If a function is continuous at all points of a region, then it is said to be *continuous in that region.* A function which is not continuous at a point is said to be *discontinuous* at that point.

Obs. *Usually, the limit is the same irrespective of the path along which the point (x, y) approaches (a, b) and*

$$\underset{x\to a}{Lt}\left\{\underset{y\to b}{Lt} f(x,y)\right\} = \underset{y\to b}{Lt}\left\{\underset{x\to a}{Lt} f(x,y)\right\}$$

But it is not always so, as the following examples show :

$$\text{Lt}\left(\frac{x-y}{x+y}\right) \text{ as } (x, y) \to (0, 0) \text{ along the line } y = mx$$

$$= \underset{x\to 0}{\text{Lt}} \frac{x-mx}{x+mx} = \frac{1-m}{1+m} \text{ which is different for lines with different slopes.}$$

Also $$\underset{x\to 0}{\text{Lt}}\left[\underset{y\to 0}{\text{Lt}}\left(\frac{x-y}{x+y}\right)\right] = \underset{x\to 0}{\text{Lt}}\left(\frac{x}{x}\right) = 1, \text{ whereas } \underset{y\to 0}{\text{Lt}}\left[\underset{x\to 0}{\text{Lt}}\left(\frac{x-y}{x+y}\right)\right] = \underset{y\to 0}{\text{Lt}}\left(\frac{-y}{y}\right) = 1.$$

$\therefore$ As (x, y) is made to approach $(0, 0)$ along different paths, $f(x, y)$ approaches different limits. Hence the two repeated limits are not equal and $f(x, y)$ is discontinuous at the origin.

Also the function is not defined at $(0, 0)$ since $f(x, y) = 0/0$ for $x = 0, y = 0$.

(4) *As in the case of functions of one variable, the following results hold :*

I. *If* $\underset{\substack{x\to a\\ y\to b}}{Lt} f(x, y) = l$ and $\underset{\substack{x\to a\\ y\to b}}{Lt} g(x, y) = m$,

then (*i*) *If* $\underset{\substack{x\to a\\ y\to b}}{Lt} [f(x, y) \pm g(x, y)] = l \pm m$ (*ii*) $\underset{\substack{x\to a\\ y\to b}}{Lt} [f(x,y) \,.\, g(x, y)] = l \,.\, m$

(*iii*) $\underset{\substack{x\to a\\ y\to b}}{Lt} [f(x,y)/g(x, y)] = l/m$ $(m \neq 0)$

II. *If $f(x, y)$, $g(x, y)$ are continuous at (a, b) then so also are the functions*

$f(x, y) \pm g(x, y)$, $f(x, y) \,.\, g(x, y)$ and $f(x, y)/g(x, y)$

provided $g(x, y) \neq 0$ in the last case.

PROBLEMS 5.1

Evaluate the following limits :

1. $\underset{\substack{x\to 1\\ y\to 2}}{\text{Lt}} \dfrac{2x^2y}{x^2+y^2+1}$
2. $\underset{\substack{x\to 0\\ y\to 0}}{\text{Lt}} \dfrac{xy}{x^2+y^2}$
3. $\underset{\substack{x\to \infty\\ y\to 2}}{\text{Lt}} \dfrac{xy+1}{x^2+2y^2}$
4. $\underset{\substack{x\to 1\\ y\to 1}}{\text{Lt}} \dfrac{x(y-1)}{y(x-1)}$
5. If $f(x, y) = \dfrac{x-y}{2x+y}$, show that $\underset{x\to 0}{\text{Lt}}\left[\underset{y\to 0}{\text{Lt}} f(x, y)\right] \neq \underset{y\to 0}{\text{Lt}}\left[\underset{x\to 0}{\text{Lt}} f(x, y)\right]$

   Also show that the function is discountinuous at the origin.
6. Show that the function $f(x, y) = x^2 + 2y$, $(x, y) \neq (1, 2)$

   $3(x, y) = (1, 2)$ $= 0$

   is discontinuous at $(1, 2)$.
7. Investigate the continuity of the function

   $f(x, y) = xy/(x^2 + y^2)$, $(x, y) \neq (0, 0)$

   $= 0$ $(x, y) = (0, 0)$

   at the origin.

Note. In whatever follows, all the functions considered are continuous and their partial derivatives (as defined below) exist.

5.2 PARTIAL DERIVATIVES

Let $z = f(x, y)$ be a function of two variables x and y.

If we keep y as constant and vary x alone, then z is a function of x only. The derivative of z with respect to x, treating y as constant, is called the *partial derivative of z with respect to x* and is denoted by one of the symbols

$$\frac{\partial z}{\partial x}, \frac{\partial f}{\partial x}, f_x(x, y), D_x f. \qquad \text{Thus} \quad \frac{\partial z}{\partial x} = \underset{\delta x \to 0}{\text{Lt}} \frac{f(x + \delta x, y) - f(x, y)}{\delta x}$$

Similarly, the derivative of z with respect to y, keeping x as constant, is called the *partial derivative of z with respect to y* and is denoted by one of the symbols.

$$\frac{\partial z}{\partial y}, \frac{\partial f}{\partial y}, f_y(x, y), D_y f. \qquad \text{Thus} \quad \frac{\partial z}{\partial y} = \underset{\partial y \to 0}{\text{Lt}} \frac{f(x, y + \delta y) - f(x, y)}{\delta y}$$

Similarly, if z is a function of three or more variables $x_1, x_2, x_3, \ldots$ *the partial derivative of z with respect to x_1, is obtained by differentiating z with respect to x_1, keeping all other variables constant* and is written as $\partial z/\partial x_1$.

In general f_x and f_y are also functions of x and y and so these can be differentiated further partially with respect to x and y.

Thus $$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \quad \text{or} \quad \frac{\partial^2 z}{\partial x^2} \quad \text{or} \quad f_{xx}, \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} \text{ or } \frac{\partial^2 f}{\partial x \partial y} \text{ or } f_{yx}^*$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} \quad \text{or} \quad \frac{\partial^2 f}{\partial y \partial x} \quad \text{or} \quad f_{xy}, \text{ and } \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} \text{ or } \frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy}.$$

It can easily be verified that, in all ordinary cases,

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}.$$

Sometimes we use the following notation

$$\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \frac{\partial^2 z}{\partial x^2} = r, \frac{\partial^2 z}{\partial x \partial y} = s, \frac{\partial^2 z}{\partial y^2} = t.$$

Example 5.1. *Find the first and second partial derivatives of $z = x^3 + y^3 - 3axy$.*

Solution. We have $z = x^3 + y^3 - 3axy$.

$$\therefore \qquad \frac{\partial z}{\partial x} = 3x^2 + 0 - 3ay(1) = 3x^2 - 3ay, \text{ and } \frac{\partial z}{\partial y} = 0 + 3y^2 - 3ax(1) = 3y^2 - 3ax$$

Also $$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(3x^2 - 3ay) = 6x, \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}(3x^2 - 3ay) = -3a$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(3y^2 - 3ax) = 6y, \text{ and } \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(3y^2 - 3ax) = -3a.$$

We observe that $\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y}$.

Example 5.2. *If $u = x^2 \tan^{-1} \dfrac{y}{x} - y^2 \tan^{-1} \dfrac{x}{y}$,*

show that $$\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}.$$ *(Mumbai, 2008 S)*

and $$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$ *(Madras, 2000)*

Solution. We have $$\frac{\partial u}{\partial y} = x^2 \cdot \frac{1}{1 + (y/x)^2} \cdot \frac{1}{x} - \left\{2y \cdot \tan^{-1} \frac{x}{y} + y^2 \cdot \frac{1}{1 + (x/y)^2} \cdot \left(-\frac{x}{y}\right)\right\}$$

$$= \frac{x^3}{x^2 + y^2} - 2y \tan^{-1} \frac{x}{y} + \frac{xy^2}{x^2 + y^2} = x - 2y \tan^{-1} \frac{x}{y}.$$

*It is important to note that in the subscript notation the subscripts are written in the same order in which we differentiate whereas in the '∂' notation the order is opposite.

$$\therefore \qquad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left\{x - 2y \tan^{-1}\frac{x}{y}\right\} = 1 - 2y \cdot \frac{1}{1+(x/y)^2} \cdot \frac{1}{y} = 1 - \frac{2y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}.$$

Similarly, $\quad \dfrac{\partial u}{\partial x} = 2x \tan^{-1} y/x - y$

and $\quad \dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left\{2x \tan^{-1}\dfrac{y}{x} - y\right\} = \dfrac{x^2-y^2}{x^2+y^2}$. Hence the result.

Example 5.3. *If $z = f(x+ct) + \phi(x-ct)$, prove that*

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$$

(J.N.T.U., 2006 ; V.T.U., 2003 S)

Solution. We have $\dfrac{\partial z}{\partial x} = f'(x+ct) \cdot \dfrac{\partial}{\partial x}(x+ct) + \phi'(x-ct)\dfrac{\partial}{\partial x}(x-ct) = f'(x+ct) + \phi'(x-ct)$

and $$\frac{\partial^2 z}{\partial x^2} = f''(x+ct) + \phi''(x-ct) \qquad ...(i)$$

Again $$\frac{\partial z}{\partial t} = f'(x+ct)\frac{\partial}{\partial t}(x+ct) + \phi'(x-ct)\frac{\partial}{\partial t}(x-ct) = cf'(x-ct) - c\phi'(x-ct)$$

and $$\frac{\partial^2 z}{\partial t^2} = c^2 f''(x+ct) + c^2\phi''(x-ct) = c^2[f''(x+ct) + \phi''(x-ct)] \qquad ...(ii)$$

From (i) and (ii), it follows that $\dfrac{\partial^2 z}{\partial t^2} = c^2 \dfrac{\partial^2 z}{\partial x^2}$.

Obs. This is an important partial differential equation, known as *wave equation* (§ 18.4).

Example 5.4. *If $\theta = t^n e^{-r^2/4t}$, what value of n will make $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial \theta}{\partial r}\right) = \dfrac{\partial \theta}{\partial t}$?*

(Nagpur, 2009 ; Kurukshetra, 2006 ; U.P.T.U., 2006)

Solution. We have $\dfrac{\partial \theta}{\partial r} = t^n \cdot e^{-r^2/4t} \cdot \left(\dfrac{-2r}{4t}\right) = -\dfrac{r}{2}t^{n-1}e^{-r^2/4t}$

$$\therefore \qquad r^2\frac{\partial \theta}{\partial r} = -\frac{r^3}{2}t^{n-1} \cdot e^{-r^2/4t}$$

and $$\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) = -\frac{3r^2}{2}t^{n-1}e^{-r^2/4t} - \frac{r^3}{2}t^{n-1} \cdot e^{-r^2/4t}\left(-\frac{2r}{4t}\right)$$

$$\therefore \qquad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) = \left(-\frac{3}{2}t^{n-1} + \frac{r^2}{4}t^{n-2}\right)e^{-r^2/4t}$$

Also $$\frac{\partial \theta}{\partial t} = nt^{n-1} \cdot e^{-r^2/4t} + t^n \cdot e^{-r^2/4t} \cdot \frac{r^2}{4t^2} = \left(nt^{n-1} + \frac{1}{4}r^2t^{n-2}\right)e^{-r^2/4t}$$

Since $\quad \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial \theta}{\partial r}\right) = \dfrac{\partial \theta}{\partial t}$,

$$\therefore \qquad \left(-\frac{3}{2}t^{n-1} + \frac{1}{4}r^2t^{n-2}\right)e^{-r^2/4t} = \left(nt^{n-1} + \frac{1}{4}r^2t^{n-2}\right)e^{-r^2/4t} \quad \text{or} \quad \left(n + \frac{3}{2}\right)t^{n-1}e^{-r^2/4t} = 0.$$

Hence $\qquad n = -3/2.$

Example 5.5. *If $v = (x^2+y^2+z^2)^{-1/2}$, prove that*

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0. \text{ (Laplace equation)*}$$

(V.T.U., 2006 ; Osmania, 2003 S)

*See footnote p. 18.

Solution. We have $\dfrac{\partial v}{\partial x} = -\dfrac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2x = -x(x^2+y^2+z^2)^{-3/2}$

and
$$\frac{\partial^2 v}{\partial x^2} = -1[1 \cdot (x^2+y^2+z^2)^{-3/2} + x(-3/2)(x^2+y^2+z^2)^{-5/2} \cdot 2x]$$
$$= -(x^2+y^2+z^2)^{-5/2}[x^2+y^2+z^2-3x^2] = (x^2+y^2+z^2)^{-5/2}(2x^2-y^2-z^2)$$

Similarly, $\dfrac{\partial^2 v}{\partial y^2} = (x^2+y^2+z^2)^{-5/2}(-x^2+2y^2-z^2)$ and $\dfrac{\partial^2 v}{\partial z^2} = (x^2+y^2+z^2)^{-5/2}(-x^2-y^2+2z^2)$

Hence $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} + \dfrac{\partial^2 v}{\partial z^2} = (x^2+y^2+z^2)^{-5/2} \cdot (0) = 0.$

Obs. *A function v satisfying the Laplace equation is said to be a* **harmonic function.**

Example 5.6. *If $u = \log(x^3+y^3+z^3-3xyz)$, show that* $\left(\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial y}+\dfrac{\partial}{\partial z}\right)^2 u = \dfrac{-9}{(x+y+z)^2}$.

(P.T.U., 2010 ; Anna, 2009 ; Bhopal, 2008 ; U.P.T.U., 2006)

Solution. We have $\dfrac{\partial u}{\partial x} = \dfrac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$, $\dfrac{\partial u}{\partial y} = \dfrac{3y^2-3zx}{x^3+y^3+z^3-3xyz}$, $\dfrac{\partial u}{\partial z} = \dfrac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$

$$\therefore \quad \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}$$
$$= \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} = \frac{3}{x+y+z} \qquad \textit{(V.T.U., 2009)}$$

Now
$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$
$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right) = \left(\frac{3}{x+y+z}\right)$$
$$= -\frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} = -\frac{9}{(x+y+z)^2}.$$

Example 5.7. *If* $\dfrac{x^2}{a^2+u}+\dfrac{y^2}{b^2+u}+\dfrac{z^2}{c^2+u} = 1$, *prove that*

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right) \qquad \textit{(U.P.T.U., 2003)}$$

Solution. We have $x^2(a^2+u)^{-1}+y^2(b^2+u)^{-1}+z^2(c^2+u)^{-1} = 1$ ...(*i*)

Differentiating (*i*) partially w.r.t. x, we get

$$2x(a^2+u)^{-1} - x^2(a^2+u)^{-2}\frac{\partial u}{\partial x} - y^2(b^2+u)^{-2}\frac{\partial u}{\partial y} - z^2(c^2+u)^{-2}\frac{\partial u}{\partial z} = 0$$

or
$$\frac{2x}{a^2+u} = \left\{\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{y^2}{(c^2+u)^2}\right\}\frac{\partial u}{\partial x}$$

or
$$\frac{\partial u}{\partial x} = \frac{2x}{(a^2+u)v} \text{ where } v = \frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{y^2}{(c^2+u)^2}$$

Similarly differentiating (*i*) partially w.r.t. y, we get

$$\frac{2y}{b^2+u} = \left\{\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right\}\frac{\partial u}{\partial y} \text{ or } \frac{\partial u}{\partial y} = \frac{2y}{(b^2+u)v}$$

Similarly, differentiating (i) partially w.r.t. z, we get

$$\frac{2z}{(b^2+u)} = \left\{\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right\}\frac{\partial u}{\partial z} \text{ or } \frac{\partial u}{\partial z} = \frac{2z}{(c^2+u)v}$$

$$\therefore \quad \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = \frac{4}{v^2}\left\{\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{y^2}{(c^2+u)^2}\right\} = \frac{4}{v} \quad ...(ii)$$

Also
$$2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right) = 2\left\{\frac{2x^2}{(a^2+u)v} + \frac{2y^2}{(b^2+u)v} + \frac{2z^2}{(c^2+u)v}\right\}$$

$$= \frac{4}{v}\left\{\frac{x^2}{(a^2+u)} + \frac{y^2}{(b^2+u)} + \frac{z^2}{(c^2+u)}\right\} = \frac{4}{v} \quad \text{[By } (i)\text{]} \ ...(iii)$$

Hence the equality of (ii) and (iii) proves the result.

Example 5.8. *If $u = x^y$, show that $\dfrac{\partial^3 u}{\partial x^2 \partial y} = \dfrac{\partial^3 u}{\partial x \partial y \partial x}$.* *(Anna, 2009)*

Solution. We have $\dfrac{\partial u}{\partial y} = x^y \log_e x$ and $\dfrac{\partial^2 u}{\partial x \partial y} = yx^{y-1} \,.\, \log x + x^y \,.\, \dfrac{1}{x} = x^{y-1}(y \log x + 1)$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2 \partial y} = \frac{\partial}{\partial x}[x^{y-1}(y \log x + 1)] \quad ...(i)$$

Again
$$\frac{\partial u}{\partial x} = yx^{y-1} \text{ and } \frac{\partial^2 u}{\partial y \partial x} = 1 \,.\, x^{y-1} + y\left(\frac{1}{x}x^y \log x\right) = x^{y-1}(1 + y \log x)$$

$$\therefore \quad \frac{\partial^3 u}{\partial x \partial y \partial x} = \frac{\partial}{\partial x}[x^{y-1}(y \log x + 1)] \quad ...(ii)$$

From (i) and (ii) follows the required result.

PROBLEMS 5.2

1. Evaluate $\partial z/\partial x$ and $\partial z/\partial y$, if
   (i) $z = x^2y - x \sin xy$; (ii) $z = \log(x^2 + y^2)$;
   (iii) $z = \tan^{-1}\{(x^2 + y^2)/(x + y)\}$; (iv) $x + y + z = \log z$.
2. If $z(x + y) = x^2 + y^2$, show that $\left(\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)^2 = 4\left(1 - \dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)$. *(V.T.U., 2003)*
3. If $z = e^{ax+by} f(ax - by)$, prove that $b\dfrac{\partial z}{\partial x} + a\dfrac{\partial z}{\partial y} = 2abz$. *(V.T.U., 2010)*
4. Given $u = e^{r\cos\theta}\cos(r\sin\theta)$, $v = e^{r\cos\theta}\sin(r\sin\theta)$; prove that $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$ and $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$.
5. If $z = \tan(y + ax) - (y - ax)^{3/2}$, show that $\dfrac{\partial^2 z}{\partial x^2} = a^2\dfrac{\partial^2 z}{\partial y^2}$. *(Mumbai, 2009)*
6. Verify that $f_{xy} = f_{yx}$, when f is equal to (i) $\sin^{-1}(y/x)$; (ii) $\log x \tan^{-1}(x^2 + y^2)$.
7. If $f(x, y) = (1 - 2xy + y^2)^{-1/2}$, show that $\dfrac{\partial}{\partial x}\left[(1 - x^2)\dfrac{\partial f}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[y^2\dfrac{\partial f}{\partial y}\right] = 0$. *(Rohtak, 2006 S)*
8. Prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ if (i) $u = \tan^{-1}\left[\dfrac{2xy}{x^2 - y^2}\right]$, (ii) $u = \log(x^2 + y^2) + \tan^{-1}(y/x)$. *(Anna, 2009)*
9. If $v = \dfrac{1}{\sqrt{t}}e^{-x^2/4a^2t}$, prove that $\dfrac{\partial v}{\partial t} = a^2\dfrac{\partial^2 v}{\partial x^2}$.

10. The equation $\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2}$ refers to the conduction of heat along a bar without radiation, show that if $u = Ae^{-\mu t^2} \sin (nt - gx)$, where A, g, n are positive constants then $g = \sqrt{(n/2\mu)}$.

11. Find the value of n so that the equation $V = r^n (3 \cos^2 \theta - 1)$ satisfies the relation

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial V}{\partial \theta}\right) = 0.$$

12. If $z = \log (e^x + e^y)$, show that $rt - s^2 = 0$ where $r = \partial^2 z/\partial x^2$, $s = \partial^2 z/\partial x \partial y$, $t = \partial^2 z/\partial y^2$.

13. If $u = \frac{y}{z} + \frac{z}{x}$, show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0$.

14. Let $r^2 = x^2 + y^2 + z^2$ and $V = r^m$, prove that $V_{xx} + V_{yy} + V_{zz} = m(m+1)r^{m-2}$. *(Raipur, 2005)*

15. If $v = \log (x^2 + y^2 + z^2)$, prove that $(x^2 + y^2 + z^2)\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = 2$.

16. If $v = x^y \cdot y^x$, prove that $x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = v(x + y + \log v)$. *(Anna, 2005)*

17. If $x^x y^y z^z = c$, show that at $x = y = z$, $\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$. *(Bhopal, 2008)*

18. If $u = e^{xyz}$, find the value of $\frac{\partial^3 u}{\partial x \partial y \partial z}$. *(Rajasthan, 2005 ; Osmania, 2003 S)*

5.3 WHICH VARIABLE IS TO BE TREATED AS CONSTANT

(1) Consider the equation $x = r \cos \theta, y = r \sin \theta$ ...(1)

To find $\partial r/\partial x$, we need a relation between r and x. Such a relation will contain one more variable θ or y, for we can eliminate only one variable out of four from the relations (1). Thus the two possible relations are

$$r = x \sec \theta \quad ...(2) \qquad \text{and} \qquad r^2 = x^2 + y^2 \quad ...(3)$$

Now we can find $\partial r/\partial x$ either from (2) by treating θ as constant or from (3) by regarding y as constant. And there is no reason to suppose that the two values of $\partial r/\partial x$ so found, are equal. *To avoid confusion as to which variable is regarded constant, we introduce the following :*

Notation : $(\partial r/\partial x)_\theta$ *means the partial derivative of r with respect to x keeping* θ *constant in* a relation expressing r as a function of x and θ.

Thus from (2), $(\partial r/\partial x)_\theta = \sec \theta$.

When no indication is given regarding the variable to be kept constant, then according to convention $(\partial/\partial x)$ always means $(\partial/\partial x)_y$ and $\partial/\partial y$ means $(\partial/\partial y)_x$. Similarly, $\partial/\partial r$ means $(\partial/\partial r)_\theta$ and $\partial/\partial \theta$ means $(\partial/\partial \theta)_r$.

(2) In thermodynamics, we come across ten variables such as p (pressure), v (volume), T (temperature), W (work), ϕ (entropy) etc. Any one of these can be expressed as a function of other two variables *e.g.*, $T = f(p, v)$, $T = g(p, \phi)$

As we shall see, these respectively give rise to the following results :

$$dT = \frac{\partial T}{\partial p} dp + \frac{\partial T}{\partial v} dv \quad ...(i)$$

$$dT = \frac{\partial T}{\partial p} dp + \frac{\partial T}{\partial \phi} d\phi \quad ...(ii)$$

Now, $\partial T/\partial p$ appearing in (i), has been obtained from T as function of p and v, treating v as constant, we write it as $(\partial T/\partial p)_v$.

Similarly, $\partial T/\partial p$ occurring in (ii), is written as $(\partial T/\partial p)_\phi$.

Example 5.9. *If $u = f(r)$ and $x = r \cos \theta$, $y = r \sin \theta$, prove that*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r).$$

(S.V.T.U., 2008 ; Rajasthan, 2006 ; U.P.T.U., 2005)

Solution. We have $\dfrac{\partial u}{\partial x} = f'(r) \cdot \dfrac{\partial r}{\partial x}$ and $\dfrac{\partial^2 u}{\partial x^2} = f''(r) \cdot \left(\dfrac{\partial r}{\partial x}\right)^2 + f'(r) \cdot \dfrac{\partial^2 r}{\partial x^2}$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = f''(r) \cdot \left(\frac{\partial r}{\partial y}\right)^2 + f'(r) \cdot \frac{\partial^2 r}{\partial y^2}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) \cdot \left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right] + f'(r)\left[\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2}\right]$$

Now to find $\dfrac{\partial r}{\partial x}, \dfrac{\partial r}{\partial y}$ etc., we write $r = (x^2 + y^2)^{1/2}$

$$\therefore \quad \frac{\partial r}{\partial x} = \frac{1}{2}(x^2 + y^2)^{-1/2} \cdot 2x = \frac{x}{r} \quad \text{and} \quad \frac{\partial^2 r}{\partial x^2} = \frac{r \cdot 1 - x \cdot \partial r/\partial x}{r^2} = \frac{r - x^2/r}{r^2} = \frac{y^2}{r^3}.$$

Similarly, $$\frac{\partial r}{\partial y} = \frac{y}{x} \text{ and } \frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3}.$$

Substituting the values of $\partial r/\partial x$ etc. in (i), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r)\left[\frac{x^2}{r^2} + \frac{y^2}{r^2}\right] + f'(r)\left[\frac{y^2}{r^3} + \frac{x^2}{r^3}\right] = f''(r) + \frac{1}{r}f'(r).$$

Example 5.10. *If $x = e^{r\cos\theta}\cos(r\sin\theta)$ and $y = e^{r\cos\theta}\sin(r\sin\theta)$, prove that $\dfrac{\partial x}{\partial \theta} = -r\dfrac{\partial y}{\partial r}, \dfrac{\partial y}{\partial \theta} = r\dfrac{\partial x}{\partial r}$.*

Hence show that $\dfrac{\partial^2 x}{\partial \theta^2} + r\dfrac{\partial x}{\partial r} + r^2\dfrac{\partial^2 x}{\partial r^2} = 0$.

Solution. We have $x = e^{r\cos\theta}\cos(r\sin\theta)$

$$\therefore \quad \frac{\partial x}{\partial \theta} = e^{r\cos\theta}(-r\sin\theta) \cdot \cos(r\sin\theta) + e^{r\cos\theta}[-\sin(r\sin\theta)] \cdot r\cos\theta$$
$$= -re^{r\cos\theta}[\sin\theta\cos(r\sin\theta) + \cos\theta\sin(r\sin\theta)]$$
$$= -re^{r\cos\theta}\sin(\theta + r\sin\theta) \qquad ...(i)$$

and $$\frac{\partial x}{\partial r} = e^{r\cos\theta} \cdot \cos\theta \cdot \cos(r\sin\theta) - e^{r\cos\theta}\sin\theta(r\sin\theta)\sin\theta$$
$$= e^{r\cos\theta}\cos(\theta + r\sin\theta) \qquad ...(ii)$$

Similarly, $y = e^{r\cos\theta}\sin(r\sin\theta)$ gives

$$\frac{\partial y}{\partial \theta} = re^{r\cos\theta}\cos(\theta + r\sin\theta) \qquad ...(iii)$$

and $$\frac{\partial y}{\partial r} = e^{r\cos\theta}\sin(\theta + r\sin\theta) \qquad ...(iv)$$

From (i) and (iv), $\dfrac{\partial x}{\partial \theta} = -r\dfrac{\partial y}{\partial r}$ ...(v)

From (ii) and (iii), $\dfrac{\partial y}{\partial \theta} = r\dfrac{\partial x}{\partial r}$ ...(vi)

From (v), $$\frac{\partial^2 x}{\partial \theta^2} = -r\frac{\partial^2 y}{\partial \theta \partial r} = -r\frac{\partial^2 y}{\partial r \partial \theta}$$

From (vi), $\dfrac{\partial x}{\partial r} = \dfrac{1}{r}\dfrac{\partial y}{\partial \theta}$ which gives $\dfrac{\partial^2 x}{\partial r^2} = -\dfrac{1}{r^2}\dfrac{\partial y}{\partial \theta} + \dfrac{1}{r}\dfrac{\partial^2 y}{\partial r \partial \theta}$

$$\therefore \quad \frac{\partial^2 x}{\partial \theta^2} + r\frac{\partial x}{\partial r} + r^2\frac{\partial^2 x}{\partial r^2} = -r\frac{\partial^2 y}{\partial r \partial \theta} + \frac{\partial y}{\partial \theta} - \frac{\partial y}{\partial \theta} + r\frac{\partial^2 y}{\partial r \partial \theta} = 0.$$

PROBLEMS 5.3

1. If $x = r\cos\theta, y = r\sin\theta$, show that (i) $\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}$ (ii) $\frac{1}{r}\frac{\partial x}{\partial \theta} = r\frac{\partial \theta}{\partial x}$, (iii) $\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = 1$. *(Burdwan, 2003)*

2. If $x^2 = au + bv, y^2 = au - bv$, prove that $\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2} = \left(\frac{\partial v}{\partial y}\right)_x \cdot \left(\frac{\partial y}{\partial v}\right)_u$.

3. If $u = lx + my, v = mx - ly$, show that $\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial u}\right)_v = \frac{l^2}{l^2+m^2}, \left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_u = \frac{l^2+m^2}{l^2}$.

4. If $x = r\cos\theta, y = r\sin\theta$, prove that

   (i) $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$ (ii) $\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$ $(x \neq 0, y \neq 0)$.

5. If $z = x\log(x+r) - r$ where $r^2 = x^2 + y^2$, prove that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{x+y}, \frac{\partial^3 z}{\partial x^3} = -\frac{x}{r^3}$. *(Mumbai, 2008)*

6. If $u = f(r)$ where $r = \sqrt{(x^2+y^2+z^2)}$, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) + \frac{2}{r}f'(r)$.

5.4 (1) HOMOGENEOUS FUNCTIONS

An expression of the form $a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \ldots + a_ny^n$ in which every term is of the nth degree, is called a homogeneous function of degree n. This can be rewritten as

$$x^n[a_0 + a_1(y/x) + a_2(y/x)^2 + \ldots + a_n(y/x)^n].$$

Thus *any function $f(x, y)$ which can be expressed in the form $x^n\phi(y/x)$, is called a* **homogeneous function** of degree n in x and y.

For instance, $x^3\cos(y/x)$ is a homogeneous function of degree 3, in x and y.

In general, a function $f(x, y, z, t, \ldots)$ is said to be a homogeneous function of degree n in $x, y, z, t, \ldots$, if it can be expressed in the form $x^n\phi(y/x, z/x, t/x, \ldots)$.

(2) Euler's theorem on homogeneous functions*. *If u be a homogeneous function of degree n in x and y, then*

$$\mathbf{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.}$$

Since u is a homogeneous function of degree n in x and y, therefore,

$$u = x^n f(y/x)$$

$$\therefore \quad \frac{\partial u}{\partial x} = nx^{n-1}f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right)\cdot y\left(-\frac{1}{x^2}\right) = nx^{n-1}f\left(\frac{y}{x}\right) - yx^{n-2}f'\left(\frac{y}{x}\right)$$

and $$\frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right)\cdot\frac{1}{x} = x^{n-1}f'\left(\frac{y}{x}\right). \text{ Hence } x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n f\left(\frac{y}{x}\right) = nu.$$

In general, if u be a homogeneous function of degree n in $x, y, z, t, \ldots$, then,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + t\frac{\partial u}{\partial t} \ldots = nu.$$

Example 5.11. *Show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u\log u$ where $\log u = (x^3+y^3)/(3x+4y)$.*

Solution. Since $$z = \log u = \frac{x^3+y^3}{3x+4y} = x^2\cdot\frac{1+(y/x)^3}{3+4(y/x)},$$

* After an enormously creative Swiss mathematician *Leonhard Euler* (1707–1783). He studied under John *Bernoulli* and became a professor of mathematics in St. Petersburg, Russia. Even after becoming totally blind in 1771, he contributed to almost all branches of mathematics.

$\therefore$ z is a homogeneous function of degree 2 in x and y.

By Euler's theorem, we get

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z \qquad ...(i)$$

But $$\frac{\partial z}{\partial x} = \frac{1}{u}\frac{\partial u}{\partial x} \text{ and } \frac{\partial z}{\partial y} = \frac{1}{u}\frac{\partial u}{\partial y}$$

Hence (i) becomes

$$x \cdot \frac{1}{u}\frac{\partial u}{\partial x} + y \cdot \frac{1}{u}\frac{\partial u}{\partial y} = 2 \log u \quad \text{or} \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u \log u.$$

Example 5.12. *If* $u = \sin^{-1}\dfrac{x+2y+3z}{x^8+y^8+z^8}$, *find the value of* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z}$. *(U.P.T.U., 2004)*

Solution. Here u is not a homogeneous function. We therefore, write

$$\omega = \sin u = \frac{x+2y+3z}{x^8+y^8+z^8} = x^{-7} \cdot \frac{1+2(y/x)+3(z/x)}{1+(y/x)^8+(z/x)^8}$$

Thus ω is a homogeneous function of degree -7 in x, y, z. Hence by Euler's theorem

$$x\frac{\partial \omega}{\partial x} + y\frac{\partial \omega}{\partial y} + z\frac{\partial \omega}{\partial z} = (-7)\,\omega \qquad ...(i)$$

But $$\frac{\partial \omega}{\partial x} = \cos u\,\frac{\partial u}{\partial x}, \frac{\partial \omega}{\partial y} = \cos u\,\frac{\partial u}{\partial y}, \frac{\partial \omega}{\partial z} = \cos u\,\frac{\partial u}{\partial z}$$

$\therefore$ (i) becomes $$x \cos u\frac{\partial u}{\partial x} + y \cos u\frac{\partial u}{\partial y} + z \cos u\frac{\partial u}{\partial z} = -7 \sin u \quad \text{or} \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -7 \tan u.$$

Example 5.13. *If* $u = \dfrac{x^3y^3z^3}{x^3+y^3+z^3} + \log\left(\dfrac{xy+yz+zx}{x^2+y^2+z^2}\right)$, *find the value of* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z}$.

(Mumbai, 2009)

Solution. Let $$v = \frac{x^3y^3z^3}{x^3+y^3+z^3} \text{ and } w = \log\left(\frac{xy+yz+zx}{x^2+y^2+z^2}\right) \qquad ...(i)$$

so that $$u = v + w$$

Since $v = x^6\,\dfrac{(y/x)^3(z/x)^3}{1+(y/x)^3+(z/x)^3}$, therefore v is a homogeneous function of degree 6 in x, y, z.

Hence by Euler's theorem $$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = 6v \qquad ...(ii)$$

Since $w = \log\left\{\dfrac{\dfrac{y}{x}+\dfrac{y}{x}\cdot\dfrac{z}{x}+\dfrac{z}{x}}{1+\left(\dfrac{y}{x}\right)^2+\left(\dfrac{z}{x}\right)^2}\right\}$ therefore w is a homogeneous function of degree zero in x, y, z.

Hence by Euler's theorem $$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = 0 \qquad ...(iii)$$

Addint (ii) and (iii), we obtain

$$x\left(\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\right) + y\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial y}\right) + z\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial z}\right) = 6v$$

or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 6\,\frac{x^3y^3z^3}{x^3+y^3+z^3} \qquad \text{[By } (i)\text{]}$$

Example 5.14. *If z is a homogeneous function of degree n in x and y, show that*

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial z^2}=n(n-1)\,z.$$ *(Anna, 2009 ; V.T.U., 2007 ; U.P.T.U., 2006)*

Solution. By Euler's theorem, $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=nz$ ...(*i*)

Differentiating (*i*) partially w.r.t. x, we get $x\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial z}{\partial x}+y\dfrac{\partial^2 z}{\partial x\partial y}=n\dfrac{\partial z}{\partial x}$

i.e., $$x\frac{\partial^2 z}{\partial x^2}+y\frac{\partial^2 z}{\partial x\partial y}=(n-1)\frac{\partial z}{\partial x}$$ ...(*ii*)

Again differentiating (*i*) partially w.r.t. y, we get $x\dfrac{\partial^2 z}{\partial y\partial x}+\dfrac{\partial z}{\partial y}+y\dfrac{\partial^2 z}{\partial y^2}=n\dfrac{\partial z}{\partial y}$

i.e., $$x\frac{\partial^2 z}{\partial x\partial y}+y\frac{\partial^2 z}{\partial y^2}=(n-1)\frac{\partial z}{\partial y}$$ ...(*iii*)

Multiplying (*ii*) by x and (*iii*) by y and adding, we get

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=(n-1)\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)=n(n-1)\,z.$$ [By (*i*)]

Example 5.15. *If $u=\sin^{-1}\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=\dfrac{1}{2}\tan u$.*

(Rajasthan, 2006 ; Calicut, 2005)

and $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-\frac{\sin u\cos 2u}{4\cos^3 u}.$$ *(P.T.U., 2006)*

Solution. Here u is not a homogeneous function but $z=\sin u=\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$ is a homogeneous function of degree 1/2 in x and y.

$\therefore$ By Euler's theorem, $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=\dfrac{1}{2}z$

or $$x\cos u\,\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}=\frac{1}{2}\sin u$$

Thus $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\tan u$$ ...(*i*)

Differentiating (*i*) w.r.t. x partially, we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=\frac{1}{2}\sec^2 u\frac{\partial u}{\partial x}\quad\text{or}\quad x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=\left(\frac{1}{2}\sec^2 u-1\right)\frac{\partial u}{\partial x}$$ ...(*ii*)

Again differentiating (*i*) w.r.t. y partially, we get

$$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=\frac{1}{2}\sec^2 u\frac{\partial u}{\partial y}\quad\text{or}\quad x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}=\left(\frac{1}{2}\sec^2 u-1\right)\frac{\partial u}{\partial x}$$ ...(*iii*)

Multiplying (*ii*) by x and (*iii*) by y and adding, we obtain

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\left(\frac{1}{2}\sec^2 u-1\right)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

or $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\left(\frac{1}{2}\sec^2 u-1\right)\left(\frac{1}{2}\tan u\right)$$ [By (*i*)]

$$=\frac{1}{4}\frac{\sin u}{\cos^3 u}-\frac{1}{2}\frac{\sin u}{\cos u}=-\frac{\sin u\,(2\cos^2 u-1)}{4\cos^3 u}$$

Hence $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-\frac{\sin u\cos 2u}{4\cos^3 u}.$$

PROBLEMS 5.4

1. Verify Euler's theorem, when (*i*) $f(x, y) = ax^2 + 2hxy + by^2$
(*ii*) $f(x, y) = x^2(x^2 - y^2)^3/(x^2 + y^2)^3$.
(*iii*) $f(x, y) = 3x^2yz + 5xy^2z + 4z^4$ (*J.N.T.U., 1999*)

2. If $u = \sin^{-1}\dfrac{x}{y} + \tan^{-1}\dfrac{y}{x}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$. (*Hazaribagh, 2009 ; Osmania, 2003 S*)

3. If $u = \sin^{-1}\dfrac{x^2 + y^2}{x + y}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u$ (*Bhopal, 2009 ; V.T.U., 2003*)

4. If $\sin u = \dfrac{x^2y^2}{x + y}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3\tan u$. (*Kottayam, 2005 ; V.T.U., 2003 S*)

5. If $u = \cos^{-1}\dfrac{x + y}{\sqrt{x} + \sqrt{y}}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = -\dfrac{1}{2}\cot u$. (*V.T.U., 2004*)

6. Show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u\log u$, where $u = e^{x^2 + y^2}$ (*P.T.U., 2010*)

7. If $z = f(y/x) + \sqrt{(x^2 + y^2)}$, show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = \sqrt{x^2 + y^2}$. (*Mumbai, 2008*)

8. If $u = \dfrac{x}{y + z} + \dfrac{y}{z + x} + \dfrac{z}{x + y}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0$. (*V.T.U., 2000 S*)

9. If $\sin u = \dfrac{x + 2y + 3z}{\sqrt{(x^8 + y^8 + z^8)}}$, show that $xu_x + yu_y + zu_z + 3\tan u = 0$. (*S.V.T.U., 2009 ; U.T.U., 2009*)

10. If $z = x\phi\left(\dfrac{y}{x}\right) + \psi\left(\dfrac{y}{x}\right)$, prove that $x^2\dfrac{\partial^2 z}{\partial x^2} + 2xy\dfrac{\partial^2 z}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 0$. (*S.V.T.U., 2009 ; U.P.T.U., 2006*)

11. If $u = \tan^{-1}\dfrac{x^3 + y^3}{x + y}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u$. (*P.T.U., 2009 S*)

and $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 2\cos 3u\sin u$. (*Mumbai, 2009 ; Bhopal, 2008 ; S.V.T.U., 2007*)

12. Given $z = x^n f_1(y/x) + y^{-n} f_2(x/y)$, prove that $x^2\dfrac{\partial^2 z}{\partial x^2} + 2xy\dfrac{\partial^2 z}{\partial x\partial y} + y^2\dfrac{\partial^2 z}{\partial y^2} + x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = n^2 z$.
(*Kurukshetra, 2009 S ; Rohtak, 2003*)

13. If $u = x^2\tan^{-1}(y/x) - y^2\tan^{-1}(x/y)$, evaluate $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2}$. (*U.T.U., 2009 ; Hissar, 2005 S*)

14. If $u = \tan^{-1}(y^2/x)$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = -\sin^2 u \,.\, \sin 2u$. (*Bhillai, 2005 ; P.T.U., 2005*)

15. If $u = \operatorname{cosec}^{-1}\left(\dfrac{x^{1/2} + y^{1/2}}{x^{1/3} + y^{1/3}}\right)^{1/2}$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = \dfrac{\tan u}{12}\left(\dfrac{13}{12} + \dfrac{\tan^2 u}{12}\right)$
(*Mumbai, 2008 ; Rohtak, 2006 S*)

5.5 (1) TOTAL DERIVATIVE

If $u = f(x, y)$, where $x = \phi(t)$ and $y = \psi(t)$, then we can express u as a function of t alone by substituting the values of x and y in $f(x, y)$. Thus we can find the ordinary derivative du/dt which is called the *total derivative* of u to distinguish it from the partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$.

Now to find du/dt without actually substituting the values of x and y in $f(x, y)$, we establish the following **Chain rule :**

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} \qquad ...(i)$$

Proof. We have $u = f(x, y)$

Giving increment δt to t, let the corresponding increments of x, y and u be δx, δy and δu respectively.

Then $\quad u + \delta u = f(x + \delta x, y + \delta y)$

Subtracting, $\quad \delta u = f(x + \delta x, y + \delta y) - f(x, y)$

$$= \{f(x + \delta x, y + \delta y) - f(x, y + \delta y)\} + \{f(x, y + \delta y) - f(x, y)\}$$

$$\therefore \quad \frac{\delta u}{\delta t} = \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x} \cdot \frac{\delta x}{\delta t} + \frac{f(x, y + \delta y) - f(x, y)}{\delta y} \cdot \frac{\delta y}{\delta t}$$

Taking limits as $\delta t \to 0$, δx and δy also $\to 0$, we have

$$\frac{du}{dt} = \underset{\delta y \to 0}{\text{Lt}} \left[\underset{\delta y \to 0}{\text{Lt}} \left\{ \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x} \right\} \right] \frac{dx}{dt} + \underset{\delta y \to 0}{\text{Lt}} \left\{ \frac{f(x, y + \delta y) - f(x, y)}{\delta y} \right\} \frac{dy}{dt}$$

$$= \underset{\delta y \to 0}{\text{Lt}} \left\{ \frac{\partial f(x, y + \delta y)}{\delta y} \right\} \cdot \frac{dx}{dt} + \frac{\partial f(x, y)}{\partial y} \cdot \frac{dy}{dt}$$

[Supposing $\partial f(x, y)/\partial x$ to be a continuous function of y]

$$= \frac{\partial f(x, y)}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f(x, y)}{\partial y} \cdot \frac{dy}{dt}$$ which is the desired formula.

Cor. Taking $t = x$, (i) becomes, $\dfrac{du}{dx} = \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} \cdot \dfrac{dy}{dx}$ ...(ii)

Obs. If $u = f(x, y, z)$, where x, y, z are all functions of a variable t, then **Chain rule** is

$$\mathbf{\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}} \qquad ...(iii)$$

(2) Differentiation of implicit functions. If $f(x, y) = c$ be an implicit relation between x and y which defines as a differentiable function of x, then (ii) becomes

$$0 = \frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

This gives the *important formula* $\mathbf{\dfrac{dy}{dx} = -\dfrac{\partial f}{\partial x} \Big/ \dfrac{\partial f}{\partial y}}$ $\left[\dfrac{\partial f}{\partial y} \neq 0\right]$

for the first differential coefficient of an implicit function.

Example 5.16. *Given $u = \sin(x/y)$, $x = e^t$ and $y = t^2$, find du/dt as a function of t. Verify your result by direct substitution.*

Solution. We have $\quad \dfrac{du}{dt} = \dfrac{\partial u}{\partial x} \cdot \dfrac{dx}{dt} + \dfrac{\partial u}{\partial y} \cdot \dfrac{dy}{dt} = \left(\cos \dfrac{x}{y}\right) \dfrac{1}{y} \cdot e^t + \left(\cos \dfrac{x}{y}\right)\left(-\dfrac{x}{y^2}\right) 2t$

$$= \cos(e^t/t^2) \cdot e^t/t^2 - 2\cos(e^t/t^2) \cdot e^t/t^3 = \{(t-2)/t^3\} e^t \cos(e^t/t^2)$$

Also $u = \sin(x/y) = \sin(e^t/t^2)$

$$\therefore \quad \frac{du}{dt} = \cos\left(\frac{e^t}{t^2}\right) \cdot \frac{t^2 e^t - e^t \cdot 2t}{t^4} = \frac{t-2}{t^3} e^t \cos\left(\frac{e^t}{t^2}\right) \text{ as before.}$$

Example 5.17. *If x increases at the rate of 2 cm/sec at the instant when $x = 3$ cm. and $y = 1$ cm., at what rate must y be changing in order that the function $2xy - 3x^2y$ shall be neither increasing nor decreasing ?*

Solution. Let $\quad u = 2xy - 3x^2y$, so that

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} = (2y - 6xy) \frac{dx}{dt} + (2x - 3x^2) \frac{dy}{dt} \qquad ...(i)$$

when $x = 3$ and $y = 1$, $dx/dt = 2$, and u is neither increasing nor decreasing, *i.e.*, $du/dt = 0$.

$\therefore$ (i) becomes $\quad 0 = (2 - 6 \times 3)\, 2 + (2 \times 3 - 3 \times 9) \dfrac{dy}{dt}$

or $\quad \dfrac{dy}{dt} = -\dfrac{32}{21}$ cm/sec. Thus y is decreasing at the rate of 32/21 cm/sec.

Example 5.18. *If $u = x \log xy$ where $x^3 + y^3 + 3xy = 1$, find du/dx.* (V.T.U., 2009)

Solution. From $f(x, y) = x^3 + y^3 + 3xy - 1$, we have

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y} = -\frac{3x^2 + 3y}{3y^2 + 3x} = -\frac{x^2 + y}{y^2 + x} \qquad ...(i)$$

Also $$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = (1 \cdot \log xy + x \cdot 1/x) + (x/y) \cdot dy/dx.$$

Hence $$du/dx = 1 + \log xy - x(x^2 + y)/y(y^2 + x) \qquad \text{[By } (i)\text{]}$$

Example 5.19. *If $u = u\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, show that $x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0$.* (U.P.T.U., 2005)

Solution. Let $$v = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y} \quad \text{and} \quad w = \frac{z-x}{xz} = \frac{1}{x} - \frac{1}{z} \qquad ...(i)$$

so that $$u = u(v, w)$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial x} = \frac{\partial u}{\partial v}\left(-\frac{1}{x^2}\right) + \frac{\partial u}{\partial w}\left(-\frac{1}{x^2}\right) \qquad \text{[Using } (i)\text{]}$$

or $$x^2 \frac{\partial u}{\partial x} = -\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w} \qquad ...(ii)$$

Also $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v} \cdot \frac{\partial u}{\partial y} + \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial y} = \frac{\partial u}{\partial v}\left(\frac{1}{y^2}\right) + \frac{\partial u}{\partial w}(0) \qquad \text{[Using } (i)\text{]}$$

or $$y^2 \frac{\partial u}{\partial y} = \frac{\partial u}{\partial v} \qquad ...(iii)$$

Similarly $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial v} \frac{\partial v}{\partial z} + \frac{\partial u}{\partial w} \frac{\partial w}{\partial z} = \frac{\partial u}{\partial v}(0) + \frac{\partial u}{\partial w}\left(\frac{1}{z^2}\right) \qquad \text{[Using } (i)\text{]}$$

or $$z^2 \frac{\partial u}{\partial z} = \frac{\partial u}{\partial w} \qquad ...(iv)$$

Adding (ii), (iii) and (iv), we have

$$x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0.$$

Example 5.20. *Formula for the second differential coefficient of an implicit function.*
If $f(x, y) = 0$, show that

$$\frac{d^2y}{dx^2} = -\frac{q^2 r - 2pqs + p^2 t}{q^3}$$

(Kurukshetra, 2006)

Solution. We have $$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{p}{q} \qquad ...(i)$$

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{d}{dx}\left(\frac{dy}{dx}\right) = -\frac{d}{dx}\left(\frac{p}{q}\right) = -\frac{q(dp/dx) - p(dq/dx)}{q^2} \qquad ...(ii)$$

Using the notations : $r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial p}{\partial x}$, $s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial q}{\partial x}$, $t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial q}{\partial y}$,

we have $$\frac{dp}{dx} = \frac{\partial p}{\partial x} + \frac{\partial p}{\partial y} \cdot \frac{dy}{dx} = r + s(-p/q) = -\frac{qr - ps}{q}$$

and $$\frac{dq}{dx} = \frac{\partial q}{\partial x} + \frac{\partial q}{\partial y} \cdot \frac{dy}{dx} = s + t(-p/q) = \frac{qs - pt}{q}$$

Substituting the values of dp/dx and dq/dx in (ii), we get

$$\frac{d^2y}{dx^2} = -\frac{1}{q^2}\left[q\left(\frac{qr-ps}{q}\right) - p\left(\frac{qs-pt}{q}\right)\right] = -\frac{q^2r - 2pqs + p^2t}{q^3}.$$

PROBLEMS 5.5

1. If $z = u^2 + v^2$ and $u = at^2$, $v = 2at$, find dz/dt. (P.T.U., 2005)
2. If $u = \tan^{-1}(y/x)$ where $x = e^t - e^{-t}$, and $y = e^t + e^{-t}$, find du/dt. (V.T.U., 2003)
3. Find the value of $\frac{du}{dt}$ given $u = y^2 - 4ax$, $x = at^2$, $y = 2at$. (Anna, 2009)
4. At a given instant the sides of a rectangle are 4 ft. and 3 ft. respectively and they are increasing at the rate of 1.5 ft./sec. and 0.5 ft./sec. respectively, find the rate at which the area is increasing at that instant.
5. If $z = 2xy^2 - 3x^2y$ and if x increases at the rate of 2 cm. per second and it passes through the value $x = 3$ cm., show that if y is passing through the value $y = 1$ cm., y must be decreasing at the rate of $2\frac{2}{15}$ cm. per second, in order that z shall remain constant.
6. If $u = x^2 + y^2 + z^2$ and $x = e^{2t}$, $y = e^{2t}\cos 3t$, $z = e^{2t}\sin 3t$. Find $\frac{du}{dt}$ as a total derivative and verify the result by direct substitution.
7. If $\phi(cx - az, cy - bz) = 0$, show that $\frac{a\partial z}{\partial x} = \frac{b\partial z}{\partial y} = c$.
8. If $f(x, y) = 0$, $\phi(y, z) = 0$, show that $\frac{\partial f}{\partial y}\cdot\frac{\partial \phi}{\partial z}\cdot\frac{dz}{dx} = \frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial y}$.
9. If the curves $f(x, y) = 0$ and $\phi(y, z) = 0$ touch, show that at the point of contact, $\frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial y} = \frac{\partial f}{\partial y}\cdot\frac{\partial \phi}{\partial y}$.
10. If $f(x, y) = 0$, show that $\left(\frac{\partial f}{\partial y}\right)^3\frac{d^2y}{dx^2} = 2\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial^2 f}{\partial x\partial y}\right) - \left(\frac{\partial f}{dy}\right)^2\frac{\partial^2 f}{\partial x^2} - \left(\frac{\partial f}{\partial x}\right)^2\left(\frac{\partial^2 f}{\partial y^2}\right)$.

5.6 CHANGE OF VARIABLES

If $\quad u = f(x, y) \quad$ ...(1)

where $\quad x = \phi(s, t)$ and $y = \Psi(s, t) \quad$ ...(2)

it is often necessary to change expressions involving $u, x, y, \partial u/\partial x, \partial u/\partial y$ etc. to expressions involving $u, s, t, \partial u/\partial s, \partial u/\partial t$ etc.

The necessary formulae for the change of variables are easily obtained. If t is regarded as a constant, then x, y, u will be functions of s alone. Therefore, by (i) of page 208, we have

$$\frac{\partial \mathbf{u}}{\partial \mathbf{s}} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}}\cdot\frac{\partial \mathbf{x}}{\partial \mathbf{s}} + \frac{\partial \mathbf{u}}{\partial \mathbf{y}}\cdot\frac{\partial \mathbf{y}}{\partial \mathbf{s}} \quad ...(3)$$

where the ordinary derivatives have been replaced by the partial derivatives because x, y are functions of two variables s and t.

$\because$ Similarly, regarding s as constant, we obtain $\frac{\partial \mathbf{u}}{\partial \mathbf{t}} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}}\cdot\frac{\partial \mathbf{x}}{\partial \mathbf{t}} + \frac{\partial \mathbf{u}}{\partial \mathbf{y}}\cdot\frac{\partial \mathbf{y}}{\partial \mathbf{t}}$ ...(4)

On solving (3) and (4) as simultaneous equations in $\partial u/\partial x$ and $\partial u/\partial y$, we get their values in terms of $\partial u/\partial s$, $\partial u/\partial t$, u, s, t.

If instead of the equations (2), s and t are given in terms of x and y, say: $s = \xi(x, y)$ and $t = \eta(x, y)$, ...(5)

then it is easier to use the formulae $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$ ...(6)

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial y} \qquad ...(7)$$

The higher derivatives of u can be found by repeated application of formulae (3) and (4) or of (6) and (7).

Example 5.21. *If $u = F(x - y, y - z, z - x)$, prove that*

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

(V.T.U., 2010 ; U.T.U. 2009 ; U.P.T.U., 2003)

Solution. Put $x - y = r$, $y - z = s$ and $z - x = t$, so that $u = f(r, s, t)$.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial r} \cdot (1) + \frac{\partial x}{\partial s} \cdot (0) + \frac{\partial u}{\partial t} \cdot (-1) = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} \qquad ...(i)$$

Similarly,

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial y} = -\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} \qquad ...(ii)$$

and

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial z} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial z} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial z} = -\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} \qquad ...(iii)$$

Adding (i), (ii) and (iii), we get the required result.

Example 5.22. *If $z = f(x, y)$ and $x = e^u \cos v$, $y = e^u \sin v$, prove that $x \frac{\partial z}{\partial v} + y \frac{\partial z}{\partial u} = e^{2u} \frac{\partial z}{\partial y}$*

and

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial u}{\partial v}\right)^2\right]$$

(Mumbai, 2009)

Solution. We have

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x}(e^u \cos v) + \frac{\partial z}{\partial y}(e^u \sin v) \qquad ...(i)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x}(-e^u \sin v) + \frac{\partial x}{\partial y}(e^u \cos v) \qquad ...(ii)$$

$$\therefore \quad x \frac{\partial x}{\partial v} + y \frac{\partial z}{\partial u} = (e^u \cos v)\left[-e^u \sin v \frac{\partial z}{\partial x} + e^u \cos v \frac{\partial z}{\partial y}\right] + (e^u \sin v)\left[e^u \cos v \frac{\partial z}{\partial x} + e^u \sin v \frac{\partial z}{\partial y}\right]$$

$$= (e^{2u} \cos^2 v + e^{2u} \sin^2 v) \frac{\partial z}{\partial y} = e^{2u} \frac{\partial z}{\partial y}$$

Now squaring (i) and (ii) and adding, we get

$$\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = e^{2u}\left(\cos v \frac{\partial z}{\partial x} + \sin v \frac{\partial z}{\partial y}\right)^2 + e^{2u}\left(-\sin v \frac{\partial z}{\partial x} + \cos v \frac{\partial z}{\partial y}\right)^2$$

or

$$e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right] = \cos^2 v\left(\frac{\partial z}{\partial x}\right)^2 + \sin^2 v\left(\frac{\partial z}{\partial y}\right)^2 + 2 \sin v \cos v \frac{\delta z}{\delta z} \frac{\delta z}{\delta y}$$

$$+ \sin^2 v\left(\frac{\partial z}{\partial x}\right)^2 + \cos^2 v\left(\frac{\partial z}{\partial y}\right)^2 - 2 \sin v \cos v \frac{\partial z}{\partial x} \frac{\partial z}{\partial y}$$

$$= (\cos^2 v + \sin^2 v)\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

Hence $$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2\right].$$

Example 5.23. *If $x+y=2e^{\theta}\cos\phi$ and $x-y=2ie^{\theta}\sin\phi$, show that*

$$\frac{\partial^2 u}{\partial\theta^2}+\frac{\partial^2 u}{\partial\phi^2}=4xy\frac{\partial^2 u}{\partial x\partial y}$$ *(Nagpur, 2009 ; U.P.T.U., 2002)*

Solution. We have $x=e^{\theta}(\cos\phi+i\sin\phi)=e^{\theta}\,.\,e^{i\phi}$ [See p. 205]

and $y=e^{\theta}(\cos\phi-i\sin\phi)=e^{\theta}\,.\,e^{-i\phi}$

Here u is a composite function of θ and ϕ.

$$\therefore\quad \frac{\partial u}{\partial\theta}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial\theta}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial\theta}$$

$$=\frac{\partial u}{\partial x}\,.\,(e^{\theta}\,.\,e^{i\phi})+\frac{\partial u}{\partial y}(e^{\theta}\,.\,e^{-i\phi})=x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$$

or $$\frac{\partial}{\partial\theta}=x\frac{\partial}{\partial u}+y\frac{\partial}{\partial y}\qquad\ldots(i)$$

Also $$\frac{\partial u}{\partial\phi}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial\phi}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial\phi}=\frac{\partial u}{\partial x}\,.\,(e^{\theta}\,.\,ie^{i\phi})+\frac{\partial u}{\partial y}(e^{\theta}\,.\,-ie^{-i\phi})=ix\frac{\partial u}{\partial x}-iy\frac{\partial u}{\partial y}$$

$$\frac{\partial}{\partial\phi}=ix\frac{\partial}{\partial x}-iy\frac{\partial}{\partial y}\qquad\ldots(ii)$$

Using the operator (i), we have

$$\frac{\partial^2 u}{\partial\theta^2}=\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial\theta}\right)=\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$=x\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)+x\frac{\partial}{\partial x}\left(y\frac{\partial u}{\partial y}\right)+y\frac{\partial}{\partial y}\left(x\frac{\partial u}{\partial x}\right)+y\frac{\partial}{\partial y}\left(y\frac{\partial u}{\partial y}\right)$$

$$=x\left(x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}\right)+xy\frac{\partial^2 u}{\partial x\partial y}+yx\frac{\partial^2 u}{\partial y\partial x}+y\left(y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}\right)$$

$$=x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\qquad\ldots(iii)$$

Similarly using (ii), $$\frac{\partial^2 u}{\partial\phi^2}=\frac{\partial}{\partial\phi}\left(\frac{\partial u}{\partial\phi}\right)=\left(ix\frac{\partial}{\partial x}-iy\frac{\partial}{\partial y}\right)\left(ix\frac{\partial u}{\partial x}-iy\frac{\partial u}{\partial y}\right)$$

$$=-x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}-y^2\frac{\partial^2 u}{\partial y^2}-x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}\qquad\ldots(iv)$$

Adding (iii) and (iv), we get $\dfrac{\partial^2 u}{\partial\theta^2}+\dfrac{\partial^2 u}{\partial\phi^2}=4xy\dfrac{\partial^2 u}{\partial x\partial y}$

Example 5.24. *Transform the equation $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$ into polar coordinates.* *(P.T.U., 2010)*

Solution. We have $x=r\cos\theta$, $y=r\sin\theta$ and $r=\sqrt{(x^2+y^2)}$, $\theta=\tan^{-1}(y/x)$

$$\frac{\partial r}{\partial x}=\frac{x}{\sqrt{(x^2+y^2)}}=\cos\theta\text{ and }\frac{\partial\theta}{\partial x}=-\frac{y}{x^2+y^2}=-\frac{\sin\theta}{r}$$

Thus, $$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial u}{\partial\theta}\cdot\frac{\partial\theta}{\partial x}=\cos\theta\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}$$

i.e., $$\frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \qquad \text{Similarly, } \frac{\partial}{\partial y} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}.$$

$\therefore$ $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta \frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$= \cos^2\theta \frac{\partial^2 u}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r \partial \theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} \qquad ...(i)$$

and $$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\sin\theta \frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$= \sin^2\theta \frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r \partial \theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial r} \qquad ...(ii)$$

Adding (*i*) and (*ii*), we get $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r}$.

Hence the transformed equation is $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0.$

PROBLEMS 5.6

1. If $z = f(x, y)$ and $x = e^u + e^{-v}$, $y = e^{-u} - e^v$, prove that $\dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial v} = x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y}$. *(V.T.U., 2006)*

2. If $u = f(r, s)$, $r = x + at$, $s = y + bt$ and x, y, t are independent variables, show that $\dfrac{\partial u}{\partial t} = a\dfrac{\partial u}{\partial x} + b\dfrac{\partial u}{\partial y}$.

3. If $\phi(z/x^3, y/x) = 0$, prove that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 3z$. *(Mumbai, 2007)*

4. If $u = f(x, y)$ and $x = r\cos\theta$, $y = r\sin\theta$, prove that $\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2 = \left(\dfrac{\partial u}{\partial r}\right)^2 + \dfrac{1}{r^2}\left(\dfrac{\partial u}{\partial \theta}\right)^2$.
(V.T.U., 2010 ; Madras 2006 ; Rohtak, 2005)

5. If $u = f(2x - 3y, 3y - 4z, 4z - 2x)$, prove that $\dfrac{1}{2}\dfrac{\partial u}{\partial x} + \dfrac{1}{3}\dfrac{\partial u}{\partial y} + \dfrac{1}{4}\dfrac{\partial u}{\partial z} = 0$. *(U.P.T.U., 2006 ; Raipur, 2005)*

6. If $u = f(e^{y-z}, e^{z-x}, e^{x-y})$, prove that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$ *(Mumbai, 2008 S)*

7. If $u = f(r, s, t)$ and $r = x/y$, $s = y/z$, $t = z/x$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0$. *(Anna, 2009 : Kurukshetra, 2006)*

8. If $x = u + v + w$, $y = vw + wu + uv$, $z = uvw$ and F is a function of x, y, z, show that
$$u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\frac{\partial F}{\partial w} = x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y} + 3z\frac{\partial F}{\partial z}.$$

9. Given that $u(x, y, z) = f(x^2 + y^2 + z^2)$ where $x = r\cos\theta\cos\phi$, $y = r\cos\theta\sin\phi$ and $z = r\sin\theta$, find $\dfrac{\partial u}{\partial \theta}$ and $\dfrac{\partial u}{\partial \phi}$.

10. If the three thermodynamic variables P, V, T are connected by a relation $f(P, V, T) = 0$, show that
$$\left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_P \left(\frac{\partial V}{\partial P}\right)_T = -1.$$

11. If by the substitution $u = x^2 - y^2$, $v = 2xy$, $f(x, y) = \theta(u, v)$, show that
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(x^2 + y^2)\left(\frac{\partial^2 \theta}{\partial u^2} + \frac{\partial^2 \theta}{\partial v^2}\right).$$ *(Anna, 2003)*

12. Transform $\dfrac{\partial^2 z}{\partial x^2} + 2xy^2\dfrac{\partial z}{\partial x} + 2(y - y^3)\dfrac{\partial z}{\partial y} + x^2y^2z = 0$ by the substitution $x = uv$, $y = 1/v$. Hence show that z is the same function of u and v as of x and y.

5.7 (1) JACOBIANS

If u and v are functions of two independent variables x and y, then the determinant

$$\begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} \text{ is called the } \textit{Jacobian}^* \text{ of } u, v \text{ with respect to } x, y$$

and is written as $\dfrac{\partial(u, v)}{\partial(x, y)}$ or $J\left(\dfrac{u, v}{x, y}\right)$.

Similarly the Jacobian of u, v, w with respect to x, y, z is

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{vmatrix}$$

Likewise, we can define Jacobians of four or more variables. An important application of Jacobians is in connection with the change of variables in multiple integrals (§ 7.7).

(2) Properties of Jacobians. We give below two of the important properties of Jacobians. For simplicity, the properties are stated in terms of two variables only, but these are evidently true in general.

I. If $J = \partial(u, v)/\partial(x, y)$ and $J' = \partial(x, y)/\partial(u, v)$ then $JJ' = 1$.

Let $u = f(x, y)$ and $v = g(x, y)$.

Suppose, on solving for x and y, we get $x = \phi(u, v)$ and $y = \psi(u, v)$.

Then

$$\left.\begin{aligned} \frac{\partial u}{\partial u} &= 1 = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial u}, \\ \frac{\partial u}{\partial v} &= 0 = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v}, \\ \frac{\partial v}{\partial u} &= 0 = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u}, \\ \frac{\partial v}{\partial v} &= 1 = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial v}, \end{aligned}\right\} \quad \ldots(i)$$

and

$$\therefore \quad JJ' = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} \times \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial y/\partial x & \partial v/\partial y \end{vmatrix} \times \begin{vmatrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{vmatrix}$$

(Interchanging rows and columns of the 2nd determinant).

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1. \qquad \text{[By virtue of } (i)\text{]}$$

II. **Chain rule for Jacobians.** *If u, v are functions of r, s and r, s are functions of x, y, then*

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(r, s)}\cdot\frac{\partial(r, s)}{\partial(x, y)}.$$

$$\frac{\partial(u, v)}{\partial(r, s)}\cdot\frac{\partial(r, s)}{\partial(x, y)} = \begin{vmatrix} \partial u/\partial r & \partial u/\partial s \\ \partial v/\partial r & \partial v/\partial s \end{vmatrix} \times \begin{vmatrix} \partial r/\partial x & \partial s/\partial x \\ \partial r/\partial y & \partial s/\partial y \end{vmatrix}$$

[Interchanging rows and columns of the 2nd det.]

$$= \begin{vmatrix} \dfrac{\partial u}{\partial r}\cdot\dfrac{\partial r}{\partial x} + \dfrac{\partial u}{\partial s}\cdot\dfrac{\partial s}{\partial x} & \dfrac{\partial u}{\partial r}\cdot\dfrac{\partial r}{\partial y} + \dfrac{\partial u}{\partial s}\cdot\dfrac{\partial s}{\partial y} \\ \dfrac{\partial v}{\partial r}\cdot\dfrac{\partial r}{\partial x} + \dfrac{\partial v}{\partial s}\cdot\dfrac{\partial s}{\partial x} & \dfrac{\partial v}{\partial r}\cdot\dfrac{\partial r}{\partial y} + \dfrac{\partial v}{\partial s}\cdot\dfrac{\partial s}{\partial y} \end{vmatrix} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial(u, v)}{\partial(x, y)}.$$

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* Called after the German mathematician *Carl Gustav Jacob Jacobi* (1804–1851), who made significant contributions to mechanics, partial differential equations, astronomy, elliptic functions and the calculus of variations.

Example 5.25. *(i) In polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, show that*

$$\frac{\partial(x, y)}{\partial(r, \theta)} = r.$$

(U.P.T.U., 2006 ; V.T.U., 2004 ; Andhra, 2000)

(ii) In cylindrical coordinates (Fig. 8.28), $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$, show that

$$\frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \rho.$$

(iii) In spherical polar coordidnates (Fig. 8.29), $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, show that

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2\sin\theta.$$

(Anna, 2009 ; Hazaribagh, 2009 ; Rohtak, 2003)

Solution. (*i*) We have $\dfrac{\partial x}{\partial r} = \cos\theta,\ \dfrac{\partial x}{\partial \theta} = -r\sin\theta$ and $\dfrac{\partial y}{\partial r} = \sin\theta,\ \dfrac{\partial y}{\partial \theta} = -r\cos\theta$

$$\therefore \quad \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

(*ii*) We have

$$\frac{\partial x}{\partial \rho} = \cos\phi,\ \frac{\partial x}{\partial \phi} = -\rho\sin\phi,\ \frac{\partial x}{\partial z} = 0,$$

$$\frac{\partial y}{\partial \rho} = \sin\phi,\ \frac{\partial y}{\partial \phi} = \rho\cos\phi,\ \frac{\partial y}{\partial z} = 0 \text{ and } \frac{\partial z}{\partial \rho} = 0,\ \frac{\partial z}{\partial \phi} = 0,\ \frac{\partial z}{\partial z} = 1$$

$$\therefore \quad \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \begin{vmatrix} \cos\phi & -\rho\sin\theta & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho.$$

(*iii*) We have

$$\frac{\partial x}{\partial r} = \sin\theta\cos\theta,\ \frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi,\ \frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi,$$

$$\frac{\partial y}{\partial r} = \sin\theta\sin\phi,\ \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi,\ \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi,$$

and

$$\frac{\partial z}{\partial r} = \cos\theta,\ \frac{\partial z}{\partial \theta} = -r\sin\theta,\ \frac{\partial z}{\partial \phi} = 0.$$

$$\therefore \quad \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta.$$

Example 5.26. *If $y_1 = \dfrac{x_2x_3}{x_1}$, $y_2 = \dfrac{x_3x_1}{x_2}$, $y_3 = \dfrac{x_1x_2}{x_3}$, show that the Jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3 is 4.*

(U.P.T.U., 2006)

Solution. We have $\dfrac{\partial y_1}{\partial x_1} = -\dfrac{x_2x_3}{x_1^2},\ \dfrac{\partial y_1}{\partial x_2} = \dfrac{x_3}{x_1},\ \dfrac{\partial y_1}{\partial x_3} = \dfrac{x_2}{x_1}$

$$\frac{\partial y_2}{\partial x_1} = \frac{x_3}{x_2},\ \frac{\partial y_2}{\partial x_2} = -\frac{x_3x_1}{x_2^2},\ \frac{\partial y_2}{\partial x_3} = \frac{x_1}{x_2}$$

and

$$\frac{\partial y_3}{\partial x_1} = \frac{x_2}{x_3},\ \frac{\partial y_3}{\partial x_2} = \frac{x_1}{x_3},\ \frac{\partial y_3}{\partial x_3} = -\frac{x_1x_2}{x_3^2}$$

$$\therefore \quad \frac{\partial(y_1y_2y_3)}{\partial(x_1x_2x_3)} = \begin{vmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} & \dfrac{\partial y_1}{\partial x_3} \\ \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} & \dfrac{\partial y_2}{\partial x_3} \\ \dfrac{\partial y_3}{\partial x_1} & \dfrac{\partial y_3}{\partial x_2} & \dfrac{\partial y_3}{\partial x_3} \end{vmatrix} \begin{vmatrix} -\dfrac{x_2x_3}{x_1^2} & \dfrac{x_3}{x_1} & \dfrac{x_2}{x_1} \\ \dfrac{x_3}{x_2} & -\dfrac{x_3x_1}{x_2^2} & \dfrac{x_1}{x_1} \\ \dfrac{x_2}{x_3} & \dfrac{x_1}{x_3} & -\dfrac{x_1x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{1}{x_1^2x_2^2x_3^2}\begin{vmatrix} -x_2x_3 & x_3x_1 & x_1x_2 \\ x_2x_3 & -x_3x_1 & x_1x_2 \\ x_2x_3 & x_3x_1 & -x_1x_2 \end{vmatrix} = -\frac{x_1^2x_2^2x_3^2}{x_1^2x_2^2x_3^2}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= -1(1-1) - 1(-1-1) + 1(1+1) = 0 + 2 + 2 = 4.$$

Example 5.27. *In $u = x + 3y^2 - z^2$, $v = 4x^2yz$, $w = 2z^2 - xy$, evaluate $\partial(u, v, w)/\partial(x, y, z)$ at $(1, -1, 0)$.* *(V.T.U., 2006)*

Solution. $$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 6y & -3z^2 \\ 8xyz & 4x^2z & 4x^2y \\ -y & -x & 4z \end{vmatrix}$$

$\therefore$ At the point $(1, -1, 0)$ $$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} 1 & -6 & 0 \\ 0 & 0 & -4 \\ 1 & -1 & 0 \end{vmatrix} = 4(-1+6) = 20.$$

Example 5.28. *If $u = x^2 - y^2$, $v = 2xy$ and $x = r\cos\theta$, $y = r\sin\theta$, find $\frac{\partial(u, v)}{\partial(r, \theta)}$.* *(V.T.U., 2009 ; Madras, 2006)*

Solution. We have $$\frac{\partial(u, v)}{\partial(r, \theta)} = \frac{\partial(u, v)}{\partial(x, y)} \times \frac{\partial(x, y)}{\partial(r, \theta)}$$

Since $u = x^2 - y^2$, $u = 2xy$

$$\therefore \quad \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2 + y^2) \qquad ...(ii)$$

Since $x = r\cos\theta$, $y = r\sin\theta$,

$$\therefore \quad \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \qquad ...(iii)$$

Hence, $$\frac{\partial(u, v)}{\partial(r, \theta)} = 4(x^2 + y^2).\, r = 4(r^2\cos^2\theta + r^2\sin^2\theta).\, r = 4r^3 \qquad \text{[Using (ii) \& (iii)]}$$

(3) Jacobian of Implicit functions. *If u_1, u_2, u_3 instead of being given explicitly in terms x_1, x_2, x_3, be connected with them equations such as*

$f_1(u_1, u_2, u_3, x_1, x_2, x_3) = 0$, $f_2(u_1, u_2, u_3, x_1, x_2, x_3) = 0$, $f_3(u_1, u_2, u_3, x_1, x_2, x_3) = 0$, then

$$\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} = (-1)^3 \frac{\partial(f_1, f_2, f_3)}{\partial(x_1, x_2, x_3)} \div \frac{\partial(f_1, f_2, f_3)}{\partial(u_1, u_2, u_3)}$$

Obs. This result can be easily generalised. It bears analogy to the result $\frac{dy}{dx} = -\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y}$, where x, y are connected by the relation $f(x, y) = 0$.

Example 5.29. *If $u = x, y, z$, $v = x^2 + y^2 + z^2$, $w = x + y + z$, find $\partial(x, y, z)/\partial(u, v, w)$.* *(U.P.T.U., 2003)*

Solution. Let $f_1 = u - xyz$, $f_2 = v - x^2 - y^2 - z^2$, $f_3 = w - x - y - z$.

We have $$\frac{\partial(x, y, z)}{\partial(u, v, w)} = (-1)^3 \frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} \div \frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)} \quad ...(i)$$

Now, $$\frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)} = \begin{vmatrix} \partial f_1/\partial x & \partial f_1/\partial y & \partial f_1/\partial z \\ \partial f_2/\partial x & \partial f_2/\partial y & \partial f_2/\partial z \\ \partial f_3/\partial x & \partial f_3/\partial y & \partial f_3/\partial z \end{vmatrix} = \begin{vmatrix} -yz & -xz & -xy \\ -2x & -2y & -2z \\ -1 & -1 & -1 \end{vmatrix}$$

$$= -2(x-y)(y-z)(z-x) \quad ...(ii)$$

and $$\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \quad ...(iii)$$

Substituting values from (ii) and (iii) in (i), we get

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = (-1) \times 1/[-2(x-y)(y-z)(z-x)] = 1/2(x-y)(y-z)(z-x).$$

(4) Functional relationship. *If u_1, u_2, u_3 be functions of x_1, x_2, x_3 then the necessary and sufficient condition for the existence of a functional relationship of the form $f(u_1, u_2, u_3) = 0$, is*

$$J\left(\frac{u_1, u_2, u_3}{x_1, x_2, x_3}\right) = 0.$$

Example 5.30. *If $u = x\sqrt{(1-y^2)} + y\sqrt{(1-x^2)}$, $v = \sin^{-1} x + \sin^{-1} y$, show that u, v are functionally related and find the relationship.* (*Kurukshetra, 2006*)

Solution. We have $$\frac{\partial u}{\partial x} = \sqrt{(1-y^2)} - \frac{xy}{\sqrt{(1-x^2)}}, \frac{\partial u}{\partial y} = \frac{-xy}{\sqrt{(1-y^2)}} + \sqrt{(1-x^2)}$$

and $$\frac{\partial v}{\partial x} = \frac{1}{\sqrt{(1-x^2)}}, \frac{\partial v}{\partial y} = \frac{1}{\sqrt{(1-y^2)}}$$

$$\therefore \quad \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \sqrt{(1-y^2)} - \frac{xy}{\sqrt{(1-x^2)}}, & \sqrt{(1-x^2)} - \frac{xy}{\sqrt{(1-y^2)}} \\ \frac{1}{\sqrt{(1-x^2)}}, & \frac{1}{\sqrt{(1-y^2)}} \end{vmatrix}$$

$$= 1 - \frac{xy}{\sqrt{[(1-x^2)(1-y^2)]}} - 1 + \frac{xy}{\sqrt{[(1-x^2)(1-y^2)]}} = 0$$

Hence u and v are functionally related *i.e.*, they are not independent.

We have $$v = \sin^{-1} x + \sin^{-1} y = \sin^{-1} [x\sqrt{(1-y^2)} + y\sqrt{(1-x^2)}]$$

i.e., $$u = \sin v$$

which is the required relationship between u and v.

PROBLEMS 5.7

1. If $x = r\cos\theta$, $y = r\sin\theta$, evaluate $\partial(r, \theta)/\partial(x, y)$, $\partial(x, y)/\partial(r, \theta)$ and prove that $[\partial(r, \theta)\,\partial(x, y)]$. $[\partial(x, y)/\partial(r, \theta)] = 1$. (*V.T.U., 2010*)
2. If $x = u(1-v)$, $y = uv$, prove that $JJ' = 1$. (*V.T.U., 2000 S*)
3. If $x = a\cosh\xi\cos\eta$, $y = a\sinh\xi\sin\eta$, show that $\partial(x, y)/\partial(\xi, \eta) = \frac{1}{2}a^2(\cosh 2\xi - \cos 2\eta)$ (*S.V.T.U., 2007*)
4. If $x = e^u \sec v$, $y = e^u \tan v$, find $J = \partial(u, v)/\partial(x, y)$, $J' = \partial(x, y)/\partial(u, v)$. Hence show $JJ' = 1$. (*V.T.U., 2007 S*)
5. If $u = x^2 - 2y^2$, $v = 2x^2 - y^2$ where $x = r\cos\theta$, $y = r\sin\theta$, show that $\frac{\partial(u, v)}{\partial(r, \theta)} = 6r^3 \sin 2\theta$.
6. If $u = x^2 + y^2 + z^2$, $v = xy + yz + zx$, $w = x + y + z$, find $\frac{\partial(u, v, w)}{\partial(x, y, z)}$. (*U.T.U., 2009 ; V.T.U., 2008*)

7. If $F = xu + v - y$, $G = u^2 + vy + w$, $H = zu - v + vw$, compute $\partial(F, G, H)/\partial(u, v, w)$.
8. If $u = x + y + z$, $uv = y + z$, $uvw = z$, show that $\partial(x, y, z)/\partial(u, v, w) = u^2v$.
(*Kurukshetra, 2009; P.T.U., 2009 S; V.T.U., 2003*)
9. If $u^3 + v^3 = x + y$ and $u^2 + v^2 = x^3 + y^3$, show that $\dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{1}{2}\dfrac{y^2 - x^2}{uv(u-v)}$ (*U.P.T.U., 2006 MCA*)
10. If $u = \dfrac{x+y}{1-xy}$ and $v = \tan^{-1}x + \tan^{-1}y$, find $\dfrac{\partial(u,v)}{\partial(x,y)}$. Are u and v functionally related. If so, find this relationship.
(*Nagpur, 2008*)
11. If $u = 3x + 2y - z$, $v = x - 2y + z$ and $w = x(x + 2y - z)$, show that they are functionally related, and find the relation.
(*Nagpur, 2009*)

5.8 (1) GEOMETRICAL INTERPRETATION

If $P(x, y, z)$ be the coordinates of a point referred to axes OX, OY, OZ then the equation $z = f(x, y)$ represents a surface. (Fig. 5.1)

Let a plane $y = b$ parallel to the XZ-plane pass through P cutting the surface along the curve APB given by

$$z = f(x, b).$$

As y remains equal to b and x varies then P moves along the curve APB and $\partial z/\partial x$ is the ordinary derivative of $f(x, b)$ w.r.t. x.

Hence *$\partial z/\partial x$ at P is the tangent of the angle which the tangent at P to the section of the surface $z = f(x, y)$ by a plane through P parallel to the plane XOZ, makes with a line parallel to the x-axis.*

Similarly, *$\partial z/\partial y$ at P is the tangent of the angle which the tangent at P to the curve of intersection of the surface $z = f(x, y)$ and the plane $x = a$, makes with a line parallel to the y-axis.*

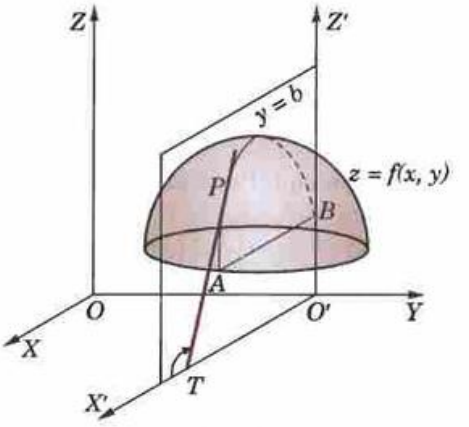

Fig. 5.1

(2) Tangent plane and Normal to a surface. Let $P(x, y, z)$ and $Q(x + \delta x, y + \delta y, z + \delta z)$ be two neighbouring points on the surface $F(x, y, z) = 0$. (Fig. 5.2) ...(*i*)

Let the arc PQ be δs and the chord PQ be δc, so that (as for plane curves)

$$\underset{Q \to P}{\text{Lt}} (\delta s/\delta c) = 1.$$

The direction cosines of PQ are $\dfrac{\delta x}{\delta c}, \dfrac{\delta y}{\delta c}, \dfrac{\delta z}{\delta c}$ *i.e.*, $\dfrac{\delta x}{\delta s}\cdot\dfrac{\delta s}{\delta c}, \dfrac{\delta y}{\delta s}\cdot\dfrac{\delta s}{\delta c}, \dfrac{\delta z}{\delta s}\cdot\dfrac{\delta s}{\delta c}$

When $\delta s \to 0$, $Q \to P$ and PQ tends to tangent line PT. Then noting that the coordinates of any point on arc PQ are functions of s only, the direction cosines of PT are

$$\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds} \qquad ...(ii)$$

Differentiating (*i*) with respect to s, we obtain $\dfrac{\partial F}{\partial x}\cdot\dfrac{dx}{ds} + \dfrac{\partial F}{\partial y}\cdot\dfrac{dy}{ds} + \dfrac{\partial F}{\partial z}\cdot\dfrac{dz}{ds} = 0.$

This shows that the tangent line whose direction cosines are given by (*ii*), is perpendicular to the line having direction ratios

$$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \qquad ...(iii)$$

Since we can take different curves joining Q to P, we get a number of tangent lines at P and the line having direction ratios (*iii*) will be perpendicular to *all* these tangent lines at P. Thus all the tangent lines at P lie in a plane through P perpendicular to line (*iii*).

Hence *the equation of the tangent plane to (i) at the point P is*

$$\frac{\partial \mathbf{F}}{\partial \mathbf{x}}(\mathbf{X} - \mathbf{x}) + \frac{\partial \mathbf{F}}{\partial \mathbf{y}}(\mathbf{Y} - \mathbf{y}) + \frac{\partial \mathbf{F}}{\partial \mathbf{z}}(\mathbf{Z} - \mathbf{z}) = \mathbf{0}$$

where (X, Y, Z) are the current coordinates of any point on this tangent plane.

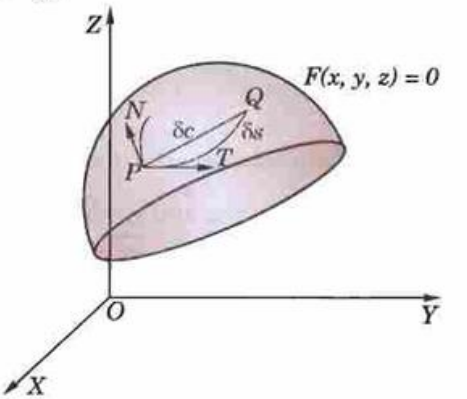

Fig. 5.2

Also *the equation of the normal to the surface at P (i.e., the line through P,* perpendicular to the tangent plane at P) is

$$\frac{\mathbf{X-x}}{\partial \mathbf{F}/\partial \mathbf{x}} = \frac{\mathbf{Y-y}}{\partial \mathbf{F}/\partial \mathbf{y}} = \frac{\mathbf{Z-z}}{\partial \mathbf{F}/\partial \mathbf{z}}.$$

Example 5.31. *Find the equations of the tangent plane and the normal to the surface $z^2 = 4(1 + x^2 + y^2)$ at (2, 2, 6).*

Solution. We have $\quad F(x, y, z) = 4x^2 + 4y^2 - z^2 + 4.$

$\therefore \quad \partial F/\partial x = 8x,\ \partial F/\partial y = 8y,\ \partial F/\partial z = -2z$, and at the point (2, 2, 6)

$\partial F/\partial x = 16,\ \partial F/\partial y = 16,\ \partial F/\partial z = -12$

Hence the equation of the tangent plane at (2, 2, 6) is $16(X-2) + 16(Y-2) - 12(Z-6) = 0$

i.e., $\quad 4X + 4Y - 3Z + 2 = 0 \qquad \ldots(i)$

Also the equation of the normal at (2, 2, 6) [being perpendicular to (i)] is

$$\frac{X-2}{4} = \frac{Y-2}{4} = \frac{Z-6}{-3}.$$

PROBLEMS 5.8

Find the equations of the tangent plane and normal to each of the following surfaces at the given points :

1. $2x^2 + y^2 = 3 - 2z$ at $(2, 1, -3)$ *(Assam, 1998)*
2. $x^3 + y^3 + 3xyz = 3$ at $(1, 2, -1)$ *(Osmania, 2003 S)*
3. $xyz = a^2$ at (x_1, y_1, z_1).
4. $2xz^2 - 3xy - 4x = 7$ at $(1, -1, 2)$.
5. Show the plane $3x + 12y - 6z - 17 = 0$ touches the conicoid $3x^2 - 6y^2 + 9z^2 + 17 = 0$. Find also the point of contact.
6. Show that the plane $ax + by + cz + d = 0$ touches the surface $px^2 + qy^2 + 2z = 0$, if $\frac{a^2}{p} + \frac{b^2}{q} + 2cd = 0$.
7. Find the equation of the normal to the surface $x^2 + y^2 + z^2 = a^2$. *(P.T.U., 2009 S)*

5.9 TAYLOR'S THEOREM FOR FUNCTIONS OF TWO VARIABLES

Considering $f(x + h, y + k)$ as a function of a single variable x, we have by Taylor's theorem*

$$f(x+h, y+k) = f(x, y+k) + h\frac{\partial f(x, y+k)}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 f(x, y+k)}{\partial x^2} + \ldots \qquad \ldots(i)$$

Now expanding $f(x, y + k)$ as a function of y only,

$$f(x, y+k) = f(x, y) + k\frac{\partial f(x, y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2} + \ldots$$

$\therefore$ (i) takes the form $f(x+h, y+k) = f(x, y) + k\dfrac{\partial f(x, y)}{\partial y} + \dfrac{k^2}{2!}\dfrac{\partial^2 f(x, y)}{\partial y^2} + \ldots$

$$+ h\frac{\partial}{\partial x}\left\{f(x, y) + k\frac{\partial f(x, y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2} + \ldots\right\} + \frac{h^2}{2!}\frac{\partial^2}{\partial x^2}\left\{f(x, y) + k\frac{\partial(x, y)}{\partial y} + \ldots\right\}$$

Hence, $\quad f(x+h, y+k) = f(x, y) + h\dfrac{\partial f}{\partial x} + k\dfrac{\partial f}{\partial y} + \dfrac{1}{2!}\left(h^2\dfrac{\partial^2 f}{\partial x^2} + 2hk\dfrac{\partial^2 f}{\partial x \partial y} + k^2\dfrac{\partial^2 f}{\partial y^2}\right) + \ldots \qquad \ldots(1)$

In symbols we write it as $f(x+h, y+k) = f(x, y) + \left(h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y}\right)f + \dfrac{1}{2!}\left(h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y}\right)^2 f + \ldots$

Taking $x = a$ and $y = b$, (1) becomes

$$f(a+h, b+k) = f(a, b) + [hf_x(a, b) + kf_y(a, b)] + \frac{1}{2!}[h^2 f_{xx}(a, b) + 2hkf_{xy}(a, b) + k^2 f_{yy}(a, b)] + \ldots$$

*See footnote on page 145.

Putting $a + h = x$ and $b + k = y$ so that $h = x - a$, $k = y - b$, we get

$$\mathbf{f(x, y) = f(a, b) + [(x - a) f_x (a, b) + (y - b) f_y (a, b)]}$$
$$\mathbf{+ \frac{1}{2!} [(x - a)^2 f_{xx} (a, b) + 2(x - a)(y - b) f_{xy} (a, b) + (y - b)^2 f_{yy}(a, b)] + \ldots} \qquad \ldots(2)$$

This is Taylor's expansion of f(x, y) in powers of (x – a) and (y – b). It is used to expand f(x, y) in the neighbourhood of (a, b).

Cor. Putting $a = 0$, $b = 0$, in (2), we get

$$\mathbf{f(x, y) = f(0, 0) + [xf_x(0, 0) + yf_y (0, 0)] + \frac{1}{2!} [x^2 f_{xx} (0, 0) + 2xy f_{xy} (0, 0) + y^2 f_{yy} (0, 0)] + \ldots} \qquad \ldots(3)$$

This is Maclaurin's expansion of $f(x, y)$.

Example 5.32. *Expand $e^x \log (1 + y)$ in powers of x and y upto terms of third degree.*
(V.T.U., 2010 ; P.T.U., 2009 ; J.N.T.U., 2006)

Solution. Here

$$\begin{aligned}
f(x, y) &= e^x \log (1 + y) & \therefore \quad f(0, 0) &= 0\\
f_x(x, y) &= e^x \log (1 + y) & f_x(0, 0) &= 0\\
f_y(x, y) &= e^x \frac{1}{1 + y} & f_y(0, 0) &= 1\\
f_{xx}(x, y) &= e^x \log (1 + y) & f_{xx}(0, 0) &= 0\\
f_{xy}(x, y) &= e^x \frac{1}{1 + y} & f_{xy}(0, 0) &= 1\\
f_{yy}(x, y) &= - e^x (1 + y)^{-2} & f_{yy}(0, 0) &= - 1\\
f_{xxx}(x, y) &= e^x \log (1 + y) & f_{xxx}(0, 0) &= 0\\
f_{xxy}(x, y) &= e^x \frac{1}{1 + y} & f_{xxy}(0, 0) &= 1\\
f_{xyy}(x, y) &= - e^x (1 + y)^{-2} & f_{xyy}(0, 0) &= - 1\\
f_{yyy}(x, y) &= 2e^x (1 + y)^{-3} & f_{yyy}(0, 0) &= 2
\end{aligned}$$

Now Maclaurin's expansion of $f(x, y)$ gives

$$f(x, y) = f(0, 0) + xf_x(0, 0) + yf_y(0, 0) + \frac{1}{2!} \{x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)\}$$
$$+ \frac{1}{3!} \{x^3 f_{xxx}(0, 0) + 3x^2 y f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)\} + \ldots$$

$$\therefore \quad e^x \log (1 + y) = 0 + x(0) + y(1) + \frac{1}{2!} \{x^2 (0) + 2xy(1) + y^2 (-1)\}$$
$$+ \frac{1}{3!} \{x^3(0) + 3x^2y(1) + 3xy^2(-1) + y^3(2)\} + \ldots$$
$$= y + xy - \frac{1}{2} y^2 + \frac{1}{2} (x^2y - xy^2) + \frac{1}{3} y^3 + \ldots$$

Example 5.33. *Expand $x^2y + 3y - 2$ in powers of $(x - 1)$ and $(y + 2)$ using Taylor's theorem.*
(P.T.U., 2010 ; V.T.U., 2008 ; U.P.T.U., 2006 ; Anna, 2005)

Solution. Taylor's expansion of $f(x, y)$ in powers of $(x - a)$ and $(x - b)$ is given by

$$f(x, y) = f(a, b) + [(x - a) f_x (a, b) + (y - b) f_y(a, b)] + \frac{1}{2!} [(x - a)^2 f_{xx} (a, b)$$
$$+ 2 (x - a)(y - b) f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b)] + \frac{1}{3!} [(x - a)^3 f_{xxx}(a, b)$$
$$+ 3 (x - a)^2 (y - b) f_{xxy}(a, b) + 3 (x - a) (y - b)^2 f_{xyy} (a, b)$$
$$+ (y - b)^3 f_{yyy} (a, b)] + \ldots \qquad \ldots(i)$$

Hence $a = 1$, $b = - 2$ and $f(x, y) = x^2y + 3y - 2$

$\therefore$ $f(1,-2) = -10, f_x = 2xy, f_x(1,-2) = -4; f_y = x^2+3, f_y(1,-2) = 4; f_{xx} = 2y,$
$f_{xx}(1,-2) = -4; f_{xy} = 2x, f_{xy}(1,-2) = 2; f_{yy} = 0, f_{yy}(1,-2) = 0; f_{xxx} = 0, f_{xxx} = (1,-2) = 0;$
$f_{xxy}(1,-2) = 2, f_{xyy}(1,-2) = 0, f_{yyy}(1,-2) = 0$

All partial derivatives of higher order vanish.

Substituting these in (i), we get

$$x^2y + 3y - 2 = -10 + [(x-1)(-4) + (y+2)4] + \frac{1}{2}[(x-1)^2(-4) + 2(x-1)(y+2)(2)$$
$$+ (y+2)^2(0)] + \frac{1}{6}[(x-1)^3(0) + 3(x-1)^2(y+2)(2) + 3(x-1)(y+2)^2(0) + (y+2)^3(0)]$$
$$= -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2).$$

Example 5.34. *Expand $f(x, y) = \tan^{-1}(y/x)$ in powers of $(x-1)$ and $(y-1)$ upto third-degree terms. Hence compute $f(1.1, 0.9)$ approximately.* (V.T.U., 2010 ; J.N.T.U., 2006 ; U.P.T.U., 2006)

Solution. Here $a = 1, b = 1$ and $f(1, 1) = \tan^{-1}(1) = \pi/4$.

$$f_x = \frac{-y}{x^2+y^2} \qquad f_x(1,1) = -\frac{1}{2}; \qquad f_y = \frac{x}{x^2+y^2}, \qquad f_y(1,1) = \frac{1}{2}$$

$$f_{xx} = \frac{2xy}{(x^2+y^2)^2}, \qquad f_{xx}(1,1) = \frac{1}{2}; \qquad f_{xy} = \frac{y^2-x^2}{(x^2+y^2)^2}, \qquad f_{xy}(1,1) = 0$$

$$f_{yy} = \frac{-2xy}{(x^2+y^2)^2}, \qquad f_{yy}(1,1) = -\frac{1}{2};$$

$$f_{xxx} = \frac{2y^3-6x^2y}{(x^2+y^2)^3}, \qquad f_{xxx}(1,1) = -\frac{1}{2}; \qquad f_{xxy} = \frac{2x^3-6xy^2}{(x^3+y^2)^3}, \qquad f_{xxy}(1,1) = -\frac{1}{2}$$

$$f_{xyy} = \frac{6x^2y-2y^3}{(x^2+y^2)^3}, \qquad f_{xyy}(1,1) = \frac{1}{2}; \qquad f_{yyy} = \frac{6xy^2-2x^3}{(x^2+y^2)^3}, \qquad f_{yyy}(1,1) = \frac{1}{2}$$

Taylor's expansion of $f(x, y)$ in powers of $(x-1)$ and $(y-1)$ is given by

$$f(x,y) = f(1,1) + \frac{1}{1!}\{(x-1)f_x(1,1) + (y-1)f_y(1,1)\} + \frac{1}{2!}\{(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1)$$
$$f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1) + \frac{1}{3!}\{(x-1)^3 f_{xxx}(1,1) + 3(x-1)^2(y-1)f_{xxy}(1,1)$$
$$+ 3(x-1)(y-1)^2 f_{xyy}(1,1) + (y-1)^3 f_{yyy}(1,1)\} + \ldots$$

$$\therefore \quad \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4} + \left\{(x-1)\left(-\frac{1}{2}\right) + (y-1)\frac{1}{2}\right\} + \frac{1}{2!}\left\{(x-1)^2\frac{1}{2} + 2(x-1)(y-1)(0) + (y-1)^2\left(-\frac{1}{2}\right)\right\}$$
$$+ \frac{1}{3!}\left\{(x-1)^3\left(-\frac{1}{2}\right) + 3(x-1)^2(y-1)\left(-\frac{1}{2}\right) + 3(x-1)(y-1)^2\frac{1}{2} + (y-1)^3\frac{1}{2}\right\} + \ldots$$
$$= \frac{\pi}{4} - \frac{1}{2}\{(x-1)-(y-1)\} + \frac{1}{4}\{(x-1)^2-(y-1)^2\} - \frac{1}{12}\{(x-1)^3 + 3(x-1)^2(y-1)$$
$$- 3(x-1)(y-1)^2 - (y-1)^3\} + \ldots$$

Putting $x = 1.1$ and $y = 0.9$, we get

$$f(1.1, 0.9) = \frac{\pi}{4} - \frac{1}{2}(0.2) + \frac{1}{4}(0) - \frac{1}{12}\{(0.1)^3 - 3(0.1)^3 - 3(0.1)^3 - (-0.1)^3\}$$
$$= 0.7854 - 0.1000 + 0.0003 = 0.6857.$$

5.10 (1) ERRORS AND APPROXIMATIONS

Let $f(x, y)$ be a continuous function of x and y. If δx and δy be the increments of x and y, then the new value of $f(x, y)$ will be $f(x + \delta x, y + \delta y)$. Hence

$$\delta f = f(x + \delta x, y + \delta y) - f(x, y).$$

Expanding $f(x + \delta x, y + \delta y)$ by Taylor's theorem and supposing δx, δy to be so small that their products, squares and higher powers can be neglected, we get

$$\delta \mathbf{f} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \delta \mathbf{x} + \frac{\partial \mathbf{f}}{\partial \mathbf{y}} \delta \mathbf{y}, \text{ approximately.}$$

Similarly if f be a function of several variables $x, y, z, t, \ldots$, then

$$\delta f = \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial z} \delta z + \frac{\partial f}{\partial t} \delta t + \ldots \text{ approximately.}$$

These formulae are very useful in correcting the effect of small errors in measured quantities.

(2) Total Differential

If u is a function of two variables x and y, the *total differential* of u is defined as

$$\mathbf{du} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \mathbf{dx} + \frac{\partial \mathbf{u}}{\partial \mathbf{y}} \mathbf{dy} \qquad \ldots(1)$$

The differentials dx and dy are respectively the increments δx and δy in x and y. If x and y are not independent variables but functions of another variable t even then the formula (1) holds and we write $dx = \frac{dx}{dt} dt$ and $dy = \frac{dy}{dt} dt$. Similar definition can be given for a function of three or more variables.

Example 5.35. *The diameter and altitude of a can in the shape of a right circular cylinder are measured as 4 cm and 6 cm respectively. The possible error in each measurement is 0.1 cm. Find approximately the maximum possible error in the values computed for the volume and the lateral surface.*

Solution. Let x be the diameter and y the height of the can. Then its volume $V = \frac{\pi}{4} x^2 y$

$$\therefore \qquad \delta V = \frac{\partial V}{\partial x} \delta x + \frac{\partial V}{\partial y} \delta y = \frac{\pi}{4} (2xy\delta x + x^2 \delta y)$$

When $x = 4$ cm., $y = 6$ cm. and $\delta x = \delta y = 0.1$ cm.

$$\therefore \qquad \delta V = \frac{\pi}{4}(2 \times 4 \times 6 \times 0.1 + 4^2 \times 0.1) = 1.6\pi \text{ cm}^3$$

Also its lateral surface $S = \pi xy$

$$\therefore \qquad \delta S = \pi(y\, \delta x + x\, \delta y)$$

When $x = 4$ cm., $y = 6$ cm. and $\delta x = \delta y = 0.1$ cm., we have $\delta S = \pi(6 \times 0.1 + 4 \times 0.1) = \pi \text{ cm}^2$.

Example 5.36. *The period of a simple pendulum is $T = 2\pi \sqrt{(l/g)}$, find the maximum error in T due to the possible error upto 1% in l and 2.5% in g.* *(U.P.T.U., 2004)*

Solution. We have $T = 2\pi \sqrt{(l/g)}$

or
$$\log T = \log 2\pi + \frac{1}{2} \log l - \frac{1}{2} \log g$$

$$\therefore \qquad \frac{1}{T} \delta T = 0 + \frac{1}{2} \frac{1}{l} \delta l - \frac{1}{2} \frac{1}{g} \delta g$$

$$\frac{\delta T}{T} 100 = \frac{1}{2}\left(\frac{\delta l}{l} 100 - \frac{\delta g}{g} 100\right) = \frac{1}{2}(1 \pm 2.5) = 1.75 \text{ or } -0.75$$

Thus the maximum error in $T = 1.75\%$

Example 5.37. *A balloon is in the form of right circular cylinder of radius 1.5 m and length 4 m and is surmounted by hemispherical ends. If the radius is increased by 0.01 m and length by 0.05 m, find the percentage change in the volume of balloon.* *(U.P.T.U., 2005)*

Solution. Let the volume of the balloon (Fig. 5.3) be V, so that

$$V = \pi r^2 h + \frac{2}{3}\pi r^3 + \frac{2}{3}\pi r^3 = \pi r^2 h + \frac{4}{3}\pi r^3$$

$$\therefore \qquad \delta V = 2\pi r \delta r h + \pi r^2 \delta h + \frac{4}{3}\pi 3 r^2 \delta r$$

or
$$\frac{\delta V}{V} = \frac{\pi r[2h\delta r + r\delta h + 4r\delta r]}{\pi r^2 h + \frac{4}{3}\pi r^3}$$

$$= \frac{2(h+2r)\,\delta r + r\delta h}{rh + \frac{4}{3}r^2} = \frac{2(4+3)(.01) + 1.5(.05)}{1.5 \times 4 + \frac{4}{3}(1.5)^2}$$

$$= \frac{0.14 + 0.075}{6+3} = \frac{0.215}{9}$$

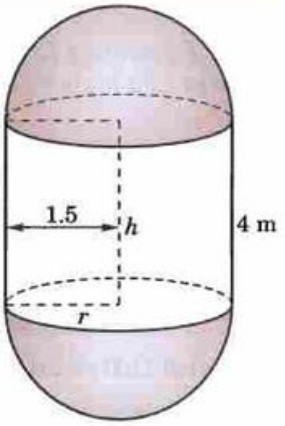

Fig. 5.3

Hence, the percentage change in $V = 100\,\dfrac{\delta V}{V} = \dfrac{21.5}{9} = 2.39\%$

Example 5.38. *In estimating the cost of a pile of bricks measured as 2 m × 15 m × 1.2 m, the tape is stretched 1% beyond the standard length. If the count is 450 bricks to 1 cu. m. and bricks cost ₹ 530 per 1000, find the approximate error in the cost.* (V.T.U., 2001)

Solution. Let x, y and z m be the length, breadth and height of the pile so that its volume $V = xyz$

or
$$\log V = \log x + \log y + \log z \;\therefore\; \frac{\delta V}{V} = \frac{\delta x}{x} + \frac{\delta y}{y} + \frac{\delta z}{z}$$

Since
$$V = 2 \times 15 \times 1.2 = 36 \text{ m}^3, \text{ and } \frac{\delta x}{x} = \frac{\delta y}{y} = \frac{\delta z}{z} = \frac{1}{100}$$

$$\therefore \qquad \delta V = 36\left(\frac{3}{100}\right) = 1.08 \text{ m}^3.$$

Number of bricks in $\delta V = 1.08 \times 450 = 486$

Thus error in the cost $= 486 \times \dfrac{530}{1000} =$ ₹ 257.58 which is a loss to the brick seller.

Example 5.39. *The height h and semi-vertical angle α of a cone are measured and from them A, the total area of the surface of the cone including the base is calculated. If h and α are in error by small quantities δh and $\delta\alpha$ respectively, find the corresponding error in the area. Show further that if $\alpha = \pi/6$, an error of + 1% in h will be approximately compensated by an error of − 0.33 degrees in α.*

Solution. If r be the base radius and l the slant height of the cone, (Fig. 5.4), then total area

$$A = \text{area of base} + \text{area of curved surface}$$
$$= \pi r^2 + \pi r l = \pi r(r + l)$$
$$= \pi h \tan\alpha\,(h \tan\alpha + h \sec\alpha)$$
$$= \pi h^2 (\tan^2\alpha + \tan\alpha \sec\alpha)$$

$$\therefore \qquad \delta A = \frac{\delta A}{\delta h}\delta h + \frac{\delta A}{\delta \alpha}\delta\alpha$$
$$= 2\pi h(\tan^2\alpha + \tan\alpha\sec\alpha)\delta h$$
$$+ \pi h^2 (2\tan\alpha\sec^2\alpha + \sec^3\alpha + \tan\alpha\sec\alpha\tan\alpha)\delta\alpha$$

which gives the error in the area A.

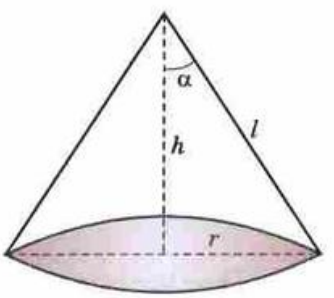

Fig. 5.4

Putting $\delta h = h/100$ and $\alpha = \pi/6$, we get

$$\delta A = 2\pi h\left[\left(\frac{1}{\sqrt{3}}\right)^2 + \frac{1}{\sqrt{3}}\cdot\frac{2}{\sqrt{3}}\right]\frac{h}{100} + \pi h^2\left[2.\frac{1}{\sqrt{3}}\cdot\frac{4}{3} + \frac{8}{3\sqrt{3}} + \frac{1}{\sqrt{3}}\cdot\frac{2}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}\right]\delta\alpha$$

$$= \frac{2\pi h^2}{100} + 2\sqrt{3}\pi h^2 \ \delta\alpha$$

The error in h will be compensated by the error in α, when

$$\delta A = 0 \ i.e., \ \frac{2\pi h^2}{100} + 2\sqrt{3}\pi h^2 \, \delta\alpha = 0$$

or $$\delta\alpha = -\frac{1}{100\sqrt{3}} \text{ radians} = -\frac{.01}{1.732} \times 57.3° = -0.33°.$$

Example 5.40. *Show that the approximate change in the angle A of a triangle ABC due to small changes δa, δb, δc in the sides a, b, c respectively, is given by*

$$\delta A = \frac{a}{2\Delta} (\delta a - \delta b \cos C - \delta c \cos B)$$

where Δ is the area of the triangle. Verify that $\delta A + \delta B + \delta C = 0$.

Solution. We know that $a^2 = b^2 + c^2 - 2bc \cos A$

so that $$2a\delta a = 2b\delta b + 2c\delta c - 2(c\delta b \cos A - b\delta c \cos A + bc \sin A \ \delta A)$$

$\therefore$ $$bc \sin A \ \delta A = a\delta a - (b - c \cos A) \ \delta b - (c - b \cos A) \ \delta c$$

or $$2 \ \Delta\delta A = a\delta a - (c \cos A + a \cos C - c \cos A) \ \delta b - (a \cos B + b \cos A - b \cos A) \ \delta c$$

$[\because \ b = c \cos A + a \cos C$ etc. ...(*i*)$]$

$$= a\delta a - a \cos C \ \delta b - a \cos B \ \delta c$$

or $$\delta A = \frac{a}{2\Delta} (\delta a - \delta b \cos C - \delta c \cos B)$$

By symmetry, we have

$$\delta B = \frac{b}{2\Delta} (\delta b - \delta c \cos A - \delta a \cos C)$$

$$\delta C = \frac{c}{2\Delta} (\delta c - \delta a \cos B - \delta b \cos A)$$

$\therefore$ $$\delta A + \delta B + \delta C = \frac{1}{2\Delta} (a - b \cos C - c \cos B) \ \delta a + (b - c \cos A - a \cos C) \ \delta b + (c - a \cos B - b \cos A)]$$

$$= \frac{1}{2\Delta} [(a - a) \ \delta a + (b - b) \ \delta b + (c - c) \ \delta c] = 0 \qquad \text{[By (i)]}$$

Example 5.41. *If the sides of a plane triangle ABC vary in such a way that its circumradius remains constant, prove that* $\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0$.

Solution. The circumradius R of ΔABC is given by

$$R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C}$$

$\therefore$ $$a = 2R \sin A \qquad [\because \ R \text{ is constant}$$

Taking differentials, $da = 2R \cos A \ dA$ or $\frac{da}{\cos A} = 2RdA$

Similarly, $$\frac{db}{\cos B} = 2RdB, \ \frac{dc}{\cos C} = 2RdC$$

$\therefore$ $$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 2R \ (dA + dB + dC)$$

Now $$A + B + C = \pi, \text{ gives } dA + dB + dC = 0 \qquad ...(i)$$

Thus $$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0 \qquad \text{[By (i)]}$$

PROBLEMS 5.9

1. Expand the following functions as far as terms of third degree :
   (*i*) $\sin x \cos y$ (*V.T.U., 2009*) (*ii*) $e^x \sin y$ at $(-1, \pi/4)$ (*Anna, 2009*)
   (*iii*) $xy^2 + \cos xy$ about $(1, \pi/2)$. (*Hissar, 2005 S ; V.T.U., 2003*)
2. Expand $f(x, y) = x^y$ in powers of $(x - 1)$ and $(y - 1)$. (*U.T.U., 2009*)
3. If $f(x, y) = \tan^{-1} xy$, compute $f(0.9, -1.2)$ approximately.
4. If the kinetic energy $k = wv^2/2g$, find approximately the change in the kinetic energy as w changes from 49 to 49.5 and v changes from 1600 to 1590. (*V.T.U., 2006*)
5. Find the possible percentage error in computing the resistance r from the formula $1/r = 1/r_1 + 1/r_2$, if r_1, r_2 are both in error by 2%.
6. The voltage V across a resistor is measured with an error h, and the resistance R is measured with an error k. Show that the error in calculating the power $W(V, R) = V^2/R$ generated in the resistor, is $VR^{-2}(2Rh - Vk)$. (*V.T.U., 2009*)
7. Find the percentage error in the area of an ellipse if one per cent error is made in measuring the major and minor axes. (*V.T.U., 2011*)
8. The time of oscillation of a simple pendulum is given by the equation $T = 2\pi\sqrt{(l/g)}$. In an experiment carried out to find the value of g, errors of 1.5% and 0.5% are possible in the values of l and T respectively. Show that the error in the calculated value of g is 0.5%. (*Cochin, 2005*)
9. If $pv^2 = k$ and the relative errors in p and v are respectively 0.05 and 0.025, show that the error in k is 10%. (*Mysore, 1999*)
10. If the H.P. required to propel a steamer varies as the cube of the velocity and square of the length. Prove that a 3% increase in velocity and 4% increase in length will require an increase of about 17% in H.P.
11. The range R of a projectile which starts with a velocity v at an elevation α is given by $R = (v^2 \sin 2\alpha)/g$. Find the percentage error in R due to an error of 1% in v and an error of $\frac{1}{2}$% in α. (*Kurukshetra, 2009*)
12. In estimating the cost of a pile of bricks measured as 6 m × 50 m × 4 m, the tape is stretched 1% beyond the standard length. If the count is 12 bricks in 1 m^3 and bricks cost ₹ 100 per 1000, find the approximate error in the cost. (*U.T.U., 2010 ; U.P.T.U., 2005*)
13. The deflection at the centre of a rod of length l and diameter d supported at its ends, loaded at the centre with a weight w varies at wl^3d^{-4}. What is the increase in the deflection corresponding to p% increase in w, q% decrease in l and r% increase in d ?
14. The work that must be done to propel a ship of displacement D for a distance s in time t is proportional to $(s^2D^{2/3}/t^2)$. Find approximately the increase of work necessary when the displacement is increased by 1%, the time is diminished by 1% and the distance diminished by 2%.
15. The indicated horse power l of an engine is calculated from the formula $l = PLAN/33{,}000$, where $A = \pi d^2/4$. Assuming that error of r per cent may have been made in measuring P, L, N and d, find the greatest possible error in l.
16. The torsional rigidity of a length of wire is obtained from the formula $N = 8\pi\, Il/t^2r^4$. If l is decreased by 2%, r is increased by 2%, t is increased by 1.5%, show that the value of N is diminished by 13% approximately. (*V.T.U., 2003*)
17. If $x^2 + y^2 + z^2 - 2xyz = 1$, show that $\dfrac{dx}{\sqrt{(1-x^2)}} + \dfrac{dy}{\sqrt{(1-y^2)}} + \dfrac{dz}{\sqrt{(1-z^2)}} = 0$.
    [**Hint.** $2(x - yz)\,dx + 2(y - zx)\,dy + 2(z - xy)\,dz = 0$. Also $(x - yz)^2 = (1 - y^2)(1 - z^2)$, ...]

