Solving these equations, we get

(*i*) *When k is positive* and is equal to p^2, say

$$X = c_1 e^{px} + c_2 e^{-px},\ Y = c_3 \cos py + c_4 \sin py$$

(*ii*) *When k is negative*, and is equal to $-p^2$, say

$$X = c_5 \cos px + c_6 \sin px,\ Y = c_7 e^{py} + c_8 e^{-py}$$

(*iii*) *When k is zero* ; $X = c_9 x + c_{10}$, $Y = c_{11} y + c_{12}$.

Thus the various possible solutions of (1) are

$$u = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py) \quad ...(3)$$

$$u = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py}) \quad ...(4)$$

$$u = (c_9 x + c_{10})(c_{11} y + c_{12}) \quad ...(5)$$

Of these we take that solution which is consistent with the given boundary conditions.

(*V.T.U., 2011 S ; Kerala, 2005*)

Temperature distribution in long plates

Example 18.15. *An infinitely long plane uniform plate is bounded by two parallel edges and an end at right angles to them. The breadth is π ; this end is maintained at a temperature u_0 at all points and other edges are at zero temperature. Determine the temperature at any point of the plate in the steady-state.*

(*P.T.U., 2005 ; J.N.T.U., 2002 S*)

Solution. In the steady state (Fig. 18.6), the temperature $u(x, y)$ at any point $P(x, y)$ satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad ...(i)$$

The boundary conditions are $u(0, y) = 0$ for all values of y ...(*ii*)

$u(\pi, y) = 0$ for all values of y ...(*iii*)

$u(x, \infty) = 0$ in $0 < x < \pi$ ...(*iv*)

$u(x, 0) = u_0$ in $0 < x < \pi$ ...(*v*)

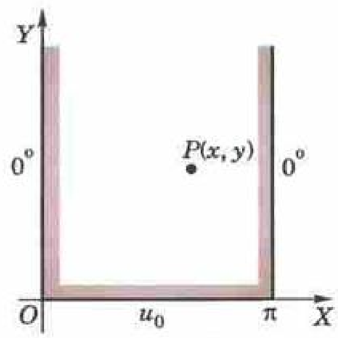


Fig. 18.6

Now the three possible solutions of (*i*) are

$$u = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py) \quad ...(vi)$$

$$u = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py}) \quad ...(vii)$$

$$u = (c_9 x + c_{10})(c_{11} y + c_{12}) \quad ...(viii)$$

Of these, we have to choose that solution which is consistent with the physical nature of the problem. The solution (*vi*) cannot satisfy the condition (*ii*) for $u \neq 0$ for $x = 0$, for all values of y. The solution (*viii*) cannot satisfy the condition (*iv*). Thus the only possible solution is (*vii*), *i.e.* of the form

$$u(x, y) = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py}) \quad ...(ix)$$

By (*ii*), $u(0, y) = C_1(C_3 e^{py} + C_4 e^{-py}) = 0$ for all y.

Hence $C_1 = 0$ and (*ix*) reduces to

$$u(x, y) = C_2 \sin px\,(C_3 e^{py} + C_4 e^{-py}) \quad ...(x)$$

By (*iii*), $u(\pi, y) = C_2 \sin p\pi (C_3 e^{py} + C_4 e^{-py}) = 0$, for all y.

This requires $\sin p\pi = 0$, *i.e.* $p\pi = n\pi$ as $C_2 \neq 0$. $\therefore$ $p = n$, an integer.

Also to satisfy the condition (*iv*), *i.e.*, $u = 0$ as $y \to \infty$, $C_3 = 0$.

Hence (*x*) takes the form $u(x, y) = b_n \sin nx \,.\, e^{-ny}$, where $b_n = C_2 C_4$.

$\therefore$ the most general solution satisfying (*ii*), (*iii*) and (*iv*) is of the form

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin nx \,.\, e^{-ny} \quad ...(xi)$$

Putting $y = 0$, $$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin nx \quad ...(xii)$$

In order that the condition (*v*) may be satisfied, (*v*) and (*xii*) must be same. This requires the expansion of u as a half-range Fourier sine series in $(0, \pi)$. Thus

$$u = \sum_{n=1}^{\infty} b_n \sin nx \text{ where } b_n = \frac{2}{\pi}\int_0^{\pi} u_0 \sin nx\, dx = \frac{2u_0}{n\pi}[1-(-1)^n]$$

i.e., $b_n = 0$, if n is even ; $= 4u_0/n\pi$, if n is odd.

Hence (*xi*) becomes $u(x, y) = \frac{4u_0}{\pi}\left[e^{-y} \sin x + \frac{1}{3}e^{-3y} \sin 3x + \frac{1}{5}e^{-5y} \sin 5x + \ldots\right]$.

Temperature distribution in finite plates

Example 18.16. *Solve the Laplace equation* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ *subject to the conditions* $u(0, y) = u(l, y) = u(x, 0) = 0$ *and* $u(x, a) = \sin n\pi x/l$. (V.T.U., 2011 ; J.N.T.U., 2006 ; Kerala M. Tech., 2005, U.P.T.U., 2004)

Solution. The three possible solutions of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \ldots(i)$$

are
$$u = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py) \quad \ldots(ii)$$
$$u = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py}) \quad \ldots(iii)$$
$$u = (c_9 x + c_{10})(c_{11} y + c_{12}) \quad \ldots(iv)$$

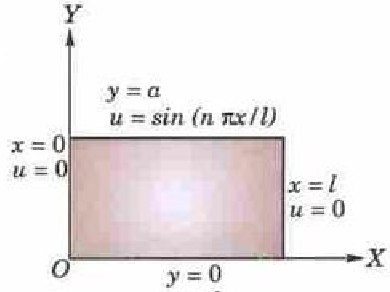


Fig. 18.7

We have to solve (*i*) satisfying the following boundary conditions

$$u(0, y) = 0 \quad \ldots(v) \qquad u(l, y) = 0 \quad \ldots(vi)$$
$$u(x, 0) = 0 \quad \ldots(vii) \qquad u(x, a) = \sin n\pi x/l \quad \ldots(viii)$$

Using (*v*) and (*vi*) in (*ii*), we get

$$c_1 + c_2 = 0, \quad \text{and} \quad c_1 e^{pl} + c_2 e^{-pl} = 0$$

Solving these equations, we get $c_1 = c_2 = 0$ which lead to trivial solution. Similarly, we get a trivial solution by using (*v*) and (*vi*) in (*iv*). Hence the suitable solution for the present problem is solution (*iii*). Using (*v*) in (*iii*), we have $c_5(c_7 e^{py} + c_8 e^{-py}) = 0$ *i.e.*, $c_5 = 0$

$\therefore$ (*iii*) becomes $$u = c_6 \sin px(c_7 e^{py} + c_8 e^{-py}) \quad \ldots(ix)$$

Using (*vi*), we have $c_6 \sin pl\,(c_7 e^{py} + c_8 e^{-py}) = 0$

$\therefore$ either $c_6 = 0$ or $\sin pl = 0$

If we take $c_6 = 0$, we get a trivial solution.

