y_0 = y_1 = 0.$ *(V.T.U., 2009)*
3. $y_{p+3} - 3y_{p+2} 3y_{p+1} - y_p = 1.$ *(Kottayam, 2005)*
4. $y_{n+2} - 2y_{n+1} + 4y_n = 6$, given that $y_0 = 0$ and $y_1 = 2$.
5. $(E^2 - 4E + 3)y = 3^x.$
6. $y_{x+2} - 4y_{x+1} + 4y_x = 3.2^x + 5.4^x.$
7. $u_{n+2} - u_n = \cos n/2.$ *(Madras, 2001 S)*
8. $y_{p+2} - \left(2\cos\frac{1}{2}\right) y_{p+1} + y_p = \sin p/2.$
9. $(E^2 - 4)y_x = x^2 - 1.$
10. $y_{n+3} + y_n = n^2 + 1 y_0 = y_1 = y_2 = 0.$ *(Tirchirapalli, 2001)*
11. $y_{n+3} - 5y_{n+2} + 3y_{n+1} + 9y_n = 2^n + 3n.$ *(Nagpur, 2009)*
12. $(4E^2 - 4E + 1)y = 2^n + 2^{-n}.$ *(Madras, 2001)*
13. $y_{n+2} + 5y_{n+1} + 6y_n = n + 2^n.$ *(Nagpur, 2006)*
14. $u_{x+2} + 6u_{x+1} + 9u_x = x2^x + 3^x + 7.$
15. $y_{n+3} + 8y_n = (2n + 3)\, 2^n$ *(Nagpur, 2005)*
16. $u_{n+2} - 4u_{n+1} 4u_n = n^2 2^n.$
17. $(E^2 - 5E + 6)y_k = 4^k (k^2 - k + 5).$
18. $(E^2 - 2E + 4)y_n = -2^n \left\{6\cos\frac{n\pi}{3} + 2\sqrt{3}\sin\frac{n\pi}{3}\right\}.$
19. A beam of length l, supported at n points carries a uniform load w per unit length. The bending moments $M_1, M_2 \ldots\ldots M_n$ at the supports satisfy the Clapeyron's equation :

$$M_{r+2} + 4M_{r+1} + M_r = -\frac{1}{2} wl^2$$

If a beam weighing 30 kg is supported at its ends and at two other supports dividing the beam into three equal parts of 1 metre length, show that the bending moment at each of the two middle supports is 1 kg metre.

31.7 SIMULTANEOUS DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS

The method used for solving simultaneous differential equations with constant coefficients also applies to simultaneous difference equations with constants coefficients. The following example illustrates the technique.

Example 31.12. *Solve the simultaneous difference equations*

$$u_{x+1} + v_x - 3u_x = x,\ 3u_x + v_{x+1} - 5v_x = 4^x$$

subject to the conditions $u_1 = 2,\ v_1 = 0.$

Solution. Given equation in symbolic form, are

$$(E - 3)\, u_x + v_x = x \qquad \ldots(i)$$
$$3u_x + (E - 5)v_x = 4^x \qquad \ldots(ii)$$

Operating the first equation with $E - 5$ and subtracting the second from it, we get

$$[(E - 5)(E - 3) - 3]u_x = (E - 5)x - 4^x$$

or
$$(E^2 - 8E + 12)u_x = 1 - 4x - 4^x$$

Its solution is
$$u_x = c_1 2^x + c_2 6^x - \frac{4}{5}x - \frac{19}{25} + \frac{4^x}{4} \qquad \ldots(iii)$$

Substituting the value of u_x from (iii) in (i), we get

$$v_x = c_1 2^x - 3c_2 6^x - \frac{3x}{5} - \frac{34}{25} - \frac{4^x}{4} \qquad \ldots(iv)$$

Taking $u_1 = 2$, $v_1 = 0$, in (iii) and (iv), we obtain

$$2c_1 + 6c_2 = \frac{64}{25},\ 2c_1 - 18c_2 = \frac{74}{25}$$

when
$$c_1 = 1.33,\ c_2 = -0.0167$$

Hence $u_x = 1.33.2^x - 0.0167.6^x - 0.8x - 0.76 + 4^{x-1}$

$u_x = 1.33.2^x - 0.05.6^x - 0.6x - 1.36 - 4^{x-1}$.

PROBLEMS 31.4

Solve the following simultaneous difference equations :

1. $y_{x+1} - z_x = 2(x+1), z_{x+1} - y_x = -2(x+1)$.
2. $y_{n+1} - y_n + 2z_{n+1} = 0, z_{n+1} - z_n - 2y_n = 2^n$.
3. $u_{n+1} + n = 3u_n + 2v_n, v_{n+1} - n = u_n + 2v_n$, given $u_0 = 0, v_0 = 3$.
4. $u_{x+1} + v_x + w_x = 1, u_x + v_{x+1} + w_x = x, u_x + v_x + w_{x+1} = 2x$.

31.8 APPLICATION TO DEFLECTION OF A LOADED STRING

Consider a light string of length l stretched tightly between A and B. Let the forces P_i be acting at its equispaced points x_i $(i = 1, 2, ..., n-1)$ and perpendicular to AB resulting in small transverse displacements y_i at these points (Fig. 31.1). Assuming the angle θ_i made by the portion between x_i and x_{i+1} with the horizontal, to be small, we have

$$\sin\theta_i = \tan\theta_i = \theta_i \text{ and } \cos\theta_i = 1$$

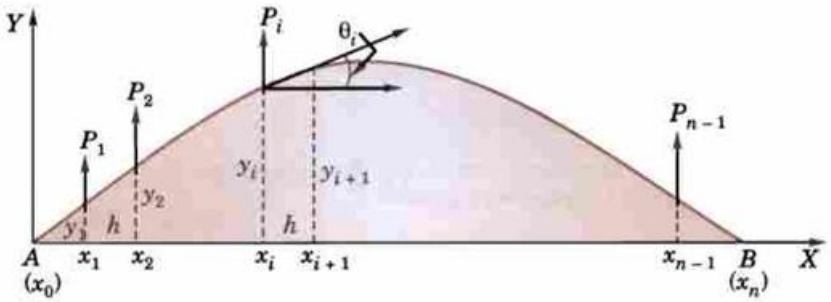


Fig. 31.1

If T be the tension of the string at x_i, then $T\cos\theta_i = T$

i.e., the tension may be taken as uniform.

Taking $x_{i+1} - x_i = h$, we have

$$y_{i+1} - y_i = h\tan\theta_i = h\theta_i \quad ...(1)$$

$$y_i - y_{i-1} = h\tan\theta_{i-1} = h\theta_{i-1} \quad ...(2)$$

Also resolving the forces in equilibrium at $(x_i, y_i) \perp$ to AB, we get

$$T\sin\theta_i - T\sin\theta_{i-1} + P_i = 0 \text{ i.e. } T(\theta_i - \theta_{i-1}) + P_i = 0 \quad ...(3)$$

Eliminating θ_i and θ_{i-1} from (1), (2) and (3), we obtain

$$y_{i+1} - 2y_i + y_{i-1} = -\frac{hP_i}{T} \quad ...(4)$$

which is a difference equation and its solution gives the displacements y_i. To obtain the arbitrary constants in the solution, we take $y_0 = y_n = 0$ as the boundary conditions, since the ends A and B of the string are fixed.

Example 31.13. *A light string stretched between two fixed nails 120 cm apart, carries 11 loads of weight 5 gm each at equal intervals and the resulting tension is 500 gm weight. Show that the sag at the mid-point is 1.8 cm.*

Solution. Taking $h = 10$ cm, $P_i = 5$ gm and $T = 500$ gm wt.,

the above equation (4) becomes $y_{i+1} - 2y_i + y_{i-1} = -1/10$

i.e., $$y_{i+2} - 2y_{i+1} + y_i = -\frac{1}{10}$$

Its A.E. is $(E-1)^2 = 0$ *i.e.* $E = 1, 1$. $\therefore$ C.F. $= c_1 + c_2 i$

and $$\text{P.I.} = \frac{1}{(E-1)^2}\left(-\frac{1}{10}\right) = -\frac{1}{10}\frac{1}{(E-1)^2}(1)^i = -\frac{1}{10}\frac{i(i-1)}{2} = \frac{1}{20}(i-i^2)$$

Thus the C.S. is $y_i = c_1 + c_2 i + \frac{1}{20}(i - i^2)$

Since $y_0 = 0, \quad \therefore \quad c_1 = 0$

and $y_{12} = 0, \quad \therefore \quad c_2 = \frac{11}{20}$.

Hence $y_i = \frac{11}{20} i + \frac{1}{20}(i - i^2)$

At the mid-point $i = 6$, we get $y_6 = 1.8$ cm.

PROBLEMS 31.5

1. A light string of length $(n+1)l$ is stretched between two fixed points with a force P. It is loaded with n equal masses m at distance l. If the system starts rotating with angular velocity ω, find the displacement y_i of the ith mass.

31.9 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 31.6

Select the correct answer or fill up the blanks in the following questions :

1. $y_n = A\,2^n + B\,3^n$, is the solution of the difference equation ..... .
2. The solution of $(E-1)^3 u_n = 0$ is ..... .
3. The solution of the difference equation $u_{n+3} - 2u_{n+2} - 5u_{n+1} + 6u_n = 0$ is ..... .
4. The solution of $y_{n+1} - y_n = 2^n$ is ..... .
5. The difference equation $y_{n+1} - 2y_n = n$ has $y_n =$ ... as its solution.
6. The difference equation corresponding to the family of curves $y = ax^2 + bx$ is ..... .
7. The particular integral of the equation $(E-2)\,y_n = 1$.
8. The solution of $4y_n = y_{n+2}$ such that $y_0 = 0, y_1 = 2$, is ..... .
9. The equation $\Delta^2 u_{n+1} + \frac{1}{2}\Delta^2 u_n = 0$ is of order ..... .
10. The difference equation satisfied by $y = a + b/x$ is ..... .
11. The order of the difference equation $y_{n+2} - 2y_{n+1} + y_n = 0$ is ..... .
12. The solution of $y_{n+2} - 4y_{n+1} + 4y_n = 0$ is ..... .
13. The particular integral of $u_{x+2} - 6u_{x+1} + 9u_x = 3$ is ..... .
14. The difference equation generated by $u_n = (a + bn)\,3^n$ is ..... .
15. Solution of $6y_{n+2} + 5y_{n+1} - 6y_n = 2^n$ is $y_n = A(2/3)^n + B(-3/2)^n + 2^n/28$. (True or False)

CHAPTER 32

Numerical Solution of Ordinary Differential Equations

1. Introduction. 2. Picard's method. 3. Taylor's series method. 4. Euler's method. 5. Modified Euler's method. 6. Runge's method. 7. Runge-Kutta method. 8. Predictor-corrector methods. 9. Milne's method. 10. Adams-Bashforth method. 11. Simultaneous first order differential equations. 12. Second order differential equations. 13. Boundary value problems. 14. Finite-difference method. 15. Objective Type of Questions.

32.1 INTRODUCTION

The methods of solution so far presented are applicable to a limited class of differential equations. Frequently differential equations appearing in physical problems do not belong to any of these familiar types and one is obliged to resort to numerical methods. These methods are of even greater importance when we realise that computing machines are now available which reduce numerical work considerably.

A number of numerical methods are available for the solution of first order differential equations of the form :

$$\frac{dy}{dx} = f(x, y), \text{ given } y(x_0) = y_0 \qquad ...(1)$$

These methods yield solutions either as a power series in x from which the values of y can be found by direct substitution, or as a set of values of x and y. The methods of Picard and Taylor series belong to the former class of solutions whereas those of Euler, Runge-Kutta, Milne, Adams-Bashforth etc. belong to the latter class. In these later methods, the values of y are calculated in short steps for equal intervals of x and are therefore, termed as *step-by-step methods.*

Euler and Runge-Kutta methods are used for computing y over a limited range of x-values whereas Milne and Adams-Bashforth methods may be applied for finding y over a wider range of x-values. These later methods require starting values which are found by Picard's or Taylor series or Runge-Kutta methods.

The initial condition in (1) is specified at the point x_0. *Such problems in which all the initial conditions are given at the initial point only are called* ***initial value problems.*** *But there are problems involving second and higher order differential equations in which the conditions may be given at two or more points. These are known as* ***boundary value problems.*** In this chapter, we shall first explain methods for solving initial value problems and then give a method of solving boundary value problems.

32.2 PICARD'S METHOD*

Consider the first order equation $dy/dx = f(x, y)$ ...(1)

* Called after the French mathematician *Emile Picard* (1856—1941) who was professor in Paris since 1881 and is famous for his researches in the theory of functions.

It is required to find that particular solution of (1) which assumes the value y_0 when $x = x_0$. Integrating (1) between limits, we get

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\,dx \quad \text{or} \quad y = y_0 + \int_{x_0}^{x} f(x, y)\,dx \qquad \ldots(2)$$

This is an integral equation equivalent to (1), for it contains the unknown y under the integral sign.

As a first approximation y_1 to the solution, we put $y = y_0$ in $f(x, y)$ and integrate (2), giving

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

For a second approximation y_2, we put $y = y_1$ in $f(x, y)$ and integrate (2), giving

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx.$$

Similarly, *a third approximation is* $y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\,dx.$

Continuing this process, a sequence of functions of x, *i.e.*, $y_1, y_2, y_3 \ldots$ is obtained each giving a better approximation of the desired solution than the preceding one.

Obs. Picard's method is of considerable theoretical value, but can be applied only to a limited class of equations in which the successive integrations can be performed easily. The method can be extended to simultaneous equations and equations of higher order (See § 32.11 and 32.12).

Example 32.1. *Using Picard's process of successive approximation, obtain a solution upto the fifth approximation of the equation $dy/dx = y + x$, such that $y = 1$ when $x = 0$. Check your answer by finding the exact particular solution.*

Solution. (*a*) We have $y = 1 + \int_0^x (y + x)\,dx.$

First approximation. Put $y = 1$, in $y + x$, giving

$$y_1 = 1 + \int_0^x (1 + x)\,dx = 1 + x + x^2/2.$$

Second approximation. Put $y = 1 + x + x^2/2$ in $y + x$, giving

$$y_2 = 1 + \int_0^x (1 + 2x + x^2/2)\,dx = 1 + x + x^2 + x^3/6.$$

Third approximation. Put $y = 1 + x + x^2 + x^3/6$ in $y + x$, giving

$$y_3 = 1 + \int_0^x (1 + 2x + x^2 + x^3/6)\,dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}.$$

Fourth approximation. Put $y = y_3$ in $y + x$, giving

$$y_4 = 1 + \int_0^x \left(1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}\right) dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}.$$

Fifth approximation. Put $y = y_4$ in $y + x$, giving

$$y_5 = 1 + \int_0^x \left(1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}\right) dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{720} \qquad \ldots(i)$$

(*b*) Given equation :

$$\frac{dy}{dx} - y = x \text{ is a Leibnitz's linear in } x.$$

Its I.F. being e^{-x}, the solution is

$$ye^{-x} = \int xe^{-x}\,dx + c = -xe^{-x} - \int (-e^{-x})\,dx + c = -xe^{-x} - e^{-x} + c \qquad \text{[Integrate by parts]}$$

$\therefore \qquad y = ce^x - x - 1.$

Since $y = 1$, when $x = 0$, $\therefore \quad c = 2.$

Thus the desired particular solution is $y = 2e^x - x - 1$ $\qquad \ldots(ii)$

Or using the series : $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \infty,$

we get $$y = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{360} + \ldots \infty \qquad \ldots(iii)$$

Comparing (i) and (iii), it is clear that (i) approximates to the exact particular solution (ii) upto the term in x^5.

Obs. At $x = 1$, the fourth approximation $y_4 = 3.433$ and the fifth approximation $y_5 = 3.434$ whereas exact value is 3.44.