5.11 (1) MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

Def. *A function $f(x, y)$ is said to have a* **maximum** *or* **minimum** *at $x = a, y = b$, according as*

$$f(a + h, b + k) < \text{or} > f(a, b),$$

for all positive or negative small values of h and k.

In other words, if $\Delta = f(a + h, b + k) - f(a, b)$, *is of the same sign for all small values of h, k, and if this sign is negative, then $f(a, b)$ is a maximum. If this sign is positive, $f(a, b)$ is a minimum.*

Considering $z = f(x, y)$ as a surface, maximum value of z occurs at the top of an elevation (*e.g.*, a dome) from which the surface descends in every direction and a minimum value occurs at the bottom of a depression (*e.g.*, a bowl) from which the surface ascends in every direction. Sometimes the maximum or minimum value may form a *ridge* such that the surface descends or ascends in all directions except that of the ridge. Besides these, we have such a point of the surface, where the tangent plane is horizontal and the surface looks like leather seat on the horse's back [Fig. 5.5 (*c*)] which falls for displacement in certain directions and rises for displacements in other directions. Such a point is called a **saddle point.**

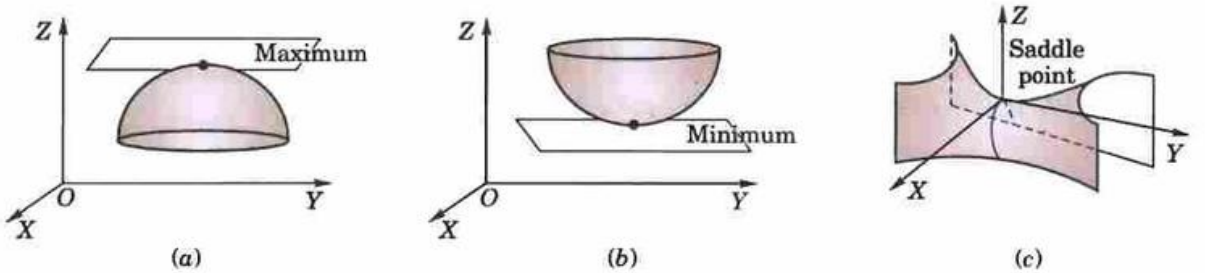

Fig. 5.5

Note. *A maximum or minimum value of a function is called its* **extreme value.**

(2) Conditions for f(x, y) to be maximum or minimum

Using Taylor's theorem page 235, we have $\Delta = f(a + h, b + k) - f(a, b)$

$$= \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)_{a,b} + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \ldots \qquad \ldots(i)$$

For small values of h amd k, the second and higher order terms are still smaller and hence may be neglected. Thus

$$\text{sign of } \Delta = \text{sign of } [hf_x(a, b) + kf_y(a, b)].$$

Taking $h = 0$ we see that the right hand side changes sign when k changes sign. Hence $f(x, y)$ cannot have a maximum or a minimum at (a, b) unless $f_y(a, b) = 0$.

Similarly taking $k = 0$, we find that $f(x, y)$ cannot have a maximum or minimum at (a, b) unless $f_x(a, b) = 0$.

Hence *the necessary conditions for $f(x, y)$ to have a maximum or minimum at (a, b) are that*

$$\mathbf{f_x(a, b) = 0, f_y(a, b) = 0.}$$

If these conditions are satisfied, then for small value of h and k, (i) gives

$$\text{sign of } \Delta = \text{sign of } \left[\frac{1}{2!}(h^2 r + 2hks + k^2 t)\right] \text{ where } r = f_{xx}(a, b),\ s = f_{xy}(a, b) \text{ and } t = f_{yy}(a, b).$$

Now $\quad h^2 r + 2hks + k^2 t = \frac{1}{r}\left[(h^2 r^2 + 2hkrs + k^2 rt)\right] = \frac{1}{r}\left[(hr + ks)^2 + k^2(rt - s^2)\right]$

Thus sign of Δ = sign of $\frac{1}{2r}\left\{(hr + ks)^2 + k^2(rt - s^2)\right\}$ $\qquad \ldots(ii)$

In (ii), $(hr + ks)^2$ is always positive and $k^2(rt - s^2)$ will be positive if $rt - s^2 > 0$. In this case, Δ will have the same sign as that of r for all values of h and k.

Hence if $rt - s^2 > 0$, then *$f(x, y)$ has a maximum or a minimum at (a, b) according as $r <$ or > 0.*

If $rt - s^2 < 0$, then Δ will change with h and k and hence there is no maximum or minimum at (a, b) *i.e.*, it is a *saddle point*.

If $rt - s^2 = 0$, further investigation is required to find whether there is a maximum or minimum at (a, b) or not.

Note. Stationary value. *$f(a, b)$ is said to be a stationary value of $f(x, y)$, if $f_x(a, b) = 0$ and $f_y(a, b) = 0$ i.e.* the function is stationary at (a, b).

Thus every extreme value is a stationary value but the converse may not be true.

(3) Working rule to find the maximum and minimum values of f(x, y)

1. Find $\partial f/\partial x$ and $\partial f/\partial y$ *and equate each to zero. Solve these as simultaneous equations in x and y.* Let (a, b), (c, d), ... *be the pairs of values.*
2. *Calculate the value of $r = \partial^2 f/\partial x^2$, $s = \partial^2 f/\partial x \partial y$, $t = \partial^2 f/\partial y^2$ for each pair of values.*

3. (i) *If* $rt - s^2 > 0$ *and* $r < 0$ *at* (a, b), $f(a, b)$ *is a max. value.*
(ii) *If* $rt - s^2 > 0$ *and* $r > 0$ *at* (a, b), $f(a, b)$ *is a min. value.*
(iii) *If* $rt - s^2 < 0$ *at* (a, b), $f(a, b)$ *is not an extreme value, i.e.,* (a, b) *is a saddle point.*
(iv) *If* $rt - s^2 = 0$ *at* (a, b), *the case is doubtful and needs further investigation.*

Similarly examine the other pairs of values one by one.

Example 5.42. *Examine the following function for extreme values :*

$$f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2.$$ *(J.N.T.U., 2003)*

Solution. We have $f_x = 4x^3 - 4x + 4y$; $f_y = 4y^3 + 4x - 4y$

and $r = f_{xx} = 12x^2 - 4$, $s = f_{xy} = 4$, $t = f_{yy} = 12y^2 - 4$ ...(i)

Now $f_x = 0, f_y = 0$ give $x^3 - x + y = 0$, ...(i) $y^3 + x - y = 0$ ...(ii)

Adding these, we get $4(x^3 + y^3) = 0$ or $y = -x$.

Putting $y = -x$ in (i), we obtain $x^3 - 2x = 0$, *i.e.* $x = \sqrt{2}, -\sqrt{2}, 0$.

$\therefore$ Corresponding values of y are $-\sqrt{2}, \sqrt{2}, 0$.

At $(\sqrt{2}, -\sqrt{2})$, $rt - s^2 = 20 \times 20 - 4^2 = +$ ve and r is also + ve. Hence $f(\sqrt{2}, -\sqrt{2})$ is a minimum value.

At $(-\sqrt{2}, \sqrt{2})$ also both $rt - s^2$ and r are + ve.

Hence $f(-\sqrt{2}, \sqrt{2})$, is also a minimum value.

At $(0, 0)$, $rt - s^2 = 0$ and, therefore, further investigation is needed.

Now $f(0, 0) = 0$ and for points along the x-axis, where $y = 0$, $f(x, y) = x^4 - 2x^2 = x^2(x^2 - 2)$, which is negative for points in the neighbourhood of the origin.

Again for points along the line $y = x$, $f(x, y) = 2x^4$ which is positive.

Thus in the neighbourhood of $(0, 0)$ there are points where $f(x, y) < f(0, 0)$ and there are points where $f(x, y) > f(0, 0)$.

Hence $f(0, 0)$ is not an extreme value *i.e.*, it is a *saddle point*.

Example 5.43. *Discuss the maxima and minima of* $f(x, y) = x^3y^2(1 - x - y)$.

(Anna, 2009 ; J.N.T.U., 2006 ; Bhopal, 2002)

Solution. We have $f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$; $f_y = 2x^3y - 2x^4y - 3x^3y^2$

and $r = f_{xx} = 6xy^2 - 12x^2y^2 - 6xy^3$; $s = f_{xy} = 6x^2y - 8x^3y - 9x^2y^2$; $t = f_{yy} = 2x^3 - 2x^4 - 6x^3y$.

When $f_x = 0, f_y = 0$, we have $x^2y^2(3 - 4x - 3y) = 0$, $x^3y(2 - 2x - 3y) = 0$

Solving these, the stationary points are $(1/2, 1/3)$, $(0, 0)$.

Now $rt - s^2 = x^4y^2[12(1 - 2x - y)(1 - x - 3y) - (6 - 8x - 9y)^2]$

At $(1/2, 1/3)$, $$rt - s^2 = \frac{1}{16} \cdot \frac{1}{9}\left[12\left(1 - 1 - \frac{1}{3}\right)\left(1 - \frac{1}{2} - 1\right) - (6 - 4 - 3)^2\right] = \frac{1}{14} > 0$$

Also $$r = 6\left(\frac{1}{2} \cdot \frac{1}{9} - \frac{2}{4} \cdot \frac{1}{9} - \frac{1}{2} \cdot \frac{1}{27}\right) = -\frac{1}{9} < 0$$

Hence $f(x, y)$ has a maximum at $(1/2, 1/3)$ and maximum value $= \frac{1}{8} \cdot \frac{1}{9}\left(1 - \frac{1}{2} - \frac{1}{3}\right) = \frac{1}{432}$.

At $(0, 0)$, $rt - s^2 = 0$ and therefore further investigation is needed.

For points along the line $y = x$, $f(x, y) = x^5(1 - 2x)$ which is positive for $x = 0.1$ and negative for $x = -0.1$ *i.e.*, in the neighbourhood of $(0, 0)$ there are points where $f(x, y) > f(0, 0)$ and there are points where $f(x, y) < f(0, 0)$. Hence $f(0, 0)$ is not an extreme value.

Example 5.44. *In a plane triangle, find the maximum value of* $\cos A \cos B \cos C$.

(V.T.U., 2010 ; Nagpur, 2009 ; Anna, 2005 S)

Solution. We have $A + B + C = \pi$ so that $C = \pi - (A + B)$.

$$\cos A \cos B \cos C = \cos A \cos B \cos [\pi - (A + B)]$$
$$= -\cos A \cos B \cos (A + B) = f(A, B), \text{ say.}$$

We get $$\frac{\partial f}{\partial A} = \cos B\,[\sin A \cos (A + B) + \cos A \sin (A + B)]$$
$$= \cos B \sin (2A + B)$$

and $$\frac{\partial f}{\partial B} = \cos A \sin (A + 2B)$$

$$\frac{\partial f}{\partial A} = 0,\ \frac{\partial f}{\partial B} = 0 \quad \text{only when } A = B = \pi/3.$$

Also $$r = \frac{\partial^2 f}{\partial A^2} = 2 \cos B \cos (2A + B),\ t = \frac{\partial^2 f}{\partial B^2} = 2 \cos A \cos (A + 2B)$$

$$s = \frac{\partial^2 f}{\partial A \partial B} = -\sin B \sin (2A + B) + \cos B \cos (2A + B) = \cos (2A + 2B)$$

When $A = B = \pi/3$, $r = -1$, $s = -1/2$, $t = -1$ so that $rt - s^2 = 3/4$.

These show that $f(A, B)$ is maximum for $A = B = \pi/3$.

Then $C = \pi - (A + B) = \pi/3$.

Hence $\cos A \cos B \cos C$ is maximum when each of the angles is $\pi/3$ *i.e.*, triangle is equilateral and its maximum value = 1/8.

5.12 LAGRANGE'S METHOD OF UNDERTERMINED MULTIPLIERS

Sometimes it is required to find the stationary values of a function of several variables which are not all independent but are connected by some given relations. Ordinarily, we try to convert the given function to the one, having least number of independent variables with the help of given relations. Then solve it by the above method. When such a procedure becomes impracticable, Lagrange's method* proves very convenient. Now we explain this method.

Let $$u = f(x, y, z) \qquad ...(1)$$

be a function of three variables x, y, z which are connected by the relation.

$$\phi(x, y, z) = 0 \qquad ...(2)$$

For u to have stationary values, it is necessary that

$$\partial u/\partial x = 0,\ \partial u/\partial y = 0,\ \partial u/\partial z = 0.$$

$$\therefore \quad \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = du = 0 \qquad ...(3)$$

Also differentiating (2), we get $$\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = d\phi = 0 \qquad ...(4)$$

Multiply (4) by a parameter λ and add to (3). Then

$$\left(\frac{\partial u}{\partial x} + \lambda \frac{\partial \phi}{\partial x}\right) dx + \left(\frac{\partial u}{\partial y} + \lambda \frac{\partial \phi}{\partial y}\right) dy + \left(\frac{\partial u}{\partial z} + \lambda \frac{\partial \phi}{\partial z}\right) dz = 0$$

This equation will be satisfied if $\frac{\partial u}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0,\ \frac{\partial u}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0,\ \frac{\partial u}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0.$

These three equations together with (2) will determine the values of x, y, z and λ for which u is stationary.

Working rule : **1.** *Write* $F = f(x, y, z) + \lambda\phi(x, y, z)$

2. *Obtain the equations* $\frac{\partial F}{\partial x} = 0,\ \frac{\partial F}{\partial y} = 0,\ \frac{\partial F}{\partial z} = 0.$

3. *Solve the above equations together with* $\phi(x, y, z) = 0$.

The values of x, y, z *so obtained will give the stationary value of* $f(x, y, z)$.

Obs. Although the Lagrange's method is often very useful in application yet *the drawback is that we cannot determine the nature of the stationary point*. This can sometimes, be determined from physical considerations of the problem.

*See footnote page 142.

Example 5.45. *A rectangular box open at the top is to have volume of 32 cubic ft. Find the dimensions of the box requiring least material for its construction.* (*Kurukshetra, 2006 ; P.T.U., 2006 ; U.P.T.U., 2005*)

Solution. Let x, y and z ft. be the edges of the box and S be its surface.

Then $$S = xy + 2yz + 2zx \quad ...(i)$$

and $$xyz = 32 \quad ...(ii)$$

Eliminating z from (i) with the help of (ii), we get $S = xy + 2(y + x)\dfrac{32}{xy} = xy + 64\left(\dfrac{1}{x} + \dfrac{1}{y}\right)$

$\therefore$ $$\partial S/\partial x = y - 64/x^2 = 0 \quad \text{and} \quad \partial S/\partial y = x - 64/y^2 = 0.$$

Solving these, we get $x = y = 4$.

Now $r = \partial^2 S/\partial x^2 = 128/x^3$, $s = \partial^2 S/\partial x \partial y = 1$, $t = \partial^2 S/\partial y^2 = 128/y^3$.

At $x = y = 4$, $rt - s^2 = 2 \times 2 - 1 = +$ ve and r is also $+$ ve.

Hence S is minimum for $x = y = 4$. Then from (ii), $z = 2$.

Otherwise (*by Lagrange's method*) :

Write $$F = xy + 2yz + 2zx + \lambda(xyz - 32)$$

Then $$\frac{\partial F}{\partial x} = y + 2z + \lambda yz = 0 \quad ...(iii)$$

$$\frac{\partial F}{\partial y} = x + 2z + \lambda zx = 0 \quad ...(iv)$$

$$\frac{\partial F}{\partial z} = 2y + 2x + \lambda xy = 0 \quad ...(v)$$

Multiplying (iii) by x and (iv) by y and subtracting, we get $2zx - 2zy = 0$ or $x = y$.

[The value $z = 0$ is neglected, as it will not satisfy (ii)]

Again multiplying (iv) by y and (v) by z and subtracting, we get $y = 2z$.

Hence the dimensions of the box are $x = y = 2z = 4$ $...(vi)$

Now let us see what happens as z increases from a small value to a large one. When z is small, the box is flat with a large base showing that S is large. As z increases, the base of the box decreases rapidly and S also decreases. After a certain stage, S again starts increasing as z increases. Thus S must be a minimum at some intermediate stage which is given by (vi). Hence S is minimum when $x = y = 4$ ft and $z = 2$ ft.

Example 5.46. *Given $x + y + z = a$, find the maximum valve of $x^m y^n z^p$.* (*Anna, 2009*)

Solution. Let $f(x, y, z) = x^m y^n z^p$ and $\phi(x, y, z) = x + y + z - a$.

Then $$F(x, y, z) = f(x, y, z) + \lambda\, \phi(x, y, z)$$
$$= x^m y^n z^p + \lambda(x + y + z - a).$$

For stationary values of F, $\dfrac{\partial F}{\partial x} = 0, \dfrac{\partial F}{\partial y} = 0, \dfrac{\partial F}{\partial z} = 0$

$\therefore$ $$mx^{m-1}y^n z^p + \lambda = 0,\ nx^m y^{n-1} z^p + \lambda = 0,\ px^m y^n z^{p-1} + \lambda = 0$$

or $$-\lambda = mx^{n-1}y^n z^p = nx^m y^{n-1} z^p = px^m y^n z^{p-1}$$

i.e. $$\frac{m}{x} = \frac{n}{y} = \frac{p}{z} = \frac{m+n+p}{x+y+z} = \frac{m+n+p}{a} \qquad [\because\ x + y + z = a$$

$\therefore$ The maximum value of f occurs when

$$x = am/(m + n + p),\ y = an/(m + n + p),\ z = ap/(m + n + p)$$

Hence the maximum value of $f(x, y, z) = \dfrac{a^{m+n+p} \cdot m^m \cdot n^n \cdot p^p}{(m + n + p)^{m+n+p}}$.

Example 5.47. *Find the maximum and minimum distances of the point (3, 4, 12) from the sphere $x^2 + y^2 + z^2 = 4$.* (*U.T.U., 2010*)

Solution. Let $P(x, y, z)$ be any point on the sphere and $A(3, 4, 12)$ the given point so that

$$AP^2 = (x - 3)^2 + (y - 4)^2 + (z - 12)^2 = f(x, y, z), \text{ say} \quad ...(i)$$

We have to find the maximum and minimum values of $f(x, y, z)$ subject to the condition

$$\phi(x, y, z) = x^2 + y^2 + z^2 - 4 = 0 \qquad ...(ii)$$

Let $$F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z)$$
$$= (x-3)^2 + (y-4)^2 + (z-12)^2 + \lambda(x^2 + y^2 + z^2 - 4)$$

Then $$\frac{\partial F}{\partial x} = 2(x-3) + 2\lambda x,\ \frac{\partial F}{\partial y} = 2(y-4) + 2\lambda y,\ \frac{\partial F}{\partial z} = 2(z-12) + 2\lambda z$$

$\therefore$ $$\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0 \text{ and } \frac{\partial F}{\partial z} = 0 \text{ give}$$

$$x - 3 + \lambda x = 0, y - 4 + \lambda y = 0, z - 12 + \lambda z = 0 \qquad ...(iii)$$

which give $$\lambda = -\frac{x-3}{x} = -\frac{y-4}{y} = -\frac{z-12}{z}$$

$$= \pm\frac{\sqrt{[(x-3)^2 + (y-4)^2 + (z-12)^2]}}{\sqrt{(x^2 + y^2 + z^2)}} = \pm\frac{\sqrt{f}}{1}$$

Substituting for λ in (iii), we get

$$x = \frac{3}{1+\lambda} = \frac{3}{1 \pm \sqrt{f}},\ y = \frac{4}{1 \pm \sqrt{f}},\ z = \frac{12}{1 \pm \sqrt{f}}$$

$\therefore$ $$x^2 + y^2 + z^2 = \frac{9 + 16 + 144}{(1 \pm \sqrt{f})^2} = \frac{169}{(1 \pm \sqrt{f})^2}$$

Using (ii), $$1 = \frac{169}{(1 \pm \sqrt{f})^2} \quad \text{or} \quad 1 \pm \sqrt{f} = \pm 13,\ \sqrt{f} = 12, 14.$$

[We have left out the negative values of $\sqrt{f}$, because $\sqrt{f} = AP$ is + ve by (i)]

Hence maximum $AP = 14$ and minimum $AP = 12$.

Example 5.48. *Show that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube.* *(Kurukshetra, 2006 ; U.P.T.U., 2004)*

Solution. Let $2x$, $2y$, $2z$ be the length, breadth and height of the rectangular solid so that its volume

$$V = 8xyz \qquad ...(i)$$

Let R be the radius of the sphere so that $x^2 + y^2 + z^2 = R^2$ $\qquad ...(ii)$

Then $$F(x, y, z) = 8xyz + \lambda(x^2 + y^2 + z^2 - R^2)$$

and $$\partial F/\partial x = 0, \partial F/\partial y = 0 \text{ and } \partial F/\partial z = 0 \quad \text{give}$$

$$8yz + 2x\lambda = 0, 8zx + 2y\lambda = 0, 8xy + 2z\lambda = 0$$

or $$2x^2\lambda = -8xyz = 2y^2\lambda = 2z^2\lambda$$

Thus for a maximum volume $x = y = z$.

i.e., the rectangular solid is a cube.

Example 5.49. *A tent on a square base of side x, has its sides vertical of height y and the top is a regular pyramid of height h. Find x and y in terms of h, if the canvas required for its construction is to be minimum for the tent to have a given capacity.*

Solution. Let V be the volume enclosed by the tent and S be its surface area (Fig. 5.6).

Then V = cuboid $(ABCD, A'B'C'D')$ + pyramid $(K, A'B'C'D')$

$$= x^2y + \frac{1}{3}x^2h = x^2(y + h/3)$$

$$S = 4(ABGF) + 4\Delta KGH = 4xy + 4\frac{1}{2}(x \cdot KM)$$

$$= 4xy + x\sqrt{(x^2 + 4h^2)} \qquad [\because\ KM = \sqrt{(KL^2 + LM^2)} = \sqrt{[h^2 + (x/2)^2]}$$

For constant V, we have

$$\delta V = 2x(y + h/3)\,\delta x + x^2(\delta y) + \frac{x^2}{3}\,\delta h = 0$$

For minimum S, we have

$$\delta S = [4y + \sqrt{(x^2 + 4h^2)} + x \cdot \frac{1}{2}(x^2 + 4h^2)^{-1/2} \cdot 2x]\,\delta x$$

$$+ 4x\delta y + x \cdot \frac{1}{2}(x^2 + 4h^2)^{-1/2} \cdot 8h\delta h = 0$$

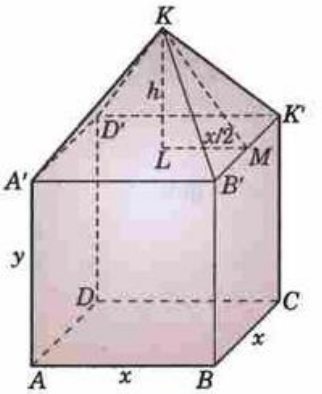

Fig. 5.6

By Lagrange's method,

$$[4y + \sqrt{(x^2 + 4h^2)} + x^2(x^2 + 4h^2)^{-1/2}] + \lambda \cdot 2x(y + h/3) = 0 \qquad ...(i)$$

$$4x + \lambda \cdot x^2 = 0 \qquad ...(ii)$$

$$4hx(x^2 + 4h^2)^{-1/2} + \lambda \cdot x^2/3 = 0 \qquad ...(iii)$$

(ii) gives $\lambda = -4/x$. Then (iii) becomes

$$4hx(x^2 + 4h^2)^{-1/2} - 4x/3 = 0 \quad \text{or} \quad x = \sqrt{5}\,h$$

Now putting $x = \sqrt{5}\,h$, $\lambda = -4/x$ in (i), we get

$$4y + 3h + \frac{5}{3}h - \frac{4}{x} \cdot 2x(y + h/3) = 0 \quad \text{or} \quad 4y + \frac{14}{3}h - 8y - \frac{8h}{3} = 0, \quad i.e., \quad y = h/2.$$

Example 5.50. *If $u = a^3x^2 + b^3y^2 + c^3z^2$ where $x^{-1} + y^{-1} + z^{-1} = 1$, show that the stationary value of u is given by $x = \Sigma a/a$, $y = \Sigma a/b$, $z = \Sigma a/c$.* *(Kerala, 2005)*

Solution. Let $\quad u = f(x, y, z) = a^3x^2 + b^3y^2 + c^3z^2$

and $\quad \phi(x, y, z) = x^{-1} + y^{-1} + z^{-1} - 1$

Let $\quad F(x, y, z) = f(x, y, z) + \lambda\,\phi(x, y, z)$

$$= a^3x^2 + b^3y^2 + c^3z^2 + \lambda(x^{-1} + y^{-1} + z^{-1} - 1)$$

Then $\dfrac{\partial F}{\partial x} = 0, \dfrac{\partial F}{\partial y} = 0$ and $\dfrac{\partial F}{\partial z} = 0$ give

$$2a^3x^2 - \lambda/x^2 = 0, \quad 2b^3y - \lambda/y^2 = 0, \quad 2c^3z - \lambda/z^2 = 0$$

or $\quad 2a^3x^3 = \lambda,\ 2b^3y^3 = \lambda,\ 2c^3z^3 = \lambda$

which give $ax = by = cz = k$ (say) *i.e.*, $x = k/a$, $y = k/b$, $z = k/c$.

Substituting these in $x^{-1} + y^{-1} + z^{-1} = 1$, we get $k = a + b + c$

Hence the stationary value of u is given by

$$x = \Sigma a/a, \quad y = \Sigma a/b \quad \text{and} \quad z = \Sigma a/c.$$

Example 5.51. *Find the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

(U.T.U., 2010 ; Anna, 2009 ; Madras, 2006)

Solution. Let the edges of the parallelopiped be $2x$, $2y$ and $2z$ which are parallel to the axes. Then its volume $V = 8xyz$.

Now we have to find the maximum value of V subject to the condition that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \qquad ...(i)$$

Write $\quad F = 8xyz + \lambda\left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} - 1\right)$

Then $\quad \dfrac{\partial F}{\partial x} = 8yz + \lambda\left(\dfrac{2x}{a^2}\right) = 0 \qquad ...(ii)$

$$\frac{\partial F}{\partial y} = 8zx + \lambda\left(\frac{2y}{b^2}\right) = 0 \quad ...(iii) \qquad \frac{\partial F}{\partial z} = 8xy + \lambda\left(\frac{2z}{c^2}\right) = 0 \quad ...(iv)$$

Equating the values of λ from (ii) and (iii), we get $x^2/a^2 = y^2/b^2$

Similarly from (iii) and (iv), we obtain $y^2/b^2 = z^2/c^2 \quad \therefore \quad x^2/a^2 = y^2/b^2 = z^2/c^2$

Substituting these in (i), we get $x^2/a^2 = \frac{1}{3}$ *i.e.* $\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = \frac{1}{3}$

These give $x = a/\sqrt{3}$, $y = b/\sqrt{3}$, $z = c/\sqrt{3}$ ...(v)

When $x = 0$, the parallelopiped is just a rectangular sheet and as such its volume $V = 0$.

As x increases, V also increases continuously.

Thus V must be greatest at the stage given by (v).

Hence the greatest volume $= \frac{8abc}{3\sqrt{3}}$.

PROBLEMS 5.10

1. Find the maximum and minimum values of
   (*i*) $x^3 + y^3 - 3axy$ (*U.P.T.U., 2005*) (*ii*) $xy + a^3/x + a^3/y$.
   (*iii*) $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$ (*Mumbai, 2007*) (*iv*) $2(x^2 - y^2) - x^4 + y^4$ (*Osmania, 2003*)
   (*v*) $\sin x \sin y \sin (x + y)$.
2. If $xyz = 8$, find the values of x, y for which $u = 5xyz/(x + 2y + 4z)$ is a maximum. (*S.V.T.U., 2007 ; Kurukshetra, 2005*)
3. Find the minimum value of $x^2 + y^2 + z^2$, given that
   (*i*) $xyz = a^3$ (*P.T.U., 2009 ; Osmania, 2003*) (*ii*) $ax + by + cz = p$. (*V.T.U., 2010 ; U.P.T.U., 2006*)
   (*iii*) $xy + yz + zx = 3a^2$ (*Anna, 2009*)
4. Find the dimensions of the rectangular box, open at the top, of maximum capacity whose surface is 432 sq. cm. (*Madras, 2000 S*)
5. The sum of three numbers is constant. Prove that their product is maximum when they are equal.
6. Find the points on the surface $z^2 = xy + 1$ nearest to the origin. (*Burdwan, 2003 ; Andhra, 2000*)
7. Show that, if the perimeter of a triangle is constant, the triangle has maximum area when it is equilateral.
8. Find the maximum and minimum distances from the origin to the curve $5x^2 + 6xy + 5y^2 - 8 = 0$.
9. The temperature T at any point (x, y, z) in space is $T = 400\,xyz^2$. Find the highest temperature on the surface of the unit sphere $x^2 + y^2 + z^2 = 1$. (*V.T.U., 2009 ; Hissar, 2005 S*)
10. Divide 24 into three parts such that the continued product of the first, square of the second and the cube of the third may be maximum. (*Bhillai, 2005*)
11. Find the stationary values of $u = x^2 + y^2 + z^2$ subject to $ax^2 + by^2 + cz^2 = 1$ and $lx + my + nz = 0$. (*S.V.T.U., 2008*)

5.13 DIFFERENTIATION UNDER THE INTEGRAL SIGN

If a function $f(x, \alpha)$ of two variables x and α (called a parameter), be integrated with respect to x between the limits a and b, then $\int_a^b f(x, \alpha)\,dx$ is a function of $\alpha : F(\alpha)$, say. To find the derivative of $F(\alpha)$, when it exists, it is not always possible to first evaluate this integral and then to find the derivative. Such problems are solved by the following rules :

(1) Leibnitz's rule*

If $f(x, \alpha)$ and $\frac{\partial f(x, \alpha)}{\partial \alpha}$ *be continuous functions of* x *and* α, *then*

$$\frac{d}{d\alpha}\left[\int_a^b f(x, \alpha)\,dx\right] = \int_a^b \frac{\partial f(x, \alpha)}{\partial \alpha}\,dx \quad \textit{where, } a, b \textit{ are constants independent of } \alpha.$$

*See foot note on p. 139.

Let $\int_a^b f(x, \alpha)dx = F(\alpha)$,

then $F(\alpha + \delta\alpha) - F(\alpha) = \int_a^b f(x, \alpha + \delta\alpha)dx - \int_a^b f(x, \alpha)dx = \int_a^b [f(x, \alpha + \delta\alpha) - f(x, \alpha)]dx$

$$= \delta\alpha \int_a^b \frac{\partial f(x, \alpha + \theta\delta\alpha)}{\partial\alpha}dx, \ (0 < \theta < 1)$$

$\left\{\because \ f(x, \alpha + h) - f(x, \alpha) = hf'(x, \alpha + \theta h)\right.$ where $0 < \theta < 1$, by Mean Value Theorem

Proceeding to limits as $\delta\alpha \to 0$, $\underset{\delta\alpha \to 0}{\text{Lt}} \dfrac{F(\alpha + \delta\alpha) - F(\alpha)}{\delta\alpha} = \int_a^b \dfrac{\partial f(x, \alpha + \theta \cdot 0)}{\partial\alpha}dx$

or $\dfrac{dF}{d\alpha} = \int_a^b \dfrac{\partial f(x, \alpha)}{\partial\alpha}dx$ which is the desired result.

Obs. 1. Leibnitz's rule enables us to derive from the value of a simple definite integral, the value of another definite integral which it may otherwise be difficult, or even impossible, to evaluate.

Obs. 2. The rule for differentiation under the integral sign of an infinite integral is the same as for a definite integral.

Example 5.52. *Evaluate* $\int_0^1 \dfrac{x^\alpha - 1}{\log x}dx$, $\alpha \geq 0$. *(V.T.U., 2010)*

Solution. Let $F(\alpha) = \int_0^1 \dfrac{x^\alpha - 1}{\log x}dx$ ...(*i*)

then $F(\alpha) = \int_0^1 \dfrac{\partial}{\partial\alpha}\left(\dfrac{x^\alpha - 1}{\log x}\right)dx = \int_0^1 \dfrac{x^\alpha \log x}{\log x}dx$

$$= \int_0^1 x^\alpha dx = \left|\frac{x^{\alpha+1}}{\alpha + 1}\right|_0^1 = \frac{1}{1 + \alpha} \qquad \left[\because \ \frac{d}{dt}(n^t) = n^t \log n\right]$$

Now integrating both sides w.r.t. α, $F(\alpha) = \log(1 + \alpha) + c$ ...(*ii*)

From (*i*), when $\alpha = 0$, $F(0) = 0$

$\therefore$ From (*ii*), $F(0) = \log(1) + c$, *i.e.*, $c = 0$. Hence (*ii*) gives, $F(\alpha) = \log(1 + \alpha)$.

Example 5.53. *Given* $\int_0^\pi \dfrac{dx}{a + b\cos x} = \dfrac{\pi}{\sqrt{(a^2 - b^2)}}$ $(a > b)$,

evaluate $\int_0^\pi \dfrac{dx}{(a + b\cos x)^2}$ *and* $\int_0^\pi \dfrac{\cos x}{(a + b\cos x)^2}dx$ *(Madras, 2006)*

Solution. We have $\int_0^\pi \dfrac{dx}{a + b\cos x} = \dfrac{\pi}{\sqrt{(a^2 - b^2)}}$ ...(*i*)

Differentiating both sides of (*i*) w.r.t. a,

$$\int_0^\pi \frac{\partial}{\partial a}\left(\frac{1}{a + b\cos x}\right)dx = \frac{\partial}{\partial a}\left\{\frac{\pi}{\sqrt{(a^2 - b^2)}}\right\}$$

i.e. $$\int_0^\pi \frac{-dx}{(a + b\cos x)^2} = \pi \cdot \left(-\frac{1}{2}\right)(a^2 - b^2)^{-3/2} \cdot 2a$$

$\therefore$ $$\int_0^\pi \frac{dx}{(a + b\cos x)^2} = \frac{\pi a}{(a^2 - b^2)^{3/2}}$$

Now differentiating both sides of (*i*) w.r.t. b,

$$\int_0^\pi -(a + b\cos x)^{-2} \cdot \cos x \, dx = \pi\left(-\frac{1}{2}\right)(a^2 - b^2)^{-3/2} \cdot (-2b)$$

$$\therefore \qquad \int_0^{\pi} \frac{\cos x}{(a + b\cos x)^2} dx = \frac{\pi b}{(a^2 - b^2)^{3/2}}.$$

(2) Leibnitz's rule *for variable limits of integration*

If $f(x, \alpha)$, $\dfrac{\partial f(x, \alpha)}{\partial \alpha}$ *be continuous functions of* x *and* α, *then*

$$\frac{\mathbf{d}}{\mathbf{d}\alpha}\left\{\int_{\phi(\alpha)}^{\psi(\alpha)} \mathbf{f(x, \alpha)\,dx}\right\} = \int_{\phi(\alpha)}^{\psi(\alpha)} \frac{\partial \mathbf{f(x, \alpha)}}{\partial \alpha}\mathbf{dx} + \frac{\mathbf{d}\psi}{\mathbf{d}\alpha}\mathbf{f}[\psi(\alpha), \alpha] - \frac{\mathbf{d}\phi}{\mathbf{d}\alpha}\mathbf{f}[\phi(\alpha), \alpha]$$

provided $\phi(\alpha)$ *and* $\psi(\alpha)$ *possesses continuous first order derivatives w.r.t.* α.

Its proof is beyond the scope of this book.

Example 5.54. *Evaluate* $\displaystyle\int_0^{\alpha} \frac{\log(1 + \alpha x)}{1 + x^2} dx$ *and hence show that*

$$\int_0^1 \frac{\log(1 + x)}{1 + x^2} dx = \frac{\pi}{8}\log_e 2$$

(Hissar, 2005 S)

Solution. Let
$$F(\alpha) = \int_0^{\alpha} \frac{\log(1 + \alpha x)}{1 + x^2} dx \qquad ...(i)$$

Then by the above rule, $F'(\alpha) = \displaystyle\int_0^{\alpha} \frac{\partial}{\partial \alpha}\left(\frac{\log(1 + \alpha x)}{1 + x^2}\right) dx + \frac{d(\alpha)}{d\alpha} \cdot \frac{\log(1 + \alpha^2)}{1 + \alpha^2} - 0$

$$= \int_0^{\alpha} \frac{x}{(1 + \alpha x)(1 + x^2)} dx + \frac{\log(1 + \alpha^2)}{1 + \alpha^2} \qquad ...(ii)$$

Breaking the integrand into partial fractions,

$$\int_0^{\alpha} \frac{x\,dx}{(1 + \alpha x)(1 + x^2)} = -\frac{\alpha}{1 + \alpha^2}\int_0^{\alpha} \frac{dx}{1 + \alpha x} + \frac{1}{2(1 + \alpha^2)}\int_0^{\alpha} \frac{2x}{1 + x^2} dx + \frac{\alpha}{1 + \alpha^2}\int_0^{\alpha} \frac{dx}{1 + x^2}$$

$$= -\frac{1}{1 + \alpha^2}\left|\log(1 + \alpha x)\right|_0^{\alpha} + \frac{1}{2(1 + \alpha^2)} \times \left|\log(1 + x^2)\right|_0^{\alpha} + \frac{\alpha}{1 + \alpha^2}\left|\tan^{-1} x\right|_0^{\alpha}$$

$$= -\frac{\log(1 + \alpha^2)}{1 + \alpha^2} + \frac{\log(1 + \alpha^2)}{2(1 + \alpha^2)} + \frac{\alpha \tan^{-1} \alpha}{1 + \alpha^2}$$

Substituting this value in (ii), $F'(\alpha) = \dfrac{\log(1 + \alpha^2)}{2(1 + \alpha^2)} + \dfrac{\alpha \tan^{-1} \alpha}{1 + \alpha^2}$

Now integrating both sides w.r.t. α,

$$F(\alpha) = \frac{1}{2}\int \log(1 + \alpha^2) \cdot \frac{1}{1 + \alpha^2} d\alpha + \int \frac{\alpha \tan^{-1} \alpha}{1 + \alpha^2} d\alpha \qquad \text{[Integrating by parts]}$$

$$= \frac{1}{2}\left[\log(1 + \alpha^2) \cdot \tan^{-1} \alpha - \int \frac{2\alpha}{1 + \alpha^2} \cdot \tan^{-1} \alpha\, d\alpha\right] + \int \frac{\alpha \tan^{-1} \alpha}{1 + \alpha^2} d\alpha + c$$

$$= \frac{1}{2}\log(1 + \alpha^2) \cdot \tan^{-1} \alpha + c \qquad ...(iii)$$

But from (i), when $\alpha = 0$, $F(0) = 0$.

$\therefore$ From (iii), $F(0) = 0 + c$, *i.e.*, $c = 0$. Hence (iii) gives, $F(\alpha) = \dfrac{1}{2}\log(1 + \alpha^2)\tan^{-1} \alpha$

Putting $\alpha = 1$, we get $\displaystyle\int_0^1 \frac{\log(1 + x)}{1 + x^2} dx = F(1) = \frac{\pi}{8}\log_e 2.$

PROBLEMS 5.11

1. Differentiating $\int_0^x \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$ under the integral sign, find the value of $\int_0^x \frac{dx}{(x^2+a^2)^2}$.

2. By successive differentiation of $\int_0^1 x^m\,dx = \frac{1}{m+1}$ w.r.t. m, evaluate $\int_0^1 x^m (\log x)^n\,dx$.

3. Evaluate $\int_0^\pi \log(1+a\cos x)\,dx$, using the method of differentiation under the sign of integration.

4. Given that $\int_0^\pi \frac{dx}{a-\cos x} = \frac{\pi}{\sqrt{(a^2-1)}}$, evaluate $\int_0^\pi \frac{dx}{(a-\cos x)^2}$. *(V.T.U., 2009)*

Prove that :

5. $\int_0^\infty e^{-x}\cdot\frac{\sin ax}{x}\,dx = \tan^{-1} a$. $\left[\textbf{Hint.}\ \text{Use } \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right]$

6. $\int_0^\infty e^{-ax}\frac{\sin x}{x}\,dx = \tan^{-1}\frac{1}{a}$. Hence show that $\int_0^\infty \frac{\sin x}{x} = \frac{\pi}{2}$. *(Rohtak, 2003)*

7. $\int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)}\,dx = \frac{\pi}{2}\log(1+a)$ where $a \ge 0$. *(V.T.U., 2010 ; S.V.T.U., 2009 ; Rohtak, 2006 S ; Anna, 2005 S)*

8. $\int_0^\infty \frac{e^{-x}}{x}(1-e^{-ax})\,dx = \log(1+a), (a > -1)$.

9. $\int_0^{\pi/2} \log(\alpha^2\cos^2\theta + \beta^2\sin^2\theta)\,d\theta = \pi\log\frac{\alpha+\beta}{2}$ *(S.V.T.U., 2008)*

10. $\int_0^{\pi/2} \frac{\log(1+y\sin^2 x)}{\sin^2 x}\,dx = \pi\{\sqrt{(1+y)}-1\}$ *(S.V.T.U., 2008)*

11. $\int_0^\pi \frac{\log(1+\alpha\cos x)}{\cos x}\,dx = \pi\sin^{-1}\alpha$. *(V.T.U., 2007)*

12. $\int_0^\infty e^{-x^2}\cos\alpha x\,dx = \frac{\sqrt{\pi}}{2}e^{-\alpha^2/4}$ *(Mumbai, 2009 S)*

13. $\frac{d}{da}\int_0^{a^2}\tan^{-1}\frac{x}{a}\,dx = 2a\tan^{-1} a - \frac{1}{2}\log(a^2+1)$. Verify your result by direct integration.

14. $\int_{\pi/2-\alpha}^{\pi/2} \sin\theta\cos^{-1}(\cos\alpha\cos\theta)\,d\theta = \frac{\pi}{2}(1-\cos\alpha)$. *(Burdwan, 2003)*

15. If $y = \int_0^x f(t)\sin[k(x-t)]\,dt$, prove that y satisfies the differential equation $\frac{d^2y}{dx^2} + k^2y = k\,f(x)$.

5.14 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 5.12

Select the correct answer or fill up the blanks in each of the following problems :

1. If $u = e^x(x\cos y - y\sin y)$, then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \ldots$

2. If $x = uv, y = u/v$, then $\frac{\partial(x,y)}{\partial(u,v)}$ is

   (a) $-2u/v$ (b) $-2v/u$ (c) 0 (d) 1. *(V.T.U., 2010)*

3. If $J_1 = \dfrac{\partial(u, v)}{\partial(x, y)}$ and $J_2 = \dfrac{\partial(x, y)}{\partial(u, v)}$, then $J_1 J_2 = \ldots$

4. If $u = f(y/x)$, then

(a) $x\dfrac{\partial u}{\partial x} - y\dfrac{\partial u}{\partial y} = 0$ (b) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$ (c) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u$ (d) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 1$.

5. If $u = x^y$, then $\partial u/\partial x$ is

(a) 0 (b) yx^{y-1} (c) $x^y \log x$.

6. If $x = r\cos\theta$, $y = r\sin\theta$, then

(a) $\dfrac{\partial x}{\partial r} = 1/\dfrac{\partial r}{\partial x}$ (b) $\dfrac{\partial x}{\partial r} = \dfrac{\partial r}{\partial x}$ (c) $\dfrac{\partial x}{\partial r} = 0$.

7. If $u = x^y$, then $\partial u/\partial y$ is

(a) yx^{y-1} (b) 0 (c) $x^y \log x$.

8. If $u = x^3 + y^3$, then $\dfrac{\partial^2 u}{\partial x \partial y}$ is equal to

(a) -3 (b) 3 (c) 0 (d) $3x + 3y$ *(V.T.U., 2010 S)*

9. If $u = x^2 + 2xy + y^2 + x + y$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$ is equal to

(a) $2u$ (b) u (c) 0 (d) none of these.

10. If $u = \log \dfrac{x^2}{y}$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$ is equal to

(a) $2u$ (b) $3u$ (c) u (d) 1. *(V.T.U., 2010 S)*

11. If $x = r\cos\theta$, $y = r\sin\theta$, then $\dfrac{\partial(x, y)}{\partial(r, \theta)}$ is equal to

(a) 1 (b) r (c) $1/r$ (d) 0. *(V.T.U., 2010 S)*

12. If $A = f_{xx}(a, b)$, $B = f_{xy}(a, b)$, $C = f_{yy}(a, b)$, then $f(x, y)$ will have a maximum at (a, b) if

(a) $f_x = 0, f_y = 0,\ AC < B^2$ and $A < 0$ (b) $f_x = 0, f_y = 0,\ AC = B^2$ and $A > 0$

(c) $f_x = 0, f_y = 0,\ AC > B^2$ and $A > 0$ (d) $f_x = 0, f_y = 0,\ AC > B^2$ and $A < 0$.

13. If $z = \sin^{-1} \dfrac{\sqrt{x^2 + y^2}}{x + y}$, then $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y}$ is

(a) 0 (b) 1/2 (c) 1 (d) 2. *(Bhopal, 2008)*

14. If $u = \sin^{-1}(x/y) + \tan^{-1}(y/x)$, then $x\,\partial u/\partial x + y\,\partial u/\partial y$ equals

(a) $\sin^{-1}(x/y) + \tan^{-1}(y/x)$ (b) $2[\sin^{-1}(x/y) + \tan^{-1}(y/x)]$

(c) $3[\sin^{-1}(x/y) + \tan^{-1}(y/x)]$ (d) zero.

15. If an error of 1% is made in measuring its length and breadth, the percentage error in the area of a rectangle is

(a) 0.2% (b) 0.02% (c) 2% (d) 1%. *(V.T.U., 2010)*

16. $\dfrac{\partial(u, v)}{\partial(x, y)} \times \dfrac{\partial(x, y)}{\partial(u, v)}$ equals

(a) -1 (b) 1 (c) zero (d) none of these.

17. $\dfrac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$ is a homogeneous function of degree .....

18. If $z = \log(x^3 + y^3 - x^2y - xy^2)$, then $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y}$ is equal to .....

19. If $r = \partial^2 f/\partial x^2$, $s = \partial^2 f/\partial x \partial y$ and $t = \partial^2 f/\partial y^2$, then the condition for the saddle point is .....

20. If $f(x, y) = \dfrac{1}{x^3} + \dfrac{1}{y^3} + \dfrac{1}{x^3 + y^3}$, then $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y}$ is

(a) 0 (b) $3f$ (c) 9 (d) $-3f$. *(V.T.U., 2009 S)*

21. If $u = x^4 + y^4 + 3x^2y^2$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \ldots\ldots$

22. If u and v are functions of r, s where r, s are functions of x, y, then $\frac{\partial(u, v)}{\partial(r, s)} \cdot \frac{\partial(r, s)}{\partial(x, y)} = \ldots\ldots$

23. The necessary conditions for a function $f(x, y)$ to have an extreme at (a, b) are .....

24. If $u = (x-y)^4 + (y-z)^4 + (z-x)^4$, then $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$ is

(*a*) 1 (*b*) u (*c*) $4u$ (*d*) 0. (*V.T.U., 2010*)

25. If u is a composite function of t, defined by the relations $u = f(x, y)$; $x = \phi(t)$, $y = \psi(t)$, then total derivative $\frac{du}{dt} = \ldots\ldots$

26. If $u = \cos^{-1}(x/y) + \tan^{-1}(y/x)$, then $x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy}$ is

(*a*) u (*b*) $2u$ (*c*) 0 (*d*) 1. (*V.T.U., 2010*)

27. If $f(x, y, z) = 0$, then $\frac{\partial x}{\partial y} \cdot \frac{\partial y}{\partial z} \cdot \frac{\partial z}{\partial x} = \ldots\ldots$

28. If $u = f(x + ay) + g(x - ay)$, then $\frac{\partial^2 u}{\partial y^2}$ equals

(*a*) $\frac{\partial^2 u}{\partial x^2}$ (*b*) $a\frac{\partial^2 u}{\partial x^2}$ (*c*) $a^2\frac{\partial^2 u}{\partial x^2}$ (*d*) $\frac{\partial^2 u}{\partial x \partial y}$. (*V.T.U., 2010*)

29. If sum of three numbers is constant, then their product is a maximum when the numbers are .....

30. $y = \cosh(\lambda x)\cosh(-\lambda at)$ is a solution of $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$. (*True or False*)

CHAPTER 6

Integral Calculus and Its Applications

1. Reduction formulae. 2. Reduction formulae for $\int \sin^n x\,dx$, $\int \cos^n x\,dx$ and evaluation of $\int_0^{\pi/2} \sin^n x\,dx$, $\int_0^{\pi/2} \cos^n x\,dx$. 3. Reduction formula for $\int \sin^m x \cos^n x\,dx$ and evaluation of $\int_0^{\pi/2} \sin^m x \cos^n x\,dx$. 4. Reduction formulae for $\int \tan^n x\,dx$ $\int \cot^n x\,dx$. 5. Reduction formulae for $\int \sec^n x\,dx$, $\int \operatorname{cosec}^n x\,dx$. 6. Reduction formulae for $\int x^n e^{ax}\,dx$, $\int x^m (\log x)^n\,dx$. 7. Reduction formulae for $\int x^n \sin mx\,dx$, $\int x^n \cos nx\,dx$ and $\int \cos^m x \sin nx\,dx$. 8. Definite integrals. 9. Integral as the limit of a sum. 10. Areas of curves. 11. Lengths of curves. 12. Volumes of revolution. 13. Surface areas of revolution. 14. Objective Type of Questions.

6.1 REDUCTION FORMULAE

The reader is already familiar with some standard methods of integrating functions of a single variable. However, there are some integrals which cannot be evaluated by the afore-said methods. In such cases, the method of reduction formulae proves useful. A reduction formula connects an integral with another of the same type but of lower order. The successive application of the reduction formula enables us to evaluate the given integral. Now we shall derive some standard reduction formulae.

6.2 (1) REDUCTION FORMULAE for

(a) $\int \sin^n x\,dx$ (b) $\int \cos^n x\,dx$.

(a) $\int \sin^n x\,dx = \int \sin^{n-1} x \cdot \sin x\,dx$ [Integrated by parts]

$$= \sin^{n-1} x \cdot (-\cos x) - \int (n-1) \sin^{n-2} x \cos x\,(-\cos x)\,dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\,(1 - \sin^2 x)\,dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\,dx - (n-1) \int \sin^n x\,dx$$

Transposing

$$n \int \sin^n x\,dx = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\,dx$$

or

$$\int \sin^n x\,dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x\,dx \qquad \ldots(i)$$

(*b*) Similarly, $$\int \cos^n x\,dx = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

Thus we have the required reduction formulae.

Obs. To integrate $\int \sin^n x\,dx$ or $\int \cos^n x\,dx$,

(*a*) *when the index of sin x is odd put cos x = t*

when the index of cos x is odd, put sin x = t

(*b*) *when the index is an even positive integer,* express the integrand as a series of cosines of multiple angles and integrate term by term if n is small, *otherwise use the method of reduction formulae.*

(2) To show that $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$

$$= \frac{(n-1)(n-3)(n-5)\ldots}{n(n-2)(n-4)\ldots} \times \left(\frac{\pi}{2}, \textit{only if } n \textit{ is even}\right)$$

From (*i*), we have

$$I_n = \int_0^{\pi/2} \sin^n x\,dx = -\left|\frac{\sin^{n-1} x \cos x}{n}\right|_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

i.e. $$I_n = \frac{n-1}{n} I_{n-2}$$

Case I. *When n is odd*

Similarly $$I_{n-2} = \frac{n-3}{n-2} I_{n-4}, \quad I_{n-4} = \frac{n-5}{n-4} I_{n-6}$$

.....

$$I_5 = \frac{4}{5} I_3, I_3 = \frac{2}{3} I_1 = \frac{2}{3}\int_0^{\pi/2} \sin x\,dx = \frac{2}{3}\left|-\cos x\right|_0^{\pi/2} = \frac{2}{3}.$$

Form these, we get $$I_n = \frac{(n-1)(n-3)(n-5)\ldots 2}{n(n-2)(n-4)\ldots 3} \qquad \ldots(ii)$$

Case II. *When n is even*

We have $$I_{n-2} = \frac{n-3}{n-2} I_{n-4}, I_{n-4} = \frac{n-5}{n-4} I_{n-6}$$

.....

$$I_4 = \frac{3}{4} I_2, I_2 = \frac{1}{2} I_0 = \int_0^{\pi/2} \sin^0 x\,dx = \frac{1}{2}\int_0^{\pi/2} dx = \frac{1}{2}\cdot\frac{\pi}{2}.$$

Form these, we obtain $$I_n = \frac{(n-1)(n-3)(n-5)\ldots 3\cdot 1}{n(n-2)(n-4)\ldots 4\cdot 2}\cdot\frac{\pi}{2} \qquad \ldots(iii)$$

Combining (*ii*) and (*iii*), we get the required result for $\int_0^{\pi/2} \sin^n x\,dx$.

Proceeding exactly as above, we get the result for $\int_0^{\pi/2} \cos^n x\,dx$.

Example 6.1. *Integrate (i) $\int \sin^4 x\,dx$ (ii) $\int_0^{\pi/2} \cos^6 x\,dx$.*

Solution. (*i*) We have the reduction formula

$$\int \sin^n x\,dx = \frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

Putting $n = 4, 2$ successively,

$$\int \sin^4 x\,dx = -\frac{\sin^3 x \cos x}{4} + \frac{3}{4}\int \sin^2 x\,dx \qquad \ldots(\alpha)$$

$$\int \sin^2 x\, dx = -\frac{\sin x \cos x}{2} + \frac{1}{2}\int (\sin x)^0\, dx$$

But $\int (\sin x)^0\, dx = \int dx = x. \quad \therefore \quad \int \sin^2 x\, dx = -\dfrac{\sin x \cos x}{2} + \dfrac{x}{2}$

Substituting this in (α), we get

$$\int \sin^4 x\, dx = -\frac{\sin^3 x \cos x}{4} + \frac{3}{4}\left(-\frac{\sin x \cos x}{2} + \frac{x}{2}\right)$$

(*ii*) We know that $\displaystyle\int_0^{\pi/2} \cos^n x\, dx = \frac{(n-1)(n-3)(n-5)\ldots}{n(n-2)(n-4)\ldots}\left(\frac{\pi}{2} \text{ if } n \text{ is even}\right)$

Putting $n = 6$, we get

$$\int_0^{\pi/2} \cos^6 x\, dx = \frac{5.3.1}{6.4.2}\frac{\pi}{2} = \frac{5\pi}{16}.$$

Example 6.2. *Evaluate*

(*i*) $\displaystyle\int_0^a \frac{x^7\, dx}{\sqrt{(a^2 - x^2)}}$ (V.T.U., 2006) (*ii*) $\displaystyle\int_0^\pi \frac{\sqrt{(1-\cos x)}}{1+\cos x}\sin^2 x\, dx$ (*iii*) $\displaystyle\int_0^\infty \frac{dx}{(a^2+x^2)^n}$.

Solution. (*i*) $\displaystyle\int_0^a \frac{x^7}{\sqrt{(a^2-x^2)}}\, dx$ | Put $x = a \sin\theta$, so that $dx = a\cos\theta\, d\theta$
Also when $x = 0$, $\theta = 0$, when $x = a$, $\theta = \pi/2$

$$= \int_0^{\pi/2} \frac{a^7 \sin^7\theta}{a\cos\theta}.\, a\cos\theta\, d\theta = a^7 \int_0^{\pi/2} \sin^7\theta\, d\theta = a^7 \cdot \frac{6.4.2}{7.5.3.1} = \frac{16}{35}a^7$$

(*ii*) Putting $x = 2\theta$, we get

$$\int_0^\pi \frac{\sqrt{(1-\cos x)}}{1+\cos x}\sin^2 x\, dx = \int_0^{\pi/2} \frac{\sqrt{(1-\cos 2\theta)}}{1+\cos 2\theta}\sin^2 2\theta \cdot 2d\theta$$

$$= 2\int_0^{\pi/2} \frac{\sqrt{2}\sin\theta}{2\cos^2\theta}\cdot(2\sin\theta\cos\theta)^2\, d\theta = 4\sqrt{2}\int_0^{\pi/2}\sin^3\theta\, d\theta = 4\sqrt{2}\cdot\frac{2}{3} = \frac{8\sqrt{2}}{3}.$$

(*iii*) $\displaystyle\int_0^\infty \frac{dx}{(a^2+x^2)^n}$ | Put $x = a\tan\theta$, so that $dx = a\sec^2\theta\, d\theta$
Also when $x = 0$, $\theta = 0$, when $x = \infty$, $\theta = \pi/2$

$$= \int_0^{\pi/2} \frac{a\sec^2\theta\, d\theta}{a^{2n}\sec^{2n}\theta} = \frac{1}{a^{2n-1}}\int_0^{\pi/2}\cos^{2n-2}\theta\, d\theta = \frac{1}{a^{2n-1}}\cdot\frac{(2n-3)(2n-5)\ldots 3.1}{(2n-2)(2n-4)\ldots 4.2}\cdot\frac{\pi}{2}.$$

Example 6.3. *Evaluate* $\displaystyle\int_0^a \frac{x^n}{\sqrt{(a^2-x^2)}}\, dx$. *Hence find the value of* $\displaystyle\int_0^1 x^n \sin^{-1} x\, dx$.

Solution. Putting $x = a\sin\theta$, we get

$$\int_0^a \frac{x^n}{\sqrt{(a^2-x^2)}}\, dx = \int_0^{\pi/2} \frac{(a\sin\theta)^n}{a\cos\theta}(a\cos\theta)\, d\theta = a^n \int_0^{\pi/2}\sin^n\theta\, d\theta$$

$$\left.\begin{aligned} &= \frac{(n-1)(n-3)\ldots.2}{n(n-2)\ldots.3}a^n, \text{ if } n \text{ is odd} \\ &= \frac{(n-1)(n-3)\ldots 1}{n(n-2)\ldots 2}\cdot\frac{\pi}{2}a^n, \text{ if } n \text{ is even} \end{aligned}\right\} \quad \ldots(i)$$

Now integrating by parts, we have

$$\int_0^1 x^n \sin^{-1} x\, dx = \left|(\sin^{-1} x)\cdot\frac{x^{n+1}}{n+1}\right|_0^1 - \int_0^1 \frac{x^{n+1}}{n+1}\frac{1}{\sqrt{(1-x^2)}}\, dx$$

$$= \frac{1}{(n+1)}\left[\frac{\pi}{2} - \int_0^1 \frac{x^{n+1}}{(1-x^2)}\right] \qquad \text{[Using (i) p. 241]}$$

$$= \frac{1}{n+1}\left\{\frac{\pi}{2} - \frac{n(n-2)(n-4)\ldots 1}{(n+1)(n-1)(n-3)\ldots 2}\frac{\pi}{2}\right\} \qquad \text{when } n \text{ is odd}$$

$$= \frac{1}{n+1}\left\{\frac{\pi}{2} - \frac{n(n-2)(n-4)\ldots 2}{(n+1)(n-1)(n-3)\ldots 3}\right\} \qquad \text{when } n \text{ is even}$$

Evaluate 6.4. *Evaluate* $I_n = \int_0^a (a^2 - x^2)^n\, dx$ *where n is a positive integer. Hence show that*

$$I_n = \frac{2n}{2n+1} a^2 I_{n-1}.$$

Solution. Putting $n = a \sin\theta$, we get

$$I_n = \int_0^a (a^2 - x^2)^n\, dx = \int_0^{\pi/2} (a^2 - a^2\sin^2\theta)^n\, a\cos\theta\, d\theta = a^{2n+1}\int_0^{\pi/2} \cos^{2n+1}\theta\, d\theta$$

$$= a^{2n+1} \cdot \frac{(2n)(2n-2)(2n-4)\ldots 4.2}{(2n+1)(2n-1)(2n-3)\ldots 5.3} \qquad [\because\ (2n+1) \text{ is always odd}]$$

Now replacing n by $n-1$, we get

$$I_{n-1} = a^{2n-1}\frac{(2n-2)(2n-4)\ldots 4.2}{(2n-1)(2n-3)\ldots 5.3} \quad \therefore \quad \frac{I_n}{I_{n-1}} = a^2 \cdot \frac{2n}{2n+1} \quad \text{or} \quad I_n = \frac{2n}{2n+1}a^2 I_{n-1}.$$

which is the second desired result.

6.3 (1) REDUCTION FORMULAE for $\int \sin^m x \cos^n x\, dx$

$$\int \sin^m x \cos^n x\, dx = \int \sin^{m-1} x \cdot \cos^n x \cdot \sin x\, dx \qquad \text{[Integrate by parts]}$$

$$= \sin^{m-1} x \cdot \left(\frac{-\cos^{n+1} x}{n+1}\right) - \int (m-1)\sin^{m-2} x \cos x \cdot \left(-\frac{\cos^{n+1} x}{n+1}\right) dx.$$

$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{m+1}\int \sin^{m-2} x(1-\sin^2 x)\cos^n x\, dx$$

$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2} x \cos^n x\, dx - \frac{m-1}{n+1}\int \sin^m x \cos^n x\, dx$$

Transposing the last term to the left and dividing by $1 + (m-1)/(n+1)$, *i.e.*, $(m+n)/(n+1)$, we obtain the reduction formula

$$\int \sin^m x \cos^n x\, dx = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} x \cos^n x\, dx \qquad \ldots(1)$$

Obs. To integrate $\int \sin^m x \cos^n x\, dx$,

(*a*) *when m is odd,* put $\cos x = t$

when n is odd, put $\sin x = t$

(*b*) *when m and n both are even integers,* express the integrand as a series of cosines of multiple angles and integrate term by term if m and n are small, *otherwise* use the method of reduction formulae.

(2) To show that

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{(m-1)(m-3)\ldots \times (n-1)(n-3)\ldots}{(m+n)(m+n-2)(m+n-4)\ldots} \times \left(\frac{\pi}{2}, \textit{ only if both } m \textit{ and } n \textit{ are even}\right)$$

From (*i*), we have

$$I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x\, dx = \left| -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} \right|_0^{\pi/2} + \frac{m-1}{m+n} \int_0^{\pi/2} \sin^{m-2} x \cos^n x\, dx$$

i.e.,
$$I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}.$$

Case I. *When m is odd*

Similarly,
$$I_{m-2,n} = \frac{m-3}{m+n-2} I_{m-4,n}, I_{m-4,n} = \frac{m-5}{m+n-4} I_{m-6,n}$$

.....

$$I_{5,n} = \frac{4}{n+5} I_{3,n}$$

Finally
$$I_{3,n} = \frac{2}{n+3} I_{1,n} = \frac{2}{n+3} \int_0^{\pi/2} \sin x \cos^n x\, dx$$

$$= \frac{2}{n+3} \left| -\frac{\cos^{n+1} x}{n+1} \right|_0^{\pi/2} = \frac{2}{(n+3)(n+1)} \qquad ...(ii)$$

From these, we obtain

$$I_{m,n} = \frac{(m-1)(m-3)(m-5)\ldots 4.2}{(m+n)(m+n-2)(m+n-4)\ldots(n+3)(n+1)}$$

Case II. *When m is even*

We have,
$$I_{m-2,n} = \frac{m-3}{m+n-2} I_{m-4,n}, \quad I_{m-4,n} = \frac{m-5}{m+n-4} I_{m-6,n}$$

.....

$$I_{4,n} = \frac{3}{n+4} I_{2,n}, I_{2,n} = \frac{1}{n+2} I_{0,n} = \frac{1}{n+2} \int_0^{\pi/2} \cos^n x\, dx$$

From these, we have $I_{m,n} = \dfrac{(m-1)(m-3)(m-5)\ldots 1}{(m+n)(m+n-2)(m+n-4)\ldots(n+2)} \displaystyle\int_0^{\pi/2} \cos^n x\, dx$

$$= \frac{(m-1)(m-3)\ldots 1}{(m+n)(m+n-2)\ldots(n+2)} \cdot \frac{(n-1)(n-3)\ldots}{n(n-2)\ldots} \times (\pi/2 \textit{ only if } n \textit{ is even}) \qquad ...(iii)$$

Combining (*ii*) and (*iii*), we get the desired result.

Example 6.5. *Integrate* (*i*) $\int \sin^4 x \cos^2 x\, dx$ *(Raipur, 2005)*

(*ii*) $\displaystyle\int_0^\infty \frac{t^6}{(1+t^2)^7} dt$ (*iii*) $\displaystyle\int_0^\infty \frac{x^2}{(1+x^2)^{7/2}} dx$ *(V.T.U., 2010 S)*

Solution. (*i*) Taking $n = 2$, in (*i*) of page 241, we have the reduction formula :

$$\int \sin^m x \cos^2 x\, dx = \frac{\sin^{m-1} x \cos^3 x}{m+2} + \frac{m-1}{m+2} \int \sin^{m-2} x \cos^2 x\, dx$$

Putting $m = 4, 2$ successively,

$$\int \sin^4 x \cos^2 x\, dx = -\frac{\sin^3 x \cos^3 x}{6} + \frac{3}{6} \int \sin^2 x \cos^2 x\, dx \qquad ...(1)$$

$$\int \sin^2 x \cos^2 x\, dx = -\frac{\sin x \cos^3 x}{4} + \frac{1}{4} \int \cos^2 x\, dx$$

But
$$\int \cos^2 x\, dx = \frac{1}{2} \int (1 + \cos 2x)\, dx = \frac{1}{2}\left(x + \frac{1}{2} \sin 2x\right)$$

$$\therefore \qquad \int \sin^2 x \cos^2 x \, dx = -\frac{\sin x \cos^3 x}{4} + \frac{1}{16}(2x + \sin 2x)$$

Substituting this in (1), we get

$$\int \sin^4 x \cos^2 x \, dx = -\frac{\sin^3 x \cos^3 x}{6} + \frac{1}{2}\left\{-\frac{\sin x \cos^3 x}{4} + \frac{1}{16}(2x + \sin 2x)\right\}$$

(*ii*) Putting $t = \tan\theta$, so that

$$\int_0^\infty \frac{t^6}{(1+t^2)^7}\, dt = \int_0^{\pi/2} \frac{\tan^6\theta}{\sec^{14}\theta} \sec^2\theta \, d\theta = \int_0^{\pi/2} \sin^6\theta \, \cos^6\theta \, d\theta = \frac{5\cdot 3\cdot 1\times 5\cdot 3\cdot 1}{12\cdot 10\cdot 8\cdot 6\cdot 4\cdot 2}\times\frac{\pi}{2} = \frac{5\pi}{2048}.$$

(*iii*) Putting $x = \tan\theta$, so that

$$\int_0^\infty \frac{x^2}{(1+x^2)^{7/2}}\, dx = \int_0^{\pi/2} \frac{\tan^2\theta}{\sec^7\theta} \sec^2\theta \, d\theta = \int_0^{\pi/2} \sin^2\theta \cos^3\theta \, d\theta = \frac{1.2}{53.1} = \frac{2}{15}.$$

Example 6.6. *Evaluate :* (*i*) $\int_0^{\pi/6} \cos^4 3\theta \sin^3 6\theta \, d\theta$ (*V.T.U., 2003 S*)

(*ii*) $\int_0^1 x^4(1-x^2)^{3/2}\, dx$ (*iii*) $\int_0^{2a} x^2\sqrt{(2ax - x^2)}\, dx.$ (*V.T.U., 2010*)

Solution. (*i*) $\int_0^{\pi/6} \cos^4 3\theta \sin^3 6\theta \, d\theta = \int_0^{\pi/6} \cos^4 3\theta \, (2\sin 3\theta \cos 3\theta)^3 \, d\theta$

$$= 8\int_0^{\pi/6} \sin^3 3\theta \cos^7 3\theta \, d\theta$$

$$= \frac{8}{3}\int_0^{\pi/2} \sin^3 x \cos^7 x \, dx$$

$$= \frac{8}{3}\cdot\frac{2\times 6\cdot 4\cdot 2}{10\cdot 8\cdot 6\cdot 4\cdot 2} = \frac{1}{15}.$$

Put $3\theta = x$ so that $3d\theta = dx$

Also when $\theta = 0, x = 0$; when $\theta = \pi/6, x = \pi/2$.

(*ii*) $\int_0^1 x^4(1-x^2)^{3/2}\, dx$

Put $x = \sin t$ so that $dx = \cos t \, dt$
When $x = 0, t = 0$; when $x = 1, t = \pi/2$

$$= \int_0^{\pi/2} \sin^4 t \, (\cos^2 t)^{3/2} \cdot \cos t \, dt = \int_0^{\pi/2} \sin^4 t \cos^4 t \, dt$$

$$= \frac{3\cdot 1\times 3\cdot 1}{8\cdot 6\cdot 4\cdot 2}\cdot\frac{\pi}{2} = \frac{3\pi}{256}.$$

(*iii*) $\int_0^{2a} x^2\sqrt{(2ax - x^2)}\, dx$

$$= \int_0^{\pi/2} x^{5/2}\sqrt{(2a - x)}\, dx$$

Put $x = 2a \sin^2\theta$
$\therefore \; dx = 4a \sin\theta \cos\theta \, d\theta$

$$= \int_0^{\pi/2} (2a\sin^2\theta)^{5/2}\sqrt{(2a)}\cos\theta \cdot 4a \sin\theta\cos\theta \, d\theta$$

$$= 2^5 a^4 \int_0^{\pi/2} \sin^6\theta \cos^2\theta \, d\theta = 32\, a^4 \cdot \frac{5\cdot 3\cdot 1\times 1}{8\cdot 6\cdot 4\cdot 2}\cdot\frac{\pi}{2} = \frac{5\pi a^4}{8}.$$

PROBLEMS 6.1

Evaluate :

1. (*i*) $\int_0^{\frac{\pi}{2}} \cos^9 x \, dx$ (*ii*) $\int_0^{\pi/6} \sin^5 3\theta \, d\theta$

2. (*i*) $\int_0^1 \frac{x^9}{\sqrt{(1-x^2)}}\, dx$ (*ii*) $\int_0^1 x^5 \sin^{-1} x \, dx$

3. (*i*) $\int_0^\infty \frac{dx}{(1+x^2)^n}\, (n > 1)$ (*V.T.U., 2008 S*) (*ii*) $\int_0^{\pi/4} \sin^2 x \cos^4 x \, dx.$ (*J.N.T.U., 2003*)

4. If $I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x\, dx$ $(m > 0, n > 0)$, show that $I_{m,n} = \dfrac{m-1}{m+n} I_{m-2,n}$.

Hence evaluate $\int_0^{\pi/2} \sin^4 x \cos^8 x\, dx$

Evaluate :

5. (*i*) $\int_0^{\pi/2} \sin^4 x \cos^6 x\, dx$ (*Cochin, 2005*) (*ii*) $\int_0^{\pi/2} \sin^{15} x \cos^3 x\, dx$

6. (*i*) $\int_0^1 x^6 \sqrt{(1-x^2)}\, dx$ (*ii*) $\int_0^{\pi/2} \cos^4 3\theta \sin^3 6\theta\, d\theta$

7. (*i*) $\int_0^{2a} x^{7/2} (2a-x)^{-1/2}\, dx$ (*ii*) $\int_0^{2a} \dfrac{x^3\, dx}{\sqrt{(2ax-x^2)}}$ (*Madras, 2000 S*)

8. (*i*) $\int_0^2 x^{5/2} \sqrt{(2-x)}\, dx$ (*ii*) $\int_0^4 x^3 \sqrt{(4x-x^2)}\, dx$ (*V.T.U., 2004*)

9. If $I_n = \int x^n \sqrt{(a-x)}\, dx$, prove that $(2n+3) I_n = 2an\, I_{n-1} - 2x^n (a-x)^{3/2}$ (*Marathwada, 2008*)

10. If n is a positive integer, show that $\int_0^{2a} x^n \sqrt{(2ax-x^2)}\, dx = \dfrac{2n+1}{(n+2)\, n!} \cdot \dfrac{a^{n+2}}{2n} \pi$ (*V.T.U., 2007*)

6.4 REDUCTION FORMULAE for (a) $\int \tan^n x\, dx$ (b) $\int \cot^n x\, dx$

(*a*) Let
$$I_n = \int \tan^n x\, dx = \int \tan^{n-2} x \cdot \tan^2 x\, dx = \int \tan^{n-2} x \cdot (\sec^2 x - 1)\, dx$$
$$= \int \tan^{n-2} x \sec^2 x\, dx - \int \tan^{n-2} x\, dx$$

Thus,
$$I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$ which is the required reduction formula.

(*b*) Let
$$I_n = \int \cot^n x\, dx = \int \cot^{n-2} x \cot^2 x\, dx = \int \cot^{n-2} x\, (\operatorname{cosec}^2 x - 1)\, dx$$
$$= \int \cot^{n-2} x \operatorname{cosec}^2 x\, dx - \int \cot^{n-2} x\, dx$$

Thus
$$I_n = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$$
which is the required reduction formula.

Example 6.7. *Evaluate (i) $\tan^5 x\, dx$ (ii) $\int \cot^6 x\, dx$.*

Solution. (*i*) Putting $n = 5, 3$ successively in the reduction formula for $\int \tan^n x\, dx$, we get

$$I_5 = \frac{1}{4} \tan^4 x - I_3\,; \quad I_3 = \frac{1}{2} \tan^2 x - I_1$$

Thus
$$I_5 = \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x + I_1$$

i.e.,
$$\int \tan^5 x\, dx = \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x + \int \tan x\, dx = \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x - \log \cos x.$$

(*ii*) Putting $n = 6, 4, 2$ successively in the reduction formula for $\int \cot^n x\, dx$, we get

$$I_6 = -\frac{1}{5} \cot^5 x - I_4\,; I_4 = -\frac{1}{3} \cot^3 x - I_2\,; I_2 = -\cot x - I_0$$

Thus
$$I_6 = \frac{-1}{5} \cot^5 x + \frac{1}{3} \cot^3 x - \cot x - \int dx$$

i.e., $$\int \cot^6 x\,dx = -\frac{1}{5}\cot^5 x + \frac{1}{3}\cot^3 x - \cot x - x.$$

Example 6.8. *If* $I_n = \int_0^{\pi/4} \tan^n \theta\, d\theta$, *prove that* $n\,(I_{n-1} + I_{n+1}) = 1$. *(V.T.U., 2003)*

Solution. The reduction formula for $\int_0^{\pi/4} \tan^n \theta\, d\theta$ is

$$I_n = \frac{1}{n-1}\Big|\tan^n x\Big|_0^{\pi/4} - I_{n-2} = \frac{1}{n-1} - I_{n-2} \quad \text{or} \quad I_n + I_{n-2} = \frac{1}{n-1}$$

Changing n to $n+1$, we obtain

$$I_{n+1} + I_{n-1} = \frac{1}{(n+1)} \quad \text{or} \quad (n+1)(I_{n+1} + I_{n-1}) = 1.$$

6.5 REDUCTION FORMULAE for (a) $\int \sec^n x\,dx$ (b) $\int \operatorname{cosec}^n x\,dx$

(*a*) Let $$I_n = \int \sec^n x\,dx = \int \sec^{n-2} x \,.\, \sec^2 x\,dx$$

Integrating by parts, we have

$$I_n = \sec^{n-2} x.\tan x - \int \{(n-2)\sec^{n-3} x \cdot \sec x \tan x\} \tan x\,dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \cdot \tan^2 x\,dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \cdot (\sec^2 x - 1)\,dx$$

$$= \sec^{n-2} x \tan x - (n-2)\,I_n + (n-2)\,I_{n-2}$$

Transposing, we have

$$(n-1)I_n = \sec^{n-2} x \tan x + (n-2)I_{n-2}$$

Thus $$I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$ which is the desired reduction formula.

(*b*) Let $$I_n = \int \operatorname{cosec}^n x\,dx = \int \operatorname{cosec}^{n-2} x \cdot \operatorname{cosec}^2 x\,dx$$

Integrating by parts, we have

$$I_n = \operatorname{cosec}^{n-2} x.(-\cot x) - \int [(n-2)\operatorname{cosec}^{n-3} x \cdot (-\operatorname{cosec} x \cot x) \cdot (-\cot x)\,dx$$

$$= -\cot x \operatorname{cosec}^{n-2} x - (n-2)\int \operatorname{cosec}^{n-2} x\,(\operatorname{cosec}^2 x - 1)\,dx$$

$$= -\cot x \operatorname{cosec}^{n-2} x - (n-2)I_n + (n-2)I_{n-2}$$

or $$[1 + (n-2)]I_n = -\cot x \operatorname{cosec}^{n-2} x + (n-2)I_{n-2}$$

Thus $$I_n = -\frac{\cot x \operatorname{cosec}^{n-2} x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

which is the required reduction formula.

Example 6.9. *Evaluate* (*i*) $\int_0^{\pi/4} \sec^4 x\,dx$ (*ii*) $\int_{\pi/3}^{\pi/2} \operatorname{cosec}^3 \theta\,d\theta$. *(V.T.U., 2008)*

Solution. (*i*) Putting $n = 4$ in the reduction formula for $\int \sec^n x\,dx$, we get $I_4 = \dfrac{\sec^2 x \tan x}{3} + \dfrac{2}{3} I_2$

$$\therefore \quad \int_0^{\pi/4} \sec^4 x\,dx = \left|\frac{\sec^2 x \tan x}{3}\right|_0^{\pi/4} + \frac{2}{3}\int_0^{\pi/4} \sec^2 x\,dx$$

$$= \frac{2}{3} + \frac{2}{3}\Big|\tan x\Big|_0^{\pi/4} = \frac{2}{3}(1+1) = 4/3.$$

(*ii*) Putting $n = 3$ in the reduction formula for $\int \operatorname{cosec}^n x\, dx$, we get

$$I_3 = -\frac{1}{2}\cot x \operatorname{cosec} x + \frac{1}{2} I_1$$

$$\therefore \quad \int_{\pi/3}^{\pi/2} \operatorname{cosec}^3 x\, dx = -\frac{1}{2}\left| \cot x \operatorname{cosec} x \right|_{\pi/3}^{\pi/2} + \frac{1}{2}\int_{\pi/3}^{\pi/2} \operatorname{cosec} x\, dx$$

$$= -\frac{1}{2}\left(0 - \frac{2}{3}\right) + \frac{1}{2}\left| \log(\operatorname{cosec} x - \cot x) \right|_{\pi/3}^{\pi/2}$$

$$= \frac{1}{3} + \frac{1}{2}\left[\log 1 - \log\left(\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right)\right] = \frac{1}{3} + \frac{1}{4}\log 3.$$

PROBLEMS 6.2

1. Evaluate (*i*) $\int \tan^6 x\, dx$ (*V.T.U., 2007*) (*ii*) $\int \cot^5 x\, dx$.
2. Show that $\int_0^{\pi/4} \tan^7 x\, dx = \frac{1}{12}(5 - 6\log 2)$
3. If $I_n = \int_0^{\pi/4} \tan^n x\, dx$, prove that $(n-1)(I_n + I_{n-2}) = 1$. (*V.T.U., 2009*)
   Hence evaluate I_5. (*Madras, 2000*)
4. If $I_n = \int_{\pi/4}^{\pi/2} \cot^n \theta\, d\theta\ (n > 2)$, prove that $I_n = \frac{1}{n-1} - I_{n-1}$. Hence evaluate I_4. (*Marathwada, 2008*)
5. Obtain the reduction formula for $\int_0^{\pi/4} \sec^n \theta\, d\theta$. (*V.T.U., 2010 S*)
6. Evaluate (*i*) $\int \sec^6 \theta\, d\theta$ (*ii*) $\int_{\pi/6}^{\pi/2} \operatorname{cosec}^5 d\theta$.
7. Evaluate $\int_0^a (a^2 + x^2)^{5/2}\, dx$.
8. If $I_n = \int \frac{t^n}{1+t^2}\, dt$, show that $I_{n+2} = \frac{t^{n+1}}{n+1} - I_n$. Hence evaluate I_6.

6.6 REDUCTION FORMULAE for

(*a*) $\int x^n e^{ax}\, dx$ **(*b*) $\int x^m (\log x)^n\, dx$.**

(*a*) Let $I_n = \int x^n e^{ax}\, dx$

Integrating by parts, we have

$$I_n = x^n \cdot \frac{e^{ax}}{a} - \int n x^{n-1} \cdot \frac{e^{ax}}{a}\, dx$$

or $$I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$$ which is the required reduction formula. (*Madras, 2006*)

(*b*) Let $I_{m,n} = \int x^m (\log x)^n\, dx = \int (\log x)^n \cdot x^m\, dx$

Integrating by parts, we have

$$I_{m,n} = (\log x)^n \cdot \frac{x^{m+1}}{m+1} - \int n(\log x)^{n-1} \cdot \frac{1}{x} \cdot \frac{x^{m+1}}{m+1}\, dx$$

$$= \frac{x^{m+1}}{m+1}(\log x)^n - \frac{n}{m+1}\int x^m (\log x)^{n-1}\, dx \quad \text{or} \quad I_{m,n} = \frac{x^{m+1}}{m+1}(\log x)^n - \frac{n}{m+1} I_{m,n-1}$$

which is the desired reduction formula.

6.7 REDUCTION FORMULAE for

(a) $\int x^n \sin mx\, dx$ (b) $\int x^n \cos mx\, dx$ (c) $\int \cos^m x \sin nx\, dx$

(a) Let $I_n = \int x^n \sin mx\, dx$

Integrating by parts, we get

$$I_n = x^n \left(\frac{-\cos mx}{m}\right) - \int n x^{n-1} \left(\frac{-\cos mx}{m}\right) dx$$

$$= -\frac{x^n \cos mx}{m} + \frac{n}{m} \int x^{n-1} \cos mx\, dx \qquad \text{[Again integrate by parts]}$$

$$= -\frac{x^n \cos mx}{m} + \frac{n}{m}\left\{x^{n-1} \cdot \frac{\sin mx}{m} - \left\{\int (n-1) x^{n-2} \cdot \frac{\sin mx}{m} dx\right\}\right\}$$

or
$$I_n = -\frac{x^n \cos mx}{m} + \frac{n}{m^2} x^{n-1} \sin mx - \frac{n(n-1)}{m^2} I_{n-2}$$

which is the desired reduction formula. *(Madras, 2003)*

(b) Let $I_n = \int x^n \cos mx\, dx$

Integrating twice by parts as above, we get

$$I_n = \frac{x^n \sin mx}{m} + \frac{n}{m^2} x^{n-1} \cos mx - \frac{n(n-1)}{m^2} I_{n-2}$$

(c) Let $I_{m,n} = \int \cos^m x \sin nx\, dx$

Integrating by parts,

$$I_{m,n} = -\cos^m x \cdot \frac{\cos nx}{n} - \int m \cos^{m-1} x (-\sin x) \cdot \left(\frac{-\cos nx}{n}\right) dx$$

$$= -\frac{1}{n} \cos^m x \cos nx - \frac{m}{n} \int \cos^{m-1} x \cdot \cos nx \sin x\, dx$$

$$\left[\begin{array}{l} \because \quad \sin(n-1)x = \sin nx \cos x - \cos nx \sin x \\ \text{or } \cos nx \sin x = \sin nx \cos x - \sin(n-1)x \end{array}\right.$$

$$= -\frac{1}{n} \cos^m x \cos nx - \frac{m}{n} \int \cos^{m-1} x \{\sin nx \cos x - \sin(n-1)x\}\, dx$$

$$= -\frac{1}{n} \cos^m x \cos nx - \frac{m}{n} (I_{m,n} - I_{m-1,n-1})$$

Transposing, we get

$$\left(1 + \frac{m}{n}\right) I_{m,n} = -\frac{1}{n} \cos^m x \cos nx + \frac{m}{n} I_{m-1,n-1}$$

or
$$I_{m,n} = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$$

which is the desired reduction formula.

Example 6.10. *Show that* $\int_0^{\pi/2} \cos^m x \cos nx\, dx = \frac{m}{m+n} \int_0^{\pi/2} \cos^{m-1} x \cos(n-1)x\, dx$

Hence deduce that $\int_0^{\pi/2} \cos^n x \cos nx\, dx = \frac{\pi}{2^{n+1}}$. *(S.V.T.U., 2008)*

Solution. Let $I_{m,n} = \int_0^{\pi/2} \cos^m x \cos nx\, dx$

Integrating by parts

$$I_{m,n} = \left| \cos^m x \cdot \frac{\sin nx}{n} \right|_0^{\pi/2} - \int_0^{\pi/2} m \cos^{m-1} x (-\sin x) \times \frac{\sin nx}{n}\, dx$$

$$= \frac{m}{n}\int_0^{\pi/2} \cos^{m-1} x \sin nx \sin x\, dx \qquad \left[\because \cos(n-1)x = \cos nx \cos x + \sin nx \sin x \text{ or } \sin nx \sin x = \cos(n-1)x - \cos nx \cos x\right]$$

$$= \frac{m}{n}\int_0^{\pi/2} \cos^{m-1} x\,[\cos(n-1)x - \cos nx \cos x]\, dx = \frac{m}{n}(I_{m-1,\,n-1} - I_{m,\,n})$$

Transposing and dividing by $(1 + m/n)$, we get

$$I_{m,\,n} = \frac{m}{m+n} I_{m-1,\,n-1}$$

which is the required result.

Putting $\quad m = n,\ I_n \left(= \int_0^{\pi/2} \cos^n x \cos nx\, dx\right) = \frac{1}{2} I_{n-1}$

Changing n to $n-1$,

$$I_{n-1} = \frac{1}{2} I_{n-2}$$

$$\therefore \qquad I_n = \frac{1}{2}\left(\frac{1}{2} I_{n-2}\right) = \frac{1}{2^2} I_{n-2} = \frac{1}{2^3} I_{n-3} \ldots\ldots = \frac{1}{2^n} I_{n-n} = \frac{1}{2^n} \cdot \int_0^{\pi/2} (\cos x)^0\, dx$$

Hence $\quad I_n = \frac{1}{2^n} \cdot \frac{\pi}{2} = \frac{\pi}{2^{n+1}}.$

Example 6.11. *Find a reduction formula for* $\int e^{ax} \sin x\, dx$. *Hence evaluate* $\int e^x \sin^3 x\, dx$.

Solution. Let $I_n = \int e^{ax} \sin^n x\, dx = \int \frac{\sin^n x}{I} \cdot \frac{e^{ax}}{II}\, dx$

Integrating by parts,

$$I_n = \sin^n x \cdot \frac{e^{ax}}{a} - \int (n \sin^{n-1} x \cos x) \cdot \frac{e^{ax}}{a}\, dx$$

$$= \frac{e^{ax} \sin^n x}{a} - \frac{n}{a}\int (\sin^{n-1} x \cos x) \cdot e^{ax}\, dx \qquad \text{[Again integrating by parts]}$$

$$= \frac{e^{ax} \sin^n x}{a} - \frac{n}{a}\left[\sin^{n-1} x \cos x \cdot \frac{e^{ax}}{a} - \int \{(n-1)\sin^{n-2} x \times \cos x \cdot \cos x + \sin^{n-1} x(-\sin x)\} \frac{e^{ax}}{a}\, dx\right]$$

$$= \frac{e^{ax} \sin^{n-1} x}{a^2}(a \sin x - n \cos x) + \frac{n}{a^2}\int [(n-1)\sin^{n-2} x \times (1 - \sin^2 x) - \sin^n x]\, e^{ax}\, dx$$

$$= \frac{e^{ax} \sin^{n-1} x}{a}(a \sin x - n \cos x) + \frac{n(n-1)}{a^2} I_{n-2} - \frac{n^2}{a^2} I_n$$

Transposing and dividing by $(1 + n^2/a^2)$, we get

$$I_n = \frac{e^{ax} \sin^{n-1} x\,(a \sin x - n \cos x)}{a^2 + n^2} + \frac{n(n-1)}{a^2 + n^2} I_{n-2}$$

which is the required reduction formula.

Putting $a = 1$ and $n = 3$, we get

$$I_3 = \frac{e^x \sin^2 x\,(\sin x - 3 \cos x)}{1^2 + 9} + \frac{3 \cdot 2}{1^2 + 9} I_1$$

But $\quad I_1 = \int e^x \sin x\, dx = \frac{e^{ax}}{\sqrt{2}} \sin(x - \tan^{-1} 1).$

$$\therefore \qquad I_3 = \frac{e^x \sin^2 x\,(\sin x - 3 \cos x)}{10} + \frac{3}{5} \cdot \frac{e^{ax}}{\sqrt{2}} \sin(x - \pi/4).$$

PROBLEMS 6.3

1. If $I_n = \int x^n e^x \, dx$, show that $I_n + n I_{n-1} = x^n e^x$. Hence find I_4. *(Madras, 2000)*

2. If $u_n = \int_0^a x^n e^{-x} \, dx$, prove that $u_n - (n+a) u_{n-1} + a(n-1) u_{n-2} = 0$. *(Madras, 2003)*

3. Obtain a reduction formula for $\int x^m (\log x)^n \, dx$. Hence evaluate $\int_0^1 x^5 (\log x)^3 \, dx$. *(S.V.T.U., 2009 ; Bhillai, 2005)*

4. If n is a positive integer, show that $\int_0^1 x^m (\log x)^n \, dx = \dfrac{(-1)^n \, n!}{(m+1)^{n+1}}$, $m > -1$.

5. If $I_n = \int_0^{\pi/2} x \sin^n x \, dx$ $(n > 1)$, prove that $n^2 I_n = n(n-1) I_{n-2} + 1$. Hence evaluate I_5.

6. If $I_n = \int_0^{\pi/2} x \cos^n x \, dx$ $(n > 1)$, prove that $I_n = \dfrac{n-1}{n} I_{n-2} - \dfrac{1}{n^2}$. Hence evaluate I_4.

7. If $u_n = \int_0^{\pi/2} x^n \sin x \, dx$, $(n > 1)$, prove that $u_n + n(n-1) u_{n-2} = n(\pi/2)^{n-1}$. Hence evaluate u_2. *(Madras, 2000 S)*

8. If $I_n = \int x^n \sin ax \, dx$, show that $a^2 I_n = -ax^n \cos ax + nx^{n-1} \sin ax - n(n-1) I_{n-2}$. *(Marathwada, 2008)*

9. Prove that $\int_0^{\pi/2} \cos^{n-2} x \sin nx \, dx = \dfrac{1}{n-1}$, $n > 1$.

10. If $I_{m,n} = \int_0^{\pi/2} \cos^m x \cos nx \, dx$, prove that $I_{m,n} = \dfrac{m(m-1)}{m^2 - n^2} I_{m-2,n}$

11. Find a reduction formula for $\int e^{ax} \cos^n x \, dx$. Hence evaluate $\int_0^{\pi/2} e^{2x} \cos^3 x \, dx$.

12. Obtain a reduction formula for $I_m = \int_0^{\infty} e^{-x} \sin^m x \, dx$ where $m \geq 2$ in the form

$(1 + m^2) I_m = m(m-1) I_{m-2}$. Hence evaluate I_4. *(Gorakhpur, 1999)*

6.8 DEFINITE INTEGRALS

Property I. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

(i.e., the value of a definite integral depends on the limits and not on the variable of integration).

Let $\int f(x)\,dx = \phi(x)$; $\therefore \int_a^b f(x)\,dx = \phi(b) - \phi(a)$.

Then $\int f(t)\,dt = \phi(t)$; $\therefore \int_a^b f(t)\,dt = \phi(b) - \phi(a)$.

Hence the result.

Property II. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(i.e., the interchange of limits changes the sign of the integral).

Let $\int f(x)\,dx = \phi(x)$; $\therefore \int_a^b f(x)\,dx = \phi(b) - \phi(a)$

and $-\int_b^a f(x)\,dx = -\left| \phi(x) \right|_b^a = -[\phi(a) - \phi(b)] = \phi(b) - \phi(a)$.

Hence the result.

Property III. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Let $\int f(x)\,dx = \phi(x)$, so that $\int_a^b f(x)\,dx = \phi(b) - \phi(a)$ ...(1)

Also $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = |\,\phi(x)\,|_a^c + |\,\phi(x)\,|_c^b$

$= [\phi(c) - \phi(a)] + [\phi(b) - \phi(c)] = \phi(b) - \phi(a)$ ...(2)

Hence the result follows from (1) and (2).

Property IV. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Put $x = a - t$, so that $dx = -dt$. Also when $x = 0, t = a$; when $x = a, t = 0$.

$\therefore \quad \int_0^a f(x)\,dx = -\int_a^0 f(a-t)\,dt = \int_0^a f(a-t)\,dt = \int_0^a f(a-x)\,dx$ [Prop. II]

Example 6.12. *Evaluate* $\int_0^{\pi/2} \frac{\sqrt{(\sin x)}}{\sqrt{(\sin x)} + \sqrt{(\cos x)}}\,dx$.

Solution. Let $I = \int_0^{\pi/2} \frac{\sqrt{(\sin x)}}{\sqrt{(\sin x)} + \sqrt{(\cos x)}}\,dx$

Then $I = \int_0^{\pi/2} \frac{\sqrt{\left[\sin\left(\frac{1}{2}\pi - x\right)\right]}}{\sqrt{\left[\sin\left(\frac{1}{2}\pi - x\right)\right]} + \sqrt{\left[\cos\left(\frac{1}{2}\pi - x\right)\right]}}\,dx$ [Prop. IV]

$= \int_0^{\pi/2} \frac{\sqrt{(\cos x)}}{\sqrt{(\cos x)} + \sqrt{(\sin x)}}\,dx$

Adding $2I = \int_0^{\pi/2} \frac{\sqrt{(\sin x)} + \sqrt{(\cos x)}}{\sqrt{(\sin x)} + \sqrt{(\cos x)}}\,dx = \int_0^{\pi/2} dx = |\,x\,|_0^{\pi/2} = \frac{\pi}{2}$.

Hence $I = \pi/4$.

Example 6.13. *Evaluate* $\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$. *(Cochin, 2005)*

Solution. Let $I = \int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$ [Put $x = \tan\theta$ so that $dx = \sec^2\theta\,d\theta$; When $x = 0, \theta = 0$; when $x = 1, \theta = \pi/4$]

$= \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{1+\tan^2\theta} \cdot \sec^2\theta\,d\theta = \int_0^{\pi/4} \log(1+\tan\theta)\,d\theta$

$= \int_0^{\pi/4} \log\left[1 + \tan\left(\frac{\pi}{4} - \theta\right)\right] d\theta = \int_0^{\pi/4} \log\left(1 + \frac{1-\tan\theta}{1+\tan\theta}\right) d\theta$ [Prop. IV]

$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right) d\theta = \log 2 \int_0^{\pi/4} d\theta - I$

Transposing, $2I = \log 2 \cdot |\,\theta\,|_0^{\pi/4} = \frac{\pi}{4}\log 2$. Hence $I = \frac{\pi}{8}\log 2$.

Example 6.14. *Evaluate* $\int_0^{\pi} \frac{x\sin^3 x}{1+\cos^2 x}\,dx$. *(Madras, 2006)*

Solution. Let $I = \int_0^{\pi} \frac{x\sin^3 x}{1+\cos^2 x}\,dx$

Then $$I = \int_0^{\pi} \frac{(\pi - x)\sin^3(\pi - x)}{1 + \cos^2(\pi - x)}\,dx \qquad \text{[Prop. IV]}$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin^3 x}{1 + \cos^2 x}\,dx = \pi\int_0^{\pi} \frac{\sin^3 x}{1 + \cos^2 x}\,dx - \mathrm{I}$$

Transposing, $$2I = \pi\int_0^{\pi} \frac{\sin^3 x}{1 + \cos^2 x}\,dx$$

$$= -\pi\int_1^{-1} (1 - t^2)\frac{dt}{1 + t^2}$$

Put $\cos x = t$ so that $-\sin x\,dx = dt$
When $x = 0, t = 1$; When $x = \pi, t = -1$;

$$= \pi\int_1^{-1} \frac{-2 + (1 + t^2)}{1 + t^2}\,dt = -2\pi\int_1^{-1} \frac{dt}{1 + t^2} + \pi\int_1^{-1} dt$$

$$= -2\pi\left|\tan^{-1} t\right|_1^{-1} + \pi\left|t\right|_1^{-1} = -2\pi\left(-\frac{\pi}{4} - \frac{\pi}{4}\right) - 2\pi. \text{ Hence, } I = \pi^2/2 - \pi.$$

Property V. $\int_{-\alpha}^{a} \mathbf{f(x)\,dx = 2\int_0^a f(x)\,dx}$**, if f(x) is an even function,**

$\mathbf{= 0}$ **if f(x) is an odd function.** *(Bhopal, 2008)*

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx \qquad \text{...(1) [Prop. I]}$$

In $\int_{-a}^{0} f(x)\,dx$, put $x = -t$, so that $dx = -dt$

$$\therefore \quad \int_{-a}^{0} f(x)\,dx = -\int_a^0 f(-t)\,dt = \int_0^a f(-t)\,dt = \int_0^a f(-x)\,dx \qquad \text{[Prop. II]}$$

Substituting in (1), we get

$$\int_{-a}^{a} f(x)\,dx = \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx \qquad \text{...(2)}$$

(*i*) *If $f(x)$ is an even function, $f(-x) = f(x)$.*

$\therefore$ from (2), $\int_{-a}^{a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$

(*ii*) *If $f(x)$ is an odd function, $f(-x) = -f(x)$.*

$\therefore$ from (2), $\int_{-a}^{a} f(x)\,dx = -\int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 0.$

Property VI. $\int_0^{2a} \mathbf{f(x)\,dx = 2\int_0^a f(x)\,dx}$**, if f (2a – x) = f(x)**

$\mathbf{= 0,}$ **if f(2a – x) = – f(x)**

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx \qquad \text{...(1) [Prop. III]}$$

In $\int_0^{2a} f(x)\,dx$, put $x = 2a - t$, so that $dx = -dt$

Also when $x = a, t = a$; when $x = 2a, t = 0$.

$$\therefore \quad \int_0^{2a} f(x)\,dx = -\int_a^0 f(2a - t)\,dt = \int_0^a f(2a - t)\,dt = \int_0^a f(2a - x)\,dx \qquad \text{[Prop. II]}$$

Subsituting in (1), we get

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a - x)\,dx \qquad \text{...(2)}$$

(*i*) *If $f(2a - x) = f(x)$*, then from (2)

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$$

(*ii*) *If* $f(2a-x) = -f(x)$, then from (2)

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx - \int_0^a f(x)\,dx = 0.$$

Cor. 1. *If n is even,* $\int_0^{\pi} \sin^m x \cos^n x\,dx = 2\int_0^{\pi/2} \sin^m x \cos^n x\,dx$ *and if n is odd,* $\int_0^{\pi} \sin^m x \cos^n x\,dx = 0.$

Cor. 2. *If m is odd,* $\int_0^{2\pi} \sin^m x \cos^n x\,dx = 0$

and *if m is even,* $\int_0^{2\pi} \sin^m x \cos^n x\,dx = 2\int_0^{\pi} \sin^m x \cos^n x\,dx$

$$= 4\int_0^{\pi/2} \sin^m x \cos^n x\,dx, \text{ if } n \text{ is even} = 0, \text{ if } n \text{ is odd.}$$

Example 6.15. *Evaluate* $\int_0^{\pi} \theta \sin^2\theta \cos^4\theta\,d\theta.$ *(V.T.U., 2009 S)*

Solution. Let $I = \int_0^{\pi} \theta \sin^2\theta \cos^4\theta\,d\theta$

Then $I = \int_0^{\pi} (\pi-\theta)\sin^2(\pi-\theta)\cos^4(\pi-\theta)\,d\theta = \pi\int_0^{\pi} \sin^2\theta\cos^4\theta\,d\theta - I$ [Prop. IV]

or $2I = \pi\int_0^{\pi} \sin^2\theta\cos^4\theta\,d\theta = 2\pi\int_0^{\pi/2} \sin^2\theta\cos^4\theta\,d\theta$ [Prop. VI Cor. 2]

$$= 2\pi \cdot \frac{1.3.1}{6.4.2} = \frac{\pi}{2} = \frac{\pi^2}{16}$$

Hence $I = \dfrac{\pi^2}{32}$

Example 6.16. *Evaluate* $\int_0^{\pi/2} \log \sin x\,dx.$ *(Anna, 2005 S)*

Solution. Let $I = \int_0^{\pi/2} \log\sin x\,dx$ ...(*i*)

then $I = \int_0^{\pi/2} \log\sin(\pi/2 - x)\,dx = \int_0^{\pi/2} \log\cos x\,dx$ ...(*ii*)

Adding (*i*) and (*ii*)

$$2I = \int_0^{\pi/2} (\log\sin x + \log\cos x)\,dx$$

$$= \int_0^{\pi/2} \log(\sin x + \cos x)\,dx = \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right)dx$$

$$= \int_0^{\pi/2} \log\sin 2x\,dx - \int_0^{\pi/2} \log 2\,dx = \int_0^{\pi/2} \log\sin 2x\,dx - \log 2\int_0^{\pi/2} dx$$

$$= \int_0^{\pi/2} \log\sin 2x\,dx - \log 2\,|x|_0^{\pi/2} = I' - \frac{\pi}{2}\log 2 \quad ...(iii)$$

where $I' = \int_0^{\pi/2} \log\sin 2x\,dx$ [Put, $2x = t$, so that $2dx = dt$; When $x = 0$, $t = 0$; when $x = \pi/2$, $t = \pi$]

$$= \frac{1}{2}\int_0^{\pi} \log\sin t\,dt = \frac{1}{2}\int_0^{\pi} \log\sin x\,dx \quad [\because \log\sin(\pi - x) = \log\sin x, \text{ Prop. IV}]$$

$$= \frac{1}{2}.2\int_0^{\pi/2} \log\sin x\,dx = I.$$

Thus from (*iii*), $2I = I - (\pi/2)\log 2$, *i.e.*, $I = -(\pi/2)\log 2$.

Obs. The following are its immediate deductions :

$$\int_0^{\pi/2} \log \sin x \, dx = \int_0^{\pi/2} \log \cos x \, dx = -\frac{\pi}{2} \log 2$$

and $$\int_0^{\pi} \log \sin x \, dx = -\pi \log 2.$$

Example 6.17. *Evaluate* $\int_0^1 \frac{\sin^{-1} x}{x} dx$.

Solution. Put $\sin^{-1} x = \theta$ or $x = \sin\theta$ so that $dx = \cos\theta \, d\theta$

Also when $x = 0, \theta = 0$; when $x = 1, \theta = \pi/2$.

$$\therefore \quad \int_0^1 \frac{\sin^{-1} x}{x} dx = \int_0^{\pi/2} \theta \cdot \frac{\cos\theta}{\sin\theta} d\theta \qquad \text{[Integrate by parts]}$$

$$= [\theta \cdot \log \sin\theta]_0^{\pi/2} - \int_0^{\pi/2} 1 \cdot \log \sin\theta \, d\theta$$

$$= -\int_0^{\pi/2} \log \sin\theta \, d\theta = -\left(-\frac{\pi}{2} \log 2\right) = \frac{\pi}{2} \log 2 \qquad \left[\operatorname*{Lt}_{x \to 0} (x \log x) = 0\right]$$

PROBLEMS 6.4

Prove that :

1. (i) $\int_0^{\pi/2} \log \tan x \, dx = 0$ (ii) $\int_0^{\pi/2} \sin 2x \log \tan x \, dx = 0$

2. (i) $\int_0^{\infty} \frac{x^7 (1 - x^{12})}{(1 + x)^{28}} dx = 0$ (ii) $\int_0^{\pi/4} \log (1 + \tan\theta) \, d\theta = \frac{\pi}{8} \log_e 2$ *(Madras, 2000)*

3. (i) $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx = \frac{\pi}{4}$ (ii) $\int_0^{a} \frac{dx}{x + \sqrt{(a^2 + x^2)}} = \frac{\pi}{4}$

4. (i) $\int_0^{\pi/2} \frac{dx}{1 + \sqrt{\cot x}} = \frac{\pi}{4}$. (ii) $\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x} dx = \frac{\pi}{4}$.

5. (i) $\int_0^{\pi/2} \frac{x \tan x}{\sec x + \cos x} dx = \frac{\pi^2}{4}$. (ii) $\int_0^{\pi} \frac{x}{1 + \sin x} dx = \pi$ *(Anna, 2002 S)*

6. (i) $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} = \frac{1}{2} \pi (\pi - 2)$. (ii) $\int_0^{\pi/2} \frac{x \, dx}{\sin x + \cos x} = \frac{\pi}{2\sqrt{2}} \log (\sqrt{2} + 1)$.

Evaluate :

7. (i) $\int_0^{\pi} \sin^4 x \, dx$ (ii) $\int_0^{2\pi} \cos^6 x \, dx$

   (iii) $\int_0^{\pi} \sin^6 x \cos^4 x \, dx$ *(V.T.U., 2001)* (iv) $\int_0^{2\pi} \sin^4 x \cos^6 x \, dx$

8. (i) $\int_0^{\pi} x \sin^7 x \, dx$ *(V.T.U., 2009)* (ii) $\int_0^{\pi} x \cos^4 x \sin^5 x \, dx$ *(Marathwada, 2008)*

Prove that :

9. (i) $\int_0^{\pi} \frac{x \, dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi^2}{2ab}$ (ii) $\int_0^{\pi/2} \frac{x \, dx}{2 \sin^2 x + \cos^2 x} = \frac{\pi^2}{2\sqrt{2}}$

10. (i) $\int_0^{\pi} \frac{x \, dx}{a^2 - \cos^2 x} = \frac{\pi^2}{2a\sqrt{(a^2 - 1)}}$ $(a > 1)$ (ii) $\int_0^{\pi} \frac{x \, dx}{1 + \sin^2 x} = \frac{\pi^2}{2\sqrt{2}}$.

11. $\int_0^{\pi} \log(1+\cos\theta)\, d\theta = -\pi \log_e 2$ *(Madras, 2003)*

12. (*i*) $\int_0^{\infty} \frac{\log(1+x^2)}{1+x^2}\, dx = \pi \log_e 2$ (*ii*) $\int_0^{\infty} \frac{\log(x+1/x)}{1+x^2}\, dx = \pi \log_e 2.$

6.9 (1) INTEGRAL AS THE LIMIT OF A SUM

We have so far considered integration as inverse of differentiation. We shall now define the definite integral as the limit of a sum :

Def. *If $f(x)$ is continuous and single valued in the interval $[a, b]$, then the definite integral of $f(x)$ between the limits a and b is defined by the equation*

$$\int_a^b f(x)\, dx = \underset{h \to 0}{\text{Lt}}\, h\,[f(a) + f(a+h) + f(a+2h) + \ldots + f(a + \overline{n-1}\, h)],$$

where $nh = b - a$. ...(1)

(2) Evaluation of limits of series

The summation definition of a definite integral enables us to express the limits of sums of certain types of series as definite integrals which can be easily evaluated. We rewrite (1) as follows :

$$\int_a^b f(x)\, dx = \underset{n \to \infty}{\text{Lt}}\, h \sum_{r=0}^{n-1} f(a + rh), \text{ where } nh = b - a.$$

Putting $a = 0$ and $b = 1$, so that $h = 1/n$, we get

$$\underset{n \to \infty}{\text{Lt}}\, \frac{1}{n} \sum_{r=0}^{n-1} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\, dx$$

Thus to express a given series as definite integral:

(*i*) Write the general term (T_r or T_{r+1} whichever involves r)

i.e., $f(r/n) \,.\, 1/n$

(*ii*) Replace r/n by x and $1/n$ by dx,

(*iii*) Integrate the resulting expression, taking

the lower limit = $\underset{n \to \infty}{\text{Lt}}\, (r/n)$ where r is as in the first term,

and the upper limit = $\underset{n \to \infty}{\text{Lt}}\, (r/n)$ where r is as in the last term.

Example 6.18. *Find the limit, when $n \to \infty$, of the series*

$$\frac{n}{n^2} + \frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \ldots + \frac{n}{n^2+(n-1)^2}$$

Solution. Here the general term $(= T_{r+1}) = \frac{n}{n^2+r^2} = \frac{n}{1+(r/n)^2} \cdot \frac{1}{n}$

$$= \frac{1}{1+x^2}\, dx \qquad \text{[Putting } r/n = x \text{ and } 1/n = dx\text{]}$$

Now for the first term $r = 0$ and for the last term $r = n - 1$

$\therefore$ the lower limit of integration $= \underset{n \to \infty}{\text{Lt}} \left(\frac{0}{n}\right) = 0$

and the upper limit of integration $= \underset{n \to \infty}{\text{Lt}} \left(\frac{n-1}{n}\right) = \underset{n \to \infty}{\text{Lt}} \left(1 - \frac{1}{n}\right) = 1.$

Hence, the required limit $= \int_0^1 \frac{dx}{1+x^2} = \left| \tan^{-1} x \right|_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \pi/4.$

[To find limit of a product by integration :

Let $P = \underset{n\to\infty}{\text{Lt}}$ (given product)

Take logs of both sides, so that

$$\log P = \underset{n\to\infty}{\text{Lt}} \text{ (a series)} = k \text{ (say). Then } P = e^k.]$$

Example 6.19. *Evaluate* $\underset{n\to\infty}{Lt}\left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\left(1+\frac{n}{n}\right)\right\}^{1/n}$. *(Bhopal, 2008)*

Solution. Let $$P = \underset{n\to\infty}{\text{Lt}}\left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\left(1+\frac{n}{n}\right)\right\}^{1/n}.$$

Taking logs of both sides,

$$\log P = \underset{n\to\infty}{\text{Lt}}\ \frac{1}{n}\left\{\log\left(1+\frac{1}{n}\right)+\log\left(1+\frac{2}{n}\right)+\ldots+\log\left(1+\frac{n}{n}\right)\right\}$$

Its general term $= \log\left(1+\frac{r}{n}\right).\frac{1}{n} = \log(1+x)\,.\,dx$ [Putting $r/n = x$ and $1/n = dx$]

Also for first term $r = 1$ and for the last term $r = n$.