Thus $\sin pl = 0$ whence $pl = n\pi$ or $p = n\pi/l$ where $n = 0, 1, 2, \ldots\ldots$

$\therefore$ (*ix*) becomes $u = c_6 \sin (n\pi x/l)(c_7 e^{n\pi y/l} + c_8 e^{-n\pi y/l})$ ...(*x*)

Using (*vii*), we have $0 = c_6 \sin n\pi x/l \,.\, (c_7 + c_8)$ *i.e.*, $c_8 = -c_7$.

Thus the solution suitable for this problem is

$$u(x, y) = b_n \sin \frac{n\pi x}{l}(e^{n\pi y/l} - e^{-n\pi y/l}) \quad \text{where} \quad b_n = c_6 c_7$$

Now using the condition (*viii*), we have

$$u(x, a) = \sin \frac{n\pi x}{l} = b_n \sin \frac{n\pi x}{l}(e^{n\pi a/l} - e^{-n\pi a/l}),$$

we get
$$b_n = \frac{1}{(e^{n\pi a/l} - e^{-n\pi a/l})}$$

Hence the required solution is

$$u(x, y) = \frac{e^{n\pi y/l} - e^{-n\pi y/l}}{e^{n\pi a/l} - e^{-n\pi a/l}} \sin \frac{n\pi x}{l} = \frac{\sinh (n\pi y/l)}{\sinh (n\pi a/l)} \sin \frac{n\pi x}{l}.$$

Example 18.17. *The function $v(x, y)$ satisfies the Laplace's equation in rectangular coordinates (x, y) and for points within the rectangle $x = 0, x = a, y = 0, y = b$, it satisfies the conditions $v(0, y) = v(a, y) = v(x, b) = 0$ and $v(x, 0) = x(a - x)$, $0 < x < a$. Show that $v(x, y)$ is given by*

$$v(x, y) = \frac{8a^2}{\pi^3}\sum_{n=0}^{\infty} \frac{\sin (2n+1)\pi x/a}{(2n+1)^3} \frac{\sinh (2n+1)\pi(b-y)/a}{\sinh (2n+1)\pi b/a}$$ (Madras, 2003)

Solution. The only possible solution of

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \qquad \text{...}(i)$$

is of the form $\quad v(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \qquad \text{...}(ii)$

The boundary conditions are

$$v(0, y) = 0\,; \quad v(a, y) = 0 \qquad \text{...}(iii)$$

$$v(x, b) = 0 \qquad \text{...}(iv)$$

$$v(x, 0) = x(a - x),\ 0 < x < a. \qquad \text{...}(v)$$

Using (*iii*) $\quad v(0, y) = c_1(c_3 e^{py} + c_4 e^{-py}) = 0 \quad i.e., \quad c_1 = 0.$

$\therefore$ (*ii*) becomes $\quad v(x, y) = c_2 \sin px\,(c_3 e^{py} + c_4 e^{-py}) \qquad \text{...}(vi)$

Again using (*iii*), $\quad v(a, y) = c_2 \sin pa\,(c_3 e^{py} + c_4 e^{-py}) = 0.$

i.e., $\quad \sin pa = 0$, *i.e.* $pa = n\pi \quad$ or $\quad p = n\pi/a$

$\therefore$ (*vi*) becomes $\quad v(x, y) = c_2 \sin \dfrac{n\pi x}{a}\left(c_3 e^{\frac{n\pi y}{a}} + c_4 e^{-\frac{n\pi y}{a}}\right)$

or $\quad v(x, y) = \sin \dfrac{n\pi x}{a}\,(Ae^{n\pi y/a} + Be^{-n\pi y/a}) \quad$ where $\quad A = c_2 c_3, B = c_2 c_4 \qquad \text{...}(vii)$

Now using (*iv*), $\quad v(x, b) = \sin \dfrac{n\pi x}{a}\left(Ae^{\frac{n\pi b}{a}} + Be^{-\frac{n\pi b}{a}}\right) = 0$

i.e., $\quad Ae^{n\pi b/a} + Be^{-n\pi b/a} = 0 \quad$ or $\quad Ae^{n\pi b/a} - Be^{-n\pi b/a} = -\dfrac{1}{2} b_n \quad$ (say)

Thus (*vii*) becomes $\quad v(x, y) = \sin \dfrac{n\pi x}{a} \cdot \dfrac{1}{2} b_n \left\{e^{n\pi(b-y)/a} - e^{-n\pi(b-y)/a}\right\}$

$$= b_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi(b-y)}{a}$$

$\therefore$ the most general solution of (*i*) is

$$v(x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi(b-y)}{a} \qquad \text{...}(viii)$$

Using the condition (*v*), we have

$$x(a - x) = v(x, 0) = \sum_{n=1}^{\infty} b_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi x}{a}$$

where $\quad b_n \sinh \dfrac{n\pi b}{a} = \dfrac{2}{a}\displaystyle\int_0^a x(a-x) \sin \frac{n\pi x}{a}\, dx$

$$= \frac{2}{a}\left| (ax - x^2)\left(\frac{-\cos n\pi x/a}{n\pi/a}\right) - (a - 2x)\left(-\frac{\sin n\pi x/a}{(n\pi/a)^2}\right) + (-2)\left\{\frac{\cos n\pi x/a}{(n\pi/a)^3}\right\}\right|_0^a$$

$$= 0 - 0 + \frac{4a^2}{n^3\pi^3}(1 - \cos n\pi)$$

$$= \frac{8a^2}{n^3\pi^3} \text{ when } n \text{ is odd, otherwise zero when } n \text{ is even.}$$

Hence from (*viii*), the required solution is

$$v(x, y) = \frac{8a^2}{\pi^3} \sum_{n=1, 3, 5, \ldots}^{\infty} \frac{\sinh n\pi(b-y)/a}{n^3 \sinh n\pi b/a} \sin \frac{n\pi x}{a}$$

or $$v(x, y) = \frac{8a^2}{\pi^3} \sum_{n=0}^{\infty} \frac{\sinh (2n+1)\pi(b-y)/a}{(2n+1)^3 \sinh (2n+1)\pi b/a} \sin \frac{(2n+1)\pi x}{a}.$$