Example 32.2. *Find the value of y for x = 0.1 by Picard's method, given that*

$$\frac{dy}{dx} = \frac{y - x}{y + x}, \; y(0) = 1. \qquad \text{(P.T.U., 2002)}$$

Solution. We have $y = 1 + \int_0^x \frac{y - x}{y + x}\, dx$

First approximation. Put $y = 1$ in the integrand, giving

$$y_1 = 1 + \int_0^x \frac{1 - x}{1 + x}\, dx = 1 + \int_0^x \left(-1 + \frac{2}{1 + x}\right) dx$$

$$= 1 + \left[-x + 2 \log (1 + x)\right]_0^x = 1 - x + 2 \log (1 + x) \qquad \ldots(i)$$

Second approximation. Put $y = 1 - x + 2 \log (1 + x)$ in the integrand, giving

$$y_2 = 1 + \int_0^x \frac{1 - x + 2 \log (1 + x) - x}{1 - x + 2 \log (1 + x) + x}\, dx = 1 + \int_0^x \left[1 - \frac{2x}{1 + 2 \log (1 + x)}\right] dx$$

which is very difficult to integrate.

Hence we use the first approximation and taking $x = 0.1$ in (i) we obtain

$$y(0.1) = 1 - (.1) + 2 \log 1.1 = 0.9828.$$

32.3 TAYLOR'S SERIES METHOD*

Consider the first order equation $dy/dx = f(x, y)$ ...(1)

Differentiating (1), we have

$$\frac{d^2y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \qquad i.e., \quad y'' = f_x + f_y f' \qquad \ldots(2)$$

Differentiating this successively, we can get y''', y^{iv} etc. Putting $x = x_0$ and $y = 0$, the values of $(y')_0$, $(y'')_0$, $(y''')_0$ can be obtained. Hence the Taylor's series

$$y(x) = y_0 + (x - x_0)(y')_0 + \frac{(x - x_0)^2}{2!}(y'')_0 + \frac{(x - x_0)^3}{3!}(y''')_0 + \ldots \qquad \ldots(3)$$

gives the values of y for every value of x for which (3) converges.

On finding the value y_1 for $x = x_1$ from (3), y', y'' can be evaluated at $x = x_1$ by means of (1), (2) etc. Then y can be expanded about $x = x_1$. In this way, the solution can be extended beyond the range of convergence of series (3).

Example 32.3. *Find by Taylor's series method the value of y at x = 0.1 and x = [illegible] to five places of decimals from dy/dx = x²y − 1, y(0) = 1.* (V.T.U., 2002; [illegible]htak, 2005)

Solution. Here $(y)_0 = 1$, $y' = x^2y - 1$, $(y')_0 = -1$

$\therefore$ Differentiating successively and substituting, we get

$$y'' = 2xy + x^2y', \qquad (y'')_0 = 0$$
$$y''' = 2y + 4xy' + x^2y'', \qquad (y''')_0 = 2$$
$$y^{iv} = 6y' + 6xy'' + x^2y''', \qquad (y^{iv})_0 = -6 \text{ etc.}$$

*See footnote p. 145.

Putting these values in the Taylor's series,

$$y(x) = y_0 + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0) + \ldots,$$

we have $$y(x) = 1 + x(-1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \ldots = 1 - x + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

Hence $y(0.1) = 0.90033$ and $y(0.2) = 0.80227$.

Example 32.4. *Employ Taylor's method to obtain approximate value of y at x = 0.2 for the differential equation $dy/dx = 2y + 3e^x$, $y(0) = 0$. Compare the numerical solution obtained with the exact solution.*

(V.T.U., 2009 ; P.T.U., 2003)

Solution. (*a*) We have $y' = 2y + 3e^x$ $\qquad y'(0) = 2y(0) + 3e^0 = 3.$

Differentiating successively and substituting $x = 0, y = 0$, we get

$$y'' = 2y' + 3e^x, \qquad y''(0) = 2y'(0) + 3 = 9$$
$$y''' = 2y'' + 3e^x, \qquad y'''(0) = 2y''(0) + 3 = 21$$
$$y^{iv} = 2y''' + 3e^x, \qquad y^{iv}(0) = 2y'''(0) + 3 = 45 \text{ etc.}$$

Putting these values in the Taylor's series, we have

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0) + \ldots$$

$$= 0 + 3x + \frac{9}{2}x^2 + \frac{21}{6}x^3 + \frac{45}{24}x^4 + \ldots = 3x + \frac{9}{2}x^2 + \frac{7}{2}x^3 + \frac{15}{8}x^4 + \ldots$$

Hence $\quad y(0.2) = 3(0.2) + 4.5(0.2)^2 + 3.5(0.2)^3 + 1.875(0.4)^4 + \ldots = 0.8110 \qquad \ldots(i)$

(*b*) Now $\dfrac{dy}{dx} - 2y = 3e^x$ is a Leibnitz's linear in x.

Its I.F. being e^{-2x}, the solution is

$$ye^{-2x} = \int 3e^x \,.\, e^{-2x}\,dx + c = -3e^{-x} + c \quad \text{or} \quad y = -3e^x + ce^{2x}$$

Since $y = 0$ when $x = 0$, $\quad \therefore \quad c = 3$.

Thus the exact solution is $y = 3(e^{2x} - e^x)$

When $x = 0.2, y = 3\,(e^{0.4} - e^{0.2}) = 0.8112 \qquad \ldots(ii)$

Comparing (*i*) and (*ii*), it is clear that (*i*) approximates to the exact value upto 3 decimal places.

Example 32.5. *Solve by Taylor's series method the equation $\dfrac{dy}{dx} = \log(xy)$ for y(1.1) and y(1.2), given y(1) = 2.*

(Hazaribagh, 2009)

Solution. We have $y' = \log x + \log y$; $\quad y'(1) = \log 2$

Differentiating w.r.t. x and substituting $x = 1, y = 2$, we get

$$y'' = \frac{1}{x} + \frac{1}{y}y' \,;\, y''(1) = 1 + \frac{1}{2}\log 2$$

$$y''' = -\frac{1}{x^2} + \frac{1}{y} + y'' + y'\left(-\frac{1}{y^1}\right)y' \,;\, y'''(1) = -1 + \frac{1}{2}\left(1 + \frac{1}{2}\log 2\right) - \frac{1}{4}(\log 2)^2$$

Substituting these values in the Taylor's series about $x = 1$, we have

$$y(x) = y(1) + (x-1)\,y'(1) + \frac{(x-1)^2}{2!}y''(1) + \frac{(x-1)^3}{3!}y'''(1) + \ldots$$

$$= 2 + (x-1)\log 2 + \frac{1}{2}(x-1)^2\left(1 + \frac{1}{2}\log 2\right) + \frac{1}{6}(x-1)^3\left[-\frac{1}{2} + \frac{1}{4}\log 2 - \frac{1}{4}(\log 2)^2\right]$$

$$\therefore \quad y(1.1) = 2 + (0.1)\log 2 + \frac{(0.1)^2}{2}\left(1 + \frac{1}{2}\log 2\right) + \frac{(0.1)^3}{6}\left[-\frac{1}{2} + \frac{1}{4}\log 2 - \frac{1}{4}(\log 2)^2\right] = 2.036$$

$$y(1.2) = 2 + (0.2)\log 2 + \frac{(0.2)^2}{2}\left(1 + \frac{1}{2}\log 2\right) + \frac{(0.2)^3}{6}\left[-\frac{1}{2} + \frac{1}{4}\log 2 - \frac{1}{4}(\log 2)^2\right] = 2.081.$$

PROBLEMS 32.1

1. Using Picard's method, solve $dy/dx = -xy$ with $x_0 = 0, y_0 = 1$ upto third approximation. *(Mumbai, 2005)*
2. Employ Picard's method to obtain, correct to four places of decimal, solution of the differential equation $dy/dx = x^2 + y^2$ for $x = 0.4$, given that $y = 0$ when $x = 0$. *(J.N.T.U., 2009)*
3. Obtain Picard's second approximate solution of the initial value problem : $y' = x^2/(y^2 + 1), y(0) = 0$. *(Marathwada, 2008)*
4. Find an approximate value of y when $x = 0.1$, if $dy/dx = x - y^2$ and $y = 1$ at $x = 0$, using
   (*a*) Picard's method (*b*) Taylor's series. *(V.T.U., 2010 ; Madras, 2006)*
5. Solve $y' = x + y$ given $y(1) = 0$. Find $y(1.1)$ and $y(1.2)$ by Taylor's method. Compare the result with its exact value. *(J.N.T.U., 2008 ; Anna, 2005)*
6. Evaluate $y(0.1)$ correct to six places of decimals by Taylor's series method if $y(x)$ satisfies
   $$y' = xy + 1, y(0) = 1.$$
7. Solve $y' = 3x + y^2, y(0) = 1$ using Taylor's series method and computer $y(0.1)$. *(Mumbai, 2007)*
8. Using Taylor series method, find $y(0.1)$ correct to 3-decimal places given that
   $$dy/dx = e^x - y^2, y(0) = 1.$$

32.4 EULER'S METHOD*

Consider the equation $\dfrac{dy}{dx} = f(x, y)$ ...(1)

given that $y(x_0) = y_0$. Its curve of solution through $P(x_0, y_0)$ is shown dotted in Fig. 32.1. Now we have to find the ordinate of any other point Q on this curve.

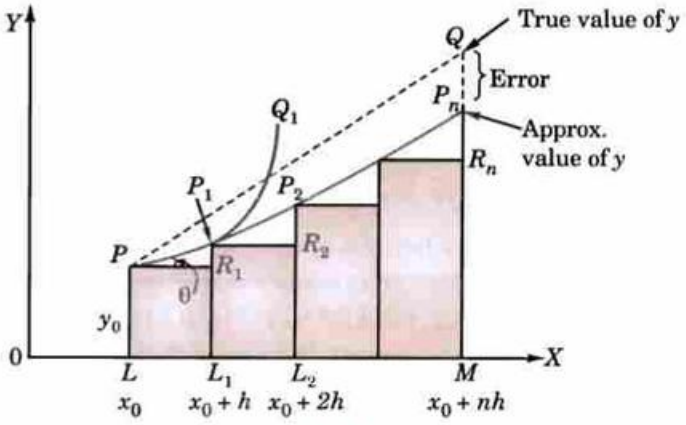


Fig. 32.1

Let us divide LM into n sub-intervals each of width h at $L_1, L_2, \ldots$ so that h is quite small. In the interval LL_1, we approximate the curve by the tangent at P. If the ordinate through L_1 meets this tangent in $P_1(x_0 + h, y_1)$, then

$$y_1 = L_1P_1 = LP + R_1P_1$$

$$= y_0 + PR_1 \tan\theta = y_0 + h\left(\frac{dy}{dx}\right)_P$$

$$= y_0 + h\, f(x_0, y_0)$$

Let P_1Q_1 be the curve of solution of (1) through P_1 and let its tangent at P_1 meet the ordinate through L_2 in $P_2(x_0 + 2h, y_2)$. Then

$$y_2 = y_1 + h\, f(x_0 + h, y_1) \qquad \ldots(2)$$

Repeating this process n times, we finally reach an approximation MP_n of MQ given by

$$y_n = y_{n-1} + h\, f(x_0 + \overline{n-1}\,h, y_{n-1})$$

This is *Euler's method* of finding an approximate solution of (1).

Obs. In Euler's method, we approximate the curve of solution by the tangent in each interval, *i.e.* by a sequence of short lines. Unless h is small, the error is bound to be quite significant. This sequence of lines may also deviate considerably from the curve of solution. Hence there is a modification of this method which is given in the next section.

Example 32.6. *Using Euler's method, find an approximate value of y corresponding to x = 1, given that dy/dx = x + y and y = 1 when x = 0.* *(Mumbai, 2005 ; Rohtak, 2003)*

*See footnote p. 302.

Solution. We take $n = 10$ and $h = 0.1$ which is sufficiently small. The various calculations are arranged as follows :

x	y	$x + y = dy/dx$	*Old* $y + 0.1\ (dy/dx) =$ *new* y
0.0	1.00	1.00	1.00 + 0.1(1.00) = 1.10
0.1	1.10	1.20	1.10 + 0.1(1.20) = 1.22
0.2	1.22	1.42	1.22 + 0.1(1.42) = 1.36
0.3	1.36	1.66	1.36 + 0.1(1.66) = 1.53
0.4	1.53	1.93	1.53 + 0.1(1.93) = 1.72
0.5	1.72	2.22	1.72 + 0.1(2.22) = 1.94
0.6	1.94	2.54	1.94 + 0.1(2.54) = 2.19
0.7	2.19	2.89	2.19 + 0.1(2.89) = 2.48
0.8	2.48	3.89	2.48 + 0.1(3.89) = 2.81
0.9	2.81	3.71	2.81 + 0.1(3.71) = 3.18
1.0	3.18		

Thus the required approximate value of $y = 3.18$.

Obs. In example 32.1, the true value of y from its exact solution at $x = 1$ is 3.44 whereas by Euler's method $y = 3.18$ and by Picard's method $y = 3.434$. In the above solution, had we chosen $n = 20$, the accuracy would have been considerably increased but at the expense of double the labour of computation. Euler's method is no doubt very simple but cannot be considered as one of the best.

Example 32.7. *Given* $\dfrac{dy}{dx} = \dfrac{y - x}{y + x}$ *with initial condition* $y = 1$ *at* $x = 0$ *; find* y *for* $x = 0.1$ *by Euler's method.*

(P.T.U., 2001)

Solution. We divide the interval (0, 0.1) into five steps *i.e.* we take $n = 5$ and $h = 0.02$. The various calculations are arranged as follows :

x	y	$(y - x)/(y + x) = dy/dx$	*Old* $y + 0.02\ (dy/dx) =$ *new* y
0.00	1.0000	1.0000	1.0000 + 0.02 (1.0000) = 1.0200
0.02	1.0200	0.9615	1.0200 + 0.02 (.9615) = 1.0392
0.04	1.0392	0.926	1.0392 + 0.02 (.926) = 1.0577
0.06	1.0577	0.893	1.0577 + 0.02 (.893) = 1.0756
0.08	1.0756	0.862	1.0756 + 0.02 (.862) = 1.0928
0.10	1.0928		

Hence the required approximate value of $y = 1.0928$.

32.5 MODIFIED EULER'S METHOD

In the Euler's method, the curve of solution in the interval LL_1 is approximated by the tangent at P (Fig. 32.1) such that at P_1, we have

$$y_1 = y_0 + h\, f(x_0, y_0) \qquad \text{...(1)}$$

Then the slope of the curve of solution through P_1 [*i.e.* $(dy/dx)_{P_1} = f(x_0 + h, y_1)$] is computed and the tangent at P_1 to P_1Q_1 is drawn meeting the ordinate through L_2 in $P_2(x_0 + 2h, y_2)$.

Now we find a better approximation $y_1^{(1)}$ of $y(x_0 + h)$ by taking the slope of the curve as the mean of the slopes of the tangents at P and P_1, *i.e.*

$$y_1^{(1)} = y_0 + \frac{h}{2}\,[f(x_0, y_0) + f(x_0 + h, y_1)] \qquad \text{...(2)}$$

As the slope of the tangent at P_1 is not known, we take y_1 as found in (1) by Euler's method and insert it on R.H.S. of (2) to obtain the first modified value $y_1^{(1)}$. The equation (1) is therefore, called the *predictor* while (2) serves as the *corrector of* y_1.

Again the corrector is applied and we find a still better value $y_1^{(2)}$ corresponding to L_1 as

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_0 + h, y_1^{(1)})]$$

We repeat this step, till two consecutive values of y agree. This is then taken as the starting point for the next interval L_1L_2.

Once y_1 is obtained to desired degree of accuracy, y corresponding to L_2 is found from the *predictor*

$$y_2 = y_1 + hf(x_0 + h, y_1)$$

and a better approximation $y_2^{(1)}$ is obtained from the *corrector*

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_0 + h, y_1) + f(x_0 + 2h, y_2)].$$

We repeat this step until y_2 becomes stationary. Then we proceed to calculate y_3 as above and so on. This is the *modified Euler's method* which is a predictor-corrector method.