$\therefore$ The lower limit of integration $= \underset{n\to\infty}{\text{Lt}}\ (1/n) = 0$ and the upper limit $= \underset{n\to\infty}{\text{Lt}}\ (n/n) = 1$

Hence $$\log P = \int_0^1 \log(1+x)\,dx = \int_0^1 \log(1+x)\ .\ 1\,dx$$ [Integrate by parts]

$$= \left|\log(1+x).x\right|_0^1 - \int_0^1 \frac{1}{1+x}.x\,dx$$

$$= \log 2 - \int_0^1 \frac{1+x-1}{1+x}\,dx = \log 2 - \int_0^1 dx + \int_0^1 \frac{dx}{1+x}$$

$$= \log 2 - \left|x\right|_0^1 + \left|\log(1+x)\right|_0^1 = \log 2 - 1 + \log 2$$

$$= \log 2^2 - \log_e e = \log(4/e). \text{ Hence, } P = 4/e.$$

PROBLEMS 6.5

Find the limit, as $n \to \infty$, of the series :

1. $\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\ldots+\frac{1}{2n}$. *(Bhopal, 2009)*
2. $\frac{1}{n^3+1}+\frac{4}{n^3+8}+\frac{9}{n^3+27}+\ldots+\frac{r^2}{n^3+r^3}+\ldots+\frac{1}{2n}$.
3. $\underset{n\to\infty}{\text{Lt}}\ \frac{\sqrt{n}}{\sqrt{n^3}}+\frac{\sqrt{n}}{\sqrt{(n+3)^3}}+\frac{\sqrt{n}}{\sqrt{(n+6)^3}}+\ldots+\frac{\sqrt{n}}{\sqrt{[n+3(n-1)]^3}}$.

Evaluate :

4. $\underset{n\to\infty}{\text{Lt}}\ \sum_{r=1}^{n-1}\frac{1}{\sqrt{(n^2-r^2)}}$ *(Bhopal, 2008)*
5. $\underset{n\to\infty}{\text{Lt}}\ \frac{[(n+1)(n+2)\ldots(n+n)]^{1/n}}{n}$.
6. $\underset{n\to\infty}{\text{Lt}}\left\{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right)\ldots\left(1+\frac{n^2}{n^2}\right)\right\}^{1/n}$ *(Bhopal, 2008)*

6.10 AREAS OF CARTESIAN CURVES

(1) *Area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$ is* $\int_a^b$ **y dx.**

Let AB be the curve $y = f(x)$ between the ordinates LA $(x = a)$ and MB $(x = b)$. (Fig. 6.1)

Let $P(x, y)$, $P'(x + \delta x, y + \delta y)$ be two neighbouring points on the curve and NP, $N'P'$ be their respective ordinates.

Let the area $ALNP$ be A, which depends on the position of P whose abscissa is x. Then the area $PNN'P' = \delta A$.

Complete the rectangles PN' and $P'N'$.

Then the area $PNN'P'$ lies between the areas of the rectangles PN' and $P'N$.

i.e., δA lies between $y\delta x$ and $(y + \delta y)\,\delta x$

$\therefore$ $\dfrac{\delta A}{\delta x}$ lies between y and $y + \delta y$.

Now taking limits as $P' \to P$ *i.e.*, $\delta x \to 0$ (and $\therefore$ $\delta y \to 0$),

$$dA/dx = y$$

Integrating both sides between the limits $x = a$ and $x = b$, we have

$$|\,A\,|_a^b = \int_a^b y\,dx$$

or (value of A for $x = b$) – (value of A for $x = a$) = $\int_a^b y\,dx$

Thus area $ALMB = \int_a^b y\,dx$.

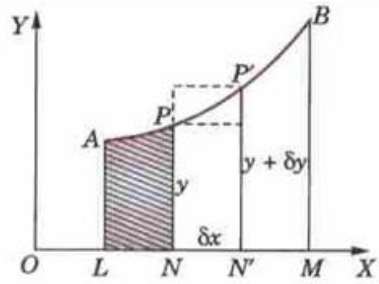

Fig. 6.1

(2) Interchanging x and y in the above formula, we see that *the area bounded by the curve $x = f(y)$, the y-axis and the abscissae $y = a$, $y = b$ is* $\int_a^b$ **x dy.** (Fig. 6.2)

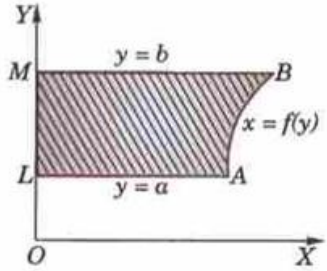

Fig. 6.2

Obs. 1. The area bounded by a curve, the x-axis and two ordinates is called the **area under the curve.** The process of finding the area of plane curves is often called **quadrature.**

Obs. 2. Sign of an area. *An area whose boundary is described in the anti-clockwise direction is considered positive and an area whose boundary is described in the clockwise direction is taken as negative.*

In Fig. 6.3, the area $ALMB$ $\left(= \int_a^b y\,dx\right)$ which is described in the anti-clockwise direction and lies above the x-axis, will give a positive result.

In Fig. 6.4, the area $ALMB$ $\left(= \int_a^b y\,dx\right)$ which is described in the clockwise direction and lies below the x-axis, will give a negative result.

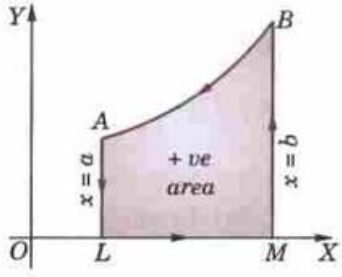

Fig. 6.3

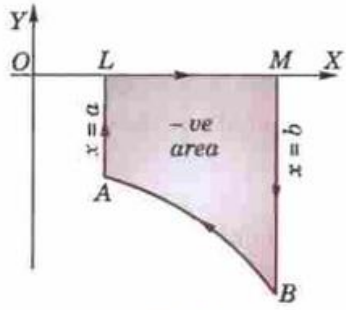

Fig. 6.4

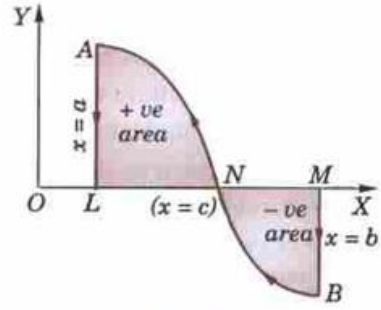

Fig. 6.5

In Fig. 6.5, the area $ALMB$ $\left(= \int_a^b y\,dx\right)$ will not consist of the sum of the area ALN $\left(= \int_a^c y\,dx\right)$ and the area NMB $\left(= \int_c^b y\,dx\right)$, but their difference.

Thus to find the total area in such cases the numerical value of the area of each portion must be evaluated separately and their results added afterwards.

Example 6.20. *Find the area of the loop of the curve $ay^2 = x^2(a - x)$.* (*S.V.T.U., 2009 ; Osmania, 2000*)

Solution. Let us trace the curve roughly to get the limits of integration.

(*i*) The curve is symmetrical about x-axis.

(*ii*) It passes through the origin. The tangents at the origin are $ay^2 = ax^2$ or $y = \pm x$. $\therefore$ Origin is a node.

(*iii*) The curve has no asymptotes.

(*iv*) The curve meets the x-axis at (0, 0) and $(a, 0)$. It meets the y-axis at (0, 0) only.

From the equation of the curve, we have $y = \dfrac{x}{\sqrt{a}}\sqrt{(a-x)}$

For $x > a$, y is imaginary. Thus no portion of the curve lies to the right of the line $x = a$. Also $x \to -\infty$, $y \to \infty$.

Thus the curve is as shown in Fig. 6.6.

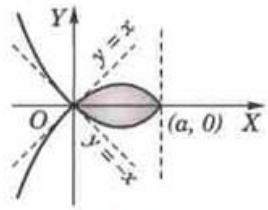

Fig. 6.6

$\therefore$ Area of the loop = 2 (area of upper half of the loop)

$$= 2\int_0^a y\,dx = 2\int_0^a x\sqrt{\left(\frac{a-x}{a}\right)}\,dx = \frac{2}{\sqrt{a}}\int_0^a [a-(a-x)]\sqrt{(a-x)}\,dx$$

$$= \frac{2}{\sqrt{a}}\int_0^a [a(a-x)^{1/2} - (a-x)^{3/2}]\,dx = 2\sqrt{a}\left|\frac{(a-x)^{3/2}}{-3/2}\right|_0^a - \frac{2}{\sqrt{a}}\left|\frac{(a-x)^{5/2}}{-5/2}\right|_0^a$$

$$= -\frac{4}{3}\sqrt{a}(0 - a^{3/2}) + \frac{4}{5\sqrt{a}}(0 - a^{5/2}) = \frac{4}{3}a^2 - \frac{4}{5}a^2 = \frac{8}{15}a^2.$$

Example 6.21. *Find the area included between the curve $y^2(2a-x) = x^3$ and its asymptote.* (*V.T.U., 2003*)

Solution. The curve is as shown in Fig. 4.23.

Area between the curve and the asymptote

$$= 2\int_0^{2a} y\,dx = 2\int_0^{2a}\sqrt{\left(\frac{x^3}{2a-x}\right)}\,dx$$

Put $x = 2a\sin^2\theta$ so that $dx = 4a\sin\theta\cos\theta\,d\theta$

$$= 2\int_0^{\pi/2}\sqrt{\left\{\frac{(2a\sin^2\theta)^3}{2a\cos^2\theta}\right\}}\cdot 4a\sin\theta\cos\theta\,d\theta$$

$$= 16a^2\int_0^{\pi/2}\sin^4\theta\,d\theta = 16a^2\cdot\frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2} = 3\pi a^2.$$

Example 6.22. *Find the area enclosed by the curve $a^2x^2 = y^3(2a-y)$.*

Solution. Let us first find the limits of integration.

(*i*) The curve is symmetrical about y-axis.

(*ii*) It passes through the origin and the tangents at the origin are $x^2 = 0$ or $x = 0, x = 0$.

$\therefore$ There is a cusp at the origin.

(*iii*) The curve has no asymptote.

(*iv*) The curve meets the x-axis at the origin only and meets the y-axis at $(0, 2a)$. From the equation of the curve, we have

$$x = \frac{y}{a}\sqrt{[y(2a-y)]}$$

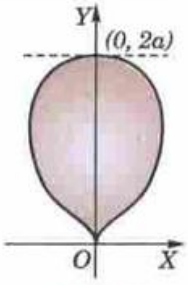

Fig. 6.7

For $y < 0$ or $y > 2a$, x is imaginary. Thus the curve entirely lies between $y = 0$ (x-axis) and $y = 2a$, which is shown in Fig. 6.7.

$\therefore$ Area of the curve $= 2\displaystyle\int_0^{2a} x\,dy = \frac{2}{a}\int_0^{2a} y\sqrt{[y(2a-y)]}\,dy$

Put $y = 2a\sin^2\theta$ $\therefore$ $dy = 4a\sin\theta\cos\theta\,d\theta$

$$= \frac{2}{a}\int_0^{\pi/2} 2a\sin^2\theta\sqrt{[2a\sin^2\theta(2a - 2a\sin^2\theta)]}\times 4a\sin\theta\cos\theta\,d\theta$$

$$= 32a^2\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta = 32a^2\,\frac{3\cdot 1\times 1}{6\cdot 4\cdot 2}\cdot\frac{\pi}{2} = \pi a^2.$$

Example 6.23. *Find the area enclosed between one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$; and its base.* (V.T.U., 2000)

Solution. To describe its first arch, θ varies from 0 to 2π *i.e.*, x varies from 0 to $2a\pi$ (Fig. 6.8).

$\therefore$ Required area $= \int_{x=0}^{2\pi a} y\,dx$

where $y = a(1 - \cos\theta)$, $dx = a(1 - \cos\theta)\,d\theta$.

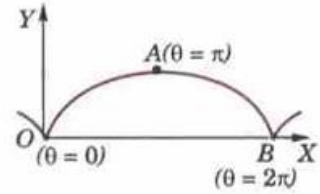

Fig. 6.8

$$= \int_{\theta=0}^{\pi/2} a(1 - \cos\theta)\cdot a(1 - \cos\theta)\,d\theta$$

$$= 2a^2 \int_0^{\pi} (1 - \cos\theta)^2\,d\theta = 8a^2 \int_0^{\pi} \sin^4 \frac{\theta}{2}\,d\theta$$

$$= 16a^2 \int_0^{\pi/2} \sin^4\phi\,d\phi, \text{ putting } \theta/2 = \phi \text{ so that } d\theta = 2d\phi.$$

$$= 16a^2 \cdot \frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2} = 3\pi a^2.$$

Example 6.24. *Find the area of the tangent cut off from the parabola $x^2 = 8y$ by the line $x - 2y + 8 = 0$.*

Solution. Given parabola is $x^2 = 8y$ ...(*i*)

and the straight line is $x - 2y + 8 = 0$ ...(*ii*)

Substituting the value of y from (*ii*) in (*i*), we get

$$x^2 = 4(x + 8) \text{ or } x^2 - 4x - 32 = 0$$

or $(x - 8)(x + 4) = 0 \quad \therefore \quad x = 8, -4.$

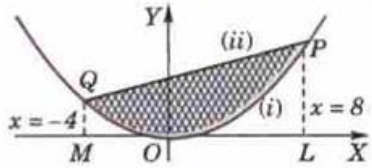

Fig. 6.9

Thus (*i*) and (*ii*) intersect at P and Q where $x = 8$ and $x = -4$. (Fig. 6.9)

$\therefore$ Required area POQ (*i.e.*, dotted area) = area bounded by straight line (*ii*) and x-axis from $x = -4$ to $x = 8$ – area bounded by parabola (*i*) and x-axis from $x = -4$ to $x = 8$.

$$= \int_{-4}^{8} y\,dx, \text{ from } (ii) - \int_{-4}^{8} y\,dx, \text{ from } (i)$$

$$= \int_{-4}^{8} \frac{x+8}{2}\,dx - \int_{-4}^{8} \frac{x^2}{8}\,dx = \frac{1}{2}\left|\frac{x^2}{2} + 8x\right|_{-4}^{8} - \frac{1}{8}\left|\frac{x^3}{3}\right|_{-4}^{8}$$

$$= \frac{1}{2}\{(32 + 64) - (-24)\} - \frac{1}{24}(512 + 64) = 36.$$

Example 6.25. *Find the area common to the parabola $y^2 = ax$ and the circle $x^2 + y^2 = 4ax$.*

Solution. Given parabola is $y^2 = ax$ ...(*i*)

and the circle is $x^2 + y^2 = 4ax.$ ...(*ii*)

Both these curves are symmetrical about x-axis. Solving (*i*) and (*ii*) for x, we have

$$x^2 + ax = 4ax \text{ or } x(x - 3a) = 0$$

or $x = 0, 3a.$

Thus the two curves intersect at the points where $x = 0$ and $x = 3a$. (Fig. 6.10).

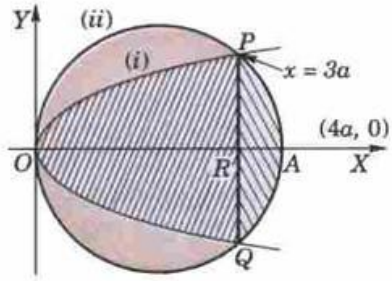

Fig. 6.10

Also (*ii*) meets the x-axis at $A(4a, 0)$.

Area common to (*i*) and (*ii*) *i.e.*, the shaded area

$$= 2[\text{Area } ORP + \text{Area } PRA] \qquad \text{(By symmetry)}$$

$$= 2\left[\int_0^{3a} y\,dx, \text{ from } (i) + \int_{3a}^{4a} y\,dx, \text{ from } (ii)\right]$$

$$= 2\left[\int_0^{3a} \sqrt{(ax)}\,dx + \int_{3a}^{4a} \sqrt{(4ax - x^2)}\,dx\right]$$

$$= 2\sqrt{a}\left|\frac{x^{3/2}}{3/2}\right|_0^{3a} + 2\int_{3a}^{4a}\sqrt{[4a^2-(x-2a)^2]}\,dx$$

$$= \frac{4\sqrt{a}}{3}(3a)^{3/2} + 2\left[\frac{1}{2}(x-2a)\sqrt{[4a^2-(x-2a)^2]} + \frac{4a^2}{2}\sin^{-1}\frac{x-2a}{2a}\right]_{3a}^{4a}$$

$$= 4\sqrt{3}\,a^2 + 2[\{0 - \frac{1}{2}a\sqrt{3}a\} + 2a^2\{\pi/2 - \pi/6\}]$$

$$= 4\sqrt{3}\,a^2 - \sqrt{3}a^2 + \frac{4}{3}\pi a^2 = \left(3\sqrt{3} + \frac{4}{3}\pi\right)a^2.$$

PROBLEMS 6.6

1. (*i*) Find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. *(Kerala, 2005 ; V.T.U., 2003 S)*

   (*ii*) Find the area bounded by the parabola $y^2 = 4ax$ and its latus-rectum.
2. Find the area bounded by the curve $y = x(x-3)(x-5)$ and the x-axis.
3. Find the area included between the curve $ay^2 = x^3$, the x-axis and the ordinates $x = a$.
4. Find the area of the loop of the curve :

   (*i*) $3ay^2 = x(x-a)^2$ *(Rajasthan, 2005)* (*ii*) $x(x^2+y^2) = a(x^2-y^2)$ *(P.T.U., 2010)*
5. Find the whole area of the curve :

   (*i*) $a^2x^2 = y^3(2a-y)$ *(Nagpur, 2009)* (*ii*) $8a^2y^2 = x^2(a^2-x^2)$ *(V.T.U., 2006)*
6. Find the area included between the curve and its asymptotes in each case :

   (*i*) $xy^2 = a^2(a-x)$. *(V.T.U., 2003)* (*ii*) $x^2y^2 = a^2(y^2-x^2)$. *(V.T.U., 2007)*
7. Show that the area of the loop of the curve $y^2(a+x) = x^2(3a-x)$ is equal to the area between the curve and its asymptote.
8. Find the whole area of the *astroid* $x^{2/3}+y^{2/3}=a^{2/3}$ or $x = a\cos^3\theta$, $y = a\sin^3\theta$. *(V.T.U., 2005)*
9. Find the area bounded by the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ and the coordinate axes.
10. Find the area included between the *cycloid* $x = a(\theta+\sin\theta)$, $y = a(1-\cos\theta)$ *and its base*. Also find the area between the curve and the x-axis. *(Gorakhpur, 1999)*
11. Find the area common to the circles $x^2+y^2=4$ and $x^2+y^2=4x$.
12. Prove that the area common to the parabolas $x^2 = 4ay$ and $y^2 = 4ax$ is $16a^2/3$. *(S.V.T.U., 2008 ; Kurukshetra, 2005)*
13. Find the area included between the circle $x^2+y^2=2ax$ and the parabola $y^2 = ax$.
14. Find the area bounded by the parabola $y^2 = 4ax$ and the line $x+y=3a$.
15. Find the area of the segment cut off from the parabola $y = 4x - x^2$ by the straight line $y = x$. *(V.T.U., 2010 ; S.V.T.U., 2008)*

(2) Areas of polar curves. *Area bounded by the curve $r = f(\theta)$ and the radii vectors*

$$\theta = \alpha,\ \theta = \beta \text{ is } \frac{1}{2}\int_\alpha^\beta \mathbf{r}^2\,\mathbf{d}\theta.$$

Let AB be the curve $r = f(\theta)$ between the radii vectors $OA(\theta=\alpha)$ and $OB(\theta=\beta)$. Let $P(r,\theta)$, $P'(r+\delta r, \theta+\delta\theta)$ be any two neighbouring points on the curve. (Fig. 6.11)

Let the area $OAP = A$ which is a function of θ. Then the area $OPP' = \delta A$. Mark circular arcs PQ and $P'Q'$ with centre O and radii OP and OP'.

Evidently area OPP' lies between the sectors OPQ and $OP'Q'$ *i.e.*, δA lies between $\frac{1}{2}r^2\,\delta\theta$ and $\frac{1}{2}(r+\delta r)^2\,\delta\theta$.

$\therefore$ $\frac{\delta A}{\delta\theta}$ lies between $\frac{1}{2}r^2$ and $\frac{1}{2}(r+\delta r)^2$.

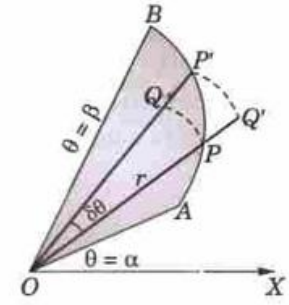

Fig. 6.11

Now taking limits as $\delta\theta \to 0$ ($\therefore$ $\delta r \to 0$), $\dfrac{dA}{d\theta} = \dfrac{1}{2} r^2$

Integrating both sides from $\theta = \alpha$ to $\theta = \beta$, we get $\mid A \mid_\alpha^\beta = \int_\alpha^\beta \frac{1}{2} r^2 \, d\theta$

or (value of A for $\theta = \beta$) – (value of A for $\theta = \alpha$) $= \dfrac{1}{2} \displaystyle\int_\alpha^\beta r^2 \, d\theta$

Hence the required area $OAB = \dfrac{1}{2} \displaystyle\int_\alpha^\beta r^2 \, d\theta$.

Example 6.26. *Find the area of the cardioid $r = a(1 - \cos\theta)$.* (V.T.U., 2004)

Solution. The curve is as shown in Fig. 6.12. Its upper half is traced from $\theta = 0$ to $\theta = \pi$.

$\therefore$ Area of the curve $= 2 \cdot \dfrac{1}{2} \displaystyle\int_0^\pi r^2 \, d\theta = a^2 \int_0^\pi (1 - \cos\theta)^2 \, d\theta$

$= a^2 \displaystyle\int_0^\pi (2 \sin^2 \theta/2)^2 \, d\theta = 4a^2 \int_0^\pi \sin^4 \theta/2 \cdot d\theta$

$= 8a^2 \displaystyle\int_0^{\pi/2} \sin^4 \phi \, d\phi$, putting $\theta/2 = \phi$ and $d\theta = 2d\phi$.

$= 8a^2 \cdot \dfrac{3 \cdot 1}{4 \cdot 2} \cdot \dfrac{\pi}{2} = \dfrac{3\pi a^2}{2}$.

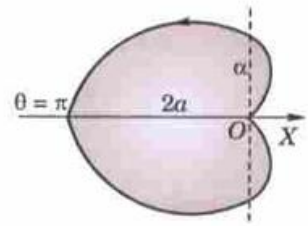

Fig. 6.12

Example 6.27. *Find the area of a loop of the curve $r = a \sin 3\theta$.*

Solution. The curve is as shown in Fig. 4.35. It consists of three loops.

Putting $r = 0$, $\sin 3\theta = 0$ $\therefore$ $3\theta = 0$ or π *i.e.*, $\theta = 0$ or $\pi/3$ which are the limits for the first loop.

$\therefore$ Area of a loop $= \dfrac{1}{2} \displaystyle\int_0^{\pi/3} r^2 \, d\theta = \dfrac{1}{2} a^2 \int_0^{\pi/3} \sin^2 3\theta \, d\theta = \dfrac{a^2}{4} \int_0^{\pi/3} (1 - \cos 6\theta) \, d\theta$

$= \dfrac{a^2}{4} \left| \theta - \dfrac{\sin 6\theta}{6} \right|_0^{\pi/3} = \dfrac{a^2}{4} \left(\dfrac{\pi}{3} - 0 \right) = \dfrac{\pi a^2}{12}$.

Obs. The limits of integration for a loop of $r = a \sin n\theta$ or $r = a \cos n\theta$ are the two consecutive values of θ when $r = 0$.

Example 6.28. *Prove that the area of a loop of the curve $x^3 + y^3 = 3axy$ is $3a^2/2$.*

Solution. Changing to polar form (by putting $x = r\cos\theta$, $y = r\sin\theta$), $r = \dfrac{3a \sin\theta \cos\theta}{\cos^3\theta + \sin^3\theta}$

Putting $r = 0$, $\sin\theta\cos\theta = 0$.

$\therefore$ $\theta = 0, \pi/2$, which are the limits of integration for its loop.

$\therefore$ Area of the loop

$= \dfrac{1}{2} \displaystyle\int_0^{\pi/2} r^2 \, d\theta = \dfrac{1}{2} \int_0^{\pi/2} \dfrac{9a^2 \sin^2\theta \cos^2\theta}{(\cos^3\theta + \sin^3\theta)^2} \, d\theta$

$= \dfrac{9a^2}{2} \displaystyle\int_0^{\pi/2} \dfrac{\tan^2\theta \sec^2\theta}{(1 + \tan^3\theta)^2} \, d\theta$ [Dividing num. and denom. by $\cos^6\theta$]

$= \dfrac{3a^2}{2} \displaystyle\int_1^\infty \dfrac{dt}{t^2}$, putting $1 + \tan^3\theta = t$ and $3\tan^2\theta\sec^2\theta \, d\theta = dt$.

$= \dfrac{3a^2}{2} \left| \dfrac{t^{-1}}{-1} \right|_1^\infty = \dfrac{3a^2}{2}(-0 + 1) = \dfrac{3a^2}{2}$.

Example 6.29. *Find the area common to the circles*

$$r = a\sqrt{2} \text{ and } r = 2a\cos\theta$$

Solution. The equations of the circles are $r = a\sqrt{2}$ ...(*i*) and $r = 2a\cos\theta$ ...(*ii*)

(*i*) represents a circle with centre at (0, 0) and radius $a\sqrt{2}$. (*ii*) represents a circle symmetrical about OX, with centre at $(a, 0)$ and radius a.

The circles are shown in Fig. 6.13. At their point of intersection P, eliminating r from (*i*) and (*ii*),

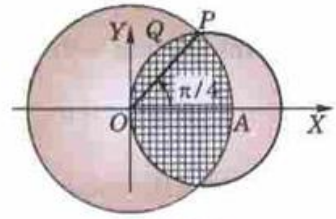

Fig. 6.13

$$a\sqrt{2} = 2a\cos\theta \text{ i.e., } \cos\theta = 1\sqrt{2}$$

or $$\theta = \pi/4$$

$\therefore$ Required area $= 2 \times$ area $OAPQ$ (By symmetry)

$$= 2(\text{area } OAP + \text{area } OPQ)$$

$$= 2\left[\frac{1}{2}\int_0^{\pi/4} r^2\, d\theta, \text{ for } (i) + \frac{1}{2}\int_{\pi/4}^{\pi/2} r^2\, d\theta, \text{ for } (ii)\right]$$

$$= \int_0^{\pi/4} (a\sqrt{2})^2\, d\theta + \int_{\pi/4}^{\pi/2} (2a\cos\theta)^2\, d\theta = 2a^2 \,|\, \theta \,|_0^{\pi/4} + 4a^2 \int_{\pi/4}^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= 2a^2(\pi/4 - 0) + 2a^2 \left| \theta + \frac{\sin 2\theta}{2} \right|_{\pi/4}^{\pi/2} = \frac{\pi a^2}{2} + 2a^2\left(\frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{2}\right) = a^2(\pi - 1).$$

Example 6.30. *Find the area common to the cardioids $r = a(1 + \cos\theta)$ and $r = a(1 - \cos\theta)$.*
(Kurukshetra, 2006 ; V.T.U., 2006)

Solution. The cardioid $r = a(1 + \cos\theta)$ is $ABCOB'A$ and the cardioid $r = a(1 - \cos\theta)$ is $OC'BA'B'O$. Both the cardioids are symmetrical about the initial line OX and intersect at B and B' (Fig. 6.14)

$\therefore$ Required area (shaded) = 2 area $OC'BCO$

$$= 2\,[\text{area } OC'BO + \text{area } OBCO]$$

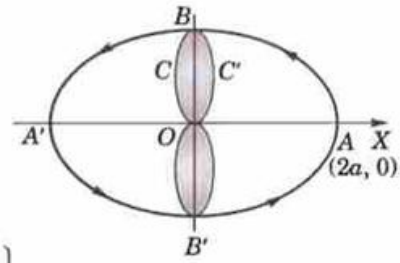

Fig. 6.14

$$= 2\left[\left\{\int_0^{\pi/2} \frac{1}{2}r^2\, d\theta\right\}_{r=a(1-\cos\theta)} + \left\{\int_{\pi/2}^{\pi} \frac{1}{2}r^2\, d\theta\right\}_{r=a(1+\cos\theta)}\right]$$

$$= a^2\int_0^{\pi/2} (1-\cos\theta)^2\, d\theta + a^2\int_{\pi/2}^{\pi} (1+\cos\theta)^2\, d\theta$$

$$= a^2\left\{\int_0^{\pi/2} (1 - 2\cos\theta + \cos^2\theta)\, d\theta + \int_{\pi/2}^{\pi} [1 + 2\cos\theta + \cos^2\theta]\, d\theta\right\}$$

$$= a^2\left\{\int_0^{\pi} (1+\cos^2\theta)\, d\theta - 2\int_0^{\pi/2} \cos\theta\, d\theta + 2\int_{\pi/2}^{\pi} \cos\theta\, d\theta\right\}$$

$$= a^2\left\{\int_0^{\pi}\left(1 + \frac{1+\cos 2\theta}{2}\right) d\theta - 2\,|\sin\theta\,|_0^{\pi/2} + 2\,|\sin\theta\,|_{\pi/2}^{\pi}\right\}$$

$$= a^2\left\{\left|\frac{3}{2}\theta + \frac{\sin 2\theta}{4}\right|_0^{\pi} - 2(1-0) + 2(0-1)\right\} = \left(\frac{3\pi}{2} - 4\right)a^2.$$

PROBLEMS 6.7

1. Find the whole area of
   (*i*) the *cardioid* $r = a(1 + \cos\theta)$ (*V.T.U., 2008*) (*ii*) the *lemniscate* $r^2 = a^2\cos 2\theta$; (*V.T.U., 2006*)
2. Find the area of one loop of the curve
   (*i*) $r = a\sin 2\theta$. (*ii*) $r = a\cos 3\theta$.
3. Show that the area included between the folium $x^3 + y^3 = 3axy$ and its asymptote is equal to the area of loop.
4. Prove that the area of the loop of the curve $x^3 + y^3 = 3axy$ is three times the area of the loop of the curve $r^2 = a^2\cos 2\theta$.
5. Find the area inside the circle $r = a\sin\theta$ and lying outside the *cardioid* $r = a(1 - \cos\theta)$. (*Anna, 2009*)
6. Find the area outside the *circle* $r = 2a\cos\theta$ and inside the *cardioid* $r = a(1 + \cos\theta)$. (*Kurukshetra, 2006*)

6.11 LENGTHS OF CURVES

(1) *The length of the arc of the curve $y = f(x)$ between the points where $x = a$ and $x = b$ is*

$$\int_a^b \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\,dx$$

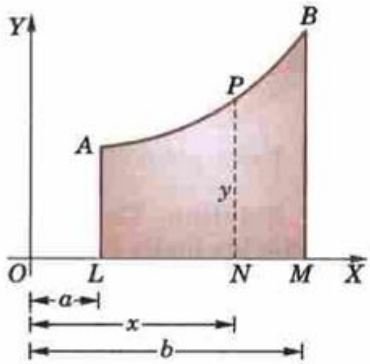

Fig. 6.15

Let AB be the curve $y = f(x)$ between the points A and B where $x = a$ and $x = b$ (Fig. 6.15)

Let $P(x, y)$ be any point on the curve and arc $AP = x$ so that it is a function of x.

Then $$\frac{ds}{dx} = \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]} \qquad \text{[(1) of p. 164]}$$

$$\therefore \quad \int_a^b \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\,dx = \int_a^b \frac{ds}{dx}\cdot dx = \left| s \right|_{x=a}^{x=b}$$

$$= (\text{value of } s \text{ for } x = b) - (\text{value of } s \text{ for } x = a) = \text{arc } AB - 0$$

Hence, the arc $AB = \displaystyle\int_a^b \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\,dx.$

(2) *The length of the arc of the curve $x = f(y)$ between the points where $y = a$ and $y = b$, is*

$$\int_a^b \sqrt{\left[1+\left(\frac{dx}{dy}\right)^2\right]}\,dy \qquad \text{[Use (2) of p. 165]}$$

(3) *The length of the arc of the curve $x = f(t)$, $y = \phi(t)$ between the points where $t = a$ and $t = b$, is*

$$\int_a^b \sqrt{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]}\,dt. \qquad \text{[Use (3) p. 165]}$$

(4) *The length of the arc of the curve $r = f(\theta)$ between the points where $\theta = \alpha$ and $\theta = \beta$, is*

$$\int_\alpha^\beta \sqrt{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]}\,d\theta. \qquad \text{[Use (1) of p. 165]}$$

(5) *The length of the arc of the curve $\theta = f(r)$ between the points where $r = a$ and $r = b$, is*

$$\int_a^b \sqrt{\left[1+\left(r\frac{d\theta}{dr}\right)^2\right]}\,dr \qquad \text{[Use (2) of p. 166]}$$

Example 6.31. *Find the length of the arc of the parabola $x^2 = 4ay$ measured from the vertex to one extremity of the latus-rectum.* *(Delhi, 2002)*

Solution. Let A be the vertex and L an extremity of the latus-rectum so that at A, $x = 0$ and at L, $x = 2a$. (Fig. 6.16).

Now $$y = x^2/4a \text{ so that } \frac{dy}{dx} = \frac{1}{4a}\cdot 2x = \frac{x}{2a}$$

$$\therefore \quad \text{arc } AL = \int_0^{2a} \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\,dx$$

$$= \int_0^{2a} \sqrt{\left[1+\left(\frac{x}{2a}\right)^2\right]}\,dx = \frac{1}{2a}\int_0^{2a} \sqrt{[(2a)^2+x^2]}\,dx$$

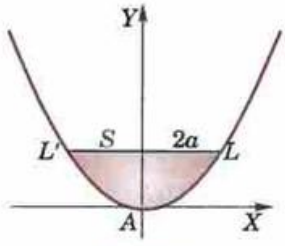

Fig. 6.16

$$= \frac{1}{2a}\left[\frac{x\sqrt{[(2a)^2 + x^2]}}{2} + \frac{(2a)^2}{2}\sinh^{-1}\frac{x}{2a}\right]_0^{2a} = \frac{1}{2a}\left[\frac{2a\sqrt{(8a)^2}}{2} + 2a^2\sinh^{-1}1\right]$$

$$= a[\sqrt{2} + \sinh^{-1}1] = a[\sqrt{2} + \log(1+\sqrt{2})] \qquad [\because \ \sinh^{-1}x = \log[x + \sqrt{(1+x^2)}]$$

Example 6.32. *Find the perimeter of the loop of the curve $3ay^2 = x(x-a)^2$.*

Solution. The curve is symmetrical about the x-axis and the loop lies between the limits $x = 0$ and $x = a$. (Fig. 6.17).

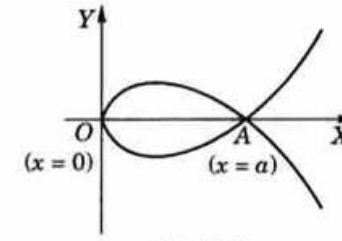

Fig. 6.17

We have $$y = \frac{\sqrt{x(x-a)}}{\sqrt{(3a)}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{(3a)}}\left[\frac{3}{2}x^{1/2} - \frac{a}{2}\cdot x^{-1/2}\right] = \frac{1}{2\sqrt{(3a)}}\,\frac{3x-a}{\sqrt{x}}$$

$\therefore$ Perimeter of the loop $= 2\int_0^a \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}\,dx$ (By symmetry)

$$= 2\int_0^a \sqrt{\left[1 + \frac{(3x-a)^2}{12ax}\right]}\,dx = 2\int_0^a \frac{\sqrt{(9x^2 + 6ax + a^2)}}{\sqrt{(12ax)}}\,dx$$

$$= \frac{1}{\sqrt{(3a)}}\int_0^a \frac{3x+a}{\sqrt{x}}\,dx = \frac{1}{\sqrt{(3a)}}\int_0^a (3x^{1/2} + ax^{-1/2})\,dx$$

$$= \frac{1}{\sqrt{(3a)}}\left|\frac{3x^{3/2}}{3/2} + a\frac{x^{1/2}}{1/2}\right|_0^a = \frac{1}{\sqrt{(3a)}}(4a^{3/2}) = \frac{4a}{\sqrt{3}}.$$

Example 6.33. *Find the length of one arch of the cycloid*
$x = a(t - \sin t)$, $y = a(1 - \cos t)$. *(P.T.U., 2009 ; V.T.U., 2004)*

Solution. As a point moves from one end O to the other end of its first arch, the parameter t increases from 0 to 2π. [see Fig. 6.8]

Also $$\frac{dx}{dt} = a(1 - \cos t), \ \frac{dy}{dt} = a\sin t.$$

$\therefore$ Length of an arch $$= \int_0^{2\pi}\sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]}\,dx$$

$$= \int_0^{2\pi}\sqrt{\{[a(1-\cos t)]^2 + (a\sin t)^2\}}\,dt = a\int_0^{2\pi}\sqrt{[2(1-\cos t)]}\,dt$$

$$= 2a\int_0^{2\pi}\sin t/2\,dt = 2a\left|-\frac{\cos t/2}{1/2}\right|_0^{2\pi} = 4a[(-\cos\pi) - (-\cos 0)] = 8a.$$

Example 6.34. *Find the entire length of the cardioid $r = a(1 + \cos\theta)$.*
(P.T.U., 2010 ; Bhopal, 2008 ; Kurukshetra, 2005)
Also show that the upper half is bisected by $\theta = \pi/3$. *(Bhillai, 2005)*

Solution. The cardioid is symmetrical about the initial line and for its upper half, θ increases from 0 to π (Fig. 6.18)

Also $$\frac{dr}{d\theta} = -a\sin\theta.$$

$\therefore$ Length of the curve $$= 2\int_0^{\pi}\sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}\,d\theta$$

$$= 2\int_0^{\pi} \sqrt{\{[a(1+\cos\theta)]^2 + (-a\sin\theta)^2\}}\, d\theta = 2a\int_0^{\pi} \sqrt{[2(1+\cos\theta)]}\, d\theta$$

$$= 4a\int_0^{\pi} \cos\theta/2\, d\theta = 4a\left|\frac{\sin\theta/2}{1/2}\right|_0^{\pi} = 8a(\sin\pi/2 - \sin 0) = 8a.$$

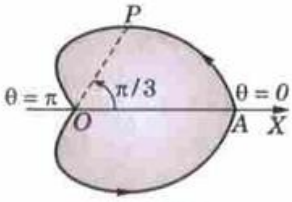

Fig. 6.18

$\therefore$ Length of upper half of the curve is $4a$. Also length of the arc AP from 0 to $\pi/3$.

$$= a\int_0^{\pi/3} \sqrt{[2(1+\cos\theta)]}\, d\theta = 2a\int_0^{\pi/3} \cos\theta/2 \cdot d\theta$$

$$= 4a\left|\sin\theta/2\right|_0^{\pi/3} = 2a = \text{half the length of upper half of the cardioid.}$$

PROBLEMS 6.8

1. Find the length of the arc of the *semi-cubical parabola* $ay^2 = x^3$ from the vertex to the ordinate $x = 5a$.
2. Find the length of the curve (*i*) $y = \log\sec x$ from $x = 0$ to $x = \pi/3$. *(V.T.U., 2010 S ; P.T.U., 2007)*
   (*ii*) $y = \log[(e^x - 1)/(e^x + 1)]$ from $x = 1$ to $x = 2$.
3. Find the length of the arc of the parabola $y^2 = 4ax$ (*i*) from the vertex to one end of the latus-rectum.
   (*ii*) cut off by the line $3y = 8x$. *(V.T.U., 2008 S ; Mumbai, 2006)*
4. Find the perimeter of the loop of the following curves :
   (*i*) $ay^2 = x^2(a - x)$ (*ii*) $9y^2 = (x-2)(x-5)^2$.
5. Find the length of the curve $y^2 = (2x-1)^2$ cut off by the line $x = 4$. *(V.T.U., 2000 S)*
6. Show that the whole length of the curve $x^2(a^2 - x^2) = 8a^2y^2$ is $\pi a\sqrt{2}$.
7. (*a*) Find the length of an arch of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$.
   (*b*) By finding the length of the curve show that the curve $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, is divided in the ratio 1 : 3 at $\theta = 2\pi/3$. *(S.V.T.U., 2009)*
8. Find the whole length of the curve $x = a\cos^3 t$, $y = a\sin^3 t$ *i.e.*, $x^{2/3} + y^{2/3} = a^{2/3}$.
   (V.T.U., 2010 ; Marathwada, 2008 ; Rajasthan, 2006)
   Also show that the line $\theta = \pi/3$ divides the length of this *astroid* in the first quadrant in the ratio 1 : 3.
   (Mumbai, 2001)
9. Find the length of the loop of the curve $x = t^2$, $y = t - t^3/3$. *(Mumbai, 2001)*
10. For the curve $r = ae^{\theta}\cot\alpha$, prove that s/r = constant, s being measured from the origin.
11. Find the length of the curve $\theta = \frac{1}{2}\left(r + \frac{1}{r}\right)$ from $r = 1$ to $r = 3$. *(Marathwada, 2008)*
12. Find the perimeter of the *cardioid* $r = a(1 - \cos\theta)$. Also show that the upper half of the curve is bisected by the line $\theta = 2\pi/3$.
13. Find the whole length of the *lemniscate* $r^2 = a^2\cos 2\theta$.
14. Find the length of the parabola $r(1 + \cos\theta) = 2a$ as cut off by the latus-rectum. *(J.N.T.U., 2003)*

6.12 (1) VOLUMES OF REVOLUTION

(*a*) **Revolution about x-axis.** *The volume of the solid generated by the revolution about the x-axis, of the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$ is*

$$\int_a^b \pi y^2\, dx.$$

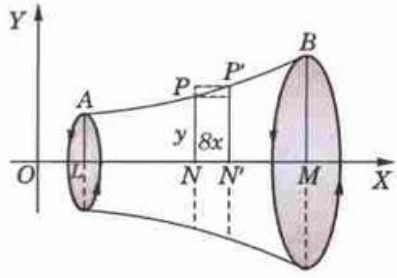

Fig. 6.19

Let AB be the curve $y = f(x)$ between the ordinates $LA(x = a)$ and $MB(x = b)$.

Let $P(x, y)$, $P'(x + \delta x, y + \delta y)$ be two neighbouring points on the curve and NP, $N'P'$ be their respective ordinates (Fig. 6.19).

Let the volume of the solid generated by the revolution about x-axis of the area $ALNP$ be V, which is clearly a function of x. Then the volume of the solid generated by the revolution of the area $PNN'P'$ is δV. Complete the rectangles PN' and $P'N$.

The δV lies between the volumes of the right circular cylinders generated by the revolution of rectangles PN' and $P'N$,

i.e., δV lies between $\pi y^2\,\delta x$ and $\pi(y+\delta y)^2\,\delta x$.

$\therefore$ $\dfrac{\delta V}{\delta x}$ lies between πy^2 and $\pi(y+\delta y)^2$.

Now taking limits as $P' \to P$, *i.e.*, $\delta x \to 0$ (and $\therefore$ $\delta y \to 0$), $\dfrac{dV}{dx} = \pi y^2$

$$\therefore \qquad \int_a^b \frac{dV}{dx}\,dx = \int_a^b \pi y^2\,dx \quad \text{or} \quad |V|_{x=a}^{b} = \int_a^b \pi y^2\,dx$$

or (value of V for $x = b$) – (value of V for $x = a$)

i.e., volume of the solid obtained by the revolution of the area $ALMB = \int_a^b \pi y^2\,dx$.

Example 6.35. *Find the volume of a sphere of radius a.* (S.V.T.U., 2007)

Solution. Let the sphere be generated by the revolution of the semi-circle ABC, of radius a about its diameter CA (Fig. 6.20)

Taking CA as the x-axis and its mid-point O as the origin, the equation of the circle ABC is $x^2 + y^2 = a^2$.

$\therefore$ Volume of the sphere = 2 (volume of the solid generated by the revolution about x-axis of the quadrant OAB)

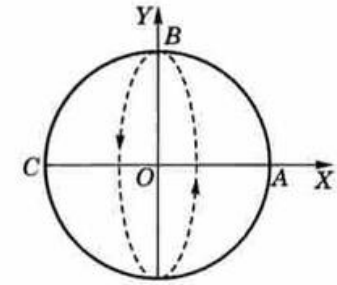

Fig. 6.20

$$= 2\int_0^a \pi y^2\,dx = 2\pi\int_0^b (a^2 - x^2)\,dx$$

$$= 2\pi\left|a^2x - \frac{x^3}{3}\right|_0^a = 2\pi\left[a^3 - \frac{a^3}{3} - (0-0)\right] = \frac{4}{3}\pi a^3.$$

Example 6.36. *Find the volume formed by the revolution of loop of the curve $y^2(a+x) = x^2(3a-x)$, about the x-axis.* (Marathwada, 2008)

Solution. The curve is symmetrical about the x-axis, and for the upper half of its loop x varies from 0 to $3a$ (Fig. 6.21)

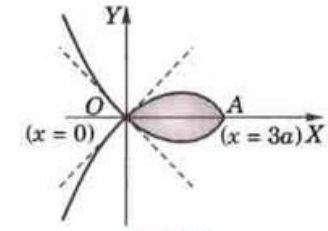

Fig. 6.21

$$\therefore \quad \text{Volume of the loop} = \int_0^{3a} \pi y^2\,dx = \pi\int_0^{3a} \frac{x^2(3a-x)}{a+x}\,dx$$

$$= \pi\int_0^{3a} \frac{-x^3 + 3ax^2}{x+a}\,dx$$

[Divide the numerator by the denominator]

$$= \pi\int_0^{3a}\left[-x^2 + 4ax - 4a^2 + \frac{4a^3}{x+a}\right]dx = \pi\left|-\frac{x^3}{3} + 4a\cdot\frac{x^2}{2} - 4a^2x + 4a^3\log(x+a)\right|_0^{3a}$$

$$= \pi\left[\frac{-27a^3}{3} + 2a\cdot 9a^2 - 4a^2\cdot 3a + 4a^3\log 4a - (4a^3\log a)\right]$$

$$= \pi a^3(-3 + 4\log 4) = \pi a^3(8\log 2 - 3).$$

Example 6.37. *Prove that the volume of the reel formed by the revolution of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ about the tangent at the vertex is $\pi^2 a^3$.* (V.T.U., 2003)

Solution. The arch AOB of the cycloid is symmetrical about the y-axis and the tangent at the vertex is the x-axis. For half the cycloid OA, θ varies from 0 to π. (Fig. 4.31).

Hence the required volume

$$= 2\int_{\theta=0}^{\theta=\pi} \pi y^2\,dx = 2\pi\int_0^{\pi} a^2(1-\cos\theta)^2 \,.\, a(1+\cos\theta)\,d\theta$$

$$= 2\pi a^3 \int_0^{\pi} (2\sin^2 \theta/2)^2 \cdot (2\cos^2 \theta/2)\, d\theta$$

$$= 16\pi a^3 \int_0^{\pi} \sin^4 \theta/2 \cdot \cos^2 \theta/2 \cdot d\theta \qquad [\text{Put } \theta/2 = \phi,\ d\theta = 2d\phi]$$

$$= 32\pi a^3 \int_0^{\pi/2} \sin^4 \phi \cos^2 \phi\, d\phi = 32\pi a^3 \cdot \frac{3.1 \times 1}{6.4.2} \cdot \frac{\pi}{2} = \pi^2 a^3.$$

Example 6.38. *Find the volume of the solid formed by revolving about x-axis, the area enclosed by the parabola $y^2 = 4ax$, its evolute $27ay^2 = 4(x - 2a)^3$ and the x-axis.*

Solution. The curve $27ay^2 = 4(x - 2a)^3$ ...(*i*)

is symmetrical about x-axis and is a semi-cubical parabola with vertex at A $(2a, 0)$. The parabola $y^2 = 4ax$ and (*i*) intersect at B and C where $27a\,(4ax) = 4(x - 2a)^3$ or $x^3 - 6ax^2 - 15a^2x - 8a^3 = 0$ which gives $x = -a, -a, 8a$. Since x is not negative, therefore we have $x = 8a$ (Fig. 6.22).

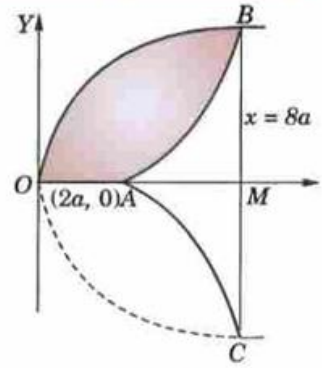

Fig. 6.22

$\therefore$ Required volume = Volume obtained by revolving the shaded area OAB about x-axis = Vol. obtained by revolving area $OMBO$ – Vol. obtained by revolving area $ADBA$

$$= \int_0^{8a} \pi y^2\, (= 4ax)\, dx - \int_{2a}^{8a} \pi y^2\, [\text{for } (i)]\, dx$$

$$= 4a\pi \left| \frac{x^2}{2} \right|_0^{8a} - \frac{4\pi}{27a} \int_{2a}^{8a} (x - 2a)^3\, dx$$

$$= 128\pi a^3 - \frac{4\pi}{27a} \left| \frac{(x - 2a)^4}{4} \right|_{2a}^{8a}$$

$$= 128\pi a^3 - \frac{\pi}{27a}(6a)^4 = 80\pi a^3.$$

(*b*) **Revolution about the y-axis.** Interchanging x and y in the above formula, we see that the *volume of the solid generated by the revolution about y-axis, of the area, bounded by the curve $x = f(y)$, the y-axis and the abscissae $y = a$, $y = b$ is*

$$\int_a^b \pi\, \mathbf{x}^2\, \mathbf{dy}.$$

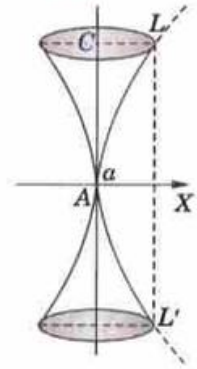

Fig. 6.23

Example 6.39. *Find the volume of the reel-shaped solid formed by the revolution about the y-axis, of the part of the parabola $y^2 = 4ax$ cut off by the latus-rectum.* (*Rohtak, 2003*)

Solution. Given parabola is $x = y^2/4a$.

Let A be the vertex and L one extremity of the latus-rectum. For the arc AL, y varies from 0 to $2a$ (Fig. 6.23).

$\therefore$ required volume = 2 (volume generated by the revolution about the y-axis of the area ALC)

$$= 2\int_0^{2a} \pi x^2\, dy = 2\pi \int_0^{2a} \frac{y^4}{16a^2} \cdot dy = \frac{\pi}{8a^2} \left| \frac{y^5}{5} \right|_0^{2a} = \frac{\pi}{40a^2}(32a^5 - 0) = \frac{4\pi a^3}{5}.$$

(*c*) **Revolution about any axis.** *The volume of the solid generated by the revolution about any axis LM of the area bounded by the curve AB, the axis LM and the perpendiculars AL, BM on the axis, is*

$$\int_{OL}^{OM} \pi(\mathbf{PN})^2\, \mathbf{d(ON)}$$

where O is a fixed point in LM and PN is perpendicular from any point P of the curve AB on LM.

With O as origin, take OLM as the x-axis and OY, perpendicular to it as the y-axis (Fig. 6.24).

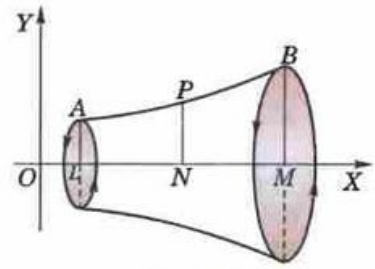

Fig. 6.24

Let the coordinates of P be (x, y) so that $x = ON$, $y = NP$

If $OL = a$, $OM = b$, then required volume $= \int_a^b \pi y^2\, dx = \int_{OL}^{OM} \pi (PN)^2\, d(ON)$.

Example 6.40. *Find the volume of the solid obtained by revolving the cissoid $y^2(2a - x) = x^3$ about its asymptote.* (V.T.U., 2000)

Solution. Given curve is $y = \dfrac{x^3}{2a - x}$ ...(*i*)

It is symmetrical about x-axis and the asymptote is $x = 2a$. (See Fig. 4.23). If $P(x, y)$ be any point on it and PN is perpendicular on the asymptote AN then $PN = 2a - x$ and

$$AN = y = \frac{x^{3/2}}{\sqrt{(2a - x)}} \qquad \text{[From (i)]}$$

$$\therefore \quad d(AN) = dy = \frac{\sqrt{(2a - x)}\,(3/2)\sqrt{x} - x^{3/2} \cdot \frac{1}{2}(2a - x)^{-1/2}(-1)}{2a - x}\, dx$$

$$= \frac{3\sqrt{x}(2a - x) + x^{3/2}}{2(2a - x)^{3/2}}\, dx = \frac{3ax^{1/2} - x^{3/2}}{(2a - x)^{3/2}}\, dx$$

$$\therefore \quad \text{Required volume} = 2\int_{x=0}^{x-2a} \pi (PN)^2\, d(AN) = 2\pi \int_0^{2a} (2a - x)^2 \cdot \frac{3ax^{1/2} - x^{3/2}}{(2a - x)^{3/2}} \cdot dx$$

$$= 2\pi \int_0^{2a} \sqrt{(2a - x)}\,(3a - x)\sqrt{x}\, dx \qquad \left[\begin{array}{l}\text{Put } x = 2a \sin^2\theta \\ \text{then } dx = 4a \sin\theta \cos\theta\, d\theta\end{array}\right]$$

$$= 2\pi \int_0^{\pi/2} \sqrt{(2a)}\, \cos\theta\,(3a - 2a\sin^2\theta)\, x\, \sqrt{(2a)}\, \sin\theta \,.\, 4a \sin\theta \cos\theta\, d\theta$$

$$= 16\pi a^3 \left[3\int_0^{\pi/2} \sin^2\theta \cos^2\theta\, d\theta - 2\int_0^{\pi/2} \sin^4\theta \cos^2\theta\, d\theta\right]$$

$$= 16\pi a^3 \left[3 \,.\, \frac{1 \times 1}{4 \cdot 2} \cdot \frac{\pi}{2} - 2 \,.\, \frac{3 \cdot 1 \times 1}{6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2}\right] = 2\pi^2 a^3.$$

(2) Volumes of revolution (polar curves). *The volume of the solid generated by the revolution of the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha$, $\theta = \beta$ (Fig. 6.25)*

(*a*) *about the initial line OX* $(\theta = 0) = \displaystyle\int_\alpha^\beta \frac{2\pi}{3} r^3 \sin\theta\, d\theta$

(*b*) *about the line OY* $(\theta = \pi/2) = \displaystyle\int_\alpha^\beta \frac{2\pi}{3} r^3 \cos\theta\, d\theta$.

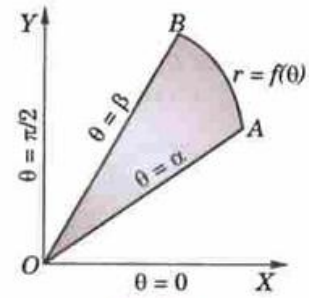

Fig. 6.25

Example 6.41. *Find the volume of the solid generated by the revolution of the cardioid $r = a(1 + \cos\theta)$ about the initial line.* (V.T.U., 2010 ; Kurukshetra, 2009 S)

Solution. The cardioid is symmetrical about the initial line and for its upper half θ varies from 0 to π. [Fig. 6.18].

$$\therefore \quad \text{required volume} = \int_0^\pi \frac{2}{3}\pi r^3 \sin\theta\, d\theta = \frac{2\pi}{3}\int_0^\pi a^3(1 + \cos\theta)^3 \sin\theta\, d\theta$$

$$= -\frac{2\pi a^3}{3}\int_0^\pi (1 + \cos\theta)^3 \cdot (-\sin\theta)\, d\theta = -\frac{2\pi a^3}{3}\left|\frac{(1 + \cos\theta)^4}{4}\right|_0^\pi = -\frac{\pi a^3}{6}[0 - 16] = \frac{8}{3}\pi a^3.$$

Example 6.42. *Find the volume of the solid generated by revolving the lemniscate $r^2 = a^2 \cos 2\theta$ about the line $\theta = \pi/2$.* (V.T.U., 2006)

Solution. The curve is symmetrical about the pole. For the upper half of the R.H.S. loop, θ varies from 0 to $\pi/4$. (Fig. 4.34).

$\therefore$ required volume = 2(volume generated by the half loop in the first quadrant)

$$= 2\int_0^{\pi/4} \frac{2}{3}\pi r^3 \cos\theta\, d\theta = \frac{4\pi}{3}\cdot\int_0^{\pi/4} a^3(\cos 2\theta)^{3/2}\cos\theta\, d\theta \qquad [\because\ r = a(\cos 2\theta)^{1/2}]$$

$$= \frac{4\pi a^3}{3}\int_0^{\pi/4} (1-2\sin^2\theta)^{3/2}\cos\theta\, d\theta \qquad \left[\begin{array}{l}\text{Put } \sqrt{2}\sin\theta = \sin\phi \\ \therefore\ \sqrt{2}\cos\theta\, d\theta = \cos\phi\, d\theta\end{array}\right.$$

$$= \frac{4\pi a^3}{3}\int_0^{\pi/2} (1-\sin^2\phi)^{3/2}\cdot\frac{1}{\sqrt{2}}\cos\phi\, d\phi = \frac{4\pi a^3}{3\sqrt{2}}\int_0^{\pi/2}\cos^4\phi\, d\phi = \frac{4\pi}{3\sqrt{2}}a^3\,\frac{3.1}{4.2}\cdot\frac{\pi}{2} = \frac{\pi a^3}{4\sqrt{2}}.$$

PROBLEMS 6.9

1. Find the volume generated by the revolution of the area bounded by x-axis, the *catenary* $y = c\cosh x/c$ and the ordinates $x = \pm c$, about the axis of x.
2. Find the volume of a spherical segment of height h cut off from a sphere of radius a.
3. Find the volume generated by revolving the portion of the parabola $y^2 = 4ax$ cut off by its latus-rectum about the axis of the parabola. *(V.T.U., 2009)*
4. Find the volume generated by revolving the area bounded by the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$, $x = 0$, $y = 0$ about the x-axis.
5. Find the volume of the solid generated by revolving the ellipse $x^2/a^2 + y^2/b^2 = 1$.
   (*i*) about the major axis. *(Bhopal, 2002 S)* (*ii*) about the minor axis. *(Bhillai, 2005)*
6. Obtain the volume of the frustrum of a right circular cone whose lower base has radius R, upper base is of radius r and altitude is h.
7. Find the volume generated by the revolution of the curve $27ay^2 = 4(x-2a)^3$ about the x-axis.
8. Find the volume of the solid formed by the revolution, about the x-axis, of the loop of the curve :
   (*i*) $y^2(a-x) = x^2(a+x)$ (*ii*) $2ay^2 = x(x-a)^2$ (*iii*) $y^2 = x(2x-1)^2$.
9. Find the volume obtained by revolving one arch of the cycloid
   (*i*) $x = a(t - \sin t)$, $y = a(1-\cos t)$, about its base. *(Kurukshetra, 2006 ; V.T.U., 2005)*
   (*ii*) $x = a(\theta + \sin\theta)$, $y = a(1+\cos\theta)$, about the x-axis.
10. Find the volume of the spindle-shaped solid generated by the revolution of the *astroid* $x^{2/3} + y^{2/3} = a^{2/3}$ about the x-axis. *(P.T.U., 2010 ; S.V.T.U., 2008)*
11. Find the volume of the solid formed by the revolution, about the y-axis, of the area enclosed by the curve $xy^2 = 4a^2(2a - x)$ and its asymptote. *(V.T.U., 2006)*
12. Prove that the volume of the solid formed by the revolution of the curve $(a^2 + x^2) = a^3$, about its asymptote is $\frac{1}{2}\pi^2 a^3$.
13. Find the volume generated by the revolution about the initial line of
    (*i*) $r = 2a\cos\theta$ (*ii*) $r = a(1-\cos\theta)$. *(P.T.U., 2006)*
14. Determine the volume of the solid obtained by revolving the *lemicon* $r = a + b\cos\theta$ $(a > b)$ about the initial line. *(Gorakhpur, 1999)*
15. Find the volume of the solid formed by revolving a loop of the lemniscate $r^2 = a^2\cos 2\theta$ about the initial line. *(J.N.T.U., 2003 ; Delhi, 2002)*

6.13 SURFACE AREAS OF REVOLUTION

(*a*) **Revolution about x-axis.** *The surface area of the solid generated by the revolution about x-axis, of the arc of the curve $y = f(x)$ from $x = a$ to $x = b$, is*

$$\int_{x=a}^{x=b} 2\pi y\, ds.$$

Let AB be the curve $y = f(x)$ between the ordinates LA $(x = a)$ and MB $(x = b)$. Let $P(x, y)$, $P'(x + \delta x y + \delta y)$ be two neighbouring points on the curve and NP, $N'P'$ be their respective ordinates (Fig. 6.19).

Let the arc $AP = s$ so that arc $PP' = \delta s$. Let the surface-area generated by the revolution about x-axis of the arc AP be S and that generated by the revolution of the arc PP' be δS.

Since δs is small, the surface area δS may be regarded as lying between the curved surfaces of the right cylinders of radii PN and $P'N'$ and of same thickness δs.

Thus δS lies between $2\pi y\ \delta s$ and $2\pi\ (y + \delta y)\ \delta s$

$\therefore\quad \dfrac{\delta S}{\delta s}$ lies between $2\pi y$ and $2\pi\ (y + \delta y)$

Taking limits as $P' \to P$, *i.e.*, $\delta s \to 0$ and $\delta y \to 0$, $dS/dx = 2\pi y$

$$\therefore \qquad \int_{x=a}^{x=b} \frac{dS}{ds}\, ds = \int_{x=a}^{x=b} 2\pi y\, ds \quad \text{or} \quad |\, S\,|_{x=a}^{x=b} = \int_{x=a}^{x=b} 2\pi y\, ds$$

or (value of S for $x = b$) – (value of S for $x = a$) = $\displaystyle\int_{x=a}^{x=b} 2\pi y\ dx$

or surface area generated by the revolution of the arc $AB - 0 = \displaystyle\int_{x=a}^{x=b} 2\pi y\ ds.$

Hence, the required surface area = $\displaystyle\int_{x=a}^{x=b} 2\pi y\ ds.$

Obs. Practical forms of the formula $S = \int 2\pi y\ ds$.

(*i*) *Cartesian form* [*for the curve* $y = f(x)$]

$$S = \int 2\pi y \frac{ds}{dx}\, dx, \text{ where } \frac{ds}{dx} = \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}$$

(*ii*) *Parametric form* [*for the curve* $x = f(t)$, $y = \phi(t)$]

$$S = \int 2\pi y \frac{ds}{dt}\, dt, \text{ where } \frac{ds}{dt} = \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]}$$

(*iii*) *Polar form* [*for the curve* $r = f(\theta)$]

$$S = \int 2\pi y \frac{ds}{d\theta}\, d\theta, \text{ where } y = r \sin\theta,\ \frac{ds}{d\theta} = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}.$$

Example 6.43. *Find the surface of the solid formed by revolving the cardioid $r = a\ (1 + \cos\theta)$ about the initial line.* (*V.T.U., 2009 ; Rajasthan, 2006 ; J.N.T.U., 2003*)

Solution. The cardioid is symmetrical about the initial line and for its upper half, θ varies from 0 to π (Fig. 6.18).

Also
$$\frac{ds}{d\theta} = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]} = \sqrt{[a^2(1 + \cos\theta)^2 + a^2 \sin^2\theta]}$$

$$= a\sqrt{[2(1 + \cos\theta)]} = a\sqrt{[2.2\cos^2\theta/2]} = 2a\cos\theta/2$$

$$\therefore \quad \text{required surface} = \int_0^\pi 2\pi y \frac{ds}{d\theta}\, d\theta = 2\pi \int_0^\pi r \sin\theta \cdot 2a\cos\theta/2\ d\theta$$

$$= 4\pi a \int_0^\pi a(1 + \cos\theta)\ \sin\theta \cdot \cos\theta/2\ d\theta = 4\pi a^2 \int_0^\pi 2\cos^2\frac{\theta}{2} \cdot 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \cdot \cos\frac{\theta}{2}\, d\theta$$

$$= 16\pi a^2 \int_0^\pi \cos^4\frac{\theta}{2}\sin\frac{\theta}{2}\, d\theta = 16\pi a^2\,(-2) \int_0^\pi \cos^4\frac{\theta}{2}\left(-\sin\frac{\theta}{2} \cdot \frac{1}{2}\right) d\theta$$

$$= -32\pi a^2 \left| \frac{\cos^5\theta/2}{5} \right|_0^\pi = \frac{-32\pi a^2}{5}(0 - 1) = \frac{32\pi a^2}{5}.$$

(*b*) **Revolution about y-axis.** Interchanging x and y in the above formula, we see that *the surface of the solid generated by the revolution about y-axis, of the arc of the curve $x = f(y)$ from $y = a$ to $y = b$ is*

$$\int_{y=a}^{y=b} 2\pi x\ ds.$$

Example 6.44. *Find the surface area of the solid generated by the revolution of the astroid $x = a\cos^3 t$, $y = a\sin^3 t$, about the y-axis.*

Solution. The astroid is symmetrical about the x-axis, and for its portion in the first quadrant t varies from 0 to $\pi/2$. (Fig. 4.29).

Also $$\frac{dx}{dt} = -3a\cos^2 t \sin t, \quad \frac{dy}{dt} = 3a \sin^2 t \cos t.$$

$$\frac{ds}{dt} = \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]} = \sqrt{[9a^2 \cos^4 t \sin^2 t + 9a^2 \sin^4 t \cos^2 t]}$$

$$= 3a \sin t \cos t \sqrt{(\cos^2 t + \sin^2 t)} = 3a \sin t \cos t$$

$$\therefore \quad \text{required surface} = 2\int_0^{\pi/2} 2\pi x \frac{ds}{dt} \cdot dt = 4\pi \int_0^{\pi/2} a \cos^3 t \cdot 3a \sin t \cos t \, dt$$

$$= 12\pi a^2 \int_0^{\pi/2} \sin t \cos^4 t \, dt = 12\pi a^2 \frac{3.1}{5.3.1} = \frac{12\pi a^2}{5}.$$

PROBLEMS 6.10

1. Find the area of the surface generated by revolving the arc of the catenary $y = c \cosh x/c$ from $x = 0$ to $x = c$ about the x-axis.
2. Find the area of the surface formed by the revolution of $y^2 = 4ax$ about its axis, by the arc from the vertex to one end of the latus-rectum.
3. Find the surface of the solid generated by the revolution of the *ellipse* $x^2/a^2 + y^2/b^2 = 1$ about the x-axis. *(Raipur, 2005 ; Bhopal, 2002 S)*
4. Find the volume and surface of the *right circular cone* formed by the revolution of a right-angled triangle about a side which contains the right angle.
5. Obtain the surface area of a *sphere* of radius a.
6. Show that the surface area of the solid generated by the revolution of the curve $x = a \cos^3 t$, $y = a \sin^3 t$ about the x-axis, is $12\pi^2/5$.
7. The arc of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ in the first quadrant revolves about x-axis. Show that the area of the surface generated is $6\pi a^2/5$.
8. Find the surface area of the solid generated by revolving the *cycloid* $x = a(t - \sin t)$, $y = a(1 - \cos t)$ about the base. *(Marathwada, 2008 ; Cochin, 2005 ; Kurukshetra, 2005)*
9. Find the surface area of the solid got by revolving the arch of the *cycloid* $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ about the base. *(V.T.U., 2010 S)*
10. Prove that the surface and volume of the solid generated by the revolution about the x-axis, of the loop of the curve $x = t^2$, $y = t - t^3/3$, [or $9y^2 = x(x-3)^2$], are respectively 3π and $3\pi/4$.
11. Prove that the surface of the solid generated by the revolution of the *tractrix* $x = a \cos t + \frac{a}{2} \log \tan^2 t/2$, $y = a \sin t$, about x-axis is $4\pi a^2$.
12. Find the surface area of the solid of revolution of the curve $r = 2a \cos\theta$ about the initial line. *(V.T.U., 2009)*
13. Find the surface of the solid generated by the revolution of the *cardioid* $r = a(1 - \cos\theta)$ about the initial line.
14. Find the surface of the solid generated by the revolution of the *lemniscate* $r^2 = a^2 \cos 2\theta$ about the initial line. *(V.T.U., 2005)*
15. The part of *parabola* $y^2 = 4ax$ cut off by the latus-rectum revolves about the tangent at the vertex. Find the curved surface of the reel thus formed.

6.14 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 6.11

Choose the correct answer or fill up the blanks in the following problems :

1. If $f(x) = f(2a - x)$, then $\int_0^{2a} f(x)\, dx$ is equal to

(a) $\int_a^0 f(2a-x)\,dx$ (b) $2\int_0^a f(x)\,dx$ (c) $-2\int_0^a f(x)\,dx$ (d) 0.

2. $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$ is equal to

(a) 0 (b) 1 (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$.

3. The value of definite integral $\int_{-a}^{a} |x|\,dx$ is equal to

(a) a (b) a^2 (c) 0 (d) $2a$.

4. $\lim_{n\to\infty}\left[\frac{n}{n^2}+\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\ldots+\frac{n}{n^2+(n-1)^2}\right]$ is equal to

(a) $-\frac{\pi}{4}$ (b) 0 (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$.

5. $\int_0^{\pi/2} \frac{\cos 2x}{\cos x+\sin x}\,dx$ equals

(a) -1 (b) 0 (c) 1 (d) 2.

6. $\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{3n}\right)$ equals

(a) $\log_e 2$ (b) $2\log_e 2$ (c) $\log_e 3$ (d) $2\log_e 3$.

7. $\int_0^{\pi} \sin^5\left(\frac{x}{2}\right)$ is equal to

(a) $\frac{16}{15}$ (b) $\frac{15}{16}\pi$ (c) $\frac{16}{15}\pi^2$ (d) $\frac{15}{16}$.

8. $\int_0^{\pi/2} \sin^{99} x\,\cos x\,dx$ is equal to

(a) $\frac{1}{99}$ (b) $\frac{\pi}{100}$ (c) $\frac{99}{100}$ (d) None of these. *(V.T.U., 2009)*

9. The value of $\int_{-\pi/2}^{\pi/2} \cos^7 x\,dx$ is

(a) $\frac{32\pi}{35}$ (b) $\frac{32}{35}$ (c) zero.

10. The length of the arc of the equiangular spiral $r = ae^{\theta\cot\alpha}$ between the points for which the radii vectors are r_1 and r_2 is

(a) $(r_2-r_1)\operatorname{cosec}\alpha$ (b) $(r_2-r_1)\cos\alpha$ (c) $(r_2-r_1)\sin\alpha$ (d) $(r_2-r_1)\sec\alpha$.

11. The area of the region in the first quadrant bounded by the y-axis and the curves $y = \sin x$ and $y = \cos x$ is

(a) $\sqrt{2}$ (b) $\sqrt{2}+1$ (c) $\sqrt{2}-1$ (d) $2\sqrt{2}-1$.

12. The value of $\int_0^1 x^{3/2}(1-x)^{3/2}\,dx$ is

(a) $\pi/32$ (b) $-\pi/32$ (c) $3\pi/128$ (d) $-3\pi/128$. *(V.T.U., 2010)*

13. The entire length of the cardioid $r = 5(1+\cos\theta)$ is

(a) 40 (b) 30 (c) 20 (d) 5. *(V.T.U., 2009)*

14. The area of the cardioid $r = a(1-\cos\theta)$ is .....

15. If S_1 and S_2 are surface areas of the solids generated by revolving the ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ about the y-axis, then

(a) $S_1 > S_2$ (b) $S_1 < S_2$ (c) $S_1 = S_2$ (d) can't predict.

16. The area of the loop of the curve $r = a\sin 3\theta$ is .....

17. If $I_n = \int_0^{\pi/4} \tan^n\theta\,d\theta$, then $n(I_{n-1}+I_{n+1}) =$ .....

18. $\int_0^2 x^3\sqrt{(2x-x^2)}\,dx =$ .....

19. $\int_0^{\pi/2} \sin 2\theta \log \tan \theta \, d\theta$ is equal to

(*a*) 1 (*b*) -1 (*c*) 0 (*d*) $\pi/2$.

20. The area of the loop of the folium of Descartes $x^3 + y^3 - 3xy = 0$ is

(*a*) π (*b*) $\pi/2$ (*c*) 1.5 (*d*) 3.

21. The volume of the frustrum of a right circular cone whose lower base has radius r_1 and upper base has radius r_2 and altitude is h = ...

22. The length of the arc of the curve $y = \log \sec x$ from $x = 0$ to $x = \pi/4$ is

(*a*) $\log_e 2$ (*b*) $\log_e 3$ (*c*) $\log_e (1 + \sqrt{2})$ (*d*) $\log_e (1 + \sqrt{3})$. (*Bhopal, 2008*)

23. If v_1 = volume of the solid generated by revolving the area included between x-axis and $x^2 + y^2 = a^2$ about x-axis ; v_2 = volume of the solid generated by revolving the entire area of the circle $x^2 + y^2 = a^2$ about x-axis, then

(*a*) $v_1 = v_2$ (*b*) $v_2 = 2v_1$ (*c*) $v_2 = 4v_1$ (*d*) $v_2 = 16v_1$.

24. If $f(r, \theta) = f(-r, \theta)$, then the curve is symmetrical about the .....

(*a*) initial line (*b*) pole (*c*) origin (*d*) tangential line. (*V.T.U., 2010*)

25. The volume generated by the revolution of the curve $y = a^3 (a^2 + x^2)^{-1}$ about its asymptote is

(*a*) $\pi^2 a^3/2$ (*b*) $\pi a^3/2$ (*c*) $\pi a^2/2$ (*d*) $\pi a/2$. (*V.T.U., 2010*)

CHAPTER 7

Multiple Integrals and Beta, Gamma Functions

1. Double integrals. 2. Change of order of integration. 3. Double integrals in Polar coordinates. 4. Areas enclosed by plane curves. 5. Triple integrals. 6. Volume of solids. 7. Change of variables. 8. Area of a curved surface. 9. Calculation of mass. 10. Centre of gravity. 11. Centre of pressure. 12. Moment of inertia. 13. Product of inertia ; Principal axes. 14. Beta function. 15. Gamma function. 16. Relation between beta and gamma functions. 17. Elliptic integrals. 18. Error function or Probability integral. 19. Objective Type of Questions.

7.1 DOUBLE INTEGRALS

The definite integral $\int_a^b f(x)\,dx$ is defined as the limit of the sum

$$f(x_1)\,\delta x_1 + f(x_2)\,\delta x_2 + \ldots + f(x_n)\,\delta x_n,$$

where $n \to \infty$ and each of the lengths $\delta x_1, \delta x_2, \ldots$ tends to zero. A double integral is its counterpart in two dimensions.

Consider a function $f(x, y)$ of the independent variables x, y defined at each point in the finite region R of the xy-plane. Divide R into n elementary areas $\delta A_1, \delta A_2, \ldots, \delta A_n$. Let (x_r, y_r) be any point within the rth elementary area δA_r. Consider the sum

$$f(x_1, y_1)\,\delta A_1 + f(x_2, y_2)\,\delta A_2 + \ldots + f(x_n, y_n)\,\delta A_n, \text{ i.e., } \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r$$

The limit of this sum, if it exists, as the number of sub-divisions increases indefinitely and area of each sub-division decreases to zero, is defined as the *double integral of f(x, y) over the region R* and is written as $\iint_R f(x, y)\,dA$.

Thus
$$\iint_R f(x, y)\,dA = \underset{\substack{n \to \infty \\ \delta A \to 0}}{\text{Lt}} \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r \qquad \ldots(1)$$

The utility of double integrals would be limited if it were required to take limit of sums to evaluate them. However, there is another method of evaluating double integrals by successive single integrations.

For purpose of evaluation, (1) is expressed as the repeated integral $\int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x, y)\,dxdy$.

Its value is found as follows :

(*i*) *When y_1, y_2 are functions of x and x_1, x_2 are constants, $f(x, y)$* is first integrated w.r.t. y keeping x fixed between limits y_1, y_2 and then resulting expression is integrated w.r.t. x within the limits x_1, x_2 *i.e.,*

$$\boxed{I_1 = \int_{x_1}^{x_2} \boxed{\int_{y_1}^{y_2} f(x, y)\,dy}\,dx}$$

where integration is carried from the inner to the outer rectangle.

Figure 7.1 illustrates this process. Here AB and CD are the two curves whose equations are $y_1 = f_1(x)$ and $y_2 = f_2(x)$. PQ is a vertical strip of width dx.

Then the inner rectangle integral means that the integration is along one edge of the strip PQ from P to Q (x remaining constant), while the outer rectangle integral corresponds to the sliding of the edge from AC to BD.

Thus the whole region of integration is the area $ABDC$.

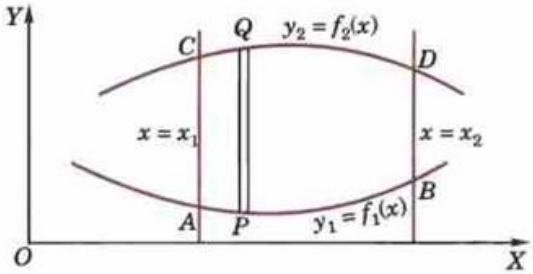

Fig. 7.1

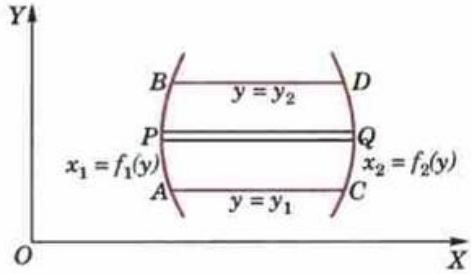

Fig. 7.2

(*ii*) *When x_1, x_2 are functions of y and y_1, y_2 are constants, $f(x, y)$* is first integrated w.r.t. x keeping y fixed, within the limits x_1, x_2 and the resulting expression is integrated w.r.t. y between the limits y_1, y_2, *i.e.*,

$$I_2 = \boxed{\int_{y_1}^{y_2} \boxed{\int_{x_1}^{x_2} f(x, y)\, dx}\, dy}$$ which is geometrically illustrated by Fig. 7.2.

Here AB and CD are the curves $x_1 = f_1(y)$ and $x_2 = f_2(y)$. PQ is a horizontal strip of width dy.

Then inner rectangle indicates that the integration is along one edge of this strip from P to Q while the outer rectangle corresponds to the sliding of this edge from AC to BD.

Thus the whole region of integration is the area $ABDC$.

(*iii*) *When both pairs of limits are constants,* the region of integration is the rectangle $ABDC$ (Fig. 7.3).

In I_1, we integrate along the vertical strip PQ and then slide it from AC to BD.

In I_2, we integrate along the horizontal strip $P'Q'$ and then slide it from AB to CD.

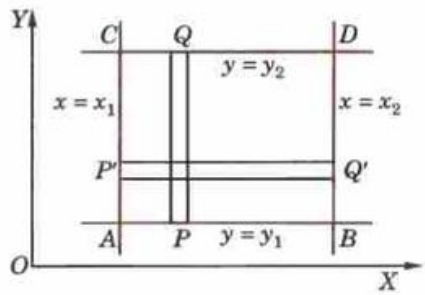

Fig. 7.3

Here obviously $I_1 = I_2$.

Thus *for constant limits, it hardly matters whether we first integrate w.r.t. x and then w.r.t. y or vice versa.*

Example 7.1. *Evaluate* $\int_0^5 \int_0^{x^2} x(x^2 + y^2)\, dxdy$.

Solution. $$I = \int_0^5 dx \int_0^{x^2} (x^3 + xy^3)\, dy = \int_0^5 \left[x^3 y + x \cdot \frac{y^3}{3}\right]_0^{x^2} dx = \int_0^5 \left[x^3 \cdot x^2 + x \cdot \frac{y^6}{3}\right] dx$$

$$= \int_0^5 \left(x^5 + \frac{x^7}{3}\right) dx = \left| \frac{x^6}{6} + \frac{x^8}{24} \right|_0^5 = 5^6 \left[\frac{1}{6} + \frac{5^2}{24}\right] = 18880.2 \text{ nearly.}$$

Example 7.2. *Evaluate* $\iint_A xy\, dx\, dy$, *where A is the domain bounded by x-axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$.*

Solution. The line $x = 2a$ and the parabola $x^2 = 4ay$ intersect at $L(2a, a)$. Figure 7.4 shows the domain A which is the area OML.

Integrating first over a vertical strip PQ, *i.e.*, w.r.t. y from $P(y = 0)$ to $Q(y = x^2/4a)$ on the parabola and then w.r.t. x from $x = 0$ to $x = 2a$, we have

$$\iint_A xy\, dx\, dy = \int_0^{2a} dx \int_4^{x^2/4a} xy\, dy = \int_0^{2a} x \left[\frac{y^2}{2}\right]_0^{x^2/4a} dx$$

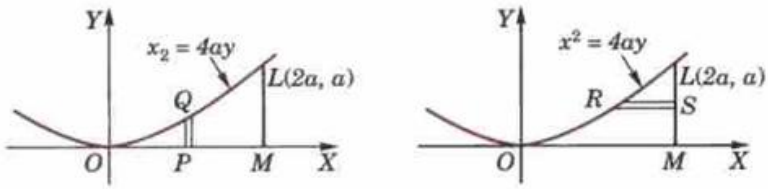

Fig. 7.4

$$= \frac{1}{32a^2}\int_0^{2a} x^5\,dx = \frac{1}{32a^2}\left|\frac{x^6}{6}\right|_0^{2a} = \frac{a^4}{3}.$$

Otherwise integrating first over a horizontal strip RS, *i.e.*, w.r.t. x from, $R\ (x = 2\sqrt{ay}\,)$ on the parabola to $S(x = 2a)$ and then w.r.t. y from $y = 0$ to $y = a$, we get

$$\iint_A xy\,dx\,dy = \int_0^a dx \int_{2\sqrt{(ay)}}^{2a} xy\,dx = \int_0^a y\left[\frac{x^2}{2}\right]_{2\sqrt{(ay)}}^{2a} dy$$

$$= 2a\int_0^a (ay - y^2)\,dy = 2a\left[\frac{ay^2}{2} - \frac{y^3}{3}\right]_0^a = \frac{a^4}{3}.$$

Example 7.3. *Evaluate* $\iint_R x^2\,dx\,dy$ *where R is the region in the first quadrant bounded by the lines* $x = y$, $y = 0$, $x = 8$ *and the curve* $xy = 16$.

Solution. The line AL $(x = 8)$ intersects the hyperbola $xy = 16$ at $A\ (8, 2)$ while the line $y = x$ intersects this hyperbola at $B\ (4, 4)$. Figure 7.5 shows the region R of integration which is the area $OLAB$. To evaluate the given integral, we divide this area into two parts OMB and $MLAB$.

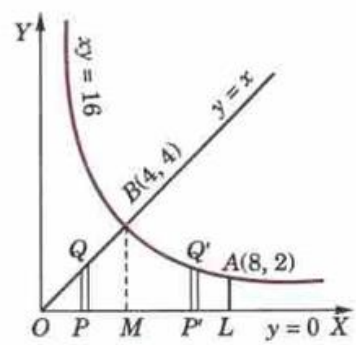

Fig. 7.5

$$\therefore \quad \iint_R x^2\,dxdy = \int_{x\text{ at }0}^{x\text{ at }M}\int_{y\text{ at }P}^{y\text{ at }Q} x^2\,dxdy + \int_{x\text{ at }M}^{x\text{ at }L}\int_{y\text{ at }P'}^{y\text{ at }Q'} x^2\,dxdy$$

$$= \int_0^4\int_0^x x^2\,dxdy + \int_4^8\int_0^{16/x} x^2\,dxdy$$

$$= \int_0^4 x^2\,dx\left|y\right|_0^x + \int_4^8 x^2\,dx\left|y\right|_0^{16/x}$$

$$= \int_0^4 x^3\,dx + \int_4^8 16x\,dx = \left|\frac{x^4}{4}\right|_0^4 + 16\left|\frac{x^2}{2}\right|_4^8 = 448$$

7.2 CHANGE OF ORDER OF INTEGRATION

In a double integral with variable limits, the change of order of integration changes the limit of integration. While doing so, sometimes it is required to split up the region of integration and the given integral is expressed as the sum of a number of double integrals with changed limits. To fix up the new limits, it is always advisable to draw a rough sketch of the region of integration.

The change of order of integration quite often facilitates the evaluation of a double integral. The following examples will make these ideas clear.

Example 7.4. *By changing the order of integration of* $\int_0^\infty\int_0^\infty e^{-xy}\sin px\,dxdy$, *show that*

$$\int_0^\infty \frac{\sin px}{x}\,dx = \frac{\pi}{2}.$$ *(U.P.T.U., 2004)*

Solution. $\int_0^\infty\int_0^\infty e^{-xy}\sin px\,dxdy = \int_0^\infty\left(\int_0^\infty e^{-xy}\sin px\,dx\right)dy$

$$= \int_0^\infty \left| -\frac{e^{-xy}}{p^2+y^2}(p\cos px + y\sin px) \right|_0^\infty dy$$

$$= \int_0^\infty \frac{p}{p^2+y^2}\,dy = \left| \tan^{-1}\left(\frac{y}{p}\right) \right|_0^\infty = \frac{\pi}{2} \qquad ...(i)$$

On changing the order of integration, we have

$$\int_0^\infty \int_0^\infty e^{-xy} \sin px\, dx\, dy = \int_0^\infty \sin px \left\{ \int_0^\infty e^{-xy}\, dy \right\} dx$$

$$= \int_0^\infty \sin px \left| \frac{e^{-xy}}{-x} \right|_0^\infty dx = \int_0^\infty \frac{\sin px}{x}\, dx \qquad ...(ii)$$

Thus from (i) and (ii), we have $\int_0^\infty \frac{\sin px}{x}\, dx = \frac{\pi}{2}$.

Example 7.5. *Change the order of integration in the integral*

$$I = \int_{-a}^{a} \int_0^{\sqrt{(a^2-y^2)}} f(x,y)\, dx\, dy.$$

Solution. Here the elementary strip is parallel to x-axis (such as PQ) and extends from $x = 0$ to $x = \sqrt{(a^2 - y^2)}$ (*i.e.*, to the circle $x^2 + y^2 = a^2$) and this strip slides from $y = -a$ to $y = a$. This shaded semi-circular area is, therefore, the region of integration (Fig. 7.6).

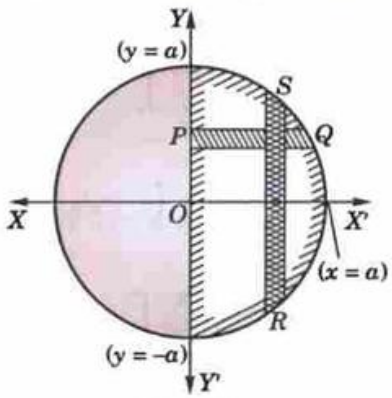

Fig. 7.6

On changing the order of integration, we first integrate w.r.t. y along a vertical strip RS which extends from R $[y = -\sqrt{(a^2 - y^2)}]$ to S $[y = \sqrt{(a^2 - y^2)}]$. To cover the given region, we then integrate w.r.t. x from $x = 0$ to $x = a$.

Thus $$I = \int_0^a dx \int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} f(x,y)\, dy$$

or $$= \int_0^a \int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} f(x,y)\, dy dx.$$

Example 7.6. *Evaluate* $\int_0^1 \int_{e^x}^{e} dydx/\log y$ *by changing the order of integration.*

Solution. Here the integration is first w.r.t. y from P on $y = e^x$ to Q on the line $y = e$. Then the integration is w.r.t. x from $x = 0$ to $x = 1$, giving the shaded region ABC (Fig. 7.7).

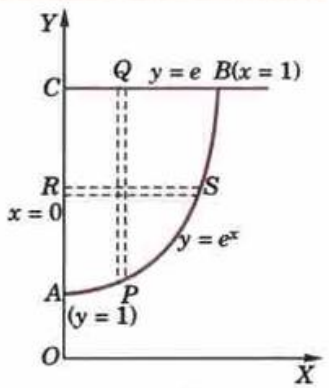

Fig. 7.7

On changing the order of integration, we first integrate w.r.t. x from R on $x = 0$ to S on $x = \log y$ and then w.r.t. y from $y = 1$ to $y = e$.

Thus $$\int_0^1 \int_{e^x}^{e} \frac{dydx}{\log y} = \int_1^e \int_0^{\log y} \frac{dxdy}{\log y}$$

$$= \int_1^e \frac{dy}{\log y} \left| x \right|_0^{\log y} = \int_1^e dy = \left| y \right|_1^e = e - 1.$$

Example 7.7. *Change the order of integration in* $I = \int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dydx$ *and hence evaluate.*

(Nagpur, 2009 ; P.T.U., 2009 S)

Solution. Here integration is first w.r.t. y and P on the parabola $x^2 = 4ay$ to Q on the parabola $y^2 = 4ax$ and then w.r.t. x from $x = 0$ to $x = 4a$ giving the shaded region of integration (Fig. 7.8).

On changing the order of integration, we first integrate w.r.t. x from R to S, then w.r.t. y from $y = 0$ to $y = 4a$

$$\therefore \quad I = \int_0^{4a} \int_{y^2/4a}^{2\sqrt{ay}} dx dy = \int_0^{4a} dy \left| x \right|_{y^2/4a}^{2\sqrt{ay}} = \int_0^{4a} (2\sqrt{ay} - y^2/4a)\, dy$$

$$= \left| 2\sqrt{a}\, \frac{y^{3/2}}{3/2} - \frac{y^3}{12a} \right|_0^{4a} = \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}.$$

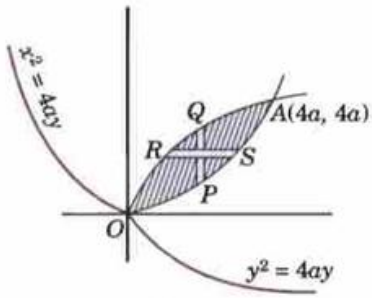

Fig. 7.8

Example 7.8. *Change the order of integration and hence evaluate*

$$I = \int_0^a \int_{\sqrt{ax}}^a \frac{y^2\, dx dy}{\sqrt{(y^4 - a^2x^2)}}$$

(S.V.T.U., 2006 S)

Solution. Here integration is first w.r.t. y from P on the parabola $y^2 = ax$ to Q on the line $y = a$, then w.r.t. x from $x = 0$ to $x = a$, giving the shaded region OAB of integration (Fig. 7.9).

On changing the order of integration, we first integrate w.r.t. x from R to S, then w.r.t. y from $y = 0$ to $y = a$.

$$\therefore \quad I = \int_0^a \int_0^{y^2/a} \frac{y^2\, dy}{\sqrt{(y^4 - a^2x^2)}}\, dx = \frac{1}{a} \int_0^a \int_0^{y^2/a} y^2\, dy \frac{dx}{\sqrt{[(y^2/a)^2 - x^2]}}\, dx$$

$$= \frac{1}{a} \int_0^a y^2 dy \left| \sin^{-1}\left(\frac{xa}{y^2}\right) \right|_0^{y^2/a} = \frac{1}{a} \int_0^a y^2 dy\ [\sin^{-1}(1) - \sin^{-1}(0)]$$

$$= \frac{\pi}{2a} \int_0^a y^2 dy = \frac{\pi}{2a} \left| \frac{y^3}{3} \right|_0^a = \frac{\pi a^2}{6}.$$

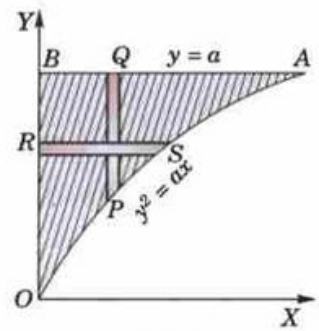

Fig. 7.9

Example 7.9. *Change the order of integration in* $I = \int_0^1 \int_{x^2}^{2-x} xy\, dx dy$ *and hence evaluate the same.*

(Bhopal, 2008 ; V.T.U., 2008 ; S.V.T.U., 2007 ; P.T.U., 2005 ; U.P.T.U., 2005)

Solution. Here the integration is first w.r.t. y along a vertical strip PQ which extends from P on the parabola $y = x^2$ to Q on the line $y = 2 - x$. Such a strip slides from $x = 0$ to $x = 1$, giving the region of integration as the curvilinear triangle OAB (shaded) in Fig. 7.10.

On changing the order of integration, we first integrate w.r.t. x along a horizontal strip $P'Q'$ and that requires the splitting up of the region OAB into two parts by the line AC ($y = 1$), *i.e.*, the curvilinear triangle OAC and the triangle ABC.

For the region OAC, the limits of integration for x are from $x = 0$ to $x = \sqrt{y}$ and those for y are from $y = 0$ to $y = 1$. So the contribution to I from the region OAC is

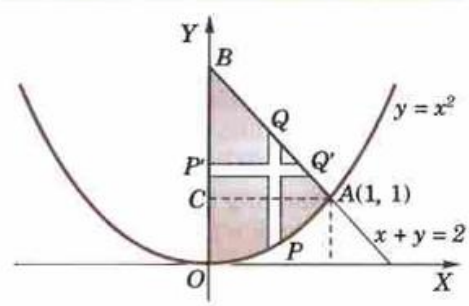

Fig. 7.10

$$I_1 = \int_0^1 dy \int_0^{\sqrt{y}} xy\, dx$$

For the region ABC, the limits of integration for x are from $x = 0$ to $x = 2 - y$ and those for y are from $y = 1$ to $y = 2$. So the contribution to I from the region ABC is

$$I_2 = \int_1^2 dy \int_0^{2-y} xy\, dx.$$

Hence, on reversing the order of integration,

$$I = \int_0^1 dy \int_0^{\sqrt{y}} xy\, dx + \int_1^2 dy \int_0^{2-y} xy\, dx$$

$$= \int_0^1 dy \left| \frac{x^2}{2} . y \right|_0^{\sqrt{y}} + \int_1^2 dy \left| \frac{x^2}{2} . y \right|_0^{2-y} = \frac{1}{2}\int_0^1 y^2 dy + \frac{1}{2}\int_1^2 y(2-y)^2\, dy = \frac{1}{6} + \frac{5}{24} = \frac{3}{8}.$$

Example 7.10. *Change the order of integration in* $I = \int_0^1 \int_x^{\sqrt{(2-x^2)}} \frac{x}{\sqrt{(x^2+y^2)}}\, dx$ *and hence evaluate it.*

(J.N.T.U., 2005 ; Rohtak, 2003)

Solution. Here the integration is first w.r.t. y along PQ which extends from P on the line $y = x$ to Q on the circle $y = \sqrt{(2 . x^2)}$. Then PQ slides from $y = 0$ to $y = 1$, giving the region of integration OAB as in Fig. 7.11.

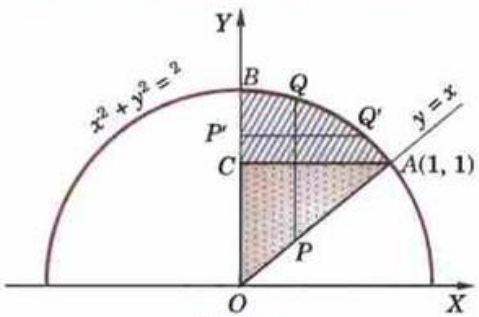

Fig. 7.11

On changing the order of integration, we first integrate w.r.t. x from P' to Q' and that requires splitting the region OAB into two parts OAC and ABC.

For the region OAC, the limits of integration for x are from $x = 0$ to $x = 1$ and those for y are from $y = 0$ to $y = 1$. So the contribution to I from the region OAC is

$$I_1 = \int_0^1 dy \int_0^y \frac{x}{\sqrt{(x^2+y^2)}}\, dx.$$

For the region ABC, the limits of integration for x are 0 to $\sqrt{(2-y^2)}$ and these for y are from 1 to $\sqrt{2}$. So the contribution to I from the region ABC is

$$I_2 = \int_1^{\sqrt{2}} dy \int_0^{\sqrt{(2-y^2)}} \frac{x}{\sqrt{(x^2+y^2)}}\, dx$$

Hence
$$I = \int_0^1 \left| (x^2+y^2)^{1/2} \right|_0^y dy + \int_1^{\sqrt{2}} \left| (x^2+y^2)^{1/2} \right|_0^{\sqrt{(2-y^2)}} dy$$

$$= \int_0^1 (\sqrt{2}-1)\, y\, dy + \int_1^{\sqrt{2}} \sqrt{(2-y)}\, dy = \frac{1}{2}(\sqrt{2}-1) + \sqrt{2}\sqrt{(2-1)} - \frac{1}{2} = 1 - 1/\sqrt{2}.$$

7.3 DOUBLE INTEGRALS IN POLAR COORDINATES

To evaluate $\int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r, \theta)\, dr\, d\theta$, we first integrate w.r.t. r between limits $r = r_1$ and $r = r_2$ keeping θ fixed and the resulting expression is integrated w.r.t. θ from θ_1 to θ_2. In this integral, r_1, r_2 are functions of θ and θ_1, θ_2 are constants.

Figure 7.12 illustrates the process geometrically.

Here AB and CD are the curves $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$ bounded by the lines $\theta = \theta_1$ and $\theta = \theta_2$. PQ is a wedge of angular thickness $\delta\theta$.

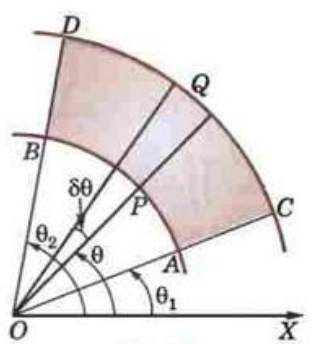

Fig. 7.12

Then $\int_{r_1}^{r_2} f(r, \theta)\, dr$ indicates that the integration is along PQ from P to Q while the integration w.r.t. θ corresponds to the turning of PQ from AC to BD.

Thus the whole region of integration is the area $ACDB$. The order of integration may be changed with appropriate changes in the limits.

Example 7.11. *Evaluate* $\iint r \sin\theta \, dr \, d\theta$ *over the cardioid* $r = a(1 - \cos\theta)$ *above the initial line.* *(Kerala, 2005)*

Solution. To integrate first w.r.t. r, the limits are from 0 ($r = 0$) to P [$r = a(1 - \cos\theta)$] and to cover the region of integration R, θ varies from 0 to π (Fig. 7.13).

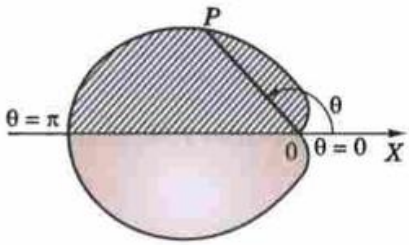

Fig. 7.13

$$\therefore \quad \iint_R r \sin\theta \, dr d\theta = \int_0^{\pi} \sin\theta \left[\int_0^{r = a(1-\cos\theta)} r dr \right] d\theta$$

$$= \int_0^{\pi} \sin\theta \, d\theta \left| \frac{r^2}{2} \right|_0^{a(1-\cos\theta)} = \frac{a^2}{2} \int_0^{\pi} (1 - \cos\theta)^2 . \sin\theta \, d\theta$$

$$= \frac{a^2}{2} \left| \frac{(1-\cos\theta)^3}{3} \right|_0^{\pi} = \frac{a^2}{2} \cdot \frac{8}{3} = \frac{4a^2}{3}.$$

Example 7.12. *Calculate* $\iint r^3 \, dr \, d\theta$ *over the area included between the circles* $r = 2 \sin\theta$ *and* $r = 4 \sin\theta$.

Solution. Given circles $r = 2 \sin\theta$ ...(*i*)

and $r = 4 \sin\theta$ ...(*ii*)

are shown in Fig. 7.14. The shaded area between these circles is the region of integration.

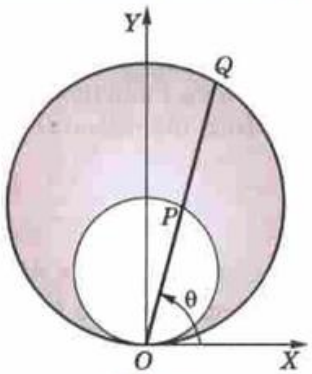

Fig. 7.14

If we integrate first w.r.t. r, then its limits are from $P(r = 2 \sin\theta)$ to $Q(r = 4 \sin\theta)$ and to cover the whole region θ varies from 0 to π. Thus the required integral is

$$I = \int_0^{\pi} d\theta \int_{2\sin\theta}^{4\sin\theta} r^3 \, dr = \int_0^{\pi} d\theta \left[\frac{r^4}{4} \right]_{2\sin\theta}^{4\sin\theta}$$

$$= 60 \int_0^{\pi} \sin^4\theta \, d\theta = 60 \times 2 \int_0^{\pi/2} \sin^4\theta \, d\theta = 120 \times \frac{3.1}{4.2} \cdot \frac{\pi}{2} = 22.5\,\pi.$$

PROBLEMS 7.1

Evaluate the following integrals (1–7) :

1. $\int_1^2 \int_1^3 xy^2 \, dxdy$.

2. $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2) \, dx \, dy$. *(V.T.U., 2000)*

3. $\int_0^1 \int_0^x e^{x/y} \, dxdy$. *(P.T.U., 2005)*

4. $\int_0^1 \int_0^{\sqrt{(1+x^2)}} \frac{dydx}{1 + x^2 + y^2}$ *(Rajasthan, 2005)*

5. $\iint xy \, dxdy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$. *(Rajasthan, 2006)*

6. $\iint (x + y)^2 dxdy$ over the area bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$. *(Kurukshetra, 2009 S ; U.P.T.U., 2004 S)*

7. $\iint xy(x + y) \, dxdy$ over the area between $y = x^2$ and $y = x$. *(V.T.U., 2010)*

Evaluate the following integrals by changing the order of integration (8–15) :

8. $\int_0^a \int_y^a \frac{xdxdy}{x^2 + y^2}$, *(Bhopal, 2008)*

9. $\int_0^3 \int_1^{\sqrt{(4-y)}} (x + y) \, dxdy$. *(V.T.U., 2005 ; Anna, 2003 S ; Delhi, 2002)*

10. $\int_0^1 \int_x^{\sqrt{(2-x^2)}} \frac{x\,dydx}{\sqrt{(x^2+y^2)}}$. (P.T.U., 2010 ; Marathwada, 2008 ; U.P.T.U., 2006)

11. $\int_0^{a/\sqrt{2}} \int_y^{\sqrt{(a^2-y^2)}} \log(x^2+y^2)\,dxdy\ (a>0)$.

12. $\int_0^1 \int_x^{\sqrt{x}} xy\,dydx$. (V.T.U., 2010)

13. $\int_0^a \int_{a-\sqrt{(a^2-y^2)}}^{a+\sqrt{(a^2-y^2)}} xy\,dx\,dy$. (Anna, 2009)

14. $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dydx$. (Bhopal, 2009 ; S.V.T.U., 2009 ; V.T.U., 2007)

15. $\int_0^\infty \int_0^x xe^{-x^2/y}\,dydx$. (S.V.T.U., 2006 ; U.P.T.U., 2005 ; V.T.U., 2004)

16. Sketch the region of integration of the following integrals and change the order of integrations,

(i) $\int_0^{2a} \int_{\sqrt{(2ax-x^2)}}^{\sqrt{(2ax)}} f(x)\,dxdy$ (Rajasthan, 2006) (ii) $\int_0^{ae^{\pi/4}} \int_{2\log(r/a)}^{\pi/2} f(r,\theta)\,r\,drd\theta$.

17. Show that $\iint_R r^2 \sin\theta\,drd\theta = 2a^2/3$, where R is the semi-circle $r = 2a\cos\theta$ above the initial line.

18. Evaluate $\iint \frac{r\,drd\theta}{\sqrt{a^2+r^2}}$ over one loop of the lemniscate $r^2 = a^2\cos 2\theta$. (Rohtak, 2006 S ; P.T.U., 2005)

19. Evaluate $\iint r^3\,drd\theta$ over the area bounded between the circles $r = 2\cos\theta$ and $r = 4\cos\theta$. (Anna, 2009 ; Madras, 2006)

7.4 AREA ENCLOSED BY PLANE CURVES

(1) *Cartesian coordinates.*

Consider the area enclosed by the curves $y = f_1(x)$ and $y = f_2(x)$ and the ordinates $x = x_1$, $x = x_2$ [Fig. 7.15 (*a*)].

Divide this area into vertical strips of width δx. If $P(x, y)$, $Q(x+\delta x, y+\delta y)$ be two neighbouring points, then the area of the small rectangle $PQ = \delta x\delta y$.

$\therefore$ area of strip $KL = \underset{\delta y \to 0}{\text{Lt}}\ \Sigma\,\delta x\,\delta y$.

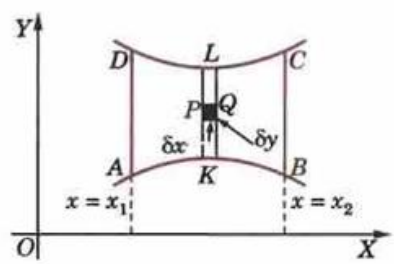

Fig. 7.15(*a*)

Since for all rectangles in this strip δx is the same and y varies from $y = f_1(x)$ to $y = f_2(x)$.

$$\therefore \quad \text{area of the strip } KL = \delta x \underset{\delta y \to 0}{\text{Lt}} \sum_{f_1(x)}^{f_2(x)} dy = \delta x \int_{f_1(x)}^{f_2(x)} dy.$$

Now adding up all such strips from $x = x_1$ to $x = x_2$, we get the area $ABCD$

$$= \underset{\delta x \to 0}{\text{Lt}} \sum_{x_1}^{x_2} \delta x \,.\, \int_{f_1(x)}^{f_2(x)} dy = \int_{x_1}^{x_2} dx \int_{f_1(x)}^{f_2(x)} dy = \int_{x_1}^{x_2} \int_{f_1(x)}^{f_2(x)} dxdy$$

Similarly, dividing the area $A'B'C'D$ [Fig. 7.15(*b*)] into horizontal strips of width δy, we get the area $A'B'C'D'$.

$$= \int_{y_1}^{y_2} \int_{f_1(y)}^{f_2(y)} dxdy$$

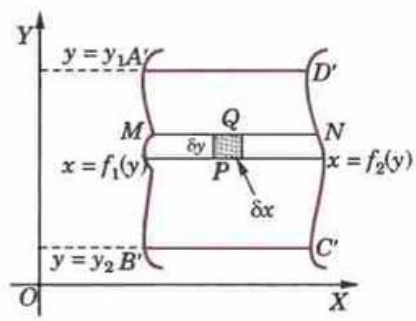

Fig. 7.15 (*b*)

(2) *Polar coordinates.*

Consider an area A enclosed by a curve whose equation is in polar coordinates.

Let $P(r, \theta)$, $Q(r+\delta r, \theta+\delta\theta)$ be two neighbouring points. Mark circular areas of radii r and $r+\delta r$ meeting OQ in R and OP (produced) in S (Fig. 7.16).

Since arc $PR = r\delta\theta$ and $PS = \delta r$.

$\therefore$ area of the curvilinear rectangle $PRQS$ is approximately $= PR \,.\, PS = r\delta\theta \,.\, \delta r$.

If the whole area is divided into such curvilinear rectangles, the sum $\Sigma\Sigma r\delta\theta\delta r$ taken for all these rectangles, gives in the limit the area A.

Hence $$A = \underset{\substack{\delta r \to 0 \\ \delta\theta \to 0}}{\text{Lt}} \Sigma\Sigma r\delta\theta\delta r = \iint r d\theta dr$$

where the limits are to be so chosen as to cover the entire area.

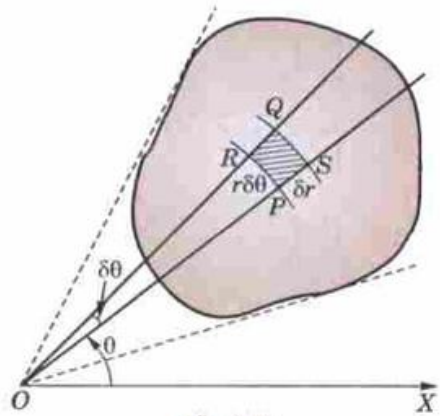

Fig. 7.16

Example 7.13. *Find the area of a plate in the form of a quadrant of the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

(V.T.U., 2001 ; Osmania, 2000 S)

Solution. Dividing the area into vertical strips of width δx, y varies from $K(y = 0)$ to $L[y = b\sqrt{(1 - x^2/b^2)}]$ and then x varies from 0 to a (Fig. 7.17).

$\therefore$ required area

$$= \int_0^a dx \int_0^{b\sqrt{(1-x^2/a^2)}} dy = \int_0^a dx\, [y]_0^{b\sqrt{(1-x^2/a^2)}}$$

$$= \frac{b}{a}\int_0^a \sqrt{(a^2 - x^2)}\, dx = \pi ab/4.$$

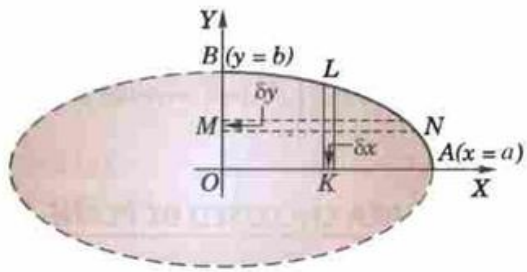

Fig. 7.17

Otherwise, dividing this area into horizontal strips of width δy, x varies from $M(x = 0)$ to $N[x = a\sqrt{(1 - y^2/b^2)}]$ and then y varies from 0 to b.

$\therefore$ required area $$= \int_0^b dy \int_0^{a\sqrt{(1-y^2/b^2)}} dx = \int_0^b dy\, [x]_0^{a\sqrt{(1-y^2/b^2)}}$$

$$= \frac{a}{b}\int_0^b \sqrt{(b^2 - y^2)}\, dy = \pi ab/4.$$

Obs. The change of the order of integration does not in any way affect the value of the area.

Example 7.14. *Show that the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$.*

(Kerala, 2005 ; Rohtak, 2003)

Solution. Solving the equations $y^2 = 4ax$ and $x^2 = 4ay$, it is seen that the parabolas intersect at $O(0, 0)$ and $A(4a, 4a)$. As such for the shaded area between these parabolas (Fig. 7.18) x varies from 0 to $4a$ and y varies from P to Q *i.e.*, from $y = x^2/4a$ to $y = 2\sqrt{(ax)}$. Hence the *required area*

$$= \int_0^{4a} \int_{x^2/4a}^{2\sqrt{(ax)}} dydx = \int_0^{4a} (2\sqrt{(ax)} - x^2/4a)\, dx$$

$$= \left| 2\sqrt{a} \,.\, \frac{2}{3}x^{3/2} - \frac{1}{4a} \,.\, \frac{x^3}{3} \right|_0^{4a} = \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2.$$

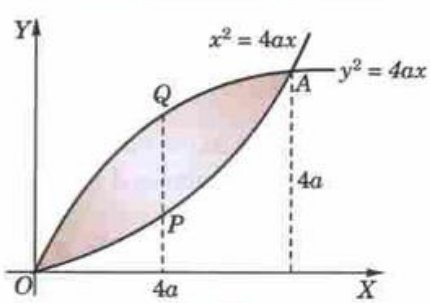

Fig. 7.18

Example 7.15. *Calculate the area included between the curve $r = a(\sec\theta + \cos\theta)$ and its asymptote.*

Solution. The curve is symmetrical about the initial line and has an asymptote $r = a \sec\theta$ (Fig. 7.19).