PROBLEMS 18.4

1. A long rectangular plate of width a cm. with insulated surface has its temperature v equal to zero on both the long sides and one of the short sides so that $v(0, y) = 0$, $v(a, y) = 0$, $v(x, \infty) = 0$, $v(x, 0) = kx$. Show that the steady-state temperature within the plate is

$$v(x, y) = \frac{2ak}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot e^{-n\pi y/a} \sin \frac{n\pi x}{a}. \qquad (J.N.T.U., 2005)$$

2. A rectangular plate with insulated surface is 8 cm. wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along one short edge $y = 0$ is given by

$$u(x, 0) = 100 \sin (\pi x/8), \quad 0 < x < 8;$$

while the two long edges $x = 0$ and $x = 8$ as well as the other short edge are kept at 0°C, show that the steady-state temperature at any point of the plane is given by

$$u(x, y) = 100e^{-\pi y/8} \sin (\pi x/8).$$

3. A rectangular plate with insulated surface is 10 cm. wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature of the short edge $y = 0$ is given by

$$u = 20\,x \qquad \text{for } 0 \le x \le 5$$

and

$$u = 20(10 - x) \qquad \text{for } 5 \le x \le 10$$

and the two long edges $x = 0$, $x = 10$ as well as the other short edge are kept at 0°C, prove that the temperature u at any point (x, y) is given by

$$u = \frac{40}{\pi^2} \sum_{1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{10} \cdot e^{-(2n-1)\pi y/10} \qquad (Anna, 2009)$$

4. Solve $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ for $0 < x < \pi$, $0 < y < \pi$, with conditions given : $u(0, y) = u(\pi, y) = u(x, \pi) = 0$, $u(x, 0) = \sin^2 x$.

5. A square plate is bounded by the lines $x = 0$, $y = 0$, $x = 20$ and $y = 20$. Its faces are insulated. The temperature along the upper horizontal edge is given by

$$u(x, 20) = x(20 - x), \text{ when } 0 < x < 20,$$

while other three edges are kept at 0°C. Find the steady state temperature in the plate. *(Madras, 2003)*

6. The temperature u is maintained at 0° along three edges of a square plate of length 100 cm. and the fourth edge is maintained at 100° until steady-state conditions prevail. Find an expression for the temperature u at any point (x, y). Hence show that the temperature at the centre of the plate

$$= \frac{200}{\pi}\left[\frac{1}{\cosh \pi/2} - \frac{1}{3 \cosh 3\pi/2} + \frac{1}{5 \cosh 5\pi/2} - \ldots\right].$$

7. A square thin metal plate of side a is bounded by the lines $x = 0$, $x = a$, $y = 0$, $y = a$. The edges $x = 0$, $y = a$ are kept at zero temperature, the edge $y = 0$ is insulated and the edge $x = a$ is kept at constant temperature T_0. Show that in the steady state conditions, the temperature $u(x, y)$ at the point (x, y) is given by

$$u(x, y) = \frac{4T_0}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sinh \dfrac{(2n-1)\pi x}{2a} \cos \dfrac{(2n-1)\pi y}{2a}}{(2n-1) \sinh \dfrac{(2n-1)\pi}{2}}.$$

8. A rectangular plate has sides a and b. Taking the side of length a as OX and that of length b as OY and other sides to be $x = a$ and $y = b$, the sides $x = 0$, $x = a$, $y = b$ are insulated and the edge $y = 0$ is kept at temperature $u_0 \cos \dfrac{\pi x}{a}$. Find the temperature $u(x, y)$ in the steady-state.

18.8 (1) LAPLACE'S EQUATION IN POLAR COORDINATES

In the study of steady-state temperature distribution in a rectangular plate, it is usually convenient to employ Cartesian coordinates as hitherto done. Sometimes Polar coordinates (r, θ) are found to be more useful and the Cartesian form of Laplace's equation is replaced by its polar form :

$$r^2 \frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0$$

(See Ex. 5.24, p. 213-214)

(2) Solution of Laplace's equation

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \quad ...(1)$$

Assume that a solution of (1) is of the form $u = R(r) \,.\, \phi(\theta)$ where R is a function of r alone and ϕ is a function of θ only.

Substituting it in (1), we get $r^2 R''\phi + rR'\phi + R\phi'' = 0$ or $\phi(r^2R'' + rR') + R\phi'' = 0$.

Separating the variables $\dfrac{r^2R'' + rR'}{R} = -\dfrac{\phi''}{\phi}$ ...(2)

Clearly the left side of (2) is a function of r only and the right side is a function of θ alone. Since r and θ are independent variables, (2) can hold good only if each side is equal to a constant k (say). Then (2) leads to the ordinary differential equations

$$r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} - kR = 0 \quad ...(3) \qquad \text{and} \qquad \frac{d^2\phi}{d\theta^2} + k\phi = 0 \quad ...(4)$$

Putting $r = e^z$, (3) reduces to $\dfrac{d^2R}{dz^2} - kR = 0$ ...(5)

Solving (5) and (4), we get

(*i*) *When k is positive* and $= p^2$, say :

$$R = c_1 e^{pz} + c_2 e^{-pz} = c_1 r^p + c_2 r^{-p},\ \phi = c_3 \cos p\theta + c_4 \sin p\theta$$

(*ii*) *When k is negative* and $= -p^2$, say

$$R = c_5 \cos pz + c_6 \sin pz = c_5 \cos (p \log r) + c_6 \sin (p \log r),\ \phi = c_7 e^{p\theta} + c_8 e^{-p\theta}$$

(*iii*) *When k is zero* :

$$R = c_9 z + c_{10} = c_9 \log r + c_{10},\ \phi = c_{11}\theta + c_{12}$$

Thus the three possible solutions of (1) are

$$u = (c_1 r^p + c_2 r^{-p})(c_3 \cos p\theta + c_4 \sin p\theta) \quad ...(6)$$
$$u = [c_5 \cos (p \log r) + c_6 \sin (p \log r)](c_7 e^{p\theta} + c_8 e^{-p\theta}) \quad ...(7)$$
$$u = (c_9 \log r + c_{10})(c_{11}\theta + c_{12}) \quad ...(8)$$

Of these solutions, we have to take that solution which is consistent with the physical nature of the problem. The general solution will consist of a sum of terms of type (6), (7) or (8). (*S.V.T.U., 2008*)

Example 18.18. *The diameter of a semi-circular plate of radius a is kept at 0°C and the temperature at the semi-circular boundary is T°C. Show that the steady state temperature in the plate is given by*

$$u(r, \theta) = \frac{4T}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \left(\frac{r}{a}\right)^{2n-1} \sin (2n-1)\theta.$$ (*Kerala M. Tech., 2005*)