Example 32.8. *Using modified Euler's method, find an approximate value of y when x = 0.3, given that dy/dx = x + y and y = 1 when x = 0.* (*Rohtak, 2005 ; Bhopal, 2002 S ; Delhi, 2002*)

Solution. Taking $h = 0.1$, the various calculations are arranged as follows :

x	$x + y = y'$	*Mean slope*	*Old y + 0.1 (mean slope) = new y*
0.0	0 + 1	—	1.00 + 0.1 (1.00) = 1.10
0.1	.1 + 1.1	$\frac{1}{2}(1 + 1.2)$	1.00 + 0.1 (1.1) = 1.11
0.1	.1 + 1.11	$\frac{1}{2}(1 + 1.21)$	1.00 + 0.1 (1.105) = 1.1105
0.1	.1 + 1.1105	$\frac{1}{2}(1 + 1.2105)$	1.00 + 0.1 (1.1052) = 1.1105
0.1	1.2105	—	1.1105 + 0.1 (1.2105) = 1.2316
0.2	.2 + 1.2316	$\frac{1}{2}(1.2105 + 1.4316)$	1.1105 + 0.1 (1.3211) = 1.2426
0.2	.2 + 1.2426	$\frac{1}{2}(1.2105 + 1.4426)$	1.1105 + 0.1 (1.3266) = 1.2432
0.2	.2 + 1.2432	$\frac{1}{2}(1.2105 + 1.4432)$	1.1105 + 0.1 (1.3268) = 1.2432
0.2	1.4432	—	1.2432 + 0.1 (1.4432) = 1.3875
0.3	.3 + 1.3875	$\frac{1}{2}(1.4432 + 1.6875)$	1.2432 + 0.1 (1.5654) = 1.3997
0.3	.3 + 1.3997	$\frac{1}{2}(1.4432 + 1.6997)$	1.2432 + 0.1 (1.5715) = 1.4003
0.3	.3 + 1.4003	$\frac{1}{2}(1.4432 + 1.7003)$	1.2432 + 0.1 (1.5718) = 1.4004
0.3	.3 + 1.4004	$\frac{1}{2}(1.4432 + 1.7004)$	1.2432 + 0.1 (1.5718) = 1.4004

Hence $y(0.3) = 1.4004$ approximately.

Obs. In example 32.6, the approximate value of y for $x = 0.3$ would be 1.53 whereas by modified Euler's method the corresponding value is 1.4004 which is nearer its true value 1.3997, obtained from its exact solution $y = 2e^x - x - 1$ by putting $x = 0.3$.

Example 32.9. *Using modified Euler's method, find y(0.2) and y(0.4) given*

$$y' = y + e^x, \; y(0) = 0.$$ (*J.N.T.U., 2009*)

Solution. We have $y' = y + e^x = f(x, y)$; $x = 0$, $y = 0$ and $h = 0.2$

The various calculations are arranged as under :

To calculate $y(0.2)$:

x	$y + e^x = y'$	*Mean slope*	*Old* $y + h$ *(mean slope) = new* y
0.0	1	—	$0 + 0.2\,(1) \quad = 0.2$
0.2	$0.2 + e^{0.2} = 1.4214$	$\frac{1}{2}(1 + 1.4214) = 1.2107$	$0 + 0.2\,(1.2107) = 0.2421$
0.2	$0.2421 + e^{0.2} = 1.4635$	$\frac{1}{2}(1 + 1.4635) = 1.2317$	$0 + 0.2\,(1.2317) = 0.2463$
0.2	$0.2463 + e^{0.2} = 1.4677$	$\frac{1}{2}(1 + 1.4677) = 1.2338$	$0 + 0.2\,(1.2338) = 0.2468$
0.2	$0.2468 + e^{0.2} = 1.4682$	$\frac{1}{2}(1 + 1.4682) = 1.2341$	$0 + 0.2\,(1.2341) = 0.2468$

Since the last two values of y are equal, we take $y(0.2) = 0.2468$.

To calculate $y(0.4)$.

x	$y + e^x = y'$	*Mean slope*	*Old* $y + h$ *(Mean slope) = new* y
0.2	$0.2468 + e^{0.2} = 1.4682$	—	$0.2468 + 0.2\,(1.4682) = 0.5404$
0.4	$0.5404 + e^{0.4} = 2.0322$	$\frac{1}{2}(1.4682 + 2.0322) = 1.7502$	$0.2468 + 0.2\,(1.7502) = 0.5968$
0.4	$0.5968 + e^{0.4} = 2.0887$	$\frac{1}{2}(1.4682 + 2.0887) = 1.7784$	$0.2468 + 0.2\,(1.7784) = 0.6025$
0.4	$0.6025 + e^{0.4} = 2.0943$	$\frac{1}{2}(1.4682 + 2.0943) = 1.78125$	$0.2468 + 0.2\,(1.78125) = 0.6030$
0.4	$0.6030 + e^{0.4} = 2.0949$	$\frac{1}{2}(1.4682 + 2.0949) = 1.7815$	$0.2468 + 0.2\,(1.7815) = 0.6031$
0.4	$0.6031 + e^{0.4} = 2.0949$	$\frac{1}{2}(1.4682 + 2.0949) = 1.7816$	$0.2468 + 0.2\,(1.7815) = 0.6031$

Since the last two value of y are equal, we take $y(0.4) = 0.6031$.

Hence $y(0.2) = 0.2468$ and $y(0.4) = 0.6031$ approximately.

Example 32.10. *Solve the following by Euler's modified method :*

$$\frac{dy}{dx} = \log(x + y),\ y(0) = 2.$$

at $x = 1.2$ and 1.4 with $h = 0.2$. *(Bhopal, 2009 ; U.P.T.U., 2007)*

Solution. The various calculations are arranged as follows :

x	$\log(x + y) = y'$	*Mean slope*	*Old* $y + 0.2$ *(mean slope) = new* y
0.0	$\log(0 + 2)$	—	$2 + 0.2\,(0.301) = 2.0602$
0.2	$\log(0.2 + 2.0602)$	$\frac{1}{2}(0.301 + 0.3541)$	$2 + 0.2\,(0.3276) = 2.0655$
0.2	$\log(0.2 + 2.0655)$	$\frac{1}{2}(0.301 + 0.3552)$	$2 + 0.2\,(0.3281) = 2.0656$
0.2	0.3552	—	$2.0656 + 0.2\,(0.3552) = 2.1366$
0.4	$\log(0.4 + 2.1366)$	$\frac{1}{2}(0.3552 + 0.4042)$	$2.0656 + 0.2\,(0.3797) = 2.1415$
0.4	$\log(0.4 + 2.1415)$	$\frac{1}{2}(0.3552 + 0.4051)$	$2.0656 + 0.2\,(0.3801) = 2.1416$

x	$\log(x+y) = y'$	*Mean slope*	*Old y + 0.2 (mean slope) = new y*
0.4	0.4051	—	2.1416 + 0.2 (0.4051) = 2.2226
0.6	log (0.6 + 2.2226)	$\frac{1}{2}(0.4051 + 0.4506)$	2.1416 + 0.2 (0.4279) = 2.2272
0.6	log (0.6 + 2.2272)	$\frac{1}{2}(0.4051 + 0.4514)$	2.1416 + 0.2 (0.4282) = 2.2272
0.6	0.4514	—	2.2272 + 0.2 (0.4514) = 2.3175
0.8	log (0.8 + 2.3175)	$\frac{1}{2}(0.4514 + 0.4938)$	2.2272 + 0.2 (0.4726) = 2.3217
0.8	log (0.8 + 2.3217)	$\frac{1}{2}(0.4514 + 0.4943)$	2.2272 + 0.2 (0.4727) = 2.3217
0.8	0.4943	—	2.3217 + 0.2 (0.4943) = 2.4206
1.0	log (1 + 2.4206)	$\frac{1}{2}(0.4943 + 0.5341)$	2.3217 + 0.2 (0.5142) = 2.4245
1.0	log (1 + 2.4245)	$\frac{1}{2}(0.4943 + 0.5346)$	2.3217 + 0.2 (0.5144) = 2.4245
1.0	0.5346	—	2.4245 + 0.2 (0.5346) = 2.5314
1.2	log (1.2 + 2.5314)	$\frac{1}{2}(0.5346 + 0.5719)$	2.4245 + 0.2 (0.5532) = 2.5351
1.2	log (1.2 + 2.5351)	$\frac{1}{2}(0.5346 + 0.5723)$	2.4245 + 0.2 (0.5534) = 2.5351
1.2	0.5723	—	2.5351 + 0.2 (0.5723) = 2.6496
1.4	log (1.4 + 2.6496)	$\frac{1}{2}(0.5723 + 0.6074)$	2.5351 + 0.2 (0.5898) = 2.6531
1.4	log (1.4 + 2.6531)	$\frac{1}{2}(0.5723 + 0.6078)$	2.5351 + 0.2 (0.5900) = 2.6531

Hence $y(1.2) = 2.5351$ and $y(1.4) = 2.6531$ approximately.

Example 32.11. *Using Euler's modified method, obtain a solution of the equation $dy/dx = x + |\sqrt{y}|$, with initial conditions $y = 1$ at $x = 0$, for the range $0 \le x \le 0.6$ in steps of 0.2.* (V.T.U., 2007)

Solution. **The various calculations are arranged as follows :**

x	$x + \lvert\sqrt{y}\rvert = y'$	*Mean slope*	*Old y + .2 (mean slope) = new y*
0.0	0 + 1 = 1	—	1 + 0.2 (1) = 1.2
0.2	$0.2 + \lvert\sqrt{(1.2)}\rvert = 1.2954$	$\frac{1}{2}(1 + 1.2954) = 1.1477$	1 + 0.2 (1.1477) = 1.2295
0.2	$0.2 + \lvert\sqrt{(1.2295)}\rvert = 1.3088$	$\frac{1}{2}(1 + 1.3088) = 1.1544$	1 + 0.2 (1.1544) = 1.2309
0.2	$0.2 + \lvert\sqrt{(1.2309)}\rvert = 1.3094$	$\frac{1}{2}(1 + 1.3094) = 1.1547$	1 + 0.2 (1.1547) = 1.2309
0.2	1.3094	—	1.2309 + 0.2 (1.3094) = 1.4927
0.4	$0.4 + \lvert\sqrt{(1.4927)}\rvert = 1.6218$	$\frac{1}{2}(1.3094 + 1.6218) = 1.4654$	1.2309 + 0.2 (1.4654) = 1.5240
0.4	$0.2 + \lvert\sqrt{(1.524)}\rvert = 1.6345$	$\frac{1}{2}(1.3094 + 1.6345) = 1.4718$	1.2309 + 0.2 (1.4718) = 1.5253
0.4	$0.4 + \lvert\sqrt{(1.5253)}\rvert = 1.6350$	$\frac{1}{2}(1.3094 + 1.6350) = 1.4721$	1.2309 + 0.2 (1.4721) = 1.5253

x	$x + \lvert\sqrt{y}\rvert = y'$	*Mean slope*	*Old y + .2 (mean slope) = new y*
0.4	1.6350	—	1.5253 + 0.2 (1.635) = 1.8523
0.6	$0.6 + \lvert\sqrt{(1.8523)}\rvert = 1.9610$	$\frac{1}{2}(1.635 + 1.961) = 1.798$	1.5253 + 0.2 (1.798) = 1.8849
0.6	$0.6 + \lvert\sqrt{(1.8849)}\rvert = 1.9729$	$\frac{1}{2}(1.635 + 1.9729) = 1.8040$	1.5253 + 0.2 (1.804) = 1.8861
0.6	$0.6 + \lvert\sqrt{(1.8861)}\rvert = 1.9734$	$\frac{1}{2}(1.635 + 1.9734) = 1.8042$	1.5253 + 0.2 (1.8042) = 1.8861

Hence $y(0.6) = 1.8861$ approximately.

PROBLEMS 32.2

1. Apply Euler's method to solve $y' = x + y$, $y(0) = 0$, choosing the step length = 0.2. (Carry out 6 steps). *(Kottayam, 2005)*
2. Using simple Euler's method solve for y at $x = 0.1$ from $dy/dx = x + y + xy$, $y(0) = 1$, taking step size $h = 0.025$.
3. Using Euler's method, find the approximate value of y when $dy/dx = x^2 + y^2$ and $y(0) = 1$ in five steps (*i.e.* $h = 0.2$). *(Mumbai, 2006)*
4. Solve $y' = 1 - y$, $y(0) = 0$ by modified Euler's method and obtain y at $x = 0.1, 0.2, 0.3$. *(Anna, 2005)*
5. Given $y' = x + \sin y$, $y(0) = 1$. Compute $y(0.2)$ and $y(0.4)$ with $h = 0.2$ using Euler's modified method. *(J.N.T.U., 2007)*
6. Given that $dy/dx = x^2 + y$ and $y(0) = 1$. Find an approximate value of $y(0.1)$ taking $h = 0.05$ by modified Euler's method. *(V.T.U., 2010)*
7. Given $\frac{dy}{dx} = \frac{y - x}{y + x}$ with boundary conditions $y = 1$ when $x = 0$, find approximately y for $x = 0.1$, by Euler's modified method (5 steps). *(V.T.U., 2007)*
8. Given that $dy/dx = 2 + \sqrt{(xy)}$ and $y = 1$ when $x = 1$. Find approximate value of y at $x = 2$ in steps of 0.2, using Euler's modified method. *(Anna, 2004)*

32.6 RUNGE'S METHOD*

Consider the differential equation,

$$\frac{dy}{dx} = f(x, y), \; y(x_0) = y_0 \qquad \ldots(1)$$

Clearly the slope of the curve through $P(x_0, y_0)$ is $f(x_0, y_0)$ (Fig. 32.2).

Integrate both sides of (1) from (x_0, y_0) to $(x_0 + h, y_0 + k)$, we have

$$\int_{y_0}^{y_0 + k} dy = \int_{x_0}^{x_0 + h} f(x, y)\, dx \qquad \ldots(2)$$

To evaluate the integral on the right, we take N as the mid-point of LM and find the values of $f(x, y)$ (*i.e.* dy/dx) at the points $x_0, x_0 + h/2, x_0 + h$. For this purpose, we first determine the values of y at these points.

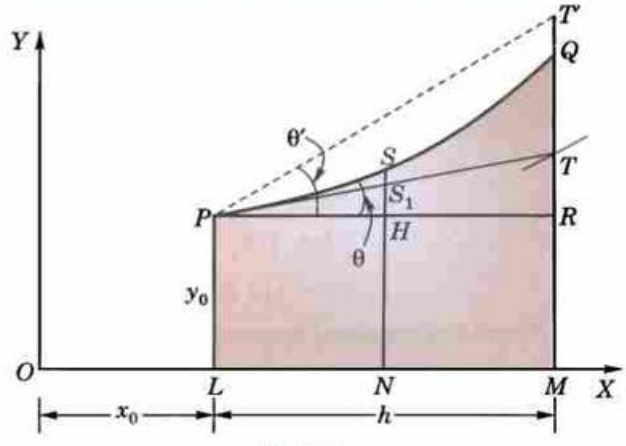


Fig. 32.2

Let the ordinate through N cut the curve PQ in S and the tangent PT in S_1. The value of y_s is given by the point S_1.

$$\therefore \qquad y_s = NS = LP + HS_1 = y_0 + PH \tan\theta$$

* Called after the German mathematician *Carl Runge* (1856–1927) who was professor at Gottingen.

$$= y_0 + \frac{h}{2}(dy/dx)_P = y_0 + \frac{h}{2} f(x_0, y_0) \qquad ...(3)$$

Also $\quad y_T = MT = LP + RT = y_0 + PR \tan\theta = y_0 + hf(x_0, y_0).$

Now the value of y_Q at $x_0 + h$ is given by the point T' where the line through P drawn with slope at $T(x_0 + h, y_T)$ meets MQ.