Draw any line OP cutting the curve at P and its asymptote at P'. Along this line, θ is constant and r varies from $a \sec\theta$ at P' to $a(\sec\theta + \cos\theta)$ at P. Then to get the upper half of the area, θ varies from 0 to $\pi/2$.

$$\therefore \quad \text{required area} = 2\int_0^{\pi/2}\int_{a\sec\theta}^{a(\sec\theta+\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{a\sec\theta}^{a(\sec\theta+\cos\theta)} d\theta$$

$$= a^2\int_0^{\pi/2}(2+\cos^2\theta)\,d\theta = 5\pi a^2/4.$$

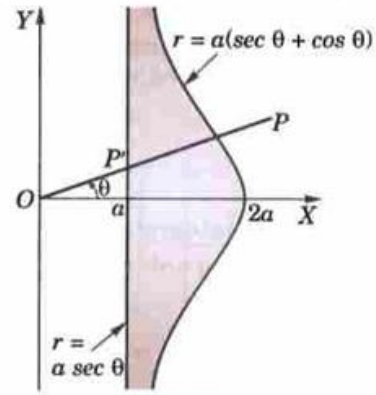

Fig. 7.19

Example 7.16. *Find the area lying inside the cardioid $r = a(1 + \cos\theta)$ and outside the circle $r = a$.*

Solution. In Fig. 7.20, $ABODA$ represents the cardioid $r = a(1 + \cos\theta)$ and $CBA'DC$ is the circle $r = a$.

Required area (shaded) = 2 (area $ABCA$)

$$= 2\int_0^{\pi/2}\int_{r=OP'}^{r=OP} r\,d\theta\,dr = 2\int_0^{\pi/2}\int_a^{a(1+\cos\theta)}(r\,dr)\,d\theta$$

$$= 2\int_0^{\pi/2}\left|\frac{r^2}{2}\right|_a^{a(1+\cos\theta)} d\theta = a^2\int_0^{\pi/2}[(1+\cos\theta)^2 - 1]\,d\theta$$

$$= a^2\int_0^{\pi/2}(\cos^2\theta + 2\cos\theta)\,d\theta = a^2\left(\frac{1}{2}\cdot\frac{\pi}{2} + 2\right) = \frac{a^2}{4}(\pi + 8).$$

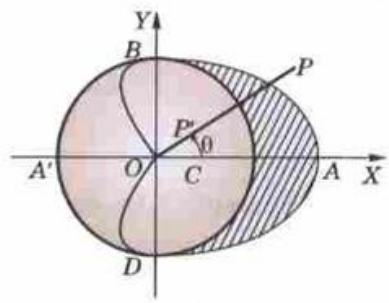

Fig. 7.20

PROBLEMS 7.2

1. Find, by double integration, the area lying between the parabola $y = 4x - x^2$ and the line $y = x$.
2. Find the area lying between the parabola $y = x^2$ and the line $x + y - z = 0$. *(Anna, 2009)*
3. By double integration, find the whole area of the curve $a^2x^2 = y^3(2a - y)$. *(U.P.T.U., 2001)*
4. Find, by double integration, the area enclosed by the curves $y = 3x/(x^2 + 2)$ and $4y = x^2$. *(J.N.T.U., 2005)*
5. Find, by double integration, the area of the lemniscate $r^2 = a^2\cos 2\theta$. *(Madras, 2000 S)*
6. Find, by double integration, the area lying inside the circle $r = a\sin\theta$ and outside the cardioid $r = a(1 - \cos\theta)$. *(Anna 2009 ; Mumbai, 2006)*
7. Find the area lying inside the cardioid $r = 1 + \cos\theta$ and outside the parabola $r(1 + \cos\theta) = 1$.
8. Find the area common to the circles $r = a\cos\theta$, $r = a\sin\theta$ by double integration. *(Mumbai, 2007)*

7.5 TRIPLE INTEGRALS

Consider a function $f(x, y, z)$ defined at every point of the 3-dimensional finite region V. Divide V into n elementary volumes $\delta V_1, \delta V_2, \ldots, \delta V_n$. Let (x_r, y_r, z_r) be any point within the rth sub-division δV_r. Consider the sum

$$\sum_{r=1}^{\infty} f(x_r, y_r, z_r)\delta V_r .$$

The limit of this sum, if it exists, as $n \to \infty$ and $\delta V_r \to 0$ is called the *triple integral of $f(x, y, z)$ over the region V* and is denoted by

$$\iiint f(x, y, z)\,dV.$$

For purposes of evaluation, it can also be expressed as the repeated integral

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x, y, z)\,dx\,dy\,dz.$$

If x_1, x_2 are constants ; y_1, y_2 are either constants or functions of x and z_1, z_2 are either constants or functions of x and y, then this integral is evaluated as follows :

First $f(x, y, z)$ is integrated w.r.t. z between the limits z_1 and z_2 keeping x and y fixed. The resulting expression is integrated w.r.t. y between the limits y_1 and y_2 keeping x constant. The result just obtained is finally integrated w.r.t. x from x_1 to x_2.

Thus
$$I = \boxed{\int_{x_1}^{x_2} \boxed{\int_{y_1(x)}^{y_2(x)} \boxed{\int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)dz} \, dy} \, dx}$$

where the integration is carried out from the innermost rectangle to the outermost rectangle.

The order of integration may be different for different types of limits.

Example 7.17. *Evaluate* $\int_{-1}^{1} \int_{0}^{z} \int_{x-z}^{x+z} (x + y + z)\, dxdydz.$ *(J.N.T.U., 2006 ; Cochin, 2005)*

Solution. Integrating first w.r.t. y keeping x and z constant, we have

$$I = \int_{-1}^{1} \int_{0}^{z} \left| xy + \frac{y^2}{2} + yz \right|_{x-z}^{x+z} dx\, dz = \int_{-1}^{1} \int_{0}^{z} \left[(x + z)(2z) + \frac{1}{2} 4xz \right] dx\, dz$$

$$= 2 \int_{-1}^{1} \left| \frac{x^2 z}{2} + z^2 x + \frac{x^2}{2} z \right|_{0}^{z} dz = 2 \int_{-1}^{1} \left(\frac{z^3}{2} + z^3 + \frac{z^3}{2} \right) dz = 4 \left| \frac{z^4}{4} \right|_{-1}^{1} = 0.$$

Example 7.18. *Evaluate* $\int_{0}^{1} \int_{0}^{\sqrt{(1-x^2)}} \int_{0}^{\sqrt{(1-x^2-y^2)}} xyz\, dxdydz.$ *(V.T.U., 2003 S)*

Solution. We have

$$I = \int_{0}^{1} x \left[\int_{0}^{\sqrt{(1-x^2)}} y \left\{ \int_{0}^{\sqrt{(1-x^2-y^2)}} z\, dz \right\} dy \right] dx = \int_{0}^{1} x \left\{ \int_{0}^{\sqrt{(1-x^2)}} y \,.\, \left| \frac{z^2}{2} \right|_{0}^{\sqrt{(1-x^2-y^2)}} dy \right\} dx$$

$$= \int_{0}^{1} x \left\{ \int_{0}^{\sqrt{(1-x^2)}} y \,.\, \frac{1}{2}(1 - x^2 - y^2)\, dy \right\} dx = \frac{1}{2} \int_{0}^{1} x \left| (1 - x^2)\frac{y^2}{2} - \frac{y^4}{4} \right|_{0}^{\sqrt{(1-x^2)}} dx$$

$$= \frac{1}{8} \int_{0}^{1} [(1 - x^2)^2 \,.\, 2x - (1 - x^2)^4 \,.\, x]\, dx = \frac{1}{8} \int_{0}^{1} (x - 2x^3 + x^5)\, dx$$

$$= \frac{1}{8} \left| \frac{x^2}{2} - \frac{2x^4}{4} + \frac{x^6}{6} \right|_{0}^{1} = \frac{1}{8}\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6} \right) = \frac{1}{48}.$$

PROBLEMS 7.3

Evaluate the following integrals :

1. $\int_{0}^{a} \int_{0}^{b} \int_{0}^{c} (x^2 + y^2 + z^2)\, dx\, dy\, dz.$ *(Anna, 2009)*

2. $\int_{-c}^{c} \int_{-b}^{b} \int_{-a}^{a} (x^2 + y^2 + z^2)\, dx\, dy\, dz$ *(S.V.T.U., 2009 ; V.T.U. 2000)*

3. $\int_{0}^{1} \int_{y^2}^{1} \int_{0}^{1-x} x\, dz\, dx\, dy$ *(Nagpur, 2009)*

4. $\int_{0}^{a} \int_{0}^{x} \int_{0}^{x+y} e^{x+y+z}\, dz\, dy\, dx.$ *(V.T.U., 2010 ; Kurukshetra, 2009 S ; J.N.T.U., 2005)*

5. $\int_{0}^{\log 2} \int_{0}^{x} \int_{0}^{x+\log y} e^{x+y+z}\, dx\, dy\, dz.$ *(Bhopal, 2008)*

6. $\int_{1}^{e} \int_{1}^{\log y} \int_{1}^{e^z} \log z\, dz\, dx\, dy.$ *(S.V.T.U., 2008 ; Rohtak, 2005)*

7. $\int_{0}^{\pi/2} \int_{0}^{a \sin\theta} \int_{0}^{\frac{a^2 - r^2}{a}} r dz\, dr\, d\theta.$ *(V.T.U., 2009)*

7.6 VOLUMES OF SOLIDS

(1) Volumes as double integrals. Consider a surface $z = f(x, y)$. Let the orthogonal projection on XY-plane of its portion S' be the area S (Fig. 7.21).

Divide S into elementary rectangles of area $\delta x\ \delta y$ by drawing lines parallel to X and Y-axes. With each of these rectangles as base, erect a prism having its length parallel to OZ.

$\therefore$ volume of this prism between S and the given surface $z = f(x, y)$ is $z\delta x\delta y$.

Hence the volume of the solid cylinder on S as base, bounded by the given surface with generators parallel to the Z-axis.

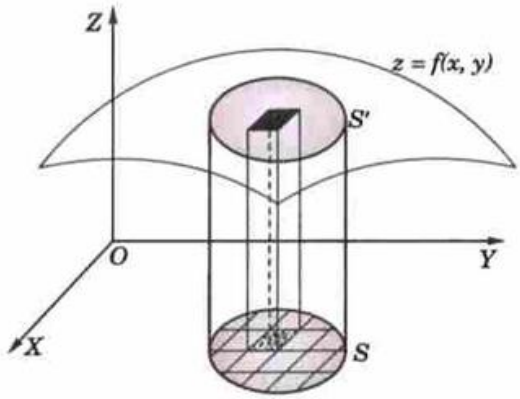

Fig. 7.21

$$= \underset{\substack{\delta x \to 0 \\ \delta y \to 0}}{\text{Lt}} \Sigma\Sigma\, z\delta x\delta y$$

$$= \iint z\, dx\, dy \quad \text{or} \quad \iint f(x, y)\, dx\, dy$$

where the integration is carried over the area S.

Obs. While using polar coordinates, divide S into elements of area $r\delta\theta\ \delta r$.

$\therefore$ replacing $dxdy$ by $r\delta\theta\delta r$, we get the required volume $= \iint zr\, d\theta\, dr$.

Example 7.19. *Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.*
(S.V.T.U., 2007 ; Cochin, 2005 ; Madras, 2000 S)

Solution. From Fig. 7.22, it is self-evident that $z = 4 - y$ is to be integrated over the circle $x^2 + y^2 = 4$ in the XY-plane. To cover the shaded half of this circle, x varies from 0 to $\sqrt{(4 - y^2)}$ and y varies from -2 to 2.

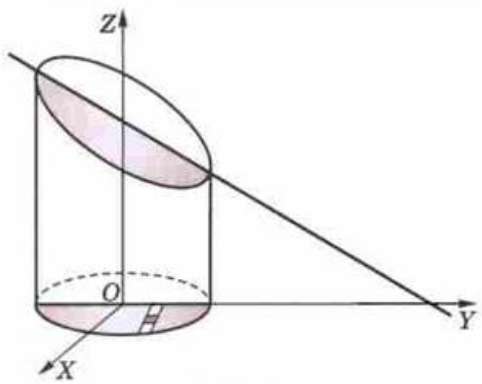

Fig. 7.22

$\therefore$ Required volume

$$= 2\int_{-2}^{2}\int_{0}^{\sqrt{(4-y^2)}} z\, dx\, dy = 2\int_{-2}^{2}\int_{0}^{\sqrt{(4-y^2)}} (4 - y)\, dx\, dy$$

$$= 2\int_{-2}^{2} (4 - y)\,[x]_0^{\sqrt{(4-y^2)}}\, dy = 2\int_{-2}^{2} (4 - y)\sqrt{(4 - y^2)}\, dy$$

$$= 2\int_{-2}^{2} 4\sqrt{(4 - y^2)}\, dy - 2\int_{-2}^{2} y\sqrt{(4 - y^2)}\, dy$$

$$= 8\int_{-2}^{2} \sqrt{(4 - y^2)}\, dy$$ [The second term vanishes as the integrand is an odd function.]

$$= 8\left| \frac{y\sqrt{(4 - y)^2}}{2} + \frac{4}{2}\sin^{-1}\frac{y}{2} \right|_{-2}^{2} = 16\pi.$$

(2) Volume as triple integral

Divide the given solid by planes parallel to the coordinate planes into rectangular parallelopipeds of volume $\delta x\ \delta y\ \delta z$ (Fig. 7.23).

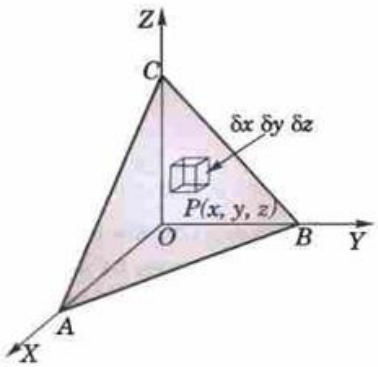

Fig. 7.23

$\therefore$ the total volume $= \underset{\substack{\delta x \to 0 \\ \delta y \to 0 \\ \delta z \to 0}}{\text{Lt}} \Sigma\Sigma\Sigma\ \delta x\ \delta y\ \delta z$

$$= \int\int\int dx\, dy\, dz$$

with appropriate limits of integration.

Example 7.20. *Calculate the volume of the solid bounded by the planes $x = 0, y = 0, x + y + z = a$ and $z = 0$.* (P.T.U., 2009)

Solution. Volume required $= \int_0^a \int_0^{a-x} \int_0^{a-x-y} dz\, dy\, dx$

$$= \int_0^a \int_0^{a-x} (a - x - y)\, dy\, dx = \int_0^a \left| (a-x)y - \frac{y^2}{2} \right|_0^{a-x} dx$$

$$= \int_0^a \left\{ (a-x)^2 - \frac{(a-x)^2}{2} \right\} dx = \frac{1}{2}\int_0^a (a-x)^2\, dx = \frac{1}{2}\left| -\frac{(a-x)^3}{3} \right|_0^a = \frac{a^3}{6}.$$

Example 7.21. *Find the volume of the ellipsoid* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

(Anna, 2009 ; P.T.U., 2006 ; Kottayam, 2005)

Solution. Let $OABC$ be the positive octant of the given ellipsoid which is bounded by the planes OAB $(z = 0)$, OBC $(x = 0)$, OCA $(y = 0)$ and the surface ABC, *i.e.*,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Divide this region R into rectangular parallellopipeds of volume $\delta x \delta y \delta z$. Consider such an element at $P(x, y, z)$. (Fig. 7.24)

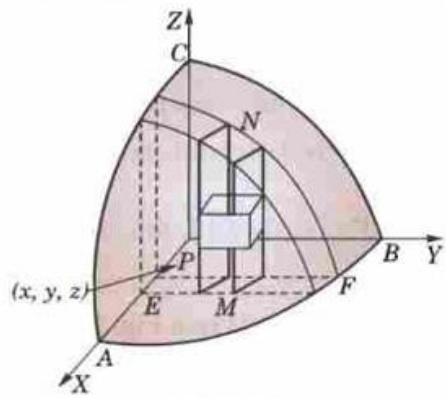

Fig. 7.24

$\therefore$ the required volume $= 8 \iiint_R dx\, dy\, dz$.

In this region R,

(*i*) z varies from 0 to MN where

$$MN = c\sqrt{(1 - x^2/a^2 - y^2/b^2)}.$$

(*ii*) y varies from 0 to EF, where $EF = b\sqrt{(1 - x^2/a^2)}$ from the equation of the ellipse OAB, *i.e.*,

$$x^2/a^2 + y^2/b^2 = 1.$$

(*iii*) x varies from 0 to $OA = a$.

Hence the volume of the whole ellipsoid

$$= 8 \int_0^a \int_0^{b\sqrt{(1-x^2/a^2)}} \int_0^{c\sqrt{(1-x^2/a^2-y^2/b^2)}} dx\, dy\, dz = 8 \int_0^a dx \int_0^{b\sqrt{(1-x^2/a^2)}} dy \,|\, z \,|_0^{c\sqrt{(1-x^2/a^2-y^2/b^2)}}$$

$$= 8c \int_0^a dx \int_0^{b\sqrt{(1-x^2/a^2)}} \sqrt{(1 - x^2/a^2 - y^2/b^2)}\, dy$$

$$= \frac{8c}{b} \int_0^a dx \int_0^{\rho} \sqrt{(\rho^2 - y^2)}\, dy \quad \text{when } \rho = b\sqrt{1 - x^2/a^2}.$$

$$= \frac{8c}{b} \int_0^a dx \left[\frac{y\sqrt{(\rho^2 - y^2)}}{2} + \frac{\rho^2}{2} \sin^{-1}\frac{y}{\rho} \right]_0^{\rho} = \frac{8c}{b} \int_0^a \frac{b^2}{2}\left(1 - \frac{x^2}{a^2}\right)\frac{\pi}{2}\, dx$$

$$= 2\pi bc \int_0^a \left(1 - \frac{x^2}{a^2}\right) dx = 2\pi bc \left| x - \frac{x^3}{3a^2} \right|_0^a = \frac{4\pi abc}{3}.$$

Otherwise. See Problem 27 page 292.

(3) Volumes of solids of revolution

Consider an elementary area $\delta x \delta y$ at the point $P(x, y)$ of a plane area A. (Fig. 7.25)

As this elementary area revolves about x-axis, we get a ring of volume

$$= \pi[(y + \delta y)^2 - y^2]\, \delta x = 2\pi y\, \delta x\, \delta y,$$

nearly to the first powers of δy.

Hence the total volume of the solid formed by the revolution of the area A about x-axis.

$$= \iint_A 2\pi y \, dxdy.$$

In polar coordinates, the above formula for the volume becomes

$$\iint_A 2\pi r \sin\theta \, . \, rd\theta dr, \text{ i.e. } \iint_A 2\pi r^2 \sin\theta \, d\theta dr$$

Similarly, the volume of the solid formed by the revolution of the area A about y-axis $= \iint_A 2\pi x \, dx \, dy$.

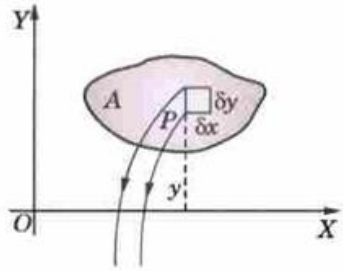

Fig. 7.25

Example 7.22. *Calculate by double integration, the volume generated by the revolution of the cardioid $r = a(1 - \cos\theta)$ about its axis.*

Solution. Required volume

$$= \int_0^{\pi} \int_0^{a(1-\cos\theta)} 2\pi r^2 \sin\theta \, dr \, d\theta$$

$$= 2\pi \int_0^{\pi} \left| \frac{r^3}{3} \right|_0^{a(1-\cos\theta)} \sin\theta \, d\theta$$

$$= \frac{2\pi a^3}{3} \int_0^{\pi} (1 - \cos\theta)^3 \, . \sin\theta \, d\theta = \frac{2\pi a^3}{3} \left| \frac{(1-\cos\theta)^4}{4} \right|_0^{\pi} = \frac{8\pi a^3}{3}.$$

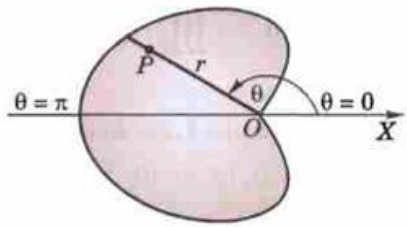

Fig. 7.26

7.7 CHANGE OF VARIABLES

An appropriate choice of co-ordinates quite often facilitates the evaluation of a double or a triple integral. By changing the variables, a given integral can be transformed into a simpler integral involving the new variables.

(1) **In a double integral,** let the variables x, y be changed to the new variables u, v by the transformation.

$$x = \phi(u, v), \; y = \psi(u, v)$$

where $\phi(u, v)$ and $\psi(u, v)$ are continuous and have continuous first order derivatives in some region R'_{uv} in the uv-plane which corresponds to the region R_{xy} in the xy-plane. Then

$$\iint_{R_{xy}} f(x, y) \, dxdy = \iint_{R'_{uv}} f[\phi(u, v), \psi(u, v)] \, | \, J \, | \, dudv \qquad ...(1)$$

where $$J = \frac{\partial(x, y)}{\partial(u, v)} \; (\neq 0)$$

is the *Jacobian of transformation** from (x, y) to (u, v) coordinates.

(2) **For triple integrals,** the formula corresponding to (1) is

$$\iiint_{R_{xyz}} f(x, y, z) \, dxdydz = \iiint_{R'_{uvw}} f[x(u, v, w), y(u, v, w), z(u, v, w)] \, | \, J \, | \, dudvdw$$

where $$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} \; (\neq 0)$$

is the *Jacobian of transformation* from (x, y, z) to (u, v, w) coordinates.

Particular cases :

(*i*) ***To change cartesian coordinates (x, y) to polar coordinates (r, θ),*** *we have* $x = r\cos\theta, y = r\sin\theta$ *and*

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = r \qquad \text{[Ex. 5.25, p. 216]}$$

$$\therefore \quad \iint_{R_{xy}} f(x, y) \, dx \, dy = \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta) \, . \, r \, dr \, d\theta.$$

* See footnote page 215.

(*ii*) ***To change rectangular coordinates (x, y, z) to cylindrical coordinates*** (ρ, ϕ, z) — *Fig. 8.27, we have*

$$x = \rho \cos \phi, y = \rho \sin \phi, z = z$$

and
$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \rho \qquad \text{[Ex. 5.25]}$$

Then $$\iiint_{R_{xyz}} f(x, y, z)\, dxdydz = \iiint_{R'_{\rho\phi z}} f(\rho \cos \phi, \rho \sin \phi, z) \,.\, \rho d\rho d\phi dz.$$

(*iii*) ***To change rectangular coordinates (x, y, z) to spherical polar coordinates (r,*** θ, ϕ)—*Fig. 8.28, we have*

$$x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta$$

and
$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin \theta \qquad \text{[Ex. 5.25]}$$

Then $$\iiint_{R_{xyz}} f(x, y, z)\, dxdydz = \iiint_{R'_{r\theta\phi}} f(r \sin \theta \cos \phi, r \sin \theta, \sin \phi, r \cos \theta) \,.\, r^2 \sin \theta\, drd\theta d\phi$$

Example 7.23. *Evaluate* $\iint_R (x + y)^2\, dxdy$, *where R is the parallelogram in the xy-plane with vertices (1, 0), (3, 1), (2, 2), (0, 1) using the transformation* $u = x + y$ *and* $v = x - 2y$. *(U.P.T.U., 2004)*

Solution. The region R, *i.e.*, parallelogram $ABCD$ in the xy-plane becomes the region R', *i.e.*, rectangle $A'B'C'D'$ in the uv-plane as shown in Fig. 7.27, by taking

$$u = x + y \quad \text{and} \quad v = x - 2y \qquad ...(i)$$

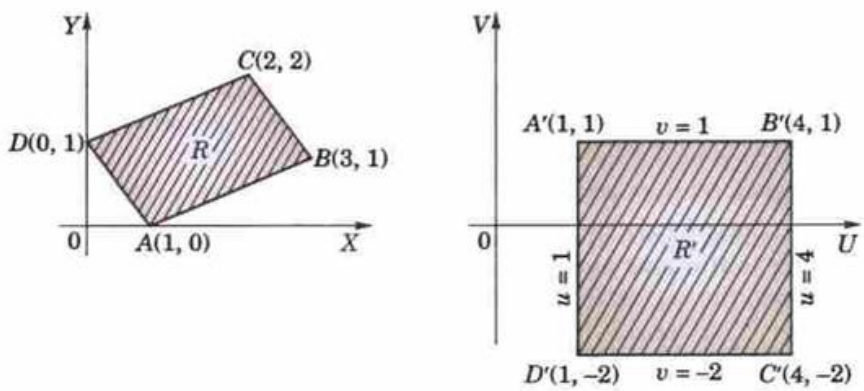

Fig. 7.27

From (*i*), we have $$x = \frac{1}{3}(2u + v), y = \frac{1}{3}(u - v)$$

$$\therefore \qquad \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = -\frac{1}{3}$$

Hence, the given integral

$$= \iint_{R'} u^2 \,|\, J \,|\, dudv = \int_1^4 \int_{-2}^1 u^2 \,.\, \frac{1}{3} \,.\, dudv = \frac{1}{3} \left| \frac{u^3}{3} \right|_1^4 . \left| v \right|_{-2}^1 = 21.$$

Example 7.24. *Evaluate* $\iint_D xy\sqrt{(1 - x - y)}\, dx\, dy$ *where D is the region bounded by* $x = 0$, $y = 0$ *and* $x + y = 1$ *using the transformation* $x + y = u$, $y = uv$. *(Marathwada, 2008)*

Solution. We have $x = u - uv$, $y = uv$

$$\therefore \qquad J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 1-v & v \\ -u & u \end{vmatrix} = u.$$

Also when $x = 0$, $u = 0$, $v = 1$; when $y = 0$, $u = 0$, $v = 0$ and when $x + y = 1$, $u = 1$

$\therefore$ the limits of u are from 0 to 1 and limits of v are from 0 to 1.

Thus
$$\iint_D xy\sqrt{(1-x-y)}\,dxdy = \int_0^1 \int_0^1 u(1-v)\,uv\,(1-u)^{1/2}\,|J|\,dudv$$

$$= \int_0^1 \int_0^1 u^3(1-u)^{1/2}\,v(1-v)\,du\,dv$$

where $u = \sin^2\theta$
$du = 2\sin\theta\cos\theta\,d\theta$
$u = 0, \theta = 0$
$u = 1, \theta = \pi/2$

$$= \int_0^1 u^3(1-u)^{1/2}\,du \times \int_0^1 v(1-v)\,dv$$

$$= \int_0^{\pi/2} \sin^6\theta\cos\theta\,.\,2\sin\theta\cos d\theta \times \left|\frac{v^2}{2} - \frac{v^3}{3}\right|_0^1$$

$$= 2\int_0^{\pi/2} \sin^7\theta\cos^2\theta\,d\theta\left(\frac{1}{6}\right) = \frac{1}{3}\cdot\frac{6\,.\,1}{9\,.\,7\,.\,5\,.\,3} = \frac{2}{945}.$$

Example 7.25. *Evaluate* $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dxdy$ *by changing to polar coordinates.* *(Anna, 2003)*

Hence show that $\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi/2}$. *(Madras, 2003 ; U.P.T.U., 2003 ; J.N.T.U., 2000)*

Solution. The region of integration being the first quadrant of the xy-plane, r varies from 0 to ∞ and θ varies from 0 to $\pi/2$. Hence,

$$I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dxdy = \int_{\theta=0}^{\pi/2} \int_{r=0}^\infty e^{-r^2}\,rdrd\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2}\left\{\int_0^\infty e^{-r^2}(-2r)\,dr\right\}d\theta = -\frac{1}{2}\int_0^{\pi/2}\left|e^{-r^2}\right|_0^\infty d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{\pi}{4}. \qquad ...(i)$$

Also
$$I = \int_0^\infty e^{-x^2}\,dx \times \int_0^\infty e^{-y^2}\,dy = \left\{\int_0^\infty e^{-x^2}\,dx\right\}^2 \qquad ...(ii)$$

Thus, from (i) and (ii), we have $\int_0^\infty e^{-x^2} = \sqrt{\pi/2}$. ...$(iii)$

Example 7.26. *Find the volume bounded by the paraboloid* $x^2 + y^2 = az$, *the cylinder* $x^2 + y^2 = 2ay$ *and the plane* $z = 0$.

Solution. The required volume is found by integrating $z = (x^2 + y^2)/a$ over the circle $x^2 + y^2 = 2ay$.

Changing to polar coordinates in the xy-plane, we have $x = r\cos\theta$, $y = r\sin\theta$ so that $z = r^2/a$ and the polar equation of the circle is $r = 2a\sin\theta$.

To cover this circle, r varies from 0 to $2a\sin\theta$ and θ varies from 0 to π. (Fig. 7.28)

Hence the required volume

$$= \int_0^\pi \int_0^{2a\sin\theta} z\,.\,r\,d\theta\,dr = \frac{1}{a}\int_0^\pi d\theta \int_0^{2a\sin\theta} r^3\,dr$$

$$= \frac{1}{a}\int_0^\pi d\theta\left|\frac{r^4}{4}\right|_0^{2a\sin\theta} = 4a^3\int_0^\pi \sin^4\theta\,d\theta = \frac{3\pi a^3}{2}.$$

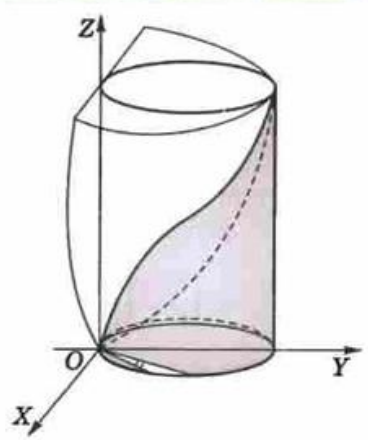

Fig. 7.28

Example 7.27. *Find, by triple integration, the volume of the sphere $x^2 + y^2 + z^2 = a^2$.*
(Bhopal, 2009 ; Madras, 2006 ; V.T.U., 2003 S)

Solution. Changing to polar spherical coordinates by putting

$$x = r \sin\theta \cos\phi,\ y = r \sin\theta \sin\phi,\ z = r \cos\theta$$

we have $dx\,dy\,dz = r^2 \sin\theta\, dr\, d\theta\, d\phi$.

Also the volume of the sphere is 8 times the volume of its portion in the positive octant for which r varies from 0 to a, θ varies from 0 to $\pi/2$ and ϕ varies from 0 to $\pi/2$.

$\therefore$ volume of the sphere

$$= 8 \int_0^a \int_0^{\pi/2} \int_0^{\pi/2} r^2 \sin\theta\, dr\, d\theta\, d\phi = 8 \int_0^a r^2\, dr \,.\, \int_0^{\pi/2} .\sin\theta\, d\theta \,.\, \int_0^{\pi/2} d\phi$$

$$= 8 \,.\, \left| \frac{r^3}{3} \right|_0^a .\, \left| -\cos\theta \right|_0^{\pi/2} .\, \frac{\pi}{2} = 4\pi \,.\, \frac{a^3}{3} \,.\, (-0 + 1) = \frac{4}{3}\pi a^3.$$

Example 7.28. *Find the volume of the portion of the sphere $x^2 + y^2 + z^2 = a^2$ lying inside the cylinder $x^2 + y^2 = ay$.* *(Rohtak, 2003)*

Solution. The required volume is easily found by changing to cylindrical coordinates (ρ, ϕ, z). We therefore, have

$$x = \rho \cos\phi,\ y = \rho \sin\phi,\ z = z$$

and
$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \rho.$$

Then the equation of the sphere becomes $\rho^2 + z^2 = a^2$ and that of cylinder becomes $\rho = a \sin\phi$.

The volume inside the cylinder bounded by the sphere is twice the volume shown shaded in the Fig. 7.29 for which z varies from 0 to $\sqrt{(a^2 - \rho^2)}$, ρ varies from 0 to $a \sin\phi$ and ϕ varies from 0 to π.

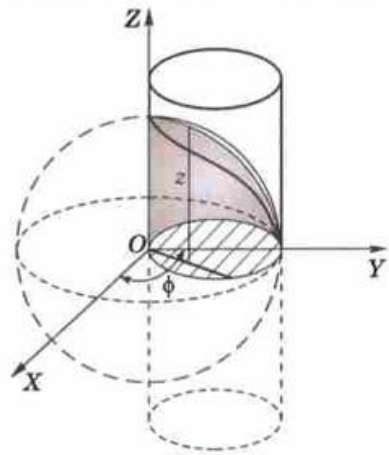

Fig. 7.29

Hence the required volume $= 2 \int_0^{\pi} \int_0^{a \sin\phi} \int_0^{\sqrt{(a^2 - \rho^2)}} \rho\, dz\, d\rho\, d\phi$

$$= 2 \int_0^{\pi} \int_0^{a \sin\phi} \rho \sqrt{(a^2 - \rho^2)}\, d\rho\, d\phi = 2 \int_0^{\pi} \left| -\frac{1}{3}(a^2 - \rho^2)^{3/2} \right|_0^{a \sin\phi} d\phi$$

$$= \frac{2a^3}{3} \int_0^{\pi} (1 - \cos^3\phi)\, d\phi = \frac{2a^3}{9}(3\pi - 4).$$

Example 7.29. *Evaluate* $\displaystyle\int_0^1 \int_0^{\sqrt{(1-x^2)}} \int_{\sqrt{(x^2+y^2)}}^1 \frac{dz\, dy\, dx}{\sqrt{(x^2 + y^2 + z^2)}}$. *(V.T.U., 2008)*

Solution. We change to spherical polar coordinates (r, θ, ϕ), so that

$$x = r \sin\theta \cos\phi,\ y = r \sin\theta \sin\phi,\ z = r \cos\theta$$

and
$$J = r^2 \sin\theta,\ x^2 + y^2 + z^2 = r^2.$$

The region of integration is common to the cone $z^2 = x^2 + y^2$ and the cylinder $x^2 + y^2 = 1$ bounded by the plane $z = 1$ in the positive octant (Fig. 7.30). Hence θ varies from 0 to $\pi/4$, r varies from 0 to $\sec\theta$ and ϕ varies from 0 to $\pi/2$.

$\therefore$ given integral becomes

$$\int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\sec\theta} \frac{1}{r} \,.\, r^2 \sin\theta\, dr\, d\theta\, d\phi = \int_0^{\pi/2} d\phi \int_0^{\pi/4} \left| \frac{r^2}{2} \right|_0^{\sec\theta} \sin\theta\, d\theta$$

$$= \frac{\pi}{2} \int_0^{\pi/4} \frac{\sec^2\theta}{2} \sin\theta\, d\theta = \frac{\pi}{4} \int_0^{\pi/4} \sec\theta \tan\theta\, d\theta = \frac{\pi}{4} \left| \sec\theta \right|_0^{\pi/4} = \frac{(\sqrt{2} - 1)\pi}{4}.$$

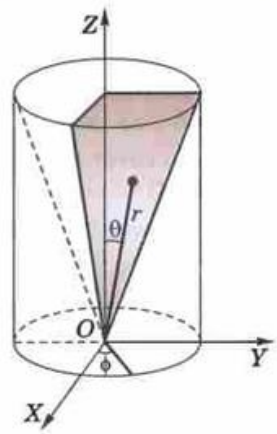

Fig. 7.30

Example 7.30. *Find the volume of the solid surrounded by the surface*
$$(x/a)^{2/3} + (y/b)^{2/3} + (z/c)^{2/3} = 1. \qquad \textit{(Hissar, 2005 S)}$$

Solution. Changing the variables, x, y, z to X, Y, Z where, $(x/a)^{1/3} = X$, $(y/b)^{1/3} = Y$, $(z/c)^{1/3} = Z$

i.e., $\quad x = aX^3, y = bY^3, z = cZ^3$ so that $J = \partial(x, y, z)/\partial(X, Y, Z) = 27\, abc\, X^2Y^2Z^2$.

$$\therefore \quad \text{required volume} = \iiint dx\, dy\, dz = 27\, abc \iiint X^2Y^2Z^2\, dX\, dY dZ$$

taken throughout the sphere $X^2 + Y^2 + Z^2 = 1$. ...(*i*)

Now change X, Y, Z to spherical polar coordinates r, θ, ϕ so that $X = r \sin\theta \cos\phi$, $Y = r \sin\theta \sin\phi$, $Z = r\cos\theta$, and $\partial(X, Y, Z)/\partial(r, \theta, \phi) = r^2 \sin\theta$. To describe the positive octant of the sphere (*i*), r varies from 0 to 1, θ from 0 to $\pi/2$ and ϕ from 0 to $\pi/2$.

$$\therefore \quad \text{required volume} = 27abc \times 8 \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} r^2 \sin^2\theta \cos^2\phi \times r^2 \sin^2\theta \sin^2\phi \,.\, r^2\cos^2\theta \,.\, r^2 \sin\theta\, dr d\theta d\phi$$

$$= 216\, abc \int_0^1 r^8\, dr \int_0^{\pi/2} \sin^4\theta \cos^2\theta\, d\theta \int_0^{\pi/2} \sin^2\phi \cos^2\phi\, d\phi = 4\pi\, abc/35.$$

PROBLEMS 7.4

Evaluate the following integrals by changing to polar co-ordinates :

1. $\int_0^1 \int_0^{\sqrt{(1-y^2)}} (x^2 + y^2)\, dy\, dx$. *(P.T.U., 2010)*

2. $\int_0^2 \int_0^{\sqrt{(2x-x^2)}} \frac{x\, dx\, dy}{x^2 + y^2}$ *(Anna, 2009)*

3. $\int_0^{4a} \int_{y^2/4a}^{y} \frac{x^2 - y^2}{x^2 + y^2} dxdy$ *(Mumbai, 2006)*

4. $\iint xy(x^2 + y^2)^{n/2}\, dxdy$ over the positive quadrant of $x^2 + y^2 = 4$, supposing $n + 3 > 0$. *(S.V.T.U., 2007)*

5. $\iint \frac{dxdy}{(1 + x^2 + y^2)^2}$ over one loop of the lemniscate $(x^2 + y^2) = x^2 - y^2$. *(Mumbai, 2007)*

6. Transform the following to cartesian form and hence evaluate $\int_0^{\pi} \int_0^{a} r^3 \sin\theta \cos\theta\, dr d\theta$. *(P.T.U., 2005)*

7. $\iint y^2\, dx\, dy$ over the area outside $x^2 + y^2 - ax = 0$ and inside $x^2 + y^2 - 2ax = 0$. *(Mumbai, 2006)*

8. By using the transformation $x + y = u$, $y = uv$, show that $\int_0^1 \int_0^{1-x} e^{y/(x+y)}\, dydx = \frac{1}{2}(e - 1)$. *(P.T.U., 2003)*

9. Transform $\int_0^{\pi/2} \int_0^{\pi/2} \sqrt{\frac{\sin\phi}{\sin\theta}}\, d\phi\, d\theta$ by the substitution $x = \sin\phi \cos\theta$, $y = \sin\phi \sin\theta$ and show that its value is π. *(U.P.T.U., 2001)*

Evaluate the following integrals by changing to spherical coordinates :

10. $\int_0^1 \int_0^{\sqrt{(1-x^2)}} \int_0^{\sqrt{(1-x^2-y^2)}} \frac{dx\, dy\, dz}{\sqrt{(1 - x^2 - y^2 - z^2)}}$. *(V.T.U., 2006 ; Kottayam, 2005)*

11. $\iiint_V \frac{dx\, dy\, dz}{x^2 + y^2 + z^2}$ where V is the volume of the sphere $x^2 + y^2 + z^2 = a^2$. *(Anna, 2009)*

12. Evaluate $\iiint \frac{dx\, dy\, dz}{(1 + x + y + z)^3}$ over the volume of the tetrahedron $x = 0, y = 0, z = 0, x + y + z = 1$. *(Mumbai, 2007)*

13. Show that $\iiint \frac{dx\, dy\, dz}{\sqrt{(a^2 - x^2 - y^2 - z^2)}} = \frac{\pi^2 a^3}{8}$, the integral being extended for all the values of the variables for which the expression is real. *(U.T.U., 2010)*

14. $\iiint z^2\, dx\, dy\, dz$, taken over the volume bounded by the surfaces $x^2 + y^2 = a^2$, $x^2 + y^2 = z$ and $z = 0$.

15. Find the volume bounded by the xy-plane, the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 3$. (I.S.M., 2001)
16. Find the volume bounded by the xy-plane, the paraboloid $2z = x^2 + y^2$ and the cylinder $x^2 + y^2 = 4$. (Raipur, 2005)
17. Find the volume cut from the sphere $x^2 + y^2 + z^2 = a^2$ by the cone $x^2 + y^2 = z^2$.
18. Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$. (S.V.T.U., 2006)
19. Find the volume cut off from the cylinder $x^2 + y^2 = ax$ by the planes $z = 0$ and $z = x$. (U.P.T.U., 2006)
20. Find the volume enclosed by the cylinders $x^2 + y^2 = 2ax$ and $z^2 = 2ax$. (Marathwada, 2008)
21. Find the volume of the cylinder $x^2 + y^2 - 2ax = 0$, intercepted between the paraboloid $x^2 + y^2 = 2az$ and the xy-plane.
22. Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the hyperboloid $x^2 + y^2 - z^2 = 1$.
23. Find the volume of the region bounded by $z = x^2 + y^2$, $z = 0$, $x = -a$, $x = a$ and $y = -a$, $y = a$.
24. Prove, by using a double integral that the volume generated by the revolution of the cardioid $r = a(1 + \cos\theta)$ about its axis is $8\pi a^3/3$. (V.T.U., 2000)
25. Evaluate $\iiint (x + y + z)\, dx\, dy\, dz$ over the tetrahedron bounded by the planes $x = 0, y = 0, z = 0$ and $x + y + z = 1$. [See Fig. 7.34]
26. Find the volume of the tetrahedron bounded by the coordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. (Burdwan, 2003)
27. Work out example 7.21 by changing the variables.

7.8 AREA OF A CURVED SURFACE

Consider a point P of the surface $S : z = f(x, y)$. Let its projection on the xy-plane be the region A. Divide it into area elements by drawing lines parallel to the axes of X and Y. (Fig. 7.31).

On the element $\delta x \delta y$ as base, erect a cylinder having generators parallel to OZ and meeting the surface S in an element of area δS.

As $\delta x \delta y$ is the projection of δS on the xy-plane,

$\therefore \quad \delta x \delta y = \delta S \,.\, \cos\gamma$, where γ is the angle between the xy-plane and the tangent plane to S at P, *i.e.*, it is the angle between the Z-axis and the normal to S at $P (= \angle Z'PN)$.

Now since the direction cosines of the normal to the surface $F(x, y, z) = 0$ proportional to

$$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}.$$

$\therefore$ the direction cosines of the normal to $S[F = f(x, y) - z]$ are proportional to $-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}$, 1 and those of the z-axis are 0, 0, 1.

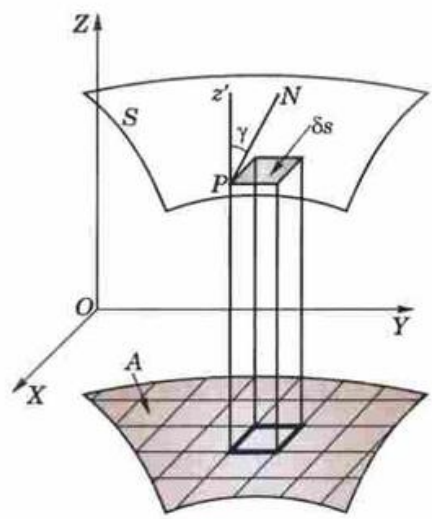

Fig. 7.31

Hence $$\cos\gamma = \frac{1}{\sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}} \quad \therefore \quad \delta S = \frac{\delta x\, \delta y}{\cos\gamma} = \sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}\, \delta x\, \delta y$$

Hence $$S = \underset{\delta S \to 0}{\text{Lt}} \sum \delta S = \iint_A \sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}\, dx dy$$

Similarly, if B and C be the projections of S on the yz-and zx-planes respectively, then

$$S = \iint_B \sqrt{\left[\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 + 1\right]}\, dy\, dz$$

and $$S = \iint_C \sqrt{\left[\left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial x}\right)^2 + 1\right]}\, dz\, dx\,.$$

Example 7.31. *Find the area of the portion of the cylinder $x^2 + z^2 = 4$ lying inside the cylinder $x^2 + y^2 = 4$.*

Solution. Figure 7.32 shows one-eighth of the required area. Its projection on the xy-plane is a quadrant circle $x^2 + y^2 = 4$.

For the cylinder $\quad x^2 + z^2 = 4, \quad \ldots(i)$

we have
$$\frac{\partial z}{\partial x} = -\frac{x}{z}, \frac{\partial z}{\partial y} = 0.$$

so that
$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1 = \frac{x^2 + z^2}{z^2} = \frac{4}{4 - x^2}.$$

Hence the required surface area = 8 (surface area of the upper portion of (i) lying within the cylinder $x^2 + y^2 = 4$ in the positive octant)

$$= 8 \int_0^2 \int_0^{\sqrt{(4-x^2)}} \frac{2}{\sqrt{(4-x^2)}}\, dx dy = 16 \int_0^2 dx = 32 \text{ sq. units.}$$

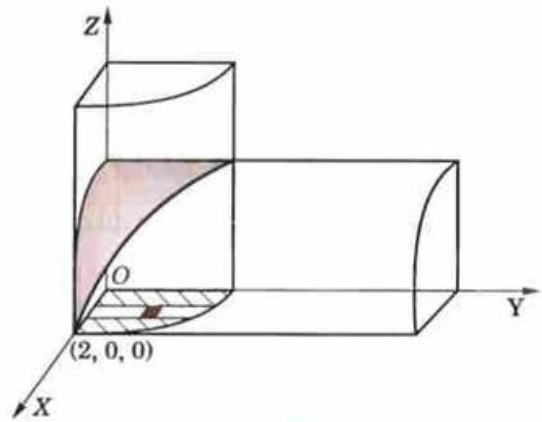

Fig. 7.32

Example 7.32. *Find the area of the portion of the sphere $x^2 + y^2 + z^2 = 9$ lying inside the cylinder $x^2 + y^2 = 3y$.*

Solution. Figure 7.33 shows one-fourth of the required area. Its projection on the xy-plane is the semi-circle $x^2 + y^2 = 3y$ bounded by the Y-axis.

For the sphere

$$x^2 + y^2 + z^2 = 9, \frac{\partial z}{\partial x} = -\frac{x}{z} \text{ and } \frac{\partial z}{\partial y} = -\frac{y}{z}$$

$$\therefore \quad \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1 = (x^2 + y^2 + z^2)/z^2$$

$$= \frac{9}{9 - x^2 - y^2} = \frac{9}{9 - r^2} \quad \text{when } x = r\cos\theta, y = r\sin\theta.$$

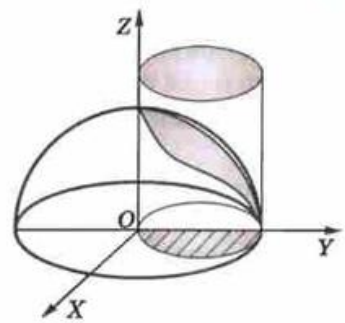

Fig. 7.33

Using polar coordinates, the required area is found by integrating $3/\sqrt{(9 - r^2)}$ over the semi-circle $r = 3\sin\theta$, for which r varies from 0 to $3\sin\theta$ and θ varies from 0 to $\pi/2$.

Hence the required surface area

$$= 4 \int_0^{\pi/2} \int_0^{3\sin\theta} \frac{3}{\sqrt{(9 - r^2)}}\, r d\theta\, dr = -6 \int_0^{\pi/2} \left| \frac{\sqrt{(9 - r^2)}}{1/2} \right|_0^{3\sin\theta} d\theta$$

$$= 36 \int_0^{\pi/2} (1 - \cos\theta)\, d\theta = 36 \left| \theta - \sin\theta \right|_0^{\pi/2} = 18(\pi - 2) \text{ sq. units.}$$

PROBLEMS 7.5

1. Show that the surface area of the sphere $x^2 + y^2 + z^2 = a^2$ is $4\pi a^2$.
2. Find the area of the portion of the cylinder $x^2 + y^2 = 4y$ lying inside the sphere $x^2 + y^2 + z^2 = 16$.
3. Find the area of the portion of the sphere $x^2 + y^2 + z^2 = a^2$ lying inside the cylinder $x^2 + y^2 = ax$.
4. Find the area of the surface of the cone $x^2 + y^2 = z^2$ cut off by the surface of the cylinder $x^2 + y^2 = a^2$ above the xy-plane.
5. Compute the area of that part of the plane $x + y + z = 2a$ which lies in the first octant and is bounded by the cylinder $x^2 + y^2 = a^2$. *(Burdwan, 2003)*

7.9 CALCULATION OF MASS

(*a*) **For a plane lamina,** if the surface density at the point $P(x, y)$ be $\rho = f(x, y)$ then the elementary mass at $P = \rho\delta x\delta y$.

$$\therefore \quad \textit{total mass of the lamina} = \iint \rho\, dx\, dy \qquad ...(i)$$

with integrals embracing the whole area of the lamina.

In polar coordinates, taking $\rho = \phi(r, \theta)$ at the point $P(r, \theta)$,

$$\textit{total mass of the lamina} = \iint \rho r\, d\theta\, dr \qquad ...(ii)$$

(*b*) **For a solid,** if the density at the point $P(x, y, z)$ be $\rho = f(x, y, z)$, then

total mass of the solid $= \iiint \rho\, dx\, dy\, dz$ with appropriate limits of integration.

Example 7.33. *Find the mass of the tetrahedron bounded by the coordinates planes and the plane* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, *the variable density* $\rho = \mu xyz$. *(Rohtak, 2003 ; U.P.T.U., 2003)*

Solution. Elementary mass at $P = \mu xyz\,.\,\delta x\,\delta y\,\delta z$.

$$\therefore \quad \text{the whole mass} = \iiint \mu xyz\, dx\, dy\, dz,$$

the integrals embracing the whole volume $OABC$ (Fig. 7.34). The limits for z are from 0 to $z = c(1 - x/a - y/b)$.

The limits for y are from 0 to $y = b(1 - x/a)$ and limits for x are from 0 to a.

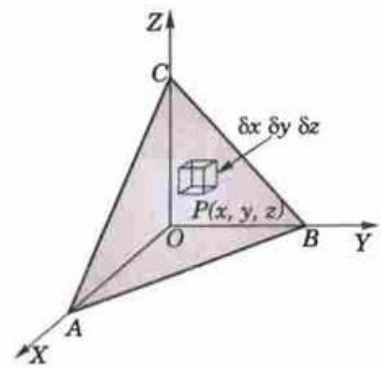

Fig. 7.34

Hence the required mass

$$= \int_0^a \int_0^{b(1-x/a)} \int_0^{c(1-x/a-y/b)} \mu xyz\, dz\, dy\, dx$$

$$= \mu \int_0^a \int_0^{b(1-x/a)} xy \left| z^2/2 \right|_0^{c(1-x/a-y/b)} dydz$$

$$= \mu \int_0^a \int_0^{b(1-x/a)} xy\,.\,\frac{c^2}{2}\left(1 - \frac{x}{a} - \frac{y}{b}\right)^2 dydx$$

$$= \frac{\mu c^2}{2} \int_0^a \int_0^{b(1-x/a)} x\,.\left[\left(1 - \frac{x}{a}\right)^2 y - 2\left(1 - \frac{x}{a}\right)\frac{y^2}{b} + \frac{y^3}{b^2}\right] dy\, dx$$

$$= \frac{\mu c^2}{2} \int_0^a x \left| \left(1 - \frac{x}{a}\right)^2 \frac{y^2}{2} - 2\left(1 - \frac{x}{a}\right)\frac{y^3}{3b} + \frac{y^4}{4b^2} \right|_0^{b(1-x/a)} dx$$

$$= \frac{\mu c^2}{2} \int_0^a x \left[\frac{b^2}{2}\left(1 - \frac{x}{a}\right)^4 - \frac{2b^2}{3}\left(1 - \frac{x}{a}\right)^4 + \frac{b^2}{4}\left(1 - \frac{x}{a}\right)^4\right] dx = \frac{\mu b^2 c^2}{24} \int_0^a x\,(1 - x/a)^4\, dx = \frac{\mu a^2 b^2 c^2}{720}.$$

7.10 CENTRE OF GRAVITY

(*a*) **To find the C.G. $(\bar{x}, \bar{y})$ of a plane lamina,** take the element of mass $\rho\delta x\delta y$ at the point $P(x, y)$. Then

$$\bar{x} = \frac{\iint x\rho\, dxdy}{\iint \rho\, dxdy},\ \bar{y} = \frac{\iint y\,\rho dxdy}{\iint \rho\, dxdy} \text{ with integrals embracing the whole lamina.}$$

While using polar coordinates, take the elementary mass as $\rho r\delta\theta\delta r$ at the point $P(r, \theta)$ so that $x = r\cos\theta$, $y = r\sin\theta$.

$$\therefore \quad \bar{x} = \frac{\iint r\cos\theta\, \rho r\, d\theta dr}{\iint \rho\, rd\theta dr},\ \bar{y} = \frac{\iint r\sin\theta\, \rho\, rd\theta dr}{\iint \rho\, rd\theta dr}$$

(*b*) **To find the C.G. $(\bar{x}, \bar{y}, \bar{z})$ of a solid,** take an element of mass $\rho\delta x\delta y\delta z$ enclosing the point $P(x, y, z)$. *Then*

$$\bar{x} = \frac{\iiint x\rho\, dxdydz}{\iiint \rho\, dxdydz}, \quad \bar{y} = \frac{\iiint y\rho\, dxdydz}{\iiint \rho\, dxdydz} \text{ and } \bar{z} = \frac{\iiint z\rho\, dxdydz}{\iiint \rho\, dxdydz}.$$

Example 7.34. *Find by double integration, the centre of gravity of the area of the cardioid $r = a\,(1 + \cos\theta)$.*

Solution. The cardioid being symmetrical about the initial line, its C.G. lies on OX, *i.e.*, $\bar{y} = 0$ (Fig. 7.35).

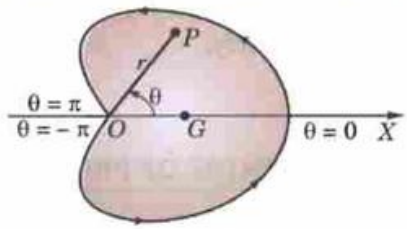

Fig. 7.35

$$\therefore \quad \bar{x} = \frac{\iint \rho\, r\cos\theta\,.\,rd\theta\, dr}{\iint \rho\, rd\theta\, dr} = \frac{\int_{-\pi}^{\pi}\int_0^{a(1+\cos\theta)} \cos\theta.\, r^2 dr\,.\,d\theta}{\int_{-\pi}^{\pi}\int_0^{a(1+\cos\theta)} rdr\,.\,d\theta}$$

$$= \frac{\int_{-\pi}^{\pi} \cos\theta \left| \frac{r^3}{3} \right|_0^{a(1+\cos\theta)} d\theta}{\int_{-\pi}^{\pi} \left| \frac{r^2}{2} \right|_0^{a(1+\cos\theta)} d\theta} = \frac{2a}{3}\cdot\frac{\int_{-\pi}^{\pi} \cos\theta\,(1+\cos\theta)^3\, d\theta}{\int_{-\pi}^{\pi} (1+\cos\theta)^2\, d\theta}$$

$$= \frac{2a}{3}\cdot\frac{2\,.\int_0^{\pi} (3\cos^2\theta + \cos^4\theta)\, d\theta}{2\,.\int_0^{\pi} (1+\cos^2\theta)\, d\theta} \qquad \left\{\because \int_{-\pi}^{\pi} \cos^n\theta\, d\theta = 2\int_0^{\pi} \cos^n\theta\, d\theta \text{ or } 0 \text{ according as } n \text{ in even or odd.}\right\}$$

$$= \frac{2a}{3}\cdot\frac{2\,.\int_0^{\pi/2} (3\cos^2\theta + \cos^4\theta)\, d\theta}{2\,.\int_0^{\pi/2} (1+\cos^2\theta)\, d\theta} \text{ (as the powers of } \cos\theta \text{ are even)} = \frac{2a}{3}\cdot\frac{3\cdot\frac{1}{2}\cdot\frac{\pi}{2} + \frac{3\,.\,1}{4\,.\,2}\cdot\frac{\pi}{2}}{\frac{\pi}{2} + \frac{1}{2}\cdot\frac{\pi}{2}} = \frac{5a}{6}$$

Hence the C.G. of the cardioid is at $G(5a/6, 0)$.

Example 7.35. *Using double integration, find the C.G. of a lamina in the shape of a quadrant of the curve $(x/a)^{2/3} + (y/b)^{2/3} = 1$, the density being $\rho = kxy$, where k is a constant.*

Solution. Let G $(\bar{x}, \bar{y})$ be the C.G. of the lamina OAB (Fig. 7.36), so that

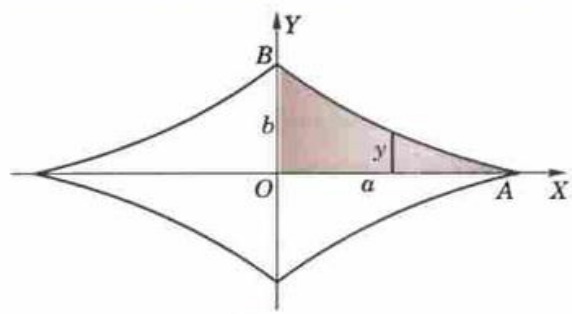

Fig. 7.36

$$\bar{x} = \frac{\iint kxy\,.\,xdxdy}{\iint kxy\,.\,dxdy} = \frac{\iint x^2 y\, dxdy}{\iint xy\, dxdy}$$

where the integrals are taken over the area OAB so that y varies from 0 to y (to be found from the equation of the curve in terms of x) and then x varies from 0 to a.

Thus

$$\bar{x} = \frac{\int_0^a \int_0^y x^2 y\, dydx}{\int_0^a \int_0^y xy\, dydx} = \frac{\int_0^a x^2 \,.\left| y^2/2 \right|_0^y dx}{\int_0^a x\,.\left| y^2/2 \right|_0^y dx} = \frac{\int_0^a x^2 y^2\, dx}{\int_0^a xy^2\, dx}$$

For any point on the curve, we have

$$x = a\cos^3\theta,\ y = b\sin^3\theta \text{ so that}$$
$$dx = -3a\cos^2\theta\sin\theta\, d\theta.$$

Also when $\quad x = 0,\ \theta = \pi/2$; when $x = a,\ \theta = 0$.

Hence
$$\bar{x} = \frac{\int_{\pi/2}^{0} a^2 \cos^6\theta \,.\, b^2 \sin^6\theta \,.\, (-3a\cos^2\theta\sin\theta)\, d\theta}{\int_{\pi/2}^{0} a\cos^3\theta \,.\, b^2\sin^6\theta \,.\, (-3a\cos^2\theta\sin\theta)\, d\theta}$$
$$= a\frac{\int_0^{\pi/2} \sin^7\theta\cos^8\theta\, d\theta}{\int_0^{\pi/2} \sin^7\theta\cos^5\theta\, d\theta} = \frac{128}{429}a$$

Similarly,
$$\bar{y} = \frac{\int_0^a \int_0^y kxy \,.\, y\, dxdy}{\int_0^a \int_0^y kxy \,.\, dxdy} = \frac{128}{429}b.$$ Hence the required C.G. is $G\left(\frac{128}{429}a, \frac{128}{429}b\right)$.

7.11 CENTRE OF PRESSURE

Consider plane area A immersed vertically in a homogeneous liquid. Take the line of intersection of the given plane with the free surface of the liquid as the x-axis and any line lying in this plane and perpendicular to it downwards as the y-axis (Fig. 7.37).

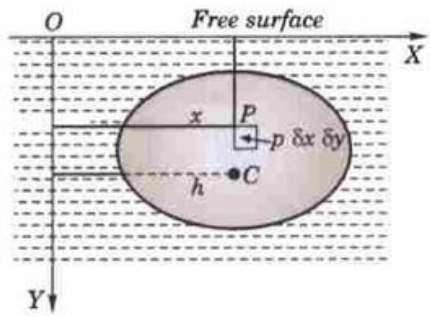

Fig. 7.37

If p be the pressure at the point $P(x, y)$ of the area A, then the pressure on an elementary area $\delta x\delta y$ at P is $p\delta x\delta y$ which is normal to the plane.

$\therefore$ the resultant pressure on $A = \iint p dxdy$.

If this resultant pressure acting at $C(h, k)$ is equivalent to pressure at various points such as $p\delta x\delta y$ distributed over the whole area A, then C is called the *centre of pressure*.

$\therefore$ taking the moment of the resultant pressure at C and the sum of the moments of the individual pressures such as $p\delta x\delta y$ at $P(x, y)$ about the y-axis, we get

$$h \iint p\, dxdy = \iint x \,.\, p dxdy, \quad i.e., h = \iint x \,.\, dxdy \Big/ \iint p dxdy$$

Similarly, taking moments about x-axis, we have

$$k = \iint y \,.\, pdxdy \Big/ \iint p dxdy$$ with integrals embracing the whole of the area A.

While using polar coordinates, replace x by $r\cos\theta$, y by $r\sin\theta$ and $dxdy$ by $r\, d\theta dr$ in the above formulae.

Example 7.36. *A horizontal boiler has a flat bottom and its ends are plane and semi-circular. If it is just full of water, show that the depth of the centre of pressure of either end is 0.7 × total depth approximately.*

Solution. Let the semi-circle $x^2 + y^2 = a^2$ represent an end of the given boiler (Fig. 7.38). By symmetry, its centre of pressure lies on OY.

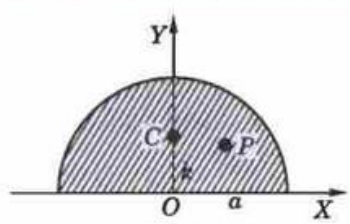

Fig. 7.38

If w be the weight of water per unit volume, then the pressure p at the point $P(x, y) = w(a - y)$.

$\therefore$ the height k of the C.P. above OX, is given by

$$k = \frac{\iint y \cdot pdxdy}{\iint pdxdy} = \frac{\int_{-a}^{a}\int_0^{\sqrt{(a^2-x^2)}} w(a-y)y\, dy \cdot dx}{\int_{-a}^{a}\int_0^{\sqrt{(a^2-x^2)}} w(a-y)\, dy \cdot dx}$$

$$= \frac{\int_{-a}^{a} \left| ay^2/2 - y^3/3 \right|_0^{\sqrt{(a^2-x^2)}} dx}{\int_{-a}^{a} \left| ay - y^2/2 \right|_0^{\sqrt{(a^2-x^2)}} dx} = \frac{\int_{-a}^{a}\left[\frac{a}{2}(a^2-x^2) - \frac{1}{3}(a^2-x^2)^{3/2}\right]dx}{\int_{-a}^{a}\left[a(a^2-x^2)^{1/2} - \frac{1}{2}(a^2-x^2)\right]dx}$$

Now put $x = a \sin\theta$, so that $dx = a \cos\theta \, d\theta$.

Also when $x = -a$, $\theta = -\pi/2$, and when $x = a$, $\theta = \pi/2$.

$$\therefore \qquad k = \frac{\int_{-\pi/2}^{\pi/2} \left[\frac{a^3}{2}\cos^2\theta - \frac{a^3}{3}\cos^3\theta\right] a\cos\theta \, d\theta}{\int_{-\pi/2}^{\pi/2} \left[a^2\cos\theta - \frac{a^2}{2}\cos^2\theta\right] a\cos\theta \, d\theta}$$

$$= \frac{a}{3} \cdot \frac{2\int_0^{\pi/2} (3\cos^3\theta - 2\cos^4\theta)\, d\theta}{2\int_0^{\pi/2} (2\cos^2\theta - \cos^3\theta)\, d\theta} = \frac{a}{4}\left(\frac{16 - 3\pi}{3\pi - 4}\right) = 0.3a \text{ nearly.}$$

Hence, the depth of the C.P. $= a - k = 0.7a$ approximately.

PROBLEMS 7.6

1. A lamina is bounded by the curves $y = x^2 - 3x$ and $y = 2x$. If the density at any point is given by λxy, find by double integration, the mass of the lamina.
2. Find the mass of a lamina in the form of cardioid $r = a(1 + \cos\theta)$ whose density at any point varies as the square of its distance from the initial line.
3. Find the mass of a solid in the form of the positive octant of the sphere $x^2 + y^2 + z^2 = 9$, if the density at any point is $2xyz$.
4. Find the centroid of the area enclosed by the parabola $y^2 = 4ax$, the axis of x and its latus-rectum.
5. The density at any point (x, y) of a lamina is $\sigma(x + y)/a$ where σ and a are constants. The lamina is bounded by the lines $x = 0, y = 0, x = a, y = b$. Find the position of its centre of gravity.
6. Find the centroid of a loop of the lemniscate $r^2 = a^2 \cos 2\theta$.
7. A plane in the form of a quadrant of the ellipse $(x/a)^2 + (y/b)^2 = 1$ is of small but varying thickness, the thickness at any point being proportional to the product of the distances of that point from the axes ; show that the coordinates of the centroid are $(8a/15, 8b/15)$. *(Nagpur, 2009)*
8. In a semi-circular disc bounded by a diameter OA, the density at any point varies as the distance from O ; find the position of the centre of gravity.
9. Find the centroid of the tetrahedron bounded by the coordinate planes and the plane $x + y + z = 1$, the density at any point varying as its distance from the face $z = 0$.
10. Find $\bar{x}$ where $(\bar{x}, \bar{y}, \bar{z})$ is the centroid of the region R bounded by the parabolic cylinder $z = 4 - x^2$ and the planes $x = 0, y = 0, y = 6, z = 0$. (Assume that the density is constant).
11. If the density at any point of the solid octant of the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$ varies as xyz, find the coordinates of the C.G. of the solid. *(P.T.U., 2005)*
12. A horizontal boiler has a flat bottom and its ends consist of a square 1 metre wide surmounted by an isosceles triangle of height 0.5 metre. Determine the depth of the centre of pressure of either end when the boiler is just full.
13. A quadrant of a circle is just, immersed vertically in a heavy homogeneous liquid with one edge in the surface. Find the centre of pressure.
14. Find the depth of the centre of pressure of a square lamina immersed in the liquid, with one vertex in the surface and the diagonal vertical.
15. Find the centre of pressure of a triangular lamina immersed in a homogeneous liquid with one side in the free surface. *(P.T.U., 2003)*
16. A uniform semi-circular is lamina immersed in a fluid with its plane vertical and its boundary diameter on the free surface. If the density at any point of the fluid varies as the depth of the point below the free surface, find the position of the centre of pressure of the lamina.

7.12 (1) MOMENT OF INERTIA

If a particle of mass m of a body be at a distance r from a given line, then mr^2 is called the *moment of inertia of the particle about the given line* and the sum of similar expressions taken for all the particles of the body, *i.e.*, Σmr^2 is called the *moment of inertia of the body about the given line* (Fig. 7.39).

If M be the total mass of the body and we write its moment of inertia $= Mk^2$, then k is called the *radius of gyration* of the body about the axis.

(2) M.I. of plane lamina. Consider the elementary mass $\rho\delta x\delta y$ at the point $P(x, y)$ of a plane area A so that its M.I. about x-axis $= \rho\delta x\delta y\, y^2$.

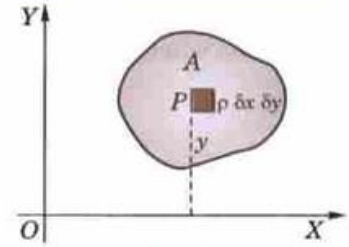

Fig. 7.39

$\therefore$ M.I. of the lamina about x-axis, *i.e.* $I_x = \iint_A \rho\, y^2\, dxdy$.

Similarly, M.I. of the lamina about y-axis' *i.e.*, $I_y = \iint_A \rho\, x^2\, dxdy$.

Also M.I. of the lamina about an axis perpendicular to the xy-plane, *i.e.*,

$$I_z = \iint_A \rho(x^2 + y^2)\, dxdy.$$

(3) M.I. of a solid. Consider an elementary mass $\rho\delta x\delta y\delta z$ enclosing a point $P(x, y, z)$ of a solid of volume V.

Distance of P from the x-axis $= \sqrt{(y^2 + z^2)}$.

$\therefore$ M.I. of this element about the x-axis $= \rho\, \delta x\delta y\delta z\, (y^2 + z^2)$.

Thus *M.I. of this solid about x-axis, i.e.*, $I_x = \iiint_V \rho(y^2 + z^2)\, dxdydz$.

Similarly, *its M.I. about y-axis, i.e.*, $I_y = \iiint_V \rho(z^2 + x^2)\, dxdydz$

and M.I. *about z-axis, i.e.*, $I_z = \iiint_V \rho(x^2 + y^2)\, dxdydz$.

(4) Sometimes we require the moment of inertia of a body about axes other than the principal axes. The following theorems prove useful for this purpose :

I. Theorem of perpendicular axis. *If the moment of inertia of a lamina about two perpendicular axes OX, OY in its plane are I_x and I_y, then the moment of inertia of the lamina about an axis OZ, perpendicular to it is given by $I_z = I_x + I_y$.*

Its proof follows from the relations giving I_x, I_y and I_z for a plane lamina [(2) above].

II. Steiner's theorem*. *If the moment of inertia of a body of mass M about an axis through its centre of gravity is I, then I′, moment of inertia about a parallel axis at a distance d from the first axis, is given by $I' = I + Md^2$.*

Its proof will be found in any text book on Dynamics of a Rigid Body.

Example 7.37. *Find the M.I. of the area bounded by the curve $r^2 = a^2 \cos 2\theta$ about its axis.*

Solution. Given curve is symmetrical about the pole and for half of the loop in the first quadrant θ varies from 0 to $\pi/4$ (Fig. 7.40).

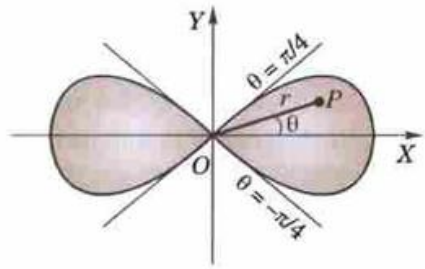

Fig. 7.40

Elementary area at $P(r, \theta) = r\, d\theta dr$.

If ρ be the surface density, then elementary mass

$$= \rho r\, d\theta dr \qquad ...(i)$$

$\therefore$ its total mass $M = 4 \int_0^{\pi/4} \int_0^{a\sqrt{(\cos 2\theta)}} \rho r\, dr d\theta$

$$= 2\rho a^2 \int_0^{\pi/4} \cos 2\theta\, d\theta = \rho a^2 \qquad ...(ii)$$

Now M.I. of the elementary mass (i) about the x-axis.

$$= \rho r\, d\theta dr \cdot y^2 = \rho r d\theta\, dr\, (r \sin\theta)^2 = \rho r^3 \sin^2\theta\, dr d\theta$$

Hence the M.I. of the whole area

$$= 4 \int_0^{\pi/4} \int_0^{a\sqrt{(\cos 2\theta)}} \rho r^3 \sin^2\theta\, dr d\theta = 4\rho \int_0^{\pi/4} \sin^2\theta \cdot \left| r^4/4 \right|_0^{a\sqrt{(\cos 2\theta)}} d\theta$$

$$= \rho a^2 \int_0^{\pi/4} \cos^2 2\theta \cdot \sin^2\theta\, d\theta = \rho a^4 \int_0^{\pi/2} \cos^2\phi \cdot \sin^2 \frac{\phi}{2} \cdot \frac{d\phi}{2} \qquad [\text{Put } 2\theta = \phi,\ d\theta = d\phi/2]$$

$$= \frac{\rho a^4}{4} \int_0^{\pi/2} (\cos^2\phi - \cos^3\phi)\, d\phi = \frac{\rho a^4}{48}(3\pi - 8) = \frac{Ma^2}{48}(3\pi - 8). \qquad [\text{By } (ii)]$$

*Named after a Swiss geometrer *Jacob Steiner* (1796–1863) who was a professor at Berlin University.

Example 7.38. *Find the moment of inertia of a hollow sphere about a diameter, its external and internal radii being 5 metres and 4 metres.*

Solution. Let ρ be the density of the given hollow sphere. Then the M.I. about the diameter, *i.e.*, x-axis is

$$I_x = \iiint_V \rho(y^2 + z^2)\, dx\, dy\, dz$$

Changing to polar spherical coordinates, we get

$$I_x = \int_0^{2\pi} \int_0^{\pi} \int_4^5 \rho\{(r \sin\theta \sin\phi)^2 + (r\cos\theta)^2\}\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \rho\left\{\int_0^{2\pi} \sin^2\phi\, d\phi \cdot \int_0^{\pi} \sin^3\theta\, d\theta \left|\frac{r^5}{5}\right|_4^5 + \int_0^{2\pi} d\phi \int_0^{\pi} \cos^2\theta \sin\theta\, d\theta \cdot \left|\frac{r^5}{5}\right|_4^5\right\}$$

$$= \frac{8\pi\rho}{15}(5^5 - 4^5) = 1120.5 \text{ m.}$$

Example 7.39. *A solid body of density ρ is in the shape of the solid formed by revolution of the centroid $r = a(1 + \cos\theta)$ about the initial line. Show that its moment of inertia about a straight line through the pole perpendicular to the initial line is $\frac{352}{105}\pi\rho a^5$.* (U.P.T.U., 2001)

Solution. An elementary area $rd\theta dr$, when revolved about OX generates a circular ring of radius $LP = r \sin\theta$ (Fig. 7.41).

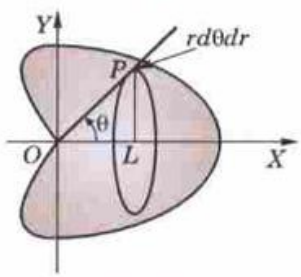

Fig. 7.41

M.I. of this ring about a diameter parallel to OY

$$= (2\pi r \sin\theta)(rd\theta dr)\rho \cdot \frac{(r\sin\theta)^2}{2}.$$

[$\because$ M.I. of a ring about a diameter $= Ma^2/2$.]

Now using Steiner's theorem, we have M.I. of the ring about OY = M.I. of the ring about a diameter LP parallel to OY + Mass of the ring $(OL)^2 (r\cos\theta)^2$

$$= 2\pi\rho r^4 \sin^3\theta\, d\theta dr + 2\pi r \sin\theta\,(rd\theta dr)(r\cos\theta)^2$$

Hence M.I. of the solid generated by revolution about OY

$$= \pi\rho \int_0^{\pi} \int_0^{r=a(1+\cos\theta)} (r^4 \sin^3\theta + 2r^4 \sin\theta\cos^2\theta)\, d\theta dr$$

$$= \pi\rho \int_0^{\pi} (\sin^3\theta + 2\sin\theta\cos^2\theta)\, d\theta \int_0^{a(1+\cos\theta)} r^4\, dr$$

$$= \frac{\pi\rho a^5}{5} \int_0^{\pi} \sin\theta\,(1 + \cos^2\theta)(1 + \cos\theta)^5\, d\theta \qquad [\text{Put } \theta = 2\phi]$$

$$= \frac{\pi\rho a^5}{5} \int_0^{\pi/2} \sin 2\phi\,(1 + \cos^2 2\phi)(1 + \cos 2\phi)^5\, 2d\phi$$

$$= \frac{\pi\rho a^5}{5} \int_0^{\pi/2} 2\sin\phi\cos\phi\,\{1 + (2\cos^2\phi - 1)^2\}(2\cos^2\phi)^5\, 2d\phi$$

$$= \frac{256\,\pi\rho a^5}{5} \int_0^{\pi/2} (\cos^{11}\phi - 2\cos^{13}\phi + 2\cos^{15}\phi)\sin\phi\, d\phi$$

$$= \frac{256\,\pi\rho a^5}{5}\left| -\frac{\cos^{12}\phi}{12} + \frac{2\cos^{14}\phi}{14} - \frac{2\cos^{16}\phi}{16}\right|_0^{\pi/2} = \frac{352\,\pi\rho a^5}{105}.$$

Example 7.40. *A hemisphere of radius R has a cylindrical hole of radius a drilled through it, the axis of the hole being along the radius normal to the plane face of the hemisphere. Find its radius of gyration about a diameter of this face.*

Solution. M.I. of the given solid about x-axis

$$= \iiint \rho(y^2 + z^2)\, dx\, dy\, dz$$

The limits of integration for z are from 0 to $z = \sqrt{(R^2 - x^2 - y^2)}$ found from the equation of the sphere $x^2 + y^2 + z^2 = R^2$. The limits for x and y are to be such as to cover the shaded area A in the xy-plane between the concentric circles of radii a and R (Fig. 7.42).

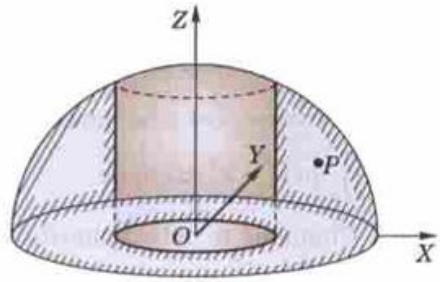

Fig. 7.42

Thus the required M.I. about x-axis

$$= \rho \iint_A \int_0^{\sqrt{(R^2 - x^2 - y^2)}} (y^2 + z^2)\, dz\, dx\, dy$$

$$= \rho \iint_A \left| y^2 z + z^3/3 \right|_0^{\sqrt{(R^2 - x^2 - y^2)}} dx\, dy = \rho \iint_A \left[y^2 (R^2 - x^2 - y^2)^{1/2} + \frac{1}{3}(R^2 - x^2 - y^2)^{3/2} \right] dx\, dy.$$

Now changing to polar coordinates, we have $x = r\cos\theta$, $y = r\sin\theta$ and $dxdy = rd\theta dr$.

Also to cover the area A, r varies from a to R and θ varies from 0 to 2π.

Hence the required M.I. about x-axis

$$= \rho \int_a^R \int_0^{2\pi} \left[r^2 \sin^2\theta \cdot (R^2 - r^2)^{1/2} + \frac{1}{3}(R^2 - r^2)^{3/2} \right] r d\theta dr$$

$$= \rho \int_a^R \int_0^{2\pi} \left[\frac{1}{2} r^2 (1 - \cos 2\theta) + \frac{1}{3}(R^2 - r^2) \right] d\theta \cdot r(R^2 - r^2)^{1/2}\, dr$$

$$= \rho \int_a^R \left| \frac{r^2}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + \frac{1}{3}(R^2 - r^2)\theta \right|_0^{2\pi} \cdot r(R^2 - r^2)^{1/2}\, dr$$

$$= \rho \int_a^R 2\pi \left(\frac{r^2}{2} + \frac{R^2 - r^2}{3} \right) . r(R^2 - r^2)^{1/2}\, dr$$

$$= \frac{\pi\rho}{3} \int_a^R (2R^2 + r^2)(R^2 - r^2)^{1/2} \cdot r dr \qquad \text{[Put } r^2 = t \text{ and } rdr = dt/2\text{]}$$

$$= \frac{\pi\rho}{6} \int_{a^2}^{R^2} (2R^2 + t)(R^2 - t)^{1/2}\, dt \qquad \text{[Integrate by parts]}$$

$$= \frac{\pi\rho}{9} \left[(2R^2 + a^2)(R^2 - a^2)^{3/2} + \frac{2}{5}(R^2 - a^2)^{5/2} \right] = \frac{2\pi\rho}{3}(R^2 - a^2)^{3/2} \times \frac{1}{10}(4R^2 + a^2)$$

$$\left[\because \text{ Mass} = \rho \int_0^{2\pi} \int_a^R \int_0^{\sqrt{(R^2 - r^2)}} dz \cdot rdr \cdot d\theta = \frac{2\pi\rho}{3}(R^2 - a^2)^{3/2} \right]$$

Hence, the radius of gyration = $[(4R^2 + a^2)/10]^{1/2}$.

7.13 (1) PRODUCT OF INERTIA

If a particle of mass m of a body be at distances x and y from two given perpendicular lines, then Σmxy is called the *product of inertia* of the body about the given lines.

Consider an elementary mass $\delta x \delta y \delta z$ enclosing the point $P(x, y, z)$ of solid of volume V. Then the product of inertia (P.I.) of this element about the axes of x and $y = \rho \delta x \delta y \delta z\, xy$.

$\therefore$ P.I. of the solid about x and y-axes, *i.e.*, $P_{xy} = \iiint_V \rho\, xy\, dxdydz$

Similarly, $P_{yz} = \iiint_V \rho\, yz\, dxdydz$ and $P_{zx} = \iiint_V \rho\, zx\, dxdydz$.

In particular, for a plane lamina of surface density ρ and covering a region A in the xy-plane,

$$P_{xy} = \iint_A \rho\, xy\, dxdy \text{ whereas } P_{yz} = P_{zx} = 0. \qquad [\because z = 0]$$

(2) Principal axes. *The principal axes* of a lamina at a given point are that pair of axes in its plane through the given point, about which the product of inertia of the lamina vanishes.

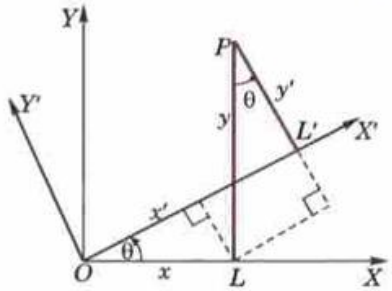

Fig. 7.43

Let $P(x, y)$ be a point of the plane area A referred to rectangular axes OX, OY. Let (x', y') be the coordinates of P referred to another pair of rectangular axes OX', OY' in the same plane and inclined at an angle θ to the first pair (Fig. 7.43).

Then $$x' = x\cos\theta + y\sin\theta$$
$$y' = y\cos\theta - x\sin\theta$$

If I_x, I_y be the moments of inetria of the area A about OX and OY and P_{xy} be its product of inertia about these axes, then

$$I_x = \iint_A \rho y^2\, dA,\ I_y = \iint_A \rho x^2\, dA,\ P_{xy} = \iint \rho xy\, dA.$$

$\therefore$ the product of inertia P'_{xy} about OX' and OY' is given by

$$P'_{xy} = \int_A \rho x'y'\, dA = \iint_A \rho(x\cos\theta + y\sin\theta)(y\cos\theta - x\sin\theta)\, dA$$

$$= \sin\theta\cos\theta \iint_A \rho(y^2 - x^2)\, dA + (\cos^2\theta - \sin^2\theta) \iint_A \rho xy\, dA$$

$$= 1/2 \sin 2\theta \cdot (I_x - I_y) + \cos 2\theta\, P_{xy}.$$

Now OX', OY' will be the principal axes of the area A if P'_{xy} vanishes.

i.e., If $$1/2 \sin 2\theta\,(I_x - I_y) + \cos 2\theta\, P_{xy} = 0$$

i.e., If $$\mathbf{\tan 2\theta = 2P_{xy}/(I_y - I_x).}$$

This gives two values of θ differing by $\pi/2$.

Example 7.41. *Show that the principal axes at the node of a half-loop of the lemniscate $r^2 = a^2 \cos 2\theta$ are inclined to the initial line at angles*

$$\frac{1}{2}\tan^{-1}\frac{1}{2} \text{ and } \frac{\pi}{2} + \frac{1}{2}\tan^{-1}\frac{1}{2}.$$

Solution. Let the element of mass at $P(r, \theta)$ be $\rho r d\theta dr$.

Then $$I_x = \rho \int_0^{\pi/4} \int_0^{a\sqrt{(\cos 2\theta)}} r^2 \sin^2\theta \cdot r d\theta dr$$ [See Fig. 7.40]

$$= \frac{\rho a^4}{4} \int_0^{\pi/4} \sin^2\theta \cos^2 2\theta\, d\theta = \frac{\rho a^4}{16}\left(\frac{\pi}{4} - \frac{2}{3}\right)$$

$$I_y = \rho \int_0^{\pi/4} \int_0^{a\sqrt{(\cos 2\theta)}} r^2 \cos^2\theta \cdot r\, d\theta dr = \frac{\rho a^4}{16}\left(\frac{\pi}{4} + \frac{2}{3}\right)$$

and $$P_{xy} = \rho \int_0^{\pi/4} \int_0^{a\sqrt{(\cos 2\theta)}} r^2 \sin\theta\cos\theta \cdot r d\theta dr = \frac{\rho a^4}{48}.$$

Hence the required direction of the principal axes at O are given by

$$\tan 2\theta = \frac{2P_{xy}}{I_y - I_x} = \frac{\rho a^4/24}{(\rho a^4/16)\times(4/3)} = \frac{1}{2}$$

or by $$\theta = \frac{1}{2}\tan^{-1}\frac{1}{2} \text{ and } \frac{\pi}{2} + \frac{1}{2}\tan^{-1}\frac{1}{2}.$$

PROBLEMS 7.7

1. Using double integrals, find the moment of inertia about the x-axis of the area enclosed by the lines
$$x = 0, y = 0, (x/a) + (y/b) = 1.$$ (*P.T.U., 2005*)
2. Find the moment of inertia of a circular plate about a tangent.
3. Find the moment of inertia of the area $y = \sin x$ from $x = 0$ to $x = 2\pi$ about OX.

4. Find the moment of inertia of a quadrant of the ellipse $(x/a)^2 + (y/b)^2 = 1$ of mass M about the x-axis, if the density at a point is proportional to xy.
5. Find the moment of inertia about the initial line of the cardioid $r = a(1 + \cos\theta)$.
6. Find the moment of inertia of a uniform spherical ball of mass M and radius R about a diameter.
7. Find the moment of inertia of a solid right circular cylinder about (*i*) its axis (*P.T.U., 2006*) (*ii*) a diameter of the base.
8. Find the M.I. of a solid right circular cone having base-radius r and height h, about (*i*) its axis, (*ii*) an axis through the vertex and perpendicular to its axis, (*iii*) a diameter of its base.
9. Find the moment of inertia of a hollow sphere about a diameter, its external and internal radii being 51 metres and 49 metres.
10. Find the moment of inertia about z-axis of a homogeneous tetrahedron bounded by the planes $x = 0, y = 0, z = x + y$ and $z = 1$.
11. Find the moment of inertia of an octant of the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$, about the x-axis.
12. Find the product of inertia of a quadrant of the ellipse $(x/a)^2 + (y/b)^2 = 1$, about the coordinate axes.
13. Show that the principal axes at the origin of the triangle enclosed by $x = 0, y = 0, (x/a) + (y/b) = 1$ are inclined to the x-axis at angles α and $\alpha + \pi/2$, where $\alpha = \frac{1}{2}\tan^{-1}[ab/(a^2 - b^2)]$ (*U.P.T.U., 2002*)
14. The lengths AB and AD of the sides of a rectangle $ABCD$ are $2a$ and $2b$. Show that the inclination to AB of one of the principal axes at A is $\frac{1}{2}\tan^{-1}\left\{\frac{3ab}{2(a^2 - b^2)}\right\}$.

7.14 BETA FUNCTION

The beta function is defined as

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad \begin{cases} m > 0 \\ n > 0 \end{cases} \qquad \text{...(1)}$$

Putting $x = 1 - y$ in (1), we get $\beta(m, n) = -\int_1^0 (1-y)^{m-1} y^{n-1}\,dy$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}\,dy = \beta(n, m)$$

Thus $\beta(m, n) = \beta(n, m)$ ...(2)

Putting $x = \sin^2\theta$ so that $dx = 2\sin\theta\cos\theta\,d\theta$, (1) becomes

$$\beta(m, n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1}\,2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta \qquad \text{...(3)}$$

which is another form of $\beta(m, n)$.

This function is also *Euler's integral of the first kind**.

7.15 (1) GAMMA FUNCTION

The gamma function is defined as

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx \ (n > 0) \qquad \text{...(i)}$$

This integral is also known as *Euler's integral of the second kinds*. It defines a function of n for positive values of n.

*After an enormously creative Swiss mathematician *Leonhard Euler (1707–1783)*. He studied under *John Bernoulli* and became a professor of mathematics in St. Petersburg, Russia. Even after becoming totally blind in 1771, he contributed to almost all branches of mathematics.

In particular, $\Gamma(1) = \int_0^\infty e^{-x}\,dx = \left| -e^{-x} \right|_0^\infty = 1.$ ...(*ii*)

(2) Reduction formula for Γ(n).

Since $\Gamma(n+1) = \int_0^\infty e^{-x} x^n\,dx$ [Integrating by parts] $= \left| -x^n e^{-x} \right|_0^\infty + n \int_0^\infty e^{-x} x^{n-1} dx$

$\therefore \quad \Gamma(n+1) = n\Gamma(n)$ ...(*iii*)

which is the reduction formula for $\Gamma(n)$. From this formula, it is clear that if $\Gamma(n)$ is known throughout a unit interval say : $1 < n \le 2$, then the values of $\Gamma(n)$ throughout the next unit interval $2 < n \le 3$ are found, from which the values of $\Gamma(n)$ for $3 < n \le 4$ are determined and so on. In this way, *the values of $\Gamma(n)$ for all positive values of $n > 1$ may be found by successive application of (iii).*

Also using (*iii*) in the form

$$\Gamma(n) = \frac{\Gamma(n+1)}{n} \qquad ...(iv)$$

We can define $\Gamma(n)$ for values of n for which the definition (1) fails. It gives the value of $\Gamma(n)$ for $0 < n \le 1$ in terms of the values of $\Gamma(n)$ for $1 < n \le 2$. Thus we can define $\Gamma(n)$ for all $n < 0$ provided its value for $1 < n \le 2$ is known. Also if $-1 < n < 0$, (4) gives $\Gamma(n)$ in terms of its values for $0 < n < 1$. Then we may find, $\Gamma(n)$ for $-2 < n < -1$ and so on.

Thus (i) and (iv) together give a complete definition of $\Gamma(n)$ for all values of n except when n is zero or a negative integer and its graph is as shown in Fig.7.44. The values of $\Gamma(n)$ for $1 < n \le 2$ are given in (Table I-Appendix 2) from which the values of $\Gamma(n)$ for values of n outside the interval $1 < n \le 2$ ($n \ne 0, -1, -2, -3$ .....) may be found.

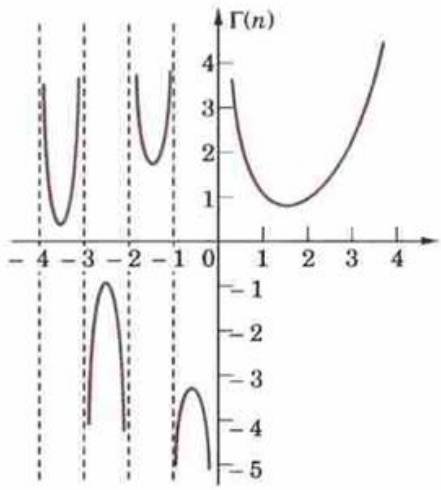

Fig. 7.44

(3) Value of Γ(n) in terms of factorial.

Using $\Gamma(n+1) = n\Gamma(n)$ successively, we get

$$\Gamma(2) = 1 \times \Gamma(1) = 1\,!$$
$$\Gamma(3) = 2 \times \Gamma(2) = 2 \times 1 = 2\,!$$
$$\Gamma(4) = 3 \times \Gamma(3) = 3 \times 2\,! = 3\,!$$
$$\ldots\ldots \qquad \ldots\ldots \qquad \ldots\ldots$$

In general $\Gamma(n+1) = n\,!$ provided n is a positive integer ...(*v*)

Taking $n = 0$, it defines $0\,! = \Gamma(1) = 1$.

(4) Value of $\Gamma\left(\frac{1}{2}\right)$. We have

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty e^{-x} x^{-1/2}\,dx \qquad [\text{Put } x = y^2 \text{ so that } dx = 2y\,dy]$$

$$= 2\int_0^\infty e^{-y^2}\,dy \text{ which is also } = 2\int_0^\infty e^{-x^2}\,dx$$

$$\therefore \quad \left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta$$

$$= 4 \cdot \frac{\pi}{2}\int_0^\infty e^{-r^2}\,r\,dr = 2\pi\left[\left(-\frac{1}{2}\right)e^{-r^2}\right]_0^\infty = \pi$$

whence $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} = 1.772$ ...(*vi*) (*V.T.U., 2006*)

7.16 RELATION BETWEEN BETA AND GAMMA FUNCTIONS

$$\beta(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

We have $\Gamma(m) = \int_0^\infty e^{-t} t^{m-1}$ [Put $t = x^2$ so that $dt = 2x\,dx$

$$= 2\int_0^\infty e^{-x^2} x^{2m-1}\, dx \qquad ...(2)$$

Similarly, $$\Gamma(n) = 2\int_0^\infty e^{-y^2} y^{2n-1}\, dy$$

$$\therefore \qquad \Gamma(m)\Gamma(n) = 4\int_0^\infty e^{-x^2} x^{2m-1}\, dx \int_0^\infty e^{-y^2} y^{2n-1}\, dy$$

$$= 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1}\, dx\, dy \qquad ...(3)$$ [$\because$ the limits of integration are constant.]

Now change to polar coordinates by writing $x = r\cos\theta$, $y = r\sin\theta$ and $dx\, dy = r d\theta dr$. To cover the region in (3) which is the entire first quadrant, r varies from 0 to ∞ and θ from 0 to $\pi/2$. Thus (3) becomes

$$\Gamma(m)\Gamma(n) = 4\int_0^\infty \int_0^{\pi/2} e^{-r^2} r^{2(m+n)-1} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta dr$$

$$= \left[2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta\right] \times \left[2\int_0^\infty e^{-r^2} r^{2(m+n)-1}\, dr\right] \qquad ...(4)$$

But by (2), $2\int_0^\infty e^{-r^2} r^{2(m+n)-1}\, dr = \Gamma(m+n)$

and by (3) of § 7.14, $2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta = \beta(m, n)$.

Thus (4) gives $\Gamma(m)\Gamma(n) = \beta(m, n)\,\Gamma(m+n)$ *(U.T.U., 2010 ; Bhopal, 2009 ; V.T.U., 2008 S)*
whence follows (1) which is extremely useful for evaluating definite integrals in terms of gamma functions.

Cor. *Rule to evaluate* $\int_0^{\pi/2} \sin^p x \cos^q x\, dx$.

$$\int_0^{\pi/2} \sin^p x \cos^q x\, dx = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) \qquad \text{[By (3) of § 7.14]}$$

$$= \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)} \qquad ...(5)$$

In particular, when $q = 0$, and $p = n$, we have

$$\int_0^{\pi/2} \sin^n x\, dx = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \cdot \frac{\sqrt{\pi}}{2}$$

Similarly, $$\int_0^{\pi/2} \cos^n x\, dx = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \cdot \frac{\sqrt{\pi}}{2} \qquad ...(6)$$

Example 7.42. *Show that*

(*a*) $\Gamma(n) = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy \ (n > 0)$. *(J.N.T.U., 2003 ; Madras, 2003 S)*

(*b*) $\beta(p, q) = \int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\, dy$ *(V.T.U., 2003 ; Gauhati, 1999)*

$= \int_0^1 \frac{x^{p-1} + x^{q-1}}{(1+x)^{p+q}}\, dx.$ *(V.T.U., 2008 ; Osmania, 2003 ; Rohtak, 2003)*

Solution. (*a*) $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx \ (n > 0)$

$$= \int_1^0 \left(\log \frac{1}{y}\right)^{n-1} y\left(-\frac{1}{y}dy\right) = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy.$$

Put $y = e^{-x}$
i.e., $x = \log(1/y)$
so that $dx = -(1/y)\,dy$

(*b*) $$\beta(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx$$

$$= \int_\infty^0 \frac{1}{(1+y)^{p-1}}\left(\frac{y}{1+y}\right)^{q-1} \frac{-1}{(1+y)^2}\,dy$$

Put $x = \dfrac{1}{1+y}$ *i.e.*, $y = \dfrac{1}{x} - 1$
so that $dx = \dfrac{-1}{(1+y)^2}\,dy$

$$= \int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\,dy = \int_0^1 \frac{y^{q-1}}{(1+y)^{p+q}}\,dy + \int_1^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\,dy$$

Now substituting $y = 1/z$ in the second integral, we get

$$\int_1^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\,dy = \int_1^0 \frac{1}{z^{q-1}} \cdot \frac{1}{(1+1/z)^{p+q}}\left(-\frac{1}{z^2}\right)dz = \int_0^1 \frac{z^{p-1}}{(1+z)^{p+q}}\,dz.$$

Hence, $$\beta(p, q) = \int_0^1 \frac{y^{q-1}}{(1+y)^{p+q}}\,dy + \int_0^1 \frac{z^{p-1}}{(1+z)^{p+q}}\,dz = \int_0^1 \frac{x^{p-1}+x^{q-1}}{(1+x)^{p+q}}\,dx.$$

Example 7.43. *Express the following integrals in terms of gamma functions :*

(*a*) $\int_0^1 \dfrac{dx}{\sqrt{(1-x^4)}}$ (*b*) $\int_0^{\pi/2} \sqrt{(\tan\theta)}\,d\theta$. (*Madras, 2006*)

(*c*) $\int_0^\infty \dfrac{x^c}{c^x}\,dx$ (*U.P.T.U., 2006*) (*d*) $\int_0^\infty a^{-bx^2}\,dx$.

(*e*) $\int_0^1 x^5\,[\log(1/x)]^3\,dx$ (*Madras, 2000*)

Solution. (*a*) $\displaystyle\int_0^1 \frac{dx}{\sqrt{(1-x^4)}}$

Put $x^2 = \sin\theta$, *i.e.*, $x = \sin^{1/2}\theta$
so that $dx = 1/2 \sin^{-1/2}\theta \cos\theta\,d\theta$

$$= \int_0^{\pi/2} \frac{1}{2} \cdot \frac{\sin^{-1/2}\theta \,.\, \cos\theta\,d\theta}{\sqrt{(1-\sin^2\theta)}} = \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta\,d\theta = \frac{1}{2} \cdot \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\left(\dfrac{-\frac{1}{2}+1}{2}\right)}{\Gamma\left(\dfrac{-\frac{1}{2}+2}{2}\right)} = \frac{\sqrt{\pi}}{4}\,\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

(*b*) $$\int_0^{\pi/2} \sqrt{(\tan\theta)}\,d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta\,d\theta$$

$$= \frac{\Gamma\left(\dfrac{\frac{1}{2}+1}{2}\right)\Gamma\left(\dfrac{-\frac{1}{2}+1}{2}\right)}{2\Gamma\left(\dfrac{\frac{1}{2}-\frac{1}{2}+2}{2}\right)} = \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{2\Gamma(1)} = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$$

(*c*) $$\int_0^\infty \frac{x^c}{c^x}\,dx = \int_0^\infty \frac{x^c}{e^{x\log c}}\,dx$$ $[\because\ c^x = e^{\log c^x} = e^{x\log c}]$

$$= \int_0^\infty e^{-x\log c}\,x^c\,dx$$ [Put $x \log c = t$ so that $dx = dt/\log c$]

$$= \int_0^\infty e^{-t}\left(\frac{t}{\log c}\right)^c \frac{dt}{\log c} = \frac{1}{(\log c)^{c+1}} \int_0^\infty t^c\, e^{-t}\, dt = \Gamma(c+1)/(\log c)^{c+1}$$

(d) $\int_0^\infty a^{-bx^2}\, dx = \int_0^\infty e^{-bx^2 \log a}\, dx$ $\quad$ [Put $(b \log a)\, x^2 = t$ so that $dx = dt/2\sqrt{(b \log a)}$]

$$= \frac{1}{2\sqrt{(b \log a)}} \int_0^\infty e^{-t}\, t^{-1/2}\, dt = \frac{\Gamma\left(\frac{1}{2}\right)}{2\sqrt{(b \log a)}} = \frac{\sqrt{\pi}}{2\sqrt{(b \log a)}}$$

(e) $\int_0^1 x^4\, [\log(1/x)]^3\, dx = \frac{1}{625} \int_0^\infty e^{-t} \,.\, t^3\, dt$ $\quad$ [Put $x = e^{-t/5}$ so that $\log(1/x) = t/5$, $dx = -\frac{1}{5} e^{-t/5}\, dt$]

$$= \frac{\Gamma(4)}{625} = \frac{6}{625}.$$

Example 7.44. *Evaluate $\int_0^\infty e^{-ax}\, x^{m-1}\, \sin bx\, dx$ in terms of Gamma function.* *(U.P.T.U., 2003)*

Solution. We have $\Gamma(m) = \int_0^\infty e^{-x}\, x^{m-1}\, dx$ $\quad$ [Put $x = ay$, $dx = a dy$]

$$= \int_0^\infty e^{-ay}\, a^m\, y^{m-1}\, dy \quad \text{or} \quad \int_0^\infty e^{-ay}\, y^{m-1}\, dy = \Gamma(m)/a^m \quad \ldots(i)$$

Then $\quad I = \int_0^\infty e^{-ax}\, x^{m-1} \sin bx\, dx = \int_0^\infty e^{-ax}\, x^{m-1}$ (Imaginary part of e^{ibx}) dx

$= \text{I.P. of} \int_0^\infty e^{-(a-ib)x}\, x^{m-1}\, dx$

$= \text{I.P. of } \{\Gamma(m)/(a-ib)^m$ $\quad$ [By (i)]

$= \text{I.P. of } \{\Gamma(m)/(r^m (\cos\theta - i \sin\theta)^m]$ $\quad$ where $a = r\cos\theta$, $b = r\sin\theta$

$= \text{I.P. of } \Gamma(m)/(r^m (\cos m\theta - i \sin m\theta)]$ $\quad$ (Using Demoivere's theorem §19.5)

$$= \text{I.P. of } \left\{\frac{\Gamma(m)\,.\,(\cos m\theta + i \sin m\theta)}{r^m (\cos m\theta + i \sin m\theta)(\cos m\theta - i \sin m\theta)}\right\}$$

$$= \frac{\Gamma(m)}{r^m} \sin m\theta \quad \text{where } r\ \sqrt{(a^2+b^2)},\ \theta = \tan^{-1} b/a.$$

Example 7.45. *Prove that* $\int_0^1 \frac{x^2 dx}{\sqrt{(1-x^4)}} \times \int_0^1 \frac{dx}{\sqrt{(1+x^4)}} = \frac{\pi}{4\sqrt{2}}$

Solution. $\int_0^1 \frac{x^2 dx}{\sqrt{(1-x^4)}} \times \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta} \cdot \frac{\cos\theta}{2\sqrt{(\sin\theta)}}\, d\theta$ $\quad$ [Putting $x^2 = \sin\theta$, $dx = \frac{\cos\theta\, d\theta}{2\sqrt{(\sin\theta)}}$]

$$= \frac{1}{2} \int_0^{\pi/2} \sqrt{(\sin\theta)}\, d\theta = \frac{1}{4}\beta\left(\frac{3}{4}, \frac{1}{2}\right) = \frac{1}{4} \frac{\Gamma(3/4)\,\Gamma(1/2)}{\Gamma(5/4)} = \frac{\Gamma(3/4)\,\Gamma(1/2)}{\Gamma(1/4)}$$

$\int_0^1 \frac{dx}{\sqrt{(1+x^4)}} = \int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{2\sqrt{(\tan\theta)}\sec\theta}$ $\quad$ [Putting $x^2 = \tan\theta$, $dx = \frac{\sec^2\theta\, d\theta}{2\sqrt{(\tan\theta)}}$]

$= \frac{1}{\sqrt{2}} \int_0^{\pi/4} \frac{d\theta}{\sqrt{(\sin 2\theta)}} = \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \sin^{-1/2}\phi\, d\phi$ $\quad$ [Putting $2\theta = \phi$, $d\theta = \frac{1}{2} d\phi$]

$$= \frac{1}{4\sqrt{2}}\beta\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{1}{4\sqrt{2}} \frac{\Gamma(1/4)\,\Gamma(1/2)}{\Gamma(3/4)}$$

$$\therefore \qquad \int_0^1 \frac{x^2\,dx}{\sqrt{(1-x^4)}} \times \int_0^1 \frac{dx}{\sqrt{(1+x^4)}} = \frac{1}{4\sqrt{2}}\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = \frac{\pi}{4\sqrt{2}}.$$

Example 7.46. *Prove that* (*i*) $\beta(m, 1/2) = 2^{2m-1}\,\beta(m, m)$ *(V.T.U., 2004)*

(*ii*) $\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma(2m)$ *(Duplication Formula)*

(V.T.U., 2010 ; Kerala, M.E., 2005 ; Madras, 2003 S)

Solution. (*i*) We know that $\beta(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$ ...(1)

Putting $n = \frac{1}{2}$, we have $\beta\left(m, \frac{1}{2}\right) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\, d\theta$ ...(2)

Again putting $n = m$ in (*i*), we get $\beta(m, m) = 2\int_0^{\pi/2} (\sin\theta\cos\theta)^{2m-1}\, d\theta$

$$= \frac{1}{2^{2m-2}}\int_0^{\pi/2} \sin^{2m-1} 2\theta\, d\theta$$

$$= \frac{1}{2^{2m-1}}\int_0^{\pi} \sin^{2m-1}\phi\, d\phi, \text{ putting } 2\theta = \phi$$

$$= \frac{1}{2^{2m-1}} \cdot 2\int_0^{\pi/2} \sin^{2m-1}\phi\, d\phi$$

or $$2^{2m-1}\,\beta(m, m) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\, d\theta = \beta\left(m, \frac{1}{2}\right) \qquad \text{[by (2)}$$

(*ii*) Rewriting the above result in terms of Γ functions, we get

$$2^{2m-1}\,\frac{\Gamma(m)\,\Gamma(m)}{\Gamma(m+m)} = \frac{\Gamma(m)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

or $$\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi\Gamma(2m)}}{2^{2m-1}}.$$

Example 7.47. *Prove that*

(*a*) $\iint_D x^{l-1}\, y^{m-1}\, dx\, dy = \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m+1)}\, h^{l+m}$ *where D is the domain* $x \ge 0$, $y \ge 0$ *and* $x + y \le h$.

(U.P.T.U., 2005)

(*b*) $\iiint_V x^{l-1}\, y^{n-1}\, z^{n-1}\, dxdydz = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}$

where V is the region $x \ge 0$, $y \ge 0$, $z \ge 0$ *and* $x + y + z \le 1$. *This important result is known as* **Dirichlet's integral***.

Solution. (*a*) Putting $x/h = X$ and $y/h = Y$, we see that the given integral

$$= \iint_{D'} (hX)^{l-1}\,(hY)^{m-1}\, h^2\; dXdY \text{ where } D' \text{ is the domain } X \ge 0,\ Y \ge 0 \text{ and } X + Y \le 1.$$

$$= h^{l+m}\int_0^1\int_0^{1-X} X^{l-1}\, Y^{m-1}\, dY\, dX = h^{l+m}\int_0^1 X^{l-1}\left|\frac{Y^m}{m}\right|_0^{1-X} dX$$

$$= \frac{h^{l+m}}{m}\int_0^1 X^{l-1}\,(1-X)^m\, dX = \frac{h^{l+m}}{m}\,\beta(l, m+1) = \frac{h^{l+m}}{m}\cdot\frac{\Gamma(l)\,\Gamma(m+1)}{\Gamma(l+m+1)}$$

*Named after a German mathematician *Peter Gustav Lejeune Dirichlet* (1805–1859) who studied under Cauchy and succeeded Gauss at Gottingen. He is known for his contributions to Fourier series and number theory.

$$= h^{l+m} \frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m+1)} \qquad \ldots(i) \; [\because \; \Gamma(m+1)/m = \Gamma(m)]$$

(*b*) Taking $y + z \le 1 - x (= h$: say), the triple integral

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x^{l-1} y^{m-1} z^{n-1} \, dz dy dx$$

$$= \int_0^1 x^{l-1} \left[\int_0^h \int_0^{h-y} y^{m-1} z^{n-1} \, dz dy \right] dx = \int_0^1 x^{l-1} \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)} h^{m+n} \, dx \qquad \ldots \text{[By } (i)]$$

$$= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)} \int_0^1 x^{l-1} (1-x)^{m+n} \, dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)} \beta(l, m+n+1)$$

$$= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)} \cdot \frac{\Gamma(l)\Gamma(m+n+1)}{\Gamma(l+m+n+1)} = \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)}.$$

Example 7.48. *Evaluate the integral* $\iiint x^{l-1} y^{m-1} z^{n-1} \, dxdydz$ *where x, y, z are all positive with condition,* $(x/a)^p + (y/b)^q + (z/c)^r \le 1$. *(U.P.T.U., 2005 S)*

Solution. Put $(x/a)^p = u$, *i.e.*, $x = au^{1/p}$ so that $dx = \frac{a}{p} u^{1/p-1} \, du$

$(y/b)^q = v$, *i.e.*, $y = bv^{1/q}$ so that $dy = \frac{b}{q} v^{1/q-1} \, dv$

and $(z/c)^r = w$, *i.e.*, $z = cw^{1/r}$ so that $dz = \frac{c}{r} w^{1/r-1} \, dw$

Then $\iiint x^{l-1} y^{m-1} z^{n-1} \, dxdydz$

$$= \iiint (au^{1/p})^{l-1} (bv^{1/q})^{m-1} (cw^{1/r})^{n-1} \left(\frac{a}{p}\right) u^{1/p-1} \left(\frac{b}{q}\right) v^{1/q-1} \left(\frac{c}{r}\right) w^{1/r-1} \, dudvdw$$

$$= \frac{a^l b^m c^n}{pqr} \iiint u^{l/p-1} v^{m/q-1} w^{n/r-1} \, du \, dv \, dw \text{ where } u + v + w \le 1.$$

$$= \frac{a^l b^m c^n}{pqr} \frac{\Gamma(l/p)\Gamma(m/q)\Gamma(n/r)}{\Gamma(l/p + m/q + n/r + 1)} \qquad \text{[By Dirichlet's integral]}$$

Example 7.49. *The plane* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ *meets the axes in A, B and C. Apply Dirichlet's integral to find the volume of the tetrahedron OABC. Also find its mass if the density at any point is kxyz.* *(U.P.T.U., 2004)*

Solution. Put $x/a = u, y/b = v, z/c = w$ then the tetrahedron $OABC$ has $u \ge 0, v \ge 0, w \ge 0$ and $u + v + w \le 1$.