Solution. Take the centre of the circle as the pole and bounding diameter as the initial line as in Fig. 18.8. Let the steady state temperature at any point $P(r, \theta)$ be $u(r, \theta)$, so that u satisfies the equation

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \quad ...(i)$$

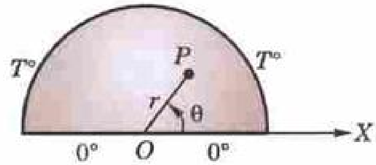


Fig. 18.8

The boundary conditions are :

$$u(r, 0) = 0 \quad \text{in } 0 \le r \le a \quad ...(ii)$$
$$u(r, \pi) = 0 \quad \text{in } 0 \le r \le a \quad ...(iii)$$

and $$u(a, \theta) = T \quad ...(iv)$$

The three possible solutions of (*i*) are

$$u = (c_1 r^p + c_2 r^{-p})(c_3 \cos p\theta + c_4 \sin p\theta) \quad ...(v)$$
$$u = [c_5 \cos (p \log r) + c_6 \sin (p \log r)](c_7 e^{p\theta} + c_8 e^{-p\theta}) \quad ...(vi)$$
$$u = (c_9 \log r + c_{10})(c_{11}\theta + c_{12}) \quad ...(vii)$$

From (*ii*) and (*iii*), $u = 0$ when $r = 0$ *i.e.*, u must be finite at the origin. Thus the solutions (*vi*) and (*vii*) are to be rejected. Hence the only suitable solution is (*v*).

By (*ii*), $u(r, \theta) = (c_1 r^p + c_2 r^{-p})\, c_3 = 0$

Hence $c_3 = 0$ and (*v*) becomes $u(r, \theta) = (c_1 r^p + c_2 r^{-p})\, c_4 \sin p\theta$ ...(*viii*)

By (*iii*), $u(r, \pi) = (c_1 r^p + c_2 r^{-p})\, c_4 \sin p\pi = 0.$

As $c_4 \neq 0$, $\sin p\pi = 0$, *i.e.*, $p = n$, where n is any integer.

Hence (*viii*) reduces to $u(r, \theta) = (c_1 r^n + c_2 r^{-n})\, c_4 \sin n\theta$ ...(*ix*)

Since $u = 0$, when $r = 0$, $\therefore$ $c_2 = 0$ and (*ix*) becomes

$u(r, \theta) = b_n r^n \sin n\theta$, where $b_n = c_1 c_4$.

$\therefore$ the most general solution of (*i*) is of the form

$$u(r, \theta) = \sum_{n=1}^{\infty} b_n r^n \sin n\theta \quad ...(x)$$

Putting $r = a$,
$$u(a, \theta) = \sum_{n=1}^{\infty} b_n a^n \sin n\theta. \quad ...(xi)$$

In order that (*iv*) may be satisfied, (*iv*) and (*xi*) must be same. This requires the expansion of T as a half-range Fourier sine series in $(0, \pi)$. Thus

$$T = \sum_{n=1}^{\infty} B_n \sin n\theta \quad \text{where} \quad B_n = \frac{2}{\pi}\int_0^{\pi} T \sin n\theta\, d\theta = \frac{2T}{n\pi}(1 - \cos n\pi) \quad \text{and } B_n = b_n a^n$$

$$\therefore \quad b_n = \frac{B_n}{a^n} = \frac{2T}{n\pi a^n}(1 - \cos n\pi)$$

i.e.,
$$b_n = 0, \text{ if } n \text{ is even}$$
$$= \frac{4T}{n\pi a^n}, \text{ if } n \text{ is odd.}$$

Hence (*x*) gives
$$u(r, \theta) = \frac{4T}{\pi}\left\{\frac{(r/a)}{1}\sin\theta + \frac{(r/a)^3}{3}\sin 3\theta + \frac{(r/a)^5}{5}\sin 5\theta + ...\right\}$$

Example 18.19. *The bounding diameter of a semi-circular plate of radius a cm is kept at 0°C and the temperature along the semi-circular boundary is given by*

$$u(a, \theta) = \begin{cases} 50\theta, \text{ when } 0 < \theta \leq \pi/2 \\ 50(\pi - \theta) \text{ when } \pi/2 < \theta < \pi \end{cases}$$

Find the steady-state temperature function u (r, θ). (Madras, 2003)

Solution. We know that $u(r, \theta)$ satisfies the equation

$$r^2 \frac{\partial^2 u}{\partial \theta^2} + r\frac{\partial u}{\partial \theta} + \frac{\partial^2 u}{\partial \theta^2} = 0 \quad ...(i)$$

The boundary conditions are $u(r, \theta) = 0$, $u(r, \pi) = 0$ ...(*ii*)

and $u(a, \theta) = 50\theta$ for $0 \leq \theta \leq \pi/2$; $u(a, \theta) = 50(\pi - \theta)$ for $\pi/2 \leq \theta \leq \pi$ ...(*iii*)

As in example 18.18, the most general solution of (*i*) satisfying the boundary conditions (*ii*) is of the form

$$u(r, \theta) = \sum_{n=1}^{\infty} b_n r^n \sin n\theta \quad ...(iv)$$

Putting $r = a$,
$$r(a, \theta) = \sum_{n=1}^{\infty} b_n a^n \sin n\theta$$

In order that the boundary condition (*iii*) is satisfied, we have $r(a, \theta) = \sum_{n=1}^{\infty} B_n \sin n\theta$

where
$$b_n a^n = B_n = \frac{2}{\pi}\left\{\int_0^{\pi/2} 50\theta \sin n\theta\, d\theta + \int_{\pi/2}^{\pi} 50(\pi - \theta)\sin n\theta\, d\theta\right\} \quad ...(v)$$

$$= \frac{100}{\pi}\left\{\left|\theta\left(\frac{-\cos n\theta}{\theta}\right)-(1)\left(\frac{-\sin n\theta}{n^2}\right)\right|_0^{\pi/2}+\left|(\pi-\theta)\left(\frac{-\cos n\theta}{n}\right)-(-1)\left(\frac{-\sin n\theta}{n^2}\right)\right|_{\pi/2}^{\pi}\right\}$$

$$= \frac{100}{\pi}\left\{-\frac{\pi}{2n}\cos\frac{n\pi}{2}+\frac{\sin n\pi/2}{n^2}+\frac{\pi}{2n}\cos\frac{n\pi}{2}+\frac{\sin n\pi/2}{n^2}\right\} = \frac{200}{\pi n^2}\sin n\pi/2.$$

When n is even $B_n = 0$, so taking $n = 1, 3, 5$ etc, (*iv*) gives

$$u(r,\theta) = \sum_{n=1,3,5,\ldots}^{\infty}\left(\frac{200}{\pi n^2}\sin\frac{n\pi}{2}\right)\frac{1}{a^n}\cdot r^n \sin n\theta$$

$$= \frac{200}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{(2m-1)^2}\left(\frac{r}{a}\right)^{2m-1}\sin(2m-1)\theta.$$

[Taking $n = 2m - 1$, $n = 1, 3, 5, \ldots$; gives $m = 1, 2, 3$, $\sin n\pi/2 = \sin(2m-1)\pi/2 = (-1)^{m-1}$. This gives the required temperature function.