$\because$ Slope at $T = \tan\theta' = f(x_0 + h, y_T) = f[x_0 + h, y_0 + hf(x_0, y_0)]$

$$\therefore \quad y_Q = MR + RT' = y_0 + PT \tan\theta' = y_0 + hf[x_0 + h, y_0 + hf(x_0, y_0)] \qquad ...(4)$$

Thus the value of $f(x, y)$ at $P = f(x_0, y_0)$,
the value of $f(x, y)$ at $S = f(x_0 + h/2, y_s)$
and the value of $f(x, y)$ at $Q = f(x_0 + h, y_Q)$
where y_S and y_Q are given by (3) and (4).

Hence from (2), we obtain

$$k = \int_{x_0}^{x_0+h} f(x, y)\, dx = \frac{h}{6}[f_P + 4f_S + f_Q] \qquad \text{[By Simpsons' rule (p. 1106)]}$$

$$= \frac{h}{6}[f(x_0, y_0) + 4f(x_0 + h/2, y_S) + f(x_0 + h, y_Q)] \qquad ...(5)$$

which gives a sufficiently accurate value of k and also of $y = y_0 + k$.

The repeated application of (5) gives the values of y for equispaced points.

Working rule *to solve (1) by Runge's method :*

Calculate successively

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_1\right)$$

$$k' = hf(x_0 + h, y_0 + k_1)$$

and $\quad k_3 = hf(x_0 + h, y_0 + k')$

Finally compute, $\quad k = \frac{1}{6}(k_1 + 4k_2 + k_3).$

(Note that k is the weighted mean of k_1, k_2 and k_3)

Example 32.12. *Apply Runge's method to find an approximate value of y when $x = 0.2$, given that $dy/dx = x + y$ and $y = 1$ when $x = 0$.*

Solution. Here we have $x_0 = 0, y_0 = 1, h = 0.2, f(x_0, y_0) = 1$

$\therefore \quad k_1 = hf(x_0, y_0) = 0.2\,(1) \qquad = 0.200$

$k_2 = hf\left(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_1\right) = 0.2\,f(0.1, 1.1) \qquad = 0.240$

$k' = hf(x_0 + h, y_0 + k_1) = 0.2\,f(0.2, 1.2) \qquad = 0.280$

and $\quad k_3 = hf(x_0 + h, y_0 + k') = 0.2\,f(0.1, 1.28) \qquad = 0.296$

$\therefore \quad k = \frac{1}{6}(k_1 + 4k_2 + k_3)$

$= \frac{1}{6}(0.200 + 0.960 + 0.296) \qquad = 0.2426$

Hence the required approximate value of y is 1.2426.

32.7 RUNGE-KUTTA METHOD*

The Taylor's series method of solving differential equations numerically is restricted by the labour involved in finding the higher order derivatives. However there is a class of methods known as Runge-Kutta methods which do not require the calculations of higher order derivatives. These methods agree with Taylor's series solution upto the terms in h^r, where r differs from method to method and is called the *order of that method. Euler's method, Modified Euler's method and Runge's method are the Runge-Kutta methods of the first, second and third order respectively.*

* See footnote p. 1017. Named after *Wilhelm Kutta* (1867—1944).

The fourth-order Runge-Kutta method is most commonly used and is often referred to as 'Runge-Kutta method' only.

Working rule *for finding the increment k of y corresponding to an increment h of x by Runge-Kutta method from*

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0 \text{ is as follows :}$$

Calculate successively

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_1\right)$$

$$k_3 = hf\left(x_0 + \tfrac{1}{2}h, y + \tfrac{1}{2}k_2\right)$$

and $$k_4 = hf(x_0 + h, y_0 + k_3)$$

Finally compute $$k = \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

which gives the required approximate value $y_1 = y_0 + k$.

(Note that k is the weighted mean of k_1, k_2, k_3 and k_4)

Obs. One of the advantages of these methods is that the operation is identical whether the differential equation is linear or non-linear.

Example 32.13. *Apply Runge-Kutta fourth order method, to find an approximate value of y when x = 0.2, given that dy / dx = x + y and y = 1 when x = 0.* *(V.T.U., 2009 ; P.T.U., 2007 ; S.V.T.U., 2007)*

Solution. Here $x_0 = 0, y_0 = 1, h = 0.2, f(x_0, y_0) = 1$

$\therefore$ $$k_1 = hf(x_0, y_0) = 0.2 \times 1 = 0.2000$$

$$k_2 = hf\left(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_1\right) = 0.2 \times f(0.1, 1.1) = 0.2400$$

$$k_3 = hf\left(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_2\right) = 0.2 \times f(0.1, 1.12) = 0.2440$$

and $$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2 \times f(0.2, 1.244) = 0.2888$$

$\therefore$ $$k = \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \tfrac{1}{6}(0.2000 + 0.4800 + 0.4880 + 0.2888) = \tfrac{1}{6} \times (1.4568) = 0.2468.$$

Hence the required approximate value of y is 1.2428.

Example 32.14. *Using Runge-Kutta method of fourth order, solve* $\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{y^2 + x^2}$ *with y(0) = 1 at x = 0.2, 0.4.* *(U.P.T.U., 2010 ; J.N.T.U., 2009 ; V.T.U., 2008)*

Solution. We have $f(x, y) = \dfrac{y^2 - x^2}{y^2 + x^2}$

To find y(0.2) :

Here $x_0 = 0, y_0 = 1, h = 0.2$

$$k_1 = h\,f(x_0, y_0) = 0.2\,f(0, 1) = 0.2000$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.2\,f(0.1, 1.1) = 0.19672$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = 0.2\,f(0.1, 1.09836) = 0.1967$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2\,f(0.2, 1.1967) = 0.1891$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \frac{1}{6}[0.2 + 2(0.19672) + 2(0.1967) + 0.1891]$$

$$= 0.19599$$

Hence $$y(0.2) = y_0 + k = 1.196.$$

To find y(0.4) :

Here $x_1 = 0.2, y_1 = 1.196, h = 0.2$

$$k_1 = h\,f(x_1, y_1) = 0.1891$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = 0.2\,f(0.3, 1.2906) = 0.1795$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = 0.2\,f(0.3, 1.2858) = 0.1793$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2\,f(0.4, 1.3753) = 0.1688$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.1891 + 2(0.1795) + 2(0.1793) + 0.1688] = 0.1792$$

Hence $y(0.4) = y_1 + k = 1.196 + 0.1792 = 1.3752.$

Example 32.15. *Apply Runge-Kutta method to find an approximate value of y for x = 0.2 in steps of 0.1, if dy/dx = x + y², given that y = 1, where x = 0.* *(V.T.U., 2009 ; Osmania, 2007 ; Madras, 2000)*

Solution. Here we take $h = 0.1$ and carry out the calculations in two steps.

Step I. $x_0 = 0, y_0 = 1, h = 0.1$

$\therefore$ $$k_1 = hf(x_0, y_0) = 0.1\,f(0, 1) = 0.1000$$

$$k_2 = hf\left(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_1\right) = 0.1\,f(0.05, 1.1) = 0.1152$$

$$k_3 = hf\left(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_2\right) = 0.1\,f(0.05, 1.1152) = 0.1168$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1\,f(0.1, 1.1168) = 0.1347$$

$\therefore$ $$k = \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \tfrac{1}{6}(0.1000 + 0.2304 + 0.2336 + 0.1347) = 0.1165$$

giving $y(0.1) = y_0 + k = 1.1165.$

Step II. $x_1 = x_0 + h = 0.1, y_1 = 1.1165, h = 0.1$

$\therefore$ $$k_1 = hf(x_1, y_1) = 0.1\,f(0.1, 1.1165) = 0.1347$$

$$k_2 = hf\left(x_1 + \tfrac{1}{2}h, y_1 + \tfrac{1}{2}k_1\right) = 0.1\,f(0.15, 1.1838) = 0.1551$$

$$k_3 = hf\left(x_1 + \tfrac{1}{2}h, y_1 + \tfrac{1}{2}k_2\right) = 0.1\,f(0.15, 1.194) = 0.1576$$

$$k_4 = hf(x_1 + h, y_2 + k_3) = 0.1\,f(0.2, 1.1576) = 0.1823$$

$\therefore$ $$k = \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.1571$$

Hence $y(0.2) = y_1 + k = 1.2736.$

Example 32.16. *Using Runge-Kutta method of fourth order, solve for y at x = 1.2, 1.4 from* $\dfrac{dy}{dx} = \dfrac{2xy + e^x}{x^2 + xe^x}$ *given $x_0 = 1, y_0 = 0$.* *(Mumbai, 2008)*

Solution. We have $f(x, y) = \dfrac{2xy + e^x}{x^2 + xe^2}$

To find y(1.2) :

Here $x_0 = 1, y_0 = 0, h = 0.2$

$\therefore$ $$k_1 = h\,f(x_0, y_0) = 0.2\,\frac{0 + e}{1 + e} = 0.1462$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2\left\{\frac{2(1 + 0.1)(0 + 0.073) + e^{1+0.1}}{(1 + 0.1)^2 + (1 + 0.1)\,e^{1+0.1}}\right\} = 0.1402$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2\left\{\frac{2(1+0.1)(0+0.07)+e^{1.1}}{(1+0.1)^2+(1+0.1)\,e^{1.1}}\right\} = 0.1399$$

$$k_4 = hf(x_0 + k, y_0 + k_3) = 0.2\left\{\frac{2(1.2)(0.1399)+e^{1.2}}{(1.2)^2+(1.2)\,e^{1.2}}\right\} = 0.1348$$

and $$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \frac{1}{6}[0.1462 + 0.2804 + 0.2798 + 0.1348]$$
$$= 0.1402.$$

Hence $y(1.2) = y_0 + k = 0 + 0.1402 = 0.1402.$

To find y(1.4) :

Here $x_1 = 1.2, y_1 = 0.1402, h = 0.2$

$$k_1 = hf(x_1, y_1) = 0.2\,f(1.2, 0) = 0.1348$$
$$k_2 = hf(x_1 + h/2, y_1 + k_1/2) = 0.2\,f(1.3, 0.2076) = 0.1303$$
$$k_3 = hf(x_1 + h/2, y_1 + k_1/2) = 0.2\,f(1.3, 0.2053) = 0.1301$$
$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2\,f(1.3, 0.2703) = 0.1260$$

$\therefore$ $$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \frac{1}{6}[0.1348 + 0.2606 + 0.2602 + 0.1260] = 0.1303$$

Hence $y(1.4) = y_1 + k = 0.1402 + 0.1303 = 0.2705.$

PROBLEMS 32.3

1. Use Runge's method to approximate y when $x = 1.1$, given that $y = 1.2$ when $x = 1$ and $dy/dx = 3x + y^2$.
2. Using Runge-Kutta method of order 4, find $y(0.2)$ given that $dy/dx = 3x + \frac{1}{2}y$, $y(0) = 1$, taking $h = 0.1$. *(V.T.U., 2004)*
3. Using Runge-Kutta method of order 4, compute $y(.2)$ and $(.4)$ from $10\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$, taking $h = 0.1$. *(Rohtak, 2003 ; Bhopal, 2002)*
4. Use Runge-Kutta method to find y when $x = 1.2$ in steps of 0.1, given that :
   $dy/dx = x^2 + y^2$ and $y(1) = 1.5$. *(Mumbai, 2007)*
5. Find $y(0.1)$ and $y(0.2)$ using Runge-Kutta 4th order formula, given that $y' = x^2 - y$ and $y(0) = 1$. *(J.N.T.U., 2006)*
6. Using 4th order Runge-Kutta method, solve the following equation, taking each step of $h = 0.1$, given $y(0) = 3$, $dy/dx = (4x/y - xy)$. Calculate y for $x = 0.1$ and 0.2. *(Anna, 2007)*
7. Use fourth order Runge-Kutta method to find y at $x = 0.1$, given that $\frac{dy}{dx} = 3e^x + 2y$, $y(0) = 0$ and $h = 0.1$. *(V.T.U., 2006)*
8. Find by Runge-Kutta method an approximate value of y for $x = 0.8$, given that $y = 0.41$ when $x = 0.4$ and $dy/dx = \sqrt{(x+y)}$. *(S.V.T.U., 2007 S)*
9. Using Runge-Kutta method of order 4, find $y(0.2)$ for the equation $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0) = 1$. Take $h = 0.2$. *(V.T.U., 2011 S)*
10. Given that $dy/dx = (y^2 - 2x)/(y^2 + x)$ and $y = 1$ at $x = 0$; find y for $x = 0.1, 0.2, 0.3, 0.4$ and 0.5. *(Delhi, 2002)*

32.8 PREDICTOR-CORRECTOR METHODS

If x_{i-1} and x_i be two consecutive mesh points, we have $x_i = x_{i-1} + h$. In the Euler's method (§ 32.4), we have

$$y_i = y_{i-1} + hf(x_0 + \overline{i-1}\,h, y_{i-1})\,;\ i = 1, 2, 3, \ldots \qquad \ldots(1)$$

The modified Euler's method (§ 32.5), gives

$$y_i = y_{i-1} + \frac{h}{2}[f(x_{i-1}, y_{i-1}) + f(x_i, y_i)] \quad ...(2)$$

The value of y_i is first estimated by using (1), then this value is inserted on the right side of (2), giving a better approximation of y_i. This value of y_i is again substituted in (2) to find a still better approximation of y_i. This step is repeated till two consecutive values of y_i agree. *This technique of refining an initially crude estimate of y_i by means of a more accurate formula is known as* **predictor-corrector method.** The equation (1) is therefore called the *predictor* while (2) serves as a *corrector* of y_i.

In the methods so far explained, to solve a differential equation over an interval (x_i, x_{i+1}) only the value of y at the beginning of the interval was required. In the *predictor-corrector* methods, four prior values are required for finding the value of y at x_{i+1}. A predictor formula is used to predict the value of y at x_{i+1} and then a corrector formula is applied to improve this value.

We now describe two such methods, namely : Milne's method and Adams-Bashforth method.

32.9 MILNE'S METHOD

Given $dy/dx = f(x, y)$ and $y = y_0, x = x_0$; to find an approximate value of y for $x = x_0 + nh$ by Milne's method, we proceed as follows :

The value $y_0 = y(x_0)$ being given, we compute

$$y_1 = y(x_0 + h), y_2 = y(x_0 + 2h), y_3 = y(x_0 + 3h),$$

by Picard's or Taylor's series method.

Next we calculate,

$$f_0 = f(x_0, y_0), f_1 = f(x_0 + h, y_1), f_2 = f(x_0 + 2h, y_2), f_3 = f(x_0 + 3h, y_3)$$

Then to find $y_4 = y(x_0 + 4h)$, we substitute Newton's forward interpolation formula

$$f(x, y) = f_0 + n\Delta f_0 + \frac{n(n-1)}{2}\Delta^2 f_0 + \frac{n(n-1)(n-2)}{6}\Delta^3 f_0 + ...$$

in the relation $y_4 = y_0 + \int_{x_0}^{x_0+4h} f(x, y)\,dx$

$$\therefore \quad y_4 = y_0 + \int_{x_0}^{x_0+4h}\left(f_0 + n\Delta f_0 + \frac{n(n-1)}{2}\Delta^2 f_0 + ...\right)dx \qquad [\text{Put } x = x_0 + nh, dx = hdn]$$

$$= y_0 + h\int_0^4\left(f_0 + n\Delta f_0 + \frac{n(n-1)}{2}\Delta^2 f_0 + ...\right)dn$$

$$= y_0 + h\left(4f_0 + 8\Delta f_0 + \frac{20}{3}\Delta^2 f_0 + \frac{8}{3}\Delta^3 f_0 + ...\right)$$

Neglecting fourth and higher order differences and expressing Δf_0, $\Delta^2 f_0$ and $\Delta^3 f_0$ in terms of the function values, we get

$$y_4^{(p)} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3) \text{ which is called a } \textit{predictor}.$$

Having found y_4, we obtain a first approximation to $f_4 = f(x_0 + 4h, y_4)$.

Then a better value of y_4 is found by Simpson's rule (p. 1106) as

$$y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4) \text{ which is called a } \textit{corrector}.$$

Then an improved value of f_4 is computed and again the corrector is applied to find a still better value of y_4. We repeat this step until y_4 remains unchanged.