$\therefore$ volume of this tetrahedron $= \iiint_D dx \, dy \, dz$

$$= \iiint_D abc \, du \, dv \, dw \qquad \left[\begin{array}{l} a \, dx = adu, dy = bdv, dz = cdw \\ \text{for } D' = u \ge 0, v \ge 0, w \ge 0 \;\&\; u + v + w \le 1. \end{array}\right.$$

$$= abc \iiint_D u^{1-1} v^{1-1} w^{1-1} \, du \, dv \, dw$$

$$= abc \frac{\Gamma(1)\Gamma(1)\Gamma(1)}{\Gamma(1+1+1+1)} = \frac{abc}{6} \qquad \text{[By Dirichlet's integral]}$$

$$\text{Mass} = \iiint kxyz dx \, dy \, dz = \iiint k(au)(bv)(cw) abc \, du \, dv \, dw$$

$$= ka^2b^2c^2 \iiint u^{2-1} v^{2-1} w^{2-1} \, du \, dv \, dw$$

$$= ka^2b^2c^2 \frac{\Gamma(2)\Gamma(2)\Gamma(2)}{\Gamma(2+2+2+1)} ka^2b^2c^2 \cdot \frac{1}{6!} = \frac{k}{720} a^2b^2c^2.$$

PROBLEMS 7.8

1. Compute :

(i) $\Gamma(3.5)$ *(Assam, 1998)* (ii) $\Gamma(4.5)$

(iii) $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$ *(S.V.T.U., 2009)* (iv) $\beta(2.5, 1.5)$ (v) $\beta\left(\frac{9}{2}, \frac{7}{2}\right)$. *(Andhra, 2000)*

2. Express the following integrals in terms of gamma functions :

(i) $\int_0^\infty e^{-x^2}\,dx$ (ii) $\int_0^\infty x^{p-1}\,e^{-kx}\,dx (k > 0)$ *(Delhi, 2002 ; V.T.U., 2000)*

(iii) $\int_0^\infty \sqrt{x}\,e^{-x^3}\,dx$ *(J.N.T.U., 2003)* (iv) $\int_0^\infty \frac{dx}{x^{p+1}.(x-1)^q} (-p < q < 1)$

3. Show that :

(i) $\int_0^\infty \frac{x^4}{4^x}\,dx = \frac{\Gamma(5)}{(\log 4)^5}$ *(Marathwada, 2008)*

(ii) $\int_0^{\pi/2} \sqrt{(\cot\theta)}\,d\theta = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$ *(Osmania, 2003 S ; V.T.U., 2001)*

(iii) $\int_0^{\pi/2} [\sqrt{(\tan\theta)} + \sqrt{(\sec\theta)}]\,d\theta = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\left\{\Gamma\left(\frac{3}{4}\right) + \sqrt{\pi/\Gamma}\left(\frac{3}{4}\right)\right\}$ *(J.N.T.U., 2000)*

(iv) $\int_0^{\pi/2} \sqrt{(\sin\theta)}\,d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{(\sin\theta)}} = \pi.$ *(V.T.U., 2007)*

4. Given $\int_0^\infty \frac{x^{n-1}}{1+x}\,dx = \frac{\pi}{\sin n\pi}$, show that $\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$. *(S.V.T.U., 2008)*

Hence evaluate $\int_0^\infty \frac{dy}{1+y^4}$. *(V.T.U., 2006 ; J.N.T.U., 2005)*

5. Prove that :

(i) $\int_0^1 \frac{x\,dx}{\sqrt{(1-x^5)}} = \frac{1}{5}\beta\left(\frac{2}{5}, \frac{1}{2}\right)$ *(Raipur, 2006)* (ii) $\int_0^1 \frac{dx}{\sqrt{(1+x^4)}} = \frac{1}{4\sqrt{2}}\beta\left(\frac{1}{4}, \frac{1}{2}\right)$. *(V.T.U., 2003)*

(iii) $\int_0^1 x^3\,(1-\sqrt{x})^5\,dx = 2\beta(8, 6)$. *(Marathwada, 2008 ; J.N.T.U., 2006)*

6. Show that (i) $\int_a^b (x-a)^{m-1}\,(b-x)^{n-1}\,dx = (b-a)^{m+n-1}\,\beta(m, n)$ *(P.T.U., 2010 ; Mumbai, 2005)*

(ii) $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,dx = \frac{1}{a^n, b^m}\,\beta(m, n)$ *(Nagpur, 2009)* (iii) $\int_0^\infty \frac{x^{10} - x^{18}}{(1+x)^{30}}\,dx = 0$ *(Mumbai, 2005)*

(iv) $\int_0^1 \frac{(1-x^4)^{3/4}}{(1+x^4)^2}\,dx = \frac{1}{2^{9/2}}\beta\left(\frac{7}{4}, \frac{1}{4}\right)$ *(Mumbai, 2007)*

7. Prove that $\int_0^1 x^m\,(\log x)^n\,dx = \frac{(-1)^n\,n!}{(m+1)^{n+1}}$, where n is a positive integer and $m > -1$. *(S.V.T.U., 2006)*

Hence evaluate $\int_0^1 x\,(\log x)^3\,dx$. *(Nagpur, 2009)*

8. Show that $\int_0^1 y^{q-1}\left(\log\frac{1}{y}\right)^{p-1} dy = \frac{\Gamma(p)}{q^p}$, where $p > 0, q > 0$. *(Rohtak, 2006 S)*

9. Express $\int_0^1 x^m\,(1-x^n)^p\,dx$ in terms of gamma functions *(Marathwada, 2008)*

Hence evaluate : (i) $\int_0^1 x(1-x^3)^{10}\,dx$. *(Bhopal, 2008)* (ii) $\int_0^1 \frac{dx}{\sqrt{(1-x^n)}}$ *(Anna, 2005)*

10. Prove that $\int_0^\infty \frac{dx}{(e^x+e^{-x})^n} = \frac{1}{4}\beta\left(\frac{n}{2}, \frac{n}{2}\right)$ and hence evaluate $\int_0^\infty \operatorname{sech}^8 x\, dx$.

11. Prove that $\beta\left(n+\frac{1}{2}, n+\frac{1}{2}\right) = \frac{\Gamma(n+1/2)\sqrt{\pi}}{2^{2n}\,\Gamma(n+1)}$. Hence show that $2^n\,\Gamma(n+1/2) = 1 \,.\, 3 \,.\, 5 \ldots (2n-1)\sqrt{\pi}$

(*Mumbai, 2007*)

12. Prove that :

(*i*) $\frac{\beta(m+1, n)}{m} = \frac{\beta(m, n+1)}{n} = \frac{\beta(m, n)}{m+n}$

(*ii*) $\beta(n, n) = \frac{\sqrt{\pi}\,\Gamma(n)}{2^{2n-1}\,\Gamma\left(n+\frac{1}{2}\right)}$

(*iii*) $\Gamma\left(n+\frac{1}{2}\right) = \frac{\Gamma(2n+1)\sqrt{\pi}}{2^{2n}\,.\,\Gamma(n+1)}$

(*iv*) $\beta(m+1) + \beta(m, n+1) = \beta(m, n)$

(*Bhopal, 2008 ; J.N.T.U., 2006 ; Madras, 2003*)

13. Show that $\iint x^{m-1}\, y^{n-1}\, dx\, dy$ over the positive quadrant of the ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \text{ is } \frac{a^m b^n}{2n}\beta\left(\frac{m}{2}, \frac{n}{2}+1\right).$$

14. Show that the area in the first quadrant enclosed by the curve $(x/a)^\alpha + (y/b)^\beta = 1$, $\alpha > 0$, $\beta > 0$, is given by

$$\frac{ab}{\alpha+\beta}\,\frac{\Gamma(1/\alpha)\,\Gamma(1/\beta)}{\Gamma(1/\alpha+1/\beta)}.$$

15. Find the mass of an octant of the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$, the density at any point being $\rho = kxyz$.

(*U.P.T.U., 2002*)

7.17 (1) ELLIPTIC INTEGRALS

In Applied Mathematics, we often come across integrals of the form $\int_0^1 e^{-x^2}\, dx$ or $\int_0^1 \sin x^2\, dx$ which cannot be evaluated by any of the standard methods of integration. In such cases, we may find the value to any desired degree of accuracy by expanding their integrands as power series. An important class of such integrals is the *elliptic integrals.*

Def. The integral $F(k, \phi) = \int_0^\phi \frac{dx}{\sqrt{(1-k^2\sin^2 x)}}\ (k^2 < 1)$ ...(*i*)

which is a function of the two variables k and ϕ, is called the *elliptic integral of the first kind with modulus k* and *amplitude* ϕ.

The integral $E(k, \phi) = \int_0^\phi \sqrt{(1-k^2\sin^2 x)}\, dx\ (k^2 < 1)$ ...(*ii*)

is called the *elliptic integral of the second kind* with *modulus k and amplitude* ϕ.

The name *elliptic integral* arose from its original application in finding the length of an elliptic arc (Fig.7.45), For instance, consider the ellipse

$$x = a\cos\phi, \qquad y = b\sin\phi, \qquad (a < b)$$

Then length of its arc

$$AP = \int_0^\phi \sqrt{\left[\left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2\right]}\, d\phi = \int_0^\phi \sqrt{[(-a\sin\phi)^2 + (b\cos\phi)^2]}\, d\phi$$

$$= \int_0^\phi \sqrt{[(b^2 + (a^2-b^2)\sin^2\phi]}\, d\phi = b\int_0^\phi \sqrt{\left[1 - \left(1 - \frac{a^2}{b^2}\right)\sin^2\phi\right]}\, d\phi$$

$$= bE(k, \phi) \text{ for } k^2 = 1 - a^2/b^2 \le 1.$$

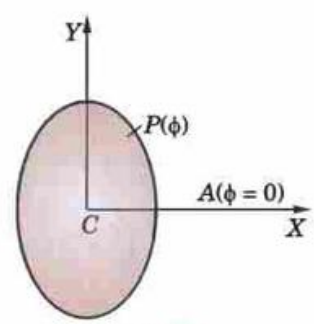

Fig. 7.45

Also the perimeter of the ellipse

$$= 4b \int_0^{\pi/2} \sqrt{(1 - k^2 \sin^2 \phi)}\, d\phi = 4bE(k, \pi/2).$$

This particular integral with upper limit $\phi = \pi/2$ is called the *complete elliptic integral of the second kind and is denoted by* $E(k)$.

Thus $$E(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 \phi)\, d\phi \ (k^2 < 1) \qquad ...(iii)$$

Similarly, the *complete elliptic integral of first kind is*

$$K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{(1 - k^2 \sin^2 \phi)}} \ (k^2 < 1) \qquad ...(iv)$$

To evaluate it, we expand the integral in the form

$$(1 - k^2 \sin^2 \phi)^{-1/2} = 1 + \frac{k^2}{2} \sin^2 \phi + \frac{3k^4}{4} \sin^4 \phi + ...$$

This series can be shown to be uniformly convergent for all k, and may, therefore, be integrated term by term [See § 9.19-II]. Then we have

$$K(k) = \int_0^{\pi/2} \left(1 + \frac{k^2}{2} \sin^2 \phi + \frac{3k^4}{8} \sin^4 \phi + \frac{5k^6}{16} \sin^6 \phi + ... \right) d\phi$$

$$= \frac{\pi}{2} \left[1 + \left(\frac{1}{2} \right)^2 k^2 + \left(\frac{1.3}{2.4} \right)^2 k^4 + \left(\frac{1.3.5}{2.4.6.} \right)^2 k^6 + ... \right] \qquad ...(v)$$

This series may be used to compute K for various values of k. In particular, if $k = \sin 10°$; we have

$$K = \frac{\pi}{2} (1 + 0.00754 + 0.00012 + ...) = 1.5828 \qquad ...(vi)$$

In this way tables of the elliptic integrals are constructed. Values of $F(k, \phi)$ and $E(k, \phi)$ are readily available for $0 \le \phi \le \pi/2$, $0 < k < 1$. (*See Peirce's short tables*).

Example 7.50. *Express* $\int_0^{\pi/6} \frac{dx}{\sqrt{(\sin x)}}$ *in terms of elliptic integral.*

Solution. Put $\cos x = \cos^2 \phi$ and $dx = \frac{2 \cos \phi\, d\phi}{\sqrt{(1 + \cos^2 \phi)}}$

Then $$I = \int_0^{\pi/2} \frac{2 \cos^2 \phi}{\sqrt{(1 + \cos^2 \phi)}} d\phi = 2 \int_0^{\pi/2} \frac{(1 + \cos^2 \phi) - 1}{\sqrt{(1 + \cos^2 \phi)}} d\phi$$

$$= 2 \left\{ \int_0^{\pi/2} \sqrt{(1 + \cos^2 \phi)}\, d\phi - \int_0^{\pi/2} \frac{d\phi}{\sqrt{(1 + \cos^2 \phi)}} \right\} = 2 \left\{ \int_0^{\pi/2} \sqrt{(2 - \sin^2 \phi)}\, d\phi - \int_0^{\pi/2} \frac{d\phi}{\sqrt{(2 - \sin^2 \phi)}} \right\}$$

$$= 2\sqrt{2} \int_0^{\pi/2} \sqrt{(1 - 1/2 \sin^2 \phi)}\, d\phi - \sqrt{2} \int_0^{\pi/2} \frac{d\phi}{\sqrt{(1 - 1/2 \sin^2 \phi)}} = 2\sqrt{2}\, E\left(\frac{1}{\sqrt{2}} \right) - \sqrt{2}\, K\left(\frac{1}{\sqrt{2}} \right)$$

(2) **Elliptic functions.** By putting $\sin x = t$ and $\sin \phi = z$, (i) becomes

$$u = \int_0^z \frac{dt}{\sqrt{[(1 - t^2)(1 - k^2 t^2)]}} \ (k^2 < 1) \qquad ...(vii)$$

This is known as *Jacobi's form of the elliptic integral of first kind** whereas (i) is the *Legendre's form*†.

If $k = 0$, (vii) gives $u = \sin^{-1} z$. By analogy, we denote (vii) $sn^{-1} z$ for a fixed non-zero value of k. This leads to the functions $sn\ u = z = \sin \phi$ and $cn\ u = \cos \phi$
which are called the *Jacobi's elliptic functions.*

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* See footnote p. 215.

† A French mathematician *Adrien Marie Legendre* (1752–1833) who made important contributions to number theory, special functions, calculus of variations and elliptic integrals.

The elliptic functions $sn\ u$ and $cn\ u$ are periodic with a period depending on k and an amplitude equal to unity. These behave somewhat like $\sin u$ and $\cos u$. For instance

$$sn(0) = 0,\ cn(1) = 1 \qquad \text{and} \qquad sn(-u) = -sn(u),\ cn(-u) = cn(u).$$

Example 7.51. *Show that* $\displaystyle\int_0^{a/2} \frac{dx}{\sqrt{(2ax - x^2)}\sqrt{(a^2 - x^2)}} = \frac{2}{3a} K\left(\frac{1}{3}\right)$.

Solution. Putting $\quad x = \dfrac{a}{2}(1 - \sin\theta),\ dx = -\dfrac{a}{2}\cos\theta\, d\theta,$

$$2ax - x^2 = \frac{a^2}{4}(1 - \sin\theta)(3 + \sin\theta) \text{ and } a^2 - x^2 = \frac{a^2}{4}(1 + \sin\theta)(3 - \sin\theta)$$

Also when $\quad x = 0,\ \theta = \pi/2$; when $x = a/2,\ \theta = 0$.

Thus the given integral

$$= \frac{4}{a^2}\int_{\pi/2}^{0} \frac{-(a/2)\cos\theta\, d\theta}{\sqrt{[(1 - \sin^2\theta)(2 - \sin^2\theta)]}} = \frac{2}{3a}\int_0^{\pi/2} \frac{d\theta}{\sqrt{[(1 - (1/3)^2\sin^2\theta]}} = \frac{2}{3a} K\left(\frac{1}{3}\right).$$

7.18 (1) ERROR FUNCTION OR PROBABILITY INTEGRAL

The error function or the probability integral is defined as

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\, dt.$$

This integral arises in the solution of certain partial differential equations of applied mathematics and occupies an important position in the probability theory.

The complementary error function $erfc(x)$ is defined as $erfc(x) = 1 - erf(x)$.

(2) Properties : (*i*) $erf(-x) = -erf(x)$; (*ii*) $erf(0) = 0$

$$(iii)\ erf(\infty) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}\, dt = \frac{2}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} = 1 \qquad \text{[By (iii), p. 289]}$$

This proves that the total area under the *Normal* or *Gaussian error function curve* is unity – § 26.16.

PROBLEMS 7.9

1. By means of the substitution $k \sin x = \sin z$, show that

   (*i*) $\displaystyle\int_0^{\pi} \frac{dx}{\sqrt{(1 - k^2\sin^2 x)}} = \frac{1}{k} F\left(\frac{1}{k}, \phi'\right)$;

   (*ii*) $\displaystyle\int_0^{\phi} \sqrt{(1 - k^2\sin^2 x)}\, dx = \left(\frac{1}{k} - k\right) F\left(\frac{1}{k}, \phi'\right) + kE\left(\frac{1}{k}, \phi'\right)$

   where $k > 1$ and $\phi' = \sin^{-1}(k \sin\phi)$.

   Express the following integrals in terms of elliptic integrals :

2. $\displaystyle\int_0^{\pi/2} \frac{dx}{\sqrt{(1 + 3\sin^2 x)}}$. *(Kerala, M.E., 2005)*
3. $\displaystyle\int_0^{\pi/2} \frac{dx}{\sqrt{(2 - \cos x)}}$.
4. $\displaystyle\int_0^{\pi/2} \sqrt{(\cos x)}\, dx$.
5. Expand $erf(x)$ in ascending powers of x. Hence evaluate $erf(0)$. *(P.T.U., 2009 S)*
6. Compute (*i*) $erf(0.3)$, (*ii*) $erf(0.5)$, correct to three decimal places.
7. Show that (*i*) $erf(x) + erf(-x) = 0$ (*ii*) $erfc(x) + erfc(-x) = 2$
8. Prove that

   (*i*) $\displaystyle\frac{d}{dx}[erf(ax)] = \frac{2a}{\sqrt{\pi}} e^{-a^2x^2}$ *(Osmania, 2003)* (*ii*) $\displaystyle\frac{d}{dx}[erfc(ax)] = -\frac{2a}{\sqrt{\pi}} e^{-a^2x^2}$.

9. Prove that $\displaystyle\int_0^\infty e^{-x^2 - 2ax}\, dx = \frac{\sqrt{\pi}}{2} e^{a^2}[1 - erf(0)]$

7.19 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 7.10

Fill up the blanks or choose the correct answer from the following problems :

1. $\int_0^2 \int_0^x (x+y)\, dx\, dy =$ .....
2. $\int_0^1 \int_0^{1-x} dx\, dy$ .....
3. $\int_0^1 e^{-x^2}\, dx =$ .....
4. $\beta\left(\frac{1}{2}, \frac{1}{2}\right) =$ ..... *(V.T.U., 2010)*
5. $\Gamma(3.5) =$ .....
6. The surface area of the sphere $x^2 + y^2 + z^2 + 2x - 4y + 8z - 2 = 0$ is .....
7. $\int_0^2 \int_1^3 \int_1^2 xy^2 z\, dz dy dx =$ .....
8. If $u = x + y$ and $v = x - 2y$, then the area-element $dxdy$ is replaced by ..... $dudv$.
9. In terms of Beta function $\int_0^{\pi/2} \sin^7 \theta \sqrt{\cos \theta}\, d\theta =$ .....
10. The value of $\beta(2, 1) + \beta(1, 2)$ is .....
11. $\int_0^1 \int_1^2 xy\, dy\, dx =$ .....
12. Volume bounded by $x \geq 0, y \geq 0, z \geq 0$ and $x^2 + y^2 + z^2 = 1$ as a triple integral integral.
13. Value of $\int_0^1 \int_0^{x^2} xe^y\, dy\, dx$ is equal to

    (*a*) $e/2$ (*b*) $e - 1$ (*c*) $1 - e$ (*d*) $e/2 - 1$. *(Bhopal, 2008)*
14. $\iint x^2 y^3\, dxdy$ over the rectangle $0 \leq x \leq 1$ and $0 \leq y \leq 3$ is .....
15. $\int_0^{\pi} \int_0^{a \sin \theta} r\, dr\, d\theta =$ .....
16. $\int_{x=0}^{x=3} \int_{y=0}^{y=1/x} ye^{xy}\, dxdy =$ .....
17. $\int_0^{\pi/2} \int_0^{\infty} \frac{r\, dr\, d\theta}{(r^2 + a^2)} =$ .....
18. $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2)\, dx\, dy =$ .....
19. To change cartesian coordinates (x, y, z) to spherical polar coordinate (r, θ, ϕ); $dx\, dy\, dz$ is replaced by .....
20. $\int_0^2 \int_0^{x^2} e^{y/x}\, dydx =$ .....
21. $\iint (x+y)^2\, dxdy$ over the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is .....
22. $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx\, dy}{1 + x^2 + y^2} =$ .....
23. $\iint xy(x+y)\, dx\, dy$ over the area between $y + x^2$ and $y = x$, is .....
24. Value of $\int_0^1 \int_x^{\infty} xy\, dx\, dy$ is

    (*a*) zero (*b*) $-1/24$ (*c*) $1/24$ (*d*) 24. *(V.T.U., 2010)*
25. $\iint dx\, dy$ over the area bounded by $x = 0, y = 0, x^2 + y^2 = 1$ and $5y = 3$, is .....
26. $\iint_R y\, dx\, dy$ where R is the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$, is .....
27. $\iint (x^2 + y^2)\, dx\, dy$ in the positive quadrant for which $x + y \leq 1$, is .....
28. Area between the parabolas $y^2 = 4x$ and $x^2 = 4y$ is .....
29. Changing the order of integration in $\int_{-a}^{a} \int_0^{\sqrt{a^2 - y^2}} f(x, y)\, dx\, dy =$ .....
30. $\Gamma(1/4)\Gamma(3/4) =$ ..... *(V.T.U., 2011)*
31. $\beta(5/2, 7/2) =$ .....
32. $\int_0^{\infty} \int_0^x xe^{-x^2/y}\, dydx =$ .....
33. On changing to polar coordinates $\int_0^{2a} \int_0^{\sqrt{(2ax - x^2)}} dx\, dy$ becomes .....

34. A square lamina is immersed in the liquid with one vertex in the surface and the diagonal of length vertical. Its centre of pressure is at a depth .....

35. The centroid of the area enclosed by the parabola $y^2 = 4x$, x-axis and its latus-rectum is .....

36. The moment of inertia of a uniform spherical ball of mass 10 gm and radius 2 cm about a diameter is .....

37. M.I. of a solid right circular cone (base-radius r and height h) about its axis is .....

38. $erf_c(-x) - erf(x) =$ .....

39. $\int_0^1 \frac{x-1}{\log x}\, dx =$ .....

40. $\Gamma\left(\frac{3}{2}\right) =$ .....

41. Value of $\int_0^a \int_0^b \int_0^c x^2y^2z^2\, dx\, dy\, dz$ is

(a) $\frac{abc}{3}$ (b) $\frac{a^2b^2c^2}{27}$ (c) $\frac{a^3b^3c^3}{27}$ (d) $\frac{a^2b^2c^2}{9}$.

42. The integral $\int_0^1 \int_0^{\sqrt{(1-x^2)}} (x+y)\, dy\, dx$ after changing the order of integration.

(a) $\int_0^2 \int_0^{\sqrt{(1-y^2)}} (x+y)\, dx\, dy$ (b) $\int_0^1 \int_0^{\sqrt{(1-y^2)}} (x+y)\, dx\, dy$

(c) $\int_0^1 \int_0^{\sqrt{(1+y^2)}} (x+y)\, dx\, dy$ (d) $\int_0^{-1} \int_0^{\sqrt{(1-y^2)}} (x+y)\, dx\, dy$. (V.T.U., 2011)

CHAPTER 8

Vector Calculus and Its Applications

1. Differentiation of vectors. **2.** Curves in space. **3.** Velocity and acceleration, Tangential and normal acceleration, Relative velocity and acceleration. **4.** Scalar and vector point functions—Vector operator del. **5.** Del applied to scalar point functions—Gradient. **6.** Del applied to vector point functions—Divergence and Curl. **7.** Physical interpretations of div **F** and curl **F**. **8.** Del applied twice to point functions. **9.** Del applied to products of point functions. **10.** Integration of vectors. **11.** Line integral—Circulation—Work. **12.** Surface integral—Flux. **13.** Green's theorem in the plane. **14.** Stoke's theorem. **15.** Volume integral. **16.** Divergence theorem. **17.** Green's theorem. **18.** Irrotational and Solenoidal fields. **19.** Orthogonal curvilinear coordinates, Del applied to functions in orthogonal curvilinear coordinates. **20.** Cylindrical coordinates. **21.** Spherical polar coordinates. **22.** Objective Type of Questions.

8.1 (1) DIFFERENTIATION OF VECTORS

If a vector **R** varies continuously as a scalar variable t changes, then **R** is said to be a function of t and is written as $\mathbf{R} = \mathbf{F}(t)$.

Just as in scalar calculus, *we define* **derivative** *of a vector function* $\mathbf{R} = \mathbf{F}(t)$ *as*

$$\underset{\delta t \to 0}{\text{Lt}} \frac{\mathbf{F}(t+\delta t) - \mathbf{F}(t)}{\delta t} \text{ and write it as } \frac{d\mathbf{R}}{dt} \text{ or } \frac{d\mathbf{F}}{dt} \text{ or } \mathbf{F}'(t).$$

(2) **General rules of differentiation** are similar to those of ordinary calculus *provided the order of factors in vector products is maintained.* Thus, if ϕ, **F, G, H** are scalar and vector functions of a scalar variable t, we have

(*i*) $\dfrac{d}{dt}(\mathbf{F}+\mathbf{G}-\mathbf{H}) = \dfrac{d\mathbf{F}}{dt} + \dfrac{d\mathbf{G}}{dt} - \dfrac{d\mathbf{H}}{dt}$ (*ii*) $\dfrac{d}{dt}(\mathbf{F}\phi) = \mathbf{F}\dfrac{d\phi}{dt} + \dfrac{d\mathbf{F}}{dt}\phi$

(*iii*) $\dfrac{d}{dt}(\mathbf{F}\,.\,\mathbf{G}) = \mathbf{F}\,.\,\dfrac{d\mathbf{G}}{dt} + \dfrac{d\mathbf{F}}{dt}\,.\,\mathbf{G}$ (*iv*) $\dfrac{d}{dt}(\mathbf{F}\times\mathbf{G}) = \mathbf{F}\times\dfrac{d\mathbf{G}}{dt} + \dfrac{d\mathbf{F}}{dt}\times\mathbf{G}$

(*v*) $\dfrac{d}{dt}[\mathbf{FGH}] = \left[\dfrac{d\mathbf{F}}{dt}\mathbf{GH}\right] + \left[\mathbf{F}\dfrac{d\mathbf{G}}{dt}\mathbf{H}\right] + \left[\mathbf{FG}\dfrac{d\mathbf{H}}{dt}\right]$

(*vi*) $\dfrac{d}{dt}[(\mathbf{F}\times\mathbf{G})\times\mathbf{H}] = \left(\dfrac{d\mathbf{F}}{dt}\times\mathbf{G}\right)\times\mathbf{H} + \left(\mathbf{F}\times\dfrac{d\mathbf{G}}{dt}\right)\times\mathbf{H} + (\mathbf{F}\times\mathbf{G})\times\dfrac{d\mathbf{H}}{dt}$

As an illustration, let us prove (*iv*), while the others can be proved similarly :

$$\frac{d}{dt}(\mathbf{F}\times\mathbf{G}) = \underset{\delta t\to 0}{\text{Lt}} \frac{(\mathbf{F}+\delta\mathbf{F})\times(\mathbf{G}+\delta\mathbf{G}) - \mathbf{F}\times\mathbf{G}}{\delta t} = \underset{\delta t\to 0}{\text{Lt}} \frac{\mathbf{F}\times\delta\mathbf{G} + \delta\mathbf{F}\times\mathbf{G} + \delta\mathbf{F}\times\delta\mathbf{G}}{\delta t}$$

$$= \underset{\delta t\to 0}{\text{Lt}}\left[\mathbf{F}\times\frac{\delta\mathbf{G}}{\delta t} + \frac{\delta\mathbf{F}}{\delta t}\times\mathbf{G} + \frac{\delta\mathbf{F}}{\delta t}\times\delta\mathbf{G}\right] = \mathbf{F}\times\frac{d\mathbf{G}}{dt} + \frac{d\mathbf{F}}{dt}\times G \qquad [\because\ \delta\mathbf{G}\to 0 \text{ as } \delta t\to 0]$$

Obs. 1. *If* $\mathbf{F}(t)$ *has a constant magnitude, then* $\mathbf{F}\,.\,\dfrac{d\mathbf{F}}{dt} = 0$

For $\mathbf{F}(t) \cdot \mathbf{F}(t) = [\mathbf{F}(t)]^2 = \text{constant}$

$$\therefore \qquad \mathbf{F} \cdot \frac{d\mathbf{F}}{dt} = 0, \quad i.e., \quad \frac{d\mathbf{F}}{dt} \perp \mathbf{F}.$$

Obs. 2. *If* $\mathbf{F}(t)$ *has constant (fixed) direction, then* $\mathbf{F} \times \dfrac{d\mathbf{F}}{dt} = \mathbf{0}$

Let $\mathbf{G}(t)$ be a unit vector in the direction of $\mathbf{F}(t)$ so that

$$\mathbf{F}(t) = f(t)\,\mathbf{G}(t) \quad \text{where} \quad f(t) = |\,\mathbf{F}(t)\,|.$$

$$\therefore \qquad \frac{d\mathbf{F}}{dt} = f\frac{d\mathbf{G}}{dt} + \frac{df}{dt}\mathbf{G} \quad \text{and} \quad \mathbf{F} \times \frac{d\mathbf{F}}{dt} = f\,\mathbf{G} \times \left[f\frac{d\mathbf{G}}{dt} + \frac{df}{dt}\mathbf{G}\right]$$

$$= f^2\,\mathbf{G} \times \frac{d\mathbf{G}}{dt} = \mathbf{0}. \qquad \text{[since } \mathbf{G} \text{ is constant, } d\mathbf{G}/dt = \mathbf{0}.]$$

Example 8.1. *If* $\mathbf{A} = 5t^2\mathbf{I} + t\mathbf{J} - t^3\mathbf{K}$, $\mathbf{B} = \sin t\mathbf{I} - \cos t\mathbf{J}$, *find (i)* $\dfrac{d}{dt}(\mathbf{A} \cdot \mathbf{B})$; *(ii)* $\dfrac{d}{dt}(\mathbf{A} \times \mathbf{B})$.

Solution. (*i*) $\dfrac{d}{dt}(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot \dfrac{d\mathbf{B}}{dt} + \dfrac{d\mathbf{A}}{dt} \cdot \mathbf{B}$

$$= (5t^2\mathbf{I} + t\mathbf{J} - t^3\mathbf{K}) \cdot [\cos t\mathbf{I} - (-\sin t)\,\mathbf{J}] + (10t\mathbf{I} + \mathbf{J} - 3t^2\mathbf{K}) \cdot (\sin t\mathbf{I} - \cos t\mathbf{J})$$

$$= (5t^2 \cos t + t \sin t) + (10t \sin t - \cos t) = 5t^2 \cos t + 11t \sin t - \cos t.$$

(*ii*) $\dfrac{d}{dt}(\mathbf{A} \times \mathbf{B}) = \mathbf{A} \times \dfrac{d\mathbf{B}}{dt} + \dfrac{d\mathbf{A}}{dt} \times \mathbf{B}$

$$= (5t^2\mathbf{I} + t\mathbf{J} - t^3\mathbf{K}) \times (\cos t\mathbf{I} + \sin t\mathbf{J}) + (10\,t\mathbf{I} + \mathbf{J} - 3t^2\mathbf{K}) \times (\sin t\mathbf{I} - \cos t\mathbf{J})$$

$$= [5t^2 \sin t\mathbf{K} + r \cos t\,(-\mathbf{K}) - t^3 \cos t\mathbf{J} - t^3 \sin t\,(-\mathbf{I})]$$

$$+ [-10t \cos t\mathbf{K} + \sin t(-\mathbf{K}) - 3t^2 \sin t\mathbf{J} + 3t^2 \cos t\,(-\mathbf{I})]$$

$$= (t^3 \sin t - 3t^2 \cos t)\,\mathbf{I} - t^2(t \cos t + 3 \sin t)\,\mathbf{J} + [(5t^2 - 1) \sin t - 11t \cos t]\,\mathbf{K}.$$

8.2 CURVES IN SPACE

(1) Tangent. Let $\mathbf{R}(t) = x(t)\mathbf{I} + y(t)\mathbf{J} + z(t)\mathbf{K}$ be the position vector of a point P. Then as the scalar parameter t takes different values, the point P traces out a *curve in space* (Fig. 8.1). If the neighbouring point Q corresponds to $t + \delta t$, then $\delta\mathbf{R} = \mathbf{R}(t + \delta t) - \mathbf{R}(t)$ or $\delta\mathbf{R}/\delta t$ is directed along the chord PQ. As $\delta t \to 0$, Lt $\delta\mathbf{R}/\delta t$ becomes the tangent (vector) to the curve at P whenever it exists and is not zero.

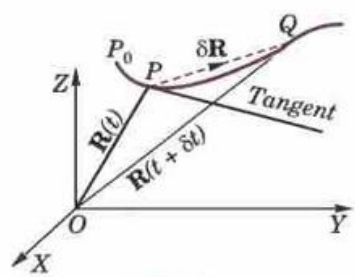

Fig. 8.1

Thus the vector $\mathbf{R}' = d\mathbf{R}/dt$ is a tangent to the space curve $\mathbf{R} = \mathbf{F}(t)$.

Let P_0 be a fixed point of this curve corresponding to $t = t_0$. If s be the length of the arc P_0P, then

$$\frac{\delta s}{\delta t} = \frac{\delta s}{|\,\delta\mathbf{R}\,|} \cdot \frac{|\,\delta\mathbf{R}\,|}{\delta t} = \frac{\text{arc } PQ}{\text{chord } PQ}\left|\frac{\delta\mathbf{R}}{\delta t}\right|$$

As $Q \to P$ along the curve QR *i.e.*, $\delta t \to 0$, then arc PQ/chord $PQ \to 1$ and

$$\frac{ds}{dt} = \left|\frac{d\mathbf{R}}{dt}\right| \quad \text{or} \quad |\,\mathbf{R}'(t)\,|.$$

If $\mathbf{R}'(t)$ is continuous, then arc P_0P is given by

$$s = \int_{t_0}^{t} |\,\mathbf{R}'\,|\,dt = \int_{t_0}^{t} \sqrt{(x')^2 + (y')^2 + (z')^2}\,dt$$

If we take s the parameter in place of t then the magnitude of the tangent vector, *i.e.*, $|\,d\mathbf{R}/ds\,| = 1$. Thus denoting the unit tangent vector by $\mathbf{T}$, we have

$$\mathbf{T} = \frac{d\mathbf{R}}{ds} \qquad \text{...(1)}$$

(2) **Principal normal.** Since $\mathbf{T}$ is a unit vector, we have

$$d\mathbf{T}/ds.\ \mathbf{T} = 0$$

i.e., $d\mathbf{T}/ds$ is perpendicular to $\mathbf{T}$. Or else $d\mathbf{T}/ds = 0$, in which case $\mathbf{T}$ is a constant vector w.r.t. the arc length s and so has a fixed direction, *i.e.*, the curve is a straight line.

If we denote a unit normal vector to the curve at P by $\mathbf{N}$ then $d\mathbf{T}/ds$ is in the direction of $\mathbf{N}$ which is known as the *principal normal* to the space curve at P. The plane of $\mathbf{T}$ and $\mathbf{N}$ is called the *osculating plane* of the curve at P (Fig. 8.2).

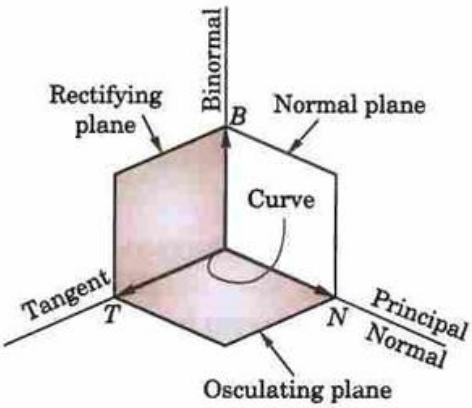

Fig. 8.2

(3) Binormal. A third unit vector $\mathbf{B}$ defined by $\mathbf{B} = \mathbf{T} \times \mathbf{N}$, is called the *binormal at P*. Since $\mathbf{T}$ and $\mathbf{N}$ are unit vectors, $\mathbf{B}$ is also a unit vector perpendicular to both $\mathbf{T}$ and $\mathbf{N}$ and hence normal to the *osculating plane at P*.

Thus at each point P of a space curve there are three mutually perpendicular unit vectors, $\mathbf{T}, \mathbf{N}, \mathbf{B}$ which form a moving trihedral such that

$$\mathbf{T} = \mathbf{N} \times \mathbf{B}, \mathbf{N} = \mathbf{B} \times \mathbf{T}, \mathbf{B} = \mathbf{T} \times \mathbf{N} \qquad ...(2)$$

This moving trihedral determines the following three fundamental planes at each point of the curve :

(*i*) The osculating plane containing $\mathbf{T}$ and $\mathbf{N}$

(*ii*) The normal plane containing $\mathbf{N}$ and $\mathbf{B}$

(*iii*) The rectifying plane containing $\mathbf{B}$ and $\mathbf{T}$.

(4) Curvature. The arc rate of turning of the tangent (*i.e.*, the magnitude of $d\mathbf{T}/ds$) is called the *curvature* of the curve and is denoted by k.

Since $d\mathbf{T}/ds$ is in the direction of the principal normal $\mathbf{N}$, therefore,

$$\frac{d\mathbf{T}}{ds} = k\mathbf{N} \qquad ...(3)$$

(5) Torsion. Since $\mathbf{B}$ is a unit vector, we have $\frac{d\mathbf{B}}{ds} \cdot \mathbf{B} = 0$

Also $\mathbf{B} \cdot \mathbf{T} = 0$, therefore $\frac{d\mathbf{B}}{ds} \cdot \mathbf{T} + \mathbf{B} \cdot \frac{d\mathbf{T}}{ds} = 0$.

or $$\frac{d\mathbf{B}}{ds} \cdot \mathbf{T} + \mathbf{B} \cdot (k\mathbf{N}) = 0, \qquad i.e., \quad \frac{d\mathbf{B}}{ds} \cdot \mathbf{T} = 0 \qquad [\because \mathbf{B} \cdot \mathbf{N} = 0]$$

Hence $d\mathbf{B}/ds$ is perpendicular to both $\mathbf{B}$ and $\mathbf{T}$ and is, therefore, parallel to $\mathbf{N}$.

The arc rate of turning of the binormal (*i.e.*, the magnitude of $d\mathbf{B}/ds$) is called *torsion* of the curve and is denoted by τ. We may, therefore, write

$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N} \qquad ...(4)$$

(The negative sign indicates that for $\tau > 0$, $d\mathbf{B}/ds$ has direction of $-\mathbf{N}$).

Finally to find $d\mathbf{N}/ds$, we differentiate $\mathbf{N} = \mathbf{B} \times \mathbf{T}$.

$$\therefore \qquad \frac{d\mathbf{N}}{ds} = \frac{d\mathbf{B}}{ds} \times \mathbf{T} + \mathbf{B} \times \frac{d\mathbf{T}}{ds} = -\tau\mathbf{N} \times \mathbf{T} + \mathbf{B} \times k\mathbf{N}$$

Using the relation (2), it reduces to $\frac{d\mathbf{N}}{ds} = \tau\mathbf{B} - k\mathbf{T}$ ...(5)

The equations (3), (4) and (5) constitute the well-known *Frenet formulae** for space curves.

Obs. 1. $\rho = 1/k$ and $\sigma = 1/\tau$ are respectively called the radii of curvature and torsion.

2. For a plane curve $\tau = 0$.

Example 8.2. *Find the angle between the tangents to the curve* $\mathbf{R} = t^2\mathbf{I} + 2t\mathbf{J} - t^3\mathbf{K}$ *at the point* $t = \pm 1$. *(V.T.U., 2010)*

Solution. The tangent at any point 't' is given by

$$\frac{d\mathbf{R}}{dt} = 2t\mathbf{I} + 2\mathbf{J} - 3t^2\mathbf{K}$$

$\therefore$ the tangents $\mathbf{T}_1$, $\mathbf{T}_2$ at $t = 1$ and $t = -1$ are respectively given by

$$\mathbf{T}_1 = 2\mathbf{I} + 2\mathbf{J} - 3\mathbf{K}\ ;\ \mathbf{T}_2 = -2\mathbf{I} + 2\mathbf{J} - 3\mathbf{K},$$

* Named after a French mathematician *Jean-Frederic Frenet* (1816–1900).

Then the required $\angle\theta$ is given by $T_1T_2 \cos\theta = \mathbf{T}_1 \cdot \mathbf{T}_2 = 2(-2) + 2 \cdot 2 + (-3)(-3)$

i.e., $\sqrt{17}\sqrt{17} \cos\theta = 9 \quad \therefore \quad \theta = \cos^{-1}(9/17)$.

Example 8.3. *Find the curvature and torsion of the curve $x = a \cos t$, $y = a \sin t$, $z = bt$.*
(This curve is drawn on a circular cylinder cutting its generators at a constant angle and is known as a *circle helix*).

Solution. The vector equation of the curve is $\mathbf{R} = a \cos t\mathbf{I} + a \sin t\mathbf{J} + bt\mathbf{K}$

$\therefore \quad d\mathbf{R}/dt = -a \sin t\mathbf{I} + a \cos t\mathbf{J} + b\mathbf{K}$

Its arc length from P_0 $(t = 0)$ to any point $P(t)$ (Fig. 8.3) is given by

$$s = \int_0^t |d\mathbf{R}/dt|\, dt = \sqrt{(a^2 + b^2)}t$$

$$\therefore \quad \frac{ds}{dt} = \sqrt{(a^2 + b^2)}$$

Then
$$\mathbf{T} = \frac{d\mathbf{R}}{ds} = \frac{d\mathbf{R}}{dt}\Big/\frac{ds}{dt} = \frac{-a \sin t\mathbf{I} + a \cos t\mathbf{J} + b\mathbf{K}}{\sqrt{(a^2 + b^2)}}$$

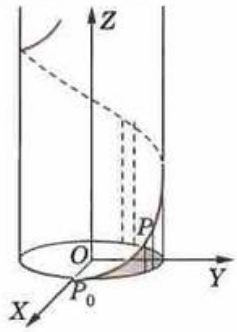

Fig. 8.3

and
$$\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{dt}\Big/\frac{ds}{dt} = \frac{-a(\cos t\mathbf{I} + \sin t\mathbf{J})}{a^2 + b^2}$$

Thus
$$k = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{a}{a^2 + b^2} \quad ...(i) \qquad \text{and} \quad \mathbf{N} = -(\cos t\mathbf{I} + \sin t\mathbf{J})$$

Also
$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = (b \sin t\mathbf{I} - b \cos t\mathbf{J} + a\mathbf{K})/\sqrt{(a^2 + b^2)}$$

$$\therefore \quad \frac{d\mathbf{B}}{ds} = \frac{d\mathbf{B}}{dt}\Big/\frac{ds}{dt} = b(\cos t\mathbf{I} + \sin t\mathbf{J})/(a^2 + b^2) = -\tau\mathbf{N} = \tau(\cos t\mathbf{I} + \sin t\mathbf{J})$$

Hence
$$\tau = \frac{b}{a^2 + b^2}. \quad ...(ii)$$

PROBLEMS 8.1

1. Show that, if $\mathbf{R} = \mathbf{A} \sin \omega t + \mathbf{B} \cos \omega t$, where $\mathbf{A}, \mathbf{B}, \omega$ are constants, then (*i*) $\dfrac{d^2\mathbf{R}}{dt^2} = -\omega^2\mathbf{R}$ *(Bhopal, 2007 S)*
   (*ii*) $\mathbf{R} \times \dfrac{d\mathbf{R}}{dt} = -\omega\mathbf{A} \times \mathbf{B}$.
2. Given $\mathbf{R} = t^m \mathbf{A} + t^n \mathbf{B}$, where $\mathbf{A}, \mathbf{B}$ are constant vectors, show that, if $\mathbf{R}$ and $d^2\mathbf{R}/dt^2$ are parallel vectors, then $m + n = 1$, unless $m = n$.
3. If $\mathbf{P} = 5t^2\mathbf{I} + t^3\mathbf{J} - t\mathbf{K}$ and $\mathbf{Q} = 2\mathbf{I} \sin t - \mathbf{J} \cos t + 5t\mathbf{K}$, find (*i*) $\dfrac{d}{dt}(\mathbf{P} \cdot \mathbf{Q})$; (*ii*) $\dfrac{d}{dt}(\mathbf{P} \times \mathbf{Q})$.
4. If $\dfrac{d\mathbf{U}}{dt} = \mathbf{W} \times \mathbf{U}$ and $\dfrac{d\mathbf{V}}{dt} = \mathbf{W} \times \mathbf{V}$, prove that $\dfrac{d}{dt}(\mathbf{U} \times \mathbf{V}) = \mathbf{W} \times (\mathbf{U} \times \mathbf{V})$. *(Mumbai, 2009)*
5. If $\mathbf{A} = x^2yz\mathbf{I} - 2xz^3\mathbf{J} + xz^2\mathbf{K}$ and $\mathbf{B} = 2z\mathbf{I} + y\mathbf{J} - x^2\mathbf{K}$, find $\dfrac{\partial^2}{\partial x \partial y}(\mathbf{A} \times \mathbf{B})$ at $(1, 0, -2)$.
6. If $\mathbf{R} = (a \cos t)\mathbf{I} + (a \sin t)\mathbf{J} + (at \tan \alpha)\mathbf{K}$, find the value of
   (*i*) $\left|\dfrac{d\mathbf{R}}{dt} \times \dfrac{d^2\mathbf{R}}{dt^2}\right|$ (*ii*) $\left|\dfrac{d\mathbf{R}}{dt}, \dfrac{d^2\mathbf{R}}{dt^2}, \dfrac{d^3\mathbf{R}}{dt^3}\right|$ *(Rohtak, 2005)*
   Also find the unit tangent vector at any point t of the curve.
7. Find the unit tangent vector at any point on the curve $x = t^2 + 2$, $y = 4t - 5$, $z = 2t^2 - 6t$, where t is any variable. Also determine the unit tangent vector at the point $t = 2$.
8. Find the equation of the tangent line to the curve $x = a \cos \theta$, $y = a \sin \theta$, $z = a\theta \tan \alpha$ at $\theta = \pi/4$.
9. Find the curvature of the (*i*) ellipse $\mathbf{R}(t) = a \cos t\mathbf{I} + b \sin t\mathbf{J}$; (*ii*) parabola $\mathbf{R}(t) = 2t\mathbf{I} + t^2\mathbf{J}$ at the point $t = 1$.

10. Find the education of the osculating plane and binormal to the curve
(*i*) $x = 2\cosh(t/2), y = 2\sinh(t/2), z = 2t$ at $t = 0$; (*ii*) $x = e^t \cos t, y = e^t \sin t, z = e^t$ at $t = 0$.
11. A circular helix is given by the equation $\mathbf{R}(t) = (2\cos t)\mathbf{I} + (2\sin t)\mathbf{J} + \mathbf{K}$. Find the curvature and torsion of the curve at any point and show that they are constant.
12. Show that for the curve $\mathbf{R} = a(3t - t^3)\mathbf{I} + 3at^2\mathbf{J} + a(3t + t^2)\mathbf{K}$, the curvature equals torsion.

8.3 (1) VELOCITY AND ACCELERATION

Let the position of a particle P at time t on a path (curve) C be $\mathbf{R}(t)$. At time $t + \delta t$, let the particle be at Q (Fig. 8.1), then $\delta\mathbf{R} = \mathbf{R}(t + \delta t) - \mathbf{R}(t)$ or $\delta\mathbf{R}/\delta t$ is directed along PQ. As $Q \to P$ along C, the line PQ becomes the tangent at P to C.

$$\therefore \qquad \underset{\delta t \to 0}{\text{Lt}} \frac{\delta\mathbf{R}}{\delta t} = \frac{d\mathbf{R}}{dt} = \mathbf{V}$$

is the tangent vector of C at P which is the *velocity* (vector) $\mathbf{V}$ of the motion and its magnitude is the *speed* $v = ds/dt$, where s is the arc length of P from a fixed point P_0 $(s = 0)$ on C.

The derivative of the velocity vector $\mathbf{V}(t)$ is called the *acceleration* (vector) $\mathbf{A}(t)$, which is given by

$$\mathbf{A}(t) = \frac{d\mathbf{V}}{dt} = \frac{d^2\mathbf{R}}{dt^2}.$$

(2) Tangential and normal accelerations. It is important to note that the magnitude of acceleration is not always the rate of change of $|\mathbf{V}|$ because $\mathbf{A}(t)$ is not always tangential to the path C. Infact

$$\mathbf{V}(t) = \frac{d\mathbf{R}}{dt} = \frac{d\mathbf{R}}{ds} \cdot \frac{ds}{dt}. \text{ where } d\mathbf{R}/ds \text{ is a unit tangent vector of } C.$$

$$\therefore \qquad \mathbf{A}(t) = \frac{d\mathbf{V}}{dt} = \frac{d}{dt}\left[\frac{ds}{dt} \cdot \frac{d\mathbf{R}}{ds}\right] = \frac{d^2s}{dt^2} \cdot \frac{d\mathbf{R}}{ds} + \left(\frac{ds}{dt}\right)^2 \frac{d^2\mathbf{R}}{ds^2}$$

Now since $d\mathbf{R}/dt \,.\, d^2\mathbf{R}/dt^2 = 0$, $d^2\mathbf{R}/dt^2$ is perpendicular to $d\mathbf{R}/dt$. Hence the acceleration $\mathbf{A}(t)$ is comprised of (*i*) the tangential component $d^2s/dt^2 \,.\, d\mathbf{R}/ds$, called the *tangential acceleration, and*

(*ii*) the normal component $(ds/dt)^2 \,.\, d^2\mathbf{R}/ds^2$, called the *normal acceleration.*

Obs. The acceleration is the time rate change of $|\mathbf{V}| = ds/dt$, if the normal acceleration is zero, for then

$$|A| = \left|\frac{d^2s}{dt^2}\right| \cdot \left|\frac{d\mathbf{R}}{ds}\right| = \left|\frac{d^2s}{dt^2}\right|.$$

(3) Relative velocity and acceleration. Let two particles P_1 and P_2 moving along the curves C_1 and C_2 have position vectors $\mathbf{R}_1$ and $\mathbf{R}_2$ at time t, (Fig. 8.4), so that

$$\mathbf{R} = \overrightarrow{P_1P_2} = \mathbf{R}_2 - \mathbf{R}_1$$

Differentiating w.r.t. t, we get $\dfrac{d\mathbf{R}}{dt} = \dfrac{d\mathbf{R}_2}{dt} - \dfrac{d\mathbf{R}_1}{dt}$ ...(*iii*)

This defines the *relative velocity* (vector) of P_2 w.r.t. P_1 and states that the *velocity (vector) of P_2 relative to P_1 = velocity (vector) of P_2 – velocity (vector) of P_1.*

Again differentiating (*iii*), we have $\dfrac{d^2\mathbf{R}}{dt^2} = \dfrac{d^2\mathbf{R}_2}{dt^2} - \dfrac{d^2\mathbf{R}_1}{dt^2}$ ...(*iv*)

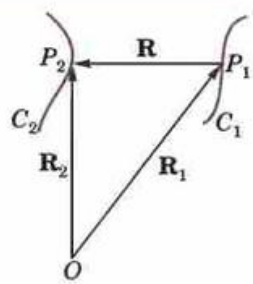

Fig. 8.4

i.e., acceleration (vector) of P_2 relative to P_1 = acceleration (vector) of P_2 – acceleration (vector) of P_1.

Example 8.4. *A particle moves along the curve $x = t^3 + 1, y = t^2, z = 2t + 3$ where t is the time. Find the components of its velocity and acceleration at $t = 1$ in the direction $\mathbf{I} + \mathbf{J} + 3\mathbf{K}$.*

Solution. *Velocity* $= \dfrac{d\mathbf{R}}{dt} = \dfrac{d}{dt}[(t^3 + 1)\mathbf{I} + t^2\mathbf{J} + (2t + 3)\mathbf{K}]$

$= 3t^2\mathbf{I} + 2t\mathbf{J} + 2\mathbf{K} = 3\mathbf{I} + 2\mathbf{J} + 2\mathbf{K}$ at $t = 1$

and *acceleration* $= \dfrac{d^2\mathbf{R}}{dt^2} = 6t\mathbf{I} + 2\mathbf{J} + 0\mathbf{K} = 6\mathbf{I} + 2\mathbf{J}$ at $t = 1$.

Now unit vector in the direction of $\mathbf{I} + \mathbf{J} + 3\mathbf{K}$ is $\dfrac{\mathbf{I} + \mathbf{J} + 3\mathbf{K}}{\sqrt{(1^2 + 1^2 + 3^2)}} = \dfrac{1}{\sqrt{11}}\ (\mathbf{I} + \mathbf{J} + 3\mathbf{K})$.

$\therefore$ component of velocity at $t = 1$ in the direction $\mathbf{I} + \mathbf{J} + 3\mathbf{K}$

$$= \frac{(3\mathbf{I} + 2\mathbf{J} + 2\mathbf{K}) \,.\, (\mathbf{I} + \mathbf{J} + 3\mathbf{K})}{\sqrt{(11)}} = \frac{3 + 2 + 6}{\sqrt{(11)}} = \sqrt{(11)}$$

and component of acceleration at $t = 1$ in the direction

$$\mathbf{I} + \mathbf{J} + 3\mathbf{K} = (6\mathbf{I} + 2\mathbf{J}) \,.\, (\mathbf{I} + \mathbf{J} + 3\mathbf{K})/\sqrt{(11)} = \frac{6 + 2}{\sqrt{(11)}} = \frac{8}{\sqrt{(11)}}.$$

Example 8.5. *A particle moves along the curve* $\mathbf{R} = (t^3 - 4t)\,\mathbf{I} + (t^2 + 4t)\mathbf{J} + (8t^2 - 3t^3)\mathbf{K}$ *where t denotes time. Find the magnitudes of acceleration along the tangent and normal at time t = 2.* *(V.T.U., 2003 S)*

Solution. Velocity $\dfrac{d\mathbf{R}}{dt} = (3t^2 - 4)\mathbf{I} + (2t + 4)\mathbf{J} + (16t - 9t^2)\mathbf{K}$

and acceleration $\dfrac{d^2\mathbf{R}}{dt^2} = 6t\mathbf{I} + 2\mathbf{J} + (16 - 18t)\mathbf{K}$

$\therefore$ at $t = 2$, velocity $\mathbf{V} = 8\mathbf{I} + 8\mathbf{J} - 4\mathbf{K}$ and acceleration $\mathbf{A} = 12\mathbf{I} + 2\mathbf{J} - 20\mathbf{K}$.

Since the velocity is along the tangent to the curve, therefore, the component of $\mathbf{A}$ along the tangent

$$= \mathbf{A} \,.\, \frac{\mathbf{V}}{|\,\mathbf{V}\,|} = (12\mathbf{I} + 2\mathbf{J} - 20\mathbf{K}) \,.\, \frac{8\mathbf{I} + 8\mathbf{J} - 4\mathbf{K}}{\sqrt{(64 + 64 + 16)}}$$

$$= \frac{12 \times 8 + 2 \times 8 + (-20) \times (-4)}{12} = 16.$$

Now the component of $\mathbf{A}$ along the normal

$$= |\ \mathbf{A} - \text{Resolved part of } \mathbf{A} \text{ along the tangent}\ |$$

$$= \left|\, 12\mathbf{I} + 2\mathbf{J} - 20\mathbf{K} - 16\,\frac{8\mathbf{I} + 8\mathbf{J} - 4\mathbf{K}}{12} \,\right| = \frac{1}{3}\ |\ 4\mathbf{I} - 26\mathbf{J} - 44\mathbf{K}\ | = 2\sqrt{73}.$$

Example 8.6. *The position vector of a particle at time t is* $\mathbf{R} = \cos(t - 1)\,\mathbf{I} + \sinh(t - 1)\,\mathbf{J} + \alpha t^3\mathbf{K}$. *Find the condition imposed on* α *by requiring that at time t = 1, the acceleration is normal to the position vector.*

Solution. *Velocity* $= \dfrac{d\mathbf{R}}{dt} = -\sin(t - 1)\,\mathbf{I} + \cosh(t - 1)\,\mathbf{J} + 3\alpha t^2\mathbf{K}$

Acceleration $= \dfrac{d^2\mathbf{R}}{dt^2} = -\cos(t - 1)\,\mathbf{I} + \sinh(t - 1)\,\mathbf{J} + 6\alpha t\mathbf{K} = -\mathbf{I} + 6\alpha\mathbf{K}$ at $t = 1$.

Also $\mathbf{R} = \mathbf{I} + \alpha\mathbf{K}$ at $t = 1$.

If $\mathbf{R}$ and acceleration at $t = 1$ are normal, then their scalar product is zero.

$\therefore$ $(-\mathbf{I} + 6\alpha\mathbf{K}) \,.\, (\mathbf{I} + \alpha\mathbf{K}) = 0$ or $-1 + 6\alpha^2 = 0$

or $\alpha^2 = 1/6$ or $\alpha = 1/\sqrt{6}$.

Example 8.7. *Find the radial and transverse acceleration of a particle moving in a plane curve.* *(Kurukshetra, 2006 ; Rajasthan, 2006)*

Solution. At any time t, let the position vector of the moving particle $P(r, \theta)$ be $\mathbf{R}$ (Fig. 8.5) so that

$$\mathbf{R} = r\,\hat{\mathbf{R}} = r(\cos\theta\mathbf{I} + \sin\theta\mathbf{J})$$

$\therefore$ its velocity $\mathbf{V} = \dfrac{d\mathbf{R}}{dt} = \dfrac{dr}{dt}\hat{\mathbf{R}} + r\dfrac{d\hat{\mathbf{R}}}{dt}$ ...(i)

As $\hat{\mathbf{R}} = \cos\theta\mathbf{I} + \sin\theta\mathbf{J}$

and $\dfrac{d\hat{\mathbf{R}}}{dt} = (-\sin\theta\mathbf{I} + \cos\theta\mathbf{J})\,\dfrac{d\theta}{dt}$

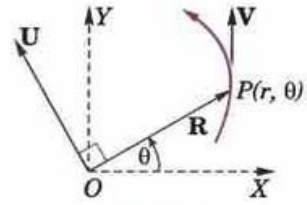

Fig. 8.5

$\therefore$ $$\frac{d\hat{\mathbf{R}}}{dt} \perp \hat{\mathbf{R}} \text{ and } \left|\frac{d\hat{\mathbf{R}}}{dt}\right| = \frac{d\theta}{dt}, \textit{ i.e., if } \mathbf{U} \text{ is a unit vector} \perp \mathbf{R}, \text{ then}$$

$$\frac{d\hat{\mathbf{R}}}{dt} = \frac{d\theta}{dt}\mathbf{U}$$

$\therefore$ (*i*) becomes, $$\mathbf{V} = \frac{dr}{dt}\hat{\mathbf{R}} + r\frac{d\theta}{dt}\mathbf{U} \qquad ...(ii)$$

Thus the radial and transverse components of the velocity are dr/dt and $r\, d\theta/dt$.

Also $$\mathbf{A} = \frac{d\mathbf{V}}{dt} = \frac{d^2r}{dt^2}\hat{\mathbf{R}} + \frac{dr}{dt}\frac{d\hat{\mathbf{R}}}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{U} + r\frac{d^2\theta}{dt^2}\mathbf{U} + r\frac{d\theta}{dt}\frac{d\mathbf{U}}{dt}$$

$$= \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{\mathbf{R}} + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\mathbf{U} \qquad \left[\because \mathbf{U} = -\sin\theta\mathbf{I} + \cos\theta\mathbf{J} \text{ gives } \frac{d\mathbf{U}}{dt} = -\frac{d\theta}{dt}\hat{\mathbf{R}}\right]$$

Thus the radial and transverse components of the acceleration are

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \text{ and } 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}.$$

Example 8.8. *A person going eastwards with a velocity of 4 km per hour, finds that the wind appears to blow directly from the north. He doubles his speed and the wind seems to come from north-east. Find the actual velocity of the wind.*

Solution. Let the actual velocity of the wind be $x\mathbf{I} + y\mathbf{J}$, where $\mathbf{I}, \mathbf{J}$ represent velocities of 1 km per hour towards the east and north respectively. As the person is going eastwards with a velocity of 4 km per hour, his actual velocity is $4\mathbf{I}$.

Then the velocity of the wind relative to the man is $(x\mathbf{I} + y\mathbf{J}) - 4\mathbf{I}$, which is parallel to $-\mathbf{J}$, as it appears to blow from the north. Hence $x = 4$. ...(*i*)

When the velocity of the person becomes $8\mathbf{I}$, the velocity of the wind relative to man is $(x\mathbf{I} + y\mathbf{J}) - 8\mathbf{I}$. But this is parallel to $-(\mathbf{I} + \mathbf{J})$.

$\therefore$ $(x - 8)/y = 1$, which by (*i*) gives $y = -4$.

Hence the actual velocity of the wind is $4(\mathbf{I} - \mathbf{J})$, *i.e.*, $4\sqrt{2}$ km. per hour towards the south-east.

PROBLEMS 8.2

1. A particle moves along a curve $x = e^{-t}, y = 2\cos 3t, z = 2\sin 3t$, where t is the time variable. Determine its velocity and acceleration vectors and also the magnitudes of velocity and acceleration at $t = 0$. *(P.T.U., 2003 ; V.T.U., 2003 S)*
2. The position vector of a particle at time t is $\mathbf{R} = \cos(t-1)\mathbf{I} + \sinh(t-1)\mathbf{J} + at^3\mathbf{K}$. Find the condition imposed on a by requiring that at time $t = 1$, the acceleration is normal to the position vector.
3. A particle moves on the curve $x = 2t^2, y = t^2 - 4t, z = 3t - 5$, where t is the time. Find the components of velocity and acceleration at time $t = 1$ in the direction $\mathbf{I} - 3\mathbf{J} + 2\mathbf{K}$. *(V.T.U., 2008)*
4. A particle moves so that its position vector is given by $\mathbf{R} = \mathbf{I}\cos\omega t + \mathbf{J}\sin\omega t$. Show that the velocity $\mathbf{V}$ of the particle is perpendicular to $\mathbf{R}$ and $\mathbf{R} \times \mathbf{V}$ is a constant vector.
5. A particle (position vector $\mathbf{R}$) is moving in a circle with constant angular velocity ω. Show by vector methods, that the acceleration is equal to $-\omega^2\mathbf{R}$.
6. (*a*) Find the tangential and normal accelerations of a point moving in a plane curve. *(Rajasthan, 2005)*

   (*b*) The position vector of a moving particle at a time t is $\mathbf{R} = 3\cos t\mathbf{I} + 3\sin t\mathbf{J} + 4t\mathbf{K}$. Find the tangent and normal components of its acceleration at $t = 1$. *(Marathwada, 2008)*
7. The velocity of a boat relative to water is represented by $3\mathbf{I} + 4\mathbf{J}$ and that of wate relative to earth is $\mathbf{I} - 3\mathbf{J}$. What is the velocity of the boat relative to the earth if $\mathbf{I}$ and $\mathbf{J}$ represent one km an hour east and north respectively.
8. A vessel A is sailing with a velocity of 11 knots per hour in the direction S.E. and a second vessel B is sailing with a velocity of 13 knots per hour in a direction 30°E of N. Find the velocity of A relative to B.
9. A person travelling towards the north-east with a velocity of 6 km per hour finds that the wind appears to blow from the north, but when he doubles his speed it seems to come from a direction inclined at an angle $\tan^{-1} 2$ to the north of east. Show that the actual velocity of the wind is $3\sqrt{2}$ km per hour towards the east.

8.4 SCALAR AND VECTOR POINT FUNCTIONS

(1) *If to each point $P(\mathbf{R})$ of a region E in space there corresponds a definite scalar denoted by $f(\mathbf{R})$, then $f(\mathbf{R})$ is called a* **scalar point function** *in E. The region E so defined is called a* **scalar field.**

The temperature at any instant, density of a body and potential due to gravitational matter are all examples of scalar point functions.

(2) *If to each point $P(\mathbf{R})$ of a region E in space there corresponds a definite vector denoted by $\mathbf{F}(\mathbf{R})$, then it is called the* **vector point function** *in E. The region E so defined is called a* **vector field.**

The velocity of a moving fluid at any instant, the gravitational intensity of force are examples of vector point functions.

Differentiation of vector point functions follows the same rules as those of ordinary calculus. Thus if $\mathbf{F}(x, y, z)$ be a vector point function, then

$$\frac{d\mathbf{F}}{dt} = \frac{\partial \mathbf{F}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{F}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{F}}{\partial z}\frac{dz}{dt}$$ (See (*iii*) p. 203]

and
$$d\mathbf{F} = \frac{\partial \mathbf{F}}{\partial x}dx + \frac{\partial \mathbf{F}}{\partial y}dy + \frac{\partial \mathbf{F}}{\partial z}dz = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy + \frac{\partial}{\partial z}dz\right)\mathbf{F} \qquad ...(i)$$

(3) **Vector operator del.** The operator on the right side of the equation (*i*) is in the form of a scalar product of $\mathbf{I}\frac{\partial}{\partial x} + \mathbf{J}\frac{\partial}{\partial y} + \mathbf{K}\frac{\partial}{\partial z}$ and $\mathbf{I}dx + \mathbf{J}dy + \mathbf{K}dz$.

If ∇ (*read as del*) *be defined by the equation* $\nabla = \mathbf{I}\frac{\partial}{\partial x} + \mathbf{J}\frac{\partial}{\partial y} + \mathbf{K}\frac{\partial}{\partial z}$ ...(*ii*)

then (*i*) may be written as $d\mathbf{F} = (\nabla \,.\, d\mathbf{R})\,\mathbf{F}$ for when $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$, $d\mathbf{R} = \mathbf{I}dx + \mathbf{J}dy + \mathbf{K}dz$.

8.5 DEL APPLIED TO SCALAR POINT FUNCTIONS—GRADIENT

(1) **Def.** *The vector function ∇f is defined as the gradient of the scalar point function f and is written as grad f.*

Thus
$$\text{grad } f = \nabla f = \mathbf{I}\frac{\partial f}{\partial x} + \mathbf{J}\frac{\partial f}{\partial y} + \mathbf{K}\frac{\partial f}{\partial z}$$

(2) **Geometrical interpretation.** Consider the scalar point function $f(\mathbf{R})$, where $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$.

If a surface $f(x, y, z) = c$ be drawn through any point $P(\mathbf{R})$ such that at each point on it, the function has the same value as at P, then such a surface is called a *level surface* of the function f through P, *e.g.*, equipotential or isothermal surface (Fig. 8.6).

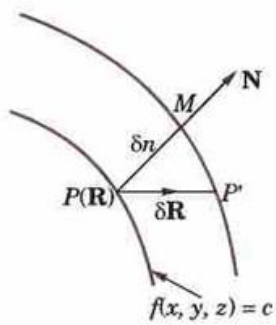

Fig. 8.6

Let $P'(\mathbf{R} + \delta\mathbf{R})$ be a point on a neighbouring level surface $f + \delta f$. Then

$$\nabla f.\,\delta\mathbf{R} = \left[\mathbf{I}\frac{\partial f}{\partial x} + \mathbf{J}\frac{\partial f}{\partial y} + \mathbf{K}\frac{\partial f}{\partial z}\right].\,(\mathbf{I}\delta x + \mathbf{J}\delta y + \mathbf{K}\delta z)$$
$$= \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z = \delta f.$$

Now if P' lies on the same level surface as P, then $\delta f = 0$, *i.e.*, $\nabla f \,.\, \delta\mathbf{R} = 0$. This means that ∇f is perpendicular to every $\delta\mathbf{R}$ lying on this surface. Thus ∇f *is normal to the surface* $f(x, y, z) = c$.

$$\therefore \qquad \nabla f = |\nabla f|\,\mathbf{N}$$

where $\mathbf{N}$ is a unit vector normal to this surface. If the perpendicular distance PM between the surfaces through P and P' be δn, then the rate of change of f normal to the surface through P

$$= \frac{\partial f}{\partial n} = \underset{\delta n \to 0}{\text{Lt}}\frac{\delta f}{\delta n} = \underset{\delta n \to 0}{\text{Lt}}\,\nabla f \,.\, \frac{\delta\mathbf{R}}{\delta n}$$
$$= |\nabla f|\,\underset{\delta n \to 0}{\text{Lt}}\frac{\mathbf{N}\,.\,\delta\mathbf{R}}{\delta n} = |\nabla f|. \qquad [\because\ \mathbf{N}\,.\,\delta\mathbf{R} = |\delta\mathbf{R}|\cos\theta = \delta n]$$

Hence the magnitude of $\nabla f = \partial f/\partial n$.

Thus *grad f is a vector normal to the surface f = constant and has a magnitude equal to the rate of change of f along this normal.*

(3) Directional derivative. If δr denotes the length PP' and $\mathbf{N}'$ is a unit vector in the direction PP', then the limiting value of $\delta f/\delta r$ as $\delta r \to 0$ (*i.e.*, $\partial f/\partial r$) is known as the *directional derivative of f at P along the direction PP'*.

Since $\qquad \delta r = \delta n/\cos\alpha = \delta n/\mathbf{N}\,.\,\mathbf{N}'$

$$\therefore \qquad \frac{\partial f}{\partial r} = \underset{\delta r \to 0}{\text{Lt}}\left[\mathbf{N}\,.\,\mathbf{N}'\,\frac{\delta f}{\delta n}\right] = \mathbf{N}'\,.\,\frac{\partial f}{\partial n}\,\mathbf{N} = \mathbf{N}'\,.\,\nabla f$$

Thus the directional derivation of f in the direction of $\mathbf{N}'$ is the resolved part of ∇f in the direction $\mathbf{N}'$.

Since $\qquad \nabla f\,.\,\mathbf{N}' = |\,\nabla f\,|\cos\alpha \le |\,\nabla f\,|$

It follows that ∇f *gives the maximum rate of change of f.*

Example 8.9. *Prove that* $\nabla r^n = nr^{n-2}\,\mathbf{R}$, *where* $\mathbf{R} + x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$.

(Bhopal, 2007 ; Anna, 2003 S ; V.T.U., 2000)

Solution. We have $\quad f(x, y, z) = r^n = (x^2 + y^2 + z^2)^{n/2}$

$$\therefore \qquad \frac{\partial f}{\partial x} = \frac{n}{2}(x^2 + y^2 + z^2)^{n/2-1}.\,2x = nxr^{n-2}.\ \text{Similarly, } \frac{\partial f}{\partial y} = nyr^{n-2} \text{ and } \frac{\partial f}{\partial z} = nzr^{n-2}$$

Thus $\qquad \nabla r^n = \mathbf{I}\,\dfrac{\partial f}{\partial x} + \mathbf{J}\,\dfrac{\partial f}{\partial y} + \mathbf{K}\,\dfrac{\partial f}{\partial z} = nr^{n-2}\,(x\mathbf{I} + y\mathbf{J} + z\mathbf{K}) = nr^{n-2}\,\mathbf{R}.$

Otherwise : The level surfaces for f = constant, *i.e.*, r^n = constant are concentric spheres with centre O and hence unit normal $\mathbf{N}$ to the level surface through P is along the radius $\mathbf{R}$

i.e., $\qquad \mathbf{N} = \hat{\mathbf{R}}.$

$$\therefore \qquad \nabla f = \frac{\partial f}{\partial n}\,.\,\mathbf{N} = \frac{df}{dr}\hat{\mathbf{R}} = nr^{n-1}\,\hat{\mathbf{R}} \qquad [\because\ f = r^n]$$

$$= nr^{n-1}\,(\mathbf{R}/r) = nr^{n-2}\mathbf{R}.$$

Example 8.10. *If* $\nabla u = 2r^4\,\mathbf{R}$, *find u.* *(Mumbai, 2008)*

Solution. We have $\nabla u = 2(x^2 + y^2 + z^2)^2\,\mathbf{R} \qquad [\because\ r = \sqrt{(x^2 + y^2 + z^2)}\,]$

$$= 2(x^2 + y^2 + z^2)^2\,(x\mathbf{I} + y\mathbf{J} + z\mathbf{K}) \qquad ...(i)$$

But $\qquad \nabla u = \dfrac{\partial u}{\partial x}\,\mathbf{I} + \dfrac{\partial u}{\partial y}\,\mathbf{J} + \dfrac{\partial u}{\partial z}\,\mathbf{K} \qquad ...(ii)$

Comparing (i) and (ii), we get

$$\frac{\partial u}{\partial x} = 2x(x^2 + y^2 + z^2)^2,\ \frac{\partial u}{\partial y} = 2y(x^2 + y^2 + z^2)^2\ ;\ \frac{\partial u}{\partial z} = 2z(x^2 + y^2 + z^2)$$

Also $\qquad du(x, y, z) = \dfrac{\partial u}{\partial x}\,dx + \dfrac{\partial u}{\partial y}\,dy + \dfrac{\partial u}{\partial z}\,dz = 2(x^2 + y^2 + z^2)^2\,(xdx + ydy + zdz)$

$$= 2t^2\,.\,\frac{dt}{2},\ \text{taking } x^2 + y^2 + z^2 = t \quad \text{and} \quad 2(xdx + ydy + zdz) = dt$$

Integrating both sides, $u = \displaystyle\int t^2\,dt + c = \frac{1}{3}\,t^3 + c = \frac{1}{3}\,(x^2 + y^2 + z^2)^3 + c$

Hence $\qquad u = \dfrac{1}{3}\,r^{3/2} + c.$

Example 8.11. *If* $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = yz + zx + xy$, *prove that grad u, grad v and grad w are coplanar.* *(U.T.U., 2010 ; U.P.T.U., 2002)*

Solution. $\qquad \text{grad } u = \left(\mathbf{I}\,\dfrac{\partial}{\partial x} + \mathbf{J}\,\dfrac{\partial}{\partial y} + \mathbf{K}\,\dfrac{\partial}{\partial z}\right)(x + y + z) = \mathbf{I} + \mathbf{J} + \mathbf{K}$

$$\text{grad } v = 2x\mathbf{I} + 2y\mathbf{J} + 2z\mathbf{K},\ \text{grad } w = (y + z)\,\mathbf{I} + (z + x)\,\mathbf{J} + (z + x)\,\mathbf{K}$$

We know that three vectors are coplanar if their scalar triple product is zero.

Here [grad u, grad v, grad w]

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 1 \\ x+y+z & y+z+x & z+x+y \\ y+z & z+x & x+y \end{vmatrix} \qquad [\text{Operate } R_2 + R_3]$$

$$= 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix} = 0.$$

Hence grad u, grad v and grad w are coplanar.

Example 8.12. *Find a unit vector normal to the surface $xy^3z^2 = 4$ at the point $(-1, -1, 2)$.* *(Mumbai, 2008)*

Solution. A vector normal to the given surface is $\nabla (xy^3z^2)$

$$= \mathbf{I}\frac{\partial}{\partial x}(xy^3z^2) + \mathbf{J}\frac{\partial}{\partial y}(xy^3z^2) + \mathbf{K}\frac{\partial}{\partial z}(xy^3z^2) = \mathbf{I}(y^3 z^2) + \mathbf{J}(3xy^2z^2) + \mathbf{K}(2xy^3z)$$

$= -4\mathbf{I} - 12\mathbf{J} + 4\mathbf{K}$ at the point $(-1, -1, 2)$.

Hence the desired unit normal to the surface

$$= \frac{-4\mathbf{I} - 12\mathbf{J} + 4\mathbf{K}}{\sqrt{[(-4)^2 + (-12)^2 + 4^2]}} = -\frac{1}{\sqrt{11}}(\mathbf{I} + 3\mathbf{J} - \mathbf{K}).$$

Example 8.13. *Find the directional derivative of $f(x, y, z) = xy^3 + yz^3$ at the point $(2, -1, 1)$ in the direction of vector $\mathbf{I} + 2\mathbf{J} + 2\mathbf{K}$.* *(Bhopal, 2008 ; Kurukshetra, 2006 ; Rohtak, 2003)*

Solution. Here $\nabla f = \mathbf{I}(y^2) + \mathbf{J}(2xy + z^3) + \mathbf{K}(3yz^2) = \mathbf{I} - 3\mathbf{J} - 3\mathbf{K}$ at the point $(2, -1, 1)$.

$\therefore$ directional derivative of f in the direction $\mathbf{I} + 2\mathbf{J} + 2\mathbf{K}$

$$= (\mathbf{I} - 3\mathbf{J} - 3\mathbf{K}) \,.\, \frac{\mathbf{I} + 2\mathbf{J} + 2\mathbf{K}}{\sqrt{(1^2 + 2^2 + 2^2)}} = (1 \,.\, 1 - 3 \,.\, 2 - 3 \,.\, 2)/3 = -3\frac{2}{3}.$$

Example 8.14. *Find the directional derivative of $f = x^2 - y^2 + 2z^2$ at the point $P(1, 2, 3)$ in the direction of the line PQ where Q is the point $(5, 0, 4)$. Also calculate the magnitude of the maximum directional derivative.*

Solution. We have $\nabla f = \left(\mathbf{I}\frac{\partial}{\partial x} + \mathbf{J}\frac{\partial}{\partial y} + \mathbf{K}\frac{\partial}{\partial z}\right)(x^2 - y^2 + 2z^2) = 2x\mathbf{I} - 2y\mathbf{J} + 4z\mathbf{K}$

$= 2\mathbf{I} - 4\mathbf{J} + 12\mathbf{K}$ at $P(1, 2, 3)$

Also $\vec{PQ} = \vec{OQ} - \vec{OP} = (5\mathbf{I} + 0\mathbf{J} + 4\mathbf{K}) - (\mathbf{I} + 2\mathbf{J} + 3\mathbf{K}) = 4\mathbf{I} - 2\mathbf{J} + \mathbf{K} = \mathbf{A}$ (say)

$\therefore$ unit vector of $\mathbf{A} = \hat{A} = \frac{\mathbf{A}}{a} = \frac{4\mathbf{I} - 2\mathbf{J} + \mathbf{K}}{\sqrt{(16 + 4 + 1)}} = \frac{4\mathbf{I} + 2\mathbf{J} + \mathbf{K}}{\sqrt{21}}$

Thus the directional derivative of f in the direction of $\vec{PQ}$

$$\nabla f \,.\, \hat{A} = (2\mathbf{I} - 4\mathbf{J} + 12\mathbf{K}) \,.\, (4\mathbf{I} - 2\mathbf{J} + \mathbf{K})/\sqrt{21}$$

$$= (8 + 8 + 12)/\sqrt{21} = 28/\sqrt{21}$$

The directional derivative of its maximum in the direction of the normal to the surface *i.e.*, in the direction of ∇f.

Hence maximum value of this directional derivative

$$= |\nabla f| = |\, 2\mathbf{I} - 4\mathbf{J} + 12\mathbf{K} \,| = (4 + 16 + 144) = \sqrt{164}\,.$$

Example 8.15. *Find the directional derivative of $\phi = 5x^2y - 5y^2z + 2.5z^2x$ at the point P(1, 1, 1) in the direction of the line $\frac{x-1}{2} = \frac{y-3}{-2} = z$.* (Bhopal, 2008 ; U.P.T.U., 2004)

Solution. We have $\nabla\phi = \mathbf{I}\frac{\partial\phi}{\partial x} + \mathbf{J}\frac{\partial\phi}{\partial y} + \mathbf{K}\frac{\partial\phi}{\partial z}$

$$= (10xy + 2.5z^2)\,\mathbf{I} + (5x^2 - 10yz)\,\mathbf{J} + (-5y^2 + 5zx)\,\mathbf{K}$$
$$= 12.5\mathbf{I} - 5\mathbf{J} \text{ at } P(1, 1, 1)$$

Also direction of the given line is $\hat{A} = \frac{2\mathbf{I} - 2\mathbf{J} + \mathbf{K}}{3}$

Hence the required directional derivative

$$= \nabla\phi \,.\, \hat{A} = (12.5\mathbf{I} - 5\mathbf{J}) \,.\, (2\mathbf{I} - 2\mathbf{J} + \mathbf{K})/3 = (25 + 10)/3 = 11\frac{2}{3}.$$

Example 8.16. *Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point (2, – 1, 2).* (V.T.U., 2010 ; Kottayam, 2005 ; U.P.T.U., 2003)

Solution. Let $f_1 = x^2 + y^2 + z^2 - 9 = 0$ and $f_2 = x^2 + y^2 - z - 3 = 0$

Then $N_1 = \nabla f_1$ at (2, – 1, 2) $= (2x\mathbf{I} + 2y\mathbf{J} + 2z\mathbf{K})$ at (2, – 1, 2) $= 4\mathbf{I} - 2\mathbf{J} + 4\mathbf{K}$

and $N_2 = \nabla f_2$ at (2, – 1, 2) $= (2x\mathbf{I} + 2y\mathbf{J} - \mathbf{K})$ at (2, – 1, 2) $= 4\mathbf{I} - 2\mathbf{J} - \mathbf{K}$

Since the angle θ between the two surfaces at a point is the angle between their normals at that point and $\mathbf{N}_1, \mathbf{N}_2$ are the normals at (2, – 1, 2) to the given surfaces, therefore

$$\cos\theta = \frac{\mathbf{N}_1 \,.\, \mathbf{N}_2}{n_1 n_2} = \frac{(4I - 2J + 4K)\,.\,(4I - 2J - K)}{\sqrt{(16+4+16)}\,\sqrt{(16+4+1)}}$$
$$= \frac{4(4) + (-2)(-2) + 4(-1)}{6\sqrt{21}} = \frac{16}{6\sqrt{21}}$$

Hence the required angle $\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$.

Example 8.17. *Find the values of a and b such that the surface $ax^2 - byz = (a + 2)x$ and $4x^2y + z^3 = 4$ cut orthogonally at (1, – 1, 2).* (Madras, 2004)

Solution. Let $f_1 = ax^2 - byz - (a + 2)x = 0$ ...(*i*)

and $f_2 = 4x^2y + z^3 - 4 = 0$ ...(*ii*)

Then $\nabla f_1 = (2ax - a - 2)\mathbf{I} - 4z\mathbf{J} - by\mathbf{K} = (a - 2)\,\mathbf{I} - 2b\mathbf{J} + b\mathbf{K}$ at (1, – 1, 2).

$\nabla f_2 = 8xy\mathbf{I} + 4x^2\mathbf{J} + 3z^2\mathbf{K} = -8\mathbf{I} + 4\mathbf{J} + 12\mathbf{K}$ at (1, – 1, 2).

The surfaces (*i*) and (*ii*) will cut orthogonally if $\nabla f_1 \,.\, \nabla f_2 = 0$, *i.e.*, $-8(a - 2) - 8b + 12b = 0$

or $-2a + b + 4 = 0$ ...(*iii*)

Also since the point (1, – 1, 2) lies on (*i*) and (*ii*),

∴ $a + 2b - (a + 2) = 0$ or $b = 1$

From (*iii*), $-2a + 5 = 0$ or $a = 5/2$.

Hence $a = 5/2$ and $b = 1$.

PROBLEMS 8.3

1. (*a*) Find $\nabla\phi$, if $\phi = \log(x^2 + y^2 + z^2)$. (*b*) Show that grad $(1/r) = -\mathbf{R}/r^3$.
2. Find a unit vector normal to the surface $x^3 + y^3 + 3xyz = 3$ at the point (1, 2, – 1). (P.T.U., 1999)
3. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point (1, – 2, 1) in the direction of the vector $2\mathbf{I} - \mathbf{J} - 2\mathbf{K}$. (V.T.U., 2007 ; Rohtak 2006 S ; J.N.T.U., 2006 ; U.P.T.U., 2006)
4. What is the directional derivative of $\phi = xy^2 + yz^3$ at the point (2, – 1, 1) in the direction of the normal to the surface $x \log z - y^2 = -4$ at (– 1, 2, 1) ? (S.V.T.U., 2009)

5. Find the values of constants a, b, c so that the directional derivative of $p = axy^2 + byz + cz^2x^3$ at $(1, 2, -1)$ has a maximum magnitude 64 in the direction parallel to the z-axis. *(Rajasthan, 2006)*
6. Find the directional derivative of $\phi = x^4 + y^4 + z^4$ at the point $A\,(1, -2, 1)$ in the direction AB where B is $(2, 6, -1)$. Also find the maximum directional derivative of ϕ at $(1, -2, 1)$. *(Mumbai, 2009)*
7. If the directional derivative of $\phi = ax^2y + by^2z + cz^2x$ at the point $(1, 1, 1)$ has maximum magnitude 15 in the direction parallel to the line $\dfrac{x-1}{2} = \dfrac{y-3}{-2} = z$, find the values of a, b and c. *(U.P.T.U., 2002)*
8. In what direction from $(3, 1, -2)$ is the directional derivative of $\phi = x^2y^2z^4$ maximum ? Find also the magnitude of this maximum. *(Rohtak, 2003)*
9. What is the greatest rate of increase of $u = xyz^2$ at the point $(1, 0, 3)$? *(Bhopal, 2008)*
10. The temperature of points in space is given by $T(x, y, z) = x^2 + y^2 - z$. A mosquito located at $(1, 1, 2)$ desires to fly in such a direction that it will get warm as soon as possible. In what direction should it move ?
11. Calculate the angle between the normals to the surface $xy = z^2$ at the points $(4, 1, 2)$ and $(3, 3, -3)$.
12. Find the angle between the tangent planes to the surfaces $x \log z = y^2 - 1$, $x^2y = 2 - z$ at the point $(1, 1, 1)$. *(Hissar, 2005 S ; J.N.T.U., 2003)*
13. Find the values of a and b so that the surface $5x^2 - 2yz - 9z = 0$ may cut the surface $ax^2 + by^3 = 4$ orthogonally at $(1, -1, 2)$. *(Nagpur, 2009)*
14. If f and $\mathbf{G}$ are point functions, prove that the components of the latter normal and tangential to the surface $f = 0$ are

$$\frac{(\mathbf{G}\,.\,\nabla f)\,\nabla f}{(\nabla f)^2} \text{ and } \frac{\nabla f \times (\mathbf{G} \times \nabla f)}{(\nabla f)^2}$$ [Cf. Ex. 3.24]

8.6 DEL APPLIED TO VECTOR POINT FUNCTIONS

(1) Divergence. *The divergence of a continuously differentiable vector point function* $\mathbf{F}$ *is denoted by div* $\mathbf{F}$ *and is defined by the equation*

$$\textbf{div } \mathbf{F} = \nabla\,.\,\mathbf{F} = \mathbf{I}\,.\,\frac{\partial \mathbf{F}}{\partial x} + \mathbf{J}\,.\,\frac{\partial \mathbf{F}}{\partial y} + \mathbf{K}\,.\,\frac{\partial \mathbf{F}}{\partial z}$$

If $\quad \mathbf{F} = f\mathbf{I} + \phi\mathbf{J} + \psi\mathbf{K}$

then $$\text{div } \mathbf{F} = \nabla\,.\,\mathbf{F} = \left(\mathbf{I}\frac{\partial}{\partial x} + \mathbf{J}\frac{\partial}{\partial y} + \mathbf{K}\frac{\partial}{\partial z}\right)\,.\,(f\mathbf{I} + \phi\mathbf{J} + \psi\mathbf{K}) = \frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial z}$$

(2) Curl. *The curl of a continuously differentiable vector point function* $\mathbf{F}$ *is defined by the equation*

$$\textbf{curl } \mathbf{F} = \nabla \times \mathbf{F} = \mathbf{I} \times \frac{\partial \mathbf{F}}{\partial x} + \mathbf{J} \times \frac{\partial \mathbf{F}}{\partial y} + \mathbf{K} \times \frac{\partial \mathbf{F}}{\partial z}$$

If $\mathbf{F} = f\mathbf{I} + \phi\mathbf{J} + \psi\mathbf{K}$ then curl $\mathbf{F} = \nabla \times \mathbf{F} = \left(\mathbf{I}\dfrac{\partial}{\partial x} + \mathbf{J}\dfrac{\partial}{\partial y} + \mathbf{K}\dfrac{\partial}{\partial z}\right) \times (f\mathbf{I} + \phi\mathbf{J} + \psi\mathbf{K})$

$$= \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f & \phi & \psi \end{vmatrix} = \mathbf{I}\left(\frac{\partial \psi}{\partial y} - \frac{\partial \phi}{\partial z}\right) + \mathbf{J}\left(\frac{\partial f}{\partial z} - \frac{\partial \psi}{\partial x}\right) + \mathbf{K}\left(\frac{\partial \phi}{\partial x} - \frac{\partial f}{\partial y}\right).$$

Example 8.18. *If* $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$, *show that*
(*i*) $\nabla\,.\,\mathbf{R} = 3$ (*ii*) $\nabla \times \mathbf{R} = 0$. *(V.T.U. 2008 ; P.T.U., 2006 ; U.P.T.U., 2006)*

Solution. (*i*) $\quad \nabla\,.\,\mathbf{R} = \dfrac{\partial}{\partial x}(x) + \dfrac{\partial}{\partial y}(y) + \dfrac{\partial}{\partial z}(z) = 3.$

(*ii*) $$\nabla \times \mathbf{R} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \mathbf{I}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) - \mathbf{J}\left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right) + \mathbf{K}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$$

$$= \mathbf{I}(0 - 0) - \mathbf{J}(0 - 0) + \mathbf{K}(0 - 0) = \mathbf{0}.$$

[Remember : *div* $\mathbf{R} = 3$; *curl* $\mathbf{R} = \mathbf{0}$**]**

Example 8.19. *Find div* **F** *and curl* **F**, *where* $\mathbf{F} = grad\ (x^3 + y^3 + z^3 - 3xyz)$.
(V.T.U., 2008 ; Kurukshetra, 2006 ; Burdwan, 2003)

Solution. If $u = x^3 + y^3 + z^3 - 3xyz$, then

$$\mathbf{F} = \nabla u = \mathbf{I}\frac{\partial u}{\partial x} + \mathbf{J}\frac{\partial u}{\partial y} + \mathbf{K}\frac{\partial u}{\partial z} = \mathbf{I}(3x^2 - 3yz) + \mathbf{J}(3y^2 - 3zx) + \mathbf{K}(3z^2 - 3xy)$$

$$\therefore \qquad \text{div } \mathbf{F} = \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3zx) + \frac{\partial}{\partial z}(3z^2 - 3xy) = 6(x + y + z)$$

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3(x^2 - yz) & 3(y^2 - zx) & 3(z^2 - xy) \end{vmatrix}$$

$$= \mathbf{I}(-3x + 3x) - \mathbf{J}(-3y + 3y) + \mathbf{K}(-3z + 3z) = \mathbf{0}.$$

8.7 (1) PHYSICAL INTERPRETATION OF DIVERGENCE

Consider the motion of the fluid having velocity $\mathbf{V} = v_x\mathbf{I} + v_y\mathbf{J} + v_z\mathbf{K}$ at a point $P(x, y, z)$. Consider a small parallelopiped with edges δx, δy, δz parallel to the axes in the mass of fluid, with one of its corners at P (Fig. 8.7).

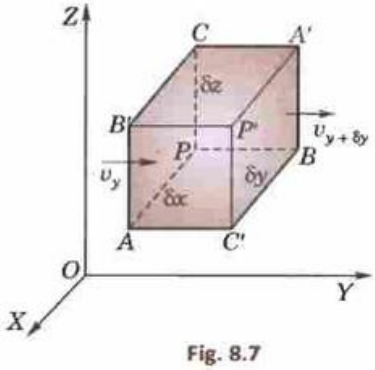

Fig. 8.7

$\therefore$ the amount of fluid entering the face PB' in unit time $= v_y\delta z\delta x$ and the amount of fluid leaving the face $P'B$ in unit time

$$= v_{y+\delta y}\delta z\delta x = \left(v_y + \frac{\partial v_y}{\partial y}\delta y\right)\delta z\delta x \quad \text{nearly}$$

$\therefore$ the net decrease of the amount of fluid due to flow across these two faces $= \dfrac{\partial v_y}{\partial y}\delta x\delta y\delta z$.

Finding similarly the contributions of other two pairs of faces, we have the total decrease of amount of fluid inside the parallelopiped per unit time $= \left(\dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} + \dfrac{\partial v_z}{\partial z}\right)\delta x\delta y\delta z$.

Thus the rate of loss of fluid per unit volume

$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \text{div } \mathbf{V}.$$

Hence *div* **V** *gives the rate at which fluid is originating at a point per unit volume.*

Similarly, if V represents an electric flux, div **V** is the amount of flux which diverges per unit volume in unit time. If **V** represents heat flux, div **V** is the rate at which heat is issuing from a point per unit volume. In general, the divergence of a vector point function representing any physical quantity gives at each point, the rate per unit volume at which the physical quantity is issuing from that point. This explains the justification for the name *divergence of a vector point function.*

If the fluid is incompressible, there can be no gain or loss in the volume element. Hence div $\mathbf{V} = 0$, which is known in Hydrodynamics as the **equation of continuity** for incompressible fluids.

Def. *If the flux entering any element of space is the same as that leaving it, i.e., div* $\mathbf{V} = 0$ *everywhere then such a point function is called a* **solenoidal vector function.**

(2) Physical interpretation of curl. Consider the motion of a rigid body rotating about a fixed axis through O. If Ω be its angular velocity, then the velocity **V** of any particle $P(\mathbf{R})$ of the body is given by $\mathbf{V} = \Omega \times \mathbf{R}$.
[See p. 91]

If $\qquad \Omega = \omega_1\mathbf{I} + \omega_2\mathbf{J} + \omega_3\mathbf{K}$ and $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$

then $$\mathbf{V} = \Omega \times \mathbf{R} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} = \mathbf{I}(\omega_2 z - \omega_3 y) + \mathbf{J}(\omega_3 x - \omega_1 z) + \mathbf{C}(\omega_1 y - \omega_2 x)$$

$$\therefore \qquad \text{curl } \mathbf{V} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y, & \omega_3 x - \omega_1 z, & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= \mathbf{I}(\omega_1 + \omega_1) + \mathbf{J}(\omega_2 + \omega_2) + \mathbf{K}(\omega_3 + \omega_3) \qquad [\because \ \omega_1, \omega_2, \omega_3 \text{ are constants.}]$$

$$= 2(\omega_1 \mathbf{I} + \omega_2 \mathbf{J} + \omega_3 \mathbf{K}) = 2\Omega. \quad \text{Hence} \quad \Omega = \frac{1}{2} \text{ curl } \mathbf{V}$$

Thus the angular velocity of rotation at any point is equal to half the curl of the velocity vector which justifies the name *rotation* used for curl.

In general, the curl of any vector point function gives the measure of the angular velocity at any point of the vector field.

Def. *Any motion in which the curl of the velocity vector is zero is said to be* **irrotational**, *otherwise* **rotational.**

Example 8.20. *Prove that div* $(r^n\mathbf{R}) = (n + 3)\, r^n$. *Hence show that* $\mathbf{R}/r^3$ *is solenoidal.*
(V.T.U., 2006 ; U.P.T.U., 2006 ; P.T.U., 2005)

Solution. We have $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$ and $r = \sqrt{(x^2 + y^2 + z^2)}$

$$\therefore \qquad div\,(r^n\mathbf{R}) = \nabla \,.\, (x^2 + y^2 + z^2)^{n/2}\,(x\mathbf{I} + y\mathbf{J} + z\mathbf{K})$$

$$= \frac{\partial}{\partial x}[x(x^2 + y^2 + z^2)^{n/2}] + \frac{\partial}{\partial y}[y(x^2 + y^2 + z^2)^{n/2}] + \frac{\partial}{\partial z}[z(x^2 + y^2 + z^2)^{n/2}]$$

$$= \Sigma\left\{1\,.\,(x^2 + y^2 + z^2)^{n/2} + x\,.\,\frac{n}{2}(x^2 + y^2 + z^2)^{\frac{n}{2}-1}\,.\,2x\right\}$$

$$= \Sigma r^n + n\,\Sigma x^2 (x^2 + y^2 + z^2)^{\frac{n}{2}-1} = 3r^n + nr^2\,.\,r^{n-2}$$

Thus $div\,(r^n\mathbf{R}) = (n + 3)\, r^n$

When $n = -3$, $div\,(\mathbf{R}/r^3) = 0$ *i.e.*, $\mathbf{R}/r^3$ is solenoidal.

Example 8.21. *Show that* $r^\alpha\mathbf{R}$ *is any irrotational vector for any value of* α *but is solenoidal if* $\alpha + 3 = 0$ *where* $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$ *and r is the magnitude of* $\mathbf{R}$. *(V.T.U., 2006 ; Kottayam, 2005)*

Solution. Let $\mathbf{A} = r^\alpha\mathbf{R} = (x^2 + y^2 + z^2)^{\alpha/2}\,(x\mathbf{I} + y\mathbf{I} + z\mathbf{K}) = \Sigma x\,(x^2 + y^2 + z^2)^{\alpha/2}\,\mathbf{I}$

$$\therefore \qquad \text{curl } \mathbf{A} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x(x^2 + y^2 + z^2)^{\alpha/2} & y(x^2 + y^2 + z^2)^{\alpha/2} & z(x^2 + y^2 + z^2)^{\alpha/2} \end{vmatrix}$$

$$= \Sigma\,\mathbf{I}\left\{\frac{\alpha z}{2}(x^2 + y^2 + z^2)^{\alpha/2-1}\,(2y) - \frac{\alpha y}{2}(x^2 + y^2 + z^2)^{\alpha/2-1}\,.\,2z\right\} = 0$$

Hence $\mathbf{A}$ is irrotational for any value of α.

But $\qquad div\,\mathbf{A} = \nabla \,.\, (r^\alpha\,\mathbf{R}) = (\alpha + 3)\, r^\alpha$

which is zero for $\alpha + 3 = 0$, *i.e.*, $\mathbf{A}$ is solenoidal if $\alpha + 3 = 0$.

8.8 DEL APPLIED TWICE TO POINT FUNCTIONS

∇f and $\nabla \times \mathbf{F}$ being vector point functions, we can form their divergence and curl whereas $\nabla . \mathbf{F}$ being a scalar point function, we can have its gradients only. Thus we have the following five formulae :

(1) $div\ grad\ f = \nabla^2 f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$

(2) $curl\ grad\ f = \nabla \times \nabla f = \mathbf{0}$

(3) $div\ curl\ \mathbf{F} = \nabla \,.\, \nabla \times \mathbf{F} = 0$

(4) *curl curl* $\mathbf{F} = grad\ div\ \mathbf{F} - \nabla^2\mathbf{F}$, *i.e.*, $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$

(5) *grad div* $\mathbf{F} = curl\ curl\ \mathbf{F} + \nabla^2\mathbf{F}$, *i.e.*, $\nabla(\nabla \cdot \mathbf{F}) = \nabla \times (\nabla \times \mathbf{F}) + \nabla^2\mathbf{F}$.

Proofs. **(1)** $\nabla^2 f = \nabla \cdot \nabla f = \nabla \cdot \left(\mathbf{I}\dfrac{\partial f}{\partial x} + \mathbf{J}\dfrac{\partial f}{\partial y} + \mathbf{K}\dfrac{\partial f}{\partial z}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f$$

$\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ is called the *Laplacian operator* and $\nabla^2 f = 0$ is called the *Laplace's equation.*

(2) $\nabla \times \nabla f = \nabla \times \left(\mathbf{I}\dfrac{\partial f}{\partial x} + \mathbf{J}\dfrac{\partial f}{\partial y} + \mathbf{K}\dfrac{\partial f}{\partial z}\right) = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix} = \Sigma\mathbf{I}\left(\dfrac{\partial^2 f}{\partial y \partial z} - \dfrac{\partial^2 f}{\partial z \partial y}\right) = \mathbf{0}$ *(V.T.U., 2007)*

(3) $\nabla \cdot \nabla \times \mathbf{F} = \left(\Sigma\mathbf{I}\dfrac{\partial}{\partial x}\right) \cdot \left(\mathbf{I} \times \dfrac{\partial \mathbf{F}}{\partial x} + \mathbf{J} \times \dfrac{\partial \mathbf{F}}{\partial y} + \mathbf{K} \times \dfrac{\partial \mathbf{F}}{\partial z}\right)$

$$= \Sigma\mathbf{I} \cdot \left(\mathbf{I} \times \frac{\partial^2 \mathbf{F}}{\partial x^2} + \mathbf{J} \times \frac{\partial^2 \mathbf{F}}{\partial x \partial y} + \mathbf{K} \times \frac{\partial^2 \mathbf{F}}{\partial x \partial z}\right)$$

$$= \Sigma\left(\mathbf{I} \times \mathbf{I} \cdot \frac{\partial^2 \mathbf{F}}{\partial x^2} + \mathbf{I} \times \mathbf{J} \cdot \frac{\partial^2 \mathbf{F}}{\partial x \partial y} + \mathbf{I} \times \mathbf{K} \cdot \frac{\partial^2 \mathbf{F}}{\partial x \partial z}\right) = \Sigma\left(\mathbf{K} \cdot \frac{\partial^2 \mathbf{F}}{\partial x \partial y} - \mathbf{J} \cdot \frac{\partial^2 \mathbf{F}}{\partial x \partial z}\right) = 0.$$

(4) $\nabla \times (\nabla \times \mathbf{F}) = \left(\Sigma\mathbf{I}\dfrac{\partial}{\partial x}\right) \times \left(\mathbf{I} \times \dfrac{\partial \mathbf{F}}{\partial x} + \mathbf{J} \times \dfrac{\partial \mathbf{F}}{\partial y} + \mathbf{K} \times \dfrac{\partial \mathbf{F}}{\partial z}\right)$

$$= \Sigma\mathbf{I} \times \left(\mathbf{I} \times \frac{\partial^2 \mathbf{F}}{\partial x^2} + \mathbf{J} \times \frac{\partial^2 \mathbf{F}}{\partial x \partial y} + \mathbf{K} \times \frac{\partial^2 \mathbf{F}}{\partial x \partial z}\right)$$

$$= \Sigma\left[\left\{\left(\mathbf{I} \cdot \frac{\partial^2 \mathbf{F}}{\partial x^2}\right)\mathbf{I} - (\mathbf{I} \cdot \mathbf{I})\frac{\partial^2 \mathbf{F}}{\partial x^2}\right\} + \left\{\left(\mathbf{I} \cdot \frac{\partial^2 \mathbf{F}}{\partial x \partial y}\right)\mathbf{J} - (\mathbf{I} \cdot \mathbf{J})\frac{\partial^2 \mathbf{F}}{\partial x \partial y}\right\} + \left\{\left(\mathbf{I} \cdot \frac{\partial^2 \mathbf{F}}{\partial x \partial z}\right)\mathbf{K} - (\mathbf{I} \cdot \mathbf{K})\frac{\partial^2 \mathbf{F}}{\partial x \partial z}\right\}\right]$$

$$= \Sigma\left[\left(\mathbf{I} \cdot \frac{\partial^2 \mathbf{F}}{\partial x^2}\right)\mathbf{I} + \left(\mathbf{I} \cdot \frac{\partial^2 \mathbf{F}}{\partial x \partial y}\right)\mathbf{J} + \left(\mathbf{I} \cdot \frac{\partial^2 \mathbf{F}}{\partial x \partial z}\right)\mathbf{K}\right] - \Sigma\frac{\partial^2 \mathbf{F}}{\partial x^2}$$

$$= \Sigma\mathbf{I}\frac{\partial}{\partial x}\left(\mathbf{I} \cdot \frac{\partial \mathbf{F}}{\partial x} + \mathbf{J} \cdot \frac{\partial \mathbf{F}}{\partial y} + \mathbf{K} \cdot \frac{\partial \mathbf{F}}{\partial z}\right) - \Sigma\frac{\partial^2 \mathbf{F}}{\partial x^2} = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}.$$ *(Madras, 2006)*

(5) is just another way of writing (4) above.

Obs. *Interpretation of* ∇ *as a vector according to rules of vector products leads to correct results so far so the repeated application of* ∇ *is concerned.*

e.g., **1.** $\nabla \cdot \nabla f = \nabla^2 f$ $\quad (\because\ \nabla \cdot \nabla = \nabla^2)$

2. $\nabla \times \nabla f = \mathbf{0}$ $\quad (\because\ \nabla \times \nabla = \mathbf{0})$

3. $\nabla \cdot \nabla \times \mathbf{F} = \mathbf{0}$ $\quad (\because\ [\nabla\nabla\mathbf{F}] = 0)$

4. $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$ by expanding it as a vector triple product.

8.9 DEL APPLIED TO PRODUCTS OF POINT FUNCTIONS

To prove that

(1) grad $(fg) = f\,(\text{grad}\ g) + g(\text{grad}\ f)$ *i.e.* $\nabla(fg) = f\,\nabla g + g\nabla f$.

(2) *div* $(f\,\mathbf{G}) = (\text{grad}\ f) \cdot \mathbf{G} + f\,(\text{div}\ \mathbf{G})$ *i.e.* $\nabla(f\,\mathbf{G}) = \nabla f \cdot \mathbf{G} + f\,\nabla \cdot \mathbf{G}$

(3) *curl* $(f\,\mathbf{G}) = (grad\ f) \times \mathbf{G} + f\,(curl\ \mathbf{G})$ *i.e.* $\nabla \times (f\,\mathbf{G}) = \nabla f \times \mathbf{G} + f\,\nabla \times \mathbf{G}$

(4) *grad* $(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\,\mathbf{G} + (\mathbf{G} \cdot \nabla)\,\mathbf{F} + \mathbf{F} \times curl\ \mathbf{G} + \mathbf{G} \times curl\ \mathbf{F}$

i.e., $\nabla\,(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\,\mathbf{G} + (\mathbf{G} \cdot \nabla)\,\mathbf{F} + \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F})$

(5) $div\,(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\,.\,(curl\,\mathbf{F}) - \mathbf{F}\,.\,(curl\,\mathbf{G})$ *i.e.,* $\nabla\,.\,(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\,.\,(\nabla\times\mathbf{F}) - \mathbf{F}\,.\,(\nabla\times\mathbf{G})$

(6) $curl\,(\mathbf{F}\times\mathbf{G}) = \mathbf{F}\,(div\,\mathbf{G}) - \mathbf{G}\,(div\,\mathbf{F}) + (\mathbf{G}\,.\,\nabla)\,\mathbf{F} - (\mathbf{F}\,.\,\nabla)\,\mathbf{G}$

i.e., $\nabla\times(\mathbf{F}\times\mathbf{G}) = \mathbf{F}(\nabla\,.\,\mathbf{G}) - \mathbf{G}\,(\nabla\,.\,\mathbf{F}) + (\mathbf{G}\,.\,\nabla)\,\mathbf{F} - (\mathbf{F}\,.\,\nabla)\,\mathbf{G}$

Proofs **(1)** $\nabla\,(fg) = \Sigma\mathbf{I}\,.\,\dfrac{\partial}{\partial x}\,(fg) = \Sigma\mathbf{I}\left(f\dfrac{\partial g}{\partial x} + g\dfrac{\partial f}{\partial x}\right)$

$$= f\,\Sigma\mathbf{I}\,\frac{\partial g}{\partial x} + g\,\Sigma\mathbf{I}\,\frac{\partial f}{\partial x} = f\nabla g + g\nabla f$$

(2) $$\nabla\,.\,(f\,\mathbf{G}) = \Sigma\mathbf{I}\,.\,\frac{\partial}{\partial x}(f\,\mathbf{G}) = \Sigma\mathbf{I}\,.\left(\frac{\partial f}{\partial x}\mathbf{G} + f\,\frac{\partial\mathbf{G}}{\partial x}\right)$$

$$= \left(\Sigma\mathbf{I}\,\frac{\partial f}{\partial x}\right)\,.\,\mathbf{G} + f\left(\Sigma\mathbf{I}\,.\,\frac{\partial\mathbf{G}}{\partial x}\right) = \nabla f\,.\,\mathbf{G} + f\,\nabla\,.\,\mathbf{G}$$ *(V.T.U., 2011)*

(3) $$\nabla\times(f\,\mathbf{G}) = \Sigma\mathbf{I}\times\frac{\partial}{\partial x}(f\,\mathbf{G}) = \Sigma\mathbf{I}\times\left(f\,\frac{\partial\mathbf{G}}{\partial x} + \frac{\partial f}{\partial x}\,\mathbf{G}\right)$$

$$= f\,\Sigma\mathbf{I}\times f\,\frac{\partial\mathbf{G}}{\partial x} + \Sigma\mathbf{I}\,\frac{\partial f}{\partial x}\times\mathbf{G} = f\,\nabla\times\mathbf{G} + \nabla f\times\mathbf{G}$$ *(V.T.U. 2008)*

(4) $$\nabla(\mathbf{F}\,.\,\mathbf{G}) = \Sigma\mathbf{I}\,\frac{\partial}{\partial x}(\mathbf{F}.\,\mathbf{G}) = \Sigma\mathbf{I}\left(\frac{\partial\mathbf{F}}{\partial x}\,.\,\mathbf{G} + \mathbf{F}\,.\,\frac{\partial\mathbf{G}}{\partial x}\right) = \Sigma\mathbf{I}\,\frac{\partial\mathbf{F}}{\partial x}\,.\,\mathbf{G} + \Sigma\mathbf{I}\left(\mathbf{F}\,.\,\frac{\partial\mathbf{G}}{\partial x}\right) \quad ...(i)$$

Now $\mathbf{G}\times\left(\mathbf{I}\times\dfrac{\partial\mathbf{F}}{\partial x}\right) = \left(\mathbf{G}\,.\,\dfrac{\partial\mathbf{F}}{\partial x}\right)\,\mathbf{I} - (\mathbf{G}\,.\,\mathbf{I})\,\dfrac{\partial\mathbf{F}}{\partial x}$

or $$\left(\mathbf{G}\,.\,\frac{\partial\mathbf{F}}{\partial x}\right)\,\mathbf{I} = \mathbf{G}\times\left(\mathbf{I}\times\frac{\partial\mathbf{F}}{\partial x}\right) + (\mathbf{G}\,.\,\mathbf{I})\,\frac{\partial\mathbf{F}}{\partial x}$$

$$\therefore\quad \Sigma\left(\mathbf{G}\,.\,\frac{\partial\mathbf{F}}{\partial x}\right)\,\mathbf{I} = \mathbf{G}\times\Sigma\mathbf{I}\times\frac{\partial\mathbf{F}}{\partial x} + \Sigma\,(\mathbf{G}\,.\,\mathbf{I})\,\frac{\partial\mathbf{F}}{\partial x} = \mathbf{G}\times(\nabla\times\mathbf{F}) + (\mathbf{G}\,.\,\nabla)\mathbf{F} \quad ...(ii)$$

Interchanging $\mathbf{F}$ and $\mathbf{G}$, $\Sigma\left(\mathbf{F}\,.\,\dfrac{\partial\mathbf{G}}{\partial x}\right)\,\mathbf{I} = \mathbf{F}\times(\nabla\times\mathbf{G}) + (\mathbf{F}\,.\,\nabla)\,\mathbf{G}$ ...(*iii*)

Substituting in (*i*) from (*ii*) and (*iii*), we get

$$\nabla(\mathbf{F}\,.\,\mathbf{G}) = (\mathbf{F}\,.\,\nabla)\mathbf{G} + (\mathbf{G}\,.\,\nabla)\mathbf{F} + \mathbf{F}\times(\nabla\times\mathbf{G}) + \mathbf{G}\times(\nabla\times\mathbf{F})$$

(5) $$\nabla\,.\,(\mathbf{F}\times\mathbf{G}) = \Sigma\mathbf{I}\,.\,\frac{\partial}{\partial x}(\mathbf{F}\times\mathbf{G}) = \Sigma\mathbf{I}\left(\frac{\partial\mathbf{F}}{\partial x}\times\mathbf{G} + F\times\frac{\partial\mathbf{G}}{\partial x}\right) = \Sigma\mathbf{I}\,.\,\frac{\partial\mathbf{F}}{\partial x}\times\mathbf{G} - \Sigma\mathbf{I}\,.\left(\frac{\partial\mathbf{G}}{\partial x}\times\mathbf{F}\right)$$

$$= \Sigma\left(I\times\frac{\partial\mathbf{F}}{\partial x}\right)\,.\,\mathbf{G} - \Sigma\left(I\times\frac{\partial\mathbf{G}}{\partial x}\right)\,.\,\mathbf{F} \qquad [\because\ \mathbf{A}\,.\,(\mathbf{B}\times\mathbf{C}) = (A\times B)\,.\,C]$$

$$= \mathbf{G}\,.\,(\nabla\times\mathbf{F}) - \mathbf{F}\,.\,(\nabla\times\mathbf{G})$$

(6) $$\nabla\times(\mathbf{F}\times\mathbf{G}) = \Sigma\mathbf{I}\times\frac{\partial}{\partial x}\,(\mathbf{F}\times\mathbf{G}) = \Sigma\mathbf{I}\times\left(\frac{\partial\mathbf{F}}{\partial x}\times\mathbf{G} + \mathbf{F}\times\frac{\partial\mathbf{G}}{\partial x}\right)$$

$$= \Sigma\left[(\mathbf{I}\,.\,\mathbf{G})\,\frac{\partial\mathbf{F}}{\partial x} - \left(\mathbf{I}\,.\,\frac{\partial\mathbf{F}}{\partial x}\right)\mathbf{G}\right] + \Sigma\left[\left(\mathbf{I}\,.\,\frac{\partial\mathbf{G}}{\partial x}\right)\mathbf{F} - (\mathbf{I}\,.\,\mathbf{F})\,\frac{\partial\mathbf{G}}{\partial x}\right]$$

$$= \Sigma\,(\mathbf{G}\,.\,\mathbf{I})\,\frac{\partial\mathbf{F}}{\partial x} - \mathbf{G}\Sigma\mathbf{I}\,.\,\frac{\partial\mathbf{F}}{\partial x} + \mathbf{F}\Sigma\mathbf{I}\,.\,\frac{\partial\mathbf{G}}{\partial x} - \Sigma(\mathbf{F}\,.\,\mathbf{I})\,\frac{\partial\mathbf{G}}{\partial x}$$

$$= \mathbf{F}\left(\Sigma\mathbf{I}\,.\,\frac{\partial\mathbf{G}}{\partial x}\right) - \mathbf{G}\Sigma\left(\mathbf{I}\,.\,\frac{\partial\mathbf{F}}{\partial x}\right) + \Sigma\,(\mathbf{G}\,.\,\mathbf{I})\,\frac{\partial\mathbf{F}}{\partial x} - \Sigma\,(\mathbf{F}\,.\,\mathbf{I})\,\frac{\partial\mathbf{G}}{\partial x}$$

$$= \mathbf{F}\,(\nabla\,.\,\mathbf{G}) - \mathbf{G}(\nabla\,.\,\mathbf{F}) + (\mathbf{G}\,.\,\nabla)\,\mathbf{F} - (\mathbf{F}\,.\,\nabla)\,\mathbf{G}$$

Rule to reproduce the above formulae easily :

(*i*) *Treating each of the factors as constants separately, expresss the results of* ∇*-operation as a sum of the two terms.*

(*ii*) *Transform each of the two terms, noting that* ∇ *always appears before a function and keeping in mind whether the result of operation is a scalar or a vector.* To carry out the simplification, we may sometimes, employ the properties of triple products.