PROBLEMS 18.5

1. A semi-circular plate of radius a has its circumference kept at temperature $u(a, \theta), = k\theta(\pi - \theta)$ while the boundary diameter is kept at zero temperature. Find the steady state temperature distribution $u(r, \theta)$ of the plate assuming the lateral surfaces of the plate to be insulated.
2. A semi-circular plate of radius 10 cm has insulated faces and heat flows in plane curves. The bounding diameter is kept at 0°C and on the circumference the temperature distribution maintained is $u(10, \theta) = (400/\pi)(\pi\theta - \theta^2)$, $0 \le \theta \le \pi$. Determine the temperature distribution $u(r, \theta)$ at any point on the plate.
3. A plate in the shape of truncated quadrant of a circle, is bounded by $r = a$, $r = b$ and $\theta = 0$, $\theta = \pi/2$. It has its faces insulated and heat flows in plane curves. It is kept at temperature 0°C along three of the edges while along the edge $r = a$, it is kept at temperature $\theta(\pi/2 - \theta)$. Determine the temperature distribution.
4. Determine the steady state temperature at the points on the sector $0 \le \theta \le \pi/4$, $0 \le r \le a$ of a circular plate, if the temperature is maintained at 0°C along the side edges and at a constant temperature k°C along the curved edges.
5. Find the steady-state temperature in a circular plate of radius a which has one-half of its circumference at 0°C and the other half at 60°C.
6. If the radii of the inner and outer boundaries of a circular annulus area 10 cm and 20 cm and
$$u(10, \theta) = 15\cos\theta,\ u(20, \theta) = 30\sin\theta,$$
find the value of $u(r, \theta)$ in the annulus. [$u(r, \theta)$ satisfies Laplace equation in the interior of the annulus.]
7. A plate in the form of a ring is bounded by the lines $r = 2$ and $r = 4$. Its surfaces are insulated and the temperature along the boundaries are
$$u(2, \theta) = 10\sin\theta + 6\cos\theta,\ u(4, \theta) = 17\sin\theta + 15\cos\theta$$
Find the steady-state temperature $u(r, \theta)$ in the ring.

18.9 (1) VIBRATING MEMBRANE—TWO DIMENSIONAL WAVE EQUATION

We shall now derive the equation for the vibrations of a tightly stretched membrane, such as the membrane of a drum. We shall assume that the membrane is uniform and the tension T in it per unit length is the same in all directions at every point.

Consider the forces acting on an element $\delta x \delta y$ of the membrane (Fig. 18.9). Forces $T\delta x$ and $T\delta y$ act on the edges along the tangent to the membrane. Let u be its small displacement perpendicular to the xy-plane, so that the forces $T\delta y$ on its opposite edges of length δy make angles α and β to the horizontal. So their vertical component

$= T\delta y \sin\beta - T\delta y \sin\alpha$

$= T\delta y (\tan\beta - \tan\alpha)$ approximately, since α and β are small

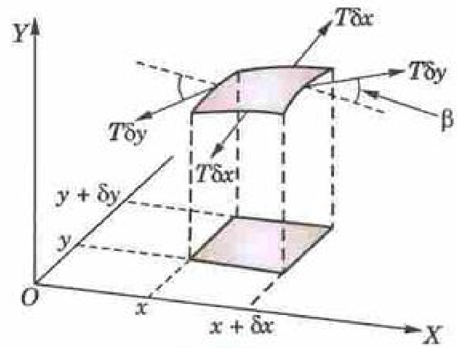


Fig. 18.9

$$= T\delta y\left\{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right\} = T\delta y \frac{\partial^2 u}{\partial x^2}\delta x, \text{ up to a first order of approximation.}$$

Similarly, the vertical component of the force $T\delta x$ acting on the edges of length δx

$$= T\delta x \frac{\partial^2 u}{\partial y^2}\delta y$$

If m be the mass per unit area of the membrane, then the equation of motion of the element is

$$m\delta x\delta y \frac{\partial^2 u}{\partial t^2} = T\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]\delta x\delta y \quad \text{or} \quad \frac{\partial^2 \mathbf{u}}{\partial \mathbf{t}^2} = \mathbf{c}^2\left[\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2}\right] \quad \text{where} \quad c^2 = T/m \qquad ...(1)$$

This is the wave equation in two dimensions.

(2) Solution of the two-dimensional wave equation - Rectangular membrane. Assume that a solution of (1) is of the form $u = X(x)Y(y)T(t)$

Substituting this in (1) and dividing by XYT, we get

$$\frac{1}{c^2 T}\frac{d^2 T}{dt^2} = \frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2}$$

This can hold good if each member is a constant. Choosing the constants suitably, we have

$$\frac{d^2 X}{dx^2} + k^2 X = 0, \quad \frac{d^2 Y}{dy^2} + l^2 Y = 0 \quad \text{and} \quad \frac{d^2 T}{dt^2} + (k^2 + l^2)\, c^2 T = 0$$

Hence a solution of (1) is

$$u = (c_1 \cos kx + c_2 \sin kx)(c_3 \cos ly + c_4 \sin ly) \times [c_5 \cos \sqrt{(k^2 + l^2)}ct + c_6 \sin \sqrt{(k^2 + l^2)}ct] \qquad ...(2)$$

Now suppose the membrane is rectangular and is stretched between the lines $x = 0, x = a, y = 0, y = b$. Then the condition $u = 0$ when $x = 0$ gives

$$0 = c_1(c_3 \cos ly + c_4 \sin ly)[c_5 \cos \sqrt{(k^2 + l^2)ct} + c_6 \sin \sqrt{(k^2 + l^2)ct}] \quad i.e., \quad c_1 = 0.$$

Then putting $c_1 = 0$ in (2) and applying the condition $u = 0$ when $x = a$, we get $\sin ka = 0$ or $k = m\pi/a$.