Once y_4 and f_4 are obtained to desired degree of accuracy, $y_5 = y(x_0 + 5h)$ is found from the *predictor* as

$$y_5^{(p)} = y_1 + \frac{4h}{3}(2f_2 - f_3 + 2f_4)$$

and $f_5 = f(x_0 + 5h, y_5)$ is calculated. Then a better approximation to the value of y_5 is obtained from the *corrector* as

$$y_5^{(c)} = y_3 + \frac{h}{3}(f_3 + 4f_4 + f_5).$$

We repeat this step till y_5 becomes stationary and we, then proceed to calculate y_6 as before.

This is *Milne's predictor-corrector method.* To ensure greater accuracy, we must first improve the accuracy of the starting values and then sub-divide the intervals.

Example 32.17. *Apply Milne's method, to find a solution of the differential equation $y' = x - y^2$ in the range $0 \le x \le 1$ for the boundary conditions $y = 0$ at $x = 0$.* (V.T.U., 2009, Anna, 2005, Rohtak, 2005)

Solution. Using Picard's method, we have

$$y = y(0) + \int_0^x f(x, y)\, dx, \text{ where } f(x, y) = x - y^2.$$

To get the first approximation, we put $y = 0$ in $f(x, y)$,

giving $$y_1 = 0 + \int_0^x x\, dx = \frac{x^2}{2}$$

To find the second approximation, we put $y = x^2/2$ in $f(x, y)$,

giving $$y_2 = \int_0^x \left(x - \frac{x^4}{4}\right) dx = \frac{x^2}{2} - \frac{x^5}{20}$$

Similarly, the third approximation is

$$y_3 = \int_0^x \left[x - \left(\frac{x^2}{2} - \frac{x^5}{20}\right)^2\right] dx = \frac{x^2}{2} - \frac{x^5}{20} + \frac{x^8}{160} - \frac{x^{11}}{4400} \qquad ...(i)$$

Now let us determine the starting values of the Milne's method from (i), by choosing $h = 0.2$.

$\therefore$ $x_0 = 0.0$, $y_0 = 0.0000$, $f_0 = 0.0000$

$x_1 = 0.2$, $y_1 = 0.020$, $f_1 = 0.1996$

$x_2 = 0.4$, $y_2 = 0.0795$, $f_2 = 0.3937$

$x_3 = 0.6$, $y_3 = 0.1762$, $f_3 = 0.5689$

Using the *predictor*, $y_4^{(p)} = y_0 \frac{4h}{3}(2f_1 - f_2 + 2f_3)$

$x = 0.8$, $y_4^{(p)} = 0.3049$, $f_4 = 0.7070$

and the *corrector*, $y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$, yields

$y_4^{(c)} = 0.3046$, $f_4 = 0.7072$ ...(ii)

Again using the *corrector*, $y_4^{(c)} = 0.3046$, which is same as in (ii)

Now using the *predictor*, $y_5^{(p)} = y_1 + \frac{4h}{3}(2f_2 - f_3 + 2f_4)$,

$x = 1.0$, $y_5^{(p)} = 0.4554$, $f_5 = 0.7926$

and the *corrector*, $y_5^{(c)} = y_3 + \frac{h}{3}(f_3 + 4f_4 + f_5)$, gives

$y_5^{(c)} = 0.4555$, $f_5 = 0.7925$

Again using the *corrector*,

$y_5^{(c)} = 0.4555$, a value which is the same as before.

Hence, $y(1) = 0.4555$.

Example 32.18. *Given $y' = x(x^2 + y^2)\, e^{-x}$, $y(0) = 1$, find y at $x = 0.1, 0.2$ and 0.3 by Taylor's series method and compute $y(0.4)$ by Milne's method.* (Anna, 2007)

Solution. Given $\quad y(0) = 1$ and $h = 0.1$

We have $\quad y'(x) = x(x^2 + y^2)\, e^{-x}; \qquad y'(0) = 0$

$$y''(x) = [(x^3 + xy^2)(-e^{-x}) + 3x^2 + y^2 + x(2y)\, y']\, e^{-x}$$

$$= e^{-x}[-x^3 - xy^2 + 3x^2 + y^2 + 2xyy']; \qquad y''(0) = 1$$

$$y'''(x) = -e^{-x}[-x^3 - xy^2 + 3x^2 + y^2 + 2xyy' + 3x^2 + y^2 + 2xyy' - 6x - 2yy' - 2xy'^2 - 2xyy'] \qquad y'''(0) = -2$$

Substitute these values in the Taylor's series,

$$y(x) = y(0) + \frac{x}{1!}y'(0) + \frac{x^2}{2!}y'(0) + \frac{x^3}{3!}y'''(0) + \ldots$$

$$y(0.1) = 1 + (0.1)(0) + \frac{1}{2}(0.1)^2(1) + \frac{1}{6}(0.1)^3(-2) + \ldots$$

$$= 1 + 0.005 - 0.0003 = 1.0047 \quad i.e., \quad 1.005$$

Now taking $\quad x = 0.1, y(0.1) = 1.005, h = 0.1$

$$y'(0.1) = 0.092, y''(0.1) = 0.849, y'''(0.1) = -1.247$$

Substituting these values in the Taylor's series about $x = 0.1$,

$$y(0.2) = y(0.1) + \frac{0.1}{1!}y'(0.1) + \frac{(0.1)^2}{2!}y''(0.1) + \frac{(0.1)^3}{3!}y'''(0.1) + \ldots$$

$$= 1.005 + (0.1)(0.092) + \frac{(0.1)^2}{2}(0.849) + \frac{(0.1)^3}{3}(-1.247) + \ldots$$

$$= 1.018$$

Now taking $\quad x = 0.2, y(0.2) = 1.018, h = 0.1$

$$y'(0.2) = 0.176, y''(0.2) = 0.77, y'''(0.2) = 0.819$$

Substituting these values in the Taylor's series

$$y(0.3) = y(0.2) + \frac{0.1}{1!}y''(0.2) + \frac{(0.1)^2}{2!}y''(0.2) + \frac{(0.1)^3}{3!}y'''(0.2) + \ldots$$

$$= 1.018 + 0.0176 + 0.0039 + 0.0001 = 1.04$$

Thus the starting values of the Milne's method with $h = 0.1$ are

$x_0 = 0.0$	$y_0 = 1$	$f_0 = y'_0 = 0$
$x_1 = 0.1$	$y_1 = 1.005$	$f_1 = 0.092$
$x_2 = 0.2$	$y_2 = 1.018$	$f_2 = 0.176$
$x_3 = 0.3$	$y_3 = 1.04$	$f_3 = 0.26$

Using the predictor, $\quad y_4^{(p)} = y_0 + \dfrac{4h}{3}(2f_1 - f_2 + 2f_3)$

$$= 1 + \frac{4(0.1)}{3}[2(0.092) - (0.176) + 2(0.26)] = 1.09$$

$\therefore \qquad x = 0.4 \qquad y_4^{(p)} = 1.09 \qquad f_4 = y'(0.4) = 0.362$

Using the corrector, $\quad y_4^{(c)} = y_2 + \dfrac{h}{3}(f_2 + 4f_3 + f_4)$

$$\therefore \qquad y_4^{(c)} = 0.018 + \frac{0.1}{3}(0.176 + 4(0.26) + 0.362) = 1.071$$

Hence $\quad y(0.4) = 1.071.$

Example 32.19. *Using Runge-Kutta method of order 4, find y for $x = 0.1, 0.2, 0.3$ given that $dy/dx = xy + y^2$, $y(0) = 1$. Continue the solution at $x = 0.4$ using Milne's method.*

(V.T.U., 2008 ; S.V.T.U., 2007 ; Madras, 2006)

Solution. We have $\quad f(x, y) = xy + y^2.$

To find $y(0.1)$:

Here $\quad x_0 = 0, y_0 = 1, h = 0.1.$

$\therefore$ $k_1 = h\,f(x_0, y_0) = (0.1)\,f(0.1) = 0.1000$

$$k_2 = hf\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_1\right) = (0.1)\,f(0.05,\ 1.05) = 0.1155$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_2\right) = (0.1)\,f(0.05,\ 1.0577) = 0.1172$$

$$k_4 = hf(x_0 + h,\ y_0 + k_3) = (0.1)\,f(0.1,\ 1.1172) = 0.13598$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.1 + 0.231 + 0.2348 + 0.13598) = 0.11687$$

Thus $y(0.1) = y_1 = y_0 + k = 1.1169$.

To find y(0.2) :

Here $x_1 = 0.1, y_1 = 1.1169, h = 0.1$.

$$k_1 = h\,f(x_1, y_1) = (0.1)\,f(0.1,\ 1.1169) = 0.1359$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h,\ y_1 + \frac{1}{2}k_1\right) = (0.1)\,f(0.15,\ 1.1848) = 0.1581$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h,\ y_1 + \frac{1}{2}k_2\right) = (0.1)\,f(0.15,\ 1.1959) = 0.1609$$

$$k_4 = hf(x_1 + h,\ y_1 + k_3) = (0.1)\,f(0.2,\ 1.2778) = 0.1888$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.1605$$

Thus $y(0.2) = y_2 = y_1 + k = 1.2773$.

To find y(0.3) :

Here $x_2 = 0.2, y_2 = 1.2773, h = 0.1$.

$$k_1 = hf(x_2, y_2) = (0.1)\,f(0.2,\ 1.2773) = 0.1887$$

$$k_2 = hf\left(x_2 + \frac{1}{2}h,\ y_2 + \frac{1}{2}k_1\right) = (0.1)\,f(0.25,\ 1.3716) = 0.2224$$

$$k_3 = hf\left(x_2 + \frac{1}{2}h,\ y_2 + \frac{1}{2}k_2\right) = (0.1)\,f(0.25,\ 1.3885) = 0.2275$$

$$k_4 = hf(x_2 + h,\ y_2 + k_3) = (0.1)\,f(0.3,\ 1.5048) = 0.2716$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.2267$$

Thus $y(0.3) = y_3 = y_2 + k = 1.504$.

Now the starting values of the Milne's method are :

$x_0 = 0.0$	$y_0 = 1.0000$	$f_0 = 1.0000$
$x_1 = 0.1$	$y_1 = 1.1169$	$f_1 = 1.3591$
$x_2 = 0.2$	$y_2 = 1.2773$	$f_2 = 1.8869$
$x_3 = 0.3$	$y_3 = 1.5049$	$f_3 = 2.7132$

Using the *predictor*,

$$y_4^{(p)} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$x_4 = 0.4$ $\quad y_4^{(p)} = 1.8344$ $\quad f_4 = 4.0988$

and the *corrector*,

$$y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4) \text{ yields}$$

$$y_4^{(c)} = 1.2773 + \frac{0.1}{3}[1.8869 + 4\,(2.7132) + 4.098]$$

$$= 1.8386 \qquad f_4 = 4.1159$$

Again using the *corrector*,

$$y_4^{(c)} = 1.2773 + \frac{0.1}{3}\,[1.8869 + 4\,(2.7132) + 4.1159]$$

$$= 1.8391 \qquad\qquad f_4 = 4.1182 \qquad\qquad ...(i)$$

Again using the *corrector*

$$y_4^{(c)} = 1.2773 + \frac{0.1}{3}\,[1.8869 + 4\,(2.7132) + 4.1182]$$

$$= 1.8392 \text{ which is same as } (i).$$

Hence $y\,(0.4) = 1.8392$.

PROBLEMS 32.4

1. Given $\frac{dy}{dx} = x^3 + y$, $y(0) = 2$. The value of $y(0.2) = 2.073$, $y(0.4) = 2.452$, and $y(0.6) = 3.023$ are got by R.K. Method of 4th order. Find $y(0.8)$ by Milne's predictor-corrector method taking $h = 0.2$. *(Anna, 2004)*
2. Given $2\,dy/dx = (1 + x^2)y^2$ and $y(0) = 1$, $y(0.1) = 1.06$, $y(0.2) = 1.12$, $y(0.3) = 1.21$, evaluate $y(0.4)$ by Milne's predictor-corrector method. *(V.T.U., 2011 S ; Madras, 2003)*
3. From the data given below, find y at $x = 1.4$, using Milne's predictor-corrector formula :

$$\frac{dy}{dx} = x^2 + \frac{y}{2}$$

x :	1	1.1	1.2	1.3
y :	2	2.2156	2.4549	2.7514

(V.T.U., 2007)

4. Using Milne's method, find $y(4.5)$ given $5xy' + y^2 - 2 = 0$ given $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$, $y(4.3) = 1.0143$, $y(4.4) = 1.0187$. *(Anna, 2007)*
5. If $\frac{dy}{dx} = 2e^x - y$, $y(0) = 2$, $y(0.1) = 2.010$, $y(0.2) = 2.04$ and $y(0.3) = 2.09$; find $y(0.4)$ using Milne's predictor-corrector method. *(V.T.U., 2010)*
6. Using Runge-Kutta method, calculate $y(0.1)$, $y(0.2)$, and $y(0.3)$ given that $\frac{dy}{dx} - \frac{2xy}{1+x^2} = 1$, $y(0) = 0$. Taking these values as starting values, find $y(0.4)$ by Milne's method.

32.10 ADAMS-BASHFORTH METHOD

Given $\frac{dy}{dx} = f(x, y)$ and $y_0 = y(x_0)$, we compute

$$y_{-1} = y(x_0 - h),\ y_{-2} = y(x_0 - 2h),\ y_{-3} = y(x_0 - 3h)$$

by Taylor's series of Euler's method or Runge-Kutta method.

Next we calculate $f_{-1} = f(x_0 - h, y_{-1}), f_{-2} = f(x_0 - 2h, y_{-2}), f_{-3} = f(x_0 - 3h, y_{-3})$.

Then to find y_1, we substitute Newton's backward interpolation formula

$$f(x, y) = f_0 + n\,\nabla f_0 + \frac{n(n+1)}{2}\nabla^2 f_0 + \frac{n(n+1)(n+2)}{6}\nabla^3 f_0 + \cdots$$

in

$$y_1 = y_0 + \int_{x_0}^{x_0+h} f(x, y)\,dx \qquad ...(1)$$

$$\therefore \qquad y_1 = y_0 + \int_{x_0}^{x_1}\left(f_0 + n\nabla f_0 + \frac{n(n+1)}{2}\nabla^2 f_0 + \cdots\right)dx \qquad [\text{Put } x = x_0 + nh,\ dx = hdn]$$

$$= y_0 + h\int_0^1\left(f_0 + n\nabla f_0 + \frac{n(n+1)}{2}\nabla^2 f_0 + \cdots\right)dn$$

$$= y_0 + h\left(f_0 + \frac{1}{2}\nabla f_0 + \frac{5}{12}\nabla^2 f_0 + \frac{3}{8}\nabla^3 f_0 + \cdots\right)$$

Neglecting fourth and higher order differences and expressing $\nabla f_0, \nabla^2 f_0$ and $\nabla^3 f_0$ in terms of function values, we get

$$y_1^{(p)} = y_0 + \frac{h}{24}\,(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3}) \qquad \text{...(2)}$$

This is called *Adams-Bashforth predictor formula.*

Having found y_1, we find $f_1 = f(x_0 + h_1, y_1)$.

Then to find a better value of y_1, we derive a *corrector formula* by substituting Newton's backward formula at f_1 *i.e.*,

$$f(x, y) = f_1 + n\nabla f_1 + \frac{n(n+1)}{2}\nabla^2 f_1 + \frac{n(n+1)(n+2)}{6}\nabla^3 f_1 + \cdots \text{ in (1).}$$

$$\therefore \qquad y_1 = y_0 + \int_{x_0}^{x_1}\left(f_1 + n\nabla f_1 + \frac{n(n+1)}{2}\nabla^2 f_1 + \cdots\right) dx \quad [\text{Put } x = x_1 + nh, dx = hdn]$$

$$= y_0 + \int_{-1}^{0}\left(f_1 + n\nabla f_1 + \frac{n(n+1)}{2}\nabla^2 f_1 + \cdots\right) dn$$

$$= y_0 + h\left(f_1 - \frac{1}{2}\nabla f_1 - \frac{1}{12}\nabla^2 f_1 - \frac{1}{24}\nabla^3 f_1 - \cdots\right)$$

Neglecting fourth and higher order differences and expressing $\nabla f_1, \nabla^2 f_1$ and $\nabla^3 f_1$ in terms of function values, we obtain

$$y_1^{(c)} = y_0 + \frac{h}{24}\,(9f_1 + 19f_0 - 5f_{-1} + f_{-2}) \qquad \text{...(3)}$$

which is called a *Adams-Moulton corrector formula.*

Then an improved value of f_1 is calculated and again the corrector (3) is applied to find a still better value of y_1. This step is repeated till y_1 remains unchanged and then proceed to calculate y_2 as above.