(*iii*) *Restore the change of treating the functions as constants.*

Let us illustrate the application of this rule to **(2)**, **(4)** and **(6)** above :

(2) $\nabla \cdot (f\mathbf{G}) = \nabla \cdot (f_c \mathbf{G} + f\mathbf{G}_c) = f_c \nabla \cdot \mathbf{G} + \mathbf{G}_c \cdot \nabla f = f \nabla \cdot \mathbf{G} + \mathbf{G} \cdot \nabla f$

(4) $\nabla(\mathbf{F} \cdot \mathbf{G}) = \nabla(\mathbf{F}_c \cdot \mathbf{G}) + \nabla(\mathbf{F} \cdot \mathbf{G}_c)$

$$= [\mathbf{F}_c \times (\nabla \times \mathbf{G}) + (\mathbf{F}_c \cdot \nabla)\mathbf{G}] + [\mathbf{G}_c \times (\nabla \times \mathbf{F}) + (\mathbf{G}_c \cdot \nabla)\mathbf{F}]$$

$$= \mathbf{F} \times (\nabla \times \mathbf{G}) + (\mathbf{F} \cdot \nabla)\mathbf{G} + \mathbf{G} \times (\nabla \times \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F}$$

(6) $\nabla \times (\mathbf{F} \times \mathbf{G}) = \nabla \times (\mathbf{F}_c \times \mathbf{G}) + \nabla \times (\mathbf{F} \times \mathbf{G}_c) = [\nabla \cdot \mathbf{G}\mathbf{F}_c - (\mathbf{F}_c \cdot \nabla)\mathbf{G}] + (\mathbf{G}_c \cdot \nabla)\mathbf{F} - \nabla \cdot \mathbf{F}\mathbf{G}_c]$

$$= \mathbf{F}(\nabla \cdot \mathbf{G}) - (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} - \mathbf{G}(\nabla \cdot \mathbf{F}).$$

Example 8.22. *Show that* $\nabla^2(r^n) = n(n+1)r^{n-2}$ *(S.V.T.U., 2006 ; J.N.T.U., 2006 ; U.P.T.U., 2005)*

Solution. $\nabla^2 r^n = \nabla \cdot (\nabla r^n)$

$$= \nabla \cdot \left(nr^{n-1}\frac{\mathbf{R}}{r}\right) = n\nabla \cdot (r^{n-2}\mathbf{R}) = n[(\nabla r^{n-2}) \cdot \mathbf{R} + r^{n-2}(\nabla \cdot \mathbf{R})] \qquad \text{[By § 8.9 (2)]}$$

$$= n\left[(n-2)r^{n-3}\frac{\mathbf{R}}{r} \cdot \mathbf{R} + r^{n-2}(3)\right] \qquad \text{[Using Ex. 8.18 (}i\text{)]}$$

$$= n[(n-2)r^{n-4}(r^2) + 3r^{n-2}] = n(n+1)r^{n-2} \qquad [\because \ \mathbf{R} \cdot \mathbf{R} = r^2]$$

Otherwise : $\nabla^2(r^n) = \dfrac{\partial^2(r^n)}{\partial x^2} + \dfrac{\partial^2(r^n)}{\partial y^2} + \dfrac{\partial^2(r^n)}{\partial z^2}$ [By § 8.8 (1)] ...(*i*)

Now $\dfrac{\partial(r^n)}{\partial x} = nr^{n-1}\dfrac{\partial r}{\partial x} = nr^{n-1}\dfrac{x}{r} = nr^{n-2}x$ $[\because \ r^2 = x^2 + y^2 + z^2]$

$$\therefore \quad \frac{\partial^2(r^n)}{\partial x^2} = n\left[r^{n-2} + (n-2)r^{n-3}\frac{\partial r}{\partial x}x\right] = n\left[r^{n-2} + (n-2)r^{n-3}\frac{x}{r}x\right]$$

$$= n\left[r^{n-2} + (n-2)r^{n-4}x^2\right] \qquad ...(ii)$$

Similarly, $\dfrac{\partial^2(r^n)}{\partial y^2} = n\left[r^{n-2} + (n-2)r^{n-4}y^2\right]$ ...(*iii*)

$$\frac{\partial^2(r^n)}{\partial z^2} = n\left[r^{n-2} + (n-2)r^{n-4}z^2\right] \qquad ...(iv)$$

Adding (*ii*), (*iii*) and (*iv*), (*i*) gives

$$\nabla^2(r^n) = n\,[3r^{n-2} + (n-2)r^{n-4}(x^2 + y^2 + z^2)]$$

$$= n\,[3r^{n-2} + (n-2)r^{n-4}r^2] = n(n+1)r^{n-2}.$$

In particular $\nabla^2(1/r) = 0$. *(U.P.T.U., 2003 ; P.T.U., 2003)*

Example 8.23. *If* $u\mathbf{F} = \nabla v$, *where* u, v *are scalar fields and* $\mathbf{F}$ *is a vector field, show that* $\mathbf{F}$. *curl* $\mathbf{F} = 0$.

Solution. Since $\mathbf{F} = \dfrac{1}{u}\nabla v \quad \therefore \quad \text{curl } \mathbf{F} = \nabla \times \left(\dfrac{1}{u}\nabla v\right)$

or $$\text{curl } \mathbf{F} = \nabla\frac{1}{u} \times \nabla v + \frac{1}{u}\nabla \times (\nabla v) \qquad \text{[By § 8.9 (3)]}$$

$$= \nabla\frac{1}{u} \times \nabla v \qquad [\because \ \nabla \times \nabla v = 0$$

Hence $\mathbf{F} \cdot \text{curl } \mathbf{F} = \dfrac{1}{u}\nabla v \cdot \left(\nabla\dfrac{1}{u} \times \nabla v\right) = 0$, for it is a scalar triple product in which two factors are equal.

Example 8.24. *If* r *and* $\mathbf{R}$ *have their usual meanings and* $\mathbf{A}$ *is a constant vector, prove that* $\nabla \times \left(\dfrac{\mathbf{A} \times \mathbf{R}}{r^n}\right) = \dfrac{2-n}{r^n}\mathbf{A} + \dfrac{n(\mathbf{A} \cdot \mathbf{R})}{r^{n+2}}\mathbf{R}$. *(Mumbai, 2009 ; Kurukshetra, 2006 ; J.N.T.U., 2005)*

Solution. $\nabla \times [r^{-n}(\mathbf{A} \times \mathbf{R})] = r^{-n}[\nabla \times (\mathbf{A} \times \mathbf{R})] + \nabla r^{-n} \times (\mathbf{A} \times \mathbf{R})]$ [By § 8.9 (3)]

$$= r^{-n}[(\nabla \cdot \mathbf{R})\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{R}] + (-nr^{-(n+1)}\mathbf{R}/r) \times (\mathbf{A} \times \mathbf{R})$$

$$= r^{-n}(3\mathbf{A} - \mathbf{A}) - nr^{-(n+2)}\,\mathbf{R} \times (\mathbf{A} \times \mathbf{R}) \qquad [\because \ \nabla . \mathbf{R} = 3, (\mathbf{A} . \nabla)\mathbf{R} = \mathbf{A}]$$

$$= 2\mathbf{A}r^{-n} - nr^{-(n+2)}[(\mathbf{R} . \mathbf{R})\mathbf{A} - (\mathbf{A} . \mathbf{R})\mathbf{R}]$$

$$= \frac{2\mathbf{A}}{r^n} - \frac{n}{r^{n+2}}[r^2\mathbf{A} - (\mathbf{A} . \mathbf{R})\mathbf{R}] = \frac{2-n}{r^n}\mathbf{A} + \frac{n(\mathbf{A} . \mathbf{R})}{r^{n+2}}\mathbf{R}.$$

Example 8.25. *If r is the distance of a point (x, y, z) from the origin, prove that* $curl\left(\mathbf{K} \times grad\frac{1}{r}\right) + grad\left(\mathbf{K} . grad\frac{1}{r}\right) = 0$, *where* **K** *is the unit vector in the direction OZ.* *(U.P.T.U., 2001)*

Solution. $$grad\ \frac{1}{r} = \left(\mathbf{I}\frac{\partial}{\partial x} + \mathbf{J}\frac{\partial}{\partial y} + \mathbf{K}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)^{-1/2} \qquad [\because \ r = \sqrt{(x^2 + y^2 + z^2)}]$$

$$= -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}(2x\mathbf{I} + 2y\mathbf{J} + 2z\mathbf{K})$$

$$= -(x^2 + y^2 + z^2)^{-3/2}(x\mathbf{I} + y\mathbf{J} + z\mathbf{K})$$

$$curl\left(\mathbf{K} \times grad\frac{1}{r}\right) = \nabla \times [-(x^2 + y^2 + z^2)^{-3/2}(x\mathbf{J} - y\mathbf{I})]$$

$$= \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y/(x^2 + y^2 + z^2)^{3/2} & -x/(x^2 + y^2 + z^2)^{3/2} & 0 \end{vmatrix}$$

$$= \mathbf{I}\frac{\partial}{\partial z}\left\{\frac{x}{(x^2 + y^2 + z^2)^{3/2}}\right\} + \mathbf{J}\frac{\partial}{\partial z}\left\{\frac{y}{(x^2 + y^2 + z^2)^{3/2}}\right\}$$

$$- \mathbf{K}\left\{\frac{\partial}{\partial x}\left[\frac{x}{(x^2 + y^2 + z^2)^{3/2}}\right] + \frac{\partial}{\partial y}\left[\frac{y}{(x^2 + y^2 + z^2)^{3/2}}\right]\right\}$$

$$= \frac{-3xz\mathbf{I} - 3yz\mathbf{J} + (x^2 + y^2 - 2z^2)\mathbf{K}}{(x^2 + y^2 + z^2)^{5/2}} \qquad ...(i)$$

$$\text{grad}\left(\mathbf{K} . \text{grad}\frac{1}{r}\right) = \nabla\left\{-\mathbf{K} . \frac{(x\mathbf{I} + y\mathbf{J} + z\mathbf{K})}{(x^2 + y^2 + z^2)^{3/2}}\right\}$$

$$= \left(\mathbf{I}\frac{\partial}{\partial x} + \mathbf{J}\frac{\partial}{\partial y} + \mathbf{K}\frac{\partial}{\partial z}\right)\left\{\frac{-z}{(x^2 + y^2 + z^2)^{3/2}}\right\}$$

$$= \frac{3xz\mathbf{I}}{(x^2 + y^2 + z^2)^{5/2}} + \frac{3yz\mathbf{J}}{(x^2 + y^2 + z^2)^{5/2}} + \frac{(3z^2 - x^2 - y^2 - z^2)\mathbf{K}}{(x^2 + y^2 + z^2)^{5/2}}$$

$$= \frac{3xz\mathbf{I} + 3yz\mathbf{J} - (x^2 + y^2 - 2z^2)\mathbf{K}}{(x^2 + y^2 + z^2)^{5/2}} \qquad ...(ii)$$

Adding (i) and (ii), we get

$$\text{curl}\left(\mathbf{K} \times \text{grad}\frac{1}{r}\right) + \text{grad}\left(\mathbf{K} . \text{grad}\frac{1}{r}\right) = \mathbf{0}.$$

Example 8.26. *In electromagnetic theory, we have* $\nabla . \mathbf{D} = \rho$, $\nabla . \mathbf{H} = 0$, $\nabla \times D = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}$, $\nabla \times \mathbf{H} = \frac{1}{c}\left(\rho\mathbf{V} + \frac{\partial \mathbf{D}}{\partial t}\right)$. *Prove that*

(*i*) $\nabla^2\mathbf{D} - \frac{1}{c^2}\frac{\partial^2\mathbf{D}}{\partial t^2} = \nabla\rho + \frac{1}{c^2}\frac{\partial}{\partial t}(\rho\mathbf{V})$ (*ii*) $\nabla^2\mathbf{H} - \frac{1}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2} = -\frac{1}{c}\nabla \times \rho\mathbf{V}$

Solution. (*i*) We have $\frac{1}{c^2}\left\{\frac{\partial^2 \mathbf{D}}{\partial t^2}+\frac{\partial}{\partial t}(\rho \mathbf{V})\right\}=\frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{D}}{\partial t}+\rho \mathbf{V}\right)$

$$=\frac{1}{c}\frac{\partial}{\partial t}(\nabla\times\mathbf{H})=\frac{1}{c}\nabla\times\frac{\partial \mathbf{H}}{\partial t} \qquad \left[\because\ \nabla\times\mathbf{H}=\frac{1}{c}\left(\rho\mathbf{V}+\frac{\partial\mathbf{D}}{\partial t}\right)\right]$$

$$=-\nabla\times(\nabla\times\mathbf{D}) \qquad \left[\because\ \nabla\times\mathbf{D}=-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}\right]$$

$$=-[\nabla(\nabla\,.\,\mathbf{D})-\nabla^2\mathbf{D}] \qquad \text{[Using § 8.8 (4)]}$$

$$=-\nabla\rho+\nabla^2\mathbf{D}$$

Hence $\nabla^2\mathbf{D}-\frac{1}{c^2}\frac{\partial^2\mathbf{D}}{\partial t^2}=\nabla\rho+\frac{1}{c^2}\frac{\partial}{\partial t}(\rho\mathbf{V})$

(*ii*) $$\text{L.H.S.}=\nabla^2\mathrm{H}-\frac{1}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2}=\nabla^2\mathbf{H}+\frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}\right)$$

$$=\nabla^2\mathbf{H}+\frac{1}{c}\frac{\partial}{\partial t}(\nabla\times\mathbf{D}) \qquad \left[\because\ \nabla\times\mathbf{D}=-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}\right.$$

$$=\nabla^2\mathbf{H}+\frac{1}{c}\left(\nabla\times\frac{\partial\mathbf{D}}{\partial t}\right) \qquad \left[\because\ \nabla\times\mathbf{H}=\frac{1}{c}\left(\rho V+\frac{\partial\mathbf{D}}{\partial t}\right)\right.$$

$$=\nabla^2\mathbf{H}+\nabla\times\left(\nabla\times\mathbf{H}-\frac{1}{C}\rho\mathbf{V}\right)=\nabla^2\mathbf{H}+\nabla\times(\nabla\times\mathbf{H})-\frac{1}{c}\nabla\times(\rho\mathbf{V})$$

$$=\nabla^2\mathbf{H}+\nabla(\nabla\,.\,\mathbf{H})-\nabla^2\mathbf{H}-\frac{1}{c}\nabla\times(\rho\mathbf{V}), \qquad \text{[Using § 8.9 (4)]}$$

$$=\nabla(\nabla\,.\,\mathbf{H})-\frac{1}{c}\nabla\times(\rho\mathbf{V}) \qquad [\because\ \nabla\,.\,\mathbf{H}=0]$$

$$=-\frac{1}{c}\nabla\times\rho V=\text{R.H.S.}$$

PROBLEMS 8.4

1. Evaluate div **F** and curl **F** at the point (1, 2, 3) given (*i*) $\mathbf{F}=x^2yz\mathbf{I}+xy^2z\mathbf{J}+xyz^2\mathbf{K}$. *(B.P.T.U., 2005)*
   (*ii*) $\mathbf{F}=3x^2\mathbf{I}+5xy^2\mathbf{J}+5xyz^3\mathbf{K}$. *(S.V.T.U., 2009)*
   (*iii*) $\mathbf{F}=\text{grad}\,[x^3y+y^3z+z^3x-x^2y^2z^2]$ *(V.T.U., 2007)*
2. If $\mathbf{V}=(x\mathbf{I}+y\mathbf{J}+z\mathbf{K})/\sqrt{(x^2+y^2+z^2)}$, show that $\nabla.\mathbf{V}=2/\sqrt{(x^2+y^2+z^2)}$ and $\nabla\times\mathbf{V}=\mathbf{0}$. *(Osmania, 2002)*
3. If $\mathbf{F}=(x+y+1)\,\mathbf{I}+\mathbf{J}-(x+y)\mathbf{K}$, show that **F**. curl **F** = 0. *(V.T.U., 2000 S)*
4. Find the value of a if the vector $(ax^2y+yz)\,\mathbf{I}+(xy^2-xz^2)\,\mathbf{J}+(2xyz-2x^2y^2)\,\mathbf{K}$ has zero divergence. Find the curl of the above vector which has zero divergence.
5. Show that each of following vectors are solenoidal :
   (*i*) $(-x^2+yz)\,\mathbf{I}+(4y-z^2x)\,\mathbf{J}+(2xz-4z)\,\mathbf{K}$ *(Delhi, 2002)*
   (*ii*) $3y^4z^2\mathbf{I}+4x^3z^2\mathbf{J}+3x^2y^2\mathbf{K}$ (*iii*) $\nabla\phi\times\nabla\psi$.
6. If **A** and **B** are irrotational, prove that $\mathbf{A}\times\mathbf{B}$ is solenoidal. *(Madras, 2003 ; V.T.U., 2001)*
7. If $u=x^2+y^2+z^2$ and $\mathbf{V}=x\mathbf{I}+y\mathbf{J}+z\mathbf{K}$, show that div $(u\mathbf{V})=5u$.
8. If $\mathbf{R}=x\mathbf{I}+y\mathbf{J}+z\mathbf{K}$ and $r\neq 0$, show that (*i*) $\nabla/(1/r^2)=-2\mathbf{R}/r^4$; $\nabla\,.\,(\mathbf{R}/r^2)=1/r^2$
   (*ii*) div $(r^n\,\mathbf{R})=(n+3)r^n$; curl $(r^n\,\mathbf{R})=\mathbf{0}$ *(P.T.U., 2006 ; Kottayam, 2005)*
   (*iii*) *grad* $\left(\text{div}\,\frac{\mathbf{R}}{r}\right)=-\frac{2\mathbf{R}}{r^3}$. *(V.T.U., 2010 S)*
9. If $\mathbf{V}_1$ and $\mathbf{V}_2$ be the vectors joining the fixed points (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively to a variable point (x, y, z), prove that
   (*i*) div $(\mathbf{V}_1\times\mathrm{V}_2)=0$, (*ii*) grad $(\mathbf{V}_1\,.\,\mathbf{V}_2)=\mathbf{V}_1+\mathbf{V}_2$,
   (*iii*) curl $(\mathbf{V}_1\times\mathbf{V}_2)=2(\mathbf{V}_1-\mathbf{V}_2)$

10. Show that (*i*) $\nabla \cdot \left[\frac{f(r)}{r}\mathbf{R}\right] = \frac{1}{r^2}\frac{d}{dr}[r^2 f(r)]$ (*Mumbai, 2008*)

(*ii*) $\nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r)$ (*U.T.U., 2010 ; Bhopal, 2008 ; S.V.T.U., 2008 ; V.T.U., 2006*)

(*iii*) $\nabla \cdot (\phi\nabla\psi - \psi\nabla\phi) = \phi\nabla^2\psi - \psi\nabla^2\phi$.

11. If **A** is a constant vector and $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$, prove that

(*i*) grad $(\mathbf{A} \cdot \mathbf{R}) = \mathbf{A}$ (*Delhi, 2002*) (*ii*) div $(\mathbf{A} \times \mathbf{R}) = 0$ (*Burdwan, 2003*)

(*iii*) curl $(\mathbf{A} \times \mathbf{R}) = 2\mathbf{A}$ (*V.T.U., 2010 S*) (*iv*) curl $[(\mathbf{A} \cdot \mathbf{R})\mathbf{R}] = \mathbf{A} \times \mathbf{R}$. (*Kurukshetra, 2009 S*)

12. Prove that (*i*) $\nabla\mathbf{A}^2 = 2(\mathbf{A} \cdot \nabla)\mathbf{A} + 2\mathbf{A} \times (\nabla \times \mathbf{A})$, where **A** is a constant vector.

(*ii*) $\nabla \times (\mathbf{R} \times \mathbf{U}) = \mathbf{R}(\nabla \cdot \mathbf{U}) - 2\mathbf{U} - (\mathbf{R} \cdot \nabla)\mathbf{U}$.

13. Calculate (*i*) curl (grad f), given $f(x, y, z) = x^2 + y^2 - z$. (*B.P.T.U., 2006*)

(*ii*) curl (curl A) given $A = x^2y\mathbf{I} + y^2z\mathbf{J} + z^2y\mathbf{K}$ (*V.T.U., 2003*)

14. (*a*) If $f = (x^2 + y^2 + z^2)^{-n}$, find div grad f and determine n if div grad $f = 0$. (*S.V.T.U., 2009 ; J.N.T.U. 2003*)

(*b*) Show that div (grad r^n) $= n(n + 1)r^{n-2}$ where $r^2 = x^2 + y^2 + z^2$. (*Bhopal, 2008 ; U.P.T.U., 2003*)

15. For a solenoidal vector **F**, show that curl curl curl curl $\mathbf{F} = \nabla^4\mathbf{F}$.

16. If $u = x^2yz$, $v = xy - 3z^2$, find (*i*) $\nabla(\nabla u \cdot \nabla v)$; (*ii*) $\nabla \cdot (\nabla u \times \nabla v)$.

17. Find the directional derivative of $\nabla \cdot (\nabla\phi)$ at the point $(1, -2, 1)$ in the direction of the normal to the surface $xy^2z = 3x + z^2$, where $\phi = 2x^3y^2z^4$. (*Raipur, 2005*)

18. Prove that $\mathbf{A} \cdot \nabla\left(\mathbf{B} \cdot \nabla\frac{1}{r}\right) = \frac{3(\mathbf{A} \cdot \mathbf{R})(\mathbf{B} \cdot \mathbf{R})}{r^5} - \frac{\mathbf{A} \cdot \mathbf{B}}{r^3}$ where **A** and **B** are constant vectors.

8.10 INTEGRATION OF VECTORS

If two vector functions $\mathbf{F}(t)$ and $\mathbf{G}(t)$ be such that

$$\frac{d\mathbf{G}(t)}{dt} = \mathbf{F}(t),$$

then $\mathbf{G}(t)$ is called an integral of $\mathbf{F}(t)$ with respect to the scalar variable t and we write

$$\int \mathbf{F}(t)\,dt = \mathbf{G}(t).$$

If **C** be an arbitrary constant vector, we have

$$\mathbf{F}(t) = \frac{d\mathbf{G}(t)}{dt} = \frac{d}{dt}[\mathbf{G}(t) + \mathbf{C}] \quad \text{then} \quad \int \mathbf{F}(t)\,dt = \mathbf{G}(t) + \mathbf{C}$$

This is called the *indefinite integral of* $\mathbf{F}(t)$ and its *definite integral is*

$$\int_a^b \mathbf{F}(t)\,dt = [\mathbf{G}(t) + \mathbf{C}]_a^b = \mathbf{G}(b) - \mathbf{G}(a).$$

Example 8.27. *Given* $\mathbf{R}(t) = 3t^2\mathbf{I} + t\mathbf{J} - t^3\mathbf{K}$, *evaluate* $\int_0^1 (R \times d^2\mathbf{R}/dt^2)\,dt$.

Solution. $$\frac{d}{dt}\left(\mathbf{R} \times \frac{d\mathbf{R}}{dt}\right) = \frac{d\mathbf{R}}{dt} \times \frac{d\mathbf{R}}{dt} + \mathbf{R} \times \frac{d^2\mathbf{R}}{dt^2} = \mathbf{R} \times \frac{d^2\mathbf{R}}{dt^2}$$

$$\therefore \quad \int\left(\mathbf{R} \times \frac{d^2\mathbf{R}}{dt^2}\right)dt = \mathbf{R} \times \frac{d\mathbf{R}}{dt}$$

$$= (3t^2\mathbf{I} + t\mathbf{J} - t^3\mathbf{K}) \times (6t\mathbf{I} + \mathbf{J} - 3t^2\mathbf{K})$$

$$= \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ 3t^2 & t & -t^3 \\ 6t & 1 & -3t^2 \end{vmatrix} = -2t^3\mathbf{I} + 3t^4\mathbf{J} - 3t^2\mathbf{K}$$

Thus $$\int_0^1 \left(\mathbf{R} \times \frac{d^2\mathbf{R}}{dt^2}\right)dt = \left| -2t^3\mathbf{I} + 3t^4\mathbf{J} - 3t^2\mathbf{K} \right|_0^1$$

$$= -2\mathbf{I} + 3\mathbf{J} - 3\mathbf{K}$$

PROBLEMS 8.5

1. Given $\mathbf{F}(t) = (5t^2 - 3t)\mathbf{I} + 6t^3\mathbf{J} - 7t\,\mathbf{K}$, evaluate $\int_{t=2}^{t=4} \mathbf{F}(t)\,dt$.

2. If $\dfrac{d^2\mathbf{P}}{dt^2} = 6t\mathbf{I} - 12t^2\mathbf{J} + 4\cos t\mathbf{K}$, find $\mathbf{P}$, Given that $\dfrac{d\mathbf{P}}{dt} = -1 - 3\mathbf{K}$ and $\mathbf{P} = 2\mathbf{I} + \mathbf{J}$ when $t = 0$.

3. The acceleration of a particle at any time $t \geq 0$ is given by $12\cos 2t\mathbf{I} - 8\sin 2t\mathbf{J} + 16t\mathbf{K}$, the velocity and acceleration are initially zero. Find the velocity and displacement at any time.

4. If $\mathbf{R}(t) = \begin{cases} 2\mathbf{I} - \mathbf{J} + 2\mathbf{K} & \text{when } t = 1 \\ 3\mathbf{I} - 2\mathbf{J} + 4\mathbf{K} & \text{when } t = 2, \end{cases}$

   show that $\int_1^2 \left(\mathbf{R} \cdot \dfrac{d\mathbf{R}}{dt}\right) dt = 10$.

8.11 (1) LINE INTEGRAL

Consider a continuous vector function $\mathbf{F}(\mathbf{R})$ which is defined at each point of curve C in space. Divide C into n parts at the points $A = P_0, P_1, \ldots, P_{i-1}, P_i \ldots, P_n = B$ (Fig. 8.8). Let their position vectors be $\mathbf{R}_0, \mathbf{R}_1, \ldots, \mathbf{R}_{i-1}, \mathbf{R}_i, \ldots, \mathbf{R}_n$. Let $\mathbf{U}_i$, be the position vector of any point on the arc $P_{i-1}P_i$.

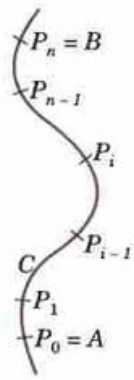

Fig. 8.8

Now consider the sum $S = \sum_{i=0}^{n} \mathbf{F}(\mathbf{U}_i) \cdot \delta\mathbf{R}_i$, *where* $d\mathbf{R}_i = \mathbf{R}_i - \mathbf{R}_{i-1}$.

The limit of this sum as $n \to \infty$ *in such a way that* $|\delta\mathbf{R}_i| \to 0$, *provided it exists, is called the* **tangential line integral** of $\mathbf{F}(\mathbf{R})$ *along* C and is symbolically written as

$$\int_C \mathbf{F}(\mathbf{R}) \cdot d\mathbf{R} \quad \text{or} \quad \int_C \mathbf{F} \cdot \frac{d\mathbf{R}}{dt}\,dt.$$

When the path of integration is a closed curve, this fact is denoted by using $\oint$ in place of $\int$.

If $\quad \mathbf{F}(\mathbf{R}) = \mathbf{I}f(x, y, z) + \mathbf{J}\phi(x, y, z) + \mathbf{K}\psi(x, y, z)$

and $\quad d\mathbf{R} = \mathbf{I}dx + \mathbf{J}dy + \mathbf{K}dz$

then $\quad \int_C \mathbf{F}(\mathbf{R}) \cdot d\mathbf{R} = \int_C (f\,dx + \phi dy + \psi dz)$.

Two other types of line integrals are $\int_C \mathbf{F} \times d\mathbf{R}$ and $\int_C f d\mathbf{R}$ which are both vectors.

(2) Circulation. *If* $\mathbf{F}$ *represents the velocity of a fluid particle then the line integral* $\int_C \mathbf{F} \cdot d\mathbf{R}$ *is called the* **circulation of F** *around the curve. When the circulation of* $\mathbf{F}$ *around every closed curve in a region E vanishes,* $\mathbf{F}$ *is said to be* **irrotational** *in E.*

(3) Work. If $\mathbf{F}$ represents the force acting on a particle moving along an arc AB then the work done during the small displacement $\delta\mathbf{R} = \mathbf{F} \cdot \delta\mathbf{R}$.

$\therefore$ *the total work done by* $\mathbf{F}$ *during the displacement from A to B is given by the line integral* $\int_A^B \mathbf{F} \cdot d\mathbf{R}$.

Example 8.28. *If* $\mathbf{F} = 3xy\mathbf{I} - y^2\mathbf{J}$, *evaluate* $\int \mathbf{F} \cdot d\mathbf{R}$, *where C is the curve in the xy-plane* $y = 2x^2$ *from (0, 0) to (1, 2).* *(V.T.U., 2010)*

Solution. Since the particle moves in the xy-plane ($z = 0$), we take $\mathbf{R} = x\mathbf{I} + y\mathbf{J}$. Then $\int_C \mathbf{F} \cdot d\mathbf{R}$, where C is the parabola $y = 2x^2$

$$= \int_C (3xy\mathbf{I} - y^2\mathbf{J}) \cdot (dx\mathbf{I} + dy\mathbf{J}) = \int_C (3xydx - y^2dy) \qquad \ldots(i)$$

Substituting $y = 2x^2$, where x goes from 0 to 1, (i) becomes

$$= \int_{x=0}^{1} [3x(2x^2)\,dx - (2x^2)^2\,d(2x^2)] = \int_0^1 (6x^3 - 16x^5)\,dx = -7/6.$$

Otherwise, let $x = t$ in $y = 2x^2$. Then the parametric equation of C are $x = t, y = 2t^2$. The points (0, 0) and (1, 2) correspond to $t = 0$ and $t = 1$ respectively. Then (i) becomes

$$= \int_{t=0}^{1} [3t(2t^2)\,dt - (2t^2)^2 d(2t^2)] = \int_0^1 (6t^3 - 16t^5)\,dt = -7/6.$$

Example 8.29. *A vector field is given by* $\mathbf{F} = \sin y\mathbf{I} + x(1 + \cos y)\,\mathbf{J}$. *Evaluate the line integral over a circular path given by* $x^2 + y^2 = a^2, z = 0$. *(Rohtak, 2006 S ; P.T.U., 2003)*

Solution. As the particle moves in xy-plane ($z = 0$), let $\mathbf{R} = x\mathbf{I} + y\mathbf{J}$ so that $d\mathbf{R} = dx\mathbf{I} + dy\mathbf{J}$. Also the circular path is $x = a\cos t, y = a\sin t, z = 0$ where t varies from 0 to 2π.

$$\therefore \quad \oint_C \mathbf{F}\,.\,d\mathbf{R} = \oint_C [\sin y\mathbf{I} + x(1 + \cos y)\mathbf{J}]\,.\,(dx\mathbf{I} + dy\mathbf{J})$$

$$= \oint_C [\sin y\,dx + x(1 + \cos y)\,dy] = \oint_C [(\sin y\,dx + x\cos y\,dy) + xdy]$$

$$= \oint [d(x\sin y) + x\,dy] = \int_0^{2\pi} [d\{a\cos t\sin(a\sin t)\} + a^2\cos^2 t\,dt]$$

$$= \left| a\cos t\sin(a\sin t)\right|_0^{2\pi} + \frac{a^2}{2}\int_0^{2\pi}(1 + \cos 2t)\,dt = \frac{a^2}{2}\left| t + \frac{\sin 2t}{2}\right|_0^{2\pi} = \pi a^2.$$

Example 8.30. *Find the work done in moving a particle in the force field* $\mathbf{F} = 3x^2\mathbf{I} + (2xz - y)\,\mathbf{J} + z\mathbf{K}$, *along*
(a) the straight line from (0, 0, 0) to (2, 1, 3). *(S.V.T.U., 2007 ; J.N.T.U., 2002)*
(b) the curve defined by $x^2 = 4y, 3x^3 = 8z$ *from* $x = 0$ *to* $x = 2$. *(Delhi, 2002)*

Solution.
$$\int_C \mathbf{F}\,.\,d\mathbf{R} = \int_C [3x^2\mathbf{I} + (2xz - y)\mathbf{J} + z\mathbf{K}]\,.\,(dx\mathbf{I} + dy\mathbf{J} + dz\mathbf{K})$$

$$= \int_C [3x^2dx + (2xz - y)dy + zdz] \qquad ...(i)$$

(a) The equations of the straight line from (0, 0, 0) to (2, 1, 3) are $x/2 = y/1 = z/3 = t$ (say)

$\therefore$ $x = 2t, y = t, z = 3t$ are its parametric equations. The points (0, 0, 0) and (2, 1, 3) correspond to $t = 0$ and $t = 1$, respectively

$$\therefore \quad \text{work done} = \int_C \mathbf{F}\,.\,d\mathbf{R} = \int_0^1 [3(2t)^2\,2dt + \{(4t)(3t) - t\}dt + (3t)\,3dt]$$

$$= \int_0^1 (36t^2 + 8t)\,dt = 16.$$

(b) Let $x = t$ in $x^2 = 4y, 3x^3 = 8z$. Then the parametric equations of C are $x = t, y = t^2/4, z = 3t^3/8$ and t varies from 0 to 2.

$$\therefore \quad \text{work done} = \int_C \mathbf{F}\,.\,d\mathbf{R} = \int_0^2 \left[3t^2\,dt + \left\{2t\left(\frac{3t^3}{8}\right) - \frac{t^2}{4}\right\} d\left(\frac{t^2}{4}\right) + \frac{3t^3}{8}\,d\left(\frac{3t^2}{8}\right)\right]$$

$$= \int_0^2 \left(3t^2 - \frac{t^3}{8} + \frac{51}{64}t^5\right)dt = \left| t^3 - \frac{t^4}{32} + \frac{17}{128}t^6\right|_0^2 = 16.$$

PROBLEMS 8.6

1. Evaluate the line integral $\int_C [(x^2 + xy)dx + (x^2 + y^2)dy]$ where C is the square formed by the lines $y = \pm 1$ and $x = \pm 1$. *(Delhi, 2002)*

2. If $\mathbf{F} = (5xy - 6x^2)\mathbf{I} + (2y - 4x)\mathbf{J}$, evaluate $\int_C \mathbf{F} \cdot d\mathbf{R}$ along the curve C in the xy-plane, $y = x^3$ from the point (1, 1) to (2, 8). *(J.N.T.U., 2006)*

3. Compute the line integral $\int_C (y^2 dx - x^2 dy)$ about the triangle whose vertices are (1, 0), (0, 1) and (– 1, 0).

4. If $A = (3x^2 + 6y)\mathbf{I} - 14yz\mathbf{J} + 20xz^2\mathbf{K}$, evaluate $\int \mathbf{A} \cdot d\mathbf{R}$ from (0, 0, 0) to (1, 1, 1) along the path $x = t, y = t^2, z = t^3$. *(V.T.U., 2001)*

5. Evaluate $\int_C (xy + z^2)\, ds$ where C is the arc of the helix $x = \cos t, y = \sin t, z = t$ which joins the points (1, 0, 0) and $(-1, 0, \pi)$.

6. Find the total work done by the force $\mathbf{F} = 3xy\mathbf{I} - y\mathbf{J} + 2zx\mathbf{K}$ in moving a particle around the circle $x^2 + y^2 = 4$. *(V.T.U., 2010)*

7. Find the total work done in moving a particle in a force field given by $\mathbf{F} = 3xy\mathbf{I} - 5z\mathbf{J} + 10x\mathbf{K}$ along the curve $x = t^2 + 1, y = 2t^2, z = t^3$ from $t = 1$ to $t = 2$. *(Bhopal, 2008)*

8. Using the line integral, compute the work done by the force $\mathbf{F} = (2y + 3)\mathbf{I} + xz\mathbf{J} + (yz - x)\mathbf{K}$ when it moves a particle from the point (0, 0, 0) to the point (2, 1, 1) along the curve $x = 2t^2, y = t, z = t^3$. *(Madras, 2000)*

9. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{R}$, where $\mathbf{F} = [2z, x, -y]$ and C is $\mathbf{R} = [\cos t, \sin t, 2t]$ from (1, 0, 0) to $(1, 0, 4\pi)$. *(B.P.T.U., 2006)*

10. If $\mathbf{F} = 2y\mathbf{I} - z\mathbf{J} + x\mathbf{K}$, evaluate $\int_C \mathbf{F} \times d\mathbf{R}$ along the curve $x = \cos t, y = \sin t, z = 2\cos t$ from $t = 0$ to $t = \pi/2$.

8.12 (1) SURFACES

As seen in § 5.8, a surface S may be represented by $F(x, y, z) = 0$.

The *parametric representation of* S is of the form $\mathbf{R}(u, v) = x(u, v)\,\mathbf{I} + y(u, v)\,\mathbf{J} + z(u, v)\mathbf{K}$ and the continuous functions $u = \phi(t)$ and $v = \psi(t)$ of a real parameter t represent a curve C on this surface S.

For example, the parametric representation of the circular cylinder $x^2 + y^2 = a^2, -1 \le z \le 1$, (radius a and height 2), is

$$\mathbf{R}(u, v) = a\cos u\mathbf{I} + a\sin u\mathbf{J} + v\mathbf{K}$$

where the parameters u and v vary in the rectangle $0 \le u \le 2\pi$ and $-1 \le v \le 1$. Also $u = t, v = bt$ represent a *circular helix* (Fig. 8.3) on this circular cylinder. The equation of the circular helix is $\mathbf{R} = a\cos t\mathbf{I} + a\sin t\mathbf{J} + bt\mathbf{K}$.

Differentiating $\mathbf{R} = \mathbf{R}(u, v)$, w.r.t. t, we get $\dfrac{d\mathbf{R}}{dt} = \dfrac{\partial \mathbf{R}}{\partial u} \cdot \dfrac{du}{dt} + \dfrac{\partial \mathbf{R}}{\partial v} \cdot \dfrac{dv}{dt}$

The vectors $\partial\mathbf{R}/\partial u$ and $\partial\mathbf{R}/\partial v$ are tangential to S at P and determine the tangent plane of S at P. $\mathbf{N} = \partial\mathbf{R}/\partial u \times \partial\mathbf{R}/\partial v\ (\neq 0)$ gives a normal vector $\mathbf{N}$ of S at P.

Def. *If S has a unique normal at each of its points whose direction depends continuously on the points of S, then the surface S is called a* **smooth surface**. If S is not smooth but can be divided into finitely many smooth portions, then it is called a **piecewise smooth surface.**

For instance, the surface of a sphere is *smooth* while the surface of a cube is *piecewise smooth.*

Def. *A surface S is said to be* **orientable or two sided** *if the positive normal direction at any point P of S can be continued in a unique and continuous way to the entire surface.* If the positive direction of the normal is reversed as we move around a curve on S passing through P, then the surface is **non-orientable** (*i.e.*, **one-sided**).

An example of a non-orientable surface is the *Mobius strip**. If we take a long rectangular strip of paper and giving a half-twist join the shorter sides so that the two points A and the two points B in Fig. 8.9 coincide, then the surface generated is non-orientable. Such a surface is a model of a Mobius strip.

Fig. 8.9

(2) Surface integral. Consider a continuous function $\mathbf{F}(\mathbf{R})$ and a surface S. Divide S into a finite number of sub-surfaces. Let the surface element surrounding any point $P(\mathbf{R})$ be $\delta\mathbf{S}$ which can be regarded as a vector ; its magnitude being the area and its direction that of the outward normal to the element.

*Named after a German mathematician *August Ferdinand Möbius (1790–1868)* who was a student of Gauss and professor of astronomy at Leipzig. His important contributions are in projective geometry, theory of surfaces and mechanics.

Consider the sum $\Sigma \mathbf{F}(\mathbf{R}) \,.\, \delta\mathbf{S}$, *where the summation extends over all the sub-surfaces. The limit of this sum as the number of sub-surfaces tends to infinity and the area of each sub-surface tends to zero, is called the* **normal surface integral** *of* $\mathbf{F}(\mathbf{R})$ *over S and is denoted by*

$$\int_S \mathbf{F} \,.\, d\mathbf{S} \quad \text{or} \quad \int_S \mathbf{F} \,.\, \mathbf{N} ds \text{ where } \mathbf{N} \text{ is a unit outward normal at } P \text{ to } S.$$

Other types of surface integrals are $\int_S \mathbf{F} \times d\mathbf{S}$ and $\int_S f \, d\mathbf{S}$ which are both vectors.

Notation : Only one integrals sign is used when there is one differential (say $d\mathbf{R}$ or $d\mathbf{S}$) and two (or three) signs when there are two (or three) differentials.

(3) **Flux across a surface.** *If* $\mathbf{F}$ *represent the velocity of a fluid particle then the total outward flux of* $\mathbf{F}$ *across a closed surface S is the surface integral* $\int_S \mathbf{F} \,.\, d\mathbf{S}$.

When the flux of $\mathbf{F}$ *across every closed surface S in a region E vanishes,* $\mathbf{F}$ *is said to be a* **solenoidal vector point function** *in E.*

It may be noted that $\mathbf{F}$ could equally well be taken as any other physical quantity *e.g.*, gravitational force, electric force and magnetic force.

Example 8.31. *Evaluate* $\int_S \mathbf{F} \,.\, \mathbf{N} ds$ *where* $\mathbf{F} = 2x^2y\mathbf{I} - y^2\mathbf{J} + 4xz^2\mathbf{K}$ *and S is the closed surface of the region in the first octant bounded by the cylinder* $y^2 + z^2 = 9$ *and the planes* $x = 0$, $x = 2$, $y = 0$ *and* $z = 0$.

Solution. The given closed surface S is piecewise smooth and is comprised of S_1 – the rectangular face $OAEB$ in xy-plane ; S_2–the rectangular face $OADC$ in xz-plane ; S_3–the circular quadrant ABC in yz-plane, S_4–the circular quadrant AED and S_5–the curved surface $BCDE$ of the cylinder in the first octant (Fig. 8.10).

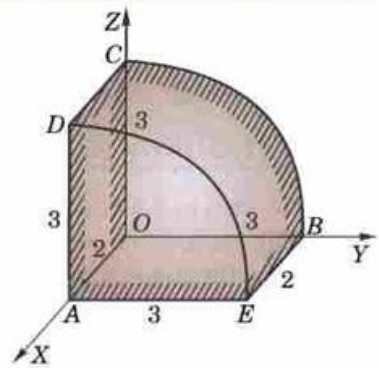

Fig. 8.10

$$\therefore \quad \int_S \mathbf{F} \,.\, \mathbf{N} ds = \int_{S_1} \mathbf{F} \,.\, \mathbf{N} \, ds + \int_{S_2} \mathbf{F} \,.\, \mathbf{N} \, ds + \int_{S_3} \mathbf{F} \,.\, \mathbf{N} \, ds + \int_{S_4} \mathbf{F} \,.\, \mathbf{N} \, ds + \int_{S_5} \mathbf{F} \,.\, \mathbf{N} \, ds \qquad ...(i)$$

Now $\int_{S_1} \mathbf{F} \,.\, \mathbf{N} ds = \int_{S_1} (2x^2y\mathbf{I} - y^2\mathbf{J} + 4xz^2\mathbf{K}) \,.\, (-\mathbf{K}) \, ds$

$$= -4 \int_{S_1} xz^2 \, ds = 0 \qquad [\because \; z = 0 \text{ in the } xy\text{–plane}]$$

Similarly, $\int_{S_2} \mathbf{F} \,.\, \mathbf{N} ds = 0$ and $\int_{S_3} \mathbf{F} \,.\, \mathbf{N} ds = 0$

$$\int_{S_4} \mathbf{F} \,.\, \mathbf{N} ds = \int_{S_4} (2x^2y\mathbf{I} - y^2 \, \mathbf{J} + 4xz^2 \, \mathbf{K}) \,.\, \mathbf{I} \, ds$$

$$= \int_{S_4} 2x^2y \, ds = \int_0^3 \int_0^{\sqrt{9-z^2}} 8y \, dydz = 4 \int_0^3 (9 - z^2) \, dz = 72$$

To find $\mathbf{N}$ in S_5, we note that $\nabla(y^2 + z^2) = 2y\mathbf{J} + 2z\mathbf{K}$

$$\therefore \qquad \mathbf{N} = \frac{2y\mathbf{J} + 2z\mathbf{K}}{\sqrt{(4y^2 + 4z^2)}} = \frac{y\mathbf{J} + z\mathbf{K}}{3} \qquad [\because \; y^2 + z^2 = 9]$$

and $\quad | \mathbf{N} \,.\, \mathbf{K} | = z/3 \qquad$ so that $ds = dxdy/(z/3)$

Thus
$$\int_{S_5} \mathbf{F} \,.\, \mathbf{N} \, ds = \int_0^2 \int_0^3 \frac{(-y^3 + 4xz^3)}{3} \,.\, dydx/(z/3) = \int_0^2 \int_0^3 \left(\frac{-y^3}{z} + 4xz^2 \right) dydx$$

$$\begin{bmatrix} \text{Put } y = 3 \sin \theta, \, z = 3 \cos \theta \\ \therefore \; dy = 3 \cos \theta \, d\theta \end{bmatrix}$$

$$= \int_0^2 \int_0^{\pi/2} \left[\frac{-27 \sin^3 \theta}{3 \cos \theta} + 4x(9 \cos^2 \theta \right] 3 \cos \theta \, d\theta dx = \int_0^2 \left[-27 \times \frac{2}{3} + 108x \times \frac{2}{3} \right] dx = 108$$

Hence (i) gives $\int_S \mathbf{F} \,.\, \mathbf{N} \, ds = 0 + 0 + 0 + 72 + 108 = 180.$

PROBLEMS 8.7

1. If velocity vector is $\mathbf{F} = y\mathbf{I} + 2\mathbf{J} + xz\mathbf{K}$ m/sec., show that the flux of water through the parabolic cylinder $y = x^2$, $0 \le x \le 3$, $0 \le z \le 2$ is 69 m³/sec.
2. Evaluate $\int_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = x\mathbf{I} + (z^2 - zx)\mathbf{J} - xy\mathbf{K}$ and S is the triangular surface with vertices (2, 0, 0), (0, 2, 0) and (0, 0, 4).
3. Evaluate $\int_S \mathbf{F} \cdot \mathbf{N}\, ds$ where $\mathbf{F} = 6z\mathbf{I} - 4\mathbf{J} + y\mathbf{K}$ and S is the portion of the plane $2x + 3y + 6z = 12$ in the first octant.
4. If $\mathbf{F} = 2y\mathbf{I} - 3\mathbf{J} + x^2\mathbf{K}$ and S is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$ and $z = 6$, show that $\int_S \mathbf{F} \cdot \mathbf{N}\, ds = 132$.

8.13 GREEN'S THEOREM IN THE PLANE*

If $\phi(x, y)$, $\psi(x, y)$, ϕ_y and ψ_x be continuous in a region E of the xy-plane bounded by a closed curve C, then

$$\int_C (\phi dx + \psi dy) = \iint_E \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dxdy \qquad ...(1)$$

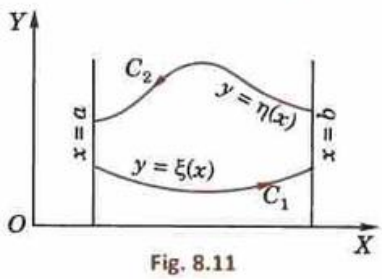

Fig. 8.11

Consider the region E bounded by a single closed curve C which is cut by any line parallel to the axes at the most in two points.

Let E be bounded by $x = a$, $y = \xi(x)$, $x = b$ and $y = \eta(x)$, where $\eta \ge \xi$, so that C is divided into curves C_1 and C_2 (Fig. 8.11).

$$\therefore \quad \iint_E \frac{\partial \phi}{\partial y} dxdy = \int_a^b dx \left[\int_\xi^\eta \frac{\partial \phi}{\partial y} dy\right] = \int_a^b dx \,|\, \phi \,|_\xi^\eta$$

$$= \int_a^b [\phi(x, \eta) - \phi(x, \xi)]\, dx = -\int_{C_2} \phi(x, y)\, dx - \int_{C_1} \phi(x, y)\, dx$$

$$= -\int_C \phi(x, y)\, dx \qquad ...(2)$$

Similarly, it can be shown that

$$\iint_E \frac{\partial \psi}{\partial x} dxdy = \int_C \psi(x, y)\, dy \qquad ...(3)$$

On subtracting (2) from (3), we get (1).

This result can be extended to regions which may be divided into a finite number of sub-regions such that the boundary of each is cut at the most in two points by any line parallel to either axis. Applying (1) to each of these sub-regions and adding the results, the surface integrals combine into an integral over the whole region ; the line integrals over the common boundaries cancel (for each is covered twice but in opposite directions), whereas the remaining line integrals combine into the line integral over the external curve C.

Obs. This theorem converts a line integral around a closed curve into a double integral and is a special case of Stoke's theorem. (See Cor. p. 342)

Example 8.32. *Verify Green's theorem for $\int_C [(xy + y^2)\, dx + x^2 dy]$, where C is bounded by $y = x$ and $y = x^2$.*

(V.T.U., 2011 ; S.V.T.U., 2009 ; Rohtak, 2003)

Solution. Here $\phi = xy + y^2$ and $\psi = x^2$

$$\therefore \quad \int_C (\phi\, dx + \psi\, dy) = \int_{C_1} + \int_{C_2}$$

*Named after the English mathematician *George Green* (1793–1841) who taught at Cambridge and is known for his work on potential theory in connection with waves, vibrations, elasticity, electricity and magnetism.

Along C_1, $y = x^2$ and x varies from 0 to 1 (Fig. 8.12)

$$\therefore \quad \int_{C_1} = \int_0^1 [\{x(x)^2 + (x^2)^2\}]\, dx + x^2 d(x^2)]$$

$$= \int_0^1 (3x^3 + x^4)\, dx = \frac{19}{20}$$

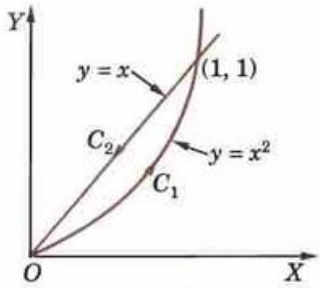

Fig. 8.12

Along C_2, $y = x$ and x varies from 1 to 0.

$$\therefore \quad \int_{C_2} = \int_1^0 [\{x(x) + (x)^2\}\, dx + x^2\, d(x)] = \int_1^0 3x^2\, dx = -1.$$

Thus $\displaystyle\int_C (\phi\, dx + \psi\, dy) = \frac{19}{20} - 1 = -\frac{1}{20}$ ...(*i*)

Also $\displaystyle\iint_E \left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right) dxdy = \iint_E \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy + y^2)\right] dxdy$

$$= \int_0^1 \int_{x^2}^{x} (2x - x - 2y)\, dydx = \int_0^1 \left[xy - y^2\right]_{x^2}^{x} dx = \int_0^1 (x^4 - x^3)\, dx = -\frac{1}{20} \quad ...(ii)$$

Hence, Green theorem is verified from the equality of (*i*) and (*ii*).

Example 8.33. *If C is a simple closed curve in the xy-plane not enclosing the origin, show that*

$$\int_C \mathbf{F} \cdot d\mathbf{R} = 0, \quad \text{where} \quad \mathbf{F} = \frac{y\mathbf{I} - x\mathbf{J}}{x^2 + y^2}$$ *(P.T.U., 2005)*

Solution. $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{R} = \int_C \frac{y\mathbf{I} - x\mathbf{J}}{x^2 + y^2}(dx\mathbf{I} + dy\mathbf{J})$ [$\because$ $\mathbf{R} = x\mathbf{I} + y\mathbf{J}$]

$$= \int_C \frac{ydx - xdy}{x^2 + y^2} = \int_C (\phi dx + \psi dy) \text{ where } \phi = \frac{y}{x^2 + y^2}, \psi = \frac{-x}{x^2 y^2}$$

$$= \iint_S \left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right) dxdy \quad \text{[By Green's theorem]}$$

$$= \iint_S \left[\frac{-(x^2 + y^2) + x(2x)}{(x^2 + y^2)^2} - \frac{(x^2 + y^2) - y(2y)}{(x^2 + y^2)^2}\right] dxdy$$

$$= \iint \left[\frac{x^2 - y^2}{(x^2 + y^2)^2} - \frac{x^2 - y^2}{(x^2 + y^2)^2}\right] dxdy = 0.$$

Example 8.34. *Using Green's theorem, evaluate* $\displaystyle\int_C [(y - \sin x)\, dx + \cos x\, dy]$ *where C is the plane triangle enclosed by the lines* $y = 0$, $x = \pi/2$ *and* $y = \frac{2}{\pi}x$. *(J.N.T.U., 2005 ; Anna, 2003)*

Solution. Here $\phi = y - \sin x$ and $\psi = \cos x$.

By Green's theorem $\displaystyle\int_C [(y - \sin x)\, dx + \cos x\, dy]$

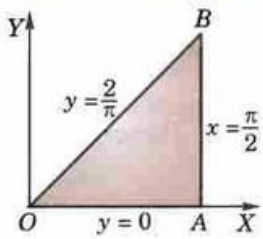

Fig. 8.13

$$= \iint_R \left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right) dx\, dy$$

$$= \int_{x=0}^{x=\pi/2} \int_{y=0}^{y=2x/\pi} (-\sin x - 1)\, dy\, dx = -\int_0^{\pi/2} (\sin x + 1) \Big|\, y\, \Big|_0^{2x/\pi} dx$$

$$= -\frac{2}{\pi}\int_0^{\pi/2} x(\sin x + 1)\, dx = -\frac{2}{\pi}\left\{\Big|\, x(-\cos x + x)\, \Big|_0^{\pi/2} - \int_0^{\pi/2} 1 \cdot (-\cos x + x)\, dx\right\}$$

$$= -\frac{2}{\pi}\left\{\frac{\pi^2}{4} - \left| -\sin x + \frac{x^2}{2} \right|_0^{\pi/2}\right\} = -\frac{\pi}{2} + \frac{2}{\pi}\left(-1 + \frac{\pi^2}{8}\right) = -\left(\frac{\pi}{4} + \frac{2}{\pi}\right)$$

Example 8.35. *Apply Green's theorem to evaluate* $\int_C [(2x^2 - y^2)dx + (x^2 + y^2)dy]$, *where C is the boundary of the area enclosed by the x-axis and the upper-half of the circle* $x^2 + y^2 = a^2$. *(U.P.T.U., 2005)*

Solution. By Green's theorem

$$\int_C [(2x^2 - y^2)dx + (x^2 + y^2)dy]$$

$$= \iint_A \left[\frac{\partial}{\partial x}(x^2 + y^2) - \frac{\partial}{\partial y}(2x^2 - y^2)\right] dxdy$$

$$= 2\iint_A (x + y)\, dxdy, \text{ where } A \text{ is the region of Fig. 8.14}$$

$$= 2\int_0^a \int_0^\pi r\,(\cos\theta + \sin\theta)\cdot r d\theta\, dr$$

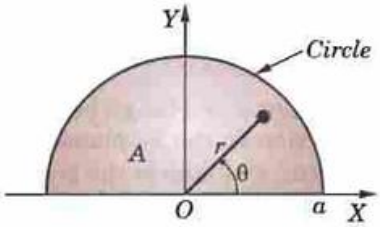

Fig. 8.14

[Changing to polar coordinates (r, θ), r varies from 0 to a and θ varies from 0 to π]

$$= 2\int_0^a r^2\, dr \int_0^\pi (\cos\theta + \sin\theta)\, d\theta = 2\cdot\frac{a^3}{3}\cdot(1+1) = \frac{4a^3}{3}.$$

PROBLEMS 8.8

1. Verify Green's theorem for $\int_C [(3x - 8y^2)\, dx + (4y - 6xy)\, dy]$ where C is the boundary of the region bounded by $x = 0, y = 0$ and $x + y = 1$. *(Nagpur, 2008 ; Kerala, 2005 ; Anna, 2003 S)*
2. Verify Green's theorem for $\int_C [(x^2 - \cosh y)dx + (y + \sin x)dy]$ where C is the rectangle with vertices $(0, 0)$, $(\pi, 0)$, $(\pi, 1)$, $(0, 1)$. *(Nagpur, 2009 ; P.T.U., 2006)*
3. Verify Green's theorem for $\int (x^2 y dx + x^2 dy)$ where C is the boundary described counter clockwise of triangle with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$. *(U.T.U., 2010)*
4. Apply Green's theorem to prove that the area enclosed by a plane curve is $\frac{1}{2}\int_C (xdy - ydx)$.
   Hence find the area of an ellipse whose semi-major and semi-minor axes are of lengths a and b. *(Kerala, 2005 ; V.T.U., 2000 S)*
5. Find the area of a circle of radius a using Green's theorem. *(Madras, 2003)*
6. Evaluate $\int_C [(x^2 + xy)dx + (x^2 + y^2)dy]$, where C is the square formed by the lines $x = \pm 1, y = \pm 1$. *(S.V.T.U., 2008 ; Marathwada, 2008)*
7. Evaluate $\int_C [(x^2 - 2xy)dx + (x^2 y + 3)dy]$, around the boundary of the region defined by $y^2 = 8x$ and $x = 2$.
8. Evaluate by Green's theorem $\int_C \mathbf{F}\cdot d\mathbf{R}$ where $\mathbf{F} = -xy(x\mathbf{I} - y\mathbf{J})$ and C is $r = a(1 + \cos\theta)$. *(Mumbai, 2006)*

8.14 STOKE'S THEOREM* (*Relation between line and surface integrals*)

If S be an open surface bounded by a closed curve C and $\mathbf{F} = f_1\mathbf{I} + f_2\mathbf{J} + f_3\mathbf{K}$ *be any continuously differentiable vector point function, then*

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \int_S curl\, \mathbf{F}\cdot \mathbf{N} ds$$

where $\mathbf{N} = \cos\alpha\mathbf{I} + \cos\beta\mathbf{J} + \cos\gamma\mathbf{K}$ *is a unit external normal at any point of S.*

* Named after an Irish mathematician *Sir George Gabriel Stokes* (1819–1903) who became professor in Cambridge. His important contributions are to infinite series, geodesy and theory of viscous fluids.

Writing $d\mathbf{R} = dx\mathbf{I} + dy\mathbf{J} + dz\mathbf{K}$, it may be reduced to the form

$$\int_C (f_1 dx + f_2 dy + f_3 dz) = \int_S \left[\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\cos\beta + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\cos\gamma\right] ds \quad ...(1)$$

Let us first prove that

$$\oint_C f_1 dx = \int_S \left(\frac{\partial f_1}{\partial z}\cos\beta - \frac{\partial f_1}{\partial y}\cos\gamma\right) ds \quad ...(2)$$

Let $z = g(x, y)$ be the equation of the surface S whose projection on the xy-plane is the region E. Then the projection of C on the xy-plane is the curve C' enclosing region E.

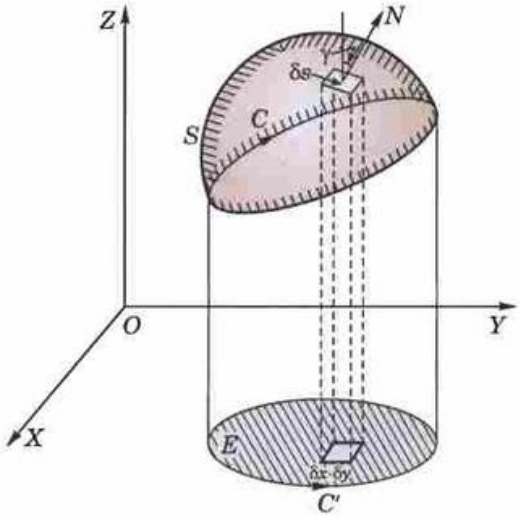

Fig. 8.15

$$\therefore \quad \int_C f_1(x, y, z)dx = \int_C f_1\{x, y, g(x, y)\}dx$$

$$= -\iint_E \frac{\partial}{\partial y} f_1(x, y, g)dxdy, \text{ by Green's theorem}$$

$$= -\iint_E \left(\frac{\partial f_1}{\partial y} + \frac{\partial f_1}{\partial z}\frac{\partial g}{\partial y}\right)dxdy \quad ...(3)$$

The direction cosines of the normal to the surface $z = g(x, y)$ are given by

$$\frac{\cos\alpha}{-\partial g/\partial x} = \frac{\cos\beta}{-\partial g/\partial y} = \frac{\cos\gamma}{1} \text{ (See p. 219)} \quad ...(4)$$

Moreover

$dxdy$ = projection of ds on the xy-plane
$= ds \cos\gamma$, *i.e.*, $ds = dxdy/\cos\gamma$.

$\therefore$ right side of (2)

$$= \iint_E \left(\frac{\partial f_1}{\partial z}\frac{\cos\beta}{\cos\gamma} - \frac{\partial f_1}{\partial y}\right)dxdy = -\iint \left(\frac{\partial f_1}{\partial y} + \frac{\partial f_1}{\partial z}\cdot\frac{\partial g}{\partial y}\right)dxdy \qquad \left[\frac{\cos\beta}{\cos\gamma} = -\frac{\partial g}{\partial y} \text{ by (4)}\right]$$

= Left side of (2), by (3).

Thus we have proved (2). Similarly, we can prove the other corresponding relations for f_2 and f_3. Adding these three results, we get (1).

Cor. Green's theorem in a plane as a special case of Stokes theorem. Let $\mathbf{F} = \phi\mathbf{I} + \psi\mathbf{J}$ be a vector function which is continuously differentiable in a region S of the xy-plane bounded by a closed curve C. Then

$$\int_C \mathbf{F}\,.\,dR = \int_C (\phi\mathbf{I} + \psi\mathbf{J})\cdot(dx\,\mathbf{I} + dy\,\mathbf{J}) = \int_C (\phi\,dx + \psi\,dy)$$

and

$$\text{curl } \mathbf{F}\cdot\mathbf{N} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \partial/\partial x & \partial/\partial y & 0 \\ \phi & \psi & 0 \end{vmatrix}\cdot\mathbf{K} = \frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}$$

Hence *Stoke's theorem* takes the form $\int_C (\phi\,dx + \psi\,dy) = \int_C \left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right) dxdy$ which is *Green's theorem* in a plane.

Example 8.36. *Verify Stoke's theorem for* $\mathbf{F} = (x^2 + y^2)\mathbf{I} - 2xy\mathbf{J}$ *taken around the rectangle bounded by the lines* $x = \pm a$, $y = 0$, $y = b$. *(Bhopal, 2008 S ; V.T.U., 2007 ; J.N.T.U., 2003 ; U.P.T.U., 2003)*

Solution. Let $ABCD$ be the given rectangle as shown in Fig. 8.16.

$$\int_{ABCD} \mathbf{F}\cdot d\mathbf{R} = \int_{AB} \mathbf{F}\cdot d\mathbf{R} + \int_{BC} \mathbf{F}\cdot d\mathbf{R} + \int_{CD} \mathbf{F}\cdot d\mathbf{R} + \int_{DA} \mathbf{F}\cdot d\mathbf{R}$$

and

$$\mathbf{F}\cdot d\mathbf{R} = [(x^2 + y^2)\mathbf{I} - 2xy\mathbf{J}]\cdot(\mathbf{I}dx + \mathbf{J}dy) = (x^2 + y^2)dx - 2xydy$$

Along AB, $x = a$ (*i.e.*, $dx = 0$) and y varies from 0 to b.

$$\therefore \quad \int_{AB} \mathbf{F}\cdot d\mathbf{R} = -2a\int_0^b y\,dy = -2a\cdot\frac{b^2}{2} = -ab^2.$$

Similarly, $$\int_{BC} \mathbf{F}\cdot d\mathbf{R} = \int_a^{-a} (x^2+b^2)dx = -\frac{2a^3}{3} - 2ab^2 \cdot$$

$$\int_{CD} \mathbf{F}\cdot d\mathbf{R} = 2a\int_b^0 y\,dy = -ab^2$$

and $$\int_{DA} \mathbf{F}\cdot d\mathbf{R} = \int_{-a}^{a} x^2\,dx = \frac{2a^3}{3}\cdot$$

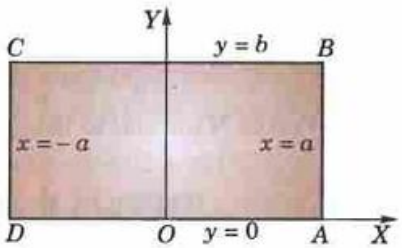

Fig. 8.16

Thus $$\int_{ABCD} \mathbf{F}\cdot d\mathbf{R} = -4ab^2 \qquad ...(i)$$

Also since $\text{curl}\,\mathbf{F} = -4\mathbf{K}y$

$$\therefore \quad \int_S \text{curl}\,\mathbf{F}\cdot\mathbf{N}\,ds = \int_0^b\int_{-a}^{a} -4\mathbf{K}y\cdot\mathbf{K}\,dxdy = -4\int_0^b\int_{-a}^{a} y\,dxdy$$

$$= -4\int_0^b |x|_{-a}^{a}\, y\,dy = -8a\left|\frac{y^2}{2}\right|_0^b = -4ab^2 \qquad ...(ii)$$

Hence Stoke's theorem is verified from the equality of (*i*) and (*ii*).

Example 8.37. *Verify Stoke's theorem for the vector field $F = (2x-y)\,\mathbf{I} - yz^2\,\mathbf{J} - y^2z\,\mathbf{K}$ over the upper half surface of $x^2+y^2+z^2 = 1$, bounded by its projection on the xy-plane.*

(Bhopal, 2008 ; Madras, 2006 ; S.V.T.U., 2006)

Solution. The projection of the upper half of given sphere on the xy-plane ($z = 0$) is the circle $c[x^2+y^2 = 1]$ (Fig. 8.17).

$$\oint_c \mathbf{F}.\,d\mathbf{R} = \oint_c [(2x-y)dx - yz^2\,dy - y^2\,zdz] = \oint_c (2x-y)dx \qquad [z = 0 \text{ in the } xy\text{-plane}]$$

$$= \int_{\theta=0}^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta\,d\theta) \qquad [\text{Putting } x = \cos\theta, y = \sin\theta]$$

$$= \int_0^{2\pi} (-\sin 2\theta + \sin^2\theta)\,d\theta = \text{x}\,0 + 4\int_0^{\pi/2} \sin^2\theta\,d\theta = \pi. \qquad ...(i)$$

Now $$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$= (-2yz + 2yz)\,\mathbf{I} + 0\,\mathbf{J} + \mathbf{K} = \mathbf{K}$$

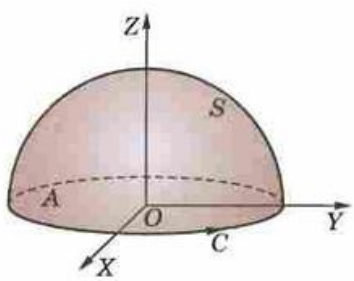

Fig. 8.17

$$\therefore \quad \int \text{curl}\,\mathbf{F}.\,Nds = \int_S K.\mathbf{N}\,ds = \int_A \mathbf{K}.\mathbf{N}\frac{dxdy}{|\mathbf{N}.\mathbf{K}|}$$

where A is the projection of S on xy-plane and $ds = dxdy/\mathbf{N}.\mathbf{K}$

$$= \int_A dx\,dy = \text{area of circle } C = \pi \qquad ...(ii)$$

Hence, the Stokes theorem is verified from the equality of (*i*) and (*ii*).

Example 8.38. *Uses Stoke's theorem evaluate $\int_C [(x+y)dx + (2x-z)dy + (y+z)dz]$ where C is the boundary of the triangle with vertices (2, 0, 0), (0, 3, 0) and (0, 0, 6).*

(Nagpur, 2009 ; Kurukshetra, 2009 S ; Kerala, 2005)

Solution. Here $$\mathbf{F} = (x+y)\,\mathbf{I} + (2x-z)\,\mathbf{J} + (y+z)\,\mathbf{K}$$

$$\therefore \quad \text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix} = 2\mathbf{I} + \mathbf{K}$$

Also equation of the plane through A, B, C (Fig. 8.18) is

$$\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=1 \text{ or } 3x+2y+z=6$$

Vector $\mathbf{N}$ normal to this plane is

$$\nabla(3x+2y+z-6)=3\mathbf{I}+2\mathbf{J}+\mathbf{K}$$

$$\therefore \quad \hat{\mathbf{N}}=\frac{3\mathbf{I}+2\mathbf{J}+\mathbf{K}}{\sqrt{(9+4+1)}}=\frac{1}{\sqrt{14}}(3\mathbf{I}+2\mathbf{J}+\mathbf{K})$$

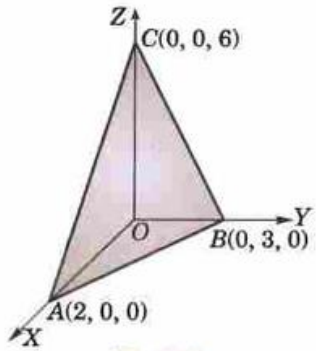

Fig. 8.18

Hence $\int_C [(x+y)dx+(2x-z)dy+(y+z)dz]=\int_C \mathbf{F}\,.\,d\mathbf{R}$

$$=\int_S \text{curl}\,\mathbf{F}\,.\,\hat{\mathbf{N}}\,ds \qquad \text{where } S \text{ is the triangle } ABC$$

$$=\int_S (2\mathbf{I}+\mathbf{K})\,.\left(\frac{3\mathbf{I}+2\mathbf{J}+\mathbf{K}}{\sqrt{14}}\right)ds=\frac{1}{\sqrt{14}}(6+1)\int_S ds$$

$$=\frac{7}{\sqrt{14}}\,(\text{Area of }\Delta ABC)=\frac{7}{\sqrt{14}}\,.\,3\sqrt{14}=21.$$

Example 8.39. *If $\mathbf{F}=3y\mathbf{I}-xz\mathbf{J}+yz^2\mathbf{K}$ and S is the surface of the paraboloid $2z=x^2+y^2$ bounded by $z=2$, evaluate $\iint_S (\nabla\times\mathbf{F})\,.\,d\mathbf{S}$ using Stoke's theorem.*

Solution. By Stokes theorem, $I=\iint_S (\nabla\times\mathbf{F})\,.\,d\mathbf{S}=\oint_C \mathbf{F}\,.\,d\mathbf{R}$

where S is the surface $2z=x^2+y^2$ bounded by $z=2$.

$$\therefore \quad I=\oint_C \mathbf{F}\,.\,d\mathbf{R}=\oint_C (3yI-xzJ+yz^2K)\,.\,(dxI+dyJ+dzK)$$

$$=\oint_C (3ydx-xzdy+yz^2dz)$$

$\because\ S\equiv x^2+y^2=4,\ z=2$

$\therefore$ Put $x=2\cos\theta,\ y=2\sin\theta$

$C\equiv x^2+y^2=4,\ \theta=0$ to 2π.

$$=\int_0^{2\pi}[6\sin\theta(-2\cos\theta\,d\theta)-4\cos\theta(2\cos\theta\,d\theta)+8\sin\theta(0)]$$

$$=-4\int_0^{2\pi}(12\sin^2\theta+8\cos^2\theta)\,d\theta$$

$$=-4\left(12\,.\,\frac{1}{2}\frac{\pi}{2}+8\,.\,\frac{1}{2}\,.\,\frac{\pi}{2}\right)=-20\pi.$$

Example 8.40. *Apply Stoke's theorem to evaluate $\int_C (ydx+zdy+xdz)$ where C is the curve of intersection of $x^2+y^2+z^2=a^2$ and $x+z=a$.* (Bhopal, 2008)

Solution. The curve C is evidently a circle lying in the plane $x+z=a$, and having $A(a, 0, 0)$, $B(0, 0, a)$ as the extremities of the diameter (Fig. 8.19).

$$\therefore \quad \int_C (y\,dx+z\,dy+x\,dz)=\int_C (y\mathbf{I}+z\mathbf{J}+x\mathbf{K})\,.\,d\mathbf{R}$$

$$=\int_S \text{curl}\,(y\mathbf{I}+z\mathbf{J}+x\mathbf{K})\,.\,\mathbf{N}ds$$

where S is the circle on AB as diameter and $\mathbf{N}=\frac{1}{\sqrt{2}}\mathbf{I}+\frac{1}{\sqrt{2}}\mathbf{K}$

$$=\int_S -(\mathbf{I}+\mathbf{J}+\mathbf{K})\,.\left(\frac{1}{\sqrt{2}}\mathbf{I}+\frac{1}{\sqrt{2}}\mathbf{K}\right)ds$$

$$=-\frac{2}{\sqrt{2}}\int_S ds=-\frac{2}{\sqrt{2}}\pi\left(\frac{a}{\sqrt{2}}\right)^2=-\frac{\pi a^2}{\sqrt{2}}.$$

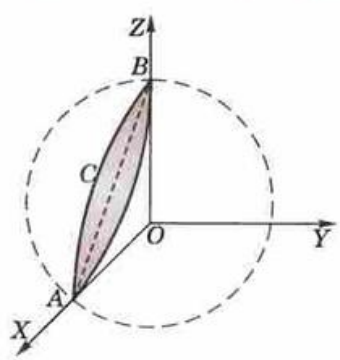

Fig. 8.19

Example 8.41. *If S be any closed surface, prove that* $\int_S \text{curl } \mathbf{F} . d\mathbf{S} = 0$.

Solution. Cut open the surface S by any plane and let S_1, S_2 denote its upper and lower portions. Let C be the common curve bounding both these portions.

$$\therefore \quad \int_S \text{curl } \mathbf{F} . d\mathbf{S} = \int_{S_1} \text{curl } \mathbf{F} . d\mathbf{S} + \int_{S_2} \text{curl } \mathbf{F} . d\mathbf{S} = \int_C \mathbf{F} . d\mathbf{R} - \int_C \mathbf{F} . d\mathbf{R} = 0,$$

on applying Stoke's theorem. The second integral is negative because it is traversed in a direction opposite to that of the first.

PROBLEMS 8.9

1. Verify Stoke's theorem for the vector field (*i*) $\mathbf{F} = (x^2 - y^2)\mathbf{I} + 2xy\mathbf{J}$ over the box bounded by the planes $x = 0, x = a$; $y = 0, y = b; z = 0, z = c$; if the face $z = 0$ is cut. *(B.P.T.U., 2006 ; Delhi, 2002)*
   (*ii*) $\mathbf{F} = (z^2, 5x, 0)$ and $S : 0 \le x \le 1, 0 \le y \le 1, z = 1$.
2. Verify Stoke's theorem for a vector field defined by $\mathbf{F} = -y^3\mathbf{I} + x^3\mathbf{J}$, in the region $x^2 + y^2 \le 1, z = 0$.
3. Evaluate $\int_c \mathbf{F} . d\mathbf{R}$ where $\mathbf{F} = (x^2 + y^2)\mathbf{I} - 2xy\,\mathbf{J}$ and c is the rectangle in the xy-plane bounded by $y = 0, x = a, y = b$, $x = 0$. *(Mumbai, 2007)*
4. Verity Stoke's theorem for $\mathbf{F} = (y - z + 2)\mathbf{I} + (yz + 4)\mathbf{J} - xz\mathbf{K}$ where S is the surface of the cube $x = 0, y = 0, z = 0$, $x = 2, y = 2, z = 2$ above the xy-plane. *(Andhra, 2000)*
5. Evaluate $\int_C \mathbf{F} . d\mathbf{R}$ where $\mathbf{F} = y\mathbf{I} + xz^3\mathbf{J} - zy^3\mathbf{K}$, C is the circle $x^2 + y^2 = 4, z = 1.5$.
6. Evaluate by Stoke's theorem $\oint_C (yz\,dz + zx\,dy + zx\,dz)$ where C is the curve $x^2 + y^2 = 1, z = y^2$. *(J.N.T.U., 2005)*
7. If S be the surface of the sphere $x^2 + y^2 + z^2 = 1$, prove that $\int_S \text{curl } \mathbf{F} . d\mathbf{S} = 0$. *(J.N.T.U., 1999)*
8. Prove that $\int_C A \times \mathbf{R} . d\mathbf{R} = 2\mathbf{A} . \int d\mathbf{S}$, $\mathbf{A}$ being any constant vector, and deduce that $\oint_C \mathbf{R} \times d\mathbf{R}$ is twice the vector area of the surface enclosed by C.
9. If ϕ is a scalar point function, use Stoke's theorem to prove that (*i*) curl (grad ϕ) = 0. (*ii*) div curl $\mathbf{F} = 0$. *(Kerala, 2005)*
10. Evaluate $\oint_C (\sin z\,dx - \cos x\,dy + \sin y\,dz)$ where C is the boundary of the rectangle $0 \le x \le \pi$. $0 \le y \le 1, z = 3$. *(Rohtak, 2005)*
11. Use Stoke's theorem to evaluate $(\nabla \times \mathbf{F}) . \mathbf{N}\,ds$, where $\mathbf{F} = y\mathbf{I} + (x - 2xz)\,\mathbf{J} - xy\mathbf{K}$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy-plane. *(Kottayam, 2005)*
12. Evaluate $\int \nabla \times \mathbf{V} . d\mathbf{S}$ over the surface of the paraboloid $z = 1 - x^2 - y^2, z \ge 0$ where $\mathbf{V} = y\mathbf{I} + z\mathbf{J} + x\mathbf{K}$.

8.15 VOLUME INTEGRAL

Consider a continuous vector function $\mathbf{F}(\mathbf{R})$ and surface S enclosing the region E. Divide E into finite number of sub-regions $E_1, E_2, \ldots, E_n$. Let δv_i be the volume of the sub-region E_i enclosing any point whose position vector is $\mathbf{R}_i$.

Consider the sum $$\mathbf{V} = \sum_{i=1}^{n} \mathbf{F}(\mathbf{R}_i)\,\delta v_i$$

The limit of this sum as $n \to \infty$ *in such a way that* $\delta v_i \to 0$, *is called the volume integral of* $\mathbf{F}(\mathbf{R})$ *over* E *and is symbolically written as* $\int_E \mathbf{F}\,dv$.

If $\mathbf{F}(\mathbf{R}) = f(x, y, z)\mathbf{I} + \phi(x, y, z)\mathbf{J} + \psi(x, y, z)\mathbf{K}$ so that $\delta v = \delta x \delta y \delta z$, then

$$\int_E \mathbf{F}\,dv = \mathbf{I} \iiint_E f dxdydz + \mathbf{J} \iiint_E \phi dxdydz + \mathbf{K} \iiint_E \psi dxdydz.$$

8.16 GAUSS DIVERGENCE THEOREM* (*Relation between surface and volume integrals*)

If **F** is a continuously differentiable vector function in the region E bounded by the closed surface S, then

$$\int_S \mathbf{F} \cdot \mathbf{N} ds = \int_E div\, \mathbf{F}\, dv$$

where **N** *is the unit external normal vector.*

If $\mathbf{F}(\mathbf{R}) = f(x, y, z)\mathbf{I} + \phi(x, y, z)\mathbf{J} + \psi(x, y, z)\mathbf{K}$

then it is required to prove that

$$\iint_S (f dydz + \phi dzdx + \psi dxdy)$$

$$= \iiint_E \left(\frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial z} \right) dxdydz \qquad ...(1)$$

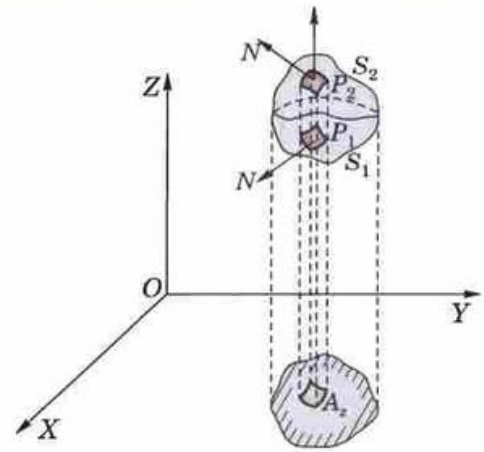

Fig. 8.20

Firstly consider such a surface S that a line parallel to z-axis cuts it in two points; say $P_1(x, y, z_1)$ and $P_2(x, y, z_2)$ $(z_1 \le z_2)$ (Fig. 8.20).

If S projects into the area A_z on the xy-plane, then

$$\iiint_E \frac{\partial \psi}{\partial z}\, dxdydz = \iint_{A_z} dxdy \int_{z_1}^{z_2} \frac{\partial \psi}{\partial z}\, dz$$

$$= \iint_{A_z} [\Psi(x, y, z_2) - \Psi(x, y, z_1)]\, dxdy = \iint_{A_z} \Psi(x, y, z_2)\, dxdy - \iint_{A_z} \Psi(x, y, z_1)\, dxdy \qquad ...(2)$$

Let S_1, S_2 be the lower and upper parts of the surface S corresponding to the points P_1 and P_2 respectively and **N** be the unit external normal vector at any point of S. As the external normal at any point of S_2 makes an acute angle with the positive direction of z-axis and that at any point of S_1 an obtuse angle, therefore

$$\iint_{A_z} \Psi(x, y, z_2)\, dxdy = \int_{S_2} \psi \mathbf{N} \cdot \mathbf{K} ds \qquad ...(3)$$

$$\iint_{A_z} \Psi(x, y, z_1)\, dxdy = -\int_{S_1} \psi \mathbf{N} \cdot \mathbf{K} ds \qquad ...(4)$$

Using (3) and (4), (2) now becomes

$$\iiint_E \frac{\partial \psi}{\partial z}\, dxdydz = \int_{S_2} \psi \mathbf{N} \cdot \mathbf{K} ds + \int_{S_1} \psi \mathbf{N} \cdot \mathbf{K} ds = \int_S \psi \mathbf{N} \cdot \mathbf{K} ds \qquad ...(5)$$

Similarly, we have

$$\iiint_E \frac{\partial f}{\partial x}\, dxdydz = \int_S f \mathbf{N} \cdot \mathbf{I} ds \qquad ...(6)$$

$$\iiint_E \frac{\partial \phi}{\partial y}\, dxdydz = \int_S \phi \mathbf{N} \cdot \mathbf{J} ds \qquad ...(7)$$

Addition of (5), (6) and (7) gives

$$\iiint_E \left(\frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial z} \right) dxdydz = \int_S (f\mathbf{I} + \phi\mathbf{J} + \psi\mathbf{K}) \cdot \mathbf{N} ds \text{ which is same as (1).}$$

Secondly, consider a general region E. Assume that it can be split up into a finite number of sub-regions each of which is met by a line parallel to any axis in only two points. Applying (1) to each of these sub-regions and adding the results, the volume integrals will combine to give the volume integral over the whole region E. Also the surface integrals over the common boundaries of two sub-regions cancel because each occurs twice and having corresponding normals in opposite directions whereas the remaining surface integrals combine to give the surface integral over the entire surface S.

Finally consider a region E bounded by two closed surfaces S_1, S_2 (S_1 being within S_2). Noting that outward normal at points of S_1 is directed inwards (*i.e.*, away from S_2) and introducing an additional surface cutting S_1, S_2 so that all parts of E are bounded by a single closed surface, the truth of the theorem follows as before. Thus theorem also holds for regions enclosed by several surfaces.

Hence the theorem is completely established.

*See footnote p. 37.

Example 8.42. *Verify Divergence theorem for* $\mathbf{F} = (x^2 - yz)\mathbf{I} + (y^2 - zx)\mathbf{J} + (z^2 - xy)\mathbf{K}$ *taken over the rectangular parallelepiped* $0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c$. *(Rohtak, 2006 S ; Madras, 2000 S)*

Solution. As
$$\text{div } \mathbf{F} = \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy)$$
$$= 2(x + y + z)$$

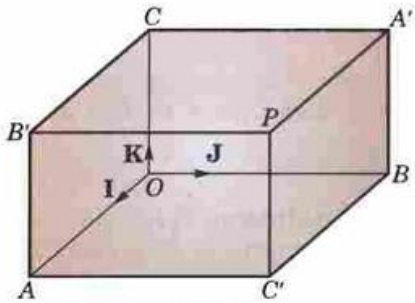

Fig. 8.21

$$\therefore \quad \int_R \text{div } \mathbf{F}\, dv = 2\int_0^c \int_0^b \int_0^a (x + y + z)\, dxdydz$$
$$= 2\int_0^c dz \int_0^b dy\left(\frac{a^2}{2} + ya + za\right)$$
$$= 2\int_0^c dz\left(\frac{a^2}{2}b + \frac{ab^2}{2} + abz\right)$$
$$= 2\left(\frac{a^2b}{2}c + \frac{ab^2}{2}c + ab\frac{c^2}{2}\right)$$
$$= abc\,(a + b + c) \qquad \ldots(i)$$

Also
$$\int_S \mathbf{F}\,.\,\mathbf{N}ds = \int_{S_1} \mathbf{F}\,.\,\mathbf{N}ds + \int_{S_2} \mathbf{F}\,.\,\mathbf{N}ds + \ldots + \int_{S_6} \mathbf{F}\,.\,\mathbf{N}ds$$
where S_1 in the face $OAC'B$, S_2 the face $CB'PA'$, S_3 the face $OBA'C$, S_4 the face $AC'PB'$, S_5 the face $OCB'A$ and S_6 the face $BAP'C'$ (Fig. 8.21).

Now
$$\int_{S_1} \mathbf{F}\,.\,\mathbf{N}ds = \int_{S_1} \mathbf{F}\,.\,(-\mathbf{K})\, ds = -\int_0^b \int_0^a (0 - xy)\, dxdy = \frac{a^2b^2}{4}$$
$$\int_{S_2} \mathbf{F}\,.\,\mathbf{N}ds = \int_{S_2} \mathbf{F}\,.\,\mathbf{K}ds = \int_0^b \int_0^a (c^2 - xy)\, dxdy = abc^2 - \frac{a^2b^2}{4}$$

Similarly,
$$\int_{S_3} \mathbf{F}\,.\,\mathbf{N}ds = \frac{b^2c^2}{4},\ \int_{S_4} \mathbf{F}\,.\,\mathbf{N}ds = a^2bc - \frac{b^2c^2}{4},$$
$$\int_{S_5} \mathbf{F}\,.\,\mathbf{N}ds = \frac{c^2a^2}{4} \text{ and } \int_{S_6} \mathbf{F}\,.\,\mathbf{N}ds = ab^2c - \frac{c^2a^2}{4}$$

Thus
$$\int_S \mathbf{F}\,.\,\mathbf{N}ds = abc(a + b + c) \qquad \ldots(ii)$$

Hence the theorem is verified from the equality of (i) and (ii).

Example 8.43. *Evaluate* $\int_S \mathbf{F}\,.\,ds$ *where* $\mathbf{F} = 4x\mathbf{I} - 2y^2\mathbf{J} + z^2\mathbf{K}$ *and S is the surface bounding the region* $x^2 + y^2 = 4$, $z = 0$ *and* $z = 3$. *(S.V.T.U., 2007 S ; Mumbai, 2006 ; J.N.T.U., 2006)*

Solution. By divergence theorem,
$$\int_S \mathbf{F}\,.\,d\mathbf{s} = \int_V div\, \mathbf{F}\, dv$$
$$= \int_V \left[\frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)\right] dv$$
$$= \iiint_V ((4 - 4y + 2z)\, dxdydz$$
$$= \int_{-2}^{2} \int_{\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} \int_0^3 (4 - 4y + 2z)\, dzdydx$$
$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} \left|4z - 4yz + z^2\right|_0^3 dydx$$
$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} (12 - 12y + 9)\, dydx$$

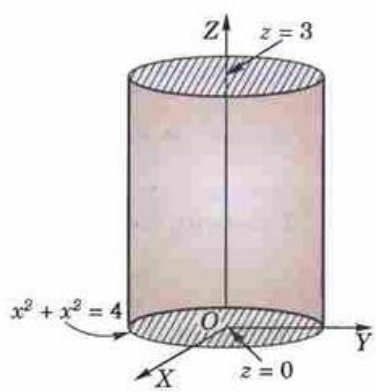

Fig. 8.22

$$= \int_{-2}^{2} \left| 21y - 6y^2 \right|_{-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} dx$$

$$= 42 \int_{-2}^{2} \sqrt{(4-x^2)}\, dx = 84 \int_{0}^{2} \sqrt{(4-x^2)}\, dx = 84 \left| \frac{x\sqrt{(4-x^2)}}{2} + \frac{4}{2} \sin^{-1} \frac{x}{2} \right|_0^2 = 84\pi.$$

Example 8.44. *Evaluate* $\int_S (yz\mathbf{I} + zx\mathbf{J} + xy\mathbf{K}) \,.\, d\mathbf{S}$ *where S is the surface of the sphere* $x^2 + y^2 + z^2 = a^2$ *in the first octant.* *(U.P.T.U., 2004 S)*

Solution. The surface of the region V: $OABC$ is piecewise smooth (Fig. 8.23) and is comprised of four surfaces (*i*) S_1 – circular quadrant OBC in the yz-plane,

(*ii*) S_2 – circular quadrant OCA in the zx-plane,

(*iii*) S_3 – circular quadrant OAB in the xy-plane,

and (*iv*) S–surface ABC of the sphere in the first octant.

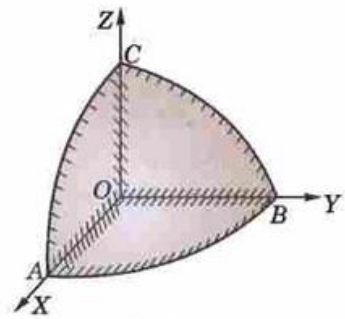

Fig. 8.23

Also $\mathbf{F} = yz\mathbf{I} + zx\mathbf{J} + xy\mathbf{K}$

By Divergence theorem,

$$\int_V div\, \mathbf{F}\, dv = \int_{S_1} \mathbf{F} \,.\, d\mathbf{S} + \int_{S_2} \mathbf{F} \,.\, d\mathbf{S} + \int_{S_3} \mathbf{F} \,.\, d\mathbf{S} + \int_S \mathbf{F} \,.\, d\mathbf{S} \quad ...(1)$$

Now $$div\, \mathbf{F} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(zx) + \frac{\partial}{\partial z}(xy) = 0.$$

For the surface S_1, $x = 0$

$$\therefore \quad \int_{S_1} \mathbf{F} \,.\, d\mathbf{S} = \int_0^a \int_0^{\sqrt{(a^2-y^2)}} (yz\mathbf{I}) \,.\, (-\, dydz\mathbf{I}) = -\int_0^a \int_0^{\sqrt{(a^2-y^2)}} yzdydz = = -\frac{a^4}{8}$$

Thus (1) becomes $0 = -\dfrac{3a^4}{8} + \int_S \mathbf{F} \,.\, d\mathbf{S}$ whence $\int_S \mathbf{F} \,.\, d\mathbf{S} = 3a^4/8$.

Example 8.45. *Apply divergence theorem to evaluate* $\int (lx^2 + my^2 + nz^2)\, ds$ *taken over the sphere* $(x-a)^2 + (y-b)^2 + (z-c)^2 = \rho^2$; *l, m, n being the direction cosines of the external normal to the sphere.*

Solution. The parametric equations of the sphere are $x = a + \rho \sin\theta \cos\phi$, $y = b + \rho \sin\theta \sin\phi$, $z = c + \rho \cos\theta$ and to cover the whole sphere, r varies from 0 to ρ, θ varies from 0 to π and ϕ from 0 to 2π.

$$\therefore \quad \int_S (lx^2 + my^2 + nz^2)\, ds = \int_S (x^2\mathbf{I} + y^2\mathbf{J} + z^2\mathbf{K}) \,.\, \mathbf{N}ds$$

$$= \int_V \text{div}\, (x^2\mathbf{I} + y^2\mathbf{J} + z^2\mathbf{K}) dv = 2 \int_V (x + y + z) dv$$

$$= 2 \int_0^{2\pi} \int_0^{\pi} \int_0^{\rho} [(a + b + c) + \rho(\sin\theta \cos\phi + \sin\theta \sin\phi + \cos\theta)] \times \rho^2 \sin\theta\, drd\theta d\phi$$

$$= 2(a + b + c) \frac{\rho^3}{3} \left| -\cos\theta \right|_0^{\pi} .\ 2\pi = \frac{8\pi}{3} (a + b + c)\, \rho^3.$$

Example 8.46. *Evaluate* $\int_S (a^2x^2 + b^2y^2 + c^2z^2)^{-1/2}\, dS$, *where S is the surface of the ellipsoid*

$$ax^2 + by^2 + cz^2 = 1.$$

Solution. Taking $\phi = ax^2 + by^2 + cz^2 - 1 = 0$, $\nabla\phi = 2ax\mathbf{I} + 2by\mathbf{J} + 2cz\mathbf{K}$

$\therefore$ Unit vector normal to the ellipsoid $= \hat{N} = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{ax\mathbf{I} + by\mathbf{J} + cz\mathbf{K}}{\sqrt{(a^2x^2 + b^2y^2 + c^2z^2)}}$

Since $\mathbf{F} \,.\, \hat{\mathbf{N}} = (a^2x^2 + b^2y^2 + c^2z^2)^{-1/2}$, $\quad \therefore \mathbf{F} \,.\, (ax\mathbf{I} + by\mathbf{J} + cz\mathbf{K}) = 1$

Obviously $\mathbf{F} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$ $[\because ax^2 + by^2 + cz^2 = 1$

$\therefore$ By Divergence theorem,

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_V div\, \mathbf{F} dv = \int_V \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) \right] dv = 3 \int_V dv = 3V$$

$$= 3 \cdot \frac{4\pi}{3} \frac{1}{\sqrt{(abc)}} = \frac{4\pi}{\sqrt{(abc)}}. \qquad \left[\because \text{ Vol. of ellipsoid} = \frac{4\pi}{3} \frac{1}{\sqrt{(abc)}}\right]$$

Example 8.47. *If the position vector of any point (x, y, z) within a closed surface S, be* **R** *measured from an origin O, then show that*

$$\iint_S \frac{\mathbf{R}}{r^3} \cdot \mathbf{N}\, ds = \begin{cases} 0, \text{ if } O \text{ lies outside } S \\ 4\pi, \text{ if } O \text{ lies inside } S \end{cases}$$

Solution. (*a*) *When O is outside S.* Here $\mathbf{F} = \mathbf{R}/r^3$ is continuously differentiable throughout the volume V enclosed by S. Hence by Divergence theorem, we have

$$\iint_S \frac{\mathbf{R}}{r^3} \cdot \mathbf{N}\, ds = \iiint_V \text{div}\left(\frac{\mathbf{R}}{r^3}\right) dV = 0 \qquad \left[\because \text{div}\left(\frac{\mathbf{R}}{r^3}\right) = 0\right.$$

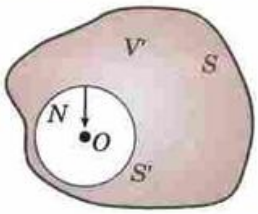

Fig. 8.24

(*b*) *When O is inside S.* Hence $F = \mathbf{R}/r^3$ has a point of discontinuity at O and as such Divergence theorem cannot be applied to the region V enclosed by S. To remove this point of discontinuity, we enclose O by a small sphere S' of radius ρ.

Now **F** is continuously differentiable throughout the region V' enclosed between S and S'. Therefore applying Divergence theorem to region V', we get

$$\iint_S \frac{\mathbf{R}}{r^3} \cdot \mathbf{N}\, ds + \iint_{S'} \frac{\mathbf{R}}{r^3} \cdot \mathbf{N}\, ds' = \iiint_{V'} \text{div}\left(\frac{\mathbf{R}}{r^3}\right) dV' = 0 \qquad \left[\because \text{div}\left(\frac{\mathbf{R}}{r^3}\right) = 0\right]$$

$$\therefore \qquad \iint_S \frac{\mathbf{R}}{r^3} \cdot \mathbf{N}\, ds = -\iint_S \frac{\mathbf{R}}{r^3} \cdot \mathbf{N}\, ds' \qquad ...(i)$$

Now the outward normal **N** on the sphere S' is directed towards the centre O. Therefore $\mathbf{N} = -\mathbf{R}/\rho$ on S' (Fig. 8.24).

$$\therefore \qquad -\iint_{S'} \frac{\mathbf{R}}{r^3} \cdot \mathbf{N}\, ds' = -\iint_{S'} \frac{\mathbf{R}}{\rho^3} \cdot \left(-\frac{\mathbf{R}}{\rho}\right) ds' \qquad [\because \text{ on } S', r = \rho]$$

$$= \iint_{S'} \frac{r^2}{\rho^4} ds' = \iint_{S'} \frac{\rho^2}{\rho^4} ds' = \frac{1}{\rho^2} \iint_{S'} ds' = \frac{1}{\rho^2} \cdot 4\pi\rho^2 = 4\pi$$

Hence from (*i*), $\iint_S \frac{\mathbf{R}}{r^3} \cdot \mathbf{N}\, ds = 4\pi.$

8.17 GREEN'S THEOREM*

If ϕ and ψ are scalar point functions possessing continuous derivatives of first and second orders, then

$$\int_E (\phi\nabla^2\psi - \psi\nabla^2\phi) dv = \int_S \left(\phi \frac{\partial\psi}{\partial n} - \psi \frac{\partial\psi}{\partial n} \right) ds \qquad ...(1)$$

where $\partial/\partial n$ denotes differentiation in the direction of the external normal to the bounding surface S enclosing the region E.

Applying Divergence theorem : $\int_S \mathbf{F} \cdot \mathbf{N} ds = \int_S \nabla \cdot \mathbf{F} dv$ to the function $\phi\nabla\psi$, we get

$$\int_S \phi\nabla\psi \cdot \mathbf{N} ds = \int_E \nabla \cdot (\phi\nabla\psi) dv = \int_E (\nabla\phi \cdot \nabla\psi + \phi\nabla^2\psi) dv \qquad \text{[By (2) page 329]}$$

$$= \int_E \nabla\phi \cdot \nabla\psi dv + \int_E \phi\nabla^2\psi dv \qquad ...(2)$$

*See footnote p. 339.

Interchanging ϕ and ψ, (*ii*) gives

$$\int_S \psi\nabla\phi \,.\, \mathbf{N}ds = \int_E \nabla\psi \,.\, \nabla\phi dv + \int_E \psi\nabla^2\phi dv \qquad ...(3)$$

Subtracting (3) from (2), we have $\int_S (\phi\nabla\psi - \psi\nabla\phi) \,.\, \mathbf{N}ds = \int_E (\phi\nabla^2\psi - \psi\nabla^2\phi)dv$

But $\nabla\psi \,.\, \mathbf{N} = \partial\psi/\partial n$ the directional derivative of ψ along the external normal at any point of S. Hence

$$\int_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)ds = \int_E (\phi\nabla^2\psi - \psi\nabla^2\phi)\,dv \text{ which is the required result (1).}$$

Obs. Harmonic function. *A scalar point function ϕ satisfying the Laplace's equation $\nabla^2\phi = 0$ at every point of a region E, is called a harmonic function in E.*

If ϕ and ϕ be both harmonic functions in E, (1) gives

$$\int_S \phi\,\frac{\partial\psi}{\partial n}ds = \int_S \psi\,\frac{\partial\phi}{\partial n}\,ds \text{ which is known as } \textbf{Green's reciprocal theorem.}$$

PROBLEMS 8.10

1. Verify divergence theorem for **F** taken over the cube bounded by $x = 0, x = 1 ; y = 0, y = 1 ; z = 0, z = 1$ where
   (*i*) $\mathbf{F} = 4xz\mathbf{I} - y^2\mathbf{J} + yz\mathbf{K}$ (*Madras, 2006*) (*ii*) $x^2\mathbf{I} + z\mathbf{J} + yz\mathbf{K}$ (*Bhopal, 2008*)
2. Verify Gauss divergence theorem for the function $\mathbf{F} = y\mathbf{I} + x\mathbf{J} + z^2\mathbf{K}$ over the cylindrical region bounded by $x^2 + y^2 = 9, z = 0$ and $z = 2$.
3. Using divergence theorem, prove that
   (*i*) $\int_S \mathbf{R} \,.\, d\mathbf{S} = 3V$ (*ii*) $\int_S \nabla r^2 \,.\, d\mathbf{S} = 6V$ (*U.P.T.U., 2003*)
   where S is any closed surface enclosing a volume V and $r^2 = x^2 + y^2 + z^2$.
4. Using divergence theorem, evaluate $\int_S \mathbf{R} \,.\, \mathbf{N}ds$ where S is the surface of the sphere $x^2 + y^2 + z^2 = 9$.
5. If S is any closed surface enclosing a volume V and $\mathbf{F} = ax\mathbf{I} + by\mathbf{J} + cz\mathbf{K}$, prove that
   $$\int_S \mathbf{F} \,.\, \mathbf{N}ds = (a + b + c)V \qquad \text{(Madras, 2003)}$$
6. For any closed surface S, prove that $\int [x(y - z)\mathbf{I} + y(z - x)\mathbf{J} + z(x - y)\mathbf{K}] \,.\, ds = 0$.
7. Use divergence theorem to evaluate $\int_S \mathbf{F} \,.\, d\mathbf{S}$, where
   (*i*) $\mathbf{F} = x^3\mathbf{I} + y^3\mathbf{J} + z^3\mathbf{K}$, and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$. (*V.T.U., 2008 ; P.T.U., 2005*)
   (*ii*) $\mathbf{F} = [e^x, e^y, e^z]$ and S is the surface of the cube $|x| \le 1, |y| \le 1, |z| \le 1$. (*B.P.T.U., 2005*)
8. Evaluate $\iint (xdydz + ydzdx + zdxdy)$ over the surface of a sphere of radius a. (*Kurukshetra, 2008 S*)
9. Evaluate $\int_S \mathbf{F} \,.\, d\mathbf{S}$ where $F = y^2z^2\mathbf{I} + z^2x^2\mathbf{J} + x^2y^2\mathbf{K}$ and S is the upper part of the sphere $x^2 + y^2 + z^2 = a^2$ above XOY plane.
10. By transforming to triple integral, evaluate $\iint_S (x^3dydz + x^2ydzdx + x^2zdxdy)$ where S is the closed surface consisting of the cylinder $x^2 + y^2 = a^2$ and the circular discs $z = 0$ and $z = b$. (*Burdwan, 2003*)
11. Evaluate $\int_S (\nabla \times \mathbf{F}) \,.\, d\mathbf{S}$, where S is the surface of the paraboloid $x^2 + y^2 + z = 4$ above the xy-plane, and $\mathbf{F} = (x^2 + y - 4)\mathbf{I} + 3xy\mathbf{J} + (2xz + z^2)\mathbf{K}$.
12. If $\mathbf{F} = (2x^2 - 3z)\mathbf{I} - 2xy\mathbf{J} - 4x\mathbf{K}$, then evaluate $\iiint_V \nabla \,.\, \mathbf{F}\, dv$, where V is bounded by $x = y = z = 0$ and $2x + 2y + z = 4$. (*Bhopal, 2008*)
13. If $\mathbf{F} = \text{grad}\,\phi$ and $\nabla^2\phi = -4\pi\rho$, prove that $\int_S \mathbf{F} \,.\, \mathbf{N}ds = -4\pi\rho \int_V dV$ where the symbol have their usual meanings.

8.18 (1) IRROTATIONAL FIELDS

An irrotational field **F** *is characterised by any one of the following conditions :*

(*i*) $\Delta \times \mathbf{F} = \mathbf{0}$. (*ii*) *Circulation* $\int \mathbf{F} \cdot d\mathbf{R}$ *along every closed surface is zero.*

(*iii*) $\mathbf{F} = \Delta\phi$, *if the domain is simply connected.**

If $\nabla \times \mathbf{F} = \mathbf{0}$, then by Stoke's theorem,

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \mathbf{0}, \textit{ i.e.,} \text{ the circulation along every closed surface is zero.}$$

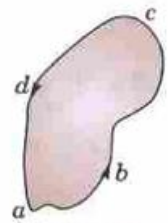

Fig. 8.25

Again since $\nabla \times \nabla\phi = 0$

$\therefore$ in an irrotational field for which $\Delta \times \mathbf{F} = 0$, the vector **F** can always be expressed as the gradient of a scalar function ϕ provided the domain is simply connected. Thus

$$\mathbf{F} = \nabla\phi.$$

Such a scalar function ϕ is called the *potential*. In a rotational field, **F** cannot be expressed as the gradient of a scalar potential.

Obs. 1. *In an irrotational field, the line integral* **F** *between two points is independent of the path of integration and is equal to the potential difference between these points.*

If *aced* be any closed contour in an irrotational field F (Fig. 8.25), then

$$\int_{abcd} \mathbf{F} \cdot d\mathbf{R} = \int_{abc} \mathbf{F} \cdot d\mathbf{R} + \int_{cda} \mathbf{F} \cdot d\mathbf{R} = 0$$

or

$$\int_{abc} \mathbf{F} \cdot d\mathbf{R} = \int_{abc} \mathbf{F} \cdot d\mathbf{R}$$

i.e. the value of the line integral is independent of the path joining the end points.

Further, substituting $\mathbf{F} = \nabla\phi$, we have

$$\int_a^c \mathbf{F} \cdot d\mathbf{R} = \int_a^c \nabla\phi \cdot d\mathbf{R} = \int_a^c \left(\mathbf{I}\frac{\partial\phi}{\partial x} + \mathbf{J}\frac{\partial\phi}{\partial y} + \mathbf{K}\frac{\partial\phi}{\partial z} \right) \cdot (\mathbf{I}dx + \mathbf{J}dy + \mathbf{K}dz)$$

$$= \int_a^c \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz \right) = \int_a^c d\phi = \phi_c - \phi_a.$$

Obs. 2. If **F** is a vector force acting on a particle, then $\oint_C \mathbf{F} \cdot d\mathbf{R}$ represents the work done in moving the particle around a closed path. [See p. 328]

When $\oint_C \mathbf{F} \cdot d\mathbf{R} = 0$, *the field is said to be* **conservative,** *i.e.*, no work is done in displacement from a point *a* to another point in the field and back to *a* and the mechanical energy is conserved.

Thus *every irrotational field is conservative.*

Obs. 3. *The well-known equations of the Poisson and Laplace hold good for every irrotational field.*

Suppose $\nabla \cdot \mathbf{F} = f(x, y, z)$. Then $\nabla \cdot \nabla\phi = f(x, y, z)$ *i.e.*, $\nabla^2\phi = f(x, y, z)$ ...(*i*)

which is known as *Poisson's equation*. Its solutions for electrostatic fields enable us to determine the potential ϕ as a function of the charge distribution $f(x, y, z)$.

If $f(x, y, z) = 0$ then (*i*) reduces to $\nabla^2\phi = 0$ which is the *Laplace's equation*. The solutions of this equation are of great importance in modern engineering and physics, some of which we'll study in § 18.11 and 18.12.

(2) **Solenoidal fields.** *A solenoidal field* **F** *is characterised by any one of the following conditions :*

(*i*) $\nabla \cdot \mathbf{F} = 0$. (*ii*) *flux* $\int \mathbf{F} \cdot \mathbf{N}\, ds$ *across every closed surface is zero.* (*iii*) $\mathbf{F} = \nabla \times \mathbf{V}$.

If $\nabla \cdot \mathbf{F} = 0$ then by the Divergence theorem,

$$\int_S \mathbf{F} \cdot \mathbf{N} ds = \int_V \nabla \cdot \mathbf{F} dv = 0, \textit{ i.e.,} \text{ the flux across every closed surface is zero.}$$

Again since $\nabla \cdot \nabla \times \mathbf{V} = 0$.

$\therefore$ in a solenoidal field for which $\nabla \cdot \mathbf{F} = 0$, the vector **F** can always be expressed as the curl of a vector function **V**; thus $\mathbf{F} = \nabla \times \mathbf{V}$.

*A domain *D* is said to be *simply connected* if every closed curve in *D* can be shrunk to any point within *D*.

Example 8.48. *A vector field is given by* $\mathbf{F} = (x^2 - y^2 + x)\mathbf{I} - (2xy + y)\mathbf{J}$.
Show that the field is irrotational and find its scalar potential.
Hence evaluate the line integral from (1, 2) to (2, 1).