(m being an integer)

Similarly, applying the conditions $u = 0$, when $y = 0$ and $y = b$, we obtain

$$c_3 = 0 \quad \text{and} \quad l = n\pi/b \qquad (n \text{ being an integer})$$

Thus the solution (2) becomes

$$u(x, y, t) = c_2 c_4 \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}(c_5 \cos p_{mn}t + c_6 \sin p_{mn}t)$$

where $p_{mn} = \pi c\sqrt{[(m/a)^2 + (n/b)^2]}$ ...(3)

[These are the solutions of the wave equation (1) which are zero on the boundary of the rectangular membrane. These functions are called **eigen functions** and the numbers p_{mn} are the **eigen values** of the vibrating membrane.]

Choosing the constants c_2 and c_4 so that $c_2 c_4 = 1$, we can write the general solution of the equation (1) as

$$u = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}(A_{mn} \cos pt + B_{mn} \sin pt) \qquad ...(4)$$

If the membrane starts from rest from the initial position $u = f(x, y)$, *i.e.*, $\frac{\partial u}{\partial t} = 0$ when $t = 0$, then (3) gives $B_{mn} = 0$.

Also using the condition $u = f(x, y)$ when $t = 0$, we get

$$f(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

This is *double Fourier series.* Multiplying both sides by $\sin(m\pi x/a)\sin(n\pi y/b)$ and integrating from $x = 0$ to $x = a$ and $y = 0$ to $y = b$, every term on the right except one, becomes zero. Hence we obtain

$$\int_0^a \int_0^b f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dydx = \frac{ab}{4} A_{mn} \quad ...(5)$$

which gives the coefficients in the solution and is called the **generalised Euler's formula.**

Rectangular Membranes

Example 18.20. *Find the deflection $u(x, y, t)$ of the square membrane with $a = b = 1$ and $c = 1$, if the initial velocity is zero and the initial deflection is $f(x, y) = A \sin \pi x \sin 2\pi y$.*

Solution. Taking $a = b = 1$ and $f(x, y) = A \sin \pi x \sin 2\pi y$, in (5), we get

$$A_{mn} = 4 \int_0^1 \int_0^1 A \sin \pi x \sin 2\pi y \sin m\pi x \sin n\pi y \, dydx$$

$$= 4A \int_0^1 \sin \pi x \sin m\pi x \, dx \left(\int_0^1 \sin 2\pi y \sin n\pi y \, dy \right) = 0, \quad \text{for } m \neq 1$$

$$= 4A \left(\frac{1}{2}\right) \int_0^1 \sin 2\pi y \sin n\pi y \, dy, \text{ for } m = 1 \qquad \left[\because \int_0^1 \sin \pi x \sin \pi x \, dx = \frac{1}{2}\right]$$

i.e.,

$$A_{mn} = 2A \int_0^1 \sin 2\pi y \sin n\pi y \, dy = 0, \text{ for } n \neq 2$$

$$= 2A \left(\frac{1}{2}\right), \text{ for } n = 2.$$

$\therefore \quad A_{12} = A$. Also from (3), $p_{mn} = \pi\sqrt{(m^2 + n^2)}$ $\quad [\because a = b = 1 = c]$

$\therefore \quad p_{12} = \pi\sqrt{(1^2 + 2^2)} = \sqrt{5}\,\pi.$

Hence from (4), the required solution is $u(x, y, t) = A \sin \pi x \sin 2\pi y \cos (\sqrt{5}\,\pi t)$.

Example 18.21. *Find the vibration $u(x, y, t)$ of a rectangular membrane $(0 < x < a, 0 < y < b)$ whose boundary is fixed given that it starts from rest and $u(x, y, 0) = hxy(a - x)(b - y)$.*

Solution. Proceeding as in § 18.9 (**2**), we have from (4),

$$u = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} (A_{mn} \cos pt + B_{mn} \sin pt) \text{ where } p = \pi c\sqrt{[(m/a)^2 + (n/b)^2}$$

Since the membrane starts from rest $\partial u/\partial t = 0$ when $t = 0$,

$$\therefore \quad \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} (-A_{mn} p \sin pt + pB_{mn} \cos pt) = 0 \text{ when } t = 0$$

This gives $B_{mn} = 0$

$$\therefore \quad u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos pt \quad ...(i)$$

Then $\quad hxy(a - x)(b - y) = u(x, y, 0) = \displaystyle\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$

where $A_{mn} = \dfrac{2}{a} \cdot \dfrac{2}{b} \displaystyle\int_0^a \int_0^b hxy(a-x)(b-y) \,.\, \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dydx$

$$= \frac{4h}{ab} \left\{ \int_0^a x(a-x) \sin \frac{m\pi x}{a} dx \right\} \left\{ \int_0^b y(b-y) \sin \frac{n\pi y}{b} dy \right\}$$

$$= \frac{4h}{ab} \left| (ax - x^2)\left(\frac{-\cos m\pi x/a}{m\pi/a}\right) - (a - 2x) \left\{ \frac{-\sin \frac{m\pi x}{a}}{(m\pi/a)^2} \right\} + (-2) \frac{\cos m\pi x/a}{(m\pi/a)^3} \right|_0^a$$

$$\times \left| (by - y^2)\left(\frac{-\cos n\pi y/b}{n\pi/b}\right) - (b - 2y) \left\{ \frac{-\sin n\pi/b}{(n\pi/b)^2} \right\} + (-2) \frac{\cos n\pi y/b}{(n\pi/b)^3} \right|_0^b$$

$$= \frac{4h}{ab} \frac{2a^3}{m^3\pi^3} \cdot \frac{2b^3}{n^3\pi^3} (1 - \cos m\pi)(1 - \cos n\pi)$$

Hence from (*i*), we get

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos pt$$

where $A_{mn} = \dfrac{16ha^2b^2}{m^3n^3\pi^6}(1 - \cos m\pi)(1 - \cos n\pi)$ and $p = \pi c\sqrt{[(m/a)^2 + (n/b)^2]}$

Circular Membranes*

Example 18.22. *A circular membrane of unit radius fixed along its boundary starts vibrating from rest and has initial deflection $u(r, 0) = f(r)$. Show that the deflection $u(r, t)$ of the membrane at any instant is given by*

$$u(r, t) = \sum_{m=1}^{\infty} A_m \cos (c\alpha_m t) \cdot J_o(\alpha_m r) \text{ where } A_m = \frac{2}{J_1^2(\alpha_m)} \int_0^1 r f(r) J_0(\alpha_m r)\, dr,$$

and $\alpha_m (m = 1, 2, ...)$ are the positive roots of the Bessel function $J_0(k) = 0$.