Obs. To apply both Milne and Adams-Bashforth methods, we require four starting values of y which are calculated by means of Picard's method or Taylor's series method or Euler's method or Runge-Kutta method. In practice, the Adams formulae (2) and (3) above together with fourth order Runge-Kutta formulae have been found to be most useful.

Example 32.20. *Given* $\frac{dy}{dx} = x^2(1+y)$ *and* $y(1) = 1, y(1.1) = 1.233, y(1.2) = 1.548, y(1.3) = 1.979$, *evaluate* $y(1.4)$ *by Adams-Bashforth method.* *(V.T.U., 2010 ; J.N.T.U., 2009 ; Anna, 2004)*

Solution. Here $f(x, y) = x^2(1 + y)$.

Starting values of the Adams-Bashforth method with $h = 0.1$, are

$$x = 1.0, y_{-3} = 1.000, f_{-3} = (1.0)^2(1 + 1.000) = 2.000$$
$$= 1.1, y_{-2} = 1.233, f_{-2} = 2.702$$
$$= 1.2, y_{-1} = 1.548, f_{-1} = 3.669$$
$$= 1.3, y_0 = 1.979, f_0 = 5.035$$

Using th

$$\frac{h}{} (55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3})$$

$$73, f_1 = 7.004$$

U

$$5f_{-1} + f_{-2})$$

$$19 \times 5.035 - 5 \times 3.669 + 2.702) = 2.575$$

Hence 5.

Example 32.21. *If $\frac{dy}{dx} = 2e^x y$, $y(0) = 0$, find $y(4)$ using Adams predictor-corrector formula by calculating $y(1)$, $y(2)$ and $y(3)$ using Euler's modified formula.* (J.N.T.U., 2006)

Solution. We have $f(x, y) = 2e^x y$.

To find 0.1 :

x	$2e^x y = y'$	*Mean slope*	*Old y + h (Mean slope) = new y*
0.0	4	—	$2 + 0.1\,(4) = 2.4$
0.1	$2e^{0.1}(2.4) = 5.305$	$\frac{1}{2}(4 + 5.305) = 4.6524$	$2 + 0.1\,(4.6524) = 2.465$
0.1	$2e^{0.1}(2.465) = 5.449$	$\frac{1}{2}(4 + 5.449) = 4.7244$	$2 + 0.1\,(4.7244) = 2.472$
0.1	$2e^{0.1}(2.4724) = 5.465$	$\frac{1}{2}(4 + 5.465) = 4.7324$	$2 + 0.1\,(4.7324) = 2.473$
0.1	$2e^{0.1}(2.473) = 5.467$	$\frac{1}{2}(4 + 5.467) = 4.7333$	$2 + 0.1\,(4.7333) = 2.473$
0.1	5.467	—	$2 + 0.1\,(5.467) = 3.0199$
0.2	$2e^{0.2}(3.0199) = 7.377$	$\frac{1}{2}(5.467 + 7.377) = 6.422$	$2.473 + 0.1\,(6.422) = 3.1155$
0.2	7.611	$\frac{1}{2}(5.467 + 7.611) = 6.539$	$2.473 + 0.1\,(6.539) = 3.127$
0.2	7.639	$\frac{1}{2}(5.467 + 7.639) = 6.553$	$2.473 + 0.1\,(6.553) = 3.129$
0.2	7.643	$\frac{1}{2}(5.467 + 7.643) = 6.555$	$2.473 + 0.1\,(6.455) = 3.129$
0.2	7.643	—	$3.129 + 0.1\,(7.643) = 3.893$
0.3	$2e^{0.3}(3.893) = 10.51$	$\frac{1}{2}(7.643 + 10.51) = 9.076$	$3.129 + 0.1\,(9.076) = 4.036$
0.3	10.897	$\frac{1}{2}(7.643 + 10.897) = 9.266$	$3.129 + 0.1\,(9.2696) = 4.056$
0.3	10.949	$\frac{1}{2}(7.643 + 10.949) = 9.296$	$3.129 + 0.1\,(9.296) = 4.058$
0.3	10.956	$\frac{1}{2}(7.643 + 10.956) = 9.299$	$3.129 + 0.1\,(9.299) = 4.0586$

To find y(0.4) by Adam's method, the starting values with $h = 0.1$ are

$x = 0.0$ $\quad y_{-3} = 2.4$ $\quad f_{-3} = 4$

$x = 0.1$ $\quad y_{-2} = 2.473$ $\quad f_{-2} = 5.467$

$x = 0.2$ $\quad y_{-1} = 3.129$ $\quad f_{-1} = 7.643$

$x = 0.3$ $\quad y_0 = 4.059$ $\quad f_{-0} = 10.956$

Using the predictor formula

$$y_1^{(p)} = y_0 + \frac{h}{24}\,(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3})$$

$$= 4.059 + \frac{0.1}{24}\,(55 \times 10.957 - 59 \times 7.643 + 37 \times 5.467 - 9 \times 4)$$

$$= 5.383$$

Now $x = 0.4$ $\quad y_1 = 5.383$ $\quad f_1 = 2e^{0.4}(5.383) = 16.061$

Using the corrector formula,

$$y_1^{(c)} = y_0 + \frac{h}{24}(9f_1 + 19f_0 - 5f_{-1} + f_{-2})$$

$$= 4.0586 + \frac{0.1}{24}(9 \times 6.061 + 19 \times 10.956 - 5 \times 7.643 + 5.467) = 5.392$$

Hence $y(0.4) = 5.392$.

Example 32.22. *Solve the initial value problem $dy/dx = x - y^2$, $y(0) = 1$ to find $y(0.4)$ by Adam's method. Starting solutions required are to be obtained using Runge-Kutta method of order 4 using step value $h = 0.1$.* (P.T.U., 2003)

Solution. We have $f(x, y) = x - y^2$.

To find y (0.1) :

Here $x_0 = 0, y_0 = 1, h = 0.1$

$\therefore$ $k_1 = hf(x_0, y_0) = (0.1) f(0, 1) = -0.1000$

$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = (0.1) f(0.05, 0.95) = -0.08525$

$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = (0.1) f(0.05, 0.9574) = -0.0867$

$k_4 = hf(x_0 + h, y_0 + k_3) = (0.1) f(0.1, 0.9137) = -0.07341$

$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = -0.0883$

Thus $y(0.1) = y_1 = y_0 + k = 1 - 0.0883 = 0.9117$

To find y(0.2) :

Here $x_1 = 0.1, y_1 = 0.9117, h = 0.1$.

$\therefore$ $k_1 = hf(x_1, y_1) = (0.1) f(0.1, 0.9117) = -0.0731$

$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = (0.1) f(0.15, 0.8751) = -0.0616$

$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = (0.1) f(0.15, 0.8809) = -0.0626$

$k_4 = hf(x_1 + h, y_1 + k_3) = (0.1) f(0.2, 0.8491) = -0.0521$

$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = -0.0623$

Thus $y(0.2) = y_2 = y_1 + k = 0.8494$.

To find y (0.3) :

Here $x_2 = 0.2, y_2 = 0.8494, y = 0.1$

$k_1 = hf(x_2, y_2) = (0.1) f(0.2, 0.8494) = -0.0521$

$k_2 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_1\right) = (0.1) f(0.25, 0.8233) = -0.0428$

$k_3 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_2\right) = (0.1) f(0.25, 0.828) = -0.0436$

$k_4 = hf(x_2 + h, y_2 + k_3) = (0.1) f(0.3, 0.8058) = -0.0349$

$k = \frac{1}{6}(k_1 + 2y_2 + 2k_3 + k_4) = -0.0438$

Thus $y(0.3) = y_3 = y_2 + k = 0.8061$

Now the starting values of Adam's method with $h = 0.1$ are :

$x = 0.0$	$y_{-3} = 1.0000$	$f_{-3} = 0.0 - (1.0)^2$	$= -1.0000$
$x = 0.1$	$y_{-2} = 0.9117$	$f_{-2} = 0.1 - (0.9117)^2$	$= -1.7312$
$x = 0.2$	$y_{-1} = 0.8494$	$f_{-1} = 0.2 - (0.8494)^2$	$= -0.5215$
$x = 0.3$	$y_0 = 0.8061$	$f_0 = 0.3 - (0.8061)^2$	$= -0.3498$

Using the *predictor*,

$$y_1^{(p)} = y_0 + \frac{h}{24}(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3})$$

$x = 0.4$ $$y_1^{(p)} = 0.8061 + \frac{0.1}{24}[55(-0.3498) - 59(-0.5215) + 37(-0.7312) - 9(-1)]$$

$$= 0.7789 \qquad f_1 = -0.2067$$

Using the *corrector*,

$$y_1^{(c)} = y_0 + \frac{h}{24}(9f_1 + 19f_0 - 5f_{-1} + f_{-2})$$

$$y_1^{(c)} = 0.8061 + \frac{0.1}{24}[9(-0.2067) + 19(-0.3498) - 5(-0.5215) - 0.7312] = 0.7785$$

Hence $y(0.4) = 0.7785$.

PROBLEMS 32.5

1. Using Adams-Bashforth method, obtain the solution of $dy/dx = x - y^2$ at $x = 0.8$, given the values

x :	0	0.2	0.4	0.6
y :	0	0.0200	0.0795	0.1762

(Bhopal, 2002)

2. Using Adams-Bashforth formulae, determine $y(0, 4)$ given the differential equation $dy/dx = \frac{1}{2}xy$ and the data

x :	0	0.1	0.2	0.3
y :	1	1.0025	1.0101	1.0228

3. Given $y' = x^2 - y$, $y(0) = 1$ and the starting values $y(0.1) = 0.90516$, $y(0.2) = 0.82127$, $y(0.3) = 0.74918$, evaluate $y(0, 4)$ using Adams-Bashforth method. *(S.V.T.U., 2007)*
4. Using Adams-Bashforth method, find $y(4.4)$ given $5xy' + y^2 = 2$, $y(4) = 1$, $y(4, 1) = 1.0049$, $y(4, 2) = 1.0097$ and $y(4.3) = 1.0143$.
5. Given the differential equation $dy/dx = x^2y + x^2$ and the data :

x :	1	1.1	1.2	1.3
y :	1	1.233	1.548488	1.978921

(Indore, 2003 S)

6. Using Adams-Bashforth method, evaluate $y(1.4)$, if y satisfies $dy/dx + y/x = 1/x^2$ and $y(1) = 1$, $y(1.1) = 0.996$, $y(1.2) = 0.986$, $y(1.3) = 0.972$. *(Madras, 2003)*

32.11 SIMULTANEOUS FIRST ORDER DIFFERENTIAL EQUATIONS

The simultaneous differential equations of the type

$$\frac{dy}{dx} = f(x, y, z) \qquad \text{...(1)}$$

and

$$\frac{dz}{dx} = \phi(x, y, z) \qquad \text{...(2)}$$

with initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$ can be solved by the methods discussed in the preceding sections, especially by Picard's or Runge-Kutta methods.

(*i*) *Picard's method gives*

$$y_1 = y_0 + \int f(x, y_0, z_0)\,dx, \; z_1 = z_0 + \int \phi(x, y_0, z_0)\,dx$$

$$y_2 = y_0 + \int f(x, y_1, z_1)\,dx, \; z_2 = z_0 + \int \phi(x, y_1, z_1)\,dx$$

$$y_3 = y_0 + \int f(x, y_2, z_2)\,dx, \; z_3 = z_0 + \int \phi(x, y_2, z_2)\,dx$$

and so on.

(*ii*) *Taylor's series* method is used as follows :

If h be the step-size, $y_1 = y(x_0 + h)$ and $z_1 = z(x_0 + h)$. Then Taylor's algorithm for (1) and (2) gives

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots \quad \ldots(3)$$

$$z_1 = z_0 + hz_0' + \frac{h^2}{2!} z_0'' + \frac{h^3}{3!} + z_0''' + \ldots \quad \ldots(4)$$

Differentiating (1) and (2) successively, we get y'', z'', etc. So the values $y_0', y_0'', y_0'''\ldots$ and $z_0', z_0'', z_0'''\ldots$ are known. Substituting these in (3) and (4), we obtain y_1, z_1 for the next step.

Similarly, we have the algorithms

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \ldots \quad \ldots(5)$$

$$z_2 = z_1 + hz_1' + \frac{h^2}{2!} z_1'' + \frac{h^3}{3!} z_1''' + \ldots \quad \ldots(6)$$

Since y_1 and z_1 are known, we can calculate $y_1', y_1'', \ldots$ and $z_1', z_1'' \ldots$. Substituting these in (5) and (6), we get y_2 and z_2.

Proceeding further, we can calculate the other values of y and z step by step.

(*iii*) *Runge-Kutta method* is applied as follows :

Starting at (x_0, y_0, z_0) and taking the step-sizes for x, y, z to be h, k, l respectively, the Runge-Kutta method gives,

$$k_1 = hf(x_0, y_0, z_0) \qquad l_1 = h\phi(x_0, y_0, z_0)$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right) \qquad l_2 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right) \qquad l_3 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3) \qquad l_4 = h\phi(x_0 + h, y_0 + k_3, z_0 + l_3)$$

Hence $\quad y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad$ and $\quad z_1 = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

To compute y_2 and z_2, we simply replace x_0, y_0, z_0 by x_1, y_1, z_1 in the above formulae.

Example 32.23. *Using Picard's method find approximate values of y and z corresponding to x = 0.1, given that y(0) = 2, z(0) = 1 and dy/dx = x + z, dz/dx = x − y².*

Solution. Here $x_0 = 0, y_0 = 2, z_0 = 1$,

$$\frac{dy}{dx} = f(x, y, z) = x + z; \quad \text{and} \quad \frac{dz}{dx} = \phi(x, y, z) = x - y^2$$

$$\therefore \quad y = y_0 + \int_{x_0}^{x} f(x, y\, z)\, dx \quad \text{and} \quad z = z_0 + \int_{x_0}^{x} \phi(x, y\, z)\, dx.$$

First approximations $\quad y_1 = y_0 + \int_{x_0}^{x} \phi(x, y_0, z_0)\, dx = 2 + \int_0^x (x+1)\, dx = 2 + x + \frac{1}{2}x^2$

$$z_1 = z_0 + \int_{x_0}^{x} \phi(x, y_0, z_0)\, dx = 1 + \int_0^x (x-4)\, dx = 1 - 4x + \frac{1}{2}x^2$$

Second approximations $\quad y_2 = y_0 + \int_{x_0}^{x} f(x, y_1, z_1)\, dx = 2 + \int_0^x \left(x + 1 - 4x + \frac{1}{2}x^2\right) dx$

$$= 2 + x - \frac{3}{2}x^2 + \frac{x^3}{6}$$

$$z_2 = z_0 + \int_{x_0}^{x} \phi(x, y_1, z_1)\, dx$$

$$= 1 + \int_{x_0}^{x} \left[x - \left(2 + x + \frac{1}{2}x^2\right)^2\right] dx = \mathbf{1 - 4x + \frac{3}{2}x^2 - x^3 - \frac{x^4}{4} - \frac{x^5}{20}}.$$

Third approximations $y_3 = y_0 + \int_{x_0}^{x} f(x, y_2, z_2)\, dx$

$$= 2 + x - \frac{3}{2}x^2 - \frac{1}{2}x^3 - \frac{1}{4}x^4 - \frac{1}{20}x^5 - \frac{1}{120}x^6$$

$$z_3 = z_0 + \int_{x_0}^{x} \phi(x, y_2, z_2)\, dx$$

$$= 1 - 4x - \frac{3}{2}x^2 + \frac{5}{3}x^3 + \frac{7}{12}x^4 - \frac{31}{60}x^5 + \frac{1}{12}x^6 - \frac{1}{252}x^7$$

and so on.