Solution. Since $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - y^2 + x & -(2xy + y) & 0 \end{vmatrix} = \mathbf{0}$

$\therefore$ this field is *irrotational* and the vector $\mathbf{F}$ can be expressed as the gradient of a scalar potential,

i.e., $(x^2 - y^2 + x)\mathbf{I} - (2xy + y)\mathbf{J} = \nabla\phi = \dfrac{\partial\phi}{\partial x}\mathbf{I} + \dfrac{\partial\phi}{\partial y}\mathbf{J}$

whence
$$\partial\phi/\partial x = x^2 - y^2 + x \qquad ...(i)$$
$$\partial\phi/\partial y = -(2xy + y) \qquad ...(ii)$$

Integrating (i) w.r.t. x, keeping y constant, we get $\phi = \dfrac{x^3}{3} - y^2x + \dfrac{x^2}{2} + f(y)$ ...(iii)

Similarly integrating (ii) w.r.t. y, keeping x constant, we obtain $\phi = -xy^2 - \dfrac{y^2}{2} + g(x)$ ...(iv)

Equating (iii) and (iv), we get $\dfrac{x^3}{3} - y^2x + \dfrac{x^2}{2} + f(y) = -xy^2 - \dfrac{y^2}{2} + g(x)$

$\therefore$ $$f(y) = -\frac{y^2}{2} \text{ and } g(x) = \frac{x^3}{3} + \frac{x^2}{2}$$

Hence $$\phi = \frac{x^3}{3} - xy^2 + \frac{x^2}{2} - \frac{y^2}{2}$$

Since the field is irrotational,

$\therefore$ $\int \mathbf{F} \cdot d\mathbf{R}$ from (1, 2) to (2, 1) $= \phi_{1,2} - \phi_{2,1} = \left(\dfrac{1}{3} - 1\times 4 + \dfrac{1}{2} - \dfrac{4}{2}\right) - \left(\dfrac{8}{3} - 2\times 1 + \dfrac{4}{2} - \dfrac{1}{2}\right) = -7\dfrac{1}{3}$.

Example 8.49. *A fluid motion is given by* $\mathbf{V} = (y + z)\mathbf{I} + (z + x)\mathbf{J} + (x + y)\mathbf{K}$.
(a) Is this motion irrotational? If so, find the velocity potential. *(U.P.T.U., 2004)*
(b) Is the motion possible for an incompressible fluid?

Solution. We have $\nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y + z & z + x & x + y \end{vmatrix} = \mathbf{I}(1 - 1) - \mathbf{J}(1 - 1) + \mathbf{K}(1 - 1) = \mathbf{0}.$

$\therefore$ this motion is irrotational and if ϕ is the velocity potential then $\mathbf{V} = \nabla\phi$. [§ 20.6]

i.e., $$(y + z)\mathbf{I} + (z + x)\mathbf{J} + (x + y)\mathbf{K} = \frac{\partial\phi}{\partial x}\mathbf{I} + \frac{\partial\phi}{\partial y}\mathbf{J} + \frac{\partial\phi}{\partial z}\mathbf{K}$$

$\therefore$ $$\frac{\partial\phi}{\partial x} = y + z, \frac{\partial\phi}{\partial y} = z + x, \frac{\partial\phi}{\partial z} = x + y$$

Integrating these, we get
$$\phi = (y + z)x + f_1(y, z) \qquad ...(i)$$
$$\phi = (z + x)y + f_2(z, x) \qquad ...(ii)$$
and
$$\phi = (x + y)z + f_3(x, y) \qquad ...(iii)$$

Equality of (i), (ii) and (iii), requires that
$$f_1(y, z) = yz, f_2(z, x) = zx, f_3(x, y) = xy.$$

Hence $$\phi = yz + zx + xy.$$

(b) The fluid motion is possible if $\mathbf{V}$ satisfies the equation of continuity which for an incompressible fluid is $\nabla \cdot \mathbf{V} = 0$. [See § 8.7 (1)

Here $$\nabla \cdot \mathbf{V} = \frac{\partial}{\partial x}(y+z) + \frac{\partial}{\partial y} = (z+x) + \frac{\partial}{\partial z}(x+y) = 0.$$

Hence, the fluid motion is possible.

Example 8.50. *Find whether* $\int_C [2xyz^2 dx + (x^2z^2 + z\cos yz)\,dy + (2x^2yz + y\cos yz)\,dz]$ *is independent of the path joining* $(0, \pi/2, 1)$ *and* $(1, 0, 1)$. *If so, evaluate this line integral.*

Solution. The line integral of $\mathbf{F}$ is independent of path of integration if $\nabla \times \mathbf{F} = \mathbf{0}$.

$$= \int_C [2xyz^2\mathbf{I} + (x^2z^2 + z\cos yz)\mathbf{J} + (2x^2yz + y\cos yz)\mathbf{K}] \cdot (\mathbf{I}dx + \mathbf{J}dy + \mathbf{K}dz) = \int_C \mathbf{F} \cdot d\mathbf{R}$$

and $$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xyz^2 & x^2z^2 + z\cos yz & 2x^2yz + y\cos yz \end{vmatrix}$$

$$= \mathbf{I}\,[2x^2z + \cos yz - yz\sin yz - (2x^2z + \cos yz - yz\sin yz)]$$
$$- \mathbf{J}[4xyz - 4xyz] + \mathbf{K}[2xz^2 - 2xz^2] = \mathbf{0}$$

$\therefore$ the given integral is independent of the path C.

Now let $\mathbf{F} = \nabla\phi$

i.e., $$(2xyz^2)\mathbf{I} + (x^2z^2 + z\cos yz)\mathbf{J} + (2x^2yz + y\cos yz)\,\mathbf{K} = \mathbf{I}\frac{\partial\phi}{\partial x} + \mathbf{J}\frac{\partial\phi}{\partial y} + \mathbf{K}\frac{\partial\phi}{\partial z}$$

$\therefore$ $$2xyz^2 = \frac{\partial\phi}{\partial x},\ x^2z^2 + z\cos yz = \frac{\partial\phi}{\partial y},\ 2x^2yz + y\cos yz = \frac{\partial\phi}{\partial z}$$

Integrating first w.r.t. x partially, we get

$$\phi = x^2y^2z^2 + \Psi_1(y, z) \qquad ...(i)$$

Integrating second w.r.t. y partially, we get

$$\phi = x^2yz^2 + \sin yz + \Psi_2(z, x) \qquad ...(ii)$$

Integrating third w.r.t. z partially, we get

$$\phi = x^2yz^2 + \sin yz + \Psi_3(x, y) \qquad ...(iii)$$

Comparing (i), (ii), (iii), we have

$\Psi_1(y, z)$ = terms in ϕ independent of $x = \sin yz$

$\Psi_2(z, x)$ = terms in ϕ independent of $y = 0$

$\Psi_3(z, x)$ = terms in ϕ independent of $z = 0$

Thus $$\phi = x^2yz^2 + \sin yz$$

Hence the value of the given integral $= \left|\phi\right|_{(0,\pi/2,1)}^{(1,0,1)}$

$$= (0 + 0) - (0 + \sin \pi/2) = -1.$$

Example 8.51. *Determine whether* $\mathbf{F} = (y^2\cos x + z^3)\mathbf{I} + (2y\sin x - 4)\mathbf{J} + (3xz^2 + 2)\mathbf{K}$ *is a conservative vector field? If so find the scalar potential* ϕ. *Also compute the work done in moving the particle from* $(0, 1, -1)$ *to* $(\pi/2, -1, 2)$. *(Mumbai, 2006)*

Solution. $\mathbf{F}$ is a conservative vector field when curl $\mathbf{F} = \mathbf{0}$. Here

$$Curl = F = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2\cos x + z^3 & 2y\sin x - 4 & 3xz^2 + 2 \end{vmatrix}$$

$$= \mathbf{I}(0 - 0) - \mathbf{J}(3z^2 - 3z^2) + \mathbf{K}(2y\cos x - 2y\cos x) = \mathbf{0}$$

$\therefore$ **F** is a conservative field.

Now let $\mathbf{F} = \nabla\phi$

i.e.,
$$(y^2 \cos x + z^3)\,\mathbf{I} + (2y \sin x - 4)\,\mathbf{J} + (3xz^2 + 2)\,\mathbf{K} = \mathbf{I}\frac{\partial\phi}{\partial x} + \mathbf{J}\frac{\partial\phi}{\partial y} + \mathbf{K}\frac{\partial\phi}{\partial z}$$

$\therefore$
$$y^2 \cos x + z^3 = \frac{\partial\phi}{\partial x},\ 2y \sin x - 4 = \frac{\partial\phi}{\partial y},\ 3xz^2 + 2 = \frac{\partial\phi}{\partial z}$$

Integrating first w.r.t. x partially, we get
$$\phi = y^2 \sin x + xz^3 + \psi_1(y, z) \qquad ...(i)$$

Integrating second w.r.t. y partially, we get
$$\phi = y^2 \sin x - 4y + \Psi_2(z, x) \qquad ...(ii)$$

Integrating third w.r.t. z partially, we obtain
$$\phi = xz^3 + 2z + \Psi_3(x, y) \qquad ...(iii)$$

Comparing (i), (ii), (iii), we get
$$\Psi_1(y, z) = \text{terms in } \phi \text{ independent of } x = -4y + 2z$$
$$\Psi_2(z, x) = \text{terms in } \phi \text{ independent of } y = xz^3 + 2z$$
$$\Psi_3(z, x) = \text{terms in } \phi \text{ independent of } z = y^2 \sin x - 4y$$

Thus
$$\phi = xz^3 + y^2 \sin x - 4y + 2z$$

In a conservative field, the work done $= \phi_B - \phi_A$
$$= \phi\left(\frac{\pi}{2}, -1, 2\right) - \phi(0, 1, -1)$$
$$= (4\pi + 1 + 4 + 4) - (-4 - 2) = 4\pi + 15.$$

PROBLEMS 8.11

1. If ϕ is a solution of the Laplace equation, prove that $\nabla\phi$ is both solenoidal and irrotational.
2. Show that the vector field defined by $\mathbf{F} = (x^2 + xy^2)\mathbf{I} + (y^2 + x^2y)\mathbf{J}$ is conservative and find the scalar potential. Hence evaluate $\int \mathbf{F} \cdot d\mathbf{R}$ from (0, 1) to (1, 2).
3. Find the work done by the variable force $\mathbf{F} = 2y\mathbf{I} + xy\mathbf{J}$ on a particle when it is displaced from the origin to the point $\mathbf{R} = 4\mathbf{I} + 2\mathbf{J}$ along the parabola $y^2 = x$.
4. Show that the vector field given by $\mathbf{A} = 3x^2y\mathbf{I} + (x^3 - 2yz^2)\mathbf{J} + (3z^2 - 2y^2z)\mathbf{K}$ is irrotational but not solenoidal. Also find $\phi(x, y, z)$ such that $\nabla\phi = \mathbf{A}$.
5. Show that the following vectors are irrotational and find the scalar potential in each case :
   (*i*) $(x^2 - yz)\mathbf{I} + (y^2 - zx)\mathbf{J} + (z^2 - xy)\mathbf{K}$
   (*ii*) $2xy\mathbf{I} + (x^2 + 2yz)\mathbf{J} + (y^2 + 1)\mathbf{K}$ *(V.T.U., 2007)*
   (*iii*) $(6xy + z^3)\mathbf{I} + (3x^2 - z)\mathbf{J} + (3xz^2 - y)\mathbf{K}$ *(Raipur, 2005 ; V.T.U., 2003 S)*
   (*iv*) $(2xy^2 + yz)\mathbf{I} + (2x^2y + xz + 2yz^2)\mathbf{J} + (2y^2z + xy)\mathbf{K}$. *(V.T.U., 2010)*
6. Fluid motion is given by $\mathbf{V} = ax\mathbf{I} + ay\mathbf{J} - 2az\mathbf{K}$.
   (*i*) Is it possible to find out the velocity potential ? If so, find it.
   (*ii*) Is the motion possible for an incompressible fluid ?
7. Show that the vector field defined by $\mathbf{F} = (y \sin z - \sin x)\mathbf{I} + (x \sin z + 2yz)\mathbf{J} + (xy \cos z + y^2)\mathbf{K}$ is irrotational and find its velocity potential. *(Kottayam, 2005)*
8. Show that $\mathbf{F} = (2xy + z^3)\mathbf{I} + x^2\mathbf{J} + 3xz^2\mathbf{K}$ is a conservative vector field and find a function ϕ such that $\mathbf{F} = \nabla\phi$. Also find the work done in moving an object in this field from (1, – 2, 1) to (3, 1, 4). *(Nagpur, 2009)*
9. If $\mathbf{F} = (x + y + az)\mathbf{I} + (bx + 2y - z)\mathbf{J} + (x + cy + 2z)\mathbf{K}$, find a, b, c such that curl $\mathbf{F} = \mathbf{0}$, then find ϕ such that $\mathbf{F} = \nabla\phi$. *(V.T.U., 2000)*
10. Find the constant a so that $\mathbf{V}$ is a conservative vector field, where
    $$\mathbf{V} = (axy - z^3)\mathbf{I} + (a - 2)x^2\mathbf{J} + (1 - a)xz^2\mathbf{K}.$$
    Calculate its scalar potential and work done in moving a particle from (1, 2, – 3) to (1, – 4, 2) in the field. *(Mumbai, 2006 ; Rajasthan, 2006)*

8.19 (1) ORTHOGONAL CURVILINEAR COORDINATES

Let the rectangular coordinates (x, y, z) of any point be expressed as functions of u, v, w so that

$$x = x(u, v, w), y = y(u, v, w), z = z(u, v, w) \quad ...(1)$$

Suppose that (1) can be solved for u, v, w in terms of x, y, z, so that

$$u = u(x, y, z), v = v(x, y, z), w = w(x, y, z) \quad ...(2)$$

We assume that the functions in (1) and (2) are single-valued and have continuous partial derivatives so that the correspondence between (x, y, z) and (u, v, w) is unique. Then (u, v, w) are called *curvilinear coordinates* of (x, y, z).

Each of u, v, w has a level surface through an arbitrary point. The surfaces $u = u_0, v = v_0, w = w_0$ are called *coordinate surfaces* through $P(u_0, v_0, w_0)$. Each pair of these coordinate surfaces intersect in curves called the *coordinate curves*. The curve of intersection of $u = u_0$ and $v = v_0$ will be called the w-curve, for only w changes along this curve. Similarly we define u and v-curves.

In vector notation, (1) can be written as $\mathbf{R} = x(u, v, w)\mathbf{I} + y(u, v, w)\mathbf{J} + z(u, v, w)\mathbf{K}$

$$\therefore \quad d\mathbf{R} = \frac{\partial \mathbf{R}}{\partial u} du + \frac{\partial \mathbf{R}}{\partial v} dv + \frac{\partial \mathbf{R}}{\partial w} dw \quad ...(3)$$

Then $\partial\mathbf{R}/\partial u$ is a tangent vector to the u-curve at P. If $\mathbf{T}_u$ is a unit vector at P in this direction, then $\partial\mathbf{R}/\partial u = h_1\mathbf{T}_v$ where $h_1 = |\partial\mathbf{R}/\partial u|$.

Similarly if $\mathbf{T}_v$ and $\mathbf{T}_w$ be unit tangent vectors to v- and w-curves at P, then

$$\frac{\partial \mathbf{R}}{\partial v} = h_2\mathbf{T}_v \text{ and } \frac{\partial \mathbf{R}}{\partial w} = h_3\mathbf{T}_w$$

where $h_2 = |\partial\mathbf{R}/\partial v|$ and $h_3 = |\partial\mathbf{R}/\partial w|$. [$h_1, h_2, h_3$ are called scalar factors.]

Then (3) can be written as

$$d\mathbf{R} = h_1 du\, \mathbf{T}_u + h_2 dv\, \mathbf{T}_v + h_3\, dw\, \mathbf{T}_w \quad ...(4)$$

Since ∇u is normal to the surface $u = u_0$ at P, therefore, a unit vector in this direction is given by $\mathbf{N}_u = \dfrac{\nabla u}{|\nabla u|}$.

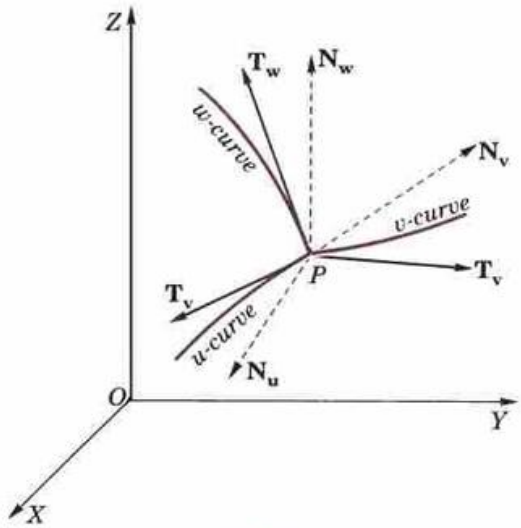

Fig. 8.26

Similarly, the unit vectors $\mathbf{N}_v = \dfrac{\nabla v}{|\nabla v|}$ and $\mathbf{N}_w = \dfrac{\nabla w}{|\nabla w|}$ are normal to the surfaces $v = v_0$ and $w = w_0$ at P respectively. Thus at each point P of a curvilinear coordinate system there exist two triads of unit vectors : $\mathbf{T}_u, \mathbf{T}_v, \mathbf{T}_w$ tangents to u, v, w-curves and $\mathbf{N}_u, \mathbf{N}_v, \mathbf{N}_w$ normals to the co-ordinates surfaces (Fig. 8.26).

In particular, when the coordinate surfaces intersect a right angles, the three coordinate curves are also mutually orthogonal and u, v, w are called the *orthogonal curvilinear coordinates*. In this case $\mathbf{T}_u, \mathbf{T}_v, \mathbf{T}_w$ and $\mathbf{N}_u, \mathbf{N}_v, \mathbf{N}_w$ are mutually perpendicular unit vector triads and hence become identical. Henceforth, we shall refer to orthogonal curvilinear coordinates only.

Multiplying (3) scalarly by ∇u, we get

$$\nabla u \,.\, d\mathbf{R} = du = \left(\nabla u \,.\, \frac{\partial \mathbf{R}}{\partial u}\right) du + \left(\nabla u \,.\, \frac{\partial \mathbf{R}}{\partial v}\right) dv + \left(\nabla u \,.\, \frac{\partial \mathbf{R}}{\partial w}\right) dw$$

whence
$$\nabla u \,.\, \frac{\partial \mathbf{R}}{\partial u} = 1, \nabla u \,.\, \frac{\partial \mathbf{R}}{\partial v} = 0, \nabla u \,.\, \frac{\partial \mathbf{R}}{\partial w} = 0$$

Similarly,
$$\nabla v \,.\, \frac{\partial \mathbf{R}}{\partial u} = 0, \nabla v \,.\, \frac{\partial \mathbf{R}}{\partial v} = 1, \nabla v \,.\, \frac{\partial \mathbf{R}}{\partial w} = 0$$

and
$$\nabla w \,.\, \frac{\partial \mathbf{R}}{\partial u} = 0, \nabla w \,.\, \frac{\partial \mathbf{R}}{\partial v} = 0, \nabla w \,.\, \frac{\partial \mathbf{R}}{\partial w} = 1.$$

These relations show that the sets $\partial\mathbf{R}/\partial u$, $\partial\mathbf{R}/\partial v$, $\partial\mathbf{R}/\partial w$ and ∇u, ∇v, ∇w constitute reciprocal system of vectors.

$$\therefore \qquad \nabla u = \frac{\dfrac{\partial \mathbf{R}}{\partial v} \times \dfrac{\partial \mathbf{R}}{\partial w}}{\left[\dfrac{\partial \mathbf{R}}{\partial u} \cdot \dfrac{\partial \mathbf{R}}{\partial v} \times \dfrac{\partial \mathbf{R}}{\partial w}\right]} = \frac{(h_2\mathbf{T_v}) \times (h_3\mathbf{T_w})}{[(h_1\mathbf{T_u}) \cdot (h_2\mathbf{T_v}) \times (h_3 T_\mathbf{w})]}$$

$$= \frac{h_2h_3\ \mathbf{T}_v \times \mathbf{T}_w}{h_1h_2h_3[\mathbf{T}_u\mathbf{T}_v\mathbf{T}_w]} = \frac{\mathbf{T}_u}{h_1} \qquad [\because\ \mathbf{T}_u\mathbf{T}_v\mathbf{T}_w = 1]$$

or $\qquad \mathbf{T}_v = h_1\ \nabla u$

Similarly $\qquad \mathbf{T}_v = h_2 \nabla v$ and $\mathbf{T}_w = h_3\ \nabla w$ $\qquad$ ...(5)

Also $\qquad = \mathbf{T}_v \times \mathbf{T}_w = h_2h_3\ \nabla v \times \nabla w$

Similarly $\qquad \mathbf{T}_v = h_3h_1\ \nabla w \times \nabla u$ and $\mathbf{T}_w = h_1h_2\ \nabla u \times \nabla v$ $\qquad$ ...(6)

Arc, area and volume elements

(*i*) *Arc element*. The element of arc length ds is determined from (4).

$$\therefore \qquad ds^2 = d\mathbf{R} \cdot d\mathbf{R} = h_1^2du^2 + h_2^2dv^2 + h_3^2dw^2 \qquad ...(7)$$

The arc length ds_1 along u-curve at P is h_1du for v and w are constants. Therefore the vector arc element along the u-curve is $d\mathbf{u} = h_1du\mathbf{T}_u$. Similarly vector arc elements along v and w curves at P are $d\mathbf{v} = h_2dv\ \mathbf{T}_v$ and $dw = h_3dw\ \mathbf{T}_w$. The arc element ds therefore corresponds to the length of the diagonal of the rectangular parallelopiped of Fig. 8.27.

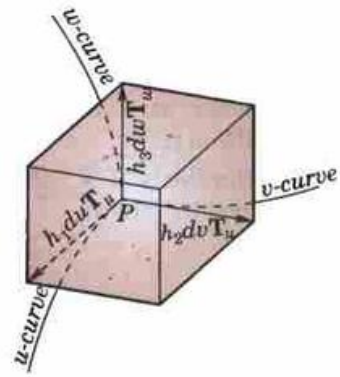

Fig. 8.27

(*ii*) *Area elements*. The area of the parallelogram formed by $d\mathbf{u}$ and $d\mathbf{v}$ is called the area element on the uv surface which is perpendicular to w-curve and we denote it by dS_w. Hence, $dS_w = |du \times dv| = h_1h_2dudv$. Similarly, $dS_u = h_2h_3\ dvdw$, $dS_v = h_3h_1dwdu$.

(*iii*) *Volume element* is the volume of the parallelopiped formed by $d\mathbf{u}$, $d\mathbf{v}$, $d\mathbf{w}$.

$$\therefore \qquad dV = [h_1du\mathbf{T}_u) \cdot (h_2dv\mathbf{T}_v) \times (h_3dw\mathbf{T}_w)]$$

$$= h_1h_2h_3\ dudvdw \qquad ...(8)\ [\because\ [\mathbf{T}_u\mathbf{T}_v\mathbf{T}_w] = 1$$

This can also be written as

$$dV = \frac{\partial \mathbf{R}}{\partial u} \cdot \frac{\partial \mathbf{R}}{\partial v} \times \frac{\partial \mathbf{R}}{\partial w}\ dudvdw = \frac{\partial(x, y, z)}{\partial(u, v, w)}\ dudvdw \qquad ...(9)$$

where $\partial(x, y, z)/\partial(u, v, w)$ is called the *Jacobian of the transformation* from (x, y, z) to (u, v, w) coordinates.

(2) Del applied to Functions in Orthogonal Curvilinear coordinates

To prove that

(1) $\nabla f = \dfrac{\mathbf{T}_u}{h_1}\dfrac{\partial f}{\partial u} + \dfrac{\mathbf{T}_v}{h_2}\dfrac{\partial f}{\partial v} + \dfrac{\mathbf{T}_w}{h_3}\dfrac{\partial f}{\partial w}$

(2) $\nabla \cdot \mathbf{F} = \dfrac{1}{h_1h_2h_3}\left[\dfrac{\partial}{\partial u}(h_2h_3f_1) + \dfrac{\partial}{\partial v}(h_3h_1f_2) + \dfrac{\partial}{\partial w}(h_1h_2f_3)\right]$

(3) $\nabla \times \mathbf{F} = \begin{vmatrix} \dfrac{\mathbf{T}_u}{h_2h_3} & \dfrac{\mathbf{T}_v}{h_3h_1} & \dfrac{\mathbf{T}_w}{h_1h_2} \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w} \\ h_1f_1 & h_2f_2 & h_3f_3 \end{vmatrix}$ where $\mathbf{F} = f_1\mathbf{T}_u + f_2\mathbf{T}_v + f_3\mathbf{T}_w$.

(1) Let $f(u, v, w)$ be any scalar point function in terms of u, v, w, the orthogonal curvilinear coordinates. Taking u, v, w as functions of x, y, z, we have

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} \qquad ...(i)$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y} \qquad ...(ii)$$

and

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial z} \qquad ...(iii)$$

Multiplying (i) by $\mathbf{I}$, (ii) by $\mathbf{J}$, (iii) by $\mathbf{K}$ and adding, we have

$$\nabla f = \frac{\partial f}{\partial u}\nabla u + \frac{\partial f}{\partial v}\nabla v + \frac{\partial f}{\partial w}\nabla w \qquad ...(iv)$$

$$= \frac{\mathbf{T}_u}{h_1}\frac{\partial f}{\partial u} + \frac{\mathbf{T}_v}{h_2}\frac{\partial f}{\partial v} + \frac{\mathbf{T}_w}{h_3}\frac{\partial f}{\partial w} \qquad \text{[By (5) p. 356]}$$

which is the required result.

(2) Let $\mathbf{F}(u, v, w)$ be a vector point function such that

$$\mathbf{F} = f_1\mathbf{T}_u + f_2\mathbf{T}_v + f_3\mathbf{T}_w = \Sigma f_1 h_2 h_3\, \nabla v \times \nabla w \qquad \text{[By (6) p. 356]}$$

$$\therefore \qquad \nabla \cdot \mathbf{F} = \Sigma \nabla \cdot \{(f_1h_2h_3)(\nabla v \times \nabla w)\}$$

$$= \Sigma[(f_1h_2h_3)\,\nabla \cdot (\nabla v \times \nabla w) + (\nabla v \times \nabla w)\,\nabla(f_1h_2h_3)] \qquad ...(v)$$

Now $\nabla \cdot (\nabla v \times \nabla w) = \nabla w \cdot \nabla \times (\nabla v) - \nabla v \cdot \nabla \times (\nabla w) = 0$ [By (5) p. 330]

and $$\nabla(f_1h_2h_3) = \frac{\partial(f_1h_2h_3)}{\partial u}\nabla u + \frac{\partial(f_1h_2h_3)}{\partial v}\nabla v + \frac{\partial(f_1h_2h_3)}{\partial w}\nabla w \qquad \text{[By } (iv) \text{ above]}$$

$\therefore$ (v) now becomes

$$\nabla \cdot \mathbf{F} = \sum(\nabla v \times \nabla w) \cdot \left\{\frac{\partial(f_1h_2h_3)}{\partial u}\nabla u + \frac{\partial(f_1h_2h_3)}{\partial v}\nabla v + \frac{\partial(f_1h_2h_3)}{\partial w}\nabla w\right\}$$

$$= [\nabla u, \nabla v, \nabla w] \sum \frac{\partial(f_1h_2h_3)}{\partial u} = \frac{1}{h_1h_2h_3}\sum \frac{\partial(f_1h_2h_3)}{\partial u}$$ which is the required result.

Cor. Laplacian. $\nabla^2 f = \nabla \cdot (\nabla f)$

$$= \nabla \cdot \left(\frac{\mathbf{T}_u}{h_1}\frac{\partial f}{\partial u} + \frac{\mathbf{T}_v}{h_2}\frac{\partial f}{\partial v} + \frac{\mathbf{T}_w}{h_3}\frac{\partial f}{\partial w}\right) = \frac{1}{h_1h_2h_3}\sum \frac{\partial}{\partial u}\left(\frac{1}{h_1}\frac{\partial f}{\partial u}h_2h_3\right)$$

(3) Let $\mathbf{F}(u, v, w)$ be a vector point function such that

$$\mathbf{F} = f_1\mathbf{T}_u + f_2\mathbf{T}_v + f_3\mathbf{T}_w = f_1h_1\nabla u + f_2h_2\nabla v + f_3h_3\,\nabla w \qquad \text{[By (5) p. 356]}$$

$$\nabla \times \mathbf{F} = \Sigma \nabla \times (f_1h_1\nabla u) \qquad \text{[Using (3) p. 329]}$$

$$= \Sigma[\nabla(f_1h_1) \times \nabla u + f_1h_1 \cdot \nabla \times \nabla u] = \sum\left[\frac{\partial(f_1h_1)}{\partial u}\nabla u + \frac{\partial(f_1h_1)}{\partial v}\nabla v + \frac{\partial(f_1h_1)}{\partial w}\nabla w\right] \times \nabla u$$

$$= \sum\left[\frac{\partial(f_1h_1)}{\partial v}\nabla v \times \nabla u + \frac{\partial(f_1h_1)}{\partial w}\nabla w \times \nabla u\right]$$

$$= \sum\left[\frac{\partial(f_1h_1)}{\partial v}\left(-\frac{\mathbf{T}_u \times \mathbf{T}_v}{h_1h_2}\right) + \frac{\partial(f_1h_1)}{\partial w}\left(\frac{\mathbf{T}_w \times \mathbf{T}_u}{h_3h_1}\right)\right]$$

$$= -\frac{\partial(f_1h_1)}{\partial v}\frac{\mathbf{T}_w}{h_1h_2} + \frac{\partial(f_1h_1)}{\partial w}\frac{\mathbf{T}_v}{h_3h_1} - \frac{\partial(f_2h_2)}{\partial w}\frac{\mathbf{T}_u}{h_2h_3} + \frac{\partial(f_2h_2)}{\partial u}\frac{\mathbf{T}_w}{h_1h_2} - \frac{\partial(f_3h_3)}{\partial u}\frac{\mathbf{T}_v}{h_3h_1} + \frac{\partial(f_3h_3)}{\partial v}\frac{\mathbf{T}_u}{h_2h_3}$$

$$= \frac{\mathbf{T}_u}{h_2h_3}\left[\frac{\partial(f_3h_3)}{\partial v} - \frac{\partial(f_2h_2)}{\partial w}\right]$$ + two similar terms, whence follows the required result.

TWO SPECIAL CURVILINEAR SYSTEMS

8.20 (1) CYLINDRICAL COORDINATES

Any point $P(x, y, z)$ whose projection on the xy-plane is $Q(x, y)$ has the *cylindrical coordinates* (ρ, ϕ, z), where $\rho = OQ$, $\phi = \angle XOQ$ and $z = QP$.

The level surfaces $\rho = \rho_0$, $\phi = \phi_0$, $z = z_0$ are respectively cylinders about the Z-axis; planes through the Z-axis and planes perpendicular to the Z-axis.

The coordinate curves for ρ are rays perpendicular to the Z-axis; for ϕ, horizontal circles with centres on the Z-axis; for z, lines parallel to the Z-axis.

From Fig. 8.28, we have

$$x = \rho\cos\phi,\ y = \rho\sin\phi,\ z = z$$

(*i*) *Arc element.*

$$\therefore \qquad (ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 = (d\rho)^2 + \rho^2 (d\phi)^2 + (dz)^2$$

so that the scale factors are $h_1 = 1$, $h_2 = \rho$, $h_3 = 1$.

(*ii*) *Area elements* $dS_\rho = \rho d\phi dz$, $dS_\phi = dz d\rho$, $dS_z = \rho d\rho d\phi$ where dS_ρ is the area element $\perp$ to ρ-direction, etc.

(*iii*) *Volume element* $dV = \rho d\rho d\phi dz$.

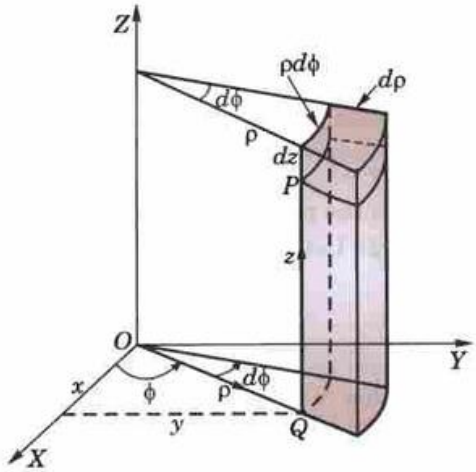

Fig. 8.28

(2) Cylindrical co-ordinate system is orthogonal

At any point P, we have $x = \rho \cos \phi$, $y = \rho \sin \phi$, $z = z$,

so that $\qquad \mathbf{R} = \rho \cos \phi \mathbf{I} + \rho \sin \phi \mathbf{J} + z\mathbf{K}$

If $\mathbf{T}_\rho$, $\mathbf{T}_\phi$, $\mathbf{T}_z$ be the unit vectors at P in the directions of the tangents to the ρ, ϕ, z-curves respectively, then

$$\mathbf{T}_\rho = \frac{\partial \mathbf{R}/\partial \rho}{|\partial \mathbf{R}/\partial \rho|} = \frac{\cos \phi \mathbf{I} + \sin \phi \mathbf{J}}{\sqrt{(\cos^2 \phi + \sin^2 \phi)}} = \cos \phi \mathbf{I} + \sin \phi \mathbf{J}$$

$$\mathbf{T}_\phi = \frac{\partial \mathbf{R}/\partial \phi}{|\partial \mathbf{R}/\partial \phi|} = \frac{-\rho \sin \phi \mathbf{I} + \rho \cos \phi \mathbf{J}}{\sqrt{[(-\rho \sin \phi)^2 + (\rho \cos \phi)^2]}} = -\sin \phi \mathbf{I} + \cos \phi \mathbf{J}$$

and $$\mathbf{T}_z = \frac{\partial \mathbf{R}/\partial z}{|\partial \mathbf{R}/\partial z|} = \mathbf{K}$$

Now $\mathbf{T}_\rho \cdot \mathbf{T}_\phi = (\cos \phi \mathbf{I} + \sin \phi \mathbf{J}) \cdot (-\sin \phi \mathbf{I} + \cos \phi \mathbf{J}) = -\cos \phi \sin \phi + \sin \phi \cos \phi = 0$,

$\mathbf{T}_\phi \cdot \mathbf{T}_z = (-\sin \phi \mathbf{I} + \cos \phi \mathbf{J}) \cdot \mathbf{K} = 0$, and $\mathbf{T}_z \cdot \mathbf{T}_\rho = \mathbf{K} \cdot (\cos \phi \mathbf{I} + \sin \phi \mathbf{J}) = 0$.

Hence the cylindrical coordinate system is orthogonal.

Also $\qquad \mathbf{T}_\rho \times \mathbf{T}_\phi = (\cos \phi \mathbf{I} + \sin \phi \mathbf{J}) \times (-\sin \phi \mathbf{I} + \cos \phi \mathbf{J}) = (\cos^2 \phi + \sin^2 \phi)\, \mathbf{I} \times \mathbf{J} = \mathbf{K} = T_z$

$\mathbf{T}_\phi \times \mathbf{T}_z = (-\sin \phi \mathbf{I} + \cos \phi \mathbf{J}) \times \mathbf{K} = \sin \phi \mathbf{J} + \cos \phi \mathbf{I} = \mathbf{T}_\rho$

$\mathbf{T}_z \times \mathbf{T}_\rho = \mathbf{K} \times (\cos \phi \mathbf{I} + \sin \phi \mathbf{J}) = \cos \phi \mathbf{J} - \sin \phi \mathbf{I} = \mathbf{T}_\phi$

These conditions satisfied by T_ρ, T_ϕ, and T_z, show that the *cylindrical coordinates system is a right handed orthogonal coordinate system.* (*V.T.U., 2008*)

(3) Del applied to functions in Cylindrical coordinates

We have $u = \rho$, $v = \phi$, $w = z$ and $h_1 = 1$, $h_2 = \rho$, $h_3 = 1$.

Let T_ρ, T_ϕ, T_z be the unit vectors in the directions of the tangents to the ρ, ϕ, z curves.

(*i*) *Expression for grad f.*

Since $$\nabla f = \frac{\mathbf{T}_u}{h_1}\frac{\partial f}{\partial u} + \frac{\mathbf{T}_v}{h_2}\frac{\partial f}{\partial v} + \frac{\mathbf{T}w}{h_3}\frac{\partial f}{\partial w}$$

$$\therefore \qquad \nabla f = \frac{\partial f}{\partial \rho}\mathbf{T}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\mathbf{T}_\phi + \frac{\partial f}{\partial z}\mathbf{T}_z$$

(*ii*) *Expression for div* $\mathbf{F}$ *where* $\mathbf{F} = f_1\mathbf{T}_u + f_2\mathbf{T}_v + f_3\mathbf{T}_w$

Since $$\nabla \cdot \mathbf{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u}(h_2h_3f_1) + \frac{\partial f}{\partial v}(h_3h_1f_2) + \frac{\partial}{\partial w}(h_1h_2f_3)\right]$$

$$\therefore \qquad \nabla \cdot \mathbf{F} = \frac{1}{\rho}\left\{\frac{\partial}{\partial \rho}(\rho f_1) + \frac{\partial f_2}{\partial \phi} + \frac{\partial}{\partial z}(\rho f_3)\right\}$$

(*iii*) *Expression for curl* $\mathbf{F}$ *where* $\mathbf{F} = f_1\mathbf{T}_u + f_2\mathbf{T}_v + f_3\mathbf{T}_w$

Since $$\nabla \times \mathbf{F} = \begin{vmatrix} \dfrac{\mathbf{T}_u}{h_2h_3} & \dfrac{\mathbf{T}_v}{h_3h_1} & \dfrac{\mathbf{T}_w}{h_1h_2} \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w} \\ h_1f_1 & h_2f_2 & h_3f_3 \end{vmatrix} = \begin{vmatrix} \mathbf{T}_\rho/\rho & \mathbf{T}_\phi & \mathbf{T}_z/\rho \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ f_1 & \rho f_2 & f_3 \end{vmatrix}$$

$$= \mathbf{T}_\rho\left(\frac{1}{\rho}\frac{\partial f_3}{\partial \phi} - \frac{\partial f_2}{\partial z}\right) + \mathbf{T}_\phi\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial \rho}\right) + \mathbf{T}_z\left(\frac{\partial f_2}{\partial \rho} - \frac{1}{\rho}\frac{\partial f_1}{\partial \phi}\right)$$

(*iv*) *Expression for* $\nabla^2 f$

Since $$\nabla^2 f = \frac{1}{h_1 h_2 h_3}\left\{\frac{\partial}{\partial u}\left(\frac{1}{h_1}\frac{\partial f}{\partial u}h_2 h_3\right) + \frac{\partial}{\partial v}\left(\frac{1}{h_2}\frac{\partial f}{\partial v}h_3 h_1\right) + \frac{\partial}{\partial w}\left(\frac{1}{h_3}\frac{\partial f}{\partial w}h_1 h_2\right)\right\}$$

$\therefore$ $$\nabla^2 f = \frac{1}{\rho}\left\{\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\rho}\frac{\partial f}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(\rho\frac{\partial f}{\partial z}\right)\right\} = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}.$$

Example 8.52. *Express the vector* $z\mathbf{I} - 2x\mathbf{J} + y\mathbf{K}$ *in cylindrical coordinates.* (*V.T.U., 2010*)

Solution. We have $x = \rho\cos\phi$, $y = \rho\sin\phi$ and $z = z$.

so that $$\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K} = \rho\cos\phi\mathbf{I} + \rho\sin\phi\mathbf{J} + z\mathbf{K}$$

If $\mathbf{T}_\rho$, $\mathbf{T}_\phi$, $\mathbf{T}_z$ be the unit vectors along the tangents to ρ, ϕ and z curves respectively, then

$$\mathbf{T}_\rho = \frac{\partial\mathbf{R}/\partial\rho}{|\partial\mathbf{R}/\partial\rho|} = \frac{\cos\phi\mathbf{I} + \sin\phi\mathbf{J}}{\sqrt{(\cos^2\phi + \sin^2\phi)}} = \cos\phi\mathbf{I} + \sin\phi\mathbf{J}$$

$$\mathbf{T}_\phi = \frac{\partial\mathbf{R}/\partial\phi}{|\partial\mathbf{R}/\partial\phi|} = \frac{-\rho\sin\phi\mathbf{I} + \rho\cos\phi\mathbf{J}}{\sqrt{[(-\rho\sin\phi)^2 + (\rho\cos\phi)^2]}} = -\sin\phi\mathbf{I} + \cos\phi\mathbf{J}$$

$$\mathbf{T}_z = \frac{\partial\mathbf{R}/\partial z}{|\partial\mathbf{R}/\partial z|} = \mathbf{K}$$

Let the expression for $\mathbf{F} = z\mathbf{I} - 2x\mathbf{J} + y\mathbf{K}$ in cylindrical coordinates be

$$\mathbf{F} = f_1\mathbf{T}_\rho + f_2\mathbf{T}_\phi + f_3\mathbf{T}_z \qquad ...(i)$$

Then $$f_1 = \mathbf{F}\cdot\mathbf{T}_\rho = z\cos\phi - 2x\sin\phi$$
$$f_2 = \mathbf{F}\cdot\mathbf{T}_\phi = -z\sin\phi - 2x\cos\phi$$
$$f_3 = \mathbf{F}\cdot\mathbf{T}_z = y$$

Substituting the values of f_1, f_2, f_3 in (*i*), we get

$$\mathbf{F} = (z\cos\phi - 2x\sin\phi)\,\mathbf{T}_\rho - (z\sin\phi + 2x\cos\phi)\,\mathbf{T}_\phi + y\mathbf{T}_z$$
$$= (z\cos\phi - \rho\sin 2\phi)\,\mathbf{T}_\rho - (z\sin\phi + 2\rho\cos^2\phi)\,\mathbf{T}_\phi + \rho\sin\phi\,\mathbf{T}_z$$

Example 8.53. *Show that* $\nabla(\log\rho)$ *and* $\nabla\phi$, $\rho \neq 0$, $\phi \neq 0$ *are solenoidal vectors.*

Solution. (*i*) $f = \log\rho$ is a function of ρ only. We have to prove that $\nabla\cdot(\nabla f)$, *i.e.*, $\nabla^2 f = 0$

$$\nabla^2 f = \frac{\partial^2 f}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial z^2} = \frac{\partial^2}{\partial\rho^2}(\log\rho) + \frac{1}{\rho}\frac{\partial(\log\rho)}{\partial\rho} + 0 + 0 = -\frac{1}{\rho^2} + \frac{1}{\rho^2} = 0$$

Hence $\nabla(\log\rho)$ is a solenoidal vector.

(*ii*) $f = \nabla\phi$ is a function of ϕ only. We have to show that $\nabla\cdot(\nabla f)$, *i.e.*, $\nabla^2 f = 0$.

$$\nabla^2 f = \frac{\partial^2 f}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial z^2} = 0 + 0 + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\phi^2} + 0 = 0.$$

Hence the result.

8.21 (1) SPHERICAL POLAR COORDINATES

Let $P(x, y, z)$ be any point whose projection on the XY-plane is $Q(x, y)$. Then the spherical polar co-ordinates of P are (r, θ, ϕ) such that $r = OP$, $\theta = \angle ZOP$ and $\phi = \angle XOQ$.

The level surfaces $r = r_0$, $\theta = \theta_0$, $\phi = \phi_0$ are respectively spheres about O, cones about the Z-axis with vertex at O and planes through the Z-axis.

The co-ordinate curves for r are rays from the origin; for θ, vertical circles with centre at O (called *meridians*); for ϕ, horizontal circles with centres on the Z-axis

From Fig. 8.29, we have

$$x = OQ\cos\phi = OP\cos(90° - \theta)\cos\phi = r\sin\theta\cos\phi,$$
$$y = OQ\sin\phi = r\sin\theta\sin\phi;\ z = r\cos\theta.$$

(*i*) *Arc element*

$$\therefore \qquad (ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 = (dr)^2 + r^2\,(d\theta)^2 + (r\sin\theta)^2\,(d\phi)^2$$

so that the scale factors are

$$h_1 = 1,\ h_2 = r,\ h_3 = r\sin\theta.$$

(*ii*) *Area elements*

$$dS_r = r^2\ \sin\theta\, d\theta d\phi,\ dS_\theta = r\sin\theta d\phi dr,\ dS_\phi = r dr d\theta$$

where dS_r is the area element perpendicular to the r-direction, etc.

(*iii*) *Volume element* $dV = r^2 \sin\theta\, dr d\theta d\phi$.

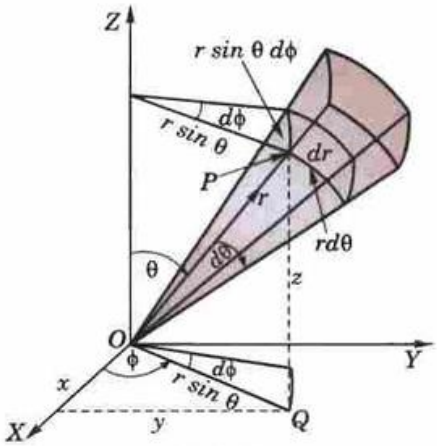

Fig. 8.29

(2) Spherical polar coordinate system is orthogonal

At any point P, we have $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$,

so that $\qquad \mathbf{R} = r\sin\theta\cos\phi\mathbf{I} + r\sin\theta\sin\phi\mathbf{J} + r\cos\theta\mathbf{K}$

If $\mathbf{T}_r$, $\mathbf{T}_\theta$, $\mathbf{T}_\phi$ be the unit vectors at P in the directions of the tangents to the r, θ, ϕ-curves respectively, then

$$\mathbf{T}_r = \frac{\partial\mathbf{R}/\partial r}{|\partial\mathbf{R}/\partial r|} = \frac{\sin\theta\cos\phi\mathbf{I} + \sin\theta\sin\phi\mathbf{J} + \cos\theta\mathbf{K}}{\sqrt{(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta)}}$$

$$= \sin\theta\cos\phi\mathbf{I} + \sin\theta\sin\phi\mathbf{J} + \cos\theta\mathbf{K}$$

$$\mathbf{T}_\theta = \frac{\partial\mathbf{R}/\partial\theta}{|\partial\mathbf{R}/\partial\theta|} = \frac{r\cos\theta\cos\phi\mathbf{I} + r\cos\theta\sin\phi\mathbf{J} - r\sin\theta\mathbf{K}}{r\sqrt{(\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta)}}$$

$$= \cos\theta\cos\phi\mathbf{I} + \cos\theta\sin\phi\mathbf{J} - \sin\theta\mathbf{K}$$

and $$\mathbf{T}_\phi = \frac{\partial\mathbf{R}/\partial\phi}{|\partial\mathbf{R}/\partial\phi|} = \frac{-r\sin\theta\sin\phi\mathbf{I} + r\sin\theta\cos\phi\mathbf{J}}{r\sin\theta} = -\sin\phi\mathbf{I} + \cos\phi\mathbf{J}$$

Now $\qquad \mathbf{T}_r\,.\,\mathbf{T}_\theta = \sin\theta\cos\theta\cos^2\phi + \sin\theta\cos\theta\sin^2\phi - \sin\theta\cos\theta = 0$

$$\mathbf{T}_\theta\,.\,\mathbf{T}_\phi = -\cos\theta\cos\phi\sin\phi + \cos\theta\sin\phi\cos\phi = 0$$

$$\mathbf{T}_\phi\,.\,\mathbf{T}_r = -\sin\theta\cos\phi\sin\phi + \sin\theta\sin\phi\cos\phi = 0$$

Also $\qquad \mathbf{T}_r \times \mathbf{T}_\theta = \sin\theta\cos\phi\cos\theta\sin\phi\mathbf{K} + \sin^2\theta\cos\phi\mathbf{J} - \sin\theta\sin\phi\cos\theta\cos\phi\mathbf{K}$

$$- \sin^2\theta\sin\phi\mathbf{I} + \cos^2\theta\cos\phi\mathbf{J} - \cos^2\theta\sin\phi\mathbf{I}$$

$$= -\sin\phi\mathbf{I} + \cos\phi\mathbf{J} = \mathbf{T}_\phi$$

$$\mathbf{T}_\theta \times \mathbf{T}_\phi = \cos\theta\cos^2\phi\mathbf{K} + \sin^2\phi\cos\theta\mathbf{K} + \sin\theta\sin\phi\mathbf{J} + \sin\theta\cos\phi\mathbf{I} = \mathbf{T}_r$$

and $$\mathbf{T}_\phi \times \mathbf{T}_r = -\sin\theta\sin^2\phi\mathbf{K} + \sin\phi\cos\theta\mathbf{J} - \sin\theta\cos^2\phi\mathbf{K} + \cos\phi\cos\theta\mathbf{I} = \mathbf{T}_\theta$$

The above conditions satisfied by $\mathbf{T}_r$, $\mathbf{T}_\theta$, and $\mathbf{T}_\phi$ show that *the spherical polar coordinate system is a right handed orthogonal coordinate system.* (*V.T.U., 2008*)

(3) Del applied to functions in spherical polar coordinates

We have $u = r$, $v = \theta$, $w = \phi$ and $h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$.

Let $\mathbf{T}_r$, $\mathbf{T}_\theta$, $\mathbf{T}_\phi$ be the unit vectors in the directions of the tangents to the r, θ, ϕ-curves.

(*i*) *Expression for grad f*

Since $$\nabla f = \frac{\mathbf{T}_u}{h_1}\frac{\partial f}{\partial u} + \frac{\mathbf{T}_v}{h_2}\frac{\partial f}{\partial v} + \frac{\mathbf{T}_w}{h_3}\frac{\partial f}{\partial w}$$

$$\therefore \qquad \nabla f = \frac{\partial f}{\partial r}\mathbf{T}_r + \frac{1}{r}\frac{\partial f}{\partial\theta}\mathbf{T}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\mathbf{T}_\phi$$

(*ii*) *Expression for div* $\mathbf{F}$ *where* $\mathbf{F} = f_1\mathbf{T}_u + f_2\mathbf{T}_v + f_3\mathbf{T}_w$

Since $$\nabla\,.\,\mathbf{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u}(h_2h_3f_1) + \frac{\partial}{\partial v}(h_3h_1f_2) + \frac{\partial}{\partial w}(h_1h_2f_3)\right]$$

$$\therefore \qquad \nabla\,.\,\mathbf{F} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta\, f_1) + \frac{\partial}{\partial\theta}(r\sin\theta\, f_2) + \frac{\partial}{\partial\phi}(rf_3)\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(f_1r^2) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(f_2\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial f_3}{\partial\phi}$$

(iii) Expression for curl $\mathbf{F}$ *where* $\mathbf{F} = f_1\mathbf{T}_u + f_2\mathbf{T}_v + f_3\mathbf{T}_w$

Since
$$\nabla \times \mathbf{F} = \begin{vmatrix} \dfrac{\mathbf{T}_u}{h_2h_3} & \dfrac{\mathbf{T}_v}{h_3h_1} & \dfrac{\mathbf{T}_w}{h_1h_2} \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w} \\ h_1f_1 & h_2f_2 & h_3f_3 \end{vmatrix}$$

$\therefore$
$$\nabla \times \mathbf{F} = \begin{vmatrix} \dfrac{\mathbf{T}_r}{r^2\sin\theta} & \dfrac{\mathbf{T}_\theta}{r\sin\theta} & \dfrac{\mathbf{T}\phi}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ f_1 & rf_2 & r\sin\theta f_3 \end{vmatrix}$$

$$= \frac{\mathbf{T}_r}{r^2\sin\theta}\left\{\frac{\partial}{\partial\theta}(r\sin\theta f_3) - \frac{\partial}{\partial\phi}(rf_2)\right\} - \frac{\mathbf{T}_\theta}{r\sin\theta}\left\{\frac{\partial}{\partial r}(r\sin\theta f_3) - \frac{\partial f_1}{\partial\phi}\right\} + \frac{\mathbf{T}_\phi}{r}\left\{\frac{\partial}{\partial r}(rf_2) - \frac{\partial f_1}{\partial\theta}\right\}$$

$$= \frac{\mathbf{T}_r}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}(f_3\sin\theta) - \frac{\partial f_2}{\partial\phi}\right\} + \frac{\mathbf{T}_\theta}{r}\left\{\frac{1}{\sin\theta}\frac{\partial f_1}{\partial\phi} - \frac{\partial}{\partial r}(rf_3)\right\} + \frac{\mathbf{T}_\phi}{r}\left\{\frac{\partial}{\partial r}(rf_2) - \frac{\partial f_1}{\partial\theta}\right\}$$

(iv) Expression for $\nabla^2 f$.

Since
$$\nabla^2 f = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial u}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_3h_1}{h_2}\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial w}\right)\right\}$$

$\therefore$
$$\nabla^2 f = \frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{r\sin\theta}{r}\frac{\partial f}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{r}{r\sin\theta}\frac{\partial f}{\partial\phi}\right)\right\}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

$$= \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial f}{\partial\theta}.$$

Example 8.54. *Express the vector field* $2y\mathbf{I} - z\mathbf{J} + 3x\mathbf{K}$ *in spherical polar coordinate system.*

Solution. We have $\quad x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$

so that $\quad \mathbf{R} = r\sin\theta\cos\phi\mathbf{I} + r\sin\theta\sin\phi\mathbf{J} + r\cos\theta\mathbf{K}.$

If $\mathbf{T}_r, \mathbf{T}_\theta, \mathbf{T}_\phi$ be the unit vectors along the tangents to r, θ, ϕ, curves respectively, then

$$\mathbf{T}_r = \frac{\partial\mathbf{R}/\partial r}{|\partial\mathbf{R}/\partial r|} = \frac{\sin\theta\cos\phi\mathbf{I} + \sin\theta\sin\phi\mathbf{J} + \cos\theta\mathbf{K}}{\sqrt{[(\sin\theta\cos\phi)^2 + (\sin\theta\sin\phi)^2 + \cos^2\theta]}}$$

$$= \sin\theta\cos\phi\mathbf{I} + \sin\theta\sin\phi\mathbf{J} + \cos\theta\mathbf{K}$$

$$\mathbf{T}_\theta = \frac{\partial\mathbf{R}/\partial\theta}{|\partial\mathbf{R}/\partial\theta|} = \frac{r\cos\theta\cos\phi\mathbf{I} + r\cos\theta\sin\phi\mathbf{J} - r\sin\theta\mathbf{K}}{\sqrt{[(r\cos\theta\cos\phi)^2 + (r\cos\theta\sin\phi)^2 + (-r\sin\theta)^2]}}$$

$$= \cos\theta\cos\phi\mathbf{I} + \cos\theta\sin\phi\mathbf{J} - \sin\theta\mathbf{K}$$

$$\mathbf{T}_\phi = \frac{\partial\mathbf{R}/\partial\phi}{|\partial\mathbf{R}/\partial\phi|} = \frac{-r\sin\theta\sin\phi\mathbf{I} + r\sin\theta\cos\phi\mathbf{J}}{\sqrt{[(-r\sin\theta\sin\phi)^2 + (r\sin\theta\cos\phi)^2]}} = -\sin\phi\mathbf{I} + \cos\phi\mathbf{J}.$$

Let the expression for $\mathbf{F} = 2y\mathbf{I} - z\mathbf{J} + 3x\mathbf{K}$ in spherical polar coordinates be

$$\mathbf{F} = f_1\mathbf{T}_r + f_2\mathbf{T}_\theta + f_3\mathbf{T}_\phi \qquad \ldots(i)$$

Then $\quad f_1 = \mathbf{F}\,.\,\mathbf{T}_r = (2r\sin\theta\sin\phi\mathbf{I} - r\cos\theta\mathbf{J} + 3r\sin\theta\cos\phi\mathbf{K})\,.\,(\sin\theta\cos\phi\mathbf{I} + \sin\theta\sin\phi\mathbf{J} + \cos\theta\mathbf{K})$

$$= 2r\sin^2\theta\sin\phi\cos\phi - r\sin\theta\cos\theta\sin\phi + 3r\sin\theta\cos\theta\cos\phi$$

$$f_2 = \mathbf{F} \cdot \mathbf{T}_\theta = (2r \sin\theta \sin\phi \mathbf{I} - r\cos\theta \mathbf{J} + 3r \sin\theta \cos\phi \mathbf{K}) \cdot (\cos\theta \cos\phi \mathbf{I} + \cos\theta \sin\phi \mathbf{J} - \sin\theta \mathbf{K})$$
$$= 2r \sin\theta \cos\theta \sin\phi \cos\phi - r\cos^2\theta \sin\phi - 3r \sin^2\theta \cos\phi.$$

and
$$f_3 = \mathbf{F} \cdot \mathbf{T}_\phi = (2r \sin\theta \sin\phi \mathbf{K} - r\cos\theta \mathbf{J} + 3r \sin\theta \cos\phi \mathbf{K}) \cdot (-\sin\phi \mathbf{I} + \cos\phi \mathbf{J})$$
$$= -2r \sin\theta \sin^2\phi - r\cos\theta \cos\phi$$

Substituting the values of f_1, f_2, f_3 in (i), we get the desired expression.

Example 8.55. *Prove that $\nabla(\cos\theta) \times \nabla\phi = \nabla(1/r)$. $r \neq 0$.*

Solution. In spherical polar coordinates,

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{T}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{T}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{T}_\phi$$

$$\therefore \quad \nabla(\cos\theta) = \frac{1}{r}\frac{\partial}{\partial\theta}(\cos\theta)\mathbf{T}_\theta = -\frac{1}{r}\sin\theta\,\mathbf{T}_\theta \qquad ...(i)$$

$$\nabla\phi = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\phi)\mathbf{T}_\phi = \frac{1}{r\sin\theta}\mathbf{T}_\phi \qquad ...(ii)$$

and
$$\nabla\left(\frac{1}{r}\right) = \frac{\partial}{\partial r}(r^{-1})\mathbf{T}_r = -\frac{1}{r^2}\mathbf{T}_r$$

Now from (i) and (ii), we get

$$\nabla(\cos\theta) \times \nabla\phi = -\frac{1}{r^2}\mathbf{T}_\theta \times \mathbf{T}_\phi = -\frac{1}{r^2}\mathbf{T}_r = \nabla\left(\frac{1}{r}\right).$$

Example 8.56. *If $F = r^2 \cos\theta\, \mathbf{T}_r - \frac{1}{r}\mathbf{T}_\theta + \frac{1}{r\sin\theta}\mathbf{T}_\phi$, find the value of $\mathbf{F} \times \text{curl}\,\mathbf{F}$.*

Solution. In spherical coordinates,

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{T}_r/r^2\sin\theta & \mathbf{T}_\theta/r\sin\theta & \mathbf{T}_\phi/r \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ f_1 & rf_2 & r\sin\theta f_3 \end{vmatrix}$$

Here $f_1 = r^2\cos\theta, f_2 = -1/r, f_3 = 1/r\sin\theta$.

$$\therefore \quad \text{curl}\,\mathbf{F} = \frac{2}{r^2\sin\theta}\begin{vmatrix} \mathbf{T}_r & r\mathbf{T}_\theta & r\sin\theta\,\mathbf{T}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ r^2\cos\theta & -1 & 1 \end{vmatrix} = r\sin\theta\mathbf{T}_\phi$$

$$\therefore \quad \mathbf{F} \times \text{curl}\,\mathbf{F} = \left(r^2\cos\theta\mathbf{T}_r - \frac{1}{r}\mathbf{T}_\theta + \frac{1}{r\sin\theta}\mathbf{T}_\phi\right) \times (r\sin\theta\mathbf{T}_\phi) = -(r^3\sin\theta\cos\theta\mathbf{T}_\theta + \sin\theta\mathbf{T}_r).$$

PROBLEMS 8.12

1. Express the following vectors in cylindrical coordinates
   (i) $2y\mathbf{I} - z\mathbf{J} + 3x\mathbf{K}$ (ii) $2x\mathbf{I} - 3y^2\mathbf{J} + zx\mathbf{K}$ *(V.T.U., 2009)*
2. Express the following vectors in spherical polar coordinates
   (i) $x\mathbf{I} + 2y\mathbf{J} + yz\mathbf{K}$ (ii) $xy\mathbf{I} + yz\mathbf{J} + zx\mathbf{K}$
3. Evaluate $\nabla\phi = xyz$ in cylindrical coordinates.
4. Show that $\nabla(r/\sin\theta) \times \nabla\theta = \nabla\phi$.
5. Prove that $\mathbf{V} = \frac{\cos\theta}{r^3}(\mathbf{T}_r/\sin\theta - \mathbf{T}_\theta/\cos\theta + r^4\mathbf{T}_\phi)$ is solenoidal.
6. Show that (i) $\nabla^2(\log r) = 1/r^2$ (ii) $\nabla \times [(\cos\theta)(\nabla\phi)] = \nabla(1/r)$.

7. Prove that $\mathbf{V} = \rho z \sin 2\phi \left[\mathbf{T}_\rho + \cot 2\phi \mathbf{T}_\phi + \frac{\rho}{2z}\mathbf{T}_z\right]$ is irrotational.

8. If u, v, w are orthogonal curvilinear coordinates with h_1, h_2, h_3 as scale factors, prove that

$$\left[\frac{\partial \mathbf{R}}{\partial u}, \frac{\partial \mathbf{R}}{\partial v}, \frac{\partial \mathbf{R}}{\partial w}\right] = \frac{1}{[\nabla u, \nabla v, \nabla w]} = h_1 h_2 h_3.$$

8.22 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 8.13

Fill up the blanks or choose the correct answer from the following problems :

1. A unit tangent vector to the surface $x = t, y = t^2, z = t^3$ at $t = 1$ is .....
2. The equation of the normal to the surface $2x^2 + y^2 + 2z = 3$ at $(2, 1, -3)$ is .....
3. If $u = u(x, y)$ and $v = v(x, y)$, then the area-element $dudv$ is related to the area-element $dxdy$ by the relation .....
4. If $A = 2x^2\mathbf{I} - 3yz\mathbf{J} + xz^2\mathbf{K}$, then $\nabla . \mathbf{A} =$ .....
5. div curl $\mathbf{F} =$ .....
6. Area bounded by a simple closed curve C is .....
7. If S is a closed surface enclosing a volume V and if $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$, then
$$\int_S \mathbf{R} . \mathbf{N}\, ds = \text{.....}$$
8. div $\mathbf{R} =$ .....; curl $\mathbf{R} =$ .....
9. If $\mathbf{A}$ is such that $\nabla \times \mathbf{A} = 0$, then $\mathbf{A}$ is called .....
10. If $\nabla . \mathbf{F} = 3$, then $\int_S \mathbf{F} . \mathbf{N}\, ds$ where S is a surface of a unit sphere, is .....
11. If $\nabla . \mathbf{F} = 0$, then $\mathbf{F}$ is called.....
12. The directional derivative of $\phi(x, y, z) = x^2yz + 4xz^2$ at the point $(1, -2, -1)$ in the direction PQ where $P = (1, 2, -1)$ and $Q = (-1, 2, 3)$ is .....
13. If $u = x^2yz$, $v = xy - 3z^2$, then $\nabla . (\nabla u \times \nabla v) =$.....
14. curl $(xy\mathbf{I} + yz\mathbf{J} + zx\mathbf{K}) =$ .....
15. If $\mathbf{F} = f_1\mathbf{I} + f_2\mathbf{J} + f_3\mathbf{K}$, then $\nabla . \mathbf{F} =$ .....
16. If $\mathbf{F}$ is a conservative force field then curl $\mathbf{F}$ is .....
17. If $\phi = 3x^2y - y^3z^2$, grad ϕ at the point $(1, -2, -1)$ is .....
18. curl $(x\mathbf{I} + y\mathbf{J} + z\mathbf{K}) =$ .....
19. Workdone by a particle along the square formed by the lines $y = \pm 1$ and $x = \pm 1$ under the force
$$\mathbf{F} = (x^2 + xy)\mathbf{I} + (x^2 + y^2)\mathbf{J} \text{ is.....}$$
20. Curl (grad ϕ) = ...
21. If $\mathbf{A}$ is a constant vector, then $div\ (\mathbf{A} \times \mathbf{R}) =$.....
22. If $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$, then $\nabla \log r =$ ..... ; $\nabla (r^n) =$ .....
23. A level surface is defined as .....
24. Unit normal vector to the surface $z = 2xy$ at the point $(2, 1, 4)$ is .....
25. If the directional derivative of $f = ax + by + cz$ at $(1, 1, 1)$ has maximum magnitude 4 in direction parallel to x-axis, then the values of a, b, c are .....
26. Maximum value of the directional derivative of $\phi = x^2 - 2y^2 + 4z^2$ at the point $(1, 1, -1)$ is .....
27. If $r^2 = x^2 + y^2 + z^2$, then $\nabla . (\mathbf{R}/r) =$ .....
28. Directional derivative of $f = xyz$ at the point $(1, -1, -2)$ in the direction of the vector $2\mathbf{I} - 2\mathbf{J} + \mathbf{K}$ is .....
29. If $\mathbf{V} = x^2\mathbf{I} + xye^z\mathbf{J} + \sin z\mathbf{K}$, then $\nabla . (\nabla \times F) =$ .....
30. If $f = \tan^{-1}(y/x)$ then div (grad f) is equal to
(a) 1 (b) -1 (c) 0 (d) 2.
31. The value of curl (grad f), where $f = 2x^2 - 3y^2 + 4z^2$ is
(a) $4x - 6y + 8z$, (b) $4x\mathbf{I} - 6y\mathbf{J} + 8z\mathbf{K}$ (c) 0 (d) 3.

32. The value of $\int \text{grad}\,(x + y - z)\, d\mathbf{R}$ from $(0, 1, -1)$ to $(1, 2, 0)$ is

(a) 0 (b) 3 (c) -1 (d) not obtainable.

33. If $\mathbf{F} = ax\mathbf{I} + by\mathbf{J} + cz\mathbf{K}$, then $\int_S \mathbf{F} \cdot d\mathbf{S}$, S being the surface of a unit sphere, is

(a) $(4/3)\pi(a + b + c)^2$ (b) 0 (c) $4\pi/3\,(a + b + c)$ (d) none of these.

34. A necessary and sufficient condition that the line integral $\int_L \mathbf{F} \cdot d\mathbf{R}$ for every closed C vanishes, is

(a) curl $\mathbf{F} = 0$ (b) div $\mathbf{F} = 0$ (c) curl $\mathbf{F} \neq 0$ (d) div $\mathbf{F} \neq 0$.

35. The value of $\iint_S (yzdydz + zxdz\,dx + xy\,dxdy)$, where S is the surface of unit sphere $x^2 + y^2 + z^2 = 1$ is

(a) 0 (b) 4π (c) $4\pi/3$ (d) 10π.

36. If $u = x^2 + y^2 + z^2$ and $\mathbf{V} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$, then $\nabla(u\mathbf{V}) =$ .....

37. For any scalar function ψ, $\nabla \times \nabla \psi =$ .....

38. $\int_C \mathbf{F} \cdot d\mathbf{R}$ is independent of the path joining any two points if and only if it is .....

39. The value of the line integral $\int_C (y^2dx + x^2dy)$ where C is the boundary of the square $-1 \le x \le 1, -1 \le y \le 1$ is

(a) 0 (b) $2(x + y)$ (c) 4 (d) 4/3. *(V.T.U., 2010)*

40. If $\mathbf{V}$ is the instantaneous velocity vector of the moving fluid at a point P, then div $\mathbf{V}$ represents .....

41. The spherical coordinate system is

(a) Orthogonal (b) Coplanar (c) Non-coplanar (d) Not orthogonal. *(V.T.U., 2010)*

42. Physical interpretation of $\nabla \phi$ is that .....

43. The magnitude of the vector drawn perpendicular to the surface $x^2 + 2y^2 + z^2 = 7$ at the point $(1, -1, 2)$ is

(a) 2/3 (b) 3/2 (c) 3 (d) 6.

44. The value of λ so that the vector $(x + 3y)\mathbf{I} + (y - 2z)\mathbf{J} + (x + \lambda z)\mathbf{K}$ is a solenoidal vector, is

(a) -2 (b) 3 (c) 1 (d) none of these.

45. The work done by the force $\mathbf{F} = yz\mathbf{I} + zx\mathbf{J} + xy\mathbf{K}$, in moving a particle from the point $(1, 1, 1)$ to the point $(3, 3, 2)$ along the path c is

(a) 17 (b) 10 (c) 0 (d) cannot be found.

46. Value of $\int_c (y^2dx + x^2dy)$ where c is the boundary of the square $-1 \le x \le 1, -1 \le y \le 1$, is

(a) 4 (b) 0 (c) $2(x + y)$ (d) 4/3.

47. The directional derivative of $f(x, y) = (x^2 - y^2)/xy$ at $(1, 1)$ is zero along a ray making an angle with the positive direction of x-axis :

(a) 45° (b) 60° (c) 135° (d) none of these.

48. The vector $\mathbf{V} = e^x \sin y\mathbf{I} + e^x \cos y\mathbf{J}$, is

(a) solenoidal (b) irrotational (c) rotational.

49. If $u = 1/r$ where $r^2 = x^2 + y^2$, then $\nabla^2 u = 0$. (True or False)

50. $\mathbf{F} = (x + 3y)\,\mathbf{I} + (z - 3y)\mathbf{J} + (x + 2z)\mathbf{K}$ is a solenoidal vector function. (True or False)

51. $\mathbf{F} = yz\mathbf{I} + zx\mathbf{J} + xy\mathbf{K}$ is irrotational. (True or False)

CHAPTER 9

Infinite Series

1. Introduction. 2. Sequences. 3. Series : Convergence. 4. General properties. 5. Series of positive terms— 6. Comparison tests. 7. Integral test. 8. Comparison of ratios. 9. D'Alembert's ratio test. 10. Raabe's test, Logarithmic test. 11. Cauchy's root test. 12. Alternating series ; Leibnitz's rule. 13. Series of positive or negative terms. 14. Power series. 15. Convergence of Exponential, Logarithmic and Binomial series. 16. Procedure for testing a series for convergence. 17. Uniform convergence. 18. Weierstrass's M-test. 19. Properties of uniformly convergent series. 20. Objective Type of Questions.

9.1 INTRODUCTION

Infinite series occur so frequently in all types of problems that the necessity of studying their convergence or divergence is very important. Unless a series employed in an investigation is convergent, it may lead to absurd conclusions. Hence it is essential that the students of engineering begin by acquiring an intelligent grasp of this subject.

9.2 SEQUENCES

(1) An ordered set of real numbers, $a_1, a_2, a_3, \ldots, a_n$ is called a *sequence* and is denoted by (a_n). If the number of terms is unlimited, then the sequence is said to be an *infinite sequence* and a_n is its *general term*.

For instance (*i*) 1, 3, 5, 7, ..., $(2n-1)$, ..., (*ii*) 1, 1/2, 1/3, ..., $1/n$, ...,

(*iii*) 1, – 1, 1, – 1, ..., $(-1)^{n-1}$, ... are infinite sequences.

(2) Limit. A sequence is said to tend to a limit l, if for every $\varepsilon > 0$, a value N of n can be found such that $|a_n - l| < \varepsilon$ for $n \geq N$.

We then write $\underset{n \to \infty}{\text{Lt}} (a_n) = l$ or simply $(a_n) \to l$ as $n \to \infty$.

(3) Convergence. *If a sequence (a_n) has a finite limit, it is called a* **convergent sequence.** If (a_n) is not convergent, it is said to be **divergent.**

In the above examples, (*ii*) is convergent, while (*i*) and (*iii*) are divergent.

(4) Bounded sequence. A sequence (a_n) is said to be bounded, if there exists a number k such that $a_n < k$ for every n.

(5) Monotonic sequence. The sequence (a_n) is said to increase steadily or to decrease steadily according as $a_{n+1} \geq a_n$ or $a_{n+1} \leq a_n$, for all values of n. Both increasing and decreasing sequences are called *monotonic sequences.*

A monotonic sequence always tends to a limit, finite or infinite. Thus, *a sequence which is monotonic and bounded is* **convergent.**

(6) Convergence, Divergence and Oscillation. If $\underset{n \to \infty}{\text{Lt}} (a_n) = l$ is finite and unique then the sequence is said to be *convergent*.

If $\underset{n \to \infty}{\text{Lt}} (a_n)$ is infinite ($\pm \infty$), the sequence is said to be *divergent*.

If $\underset{n \to \infty}{\text{Lt}} (a_n)$ is not unique, then (a_n) is said to be *oscillatory*.

Example 9.1. *Examine the following sequences for convergence :*

(*i*) $a_n = \dfrac{n^2 - 2n}{3n^2 + n}$ (*ii*) $a_n = 2^n$ (*iii*) $a_n = 3 + (-1)^n$.

Solution. (*i*) $\underset{n \to \infty}{\text{Lt}} \left(\dfrac{n^2 - 2n}{3n^2 + n} \right) = \underset{n \to \infty}{\text{Lt}} \dfrac{1 - 2/n}{3 + 1/n} = 1/3$ which is finite and unique. Hence the sequence (a_n) is convergent.

(*ii*) $\underset{n \to \infty}{\text{Lt}} (2^n) = \infty$. Hence the sequence (a_n) is divergent.

(*iii*) $\underset{n \to \infty}{\text{Lt}} [3 + (-1)^n] = 3 + 1 = 4$ when n is even

$= 3 - 1 = 2$, when n is odd

i.e., this sequence doesn't have a unique limit. Hence it oscillates.

PROBLEMS 9.1

Examine the convergence of the following sequences :

1. $a_n = \dfrac{3n - 1}{1 + 2n}$
2. $a_n = 1 + 2/n$
3. $a_n = [n + (-1)^n]^{-1}$
4. $a_n = \sin n$
5. $a_n = 1/2n$
6. $a_n = 1 + (-1)^n/n$
7. $\left(\dfrac{n}{n-1} \right)^2$
8. $a_n = 2n$.

9.3 SERIES

(1) Def. *If $u_1, u_2, u_3, \ldots, u_n, \ldots$ be an infinite sequence of real numbers, then*

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots \infty$$

is called an infinite series. An infinite series is denoted by Σu_n and the sum of its first n terms is denoted by s_n.

(2) Convergence, divergence and oscillation of a series.

Consider the infinite series $\Sigma u_n = u_1 + u_2 + u_3 + \ldots + u_n + \ldots \infty$

and let the sum of the first n terms be $s_n = u_1 + u_2 + u_3 + \ldots + u_n$

Clearly, s_n is a function of n and as n increases indefinitely three possibilities arise :

(*i*) If s_n tends to a finite limit as $n \to \infty$, the series Σu_n is said to be *convergent*.

(*ii*) If s_n tends to $\pm \infty$ as $n \to \infty$, the series Σu_n is said to be *divergent*.

(*iii*) If s_n does not tend to a unique limit as $n \to \infty$, then the series Σu_n is said to be *oscillatory* or *non-convergent*.

Example 9.2. *Examine for convergence the series (i) $1 + 2 + 3 + \ldots + n + \ldots \infty$.*
(ii) $5 - 4 - 1 + 5 - 4 - 1 + 5 - 4 - 1 + \ldots \infty$

Solution. (*i*) Here $s_n = 1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$

$\therefore$ $\underset{n \to \infty}{\text{Lt}} s_n = \dfrac{1}{2} \underset{n \to \infty}{\text{Lt}} n(n+1) \to \infty$. Hence this series is *divergent*.

(*ii*) Here $s_n = 5 - 4 - 1 + 5 - 4 - 1 + 5 - 4 - 1 + \ldots n$ terms

$= 0, 5$ or 1 according as the number of terms is $3m, 3m + 1, 3m + 2$.

Clearly in this case, s_n does not tend to a unique limit. Hence the series is *oscillatory*.

Examples 9.3. Geometric series. *Show that the series $1 + r + r^2 + r^3 + \ldots \infty$ (i) converges if $|r| < 1$, (ii) diverges if $r \geq 1$, and (iii) oscillates if $r \leq -1$.*

Solution. Let $$s_n = 1 + r + r^2 + \ldots + r^{n-1}$$

Case I. When $|r| < 1$, $\underset{n \to \infty}{\text{Lt}}\ r^n = 0$.

Also $$s_n = \frac{1 - r^n}{1 - r} = \frac{1}{1 - r} - \frac{r^n}{1 - r} \text{ so that } \underset{n \to \infty}{\text{Lt}}\ s_n = \frac{1}{1 - r}$$

$\therefore$ the series is convergent.

Case II. (*i*) When $r > 1$, $\underset{n \to \infty}{\text{Lt}}\ r^n \to \infty$.

Also $$s_n = \frac{r^n - 1}{r - 1} = \frac{r^n}{r - 1} - \frac{1}{r - 1} \text{ so that } \underset{n \to \infty}{\text{Lt}}\ s_n \to \infty$$

$\therefore$ the series is divergent.

(*ii*) When $r = 1$, then $s_n = 1 + 1 + 1 \ldots\ldots + 1 = n$

and $$\underset{n \to \infty}{\text{Lt}}\ s_n \to \infty \quad \therefore \text{ The series is divergent.}$$

Case III. (*i*) When $r = -1$, then the series becomes $1 - 1 + 1 - 1 + 1 - 1 \ldots$ which is an oscillatory series.

(*ii*) When $r < -1$, let $r = -\rho$ so that $\rho > 1$. Then $r^n = (-1)^n \rho^n$

and $$s_n = \frac{1 - r^n}{1 - r} = \frac{1 - (-1)^n \rho^n}{1 + \rho} \text{ as } \underset{n \to \infty}{\text{Lt}}\ \rho^n \to \infty.$$

$\therefore$ $\underset{n \to \infty}{\text{Lt}}\ s_n \to -\infty$ or $+\infty$ according as n is even or odd. Hence the series oscillates.

PROBLEMS 9.2

Examine the following series for convergence :

1. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \infty$.
2. $1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \frac{1}{3^4} - \ldots \infty$.
3. $6 - 10 + 4 + 6 - 10 + 4 + 6 - 10 + 4 + \ldots \infty$.
4. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots \infty$. *(V.T.U., 2006)*
5. A ball is dropped from a height h metres. Each time the ball hits the ground, it rebounds a distance r times the distance fallen where $0 < r < 1$. If $h = 3$ metres and $r = 2/3$, find the total distance travelled by the ball.

9.4 GENERAL PROPERTIES OF SERIES

The truth of the following properties is self-evident and these may be regarded as axioms :

1. *The convergence or divergence of an infinite series remains unaffected by the addition or removal of a finite number of its terms ;* for the sum of these terms being the finite quantity does not on addition or removal alter the nature of its sum.

2. *If a series in which all the terms are positive is convergent, the series remains convergent even when some or all of its terms are negative ;* for the sum is clearly the greatest when all the terms are positive.

3. *The convergence or divergence of an infinite series remains unaffected by multiplying each term by a finite number.*

9.5 SERIES OF POSITIVE TERMS

1. *An infinite series in which all the terms after some particular terms are positive, is a positive term series.* *e.g.*, $-7 - 5 - 2 + 2 + 7 + 13 + 20 + \ldots$ is a positive term series as all its terms after the third are positive.

2. *A series of positive terms either converges or diverges to $+\infty$;* for the sum of its first n terms, omitting the negative terms, tends to either a finite limit or $+\infty$.

3. **Necessary condition for convergence.** *If a positive term series Σu_n is convergent, then* $\underset{n\to\infty}{Lt}\, u_n = 0$.

(*P.T.U., 2009*)

Let $\quad s_n = u_1 + u_2 + u_3 + \ldots + u_n$. Since Σu_n is given to be convergent.

$\therefore \quad \underset{n\to\infty}{\text{Lt}}\, s_n =$ a finite quantity k (say). Also $\underset{n\to\infty}{\text{Lt}}\, s_{n-1} = k$

But $\quad u_n = s_n - s_{n-1} \quad \therefore \quad \underset{n\to\infty}{\text{Lt}}\, u_n = \underset{n\to\infty}{\text{Lt}}\, (s_n - s_{n-1}) = 0.$

Hence the result.

Obs. 1. *It is important to note that the converse of this result is not true.*

Consider, for instance, the series $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} + \ldots \infty$

Since the term go on descending,

$$\therefore \quad s_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \ldots + \frac{1}{\sqrt{n}} = \frac{n}{\sqrt{n}} \quad i.e., \ \sqrt{n}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\, s_n = \underset{n\to\infty}{\text{Lt}}\, \sqrt{n} \to \infty$$

Thus the series is divergent even though $\underset{n\to\infty}{\text{Lt}}\, u_n = \underset{n\to\infty}{\text{Lt}}\, \frac{1}{\sqrt{n}} = 0$

Hence $\underset{n\to\infty}{Lt}\, u_n = 0$ *is a necessary but not sufficient condition for convergence of* Σn_n.

Obs. The above result leads to a simple test for divergence :

If $\underset{n\to\infty}{Lt}\, u_n \neq 0$, *the series* Σu_n *must be divergent.*

9.6 COMPARISON TESTS

I. If two positive term series Σu_n and Σv_n be such that

(i) Σv_n converges, *(ii) $u_n \le v_n$ for all values of n, then Σu_n also converges.*

Proof. Since Σv_n is convergent,

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\, (v_1 + v_2 + v_3 + \ldots + v_n) = \text{a finite quantity } k \text{ (say)}$$

Also since $u_1 \le v_1, u_2 \le v_2, \ldots, u_n \le v_n$

$\therefore$ Adding, $\quad u_1 + u_2 + \ldots + u_n \le v \ + v_2 + \ldots + v_n$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\, (u_1 + u_2 + \ldots + u_n) \le \underset{n\to\infty}{\text{Lt}}\, (v_1 + v_2 + \ldots + v_n) = k.$$

Hence the series Σu_n also converges.

Obs. If, however, the relation $u_n \le v_n$ holds for values of n greater than a fixed number m, then the first m terms of both the series can be ignored without affecting their convergence or divergence.

II. If two positive term series Σu_n and Σv_n be such that :

(i) Σv_n diverges, *(ii) $u_n \ge v_n$ for all values of n, then Σu_n also diverges.*

Its proof is similar to that of Test *I*.

III. **Limit form**

If two positive term series Σu_n and Σv_n be such that

$$\underset{n\to\infty}{Lt}\, \frac{u_n}{v_n} = \textit{finite quantity } (\neq 0), \textit{ then } \Sigma u_n \textit{ and } \Sigma v_n \textit{ converge or diverge together.}$$

Proof. Since $\underset{n\to\infty}{\text{Lt}}\, \frac{u_n}{v_n} = l$, a finite number $(\neq 0)$

By definition of a limit, there exists a positive number ε, however small, such that

$$\left| \frac{u_n}{v_n} - l \right| < \varepsilon \qquad \text{for } n \ge m$$

or $$-\varepsilon < \frac{u_n}{v_n} - l < \varepsilon \qquad \text{for } n \geq m$$

or $$l - \varepsilon < \frac{u_n}{v_n} < l + \varepsilon \qquad \text{for } n \geq m$$

Omitting the first m terms of both the series, we have

$$l - \varepsilon < \frac{u_n}{v_n} < l + \varepsilon \qquad \text{for all } n \qquad ...(1)$$

Case I. When Σv_n is convergent, then

$$\underset{n \to \infty}{\text{Lt}} (v_1 + v_2 + ... + v_n) = k, \text{ a finite number} \qquad ...(2)$$

Also from (1), $\frac{u_n}{v_n} < l + \varepsilon$, *i.e.*, $u_n < (l + \varepsilon)v_n$ for all n.

$$\therefore \quad \underset{n \to \infty}{\text{Lt}} (u_1 + u_2 + ... + u_n) < (l + \varepsilon) \underset{n \to \infty}{\text{Lt}} (v_1 + v_2 + ... + v_n) = (l + \varepsilon)k \qquad \text{[By (2)]}$$

Hence Σu_n is also convergent.

Case II. When Σv_n is divergent, then

$$\underset{n \to \infty}{\text{Lt}} (v_1 + v_2 + ... + v_n) \to \infty \qquad ...(3)$$

Also from (1) $\quad l - \varepsilon < \frac{u_n}{v_n}$ or $u_n > (l - \varepsilon)v_n$ for all n

$$\therefore \quad \underset{n \to \infty}{\text{Lt}} (u_1 + u_2 + ... + u_n) > (l - \varepsilon) \underset{n \to \infty}{\text{Lt}} (v_1 + v_2 + ... + v_n) \to \infty \qquad \text{[By (3)]}$$

Hence Σu_n is also divergent.

9.7 INTEGRAL TEST

A positive term series $f(1) + f(2) + ... + f(n) + ...$, where $f(n)$ decreases as n increases, converges or diverges according as the integral

$$\int_1^\infty f(x)\,dx \qquad ...(1) \textit{ is finite or infinite.}$$

The area under the curve $y = f(x)$, between any two ordinates lies between the set of inscribed and escribed rectangles formed by ordinates at $x = 1, 2, 3, ...$ as in Fig. 9.1. Then

$$f(1) + f(2) + ... + f(n) \geq \int_1^{n+1} f(x)\,dx \geq f(2) + f(3) + ... + f(n+1)$$

or $$s_n \geq \int_1^{n+1} f(x)\,dx \geq s_{n+1} - f(1)$$

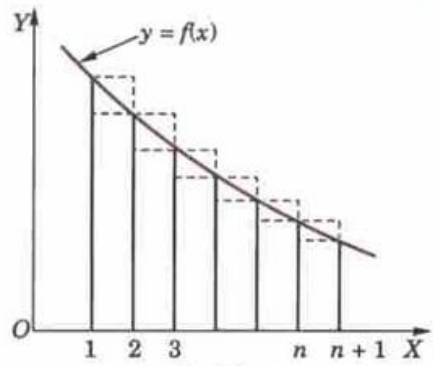

Fig. 9.1

Taking limits as $n \to \infty$, we find from the second inequality that $\text{Lt}\, s_{n+1} \leq \int_1^\infty f(x)\,dx + f(1)$.

Hence if integral (1) is finite, so is $\text{Lt}\, s_{n+1}$. Similarly, from the first inequality, we see that if the integral (1) is infinite, so is $\text{Lt}\, s_n$. But the given series either converges or diverges to $+\infty$, *i.e.*, $\text{Lt}\, s_n$ is either finite or infinite as $n \to \infty$.

Hence the result follows.

Example 9.4. Test for Comparison. *Show that the p-series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + ... \infty$$

(i) converges for $p > 1$ (ii) diverges for $p \leq 1$. *(P.T.U., 2009 ; V.T.U., 2006 ; Rohtak, 2003)*

Solution. By the above test, this series will converge or diverge according as $\int_1^\infty \frac{dx}{x^p}$ is finite or infinite.

If $p \neq 1$,
$$\int_1^\infty \frac{dx}{x^p} = \underset{m\to\infty}{\text{Lt}} \int_1^m \frac{dx}{x^p} = \underset{m\to\infty}{\text{Lt}} \left(\frac{m^{1-p}-1}{1-p}\right)$$
$$= \frac{1}{p-1}, \text{ i.e. finite} \quad \text{for } p > 1$$
$$\to \infty \quad \text{for } p < 1$$

If $p = 1$, $\int_1^\infty \frac{dx}{x} = \int_1^\infty \log x \to \infty$, this proves the result.

Obs. Application of comparison tests. Of all the above tests the 'limit form' is the most useful. To apply this comparison test to a given series Σu_n, the auxiliary series Σv_n must be so chosen that Lt (u_n/v_n) is non-zero and finite. To do this, we take v_n equal to that term of u_n which is of the highest degree in $1/n$ and the convergence or divergence of v_n is known with the help of the above series.

Example 9.5. *Test for convergence the series*

(*i*) $\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \ldots \infty$ (*P.T.U., 2009*)

(*ii*) $\frac{1}{4.7.10} + \frac{4}{7.10.13} + \frac{9}{10.13.16} + \ldots \infty$ (*V.T.U., 2010*)

(*iii*) $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \ldots \infty$

Solution. (*i*) We have $u_n = \frac{2n-1}{n(n+1)(n+2)} = \frac{1}{n^2}\frac{2-1/n}{(1+1/n)(1+2/n)}$

Take $v_n = 1/n^2$; then
$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{2-1/n}{(1+1/n)(1+2/n)} = \frac{2-0}{(1+0)(1+0)}$$
$$= 2, \text{ which is finite and non-zero}$$

$\therefore$ both Σu_n and Σv_n converge or diverge together.

But $\Sigma v_n = \Sigma 1/n^2$ is known to be convergent.

Hence Σu_n is also convergent.

(*ii*) Here
$$u_n = \frac{n^2}{(3n+1)(3n+4)(3n+7)} = \frac{1}{n\left(3+\frac{1}{n}\right)\left(3+\frac{4}{n}\right)\left(3+\frac{7}{n}\right)}$$

Taking $v_n = \frac{1}{n}$, we find that
$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{1}{\left(3+\frac{1}{n}\right)\left(3+\frac{4}{n}\right)\left(3+\frac{7}{n}\right)} = \frac{1}{27} \neq 0$$

Now since Σv_n is divergent, therefore Σu_n is also divergent.

(*iii*) Here $u_n = \frac{n^n}{(n+1)n+1} = \frac{1}{n+1}.\left(\frac{n}{n+1}\right)^n$, ignoring the first term.

Taking $v_n = 1/n$, we have
$$\underset{n\to\infty}{\text{Lt}} \left(\frac{u_n}{v_n}\right) = \underset{n\to\infty}{\text{Lt}} \frac{n}{n+1}.\underset{n\to\infty}{\text{Lt}} \left(\frac{n}{n+1}\right)^n$$
$$= \underset{n\to\infty}{\text{Lt}} \left(\frac{1}{1+1/n}\right).\underset{n\to\infty}{\text{Lt}} \frac{1}{(1+1/n)^n} = 1.\frac{1}{e} \neq 0$$

Now since Σv_n is divergent, therefore Σu_n is also divergent.

Example 9.6. *Test the convergence of the series :*

(i) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{(n+1)}}$ (V.T.U., 2008) (ii) $\sum_{n=1}^{\infty} \frac{1}{x^n + x^{-n}}$ (iii) $\sum_{n=1}^{\infty} \sqrt{\frac{3^n-1}{2^n+1}}$ (V.T.U., 2000 S)

Solution. (*i*) We have $u_n = \frac{\sqrt{(n+1)}-\sqrt{n}}{[\sqrt{(n+1)}+\sqrt{n}][\sqrt{(n+1)}-\sqrt{n}]} = \sqrt{(n+1)}-\sqrt{n}$

$$= \sqrt{n}\,[(1+1/n)^{1/2}-1] \qquad \text{(Expanding by Binomial Theorem)}$$

$$= \sqrt{n}\left\{\left(1+\frac{1}{2n}-\frac{1}{8n^2}+\ldots\right)-1\right\} = \sqrt{n}\left(\frac{1}{2n}-\frac{1}{8n^2}+\ldots\right) = \frac{1}{\sqrt{n}}\left(\frac{1}{2}-\frac{1}{8n}+\ldots\right)$$

Taking $v_n = 1/\sqrt{n}$, we have

$$\underset{u\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}}\left(\frac{1}{2}-\frac{1}{8n}+\ldots\right) = \frac{1}{2}, \text{ which is finite and non-zero.}$$

$\therefore$ both Σu_n and Σv_n converge or diverge together.

But $\Sigma v_n = \Sigma 1/\sqrt{n}$ is known to be divergent. Hence Σu_n is also divergent.

(*ii*) *When* $x < 1$, comparing the given series Σu_n with $\Sigma v_n = \Sigma x^n$,

we get $$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}}\left(\frac{1}{x^n+x^{-n}}\cdot\frac{1}{x^n}\right) = \underset{n\to\infty}{\text{Lt}} \frac{1}{x^{2n}+1} = 1 \qquad [\because\ x^{2n}\to 0 \text{ as } n\to\infty]$$

But Σv_n is convergent, so Σu_n is also convergent.

When $x > 1$, comparing Σu_n with $\Sigma w_n = \Sigma x^{-n}$, we get

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{w_n} = \underset{n\to\infty}{\text{Lt}}\left(\frac{1}{x^n+x^{-n}}\cdot x^n\right) = \underset{n\to\infty}{\text{Lt}} \frac{1}{1+x^{-2n}} = 1. \qquad [\because\ x^{-2n}\to 0 \text{ as } n\to\infty]$$

But Σw_n is convergent, so Σu_n is also convergent.

When $x = 1$, $\Sigma u_n = \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots\infty$ which is divergent.

Hence, Σu_n converges for $x < 1$ and $x > 1$ but diverges for $x = 1$.

(*iii*) Here $$u_n = \sqrt{\frac{3^n-1}{2^n+1}} = \left(\frac{3}{2}\right)^{n/2}\sqrt{\left(\frac{1-1/3^n}{1+1/2^n}\right)}$$

Taking $$v_n = \left(\frac{3}{2}\right)^{n/2}, \text{ we get}$$

$$\underset{n\to\infty}{\text{Lt}}\left(\frac{u_n}{v_n}\right) = \underset{n\to\infty}{\text{Lt}}\sqrt{\left(\frac{1-1/3^n}{1+1/2^n}\right)} = 1 \neq 0$$

Also since $\Sigma v_n = r^n$ where $r = \sqrt{3/2}$ is a geometric series having $r > 1$, is divergent.

$\therefore$ Σu_n is also divergent.

Example 9.7. *Determine the nature of the series :*

(i) $\frac{\sqrt{2}-1}{3^3-1}+\frac{\sqrt{3}-1}{4^3-1}+\frac{\sqrt{4}-1}{5^3-1}+\ldots\infty$ (ii) $\sum \frac{1}{n}\sin\frac{1}{n}$

(iii) $\sum_{1}^{\infty} \frac{(\log n)^2}{n^{3/2}}$ (iv) $\sum_{2}^{\infty} \frac{1}{n(\log n)^p}\ (p > 0)$ (P.T.U., 2010)

Solution. (*i*) We have $u_n = \frac{\sqrt{(n+1)}-1}{(n+2)^3-1} = \frac{\sqrt{n}\,[(1+1/n)-1/\sqrt{n}]}{n^3\,[(1+2/n)^3-1/n^3]}$

Taking $v_n = \dfrac{1}{n^{5/2}}$, we find that

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{[\sqrt{(1+1/n)} - 1/\sqrt{n}]}{[(1+2/n)^3 - 1/n^3]} = 1 \neq 0$$

Since Σv_n is convergent, therefore Σu_n is also convergent.

(*ii*) Here $$u_n = \frac{1}{n}\sin\frac{1}{n} = \frac{1}{n}\left[\frac{1}{n} - \frac{1}{3!\,n^3} + \frac{1}{5!\,n^5} - \ldots\right] = \frac{1}{n^2}\left[1 - \frac{1}{3!\,n^2} + \frac{1}{5!\,n^4} - \ldots\right]$$

Taking $v_n = \dfrac{1}{n^2}$, we have

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \left[1 - \frac{1}{3!\,n^2} + \frac{1}{5!\,n^4} \ldots\right] = 1 \neq 0$$

Since Σv_n is convergent, therefore Σu_n is also convergent.

(*iii*) We have $\underset{n\to\infty}{\text{Lt}} \dfrac{(\log n)^2}{n^{1/4}} = 0$, *i.e.*, $\dfrac{(\log n)^2}{n^{1/4}} < 1$ or $(\log n)^2 < n^{1/4}$

$$\therefore \qquad u_n = \frac{(\log n)^2}{n^{3/2}} < \frac{n^{1/4}}{n^{3/2}} = \frac{1}{n^{5/4}}$$

Since $\Sigma 1/n^{5/4}$ converges by p-series. $(\because \quad p = 5/4 > 1)$

Hence by comparison test, Σu_n also converges.

(*iv*) Let $$f(n) = \frac{1}{n(\log n)^p} \text{ so that } f(x) = \frac{(\log x)^{-p}}{x}$$

$$\therefore \qquad f'(x) = \frac{-p}{x}(\log x)^{-p-1} \cdot \frac{1}{x} + (\log x)^{-p} \cdot \left(-\frac{1}{x^2}\right) = -\frac{1}{x^2}\left\{\frac{p}{(\log x)^{p+1}} + \frac{1}{(\log x)^p}\right\} < 0$$

i.e., $f(x)$ is a decreasing function.

Also $$\int_2^\infty f(x)\,dx = \int_2^\infty \frac{dx}{x(\log x)^p} = \left|\frac{(\log x)^{-p+1}}{-p+1}\right|_2^\infty$$

If $p > 1$, then $p - 1 = k$(say) > 0

$$\therefore \qquad \int_2^\infty f(x)\,dx = \left|\frac{(\log x)^{-k}}{-k}\right|_2^\infty = \frac{1}{k}[0 + (\log 2)^{-k}] \text{ which is finite}$$

Thus by integral test, *the given series converges for* $p > 1$.

If $p < 1$, then $1 - p > 0$ and $(\log x)^{1-p} \to \infty$ as $x \to \infty$.

$$\therefore \qquad \int_2^\infty f(x)\,dx \to \infty.$$

Thus *the given series diverges for* $p < 1$.

If $p = 1$, then $$\int_2^\infty f(x)\,dx = \int_2^\infty \frac{dx}{x\log x} = |\log(\log x)|_2^\infty \to \infty$$

Thus *the given series diverges for* $p = 1$.

PROBLEMS 9.3

Test the following series for convergence :

1. $1 - \dfrac{1}{3} + \dfrac{1}{3^2} - \dfrac{1}{3^3} + \dfrac{1}{3^4} - \ldots\infty$ (*J.N.T.U., 2000*)
2. $\dfrac{3}{5} + \dfrac{4}{5^2} + \dfrac{3}{5^3} + \dfrac{4}{5^4} + \ldots\infty$
3. $\dfrac{1}{1.2} + \dfrac{2}{3.4} + \dfrac{3}{5.6} + \ldots\infty$ (*Cochin, 2001*)
4. $\dfrac{1}{1.3} + \dfrac{2}{3.5} + \dfrac{3}{5.7} + \ldots\infty$ (*P.T.U., 2009*)

5. $1+\frac{2^2}{2!}+\frac{3^2}{3!}+\frac{4^2}{4!}+\ldots\infty$

6. $\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\frac{5}{4^p}+\ldots\infty$

7. $\frac{1}{1^2}+\frac{1+2}{1^2+2^2}+\frac{1+2+3}{1^2+2^2+3^2}+\ldots\infty$

8. $\frac{1}{1.3.5}+\frac{2}{3.5.7}+\frac{3}{5.7.9}+\ldots\infty$ (V.T.U., 2009 S)

9. $\frac{3}{1}+\frac{4}{8}+\frac{5}{27}+\frac{6}{64}+\ldots\infty$

10. $\sum\frac{\sqrt{n}}{n^2+1}$ (Osmania, 2000 S)

11. $\sum_{n=0}^{\infty}\frac{2n^3+5}{4n^5+1}$

12. $\sum\frac{(n+1)(n+2)}{n^2\sqrt{n}}$ (J.N.T.U., 2006 S)

13. $\sum_{n=1}^{\infty}[\sqrt{(n^2+1)}-n]$ (V.T.U., 2010 ; P.T.U., 2009)

14. $\sum[\sqrt[3]{(n^3+1)}-n]$ (P.T.U., 2007 ; Rohtak 2003)

15. $\sum[\sqrt{(n^4+1)}-\sqrt{(n^4-1)}]$

16. $\sum\frac{1}{\sqrt{n}}\sin\frac{1}{n}$

17. $\sum_{n=2}^{\infty}\frac{1}{\sqrt{n\log n}}$

18. $\sum_{n=1}^{\infty}\frac{\sqrt{(n+1)}-1}{(n+2)^3-1}$ (J.N.T.U., 2003)

9.8 COMPARISON OF RATIOS

If Σu_n and Σv_n be two positive term series, then Σu_n converges if (i) Σv_n converges, and (ii) from and after some particular term,

$$\frac{u_{n+1}}{u_n}<\frac{u_{n+1}}{v_n}$$

Let the two series beginning from the particular term be $u_1+u_2+u_3+\ldots$ and $v_1+v_2+v_3+\ldots$

If $\frac{u_2}{u_1}<\frac{v_2}{v_1}, \frac{u_3}{u_2}<\frac{v_3}{v_2}, \ldots$

then $$u_1+u_2+u_3+\ldots=u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_1}+\ldots\right)$$

$$=u_1\left(1+\frac{u_2}{u_1}+\frac{u_2}{u_1}.\frac{u_3}{u_2}+\ldots\right)<u_1\left(1+\frac{v_2}{v_1}+\frac{v_2}{v_1}.\frac{v_3}{v_2}+\ldots\right)<\frac{u_1}{v_1}(v_1+v_2+v_3+\ldots).$$

Hence, if Σv_n converges, Σu_n also converges.

Obs. A more convenient form of the above test to apply is as follows :

Σu_n converges if (i) Σv_n converges and (ii) from and after a particular term $\frac{u_n}{u_{n+1}}>\frac{v_n}{v_{n+1}}$.

Similarly, Σu_n diverges, if (i) Σv_n diverges and (ii) from and after a particular term $\frac{u_n}{u_{n+1}}<\frac{v_n}{v_{n+1}}$.

9.9 D'ALEMBERT'S RATIO TEST*

In a positive term series Σu_n, if

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\lambda, \text{ then the series converges for } \lambda<1 \text{ and diverges for } \lambda>1.$$

Case I. *When* $\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\lambda<1.$

*Called after the French mathematician *Jean le-Rond d'Alembert* (1717–1783), who also made important contributions to mechanics.

By definition of a limit, we can find a positive number r (< 1) such that $\frac{u_{n+1}}{u_n} < r$ for all $n > m$

Leaving out the first m terms, let the series be $u_1 + u_2 + u_3 + \ldots$

so that $\frac{u_2}{u_1} < r, \frac{u_3}{u_2} < r, \frac{u_4}{u_3} < r, \ldots$ and so on. Then $u_1 + u_2 + u_3 + \ldots \infty$

$$= u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \frac{u_4}{u_3}\cdot\frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \ldots \infty\right) < u_1(1 + r + r^2 + r^3 + \ldots \infty)$$

$$= \frac{u_1}{1-r}, \text{ which is finite quantity. Hence } \Sigma u_n \text{ is convergent.} \qquad [\because \ r < 1]$$

Case II. *When* $\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \lambda > 1$

By definition of limit, we can find m, such that $\frac{u_{n+1}}{u_n} \geq 1$ for all $n \geq m$.

Leaving out the first m terms, let the series be

$$u_1 + u_2 + u_3 + \ldots \text{ so that } \frac{u_2}{u_1} \geq 1, \quad \frac{u_3}{u_2} \geq 1, \quad \frac{u_4}{u_3} \geq 1 \text{ and so on.}$$

$$\therefore \qquad u_1 + u_2 + u_3 + u_4 + \ldots + u_n = u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \ldots\right)$$

$$\geq u_1(1 + 1 + 1 + \ldots \text{ to } n \text{ terms}) = nu_1$$

$$\therefore \qquad \underset{n\to\infty}{\text{Lt}} (u_1 + u_2 + \ldots + u_n)$$

$$\geq \underset{n\to\infty}{\text{Lt}} (nu_1), \text{ which tends to infinity. Hence } \Sigma u_n \text{ is divergent.}$$

Obs. 1. *Ratio test fails when $\lambda = 1$. Consider, for instance, the series $\Sigma u_n = \Sigma 1/n^p$.*

Here $$\lambda = \underset{n\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{\text{Lt}} \left[\frac{1}{(n+1)^p}\cdot\frac{n^p}{1}\right] = \underset{n\to\infty}{\text{Lt}} \frac{1}{(1+1/n)^p} = 1.$$

Then for all values of p, $\lambda = 1$; whereas $\Sigma 1/n^p$ converges for $p > 1$ and diverges for $p < 1$.

Hence $\lambda = 1$ both for convergence and divergence of Σu_n, which is absurd.

Obs. 2. *It is important to note that this test makes no reference to the magnitude of u_{n+1}/u_n but concerns only with the limit of this ratio.*

For instance in the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} + \ldots$, the ratio $\frac{u_{n+1}}{u_n} = \frac{n}{n+1} < 1$ for all finite values of n, but tends to unity as $n \to \infty$. Hence the Ratio test fails although this series is divergent.

Practical form of Ratio test. Taking reciprocals, the ratio test can be stated as follows :

In the positive term series Σu_n, if $\underset{n\to\infty}{Lt} \frac{u_n}{u_{n+1}} = k$, then the series converges for $k > 1$ and diverges for $k < 1$ but fails for $k = 1$.

Example. 9.8. *Test for convergence the series*

(*i*) $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \ldots \infty.$ *(P.T.U., 2005 ; V.T.U., 2003 ; I.S.M., 2001)*

(*ii*) $1 + \frac{2}{5}x + \frac{6}{9}x^2 + \frac{14}{17}x^3 + \ldots + \frac{2^n - 2}{2^n + 1}x^{n-1} + \ldots (x > 0).$ *(P.T.U., 2009 ; V.T.U., 2004)*

Solution. (*i*) We have $u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}}$ and $u_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{(n+1)}}$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{x^{2n-2}}{(n+1)\sqrt{n}} \cdot \frac{(n+2)\sqrt{(n+1)}}{x^{2n}}$$

$$= \underset{n\to\infty}{\text{Lt}} \left[\frac{n+2}{n+1}\left(\frac{n+1}{n}\right)^{1/2}\right] x^{-2} = \underset{n\to\infty}{\text{Lt}} \left[\frac{1+2/n}{1+1/n} \cdot \sqrt{(1+1/n)}\right] x^{-2} = x^{-2}.$$

Hence Σu_n converges if $x^{-2} > 1$, *i.e.*, for $x^2 < 1$ and diverges for $x^2 > 1$.

If $x^2 = 1$, then, $u_n = \dfrac{1}{(n+1)\sqrt{n}} = \dfrac{1}{n^{3/2}} \cdot \dfrac{1}{1+1/n}$

Taking $v_n = \dfrac{1}{n^{3/2}}$, we get $\underset{n\to\infty}{\text{Lt}} \dfrac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \dfrac{1}{1+1/n} = 1$, a finite quantity.

$\therefore$ Both Σu_n and Σv_n converge or diverge together. But $\Sigma v_n = \sum \dfrac{1}{n^{3/2}}$ is a convergent series.

$\therefore$ Σu_n is also convergent. Hence the given series converges if $x^2 \le 1$ and diverges if $x^2 > 1$.

(*ii*) Here
$$\frac{u_n}{u_{n+1}} = \frac{2^n - 2}{2^n + 1} x^{n-1} \cdot \frac{2^{n+1}+1}{2^{n+1}-2} \frac{1}{x^n} = \frac{1 - \dfrac{2}{2^n}}{1 + \dfrac{1}{2^n}} \cdot \frac{2 + \dfrac{1}{2^n}}{2 - \dfrac{2}{2^n}} \frac{1}{x}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{1-0}{1+0} \cdot \frac{2+0}{2-0} \frac{1}{x} = \frac{1}{x}$$

Thus by Ratio test, Σu_n converges for $x^{-1} > 1$ *i.e.*, for $x < 1$ diverges for $x > 1$. But it fails for $x = 1$.

When $x = 1$,
$$\underset{n\to\infty}{\text{Lt}}\, u_n = \underset{n\to\infty}{\text{Lt}} \frac{2^n - 2}{2^n + 1} = \underset{n\to\infty}{\text{Lt}} \frac{1 - \dfrac{2}{2^n}}{1 + \dfrac{1}{2^n}} = 1 \ne 0$$

$\therefore$ Σu_n diverges for $x = 1$. Hence the given series converges for $x < 1$ and diverges for $x \ge 1$.

Example 9.9. *Discuss the convergence of the series*

(*i*) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(n^n)^2}$ (*P.T.U., 2010*) (*ii*) $1 + \dfrac{2!}{2^2} + \dfrac{3!}{3^3} + \dfrac{4!}{4^4} + \ldots \infty$ (*V.T.U., 2008 S*)

Solution. (*i*) We have $u_n = \dfrac{n!}{(n^n)^2}$ and $u_{n+1} = \dfrac{(n+1)!}{[(n+1)^{n+1}]^2}$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{n!}{n^{2n}} \times \frac{(n+1)^{2(n+1)}}{(n+1)!}$$

$$= \underset{n\to\infty}{\text{Lt}} \frac{(n+1)^{2n+1}}{n^{2n}} = \underset{n\to\infty}{\text{Lt}} \left(\frac{n+1}{n}\right)^{2n} . (n+1)$$

$$= \underset{n\to\infty}{\text{Lt}} \left[\left(1 + \frac{1}{n}\right)^n\right]^2 . (n+1) = e. \underset{n\to\infty}{\text{Lt}} (n+1) \to \infty$$

Hence the given series is convergent.

(*ii*) Given series is $\Sigma u_n = \displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$. Here $\dfrac{u_n}{u_{n+1}} = \dfrac{n!}{n^n} \cdot \dfrac{(n+1)^{n+1}}{(n+1)!} = \dfrac{(n+1)^n}{n^n} = \left(1 + \dfrac{1}{n}\right)^n$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \left(1 + \frac{1}{n}\right)^n = e,$$ which is > 1. Hence the given series is convergent.

Example 9.10. *Examine the convergence of the series :*

(*i*) $\dfrac{x}{1+x}+\dfrac{x^2}{1+x^2}+\dfrac{x^3}{1+x^3}+\ldots\infty$

(*ii*) $1+\dfrac{a+1}{b+1}+\dfrac{(a+1)(2a+1)}{(b+1)(2b+1)}+\dfrac{(a+1)(2a+1)(3a+1)}{(b+1)(2b+1)(3b+1)}+\ldots\infty$

Solution. (*i*) Here $\quad u_n=\dfrac{x^n}{1+x^n}$ and $u_{n+1}=\dfrac{x^{n+1}}{1+x^{n+1}}$

$$\therefore \qquad \underset{n\to\infty}{\text{Lt}}\frac{u_n}{u_{n+1}}=\underset{n\to\infty}{\text{Lt}}\left(\frac{x^n}{x^{n+1}}\cdot\frac{1+x^{n+1}}{1+x^n}\right)=\underset{n\to\infty}{\text{Lt}}\left(\frac{1+x^{n+1}}{x+x^{n+1}}\right)$$

$$=\frac{1}{x}, \text{ if } x<1. \qquad [\because\ x^{n+1}\to 0 \text{ as } n\to\infty]$$

Also $$\underset{n\to\infty}{\text{Lt}}\frac{u_n}{u_{n+1}}=\underset{n\to\infty}{\text{Lt}}\left(\frac{1+1/x^{n+1}}{1+x/x^{n+1}}\right)=1 \text{ if } x>1.$$

$\therefore$ by *Ratio test*, Σu_n converges for $x<1$ and fails for $x\geq 1$.

When $x=1$, $\Sigma u_n=\dfrac{1}{2}+\dfrac{1}{2}+\dfrac{1}{2}+\ldots+\infty$, which is divergent.

Hence the given series converges for $x<1$ and diverges for $x\geq 1$.

(*ii*) Neglecting the first term, we have

$$u_{n+1}=u_n\cdot\frac{n_{a+1}}{n_{b+1}}$$

$$\therefore \qquad \underset{n\to\infty}{\text{Lt}}\frac{u_n}{u_{n+1}}=\underset{n\to\infty}{\text{Lt}}\frac{n_{b+1}}{n_{a+1}}=\underset{n\to\infty}{\text{Lt}}\frac{b+1/n}{a+1/n}=\frac{b}{a}.$$

By *Ratio test*, Σu_n converges for $b/a>1$ or $a<b$, and diverges for $a>b$.

When $a=b$, the series becomes $1+1+1+\ldots\infty$, which is divergent.

Hence the given series converges for $0<a<b$ and diverges for $0<b\leq a$.