Solution. The vibrations of a plane circular membrane are governed by 2-dimensional wave equation in polar coordinates *i.e.*,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right)$$

For a radially symmetric membrane (in which u does not depend on θ) the above equation reduces to

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \right) \qquad ...(i)$$

For the given membrane fixed along its boundary, the boundary condition is

$$u(1, t) = 0 \quad \text{for all } t \geq 0 \qquad ...(ii)$$

For solutions not depending on θ,

initial deflection $u(r, 0) = f(r)$ ...(*iii*)

and initial velocity $\left(\dfrac{\partial u}{\partial t}\right)_{t=0} = 0$ ...(*iv*)

which are the initial conditions. We find the solutions $u(r, t) = R(r)T(t)$ ...(*v*)
satisfying the boundary condition (*ii*).

Differentiating and substituting (*v*) in (*i*), we get

$$\frac{\partial^2 T/\partial t^2}{c^2 T} = \frac{1}{R}\left(\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} \right) = -k^2 \text{ (say)}$$

This leads to $\dfrac{\partial^2 T}{\partial t^2} + p^2 T = 0$ where $p = ck$ ...(*vi*)

and $$\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} + k^2 R = 0 \qquad ...(vii)$$

Now putting $s = kr$, (*vii*) transforms to $\dfrac{d^2R}{ds^2} + \dfrac{1}{s}\dfrac{dR}{ds} + R = 0$ which is Bessel's equation. Its general solution $R = aJ_0(s) + bY_0(s)$ where J_0 and Y_0 are Bessel's functions of the first and second kind of order zero.

Since the deflection of the membrane is always finite, we must have $b = 0$. Then taking $a = 1$, we get

$$R(r) = J_0(s) = J_0(kr)$$

On the boundary of the circular membrane, we must have $J_0(k) = 0$, which is satisfied for

$$k = \alpha_m,\ m = 1, 2, ...$$

*Drums, telephones and microphones provide examples of circular membrane and as such are quite useful in engineering.

Thus the solutions of (*vii*) are $R(r) = J_0(\alpha_m r)$, $m = 1, 2, \ldots$ and the corresponding solutions of (*vi*) are $T(t) = A_m \cos p_m t + B_m \sin p_m t$, where $p_m = ck_m = c\alpha_m$.

Hence the general solution of (*i*) satisfying (*ii*) are

$$u(r, t) = (A_m \cos p_m t + B_m \sin p_m t)\, J_0(\alpha_m r)$$

which are the *eigen functions* of the problem and the corresponding *eigen values* are p_m.

To find that solution which also satisfies the initial conditions (*iii*) and (*iv*), consider the series

$$u(r, t) = \sum_{m=1}^{\infty} (A_m \cos p_m t + B_m \sin p_m t)\, J_0(\alpha_m r)$$

Putting $t = 0$ and using (*iii*), we get $u(r, 0) = \sum_{m=1}^{\infty} A_m J_0(\alpha_m r) = f(r)$

Here, the constants A_m must be the coefficients of Fourier-Bessel series [(8) page 560] with $m = 0$, *i.e.*,

$$A_m = \frac{2}{J_1^2(\alpha_m)} \int_0^1 r f(r)\, J_0(\alpha_m r)\, dr$$

Using (*iv*), we get $B_m = 0$. Hence the result.

PROBLEMS 18.6

1. A tightly stretched unit square membrane starts vibrating from rest and its initial displacement is $k \sin 2\pi x \sin \pi y$. Show that the deflection at any instant is $k \sin 2\pi x \sin \pi y \cos(\sqrt{5}\, \pi ct)$.
2. Find the deflection $u(r, t)$ of the circular membrane of unit radius if $c = 1$, the initial velocity is zero and the initial deflection is $0.25\,(1 - r^2)$.

18.10 TRANSMISSION LINE

Consider a cable l km in length, carrying an electric current with resistance R ohms/km, inductance L henries/km ; capacitance C farads/km and leakance G mhos/km (Fig. 18.10).

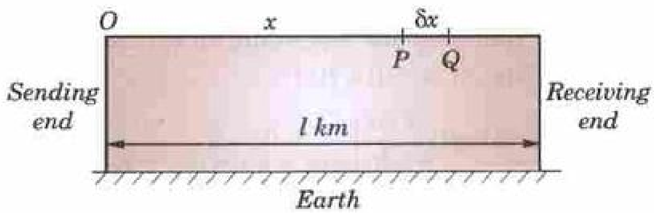


Fig. 18.10

Let the instantaneous voltage and current at any point P, distant x km from the sending end O, and at time t sec be $v(x, t)$ and $i(x, t)$ respectively. Consider a small length $PQ (= \delta x)$ of the cable.

Now since the voltage drop across the segment δx

= voltage drop due to resistance + voltage drop due to inductance

$$\therefore \qquad -\delta v = iR\delta x + L\delta x \cdot \frac{\partial i}{\partial t}$$

and dividing by δx and taking limits as $\delta x \to 0$, we get

$$-\frac{\partial v}{\partial x} = Ri + L\frac{\partial i}{\partial t} \qquad \ldots(1)$$

Similarly the current loss between P and Q

= current lost due to capacitance and leakance

$\therefore \qquad -\delta i = C\dfrac{\partial v}{\partial t}\delta x + Gv\delta x$ from which as before, we get

$$-\frac{\partial i}{\partial x} = C\frac{\partial v}{\partial t} + Gv \qquad \ldots(2)$$

Rewriting the simultaneous partial differential equations (1) and (2) as

$$\left(R + L\frac{\partial}{\partial t}\right) i + \frac{\partial v}{\partial x} = 0 \qquad \ldots(3)$$

and $$\frac{\partial i}{\partial x}+\left(C\frac{\partial}{\partial t}+G\right)v=0, \quad \text{...(4)}$$

we shall eliminate i and v in turn.

$\therefore$ operating (3) by $\frac{\partial}{\partial x}$ and (4) by $\left(R+L\frac{\partial}{\partial t}\right)$ and subtracting

$$\frac{\partial^2 v}{\partial x^2}-\left(R+L\frac{\partial}{\partial t}\right)\left(C\frac{\partial}{\partial t}+G\right)v=0$$

or $$\frac{\partial^2 v}{\partial x^2}=LC\frac{\partial^2 v}{\partial t^2}+(LG+RC)\frac{\partial v}{\partial t}+RGv \quad \text{...(5)}$$

Again operating (3) by $\left(C\frac{\partial}{\partial t}+G\right)$ and (4) by $\frac{\partial}{\partial x}$ and subtracting

$$\left(C\frac{\partial}{\partial t}+G\right)\left(R+L\frac{\partial}{\partial t}\right)i-\frac{\partial^2 i}{\partial x^2}=0$$

or $$\frac{\partial^2 i}{\partial x^2}=LC\frac{\partial^2 i}{\partial t^2}+(LG+RC)\frac{\partial i}{\partial t}+RGi \quad \text{...(6)}$$

which is (5) with v replaced by i. The equations (5) and (6) are called the *telephone equations.*

Cor. 1. *If* $L=G=0$, the equations (5) and (6) become

$$\frac{\partial^2 v}{\partial x^2}=RC\frac{\partial v}{\partial t} \quad \text{...(7)} \qquad \frac{\partial^2 i}{\partial x^2}=RC\frac{\partial i}{\partial t} \quad \text{...(8)}$$

which are known as the *telegraph equations.*

Rewriting (7) as $\frac{\partial v}{\partial t}=\frac{1}{RC}\frac{\partial^2 v}{\partial x^2}$, we observe that it is similar to the heat equation [(1) p. 611].