When $x = 0.1$,

$y_1 = 2.105, y_2 = 2.08517, y_3 = 2.08447$

$z_1 = 0.605, z_2 = 0.58397, z_3 = 0.58672.$

Hence $y(0.1) = 2.0845, z(0.1) = 0.5867$

correct to four decimal places.

Example 32.24. *Solve the differential equations*

$$\frac{dy}{dx} = 1 + xz, \quad \frac{dz}{dx} = -xy \text{ for } x = 0.3,$$

using fourth order Runge-Kutta method. Initial values are $x = 0, y = 0, z = 1$.

Solution. Here $f(x, y, z) = 1 + xz$, $\phi(x, y, z) = -xy$

$x_0 = 0, y_0 = 0, z_0 = 1$. Let us take $h = 0.3$.

$\therefore$ $k_1 = h\, f(x_0, y_0, z_0) = 0.3\, f(0, 0, 1) = 0.3\,(1 + 0) = 0.3$

$l_1 = h\, \phi(x_0, y_0, z_0) = 0.3\,(-0 \times 0) = 0$

$$k_2 = h\, f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$

$$= (0.3)\, f(0.15, 0.15, 1) = 0.3\,(1 + 0.15) = 0.345$$

$$l_2 = h\, \phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$

$$= 0.3\,[-(0.15)(0.15)] = -0.00675.$$

$$k_3 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$= (0.3)\, f(0.15, 0.1725, 0.996625)$$

$$= 0.3\,[1 + 0.996625 \times 0.15] = 0.34485$$

$$l_3 = h\, \phi\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$= 0.3\,[-(0.15)(0.1725)] = -0.007762$$

$$k_4 = h\, f(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$= (0.3)\, f(0.3, 0.34485, 0.99224) = 0.3893$$

$$l_4 = h\, \phi(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$= 0.3\,[-(0.3)(0.34485)] = -0.03104$$

Hence $$y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

i.e., $$y(0.3) = 0 + \frac{1}{6}[0.3 + 2(0.345) + 2(0.34485) + 0.3893] = 0.34483$$

and $$z(x_0 + h) = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

i.e., $$z(0.3) = 1 + \frac{1}{6}[0 + 2 + (-0.00675) + 2(-0.0077625) + (-0.03104)] = 0.98999$$

32.12 SECOND ORDER DIFFERENTIAL EQUATIONS

Consider the second order differential equation $\dfrac{d^2y}{dx^2} = f\left(x, y, \dfrac{dy}{dx}\right)$

By writing $dy/dx = z$, it can be reduced to two first order simultaneous differential equations

$$\frac{dy}{dx} = z, \qquad \frac{dz}{dx} = f(x, y, z)$$

These equations can be solved as explained above.

Example 32.25. *Using Runge-Kutta method, solve $y'' = xy'^2 - y^2$ for $x = 0.2$ correct to 4 decimal places. Initial conditions are $x = 0$, $y = 1$, $y' = 0$.* (Delhi, 2002)

Solution. Let $dy/dx = z = f(x, y, z)$. Then $dz/dx = xz^2 - y^2 = \phi(x, y, z)$

We have $x_0 = 0, y_0 = 1, z_0 = 0, h = 0.2$.

Using $k_1, k_2 \ldots$ for $f(x, y, z)$ and $l_1, l_2, \ldots$ for $\phi(x, y, z)$, Runge-Kutta formulae become

$$k_1 = hf(x_0, y_0, z_0) = 0.2\,(0) = 0$$

$$k_2 = hf(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_1, z_0 + \tfrac{1}{2}l_1) = 0.2\,(-0.1) = -0.02$$

$$k_3 = hf(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_2, z_0 + \tfrac{1}{2}l_2) = 0.2\,(-0.0999) = -0.02$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3) = 0.2\,(-0.1958) = -0.0392$$

$$\therefore \quad k = \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = -0.0199$$

$$l_1 = h\phi(x_0, y_0, z_0) = 0.2(-1) = -0.2$$

$$l_2 = h\phi(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_1, z_0 + \tfrac{1}{2}l_1) = 0.2(-0.999) = -0.1998$$

$$l_3 = h\phi(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_2, z_0 + \tfrac{1}{2}l_2) = 0.2(-0.9791) = -0.1958$$

$$l_4 = h\phi(x_0 + h, y_0 + k_3, z_0 + l_3) = 0.2(0.9527) = -0.1905$$

$$l = \tfrac{1}{6}(l_1 + 2l_2 + 2l_3 + 2l_4) = -0.1970$$

Hence at $x = 0.2$,

$$y = y_0 + k = 1 - 0.0199 = 0.9801$$

and $$y' = z = z_0 + l = 0 - 0.1970 = -0.1970.$$

Example 32.26. *Given $y'' + xy' + y = 0$, $y(0) = 1$, $y'(0) = 0$, obtain y for $x = 0(0.1)\,0.3$ by any method. Further, continue the solution by Milne's method to calculate $y(0.4)$.* (Anna, 2004; Madras, 2003 S)

Solution. Putting $y' = z$, the given equation reduces to the simultaneous equations

$$z' + xz + y = 0, \; y' = z \qquad \ldots(i)$$

We employ Taylor's series method to find y.

Differentiating the given equation n times, we get

$$y_{n+2} + xy_{n+1} + ny_n + y_n = 0$$

At $$x = 0, (y_{n+2})_0 = -(n+1)(y_n)_0$$

$\therefore$ $$y(0) = 1, \text{ gives } y_2(0) = -1, y_4(0) = 32, y_6(0) = -5 \times 3, \ldots$$

and $$y_1(0) = 0 \text{ yields } y_3(0) = y_5(0) = \ldots = 0.$$

Expanding $y(x)$ by Taylor's series, we have

$$y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \ldots$$

$$\therefore \quad y(x) = 1 - \frac{x^2}{2!} + \frac{3}{4!}x^4 - \frac{5 \times 3}{6!}x^6 + \ldots \qquad \ldots(ii)$$

and $$z(x) = y'(x) = -x + \frac{1}{2}x^3 - \frac{1}{8}x^5 = \ldots = -xy \qquad \ldots(iii)$$

From (ii), we have

$$y(0.1) = 1 - \frac{(0.1)^2}{2} + \frac{1}{8}(0.1)^4 - \ldots = 0.995$$

$$y(0.2) = 1 - \frac{(0.2)^2}{2} + \frac{(0.2)^4}{8} - \ldots = 0.9802$$

$$y(0.3) = 1 - \frac{(0.3)^2}{2} + \frac{(0.3)^4}{8} - \frac{(0.3)^6}{48} + \ldots = 0.956$$

From (*iii*), we have

$$z(0.1) = -0.0995,\ z(0.2) = -0.196,\ z(0.3) = -0.2863.$$

Also from (*i*), $z'(x) = -(xz + y)$ $\therefore$ $z'(0.1) = 0.985,\ z'(.2) = -0.941,\ z'(0.3) = -0.87.$

Applying Milne's predictor formula, first to z and then to y, we obtain

$$z(0.4) = z(0) + \frac{4}{3}(0.1)\{2z'(0.1) - z'(0.2) + 2z'(0.3)\}$$

$$= 0 + \left(\frac{0.4}{3}\right)\{-1.79 + 0.941 - 1.74\} = -0.3692$$

and

$$y(0.4) = y(0) + \frac{4}{3}(0.1)\{2y'(0.1) - y'(0.2) + 2y'(0.3)\}$$

$$= 0 + \left(\frac{0.4}{3}\right)\{-0.199 + 0.196 - 0.5736\} = 0.9231 \qquad [\because\ y' = z]$$

Also $z'(0.4) = -\{x(0.4)z(0.4) + y(0.4)\} = \{0.4(-0.3692) + 0.9231\} = -0.7754.$

Now applying Milne's corrector formula, we get

$$z(0.4) = z(0.2) + \frac{h}{3}\{z'(0.2) + 4z'(0.3) + z'(0.4)\}$$

$$= -0.196 + \left(\frac{0.1}{3}\right)\{-0.941 - 3.48 - 0.7754\} = -0.3692$$

and

$$y(0.4) = y(0.2) + \frac{h}{3}\{y'(0.2) + 4y'(0.3) + y'(0.4)\}$$

$$= 0.9802 + \left(\frac{0.1}{3}\right)\{-0.196 - 1.1452 - 0.3692\} = 0.9232$$

Hence $y(0.4) = 0.9232$ and $z(0.4) = -0.3692.$

PROBLEMS 32.6

1. Apply Picard's method to find the third approximation to the values of y and z, given that $dy/dx = z$, $dz/dx = x^3(y + z)$, given $y = 1$, $z = \frac{1}{2}$ when $x = 0$.
2. Solve the following differential equations using Taylor series method of the 4th order, for $x = 0.1$ and 0.2, $\frac{dy}{dx} = xz + 1$, $\frac{dz}{dy} = -xy$; $y(0) = 0$ and $z(0) = 1$.
3. Find $y(0.1)$, $z(0.1)$, $y(0.2)$ and $z(0.2)$ from the system of equations $y' = x + z$, $z' = x - y^2$ given $y(0) = 0$, $z(0) = 1$ using Runge-Kutta of 4th order. (*J.N.T.U., 2009*)
4. Using Picard's method, obtain the second approximation to the solution of
$$\frac{d^2y}{dx^2} = x^3\frac{dy}{dx} + x^3y \text{ so that } y(0) = 1,\ y'(0) = \frac{1}{2}.$$
5. Use Picard's method to approximate y when $x = 0.1$, given that $\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + y = 0$ and $y = 0.5$, $\frac{dy}{dx} = 0.1$, when $x = 0$.
6. Using Runge-Kutta method of order four, solve $y'' = y + xy'$, $y(0) = 1$, $y'(0) = 0$ to find $y(0.2)$ and $y'(0.2)$.
7. Consider the second order value problem $y'' - 2y' + 2y = e^{2t}\sin t$ with $y(0) = -0.4$ and $y'(0) = -0.6$. Using the fourth order Runge-Kutta method, find $y(0.2)$. (*Anna, 2003*)

8. The angular displacement θ of a simple pendulum is given by the equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{l} \sin \theta = 0$$

where $l = 98$ cm and $g = 980$ cm/sec^2. If $\theta = 0$ and $d\theta/dt = 4.472$ at $t = 0$, use Runge-Kutta method to find θ and $d\theta/dt$ when $t = 0.2$ sec.

32.13 BOUNDARY VALUE PROBLEMS

Such a problem requires the solution of a differential equation in a region R subject to the various conditions on the boundary of R. Practical applications give rise to many such problems. We shall discuss two-point linear boundary value problems of the following types :

(*i*) $\dfrac{d^2y}{dx^2} + \lambda(x)\dfrac{dy}{dx} + \mu(x)y = \gamma(x)$ with the conditions $y(x_0) = a$, $y(x_n) = b$.

(*ii*) $\dfrac{d^4y}{dx^4} + \lambda(x)y = \mu(x)$ with the conditions $y(x_0) = y'(x_0) = a$ and $y(x_n) = y'(x_n) = b$.

While there exist many numerical methods for solving such boundary value problems, the method of finite-differences is most commonly used. We shall explain this method in the next section.

32.14 FINITE-DIFFERENCE METHOD

In this method, the derivatives appearing in the differential equation and the boundary conditions are replaced by their finite-difference approximations and the resulting linear system of equations are solved by any standard procedure. These roots are the values of the required solution at the pivotal points.

The finite-difference approximations to the various derivatives are derived as under :

If $y(x)$ and its derivatives are single-valued continuous functions of x then by Taylor's expansion, we have

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \frac{h^2}{3!}y'''(x) + \ldots \qquad \ldots(1)$$

and

$$y(x-h) = y(x) - hy'(x) + \frac{h^2}{2!}y''(x) - \frac{h^3}{3!}y'''(x) + \ldots \qquad \ldots(2)$$

Equation (1) gives $y'(x) = \dfrac{1}{h}[y(x+h) - y(x)] - \dfrac{h}{2}y''(x) - \ldots$

i.e.,

$$y'(x) = \frac{1}{h}[y(x+h) - y(x)] + O(h)$$

which is the *forward difference approximation of* $y'(x)$ with an error of the order h.

Similarly (2) gives $y'(x) = \dfrac{1}{h}[y(x) - y(x-h)] + O(h)$

which is the *backward difference approximation of* $y'(x)$ with an error of the order h.

Subtracting (2) from (1), we obtain

$$y'(x) = \frac{1}{2h}[y(x+h) - y(x-h)] + O(h^2)$$

which is the *central-difference approximation of* $y'(x)$ with an error of the order h^2. Clearly this central difference approximation to $y'(x)$ is better than the forward or backward difference approximations and hence should be preferred.

Adding (1) and (2), we get

$$y''(x) = \frac{1}{h^2}[y(x+h) - 2y(x) + y(x-h)] + O(h^2)$$

which is the central difference approximation of $y''(x)$. Similarly we can derive central difference approximations to higher derivatives.

Hence *the working expressions for the central difference approximations to the first four derivatives of* y_i *are as under :*

$$y'_i = \frac{1}{2h}(y_{i+1} - y_{i-1}) \quad ...(3)$$

$$y''_i = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1}) \quad ...(4)$$

$$y'''_i = \frac{1}{2h^3}(y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}) \quad ...(5)$$

$$y_i^{iv} = \frac{1}{h^4}(y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i+1} + y_{i-2}) \quad ...(6)$$

Obs. The accuracy of this method depends on the size of the sub-interval h and also on the order of approximation. As we reduce h, the accuracy improves but the number of equations to be solved also increases.

Example 32.27. *Solve the equation* $y'' = x + y$ *with the boundary conditions* $y(0) = y(1) = 0$. *(Calicut, 1999)*

Solution. We divide the interval (0, 1) into four sub-intervals so that $h = 1/4$ and the pivot points are $x_0 = 0, x_1 = 1/4, x_2 = 1/2, x_3 = 3/4$ and $x_4 = 1$.

The differential equation is approximated as

$$\frac{1}{h^2}[y_{i+1} - 2y_i + y_{i-1}] = x_i + y_i$$

or

$$16y_{i+1} - 33y_i + 16y_{i-1} = x_i, \; i = 1, 2, 3.$$

Using $y_0 = y_4 = 0$, we get the system of equations

$$16y_2 - 33y_1 = \frac{1}{4}$$

$$16y_3 - 33y_2 + 16y_1 = \frac{1}{2}$$

$$-33y_3 + 16y_2 = \frac{3}{4}$$

Their solution gives

$$y_1 = -0.03488, y_2 = -0.05632, y_3 = -0.05003.$$

Obs. The exact solution being $y(x) = \dfrac{\sinh x}{\sinh 1} - x$, the error at each nodal point is given in the table:

x	*Computed value y(x)*	*Exact value y(x)*	*Error*
0.25	– 0.03488	– 0.03505	0.00017
0.5	– 0.05632	– 0.05659	0.00027
0.75	– 0.05003	– 0.05028	0.00025

Example 32.28. *Determine values of y at the pivotal points of the interval (0, 1), if y satisfies the boundary value problem* $y^{iv} + 81y = 81x^2$, $y(0) = y(1) = y''(0) = y''(1) = 0$. *(Take n = 3).*

Solution. Here $h = 1/3$ and the pivotal points are $x_0 = 0, x_1 = 1/3, x_2 = 2/3, x_3 = 1$. The corresponding y-values are $y_0(= 0), y_1, y_2, y_3 (= 0)$.