PROBLEMS 9.4

Test for convergence the following series :

1. $x+\dfrac{x^2}{2}+\dfrac{x^3}{3}+\ldots\infty.$

2. $\sqrt{\dfrac{1}{2}}x+\sqrt{\dfrac{2}{3}}x^2+\sqrt{\dfrac{3}{4}}x^3+\ldots\infty$

3. $1+\dfrac{x}{2}+\dfrac{x^2}{5}+\dfrac{x^3}{10}+\ldots+\dfrac{x^n}{n^2+1}\ldots\infty$

4. $\displaystyle\sum_{n=2}^{\infty}\frac{x^n}{n(n-1)(n-2)}$ (*J.N.T.U., 2006*)

5. $1+\dfrac{2^p}{2!}+\dfrac{3^p}{3!}+\dfrac{4^p}{4!}+\ldots\infty$ (*Kurukshetra, 2005*)

6. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n^2}{2^n}+\frac{1}{n^2}\right)$ (*Rohtak, 2005*)

7. $\displaystyle\sum_{n=1}^{\infty}\frac{n!\,3^n}{n^n}$ (*Kerala, 2005*)

8. $\displaystyle\sum_{n=1}^{\infty}\frac{n^3+a}{2^n+a}$

9. $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}}{\sqrt{(n^2+1)}}x^n$ (*P.T.U., 2006*)

10. $\displaystyle\sum_{1}^{\infty}\frac{n^3-n+1}{n!}$ (*Madras, 2000*)

11. $\dfrac{2}{3.4}+\dfrac{2.4}{3.5.6}+\dfrac{2.4.6}{3.5.7.8}+\ldots$ (*V.T.U., 2010*)

12. $\left(\dfrac{1}{3}\right)^2+\left(\dfrac{1.2}{3.5}\right)^2+\left(\dfrac{1.2.3}{3.5.7}\right)^2+\ldots$

13. $1+\frac{1^2.2^2}{1.3.5}+\frac{1^2.2^2.3^2}{1.3.5.7.9}+\ldots\infty$ *(Delhi, 2002)*

14. $\frac{4}{18}+\frac{4.12}{18.27}+\frac{4.12.20}{18.27.36}+\ldots\infty$ *(Madras, 2000)*

15. $\frac{1}{1^p}+\frac{x}{3^p}+\frac{x^2}{5^p}+\ldots+\frac{x^{n-1}}{(2n-1)^p}+\ldots\infty$ *(J.N.T.U., 2006)*

16. $\sum_{n=1}^{\infty}\frac{3.6.9\ldots3n}{4.7.10\ldots(3n+1)}.\frac{5^n}{3n+2}$ *(V.T.U., 2004)*

17. $1+\frac{1+\alpha}{1+\beta}+\frac{(1+\alpha)(1+2\alpha)}{(1+\beta)(1+2\beta)}+\frac{(1+\alpha)(1+2\alpha)(1+3\alpha)}{(1+\beta)(1+2\beta)(1+3\beta)}+\ldots$

9.10 FURTHER TESTS OF CONVERGENCE

When the Ratio test fails, we apply the following tests :

(1) Raabe's test*. *In the positive term series* Σu_n, if $\underset{n\to\infty}{\text{Lt}}\, n\left(\frac{u_n}{u_{n+1}}-1\right)=k$,

then the series converges for $k>1$ *and diverges for* $k<1$, but the test fails for $k=1$.

When $k>1$, choose a number p such that $k>p>1$, and compare Σu_n with the series $\sum\frac{1}{n^p}$ which is convergent since $p>1$.

$\therefore$ Σu_n will converge, if from and after some term,

$$\frac{u_n}{u_{n+1}}>\frac{(n+1)^p}{n^p} \text{ or } \left(1+\frac{1}{n}\right)^p \quad \text{or if,} \quad \frac{u_n}{u_{n+1}}>1+\frac{p}{n}+\frac{p(p-1)}{2n^2}+\ldots$$

or if, $$n\left(\frac{u_n}{u_{n+1}}-1\right)>p+\frac{p(p-1)}{2n}+\ldots \quad \text{or if,} \quad \underset{n\to\infty}{\text{Lt}}\, n\left(\frac{u_n}{u_{n+1}}-1\right)>\underset{n\to\infty}{\text{Lt}}\left[p+\frac{p(p-1)}{2n}+\ldots\right]$$

i.e., if $k>p$, which is true. Hence Σu_n is convergent.

The other case when $k<1$ can be proved similarly.

(2) Logarithmic test. *In the positive term series* Σu_n *if* $\underset{n\to\infty}{Lt}\left(n\log\frac{u_n}{u_{n+1}}\right)=k$,

then the series converges for $k>1$, *and diverges for* $k<1$, but the test fails for $k=1$.

Its proof is similar to that of Raabe's test.

Obs. 1. *Logarithmic test is a substitute for Raabe's test and should be applied when* **either** *n occurs as an exponent in* u_n/u_{n+1}, **or** *evaluation of* $\underset{n\to\infty}{Lt}$ *becomes easier on taking logarithm of* u_n/u_{n+1}.

Obs. 2. If u_n/u_{n+1} does not involve n as an exponent or a logarithm, the series Σu_n diverges.

Example 9.11. *Test for convergence the series*

(*i*) $\sum\frac{4.7\ldots(3n+1)}{1.2\ldots n}x^n$ *(V.T.U., 2009 ; P.T.U., 2006 S)* (*ii*) $\sum\frac{(n!)^2}{(2n)!}x^{2n}$.

Solution. (*i*) Here $\frac{u_n}{u_{n+1}}=\frac{4.7\ldots(3n+1)}{1.2\ldots n}x^n\div\frac{4.7\ldots(3n+4)}{1.2\ldots(n+1)}x^{n+1}=\frac{n+1}{3n+4}.\frac{1}{x}=\left[\frac{1+1/n}{3+4/n}\right]\frac{1}{x}$

$\therefore$ $$\underset{n\to\infty}{\text{Lt}}\frac{u_n}{u_{n+1}}=\frac{1}{3x}.$$

*Called after the Swiss mathematician *Joseph Ludwig Raabe* (1801–1859).

Thus by *Ratio test*, the series converges for $\frac{1}{3x} > 1$, *i.e.*, for $x < \frac{1}{3}$ and diverges for $x > \frac{1}{3}$. But it fails for $x = \frac{1}{3}$. $\therefore$ Let us try the *Raabe's test.*

Now $$\frac{u_n}{u_{n+1}} = \left(1+\frac{1}{n}\right)\left(1+\frac{4}{3n}\right)^{-1} \qquad \text{[Expand by Binomial Theorem]}$$

$$= \left(1+\frac{1}{n}\right)\left(1-\frac{4}{3n}+\frac{16}{9n^2}-\ldots\right) = 1-\frac{1}{3n}+\frac{4}{9n^2}+\ldots$$

$\therefore$ $$n\left(\frac{u_n}{u_{n+1}}-1\right) = -\frac{1}{3}+\frac{4}{9n}+\ldots \quad \therefore \quad \underset{n\to\infty}{\text{Lt}}\, n\left(\frac{u_n}{u_{n+1}}-1\right) = -\frac{1}{3} \text{ which} < 1.$$

Thus by *Raabe's test*, the series diverges.

Hence the given series converges for $x < \frac{1}{3}$ and diverges for $x \geq \frac{1}{3}$.

(*ii*) Here $$\frac{u_n}{u_{n+1}} = \left(\frac{n!}{(n+1)!}\right)^2 \frac{[2(n+1)]!}{(2n)!}\cdot\frac{x^{2n}}{x^{2(n+1)}} = \frac{(2n+1)(2n+2)}{(n+1)^2}\,\frac{1}{x^2} = \frac{2(2n+1)}{n+1}\,\frac{1}{x^2}$$

$\therefore$ $$\underset{n\to\infty}{\text{Lt}}\,\frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}}\,\frac{2(2+1/n)}{1+1/n}\cdot\frac{1}{x^2} = \frac{4}{x^2}$$

Thus by *Ratio Test*, the series converges for $x^2 < 4$ and diverges for $x^2 > 4$. But fails for $x^2 = 4$.

When $x^2 = 4$, $$n\left(\frac{u_n}{u_{n+1}}-1\right) = n\left(\frac{2n+1}{2n+2}-1\right) = -\frac{n}{2n+2}$$

$\therefore$ $$\underset{n\to\infty}{\text{Lt}}\, n\left(\frac{u_n}{u_{n+1}}-1\right) = -\frac{1}{2} < 1$$

Thus by *Raabe's test*, the series diverges.

Hence the given series converges for $x^2 < 4$ and diverges for $x^2 \geq 4$.

Example 9.12. *Discuss the convergence of the series*

$$x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \frac{4^4x^4}{4!} + \frac{5^5x^5}{5!} + \ldots\infty \qquad \textit{(P.T.U., 2008 ; Cochin, 2005 ; Rohtak, 2003)}$$

Solution. Here $$\frac{u_n}{u_{n+1}} = \frac{n^n x^n}{n!} \div \frac{(n+1)^{n+1}\,x^{n+1}}{(n+1)!} = \frac{n^n}{(n+1)^n\,x} = \frac{1}{(1+1/n)^n}\cdot\frac{1}{x}$$

$\therefore$ $$\underset{n\to\infty}{\text{Lt}}\,\frac{u_n}{u_{n+1}} = \frac{1}{ex}.$$

Thus by Ratio test, the series converges for $x < 1/e$ and diverges for $x > 1/e$. But it fails for $x = 1/e$. Let us try the *log-test*.

Now $$\frac{u_n}{u_{n+1}} = \frac{e}{(1+1/n)^n}$$

$\therefore$ $$\log\frac{u_n}{u_{n+1}} = \log_e e - n\log\left(1+\frac{1}{n}\right) = 1 - n\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\ldots\right) = \frac{1}{2n}-\frac{1}{3n^2}+\ldots$$

$\therefore$ $$\underset{n\to\infty}{\text{Lt}}\left(n\log\frac{u_n}{u_{n+1}}\right) = \frac{1}{2},$$ which < 1. Thus by the *log-test*, the series diverges.

Hence the given series converges for $x < 1/e$ and diverges for $x \geq 1/e$.

Example 9.13. *Discuss the convergence of the* ***hypergeometric series***

$$1 + \frac{\alpha.\beta}{1.\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3 + \ldots\infty. \qquad \textit{(Kurukshetra, 2005)}$$

Solution. Neglecting the first term, we have

$$u_{n+1} = u_n \frac{(\alpha+n)(\beta+n)}{(n+1)(\gamma+n)} x$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)} \cdot \frac{1}{x} = \underset{n\to\infty}{\text{Lt}} \frac{(1+1/n)(1+\gamma/n)}{(1+\alpha/n)(1+\beta/n)} \cdot \frac{1}{x} = \frac{1}{x}$$

$\therefore$ by *Ratio test,* the series converges for $1/x > 1$, *i.e.*, for $x < 1$, and diverges for $x > 1$. But it fails for $x = 1$.

$\therefore$ let us try the *Raabe's test.*

$$\underset{n\to\infty}{\text{Lt}}\, n\left(\frac{u_n}{u_{n+1}} - 1\right) = \underset{n\to\infty}{\text{Lt}}\, n\left\{\frac{(n+1)(n+\gamma)}{(n+\alpha)(n+\beta)} - 1\right\} = \underset{n\to\infty}{\text{Lt}}\, n\left\{\frac{n(1+\gamma-\alpha-\beta)+\gamma-\alpha\beta}{n^2+n(\alpha+\beta)+\alpha\beta}\right\}$$

$$= \underset{n\to\infty}{\text{Lt}} \left\{\frac{(1+\gamma-\alpha-\beta)+(\gamma-\alpha\beta)\frac{1}{n}}{1+(\alpha+\beta)\frac{1}{n}+\alpha\beta\cdot\frac{1}{n^2}}\right\} = 1+\gamma-\alpha-\beta$$

Thus the series converges for $1+\gamma-\alpha-\beta > 1$, *i.e.*, for $\gamma > \alpha+\beta$ and diverges for $\gamma < \alpha+\beta$. But it fails for $\gamma = \alpha+\beta$. Since u_n/u_{n+1} does not involve n as an exponent or a logarithm, the series Σu_n diverges for $\gamma = \alpha+\beta$.

Hence the series converges for $x < 1$ and diverges for $x > 1$. When $x = 1$, the series converges for $\gamma > \alpha+\beta$ and diverges for $\gamma \le \alpha+\beta$.

PROBLEMS 9.5

Test the following series for convergence :

1. $\frac{x}{1.2}+\frac{x^2}{3.4}+\frac{x^3}{5.6}+\frac{x^4}{7.8}+\ldots\infty\ (x>0)$ *(Mumbai, 2009)*

2. $\frac{x}{1.2}+\frac{x^2}{2.3}+\frac{x^3}{3.4}+\frac{x^4}{4.5}+\ldots\infty$ *(V.T.U., 2008 ; J.N.T.U., 2003)*

3. $1+\frac{1}{2}x+\frac{1.3}{2.4}x^2+\frac{1.3.5}{2.4.6}x^3+\ldots\infty\ (x>0)$ *(Raipur, 2005)*

4. $1+\frac{2}{3}x+\frac{2.3}{3.5}x^2+\frac{2.3.4}{3.5.7}x^3+\ldots\infty$ *(V.T.U., 2009 S)*

5. $1+\frac{x}{2}+\frac{2!}{3^2}x^2+\frac{3!}{4^3}x^3+\frac{4!}{5^4}x^4+\ldots\infty$

6. $1+\frac{3}{7}x+\frac{3.6}{7.10}x^2+\frac{3.6.9}{7.10.13}x^3+\frac{3.6.9.12}{7.10.13.16}x^4+\ldots\infty$

7. $\frac{x}{1}+\frac{1}{2}.\frac{x^3}{3}+\frac{1.3}{2.4}.\frac{x^5}{5}+\frac{1.3.5}{2.4.6}.\frac{x^7}{7}+\ldots\infty.\ (x>0)$ *(V.T.U., 2007 ; Raipur, 2005)*

8. $1+\frac{1}{2}.\frac{x^2}{4}+\frac{1.3.5}{2.4.6}.\frac{x^4}{8}+\frac{1.3.5.7.9}{2.4.6.8.10}.\frac{x^6}{12}+\ldots\infty$ *(Rohtak, 2006 S ; Roorkee, 2000)*

9. $1+\frac{(1!)^2}{2!}x^2+\frac{(2!)^2}{4!}x^4+\frac{(3!)^2}{6!}x^6+\ldots\infty\ (x>0)$

10. $\frac{1^2}{4^2}+\frac{1^2.5^2}{4^2.8^2}+\frac{1^2.5^2.9^2}{4^2.8^2.12^2}+\ldots\infty$

11. $\frac{a+x}{1!}+\frac{(a+2x)^2}{2!}+\frac{(a+3x)^3}{3!}+\ldots\infty$

12. $x^2(\log 2)^q+x^3(\log 3)^q+x^4(\log 4)^q+\ldots\infty$

13. $\frac{1}{1^2}+\frac{1+2}{1^2+2^2}+\frac{1+2+3}{1^2+2^2+3^2}+\ldots$ *(V.T.U., 2000)*

14. $1+\frac{a}{b}x+\frac{a(a+1)}{b(b+1)}x^2+\frac{a(a+1)(a+2)}{b(b+1)(b+2)}x^3+\ldots\infty\ (a, b>0, x>0).$

9.11 CAUCHY'S ROOT TEST*

In a positive series Σu_n, if $\underset{n\to\infty}{Lt}\,(u_n)^{1/n} = \lambda$,

then the series converges for $\lambda < 1$, and diverges for $\lambda > 1$.

Case I. When $\underset{n\to\infty}{\text{Lt}}\,(u_n)^{1/n} = \lambda < 1$.

By definition of a limit, we can find a positive number r ($\lambda < r < 1$) such that

$$(u_n)^{1/n} < r \text{ for all } n > m, \quad \text{or} \quad u_n < r^n \text{ for all } n > m.$$

Since $r < 1$, the geometric series Σr^n is convergent. Hence, by comparison test, Σu_n is also convergent.

Case II. When $\text{Lt}\,(u_n)^{1/n} = \lambda > 1$.

By definition of a limit, we can find a number m, such that

$$(u_n)^{1/n} > 1 \text{ for all } n > m, \quad \text{or} \quad u_n > 1 \text{ for all } n > m.$$

Omitting the first m terms, let the series be $u_1 + u_2 + u_3 + \ldots$ so that $u_1 > 1, u_2 > 1, u_3 > 1$ and so on.

$$\therefore \qquad u_1 + u_2 + u_3 + \ldots + u_n > n \quad \text{and} \quad \underset{n\to\infty}{\text{Lt}}\,(u_1 + u_2 + \ldots + u_n) \to \infty$$

Hence the series Σu_n is divergent.

Obs. *Cauchy's root test fails when $\lambda = 1$.*

Example 9.14. *Test for convergence the series*

(*i*) $\Sigma \dfrac{n^3}{3^n}$ (*ii*) $\Sigma (\log n)^{-2n}$ (*iii*) $\Sigma (1 + 1/\sqrt{n})^{-n^{3/2}}$ *(P.T.U., 2009 ; Kurukshetra, 2005)*

Solution. (*i*) We have $u_n = n^3/3^n$.

$$\therefore \qquad \underset{n\to\infty}{\text{Lt}}\,(u_n)^{1/n} = \underset{n\to\infty}{\text{Lt}}\left(\frac{n^{3/n}}{3}\right) = \underset{n\to\infty}{\text{Lt}}\,\frac{(n^{1/n})^3}{3} = \frac{1}{3}\ (< 1) \qquad \left[\because \ \underset{n\to\infty}{\text{Lt}}\, n^{1/n} = 1\right]$$

Hence the given series converges by Cauchy's root test.

(*ii*) Here $\qquad u_n = (\log n)^{-2n}$

$$\therefore \qquad \underset{n\to\infty}{\text{Lt}}\,(u_n)^{1/n} = \underset{n\to\infty}{\text{Lt}}\,(\log n)^{-2} = 0\ (< 1) \qquad [\because \ \underset{n\to\infty}{\text{Lt}}\, \log n = 0]$$

Hence, by Cauchy's root test, the given series converges.

(*iii*) Here $\qquad u_n = (1 + 1/\sqrt{n})^{-n^{3/2}}$

$$\therefore \qquad (u_n)^{1/n} = \left[\frac{1}{(1 + 1/\sqrt{n})^{n^{3/2}}}\right]^{1/n} = \frac{1}{(1 + 1/\sqrt{n})^{\sqrt{n}}}$$

$$\therefore \qquad \underset{n\to\infty}{\text{Lt}}\,(u_n)^{1/n} = \underset{n\to\infty}{\text{Lt}}\,\frac{1}{(1 + 1/\sqrt{n})^{\sqrt{n}}} = \frac{1}{e}, \text{ which is } < 1. \text{ Hence the given series is convergent.}$$

Example 9.15. *Discuss the nature of the following series :*

(*i*) $\dfrac{1}{2} + \dfrac{2}{3}x + \left(\dfrac{3}{4}\right)^2 x^2 + \left(\dfrac{4}{5}\right)^3 x^3 + \ldots \infty\ (x > 0)$ *(J.N.T.U., 2006)*

(*ii*) $\Sigma \dfrac{(n+1)^n\, x^n}{x^{n+1}}$

(*iii*) $\left(\dfrac{2^2}{1^2} - \dfrac{2}{1}\right)^{-1} + \left(\dfrac{3^3}{2^3} - \dfrac{3}{2}\right)^{-2} + \left(\dfrac{4^4}{3^4} - \dfrac{4}{3}\right)^{-3} + \ldots \infty$ *(V.T.U., 2006)*

*See footnote p. 144.

Solution. (*i*) After leaving the first term, we find that $u_n = \left(\frac{n+1}{n+2}\right)^n x^n$, so that

$$\underset{n\to\infty}{\text{Lt}}\,(u_n)^{1/n} = \underset{n\to\infty}{\text{Lt}}\left(\frac{1+1/n}{1+2/n}\right)x = x$$

$\therefore$ By Cauchy's root test, the given series converges for $x < 1$ and diverges for $x > 1$.

When $x = 1$, $\quad u_n = \left(\frac{n+1}{n+2}\right)^n = \dfrac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}}\left(1+\frac{1}{n+1}\right)$

$\therefore \quad \underset{n\to\infty}{\text{Lt}}\, u_n = \frac{1}{e} \neq 0$. Since u_n does not tend to zero, Σu_n is divergent.

Thus the given series converges for $x < 1$ and diverges for $x \geq 1$.

(*ii*) Here $\quad (u_n^{1/n}) = \dfrac{n+1}{n^{1+1/n}}x$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\,(u_n)^{1/n} = \underset{n\to\infty}{\text{Lt}}\,\frac{n+1}{n}\cdot\frac{1}{n^{1/n}}x = \underset{n\to\infty}{\text{Lt}}\left(1+\frac{1}{n}\right)\left(\frac{1}{n^{1/n}}\right)x = x \qquad \left[\because \underset{n\to\infty}{\text{Lt}}\, n/n = 1\right]$$

$\therefore$ The given series converges for $x < 1$ and diverges for $x > 1$.

When $x = 1$, $\quad u_n = \dfrac{(n+1)^n}{n^{n+1}} = \dfrac{1}{n}\left(1+\dfrac{1}{n}\right)^n$

Taking $\quad v_n = \dfrac{1}{n}$, $\underset{n\to\infty}{\text{Lt}}\left(\dfrac{u_n}{v_n}\right) = \underset{n\to\infty}{\text{Lt}}\left(1+\dfrac{1}{n}\right)^n = e \neq 0$ and finite.

$\therefore$ By comparison test both Σu_n and Σv_n behave alike.

But $\Sigma v_n = \Sigma\dfrac{1}{n}$ is divergent ($\because \; p = 1$). $\therefore$ Σu_n also diverges. Hence the given series converges for $x < 1$ and diverges for $x \geq 1$.

(*iii*) Here $\quad u_n = \left[\dfrac{(n+1)^{n+1}}{n^{n+1}} - \dfrac{n+1}{n}\right]^{-n}$

$$\therefore \quad (u_n)^{1/n} = \left(\frac{n+1}{n}\right)^{-1}\left[\left(\frac{n+1}{n}\right)^n - 1\right]^{-1} = \left(1+\frac{1}{n}\right)^{-1}\left[\left(1+\frac{1}{n}\right)^n - 1\right]^{-1}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\,(u_n)^{1/n} = 1\cdot(e-1)^{-1} = \frac{1}{e-1}\,9 < 1 \qquad [\because \; e > 1]$$

Thus the given series converges.

PROBLEMS 9.6

Discuss the convergence of the following series :

1. $\sum \dfrac{1}{n^n}$

2. $\sum \dfrac{1}{(\log n)^n}$ (*P.T.U., 2005*)

3. $\sum \left(\dfrac{n}{n+1}\right)^{n^2}$ (*P.T.U., 2010*)

4. $1 + \dfrac{x}{2} + \dfrac{x^2}{3^2} + \dfrac{x^3}{4^3} + \ldots + (x > 0)$

5. $\sum \left(\dfrac{n+2}{n+3}\right)^n x^n$

6. $\sum \dfrac{[(2n+1)x]^n}{n^{n+1}}$, $x > 0$

7. $\dfrac{3}{4}x + \left(\dfrac{4}{5}\right)^2 x^2 + \left(\dfrac{5}{6}\right)^3 x^3 + \ldots - \infty\,(x > 0)$ (*V.T.U., 2007*)

9.12 ALTERNATING SERIES

(1) Def. *A series in which the terms are alternately positive or negative is called an alternating series.*

(2) Leibnitz's series. *An alternating series* $u_1 - u_2 + u_3 - u_4 + \ldots$ *converges if (i) each term is numerically less than its preceding term, and (ii)* $\underset{n \to \infty}{Lt}\, u_n = 0$.

If $\underset{n \to \infty}{Lt}\, u_n \neq 0$, *the given series is oscillatory.*

The given series is $u_1 - u_2 + u_3 - u_4 + \ldots$

Suppose $u_1 > u_2 > u_3 > u_4 \ldots > u_{n+1} \ldots$ ...(1)

and $\underset{n \to \infty}{\text{Lt}}\, u_n = 0$ ...(2)

Consider the sum of $2n$ terms. It can be written as

$$s_{2n} = (u_1 - u_2) + (u_3 - u_4) + \ldots + (u_{2n-1} - u_{2n}) \quad \ldots(3)$$

or as

$$s_{2n} = u_1 - (u_2 - u_3) - (u_4 - u_5) \ldots - u_{2n} \quad \ldots(4)$$

By virtue of (1), the expressions within the brackets in (3) and (4) are all positive.

$\therefore$ It follows from (3) that s_{2n} is positive and increases with n.

Also from (4), we note that s_{2n} always remains less than u_1.

Hence s_{2n} must tend to a finite limit.

Moreover $\underset{n \to \infty}{\text{Lt}}\, s_{2n+1} = \underset{n \to \infty}{\text{Lt}}\, (s_{2n} + u_{2n+1}) = \underset{n \to \infty}{\text{Lt}}\, s_{2n} + 0$ [by (2)

Thus $\underset{n \to \infty}{\text{Lt}}\, s_n$ tends to the same finite limit whether n is even or odd.

Hence the given series is convergent.

When $\underset{n \to \infty}{\text{Lt}}\, u_n \neq 0$, $\underset{n \to \infty}{\text{Lt}}\, s_{2n} \neq \underset{n \to \infty}{\text{Lt}}\, s_{2n+1}$. $\therefore$ The given series is oscillatory.

Example 9.16. *Discuss the convergence of the series*

(*i*) $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ldots$ (*ii*) $\frac{5}{2} - \frac{7}{4} + \frac{9}{6} - \frac{11}{8} + \ldots$

(*iii*) $\frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \frac{1}{\log 5} + \ldots$ *(P.T.U., 2010)*

Solution. (*i*) The terms of the given series are alternately positive and negative ; each term is numerically less than its preceding term $\left[\because\ u_n - u_{n-1} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n-1}} < 0\right]$

Also $\underset{n \to \infty}{\text{Lt}}\, u_n = \underset{n \to \infty}{\text{Lt}}\, (1/\sqrt{n}) = 0$. Hence by Leibnitz's rule, the given series is convergent.

(*ii*) The terms of the given series are alternately positive and negative and

$$u_n - u_{n-1} = \frac{2n+3}{2n} - \frac{2n+1}{2n-2} = \frac{-6}{4n(n-1)} < 0 \text{ for } n > 1.$$

i.e., $u_n < u_{n-1}$ for $n > 1$. Also $\underset{n \to \infty}{\text{Lt}}\, u_n = \underset{n \to \infty}{\text{Lt}}\, \frac{2n+3}{2n} = 1 \neq 0$

Hence by Leibnitz's rule, the given series is oscillatory.

(*iii*) The terms of the given series are alternately positive and negative.

Also $n + 2 > n + 1$, *i.e.*, $\log(n+2) > \log(n+1)$

i.e., $\frac{1}{\log(n+2)} < \frac{1}{\log(n+1)}$, *i.e.*, $u_{n+1} < u_n$.

and $\underset{n \to \infty}{\text{Lt}}\, u_n = \underset{n \to \infty}{\text{Lt}}\, \frac{1}{\log(n+1)} = 0$

Hence the given series is convergent.

Example 9.17. *Examine the character of the series*

(i) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{2n-1}$. (ii) $\sum_{n=2}^{\infty} \frac{(-1)^{n-1} x^n}{n(n-1)}, 0 < x < 1.$

Solution. (*i*) The terms of the given series are alternately positive and negative ; each term is numerically less than its preceding term.

$$\left[\because \quad u_n - u_{n-1} = \frac{n}{2n-1} - \frac{n-1}{2n-3} = \frac{-1}{(2n-1)(n-3)} < 0\right.$$

But $\quad \underset{n\to\infty}{\text{Lt}}\, u_n = \underset{n\to\infty}{\text{Lt}} \frac{n}{2n-1} = \underset{n\to\infty}{\text{Lt}} \frac{1}{2-1/n} = \frac{1}{2}$ which is not zero.

Hence the given series is oscillatory.

(*ii*) The terms of the given series are alternately positive and negative

$$u_n - u_{n-1} = \frac{x^n}{n(n-1)} - \frac{x^{n-1}}{(n-1)(n-2)} = \frac{x^{n-1}[(n-2)x - n]}{n(n-1)(n-2)} < 0 \quad \text{for} \quad n \geq 2, \qquad (\because \quad 0 < x < 1)$$

i.e., $\quad u_n < u_{n-1}$ for $n \geq 2$. Also $\underset{n\to\infty}{\text{Lt}}\, u_n = \underset{n\to\infty}{\text{Lt}} \frac{x^n}{n(n-1)} = 0 \qquad (\because \quad 0 < x < 1)$

Hence the given series is convergent.

PROBLEMS 9.7

Discuss the convergence of the following series :

1. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \infty$. (*P.T.U., 2009*)
2. $1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \ldots \infty$. (*V.T.U., 2010*)
3. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ (*Delhi, 2002*)
4. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$.
5. $\frac{1}{1\cdot 2} - \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} - \frac{1}{7\cdot 8} + \ldots \infty$ (*Osmania, 2003*)
6. $\frac{1}{6} - \frac{2}{11} + \frac{3}{16} - \frac{4}{21} + \frac{5}{26} - \ldots \infty$.
7. $1 - 2x + 3x^2 - 4x^3 + \ldots + \infty, \left(x < \frac{1}{2}\right)$. (*Cochin, 2005*)
8. $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2+1}$.
9. $\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \frac{x^4}{1+x^4} + \ldots \infty \; (0 < x < 1)$. (*V.T.U., 2004 ; Delhi, 2002*)
10. $\left(\frac{1}{2} - \frac{1}{\log 2}\right) - \left(\frac{1}{2} - \frac{1}{\log 3}\right) + \left(\frac{1}{2} - \frac{1}{\log 4}\right) - \left(\frac{1}{2} - \frac{1}{\log 5}\right) + \ldots \infty$.

9.13 SERIES OF POSITIVE AND NEGATIVE TERMS

The series of positive terms and the alternating series are special types of these series with arbitrary signs.

Def. (1) *If the series of arbitrary terms* $u_1 + u_2 + u_3 + \ldots + u_n + \ldots$
be such that the series $\quad |u_1| + |u_2| + |u_3| + \ldots + |u_n| + \ldots$
is convergent, then the series Σu_n *is said to be* **absolutely convergent.**

(2) *If* $\Sigma |u_n|$ *is divergent but* Σu_n *is convergent, then* Σu_n *is said to be* **conditionally convergent.**

For instance, the series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \ldots$ *is absolutely convergent,* since the series

$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots$ is known to be convergent.

Again, since the alternating series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$ is convergent, and the series of absolute values $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$ is divergent, so the original series is conditionally convergent.

Obs. 1. *An absolutely convergent series is necessarily convergent but not conversely.*

Let Σu be an absolutely convergent series.

Clearly $\quad u_1 + u_2 + u_3 + \ldots + u_n + \ldots$

$\leq |u_1| + |u_2| + |u_3| + \ldots + |u_n| + \ldots$ which is known to be convergent.

Hence the series Σu_n is also convergent.

Obs. 2. As the series $\Sigma |u_n|$ is of positive terms, the tests already established for positive term series can be applied to examine Σu_n for its absolute convergence. For instance, Ratio test can be restated as follows :

The series Σu_n *is absolutely convergent if* $\underset{n \to \infty}{\text{Lt}} \frac{|u_{n+1}|}{|u_n|} < 1$,

and is divergent if $\underset{n \to \infty}{\text{Lt}} \frac{|u_{n+1}|}{|u_n|} > 1$. This test fails when the limit is unity.

Example 9.18. *Examine the following series for convergence :*

(*i*) $1 + \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} - \frac{1}{8^2} + \ldots \infty$ (*V.T.U., 2006*)

(*ii*) $\frac{1}{2^3} - \frac{1}{3^3}(1+2) + \frac{1}{4^3}(1+2+3) - \frac{1}{5^3}(1+2+3+4) + \ldots \infty$.

Solution. (*i*) The series of absolute terms is $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \ldots$ which is, evidently convergent.

$\therefore$ the given series is absolutely convergent and hence it is convergent.

(*ii*) Here
$$u_n = (-1)^{n-1} \frac{(1+2+3+\ldots+n)}{(n+1)^3}$$
$$= (-1)^{n-1} \frac{n(n+1)}{2(n+1)^3} = (-1)^{n-1} \frac{n}{2(n+1)^2} = (-1)^{n-1} a_n \text{ (Say).}$$

Then
$$a_n - a_{n+1} = \frac{1}{2}\left[\frac{n}{(n+1)^2} - \frac{n+1}{(n+2)^2}\right] = \frac{1}{2} \frac{n^2+n-1}{(n+1)^2(n+2)^2} > 0.$$

i.e.,
$$a_{n+1} < a_n. \text{ Also } \underset{n \to \infty}{\text{Lt}}\, a_n = \frac{1}{2} \underset{n \to \infty}{\text{Lt}} \frac{n}{(n+1)^2} = 0.$$

Thus by Leibnitz's rule, Σa_n and therefore Σu_n is convergent.

Also
$$|u_n| = \frac{1}{2} \frac{n}{n^2+1}. \text{ Taking } v_n = \frac{1}{n}, \text{ we note that}$$
$$\underset{n \to \infty}{\text{Lt}} \frac{|u_n|}{v_n} = \frac{1}{2} \underset{n \to \infty}{\text{Lt}} \frac{n^2}{n^2+1} = \frac{1}{2} \neq 0$$

Since Σv_n is divergent, therefore $\Sigma |u_n|$ is also divergent.

i.e., Σu_n is convergent but $\Sigma |u_n|$ is divergent.

Thus the given series Σu_n is conditionally convergent.

Example 9.19. *Test whether the following series are absolutely convergent or not ?*

(*i*) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$ (*ii*) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\log n)^2}$.

Solution. (*i*) Given series is $\Sigma u_n = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \infty$

This is an alternating series of which terms go on decreasing and $\underset{n\to\infty}{\text{Lt}}\, u_n = \underset{n\to\infty}{\text{Lt}} \dfrac{1}{2n-1} = 0$

$\therefore$ by Leibnitz's rule, Σu_n converges.

The series of absolute terms is $1 + \dfrac{1}{3} + \dfrac{1}{5} + \dfrac{1}{7} + \ldots \infty$

Here $u_n = \dfrac{1}{2n-1}$. Taking $v_n = \dfrac{1}{n}$, we have

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \left(\frac{n}{2n-1}\right) = \underset{n\to\infty}{\text{Lt}} \left(\frac{1}{2-\frac{1}{n}}\right) = \frac{1}{2} \neq 0 \text{ and finite.}$$

$\therefore$ by Comparison test, Σu_n diverges [$\because$ Σv_n diverges.

Hence the given series converges and the series of absolute terms diverges, therefore the given series converges conditionally.

(*ii*) The terms of given series are alternately positive and negative. Also each term is numerically less than the preceding term and $\underset{n\to\infty}{\text{Lt}} |u_n| = \underset{n\to\infty}{\text{Lt}} [1/n(\log n)^2] = 0$.

$\therefore$ by Leibnitz's rule, the given series converges.

Also $$\int_2^\infty \frac{dx}{x(\log x)^2} = \left| -\frac{1}{\log x} \right|_2^\infty = \frac{1}{\log 2} = 0 \text{ and finite.}$$

i.e., the series of absolute terms converges.

Hence, the given series converges absolutely.

9.14 POWER SERIES

(1) Def. *A series of the form* $a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots$ ...(*i*)
where the a's are independent of x, is called a **power series** *in x.* Such a series may converge for some or all values of x.

(2) Interval of convergence

In the power series (*i*), $u_n = a_nx^n$.

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{\text{Lt}} \frac{a_{n+1}x^{n+1}}{a_nx^n} = \underset{n\to\infty}{\text{Lt}} \left(\frac{a_{n+1}}{a_n}\right) . x$$

If $\underset{n\to\infty}{\text{Lt}} \left(\dfrac{a_{n+1}}{a_n}\right) = l$, then by Ratio test, the series (*i*) converges, when lx is numerically less than 1, *i.e.*, when $|x| < 1/l$ and diverges for other values.

Thus the power series (*i*) has an interval $-1/l < x < 1/l$ within which it converges and diverges for values of x outside this interval. Such an interval is called the *interval of convergence of the power series.*

Example 9.20. *State the values of x for which the following series converge :*

(*i*) $x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dfrac{x^5}{5} - \ldots \infty$, (*ii*) $\dfrac{1}{1-x} + \dfrac{1}{2(1-x)^2} + \dfrac{1}{3(1-x)^3} + \ldots \infty$.

Solution. (*i*) Here $u_n = (-1)^{n-1}\dfrac{x^n}{n}$ and $u_{n+1} = (-1)^n \dfrac{x^{n+1}}{n+1}$

$$\therefore \quad \frac{u_{n+1}}{u_n} = -\frac{n}{n+1}x \quad \text{and} \quad \underset{n\to\infty}{\text{Lt}} \left|\frac{u_{n+1}}{u_n}\right| = \left(\underset{n\to\infty}{\text{Lt}} \frac{1}{1+1/n}\right) |x| = |x|$$

$\therefore$ by Ratio test the given series converges for $|x| < 1$ and diverges for $|x| > 1$.

Let us examine the series for $x = \pm 1$.

For $x = 1$, the series reduces to $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \ldots$

which is an alternating series and is convergent.

For $x = -1$, the series becomes $-\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots\right)$

which is a divergent series as can be seen by comparison with p-series when $p = 1$.

Hence the given series converges for $-1 < x \le 1$.

(*ii*) Here $\qquad u_n = \dfrac{1}{n(1-x)^n}$

$$\underset{n\to\infty}{\text{Lt}} \left|\frac{u_{n+1}}{u_n}\right| = \underset{n\to\infty}{\text{Lt}} \left|\frac{1}{(n+1)(1-x)^{n+1}} \cdot n(1-x)^n\right| = \left|\frac{1}{1-x}\right| \underset{n\to\infty}{\text{Lt}} \frac{n}{n+1} = \left|\frac{1}{1-x}\right|$$

By Ratio test, Σu_n converges for $\left|\dfrac{1}{1-x}\right| < 1$, *i.e.*, $|1-x| > 1$

i.e., for $-1 > 1 - x > 1$ or $x < 0$ and $x > 2$.

Let us examine the series for $x = 0$ and $x = 2$.

For $x = 0$, the given series becomes $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}$ which is a divergent harmonic series.

For $x = 2$, the given series becomes $-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \ldots + \frac{(-1)^n}{n} + \ldots$

It is an alternating series which is convergent by Leibnitz's rule

$$[\because \quad u_n < u_{n-1} \text{ for all } n \text{ and } \underset{n\to\infty}{\text{Lt}}\, u_n = 0.]$$

Hence the given series converges for $x < 0$ and $x \ge 2$.

Example 9.21. *Test the series $\frac{x}{\sqrt{3}} - \frac{x^2}{\sqrt{5}} + \frac{x^3}{\sqrt{7}} - \ldots$ for absolute convergence and conditional convergence.* *(V.T.U., 2010)*

Solution. We have $u_n = (-1)^{n-1} \dfrac{x^n}{\sqrt{(2n+1)}}$ and $u_{n+1} = \dfrac{(-1)^n x^{n+1}}{\sqrt{(2n+3)}}$

$$\therefore \qquad \underset{n\to\infty}{\text{Lt}} \left|\frac{u_{n+1}}{u_n}\right| = \underset{n\to\infty}{\text{Lt}} \left|\frac{\dfrac{(-1)^n x^{n+1}}{\sqrt{(2n+3)}} \cdot \sqrt{(2n+1)}}{(-1)^{n-1} x^n}\right| = \underset{n\to\infty}{\text{Lt}} \left|(-1)\sqrt{\left(\frac{2n+1}{2n+3}\right)} x\right|$$

$$= \underset{n\to\infty}{\text{Lt}} \left|\sqrt{\left(\frac{2+1/n}{2+3/n}\right)} x\right| = |x|$$

Hence *the given series is absolutely convergent for $|x| < 1$ and is divergent for $|x| > 1$* and the test fails for $|x| = 1$.

For $x = 1$, $\qquad u_n = \dfrac{(-1)^{n-1}}{\sqrt{(2n+1)}}$. Since $2n + 1 < 2n + 3$ or $(2n+1)^{-1/2} > (2n+3)^{-1/2}$

i.e., $\qquad u_n > u_{n+1}$. Also $\underset{n\to\infty}{\text{Lt}}\, u_n = \underset{n\to\infty}{\text{Lt}} \dfrac{1}{\sqrt{(2n+1)}} = 0$.

$\therefore$ the series is convergent by Leibnitz's test.

But $\qquad \dfrac{1}{\sqrt{3}} + \dfrac{1}{\sqrt{5}} + \dfrac{1}{\sqrt{7}} + \ldots$ has $u_n = \dfrac{1}{\sqrt{(2n+1)}} = \dfrac{1}{\sqrt{n}} \cdot \dfrac{1}{\sqrt{(2+1/n)}}$

On comparing it with $v_n = \frac{1}{\sqrt{n}}$, Σu_n is divergent.

Hence the *given series is conditionally convergent for x = 1.*

For x = – 1, the series becomes $-\left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} + \ldots\right)$

But we have seen that the series $\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} + \ldots$ is divergent.

Hence, *the given series is divergent when x = – 1.*

9.15 (1) CONVERGENCE OF EXPONENTIAL SERIES

The series $1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots \infty$ *is convergent for all values of x.* (*J.N.T.U., 2006*)

Here $\quad \underset{n\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{\text{Lt}} \left[\frac{x^n}{n!} \div \frac{x^{n-1}}{(n-1)!}\right] = \underset{n\to\infty}{\text{Lt}} \frac{x}{n} = 0$

Hence the series converges, whatever be the value of x.

(2) Convergence of logarithmic series

The series $x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n \frac{x^n}{n} + \ldots \infty$ *is convergent for* $-1 < x \le 1$.

Here $\quad \underset{n\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{\text{Lt}} \frac{(-1)^{n+1}x^{n+1}}{n+1} \cdot \frac{n}{(-1)^n x^n} = -x \underset{n\to\infty}{\text{Lt}} \frac{n}{n+1} = -x \underset{n\to\infty}{\text{Lt}} \left\{\frac{1}{1+1/n}\right\} = -x.$

Hence the series converges for $|x| < 1$ and diverges for $|x| > 1$.

When $x = 1$, the series being $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$, is convergent.

When $x = -1$, the series being $-\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\right)$, is divergent.

Hence the series converges for $-1 < x \le 1$.

(3) Convergence of binomial series

The series $1 + nx + \frac{n(n-1)}{2!} x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!} x^r + \ldots \infty$

converges for $|x| < 1$.

Here $\quad u_r = \frac{n(n-1)\ldots(n-r)}{(r-1)!} x^{r-1}$ and $u_{r+1} = \frac{n(n-1)\ldots(n-r+1)}{r!} x^r$

$\therefore \quad \underset{r\to\infty}{\text{Lt}} \frac{u_{r+1}}{u_r} = \underset{r\to\infty}{\text{Lt}} \frac{n-r+1}{r} x = \underset{r\to\infty}{\text{Lt}} \left(\frac{n+1}{r} - 1\right) x = -x$ for $r > n + 1$.

Hence, the series converges for $|x| < 1$.

PROBLEMS 9.8

1. Test the following series for conditional convergence : (*i*) $\sum \frac{(-1)^{n-1}}{\sqrt{n}}$ (*ii*) $\sum \frac{(-1)^{n-1} n}{n^2+1}$.
2. Prove that the series $\frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \ldots$ converges absolutely. (*Rohtak, 2006 S*)
3. Test the following series for conditional convergence :

   (*i*) $1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \ldots \infty$ (*ii*) $1 - \frac{1}{2} + \frac{1.3}{2.4} - \frac{1.3.5}{2.4.6} + \ldots \infty$

4. Discuss the absolute convergence of (*i*) $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$ (*Hissar, 2005 S*)

(*ii*) $x - \frac{x^2}{4} + \frac{x^3}{9} - \frac{x^4}{16} + \ldots \infty$ (*iii*) $\frac{1}{\sqrt{(1^3+1)}} - \frac{1}{\sqrt{(2^3+1)}}x + \frac{1}{\sqrt{(3^3+1)}}x^2 - \ldots \infty$

5. Find the nature of the series $\frac{x}{1\cdot 2} - \frac{x^2}{2\cdot 3} + \frac{x^3}{3\cdot 4} - \frac{x^4}{4\cdot 5} + \ldots \infty$ (*V.T.U., 2009*)

6. For what values of x are the following series convergent :

(*i*) $x - \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} - \frac{x^4}{\sqrt{4}} + \ldots \infty$ (*P.T.U., 2009 S ; V.T.U., 2008*)

(*ii*) $x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \ldots \infty$

7. Find the radius of convergence of the series $\sum \frac{n!}{n^n} x^n$. (*Calicut, 2005*)

8. Prove that $\frac{1}{a} + \frac{1}{a+1} - \frac{1}{a+2} + \frac{1}{a+3} + \frac{1}{a+4} - \frac{1}{a+5} + \ldots$ is a divergent series.

9. Test the series $1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}}$ for

(*i*) absolute convergence and (*ii*) conditional convergence. (*V.T.U., 2007 ; Rohtak, 2005*)

9.16 PROCEDURE FOR TESTING A SERIES FOR CONVERGENCE

First see whether the given series is

(*i*) a series with terms alternately positive and negative ;

(*ii*) a series of positive terms excluding power series ;

or (*iii*) a power series.

For alternating series (*i*), apply the Leibnitz's rule (§ 9.12).

For series (*ii*), first find u_n and if possible evaluate Lt u_n. If Lt $u_n \neq 0$, the series is divergent. If Lt $u_n = 0$, compare Σu_n with $\Sigma 1/n^p$ and apply the comparison tests (§ 9.6).

If the comparison tests are not applicable, apply the Ratio test (§ 9.9). If Lt $u_n/u_{n+1} = 1$, *i.e.*, the ratio test fails, apply Raabe's teat (§ 9.10). If Raabe's test fails for a similar reason, apply Logarithmic test (§ 9.10). If this also fails, apply Cauchy's root test (§ 9.11).

For the power series (*iii*), apply the Ratio test as in § 9.14. If the Ratio test fails, examine the series as in case (*ii*) above.

PROBLEMS 9.9

Test the convergence of the following series :

1. $\sum_{n=1}^{\infty} \frac{2^n - 2}{2^n + 1} . x^{n-1} (x > 0)$. (*Osmania, 1999*)

2. $\sum \left(\frac{1}{\sqrt{n}} - \sqrt{\frac{n}{n+1}} \right)$.

3. $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \ldots$

4. $\sum_{n=1}^{\infty} \sqrt{\left\{ \frac{2^n + 1}{3^n + 1} \right\}}$

5. $\frac{1}{1+\sqrt{2}} + \frac{2}{1+2\sqrt{3}} + \frac{3}{1+3\sqrt{4}} + \ldots \infty$.

6. $\frac{x}{1+\sqrt{1}} + \frac{x^2}{2+\sqrt{2}} + \frac{x^3}{3+\sqrt{3}} + \ldots \infty$.

7. $1 + \frac{2^2}{3^2}x + \frac{2^2 . 4^2}{3^2 . 5^2}x^2 + \frac{2^2 . 4^2 . 6^2}{3^2 . 5^2 . 7^2}x^3 + \ldots \infty$.

8. $\sum_{1}^{\infty} \frac{n x^n}{(n+1)(n+2)} (x > 0)$.

9. $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} x^n$.

10. $\sum_{n=1}^{\infty} \frac{x^n}{(2n-1)^2 2^n}$.

11. $\sum_{0}^{\infty} \frac{(3x+5)^n}{(n+1)!}$.

12. $\sum_{n=1}^{\infty} \frac{(x+2)^n}{3^n n}$.

13. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\log n}$.

14. $\sum \frac{(-1)^{n-1} \sin nx}{n^3}$.

15. $\frac{\sqrt{2}-1}{3^3-1} + \frac{\sqrt{3}-1}{4^3-1} + \frac{\sqrt{4}-1}{5^3-1} + \ldots \infty$. *(V.T.U., 2003)*

16. $\sum_{n=2}^{\infty} \frac{1}{(n \log n)(\log \log n)^p}$.

9.17 UNIFORM CONVERGENCE

Let
$$u_1(x) + u_2(x) + \ldots \infty = \sum_{n=1}^{\infty} u_n(x) \qquad \ldots(1)$$

be an infinite series of functions each of which is defined in the interval (a, b). Let $s_n(x)$ be the sum of its first n terms, *i.e.*, $s_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x)$

At some point $x = x_1$, if $\underset{n \to \infty}{\text{Lt}}\, s_n(x_1) = s(x_1)$,

then the series (1) is said to converge to sum $s(x_1)$ at that point. This means at $x = x_1$ given a positive number ε, we can find a number N such that $|\, s(x_1) - s_n(x_1)\, | < \varepsilon$ for $n > N$ ...(2)

Evidently N will depend on ε but generally it will also depend on x_1. Now if we keep the same ε but take some other value x_2 of x for which (1) is convergent, then we may have to change N for the inequality (2) to hold. If we wish to approximate the sum $s(x)$ of the series by its partial sums $s_n(x)$, we shall require different partial sums at different points of the interval and the problem will become quite complicated. If, however, we choose an N which is independent of the values of x, the problem becomes simpler. Then the partial sum $s_n(x)$, $(n > N)$ approximates to $s(x)$ for all values of x in the interval (a, b) and ε is uniform throughout this interval. Thus we have

Definition. *The series $\sum u_n(x)$ is said to be uniformly convergent in the interval (a, b), if for a given $\varepsilon > 0$, a number N can be found* **independent of x,** *such that for every x in the interval (a, b),*

$$|\, s(x) - s_n(x)\, | < \varepsilon \text{ for all } n > N.$$

Example 9.21. *Examine the geometric series $1 + x + x^2 + \ldots + x^{n-1} + \ldots \infty$ for uniform convergence in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$.*

Solution. We have
$$s_n(x) = 1 + x + x^2 + \ldots + x^{n-1} = \frac{1-x^n}{1-x}.$$

and
$$s(x) = \underset{n \to \infty}{\text{Lt}} \frac{1-x^n}{1-x} = \frac{1}{1-x} \text{ for } |\, x\, | < 1$$

$\therefore$
$$|\, s(x) - s_n(x)\, | = \left| \frac{x^n}{1-x} \right| = \frac{|\, x^n\, |}{1-x} = \frac{|\, x\, |^n}{1-x} \text{ which will be } < \varepsilon, \text{ if } |\, x\, |^n < \varepsilon\,(1-x).$$

Choose N such that $|\, x\, |^N = \varepsilon\,(1-x)$

or
$$N = \log\,[\varepsilon\,(1-x)]/\log |\, x\, | \qquad \ldots(i)$$

Evidently N increases with the increase of $|\, x\, |$ and in the interval $-\frac{1}{2} \le x \le \frac{1}{2}$, it assumes a maximum value $N' = \log(\varepsilon/2)/\log \frac{1}{2}$ at $x = \frac{1}{2}$ for a given ε.

Thus $|\, s(x) - s_n(x)\, | < \varepsilon$ for all $n \ge N'$ for every value of x in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Hence the geometric series converges uniformly in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Obs. *The geometric series though convergent in the interval $(-1, 1)$, is not uniformly convergent in this interval,* since we cannot find a fixed number N for every x in this interval

($\because$ N given by $(i) \to \infty$ as $|\, x\, | \to 1$).

9.18 WEIERSTRASS'S M-TEST*

A series $\Sigma u_n(x)$ is uniformly convergent in an interval (a, b), if there exists a convergent series ΣM_n of positive constants such that $|u_n(x)| \le M_n$ for all values of x in (a, b).

Since ΣM_n is convergent, therefore, for a given $\varepsilon > 0$, we can find a number N, such that $|s - s_n| < \varepsilon$ for every $n > N$,

where $\quad s = M_1 + M_2 + \ldots + M_n + M_{n+1} + \ldots$ and $s_n = M_1 + M_2 + \ldots + M_n$

This implies that $|M_{n+1} + M_{n+2} + \ldots| < \varepsilon$ for every $n > N$.

Since $\quad |u_n(x)| \le M_n$

$$\therefore \quad |u_{n+1}(x) + u_{n+2}(x) + \ldots| \le |u_{n+1}(x)| + |u_{n+2}(x)| + \ldots$$
$$\le M_{n+1} + M_{n+2} + \ldots < \varepsilon \text{ for every } n > N.$$

i.e., $\quad |s(x) - s_n(x)| < \varepsilon$ for every $n > N$, where $s(x)$ is the sum of the series $\Sigma u_n(x)$.

Since N does not depend on x, the series $\Sigma u_n(x)$ converges uniformly in (a, b).

Obs. *$\Sigma u_n(x)$ is also absolutely convergent for every x, since $|u_n(x)| \le M_n$.*

Example 9.22. *Show that the following series converges uniformly in any interval :*

(*i*) $\sum \dfrac{\cos nx}{n^p}$ (*Andhra, 1999*) (*ii*) $\sum \dfrac{1}{n^3 + n^4x^2}$.

Solution. (*i*) $\left|\dfrac{\cos nx}{n^p}\right| = \dfrac{|\cos nx|}{n^p} \le \dfrac{1}{n^p}$ $(= M_n)$ for all values of x.

Since $\displaystyle\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$,

$\therefore$ By M-test, the given series converges uniformly for all real values of x and $p > 1$.

(*ii*) For all values of x, $n^3 + n^4x^2 > n^3$

$\therefore \quad \left|\dfrac{1}{n^3 + n^4x^2}\right| < \dfrac{1}{n^3}$ $(= M_n)$. But ΣM_n being p-series with $p > 1$, is convergent.

$\therefore$ By M-test, the given series converges uniformly in any interval.

Example 9.23. *Examine the following series for uniform convergence :*

(*i*) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(nx + x^2)}{n(n+2)}$ (*P.T.U., 2009*) (*ii*) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p + n^q x^2}$ (*P.T.U., 2005 S*)

Solution. (*i*) $\left|\dfrac{\sin(nx + x^2)}{n(n+2)}\right| = \dfrac{|\sin(nx + x^2)|}{n^2 + 2n} \le \dfrac{1}{n^2}$ $(= M_n)$ for all real x.

Since $\displaystyle\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, therefore, by M-test, the given series is uniformly convergent for all real values of x.

(*ii*) For all real values of x, $x^2 \ge 0$, *i.e.*, $n^q x^2 \ge 0$

i.e., $\quad n^p + n^q x^2 \ge n^p \quad$ or $\quad \dfrac{1}{n^p + n^q x^2} \le \dfrac{1}{n^p}$ $(= M_n)$

Since $\displaystyle\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p > 1$,

$\therefore$ by M-test, the given series is uniformly convergent for all real values of x and $p > 1$.

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* Named after the great German mathematician *Karl Weierstrass* (1815–1897) who made basic contributions to Calculus, Approximation theory, Differential geometry and Calculus of variations. He was also one of the founders of Complex analysis.

9.19 PROPERTIES OF UNIFORMLY CONVERGENT SERIES

I. If the series $\Sigma u_n(x)$ converges uniformly to sum $s(x)$ in the interval (a, b) and each of the functions $u_n(x)$ is continuous in this interval, then the sum $s(x)$ is also continuous in (a, b).

II. If the series $\Sigma u_n(x)$ converges uniformly in the interval (a, b) and each of the functions $u_n(x)$ is continuous in this interval, then the series can be integrated term by term

i.e.,
$$\int_a^b [u_1(x) + u_2(x) + \ldots]\, dx = \int_a^b u_1(x)\, dx + \int_a^b u_2(x)\, dx + \ldots$$

III. If $\Sigma u_n(x)$ is a convergent series having continuous derivatives of its terms, and the series $\Sigma u_n(x)$ converges uniformly, then the series can be differentiated term by term

$$\frac{d}{dx}[u_1(x) + u_2(x) + \ldots] = u_1'(x) + u_2'(x) + \ldots.$$

Example 9.24. *Prove that* $\int_0^1 \left(\sum \frac{x^n}{n^2}\right) dx = \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}$.

Solution. $|x^n| \le 1$ for $0 \le x \le 1$

$\therefore$ $\left|\frac{x^n}{n^2}\right| \le \frac{1}{n^2}$ $(= M_n)$ for $0 \le x \le 1$. But ΣM_n is a convergent series.

$\therefore$ by M-test, the series $\Sigma(x^n/n^2)$ is uniformly convergent in $0 \le x \le 1$. Also x^n/n^2 is continuous in this interval.

$\therefore$ the series $\Sigma(x^n/n^2)$ can be integrated term by term in the interval $0 \le x \le 1$.

i.e.,
$$\int_0^1 \left(\sum \frac{x^n}{n^2}\right) dx = \sum\left(\int_0^1 \frac{x^n}{n^2}\, dx\right) = \sum\left(\frac{1}{n^2}\int_0^1 x^n\, dx\right) = \sum \frac{1}{n^2(n+1)}.$$

Imp. Obs. *There is no relation between absolute and uniform convergence.* In fact, a series may converge absolutely but not uniformly while another series may converge uniformly but not absolutely.

For instance, the series

$\frac{1}{x^2+1} - \frac{1}{x^2+2} + \frac{1}{x^2+3} - \ldots$ can be seen to converge uniformly but not absolutely, while the series

$x^2 + \frac{x^2}{x^2+1} + \frac{x^2}{(x^2+1)^2} + \frac{x^2}{(x^2+1)^3} + \ldots$ can be shown to converge absolutely but not uniformly.

PROBLEMS 9.10

Test for uniform convergence the series :

1. $\sum_{n=1}^{\infty} \frac{x^n}{n^{3/2}}$.
2. $\sum \frac{\cos nx}{2^n}$.
3. $\frac{\sin x}{1^2} + \frac{\sin 2x}{2^2} + \frac{\sin 3x}{3^2} + \frac{\sin 4x}{4^2} + \ldots \infty$. *(P.T.U., 2003 ; Andhra, 2000)*
4. $\sin x - \frac{\sin 2x}{2\sqrt{2}} + \frac{\sin 3x}{3\sqrt{3}} - \frac{\sin 4x}{4\sqrt{4}} + \ldots \infty$.
5. $\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots \infty$.
6. $\frac{ax}{2} + \frac{a^2x^2}{5} + \frac{a^3x^3}{10} + \ldots + \frac{a^nx^n}{n^2+1} + \ldots \infty$.
7. Show that the series $\Sigma r^n \sin n\theta$ and $\Sigma r^n \cos n\theta$ converge uniformly for all real values of θ if $0 < r < 1$.
8. Show that $\frac{1}{1+x^2} - \frac{1}{2+x^2} + \frac{1}{3+x^2} - \frac{1}{4+x^2} + \ldots$ converges uniformly in the interval $x \ge 0$ but not absolutely.
9. Prove that $\sum \frac{x}{n(1+nx^2)}$ is uniformly convergent for all real values of x.

10. Examine the following series for uniform convergence :

(i) $\sum_{n=1}^{\infty} \frac{1}{n^4 + n^3 x^2}$ (ii) $\sum_{n=1}^{\infty} \frac{\cos(x^2 + n^2 x)}{n(n^2+2)}$.

11. Show that

(i) $\int_0^1 \left(\sum \frac{\sin x}{x}\right) dx = 1 - \frac{1}{3.3!} + \frac{1}{5.5!} - \frac{1}{7.7!} + \ldots \infty$; (ii) $\int_0^{\pi} \left(\sum \frac{\sin n\theta}{n^3}\right) d\theta = 2 \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$.

9.20 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 9.11

Choose the correct answer or fill up the blanks in each of the following problems :

1. The series $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots$ converges if

(a) $p > 0$ (b) $p < 1$ (c) $p > 1$ (d) $p \le 1$.

2. The series $\sum_{n=0}^{\infty} (2x)^n$ converges if

(a) $-1 \le x \le 1$ (b) $-\frac{1}{2} < x < \frac{1}{2}$ (c) $-2 < x < 2$ (d) $-\frac{1}{2} \le x \le \frac{1}{2}$.

3. The series $\frac{2}{1^2} - \frac{3}{2^2} + \frac{4}{3^2} - \frac{5}{4^2} + \ldots$ is

(a) conditionally convergent (b) absolutely convergent
(c) divergent (d) none of the above.

4. Which one of the following series is *not* convergent ?

(a) $\frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \ldots \infty$ (b) $1\frac{1}{2} - 1\frac{1}{3} + 1\frac{1}{4} - 1\frac{1}{5} + \ldots \infty$

(c) $\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \ldots \infty$ (d) $x + x^2 + x^3 + x^4 + \ldots \infty$ where $|x| < 1$.

5. The sum of the alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ is

(a) zero (b) infinite (c) $\log 2$
(d) not defined as the series is not convergent.

6. Let Σu_n be a series of positive terms. Given that Σu_n is convergent and also

$\underset{n \to \infty}{\text{Lt}} \frac{u_{n+1}}{u_n}$ exists, then the said limit is

(a) necessarily equal to 1 (b) necessarily greater than 1
(c) may be equal to 1 or less than 1 (d) necessarily less than 1.

7. $\sum \left(1 + \frac{1}{n}\right)^{-n^2}$ is

(a) convergent (b) oscillatory (c) divergent.

8. $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ldots$ is

(a) oscillatory (b) conditionally convergent
(c) divergent (d) absolutely convergent.

9. $1 + \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} - \frac{1}{8^2} + \ldots \infty$ is

(a) conditionally convergent (b) convergent

(c) oscillatory (d) divergent.

10. $\int_0^1 \left(\sum \frac{x^n}{n^2} \right) dx =$

(a) $\sum_{n=0}^{\infty} \frac{1}{n(n+1)}$ (b) $\sum_{n=1}^{\infty} \frac{1}{n^2(n-1)}$ (c) $\sum_{n=0}^{\infty} \frac{1}{n(n-1)}$ (d) $\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}$.

11. If Σu_n is a convergent series of positive terms, then $\underset{n \to \infty}{\text{Lt}}\, u_n$ is

(a) 1 (b) ± 1 (c) 0 (d) 0. (V.T.U., 2010)

12. Geometric series $1 + x + x^2 + \ldots + x^{n-1} + \ldots \infty$

(a) converges in the interval ..... (b) converges uniformly in the interval .....

13. The series $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \infty$ converges in the interval .....

14. If $\underset{n \to \infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = k$, then Σu_n converges for k .....

15. A sequence (a_n) is said to be bounded, if there exists a number k such that for every n, a_n is .....

16. The series $2 - 5 + 3 + 2 - 5 + 3 - 5 + \ldots \infty$ is ..... (Convergent etc.)

17. The series $1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \infty$ converges for .....

18. If $\underset{n \to \infty}{\text{Lt}}\, n \left\{ \frac{u_n}{u_{n+1}} - 1 \right\} = k$, then Σu_n diverges for k .....

19. A sequence which is monotonic and bounded is .....

20. The series $\frac{1}{1.2} + \frac{2}{3.4} + \frac{3}{5.6} + \ldots \infty$ is ..... (Convergent etc.)

21. The series $\frac{2^p}{1^q} + \frac{3^p}{2^q} + \frac{4^p}{3^q} + \ldots \infty$ converges for .....

22. The series $\sqrt{\frac{1}{4}} + \sqrt{\frac{2}{6}} + \sqrt{\frac{3}{8}} + \ldots \infty$ is ..... (Convergent etc.)

23. The series $\sqrt{\left(\frac{2^n - 1}{3^n - 1} \right)}$ is ... (Convergent etc.)

24. The series $1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 \ldots + \left(-\frac{1}{2} \right)^n (x-2)^n + \ldots \infty$ converges in the interval .....

25. Is the series $\sum_{n=1}^{\infty} \frac{8 \tan^{-1} n}{1 + n^2}$ convergent ?

26. The exponential series $1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots \infty$ is absolutely convergent. (True/False)

27. The series $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{n(n+1)} + \ldots \infty$, is ..... (Convergent/divergent/oscillatory)

28. Is the series $\Sigma n \tan 1/n$ convergent ?

29. The series $\sum \frac{1}{nx^n}$ converges for x .....

30. The series $\sum_{n=1}^{\infty} \frac{x^n}{n^3}$ converges uniformly when x lies in the interval .....

31. Every absolutely convergent series is necessarily
(*a*) divergent (*b*) convergent
(*c*) conditionally convergent (*d*) none of these. (*V.T.U., 2009*)

32. The convergence of the series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ is tested by
(*a*) Ratio test (*b*) Raabe's test (*c*) Leibnitz's (*d*) Cauchy root test. (*V.T.U., 2009*)

33. The series $\sum \frac{x^n}{(n+1)^n}, x > 0$ is
(*a*) divergent (*b*) convergent (*c*) oscillatory (*d*) none of these. (*V.T.U., 2010*)

34. $\sum \sin\left(\frac{1}{n}\right)$ is
(*a*) convergent (*b*) divergent (*c*) oscillatory (*d*) none of these.

35. $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$ is convergent. (True or False)

CHAPTER 10

Fourier Series

1. Introduction. 2. Euler's Formulae. 3. Conditions for a Fourier expansion. 4. Functions having points of discontinuity. 5. Change of interval. 6. Odd and even function—Expansions of odd or even periodic functions. 7. Half-range series. 8. Typical wave-forms. 9. Parseval's formula. 10. Complex form of F-series. 11. Practical Harmonic Analysis. 12. Objective Type of Questions.

10.1 INTRODUCTION

In many engineering problems, especially in the study of periodic phenomenae* in conduction of heat, electro-dynamics and acoustics, it is necessary to express a function in a series of sines and cosines. Most of the single-valued functions which occur in applied mathematics can be expressed in the form.

$$\tfrac{1}{2}a_0\dagger + a_1 \cos x + a_2 \cos 2x + \ldots\ldots\dagger$$
$$+ b_1 \sin x + b_2 \sin 2x + \ldots\ldots$$

within a desired range of values of the variable. Such a series is known as the **Fourier series**$.

10.2 EULER'S FORMULAE

The Fourier series for the function f(x) in the interval $\alpha < x < \alpha + 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx \end{aligned}\right\} \qquad \ldots(1)$$

These values of a_0, a_n, b_n are known as *Euler's formulae***.

*__Periodic functions.__ If at equal intervals of abscissa x, the value of each ordinate $f(x)$ repeats itself, *i.e.*, $f(x) = f(x + a)$, for all x, then $y = f(x)$ is called a *periodic function* having **period** α, *e.g.*, $\sin x$, $\cos x$ are periodic functions having a period 2π.

† To write $a_0/2$ instead of a_0 is a conventional device to be able to get more symmetric formulae for the coefficients.

$ Named after the French mathematician and physicist *Jacques Fourier* (1768–1830) who was first to use Fourier series in his memorable work *'Theorie Analytique de la Chaleur'* in which he developed the theory of heat conduction. These series had a deep influence in the further development of mathematics and mathematical physics.

**See footnote p. 205.

To establish these formulae, the following definite integrals will be required :

1. $\int_{\alpha}^{\alpha+2\pi} \cos nx\, dx = \left| \frac{\sin nx}{n} \right|_{\alpha}^{\alpha+2\pi} = 0$ $(n \neq 0)$

2. $\int_{\alpha}^{\alpha+2\pi} \sin nx\, dx = -\left| \frac{\cos nx}{n} \right|_{\alpha}^{\alpha+2\pi} = 0$ $(n \neq 0)$

3. $\int_{\alpha}^{\alpha+2\pi} \cos mx \cos nx\, dx$ $(n \neq 0)$

$$= \frac{1}{2} \int_{\alpha}^{\alpha+2\pi} [\cos (m+n)x + \cos (m-n)x]\, dx$$

$$= \frac{1}{2} \left| \frac{\sin (m+n)x}{m+n} + \frac{\sin (m-n)x}{m-n} \right|_{\alpha}^{\alpha+2\pi} = 0 \qquad (m \neq n)$$

4. $\int_{\alpha}^{\alpha+2\pi} \cos^2 nx\, dx = \left| \frac{x}{2} + \frac{\sin 2nx}{4n} \right|_{\alpha}^{\alpha+2\pi} = \pi$ $(n \neq 0)$

5. $\int_{\alpha}^{\alpha+2\pi} \sin mx \cos nx\, dx = -\frac{1}{2}\left[\frac{\cos (m-n)x}{m-n} + \frac{\cos (m+n)x}{m+n} \right] = 0$ $(m \neq n)$

6. $\int_{\alpha}^{\alpha+2\pi} \sin nx \cos nx\, dx = \left| \frac{\sin^2 nx}{2n} \right|_{\alpha}^{\alpha+2\pi} = 0$ $(n \neq 0)$

7. $\int_{\alpha}^{\alpha+2\pi} \sin mx \sin nx\, dx = \frac{1}{2} \left| \frac{\sin (m-n)x}{m-n} - \frac{\sin (m+n)x}{m+n} \right|_{\alpha}^{\alpha+2\pi} = 0$ $(m \neq n)$

8. $\int_{\alpha}^{\alpha+2\pi} \sin^2 nx\, dx = \left| \frac{x}{2} - \frac{\sin 2nx}{4n} \right|_{\alpha}^{\alpha+2\pi} = \pi.$ $(n \neq 0)$

Proof. Let $f(x)$ be represented in the interval $(\alpha, \alpha + 2\pi)$ by the Fourier series :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad ...(i)$$

To find the coefficients a_0, a_n, b_n, we assume that the series (i) can be integrated term by term from $x = \alpha$ to $x = \alpha + 2\pi$.

To find a_0, integrate both sides of (i) from $x = \alpha$ to $x = \alpha + 2\pi$. Then

$$\int_{\alpha}^{\alpha+2\pi} f(x)\, dx = \frac{1}{2} a_0 \int_{\alpha}^{\alpha+2\pi} dx + \int_{\alpha}^{\alpha+2\pi} \left(\sum_{n=1}^{\infty} a_n \cos nx \right) dx + \int_{\alpha}^{\alpha+2\pi} \left(\sum_{n=1}^{\infty} b_n \sin nx \right) dx$$

$$= \frac{1}{2} a_0 (\alpha + 2\pi - \alpha) + 0 + 0 = a_0 \pi \qquad \text{[By integrals (1) and (2) above]}$$

Hence $$a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x)\, dx.$$

To find a_n, multiply each side of (i) by $\cos nx$ and integrate from $x = \alpha$ to $x = \alpha + 2\pi$. Then

$$\int_{\alpha}^{\alpha+2\pi} f(x) \cos nx\, dx = \frac{1}{2} a_0 \int_{\alpha}^{\alpha+2\pi} \cos nx\, dx + \int_{\alpha}^{\alpha+2\pi} \left(\sum_{n=1}^{\infty} a_n \cos nx \right) \cos nx\, dx$$

$$+ \int_{\alpha}^{\alpha+2\pi} \left(\sum_{n=1}^{\infty} b_n \sin nx \right) \cos nx\, dx$$

$$= 0 + \pi a_n + 0 \qquad \text{[By integrals (1), (3), (4), (5) and (6)]}$$

Hence $$a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx\, dx.$$

To find b_n, multiply each side of (i) by $\sin nx$ and integrate from $x = \alpha$ to $x = \alpha + 2\pi$. Then

$$\int_\alpha^{\alpha+2\pi} f(x)\sin nx\,dx = \frac{1}{2}a_0\int_\alpha^{\alpha+2\pi}\sin nx\,dx + \int_\alpha^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} a_n\cos nx\right)\sin nx\,dx$$

$$+\int_\alpha^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} b_n\sin nx\right)\sin nx\,dx$$

$$= 0 + 0 + \pi b_n \qquad \text{[By integrals (2), (5), (6), (7) and (8)]}$$

Hence $$b_n = \frac{1}{\pi}\int_\alpha^{\alpha+2\pi} f(x)\sin nx\,dx.$$

Cor. 1. Making $\alpha = 0$, *the interval becomes* $0 < x < 2\pi$, and the formulae (I) reduce to

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \end{aligned}\right\} \qquad \text{...(II)}$$

Cor. 2. Putting $\alpha = -\pi$, *the interval becomes* $-\pi < x < \pi$ and the formulae (I) take the form :

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \end{aligned}\right\} \qquad \text{...(III)}$$

Example 10.1. *Obtain the Fourier series for* $f(x) = e^{-x}$ *in the interval* $0 < x < 2\pi$. (S.V.T.U., 2007)

Solution. Let $$e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx \qquad \text{...}(i)$$

Then $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx = \frac{1}{\pi}\Big|-e^{-x}\Big|_0^{2\pi} = \frac{1-e^{-2\pi}}{\pi}$$

and $$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$$= \frac{1}{\pi(n^2+1)}\Big|e^{-x}(-\cos nx + n\sin nx)\Big|_0^{2\pi} = \left(\frac{1-e^{-2\pi}}{\pi}\right)\cdot\frac{1}{n^2+1}$$

$\therefore$ $$a_1 = \left(\frac{1-e^{-2\pi}}{\pi}\right)\frac{1}{2},\ a_2 = \left(\frac{1-e^{-2\pi}}{\pi}\right)\cdot\frac{1}{5}\ \text{etc.}$$

Finally, $$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$$

$$= \frac{1}{\pi(n^2+1)}\Big|e^{-x}(-\sin nx - n\cos nx)\Big|_0^{2\pi} = \left(\frac{1-e^{-2\pi}}{\pi}\right)\cdot\frac{n}{n^2+1}$$

$\therefore$ $$b_1 = \frac{1-e^{-2\pi}}{\pi}\cdot\frac{1}{2},\ b_2 = \left(\frac{1-e^{-2\pi}}{\pi}\right)\cdot\frac{2}{5}\ \text{etc.}$$

Substituting the values of a_0, a_n, b_n in (i), we get

$$e^{-x} = \frac{1-e^{-2\pi}}{\pi}\left\{\frac{1}{2} + \left(\frac{1}{2}\cos x + \frac{1}{5}\cos 2x + \frac{1}{10}\cos 3x + \ldots\right) + \left(\frac{1}{2}\sin x + \frac{2}{5}\sin 2x + \frac{3}{10}\sin 3x + \ldots\right)\right\}.$$

Example 10.2. *Find a Fourier series to represent $x - x^2$ from $x = -\pi$ to $x = \pi$.*
(V.T.U., 2011 ; Madras, 2006)

Solution. Let $$x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad ...(i)$$

Then $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\, dx = \frac{1}{\pi}\left| \frac{x^2}{2} - \frac{x^3}{3} \right|_{-\pi}^{\pi} = -\frac{2\pi^2}{3}.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2) \cos nx\, dx^*$$

$$= \frac{1}{\pi}\left[(x - x^2)\frac{\sin nx}{n} - (1 - 2x) \times \left(-\frac{\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{-4(-1)^n}{n^2} \qquad [\because \cos n\pi = (-1)^n]$$

$\therefore$ $a_1 = 4/1^2, a_2 = -4/2^2, a_3 = 4/3^2, a_4 = -4/4^2$ etc.

Finally, $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2) \sin nx\, dx$$

$$= \frac{1}{\pi}\left[(x - x^2)\left(-\frac{\cos nx}{n}\right) - (1 - 2x) \times \left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi} = -2(-1)^n/n$$

$\therefore$ $b_1 = 2/1, b_2 = -2/2, b_3 = 2/3, b_4 = -2/4$ etc.

Substituting the values of a's and b's in (i), we get

$$x - x^2 = -\frac{\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + ...\right] + 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + ...\right]$$

Obs. Putting $x = 0$, we find another interesting series $0 = -\frac{\pi^2}{3} + 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ...\right)$

i.e., $$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ... = \frac{\pi^2}{12}.$$ *(V.T.U., 2011)*

Note. In the above example, we have used the results $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$

Also $\sin\left(n + \frac{1}{2}\right)\pi = (-1)^n$ and $\cos\left(n + \frac{1}{2}\right)\pi = 0$. *The reader should remember these results.*

Example 10.3. *Expand $f(x) = x \sin x$ as a Fourier series in the interval $0 < x < 2\pi$.*
(S.V.T.U., 2009 ; Bhopal, 2009 ; Rohtak, 2006)

Solution. Let $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad ...(i)$$

Then $$a_0 = \frac{1}{\pi}\int_0^{2\pi} x \sin x\, dx = \frac{1}{\pi}\left| x(-\cos x) - 1.(-\sin x) \right|_0^{2\pi} = -2.$$

---

* Apply the *general rule of integration by parts* which states that if u, v be two functions of x and dashes denote differentiations and suffixes integrations w.r.t. x, then

$$\int uv\, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + ...$$

In other words : *Integral of the product of two functions*

= 1st function × integral of 2nd – go on differentiating 1st, integrating 2nd signs alternately +ve and –ve.

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos nx \, dx = \frac{1}{2\pi}\int_0^{2\pi} x\,(2\cos nx \sin x)\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\,[\sin(n+1)x - \sin(n-1)x]\, dx$$

$$= \frac{1}{2\pi}\left[x\left\{\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - 1\left\{-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left\{-\frac{\cos 2(n+1)\pi}{n+1} + \frac{\cos 2(n-1)\pi}{n-1}\right\}\right] = \frac{2}{n^2-1}.\ (n \neq 1)$$

When $n = 1$, $a_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x \, dx = \frac{1}{2\pi}\int_0^{2\pi} x \sin 2x \, dx$

$$= \frac{1}{2\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - 1.\left(-\frac{\sin 2x}{4}\right)\right]_0^{2\pi} = -\frac{1}{2}.$$

Finally, $b_n = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin nx \, dx = \frac{1}{2\pi}\int_0^{2\pi} x\,[\cos(n-1)x - \cos(n+1)x]\,dx$

$$= \frac{1}{2\pi}\left[x\left\{\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)}{n+1}\right\} - 1.\left\{-\frac{\cos(n-1)x}{(n-1)^2} + \frac{\cos(n+1)x}{(n+1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\cos 2(n-1)\pi}{(n-1)^2} - \frac{\cos 2(n+1)\pi}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right] = 0\ (n \neq 1)$$

When $n = 1$, $b_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin x \, dx = \frac{1}{2\pi}\int_0^{2\pi} x\,(1 - \cos 2x)\, dx$

$$= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - 1.\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{2\pi} = \pi$$

Substituting the values of a's and b's, in (i), we get

$$x \sin x = -1 + \pi \sin x - \frac{1}{2}\cos x + \frac{2}{2^2-1}\cos 2x + \frac{2}{3^2-1}\cos 3x + \ldots$$

Example 10.4. *Expand $f(x) = \sqrt{(1-\cos x)}$, $0 < x < 2\pi$ in a Fourier series. Hence evaluate*

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots\ldots$$

(Mumbai, 2006 ; J.N.T.U., 2006)

Solution. We have $f(x) = \sqrt{(1-\cos x)} = \sqrt{2\sin x/2}$.

Let
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots(i)$$

Then
$$a_0 = \frac{1}{\pi}\int_0^{2\pi}\sqrt{2\sin x/2}\, dx = \frac{\sqrt{2}}{\pi}\left| -2\cos\frac{\pi}{2}\right|_0^{2\pi} = \frac{4\sqrt{2}}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi}\sqrt{2}\sin\frac{x}{2}\cos nx\, dx = \frac{\sqrt{2}}{2\pi}\int_0^{2\pi} 2\cos nx \sin x/2\, dx$$

$$= \frac{1}{\sqrt{2}\pi}\int_0^{2\pi}\sin\left(n+\frac{1}{2}\right)x - \sin\left(n-\frac{1}{2}\right)x\Big]dx$$

$$= \frac{1}{\sqrt{2}\pi}\left| -\frac{2}{2n+1}\cos\left(\frac{2n+1}{2}\right) + \frac{2}{2n-1}\cos\frac{2n-1}{2}x\right|_0^{2\pi}$$

$$= \frac{2}{\sqrt{2}\pi}\left\{-\frac{1}{2n+1}[\cos(2n+1)\pi - 1] + \frac{1}{2n-1}[\cos(2n-1)\pi - 1]\right\}$$

$$= \frac{\sqrt{2}}{\pi}\left(\frac{2}{2n+1} - \frac{2}{2n-1}\right) = -\frac{4\sqrt{2}}{\pi(4n^2-1)} \qquad [\because \cos(2n+1)\pi = \cos(2n-1)\pi = -1]$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\sin\frac{x}{2}\sin nx\,dx = \frac{\sqrt{2}}{2\pi}\int_0^{2\pi} 2\sin nx \sin x/2\,dx$$

$$= \frac{1}{\sqrt{2}\pi}\int_0^{2\pi}\left[\cos\left(n-\frac{1}{2}\right)x - \cos\left(n+\frac{1}{2}\right)x\right]dx$$

$$= \frac{1}{\sqrt{2}\pi}\left|\frac{2}{2n-1}\sin\left(\frac{2n-1}{2}\right)x - \frac{2}{2n+1}\sin\left(\frac{2n+1}{2}\right)x\right|_0^{2\pi}$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{1}{2n-1}\{\sin(2n-1)\pi - 0\} - \frac{1}{2n+1}\{\sin(2n+1)\pi - 0\}\right] = 0$$

Substituting the values of a's and b's in (i), we get

$$\sqrt{(1-\cos x)} = \frac{2\sqrt{2}}{\pi} - \sum_{n=1}^{\infty}\frac{4\sqrt{2}}{(4n^2-1)\pi}\cos nx$$

When $x = 0$, we have

$$0 = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)} \qquad i.e., \quad \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots\ldots\infty = \frac{1}{2}.$$

PROBLEMS 10.1

1. Obtain a Fourier series to represent e^{-ax} from $x = -\pi$ to $x = \pi$. Hence derive series for $\pi/\sinh\pi$.

2. Prove that $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}$, $-\pi < x < \pi$. *(P.T.U., 2009 ; Bhopal, 2008 ; B.P.T.U., 2006)*

   Hence show that (i) $\sum\frac{1}{n^2} = \frac{\pi^2}{6}$. *(Anna, 2009 ; P.T.U., 2009 ; Osmania, 2003)*

   (ii) $\sum\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$ (iii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}$ *(S.V.T.U., 2008)*

   (iv) $\sum\frac{1}{n^4} = \frac{\pi^4}{90}$. *(Bhopal, 2008)*

3. If $f(x) = \left(\frac{n-x}{2}\right)^2$ in the range 0 to 2π, show that $f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$. *(Delhi, 2002 ; Madras, 2000)*

4. Prove that in the range $-\pi < x < \pi$, $\cosh ax = \frac{2a^2}{\pi}\sinh a\pi\left[\frac{1}{2a^2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+a^2}\cos nx\right]$.

5. $f(x) = x + x^2$ for $-\pi < x < \pi$ and $f(x) = \pi^2$ for $x = \pm\pi$. Expand $f(x)$ in Fourier series. *(Kurukshetra, 2005 ; U.P.T.U., 2003)*

   Hence show that $x + x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n\left\{\frac{4}{n^2}\cos nx - \frac{2}{n}\sin nx\right\}$

   and $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\frac{\pi^2}{8}$ *(V.T.U., 2008)*

10.3 CONDITIONS FOR A FOURIER EXPANSION

The reader must not be misled by the belief that the Fourier expansion of $f(x)$ in each case shall be valid. The above discussion has merely shown that if $f(x)$ has an expansion, then the coefficients are given by Euler's formulae. The problems concerning the possibility of expressing a function by Fourier series and convergence

of this series are many and cumbersome. Such questions should be left to the curiosity of a pure-mathematician. However, almost all engineering applications are covered by the following well-known **Dirichlet's conditions***:

Any function f(x) can be developed as a Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ *where* a_0, a_n, b_n *are constants, provided :*

(i) f (x) is periodic, single-valued and finite;

(ii) f (x) has a finite number of discontinuties in any one period;

(iii) f (x) has at the most a finite number of maxima and minima. *(Anna, 2009 ; P.T.U., 2009)*

In fact the problem of expressing any function $f(x)$ as a Fourier series depends upon the evaluation of the integrals.

$$\frac{1}{\pi}\int f(x)\cos nx\,dx\,;\ \frac{1}{\pi}\int f(x)\sin nx\,dx$$

within the limits $(0, 2\pi)$, $(-\pi, \pi)$ or $(\alpha, \alpha + 2\pi)$ according as $f(x)$ is defined for every value of x in $(0, 2\pi)$, $(-\pi, \pi)$ or $(\alpha, \alpha + 2\pi)$.

PROBLEMS 10.2

State giving reasons whether the following functions can be expanded in Fourier series in the interval $-\pi \le x \le \pi$.

1. cosec x
2. $\sin 1/x$ *(P.T.U., 2002)*
3. $f(x) = (m + 1)/m$, $\pi/(m + 1) < |x| \le \pi/m$, $m = 1, 2, 3, \ldots \infty$.

10.4 FUNCTIONS HAVING POINTS OF DISCONTINUITY

In deriving the Euler's formulae for a_0, a_n, b_n, it was assumed that $f(x)$ was continuous. Instead a function may have a finite number of points of finite discontinuity *i.e.*, its graph may consist of a finite number of different curves given by different equations. Even then such a function is expressible as a Fourier series.

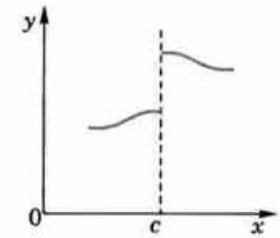


Fig. 10.1

For instance, if in the interval $(\alpha, \alpha + 2\pi)$, $f(x)$ is defined by

$$f(x) = \phi(x),\ \alpha < x < c.$$
$$= \psi(x),\ c < x < \alpha + 2\pi,\ \textit{i.e.},\ c \text{ is the point of discontinuity, then}$$

$$a_0 = \frac{1}{\pi}\left[\int_\alpha^c \phi(x)\,dx + \int_c^{\alpha+2\pi} \psi(x)\,dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_\alpha^c \phi(x)\cos nx\,dx + \int_c^{\alpha+2\pi} \psi(x)\cos nx\,dx\right]$$

and
$$b_n = \frac{1}{\pi}\left[\int_\alpha^c \phi(x)\sin nx\,dx + \int_c^{\alpha+2\pi} \psi(x)\sin nx\,dx\right]$$

At a point of finite discontinuity $x = c$, there is a finite jump in the graph of function (Fig. 10.1). Both the limit on the left [*i.e.*, $f(c - 0)$] and the limit on the right [*i.e.*, $f(c + 0)$] exist and are different. At such a point, Fourier series gives the value of $f(x)$ as the arithmetic mean of these two limits,

i.e., at $x = c$, $\quad f(x) = \frac{1}{2}[f(c - 0) + f(c + 0)]$.

Example 10.5. *Find the Fourier series expansion for f (x), if*

$$f(x) = -\pi,\ -\pi < x < 0$$
$$x,\ 0 < x < \pi.$$ *(Bhopal, 2008 S)*

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$. *(Kottayam, 2005)*

*See footnote p. 307.

Solution. Let $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad ...(i)$$

Then $$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\, dx + \int_0^{\pi} x\, dx\right] = \frac{1}{\pi}\left[-\pi \,|\, x \,|_{-\pi}^{0} + \left| x^2/2 \right|_0^{\pi}\right] = \frac{1}{\pi}\left(-\pi^2 + \frac{\pi^2}{2}\right) = -\frac{\pi}{2};$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi) \cos nx\, dx + \int_0^{\pi} x \cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[-\pi \left| \frac{\sin nx}{n} \right|_{-\pi}^{0} + \left| \frac{x \sin nx}{n} + \frac{\cos nx}{n^2} \right|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{1}{n^2}\cos n\pi - \frac{1}{n^2}\right] = \frac{1}{\pi n^2}(\cos n\pi - 1)$$

$\therefore$ $$a_1 = \frac{-2}{\pi \,.\, 1^2},\ a_2 = 0,\ a_3 = -\frac{2}{\pi \,.\, 3^2},\ a_4 = 0,\ a_5 = -\frac{2}{\pi \,.\, 5^2} \text{ etc.}$$

Finally, $$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi) \sin nx\, dx + \int_0^{\pi} x \sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\left| \frac{\pi \cos nx}{n} \right|_{-\pi}^{0} + \left| -x\frac{\cos nx}{n} + \frac{\sin nx}{n^2} \right|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1 - \cos n\pi) - \frac{\pi}{n}\cos n\pi\right] = \frac{1}{n}(1 - 2\cos n\pi)$$

$\therefore$ $$b_1 = 3,\ b_2 = -\frac{1}{2},\ b_3 = 1,\ b_4 = -\frac{1}{4}, \text{ etc.}$$

Hence substituting the values of a's and b's in (i), we get

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + ...\right) + 3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \frac{\sin 4x}{4} + ... \qquad ...(ii)$$

which is the required result.

Putting $x = 0$ in (ii), we obtain $f(0) = -\frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + ... \infty\right)$ $\qquad ...(iii)$

Now $f(x)$ is discontinuous at $x = 0$. As a matter of fact

$$f(0-0) = -\pi \text{ and } f(0+0) = 0 \quad \therefore \quad f(0) = \frac{1}{2}[f(0-0) + f(0+0)] = -\pi/2.$$

Hence (iii) takes the form $-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...\right]$ whence follows the result.

Example 10.6. *If* $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ \sin x, & 0 \le x \le \pi \end{cases}$, *prove that* $f(x) = \frac{1}{\pi} + \frac{\sin x}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2 - 1}$.

Hence show that $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - ... - \infty = \frac{1}{4}(\pi - 2)$ *(Bhopal, 2008 ; Mumbai, 2005 S ; Rohtak, 2005)*

Solution. Let $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Then $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\, dx + \int_0^{\pi} \sin x\, dx\right] = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 \,.\, dx + \int_0^{\pi} \sin x \cos nx\, dx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi} [\sin(n+1)x - \sin(n-1)x]\, dx = \frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] \qquad (n \neq 1)$$

$$= \frac{1}{2\pi}\left\{\frac{1-(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}-1}{n-1}\right\} = 0, \text{ when } n \text{ is odd}$$

$$= -\frac{2}{\pi(n^2-1)}, \text{ when } n \text{ is even.}$$

When $n = 1$, $a_1 = \frac{1}{\pi}\int_0^\pi \sin x \cos x\, dx = \frac{1}{2\pi}\int_0^\pi \sin 2x\, dx = \frac{1}{2\pi}\left|-\frac{\cos 2x}{2}\right|_0^\pi = 0$

Finally, $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,.\,dx + \int_0^\pi \sin x \sin nx\, dx\right]$$

$$= \frac{1}{2\pi}\int_0^\pi [\cos \overline{n-1}x - \cos \overline{n+1}x]\, dx = \frac{1}{2\pi}\left[\frac{\sin \overline{n-1}x}{n-1} - \frac{\sin \overline{n+1}x}{n+1}\right]_0^\pi = 0 \quad (n \neq 1)$$

When $n = 1$, $b_1 = \frac{1}{\pi}\int_0^\pi \sin x \sin x\, dx = \frac{1}{2\pi}\int_0^\pi (1-\cos 2x)\, dx = \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^\pi = \frac{1}{2}$

Hence $$f(x) = \frac{1}{\pi} - \frac{2}{\pi}\left[\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \ldots\right] + \frac{1}{2}\sin x \qquad \ldots(i)$$

Putting $x = \frac{\pi}{2}$ in (i), we get $1 = \frac{1}{\pi} - \frac{2}{\pi}\left(-\frac{1}{1.3} + \frac{1}{3.5} - \frac{1}{5.7} + \ldots \infty\right) + \frac{1}{2}$

Whence $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots \infty = \frac{1}{4}(\pi - 2)$.

Example 10.7. *Find the Fourier series for the function*

$$f(t) = \begin{cases} -1 & \text{for } -\pi < t < -\pi/2 \\ 0 & \text{for } -\pi/2 < t < \pi/2 \\ 1 & \text{for } \pi/2 < t < \pi \end{cases}$$

Solution. Let $$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt \qquad \ldots(i)$$

Then $$a_0 = \frac{1}{\pi}\left\{\int_{-\pi}^{-\pi/2} (-1)dt + \int_{-\pi/2}^{\pi/2} (0)dt + \int_{\pi/2}^{\pi} (1)\, dt\right\}$$

$$= \frac{1}{\pi}\left\{\left| -x \right|_{-\pi}^{-\pi/2} + \left| x \right|_{\pi/2}^{\pi}\right\} = \frac{1}{\pi}(\pi/2 - \pi + \pi - \pi/2) = 0$$

$$a_n = \frac{1}{\pi}\left\{\int_{-\pi}^{-\pi/2} (-1)\cos nt\, dt + \int_{-\pi/2}^{\pi/2} (0)\cos nt\, dt + \int_{\pi/2}^{\pi} (1)\cos nt\, dt\right\}$$

$$= \frac{1}{\pi}\left\{\left| -\frac{\sin nt}{n}\right|_{-\pi}^{-\pi/2} + \left|\frac{\sin nt}{n}\right|_{\pi/2}^{\pi}\right\} = \frac{1}{n\pi}\left(\frac{\sin n\pi}{2} - \frac{\sin n\pi}{2}\right) = 0$$

and $$b_n = \frac{1}{\pi}\left\{\int_{-\pi}^{-\pi/2} (-1)\sin nt\, dt + \int_{-\pi/2}^{\pi/2} (0)\sin nt\, dt + \int_{\pi/2}^{\pi} (1)\sin nt\, dt\right\}$$

$$= \frac{1}{\pi}\left\{\left|\frac{\cos nt}{n}\right|_{-\pi}^{-\pi/2} + \left| -\frac{\cos nt}{n}\right|_{\pi/2}^{\pi}\right\} = \frac{2}{n\pi}\left(\cos\frac{n\pi}{2} - \cos n\pi\right)$$

$$\therefore \qquad b_1 = \frac{2}{\pi}, b_2 = -\frac{2}{\pi}, b_3 = \frac{2}{3\pi} \text{ etc.}$$

Hence substituting the values of a's and b's in (i), we get $f(t) = \frac{2}{\pi}\left(\sin t - \sin 2t + \frac{1}{3}\sin 3t + \ldots\right)$.

PROBLEMS 10.3

1. Find the Fourier series to represent the function $f(x)$ given by

$$f(x) = x \text{ for } 0 \le x \le \pi, \text{ and } = 2\pi - x \text{ for } \pi \le x \le 2\pi.$$ *(S.V.T.U., 2008 ; B.P.T.U., 2005 S)*

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \infty = \frac{\pi^2}{8}$. *(Madras 2000 S ; V.T.U., 2000 S)*

2. An alternating current after passing through a rectifier has the form

$$i = I_0 \sin x \qquad \text{for } 0 \le x \le \pi$$
$$= 0 \qquad \text{for } \pi \le x \le 2\pi$$

where I_0 is the maximum current and the period is 2π (Fig. 10.2). Express i as a Fourier series and evaluate

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots \infty$$ *(V.T.U., 2007 ; Calicut, 2005)*

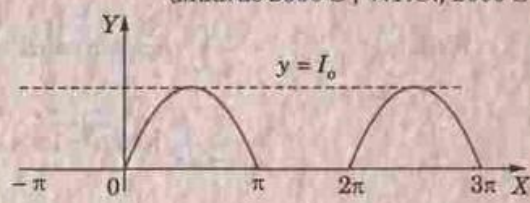


Fig. 10.2

3. Draw the graph of the function $f(x) = 0, -\pi < x < 0$
$= x^2, 0 < x < \pi.$

If $f(2\pi + x) = f(x)$, obtain Fourier series of $f(x)$.

4. Find the Fourier series of the following function :

$$f(x) = x^2, \qquad 0 \le x \le \pi,$$ *(Mumbai, 2009)*
$$= -x^2, \qquad -\pi \le x \le 0.$$ *(Hissar, 2007)*

5. Find a Fourier series for the function defined by

$$f(x) = \begin{cases} -1, \text{ for } -\pi < x < 0 \\ 0, \text{ for } x = 0 \\ 1, \text{ for } 0 < x < \pi \end{cases}$$

Hence prove that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \infty = \frac{\pi}{4}$. *(U.P.T.U., 2005)*

10.5 CHANGE OF INTERVAL

In many engineering problems, the period of the function required to be expanded is not 2π but some other interval, say : $2c$. In order to apply the foregoing discussion to functions of period $2c$, this interval must be converted to the length 2π. This involves only a proportional change in the scale.

Consider the periodic function $f(x)$ defined in $(\alpha, \alpha + 2c)$. To change the problem to period 2π

put $\quad z = \pi x/c \quad$ or $\quad x = cz/\pi$ ...(1)

so that when $\quad x = \alpha, \quad z = \alpha\pi/c = \beta$ (say)

when $\quad x = \alpha + 2c, \quad z = (\alpha + 2c)\pi/c = \beta + 2\pi.$

Thus the function $f(x)$ of period $2c$ in $(\alpha, \alpha + 2c)$ is transformed to the function $f(cz/\pi)$ [$= F(z)$ say] of period 2π in $(\beta, \beta + 2\pi)$. Hence $f(cz/\pi)$ can be expressed as the Fourier series

$$f\left(\frac{cz}{\pi}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nz + \sum_{n=1}^{\infty} b_n \sin nz \qquad \ldots(2)$$

where

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{\beta}^{\beta+2\pi} f\left(\frac{cz}{\pi}\right) dz \\ a_n &= \frac{1}{\pi}\int_{\beta}^{\beta+2\pi} f\left(\frac{cz}{\pi}\right) \cos nz\, dz \\ b_n &= \frac{1}{\pi}\int_{\beta}^{\beta+2\pi} f\left(\frac{cz}{\pi}\right) \sin nz\, dz \end{aligned}\right\} \qquad \ldots(3)$$

Making the inverse substitutions $z = \pi x/c$, $dz = (\pi/c)\,dx$ in (2) and (3) *the Fourier expansion of $f(x)$ in the interval $(\alpha, \alpha + 2c)$ is given by*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c}$$

where

$$\left.\begin{aligned} a_0 &= \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\,dx \\ a_n &= \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\cos\frac{n\pi x}{c}\,dx \\ b_n &= \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\sin\frac{n\pi x}{c}\,dx \end{aligned}\right\} \qquad ...(4)$$

Cor. Putting $\alpha = 0$ in (4), we get the results for the interval $(0, 2c)$ and putting $\alpha = -c$ in (4), we get results for the interval $(-c, c)$.

Example 10.8. *Expand $f(x) = e^{-x}$ as a Fourier series in the interval $(-l, l)$.*

(Kerala, 2005 ; V.T.U., 2004)

Solution. The required series is of the form

$$e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad ...(i)$$

Then $$a_0 = \frac{1}{l}\int_{-l}^{l} e^{-x}\,dx = \frac{1}{l}\Big|-e^{-x}\Big|_{-l}^{l} = \frac{1}{l}(e^{l} - e^{-l}) = \frac{2\sinh l}{l}$$

and $$a_n = \frac{1}{l}\int_{-l}^{l} e^{x}\cos\frac{n\pi x}{l}\,dx \qquad \left[\because \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right]$$

$$= \frac{1}{l}\left|\frac{e^{-x}}{1+(n\pi/l)^2}\left(-\cos\frac{n\pi x}{l} + \frac{n\pi}{l}\sin\frac{n\pi x}{l}\right)\right|_{-l}^{l} = \frac{2l(-1)^n\sinh l}{l^2+(n\pi)^2} \qquad [\because \cos n\pi = (-1)^n$$

$$\therefore \quad a_1 = \frac{-2l\sinh l}{l^2+\pi^2},\ a_2 = \frac{2l\sinh l}{l^2+2^2\pi^2},\ a_3 = \frac{2l\sinh l}{l^2+3^2\pi^2} \text{ etc.}$$

Finally, $$b_n = \frac{1}{l}\int_{-l}^{l} e^{-x}\sin\frac{n\pi x}{l}\,dx \qquad \left[\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

$$= \frac{1}{l}\left|\frac{e^{-x}}{1+(n\pi/l)^2}\left(-\sin\frac{n\pi x}{l} - \frac{n\pi}{l}\cos\frac{n\pi x}{l}\right)\right|_{-l}^{l} = \frac{2n\pi(-1)^n\sinh l}{l^2+(n\pi)^2}$$

$$\therefore \quad b_1 = \frac{-2\pi\sinh l}{l^2+\pi^2},\ b_2 = \frac{4\pi\sinh l}{l^2+2^2\pi^2},\ b_2 = \frac{-6\pi\sinh l}{l^2+3^2\pi^2} \text{ etc.}$$

Substituting the values of a's and b's in (i), we get

$$e^{-x} = \sinh l\left\{\frac{1}{l} - 2l\left(\frac{1}{l^2+\pi^2}\cos\frac{\pi x}{l} - \frac{1}{l^2+2^2\pi^2}\cos\frac{2\pi x}{l} + \frac{1}{l^2+3^2\pi^2}\cos\frac{3\pi x}{l} - \ldots\right)\right.$$
$$\left. - 2\pi\left(\frac{1}{l^2+\pi^2}\sin\frac{n\pi}{l} - \frac{2}{l^2+2^2\pi^2}\sin\frac{2\pi x}{l} + \frac{3}{l^2+3^2\pi^2}\sin\frac{3\pi x}{l} - \ldots\right)\right\}$$

Example 10.9. *Find the Fourier series expansion of $f(x) = 2x - x^2$ in $(0, 3)$ and hence deduce that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots - \infty = \frac{\pi}{12}$.*

(Mumbai, 2005)

Solution. The required series is of the form

$$2x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x \quad \text{where } l = 3/2. \qquad ...(i)$$

Then
$$a_0 = \frac{1}{l}\int_0^{2l} (2x - x^2)dx = \frac{2}{3}\left| x^2 - \frac{x^3}{3}\right|_0^3 = 0$$

$$a_n = \frac{1}{l}\int_0^{2l} (2x - x^2)\cos\frac{n\pi x}{l}dx = \frac{2}{3}\int_0^3 (2x - x^2)\cos\frac{2n\pi x}{3}dx$$

$$= \frac{2}{3}\left[(2x - x^2)\frac{\sin 2n\pi x/3}{2n\pi/3} - (2 - 2x)\frac{-\cos 2n\pi x/3}{(2n\pi/3)^2} + (-2)\frac{-\sin 2n\pi x/3}{(2n\pi/3)^3}\right]_0^3$$

$$= \frac{2}{3}\cdot\frac{9}{4n^2\pi^2}[(2 - 6)\cos 2n\pi - 2] = -\frac{9}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} (2x - x^2)\sin\frac{n\pi x}{l}dx = \frac{2}{3}\int_0^3 (2x - x^2)\sin\frac{2n\pi x}{3}dx$$

$$= \frac{2}{3}\left[(2x - x^2)\frac{-\cos 2n\pi x/3}{2n\pi/3} - (2 - 2x)\frac{-\sin 2n\pi x/3}{(2n\pi/3)^2} + (-2)\frac{\cos 2n\pi x/3}{(2n\pi/3)^3}\right]_0^3$$

$$= \frac{2}{3}\left\{-\frac{6}{n^2\pi^2}\cos 2n\pi - \frac{27}{4n^3\pi^3}(\cos 2n\pi - 1)\right\} = \frac{3}{n\pi}$$

Substituting the values of a_0, a_n, b_n in (i), we get

$$2x - x^2 = -\sum_{n=1}^{\infty}\frac{9}{n^2\pi^2}\cos\frac{2n\pi x}{3} + \sum_{n=1}^{\infty}\frac{3}{n\pi}\sin\frac{2n\pi x}{3}$$

Putting $x = 3/2$, we get

$$3 - \frac{9}{4} = -\sum_{n=1}^{\infty}\frac{9}{n^2\pi^2}\cos n\pi \quad \text{or} \quad -\sum_{n=1}^{\infty}\frac{\cos n\pi}{n^2} = \frac{\pi^2}{9}\cdot\frac{3}{4}$$

or
$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - ... \infty = \frac{\pi^2}{12}.$$

Example 10.10. *Obtain Fourier series for the function*

$$f(x) = \begin{cases} \pi x, & 0 \le x \le 1 \\ \pi(2 - x), & 1 \le x \le 2 \end{cases}$$

(V.T.U., 2011 ; Bhopal, 2008 ; Mumbai, 2007)

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... \infty = \frac{\pi^2}{8}$.

Solution. The required series is of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

Then
$$a_0 = \int_0^1 \pi x dx + \int_1^2 \pi(2 - x)dx = \pi\left|\frac{x^2}{2}\right|_0^1 + \pi\left|2x - \frac{x^2}{2}\right|_1^2 = \pi\left(\frac{1}{2}\right) + \pi\left\{(4 - 2) - \left(2 - \frac{1}{2}\right)\right\} = \pi$$

$$a_n = \int_0^1 \pi x \cos n\pi x dx + \int_1^2 \pi(2 - x)\cos n\pi x dx$$

$$= \left|\pi x\cdot\frac{\sin n\pi x}{n\pi} - \pi\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_0^1 + \left|\pi(2 - x)\frac{\sin n\pi x}{n\pi} - (-\pi)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_1^2$$

$$= \left(\frac{\cos n\pi}{n^2\pi} - \frac{1}{n^2\pi^2}\right) - \left(\frac{\cos 2n\pi}{n^2\pi} - \frac{\cos n\pi}{n^2\pi}\right) = \frac{2}{n^2\pi}[(-1)^n - 1]$$

$$= 0 \text{ when } n \text{ is even}; -\frac{4}{n^2\pi} \text{ when } n \text{ is odd.}$$

$$b_n = \int_0^1 \pi x \sin n\pi x\, dx + \int_1^2 \pi(2-x)\sin n\pi x\, dx$$

$$= \left| \pi x\left(-\frac{\cos n\pi x}{n\pi}\right) - \pi\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_0^1 + \left| \pi(2-x)\left(\frac{-\cos n\pi x}{n\pi}\right) - (-\pi)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_1^2$$

$$= \left(-\frac{\cos n\pi}{n}\right) + \left(\frac{\cos n\pi}{n}\right) = 0$$

Hence $\quad f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \ldots \infty\right)$

Putting $x = 2$, $0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos 2\pi}{1^2} + \frac{\cos 6\pi}{3^2} + \frac{\cos 10\pi}{5^2} + \ldots \infty\right)$

Whence $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \infty = \frac{\pi^2}{8}$.

Example 10.11. *Find the Fourier series for*

$$f(t) = 0, -2 < t < -1$$
$$= 1 + t, -1 < t < 0$$
$$= 1 - t, 0 < t < 1$$
$$= 0, \quad 1 < t < 2.$$

Solution. Let $\quad f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{2} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{2} \qquad \ldots(i)$

[$\because \quad 2c = 2 - (-2)$ so that $c = 2$]

Then $\quad a_0 = \frac{1}{2}\left\{\int_{-2}^{-1}(0)dt + \int_{-1}^{0}(1+t)dt + \int_0^1 (1-t)dt + \int_1^2 (0)dt\right\} = \frac{1}{2}\left\{\left|t + \frac{t^2}{2}\right|_{-1}^{0} + \left|t - \frac{t^2}{2}\right|_0^1\right\}$

$$= \frac{1}{2}\left\{-\left(-1 + \frac{1}{2}\right) + \left(1 - \frac{1}{2}\right)\right\} = \frac{1}{2}$$

$$a_n = \frac{1}{2}\left\{\int_{-1}^{0}(1+t)\cos\frac{n\pi t}{2}dt + \int_0^1 (1-t)\cos\frac{n\pi t}{2}dt\right\} \qquad \text{[Integrate by parts]}$$

$$= \frac{1}{2}\left\{\left|(1+t)\left(\sin\frac{n\pi t}{2}\right)\frac{2}{n\pi} - (1)\left(-\cos\frac{n\pi t}{2}\right)\frac{4}{n^2\pi^2}\right|_{-1}^{0}\right.$$

$$\left. + \left|(1-t)\left(\sin\frac{n\pi t}{2}\right)\frac{2}{n\pi} - (-1)\left(-\cos\frac{n\pi t}{2}\right)\frac{4}{n^2\pi^2}\right|_0^1\right\}$$

$$= \frac{4}{n^2\pi^2}(1 - \cos n\pi/2)$$

$$b_n = \frac{1}{2}\left\{\int_{-1}^{0}(1+t)\sin\frac{n\pi t}{2}dt + \int_0^1 (1-t)\sin\frac{n\pi t}{2}dt\right\}$$

$$= \frac{1}{2}\left\{\left|(1+t)\left(-\cos\frac{n\pi t}{2}\right)\frac{2}{n\pi} - 1\left(-\sin\frac{n\pi t}{2}\right)\frac{4}{n^2\pi^2}\right|_{-1}^{0}\right.$$

$$\left. + \left|(1-t)\left(-\cos\frac{n\pi t}{2}\right)\frac{2}{n\pi} - (-1)\left(-\sin\frac{n\pi t}{2}\right)\frac{4}{n^2\pi^2}\right|_0^1\right\}$$

$$= \frac{1}{2}\left\{\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right) - \left(\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right\} = 0$$

Substituting the values of a's and b's in (i), we get

$$f(t) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left(1 - \cos\frac{n\pi}{2}\right)\cos\frac{n\pi t}{2}.$$

PROBLEMS 10.4

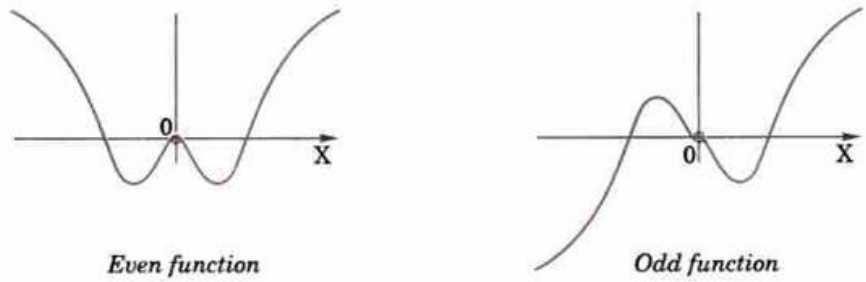
1. Obtain the Fourier series for $f(x) = \pi x$ in $0 \le x \le 2$.
2. (i) Find the Fourier series to represent x^2 in the interval $(0, a)$. *(Mumbai, 2009)*
   (ii) Find a Fourier series for $f(t) = 1 - t^2$ when $-1 \le t \le 1$. *(Mumbai, 2006)*
3. If $f(x) = 2x - x^2$ in $0 \le x \le 2$, show that $f(x) = \frac{2}{3} - \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\cos n\pi x$. *(V.T.U., 2006)*
4. Find the Fourier series for $f(x) = \begin{cases} x & \text{in } 0 \le x \le 3 \\ 6 - x & \text{in } 3 \le x \le 6 \end{cases}$ *(Anna, 2008)*
5. A sinusoidal voltage $E \sin \omega t$ is passed through a half-wave rectifier which clips the negative portion of the wave. Develop the resulting periodic function
   $$U(t) = 0 \qquad \text{when } -T/2 < t < 0$$
   $$= E \sin \omega t \qquad \text{when } 0 < t < T/2,$$
   and $T = 2\pi/\omega$, in a Fourier series. *(Calicut, 1999)*
6. Find the Fourier series of the function $f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ 0, & x = 1 \\ \pi(x-2), & 1 < x < 2 \end{cases}$

   Hence show that $\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ *(Mumbai, 2008)*

10.6 (1) EVEN AND ODD FUNCTIONS

A function $f(x)$ is said to be **even** *if $f(-x) = f(x)$,*
e.g., $\cos x$, $\sec x$, x^2 are all even functions. Graphically an even function is symmetrical about the y-axis.

A function $f(x)$ is said to be **odd** *if $f(-x) = -f(x)$,*



Even function *Odd function*

Fig. 10.3

e.g. $\sin x$, $\tan x$, x^3 are odd functions. Graphically, an odd function is symmetrical about the origin.

We shall be using the following property of definite integrals in the next paragraph :

$$\int_{-c}^{c} \mathbf{f(x)\,dx} = \mathbf{2}\int_{0}^{c} \mathbf{f(x)\,dx}, \text{ when } f(x) \text{ is an even function.}$$

$$= \mathbf{0}, \text{ when } f(x) \text{ is an odd function.}$$

(2) Expansions of even or odd periodic functions. We know that a periodic function $f(x)$ defined in $(-c, c)$ can be represented by the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c},$$