Replacing y^{iv} by its central difference approximation, the differential equation becomes

$$\frac{1}{h^4}(y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}) + 81y_i = 81x_i^2$$

or

$$y_{i+2} - 4y_{i+1} + 7y_i - 4y_{i-1} + y_{i-2} = x_i^2, \; i = 1, 2$$

At $i = 1$, $\quad y_3 - 4y_2 + 7y_1 - 4y_0 + y_{-1} = 1/9$

At $i = 2$, $\quad y_4 - 4y_3 + 7y_2 - 4y_1 + y_0 = 4/9$

Using $y_0 = y_3 = 0$, we get $\quad -4y_2 + 7y_2 + y_{-1} = 1/9 \quad ...(i)$

$$y_4 + 7y_1 - 4y_1 = 4/9 \quad ...(ii)$$

Regarding the conditions $y''_0 = y''_3 = 0$, we know that

$$x_i'' = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1})$$

At $i = 0$, $\quad y_0'' = 9(y_1 - 2y_0 + y_{-1})$

or $\quad y_{-1} = -y_1 \qquad [\because \; y_0 = y_0'' = 0] \quad ...(iii)$

At $i = 3$, $\quad y_3'' = 9(y_4 - 2y_3 + y_2)$

$$y_4 = -y_2 \qquad [\because \; y_3 = y_3'' = 0] \quad ...(iv)$$

Using (iii), the equation (i) becomes

$$-4y_2 + 6y_1 = 1/9 \qquad ...(v)$$

Using (iv), the equation (ii) reduces to

$$6y_2 - 4y_1 = 4/9 \qquad ...(vi)$$

Solving (v) and (vi), we obtain

$$y_1 = 11/90 \text{ and } y_2 = 7/45.$$

Hence $y(1/3) = 0.1222$ and $y(2/3) = 0.1556$.

Example 32.29. *The deflection of a beam is governed by the equation*

$$\frac{d^4y}{dx^4} + 81y = \phi(x)$$

where $\phi(x)$ is given by the table

x	:	1/3	2/3	1,
$\phi(x)$	:	81	162	243,

and boundary condition $y(0) = y'(0) = y''(1) = y'''(1) = 0$. Evaluate the deflection at the pivotal points of the beam using three sub-intervals.

Solution. Here $h = 1/3$ and the pivotal points are $x_0 = 0, x_1 = 1/3, x_2 = 2/3, x_3 = 1$. The corresponding y-values are $y_0(= 0), y_1, y_2, y_3$.

The given differential equation is approximated to

$$\frac{1}{h^4}(y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}) + 81y_i = \phi(x_i)$$

At $i = 1$, $\quad y_3 - 4y_2 + 7y_1 - 4y_0 + y_{-1} = 1 \qquad ...(i)$

At $i = 2$, $\quad y_4 - 4y_3 + 7y_2 - 4y_1 + y_0 = 2 \qquad ...(ii)$

At $i = 3$, $\quad y_5 - 4y_4 + 7y_3 - 4y_2 + y_1 = 3 \qquad ...(iii)$

We have $\quad y_0 = 0 \qquad ...(iv)$

Since $\quad y_i' = \dfrac{1}{2h}(y_{i+1} - y_{i-1})$

$\therefore$ for $i = 0$, $\quad 0 = y_0' = \dfrac{1}{2h}(y_1 - y_{-1})$ *i.e.* $y_{-1} = y_1 \qquad ...(v)$

Since $\quad y_i'' = \dfrac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1})$

$\therefore$ for $i = 3$, $\quad 0 = y_3'' = \dfrac{1}{h^2}(y_4 - 2y_3 + y_2)$, *i.e.* $y_4 = 2y_3 - y_2 \qquad ...(vi)$

Also $\quad y_i''' = \dfrac{1}{2h^3}(y_{i+2} - 2y_{i+1}\, 2y_{i-1} - y_{i-2})$

$\therefore$ for $i = 3$, $\quad 0 = y_3''' = \dfrac{1}{2h^3}(y_5 - 2y_4 + 2y_2 - y_1)$

$$y_5 = 2y_4 - 2y + y_1 \qquad ...(vii)$$

Using (iv) and (v), the equation (i) reduces to

$$y_3 - 4y_2 + 8y_1 = 1 \qquad ...(viii)$$

Using (iv) and (vi), the equation (ii) becomes

$$-y_3 + 3y_2 - 2y_1 = 1 \qquad ...(ix)$$

Using (*vi*) and (*vii*), the equation (*iii*) reduces to

$$3y_3 - 4y_2 + 2y_1 = 3 \qquad \ldots(x)$$

Solving (*viii*), (*ix*) and (*x*), we get

$$y_1 = 8/13, y_2 = 22/13, y_3 = 37/13.$$

Hence $\quad y(1/3) = 0.6154, y(2/3) = 1.6923, y(1) = 2.8462.$

PROBLEMS 32.7

1. Solve the boundary value problem for $x = 0.5$:
$$\frac{d^2y}{dx^2} + y + 1 = 0, y(0) = y(1) = 0. \qquad \text{(Take } n = 4)$$
2. Find an approximate solution of the boundary value problem :
$$y'' + 8(\sin^2 \pi y)y = 0, 0 \le x \le 1, y(0) = y(1) = 1. \qquad \text{(Take } n = 4)$$
3. Solve the boundary value problem
$$xy'' + y = 0, y(1) = 1, y(2) = 2. \qquad \text{(Take } n = 4)$$
4. Solve the equation
$$y'' - 4y' + 4y = e^{3x}, \text{ with the conditions } y(0) = 0, y(1) = -2, \text{ taking } n = 4.$$
5. Solve the boundary value problem $y'' - 64y + 10 = 0$ with $y(0) = y(1) = 0$ by the finite difference method. Compute the value of $y(0.5)$ and compare with the true value.
6. Solve the boundary value problem
$$y'' + xy' + y = 3x^2 + 2, y(0) = 0, y(1) = 1.$$
7. The boundary value problem governing the deflection of a beam of length 3 metres is given by
$$\frac{d^4y}{dx^4} + 2y = \frac{1}{9}x^2 + \frac{2}{3}x + 4, y(0) = y'(0) = y(3) = y'(3) = 0.$$
The beam is built-in at the left end ($x = 0$) and simply supported at the right end ($x = 3$). Determine y at the pivotal points $x = 1$ and $x = 2$.
8. Solve the boundary value problem,
$$\frac{d^4y}{dx^4} + 81y = 729x^2, y(0) = y'(0) = y''(1) = y'''(1) = 0. \quad \text{(Use } n = 3)$$
9. Solve the equation $y^{(iv)} - y''' + y = x^2$ subject to the boundary conditions
$$y(0) = y'(0) = 0 \text{ and } y(1) = 2, y'(1) = 0. \qquad \text{(Take } n = 5)$$

32.15 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 32.8

Select the correct answer or fill up the blanks in the following questions :

1. Which of the following is a step by step method :
(*a*) Taylor' s (*b*) Adams-Bashforth (*c*) Picard's (*d*) None.
2. The finite difference scheme for the equation $2y'' + y = 5$ is ..... .
3. If $y'' = x + y, y(0) = 1$ and $y^{(1)}(x) = 1 + x + x^2/2$, then by Picard's method, the value of $y^{(2)}(x)$ is ..... .
4. The iterative formula of Euler's method for solving $y' = f(x, y)$ with $y(x_0) = y_0$, is ..... .
5. Taylor's series for solution of first order ordinary differential equations is ..... .
6. Using Runge-Kutta method of order four, the value of $y(0.1)$ for $y' = x - 2y, y(0) = 1$ taking $h = 0.1$ is
(*a*) 0.813 (*b*) 0.825 (*c*) 0.0825 (*d*) none.
7. Given y_0, y_1, y_2, y_3, Milne's corrector formula to find y_4 for $dy/dx = f(x, y)$, is ..... .
8. The second order Runge-Kutta formula is ..... .
9. Adams-Bashforth predictor formula to solve $y' = f(x, y)$ given $y_0 = y(x_0)$ is ..... .
10. The multi-step methods available for solving ordinary differential equations are ..... .
11. To predict Adam's method atleast ..... values of y, prior to the desired value, are required.
12. Taylor's series solution of $y' = -xy, y(0) = 1$ upto x^4 is ..... .

13. Using modified Euler's method, the value of $y(0.1)$ for $\frac{dy}{dx} = x - y$, $y(0) = 1$ is ..... .

(*a*) 0.809 (*b*) 0.909 (*c*) 0.0809 (*d*) none.

14. Milne's Predictor formula is ..... .

15. Adam's corrector formula is ..... .

16. Using Euler's method, $dy/dx = (y - 2x)/y$, $y(0) = 1$; gives $y(0.1) =$ ..... .

17. $\frac{d^2y}{dx^2} + y^2 \frac{dy}{dx} + y = 0$ is equivalent to a set of two first order differential equations ..... and ..... .

18. The formula for the 4th order Runge-Kutta method is ..... .

19. Taylor's series method will be useful to give some ..... of Milne's method.

20. The name of two self-starting methods to solve $y' = f(x, y)$ given $y(x_0) = y_0$ are .....

21. In the derivation of fourth order Runge-Kutta formula, it is called fourth order because .....

22. If $y' = x$, $y(0) = 1$ then by Picard's method, the value of $y(1)$ is .....

(*a*) 0.915 (*b*) 0.905 (*c*) 0.981 (*d*) none.

23. The finite difference scheme of the differential equation $y'' + 2y = 0$ is .....

24. If $y' = -y$, $y(0) = 1$, the Euler's method, the value of $y(1)$ is

(*a*) 0.99 (*b*) 0.999 (*c*) 0.981 (*d*) none.

25. In Euler's method if h is small the method is too slow, if h is large, it gives inaccurate value. (True or False)

26. Runge-Kutta method is a self-starting method. (True or False)

27. Predictor-corrector methods are self-starting methods. (True or False)

CHAPTER 33

Numerical Solution of Partial Differential Equations

1. Introduction. 2. Classification of second order equations. 3. Finite difference approximation to derivatives. 4. Elliptic equations. 5. Solution of Laplace's equation. 6. Solution of Poisson's equations. 7. Parabolic equations. 8. Solution of heat equation. 9. Hyperbolic equations. 10. Solution of wave equation. 11. Objective Type of Questions.

33.1 INTRODUCTION

There are many boundary value problems which involve partial differential equations. Only a few of these equations can be solved by analytical methods. In most cases, we depend on the numerical solution of such partial differential equations. Of the various numerical methods available for solving these equations, the method of finite differences is most commonly used.

In this method, the derivatives appearing in the equation and the boundary conditions are replaced by their finite-difference approximations. Then the given equation is changed to a difference equation which is solved by iterative procedures. This process is slow but gives good results of boundary value problems. An added advantage of this method is that the computation can be done by electronic computer. Here we shall apply this method to the solution of important applied partial differential equations. For a detailed study, the reader should refer to author's book '*Numerical Methods in Engineering and Science*'.

33.2 CLASSIFICATION OF SECOND ORDER EQUATIONS

The general linear partial differential equation of the second order in two independent variables is of the form.

$$A(x, y)\frac{\partial^2 u}{\partial x^2} + B(x, y)\frac{\partial^2 u}{\partial x \partial y} + C(x, y)\frac{\partial^2 u}{\partial y^2} + F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

Such a partial differential equation is said to be

(*i*) **elliptic,** *if* $B^2 - 4AC < 0$,

(*ii*) **parabolic,** *if* $B^2 - 4AC = 0$,

and (*iii*) **hyperbolic,** *if* $B^2 - 4AC > 0$.

Example 33.1. *Classify the following equations :*

(*i*) $\dfrac{\partial^2 u}{\partial x^2} + 4\dfrac{\partial^2 u}{\partial x \partial y} + 4\dfrac{\partial^2 u}{\partial y^2} - \dfrac{\partial u}{\partial x} + 2\dfrac{\partial u}{\partial y} = 0$ (*ii*) $x^2\dfrac{\partial^2 u}{\partial x^2} + (1 - y^2)\dfrac{\partial^2 u}{\partial y^2} = 0, -\infty < x < \infty, -1 < y < 1$

(*Madras, 2003*)

(*iii*) $(1 + x^2)\dfrac{\partial^2 u}{\partial x^2} + (5 + 2x^2)\dfrac{\partial^2 u}{\partial x \partial t} + (4 + x^2)\dfrac{\partial^2 u}{\partial t^2} = 0.$

Solution. (*i*) Comparing this equation with (1) above, we find that

$$A = 1, B = 4, C = 4$$

$$\therefore \quad B^2 - 4AC = (4)^2 - 4 \times 1 \times 4 = 0$$

So the equation is parabolic.

(*ii*) Here $A = x^2, B = 0, C = 1 - y^2$

$$\therefore \quad B^2 - 4AC = 0 - 4x^2(1 - y^2) = 4x^2(y^2 - 1)$$

For all x between $-\infty$ and ∞, x^2 is positive

For all y between -1 and 1, $y^2 < 1 \quad \therefore \quad B^2 - 4AC < 0$

Hence the equation is elliptic.

(*iii*) Here $A = 1 + x^2, B = 5 + 2x^2, C = 4 + x^2$

$$\therefore \quad B^2 - 4AC = (5 + 2x^2)^2 - 4(1 + x^2)(4 + x^2) = 9 \text{ } i.e. > 0$$

So the equation is hyperbolic.

PROBLEMS 33.1

1. What is the classification of the equation $f_{xx} + 2f_{xy} + f_{yy} = 0$.
2. Determine whether the following equation is elliptic or hyperbolic ?
$$(x + 1)u_{xx} - 2(x + 2)u_{xy} + (x + 3)u_{yy} = 0.$$
3. Classify the equations (*i*) $y^2 u_{xx} - 2xy\, u_{xy} + x^2 u_{yy} + 2u_x - 3u = 0$ *(Madras, 2000 S)*

   (*ii*) $x^2 \dfrac{\partial^2 u}{\partial x^2} + y^2 \dfrac{\partial^2 u}{\partial y^2} = x\dfrac{\partial u}{\partial x} - y\dfrac{\partial u}{\partial y}$ *(P.T.U., 2009 S)*

   (*iii*) $\dfrac{3\partial^2 u}{\partial x^2} + 4\dfrac{\partial^2 u}{\partial x \partial y} + 6\dfrac{\partial^2 u}{\partial y^2} - 2\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} - u = 0.$ *(Anna, 2008)*
4. In which parts of the (x, y) plane is the following equation elliptic ?
$$\partial^2 u / \partial x^2 + \partial^2 u / \partial x \partial y + (x^2 + 4y^2)\, \partial^2 u / \partial y^2 = 2 \sin(xy).$$

33.3 FINITE-DIFFERENCE APPROXIMATIONS TO DERIVATIVES

Consider a rectangular region R in the x-y plane. Divide this region into a rectangular network of sides $\Delta x = h$ and $\Delta y = k$ as shown in Fig. 33.1. The points of intersection of the dividing lines are called *mesh points, nodal points* or *grid points.*

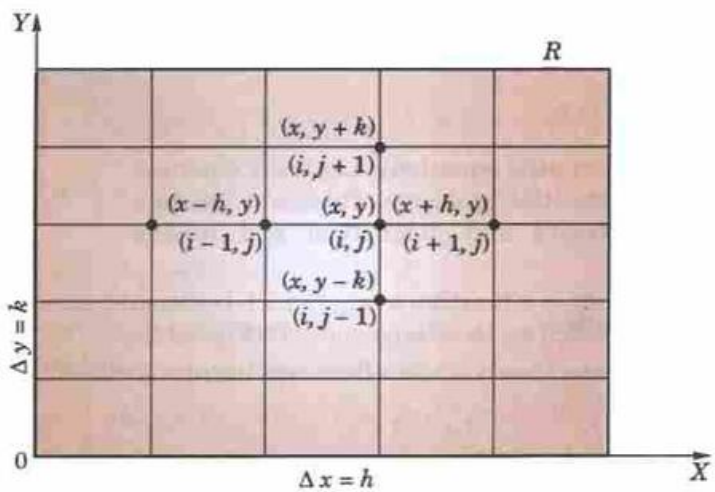


Fig. 33.1

Then we have the finite difference approximations for the partial derivatives in x-direction (§ 32.12) :

$$\frac{\partial u}{\partial x} = \frac{u(x + h, y) - u(x, y)}{h} + O(h)$$