Cor. 2. *If* $R=G=0$, the equations (5) and (6) become

$$\frac{\partial^2 v}{\partial x^2}=LC\frac{\partial^2 v}{\partial t^2} \quad \text{...(9)} \qquad \frac{\partial^2 i}{\partial x^2}=LC\frac{\partial^2 i}{\partial t^2} \quad \text{...(10)}$$

which are called the *radio equations.*

Rewriting (9) as $\frac{\partial^2 v}{\partial t^2}=k^2\frac{\partial^2 v}{\partial x^2}$ where $k^2=\frac{1}{LC}$,

its general solution is $v(x,t)=f_1(x+kt)+f_2(x-kt)$. [See (4) p. 609]

Similarly from (10), $i(x,t)=F_1(x+kt)+F_2(x-kt)$.

Thus the voltage $v(x, t)$ for the current $i(x, t)$ at any point along the lossless transmission line can be obtained by the superposition of a progressive wave and a receding wave travelling with equal velocities (k). This is the case of oscillations of $v(x, t)$ and $i(x, t)$ at high frequencies.

Cor. 3. *If* $L=C=0$, *e.g.*, in the case of a submarine cable, then (5) gives

$$\frac{\partial^2 v}{\partial x^2}=GRv, \text{ i.e. } (D^2-GR)v=0$$

$\therefore$ $$v(x)=A\cosh(\sqrt{GR}\,.\,x)+B\sinh(\sqrt{GR}\,.\,x) \quad \text{...(11)}$$

Since by (1), $$Ri=-\frac{\partial v}{\partial x}=-\sqrt{GR}\,[A\sinh(\sqrt{GR}\,.\,x)+B\cosh(\sqrt{GR}\,.\,x)]$$

$\therefore$ $$i(x)=-\sqrt{G/R}\,[A\sinh(\sqrt{GR}\,.\,x)+B\cosh(\sqrt{GR}\,.\,x)] \quad \text{...(12)}$$

If $v(0)=v_0$ and $i(0)=i_0$, then $v_0=A$ and $i_0=-\sqrt{(G/R)}B$.

Hence writing $\sqrt{(GR)}=\gamma$ and $\sqrt{(R/G)}=z_0$, (11) and (12) give

$$v(x)=v_0\cosh\gamma x-i_0 z_0\sinh\gamma x \quad \text{...(13)}$$

and $$i(x)=i_0\cosh\gamma x-\frac{v_0}{z_0}\sinh\gamma x. \quad \text{...(14)}$$

Obs. Steady-state solutions. We have so far considered the transient state solutions only. The steady-state solutions of transmission lines are however, obtained by assuming $v = Ve^{i\omega t}$ and $i = Ie^{i\omega t}$, where V and I are complex functions of x only. Substituting these in (5) and (6), we get two ordinary linear differential equations of the second order which can be solved at once.

Example 18.23. *Neglecting R and G, find the e.m.f. $v(x, t)$ in a line of length l, t seconds after the ends were suddenly grounded, given that $i(x, 0) = i_0$ and $v(x, 0) = e_1 \sin \dfrac{\pi x}{l} + e_5 \sin \dfrac{5\pi x}{l}$.* (S.V.T.U., 2008)

Solution. Since R and G are negligible, we use the *Radio equation* $\dfrac{\partial^2 v}{\partial x^2} = LC \dfrac{\partial^2 v}{\partial t^2}$ ...(*i*)

Since the ends are suddenly grounded, we have the boundary conditions

$$v(0, t) = 0, v(l, t) = 0 \quad ...(ii)$$

Also the initial conditions are $i(x, 0) = i_0$

and
$$v(x, 0) = e_1 \sin \pi x/l + e_5 \sin 5\pi x/l \quad ...(iii)$$

$$\therefore \quad \frac{\partial i}{\partial x} = -c\frac{\partial v}{\partial t} \quad \text{gives} \quad \frac{\partial v}{\partial t}(x, 0) = 0 \quad ...(iv)$$

Let $v = X(x)T(t)$ be the solution of (*i*).

$\therefore$ (*i*) gives
$$X''T = LCXT''$$

$$\frac{X''}{X} = LC\frac{T''}{T} = -k^2 \text{ (say)}$$

$$\therefore \quad X'' + k^2X = 0 \quad \text{and} \quad T'' + (k^2/LC)T = 0$$

Solving these equations, we get

$$v = (c_1 \cos kx + c_2 \sin kx)\left(c_3 \cos \frac{k}{\sqrt{LC}} . t + c_4 \sin \frac{k}{\sqrt{LC}} t\right)$$

Using the boundary conditions (*ii*), we get

$$c_1 = 0 \quad \text{and} \quad k = n\pi/l.$$

$$\therefore \quad v = \sin \frac{n\pi x}{l}\left(a_n \cos \frac{n\pi}{l\sqrt{LC}} t + b_n \sin \frac{n\pi}{l\sqrt{LC}} t\right)$$

Using the initial condition (*iv*), we get $b_n = 0$

$$\therefore \quad v = a_n \sin \frac{n\pi x}{l} \cos \frac{n\pi}{l\sqrt{LC}} t$$

Thus the most general solution of (*i*) is

$$v = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \cos \frac{n\pi t}{l\sqrt{LC}}$$

Finally by the initial condition (*iii*), we have

$$e_1 \sin \frac{\pi x}{l} + e_5 \sin \frac{5\pi x}{l} = \sum a_n \sin \frac{n\pi x}{l}$$

$$\therefore \quad a_1 = e_1 \quad \text{and} \quad a_5 = e_5$$ while all other a's are zero.

Hence
$$v = e_1 \sin \frac{\pi x}{l} \cos \frac{\pi t}{l\sqrt{LC}} + e_5 \sin \frac{5\pi x}{l} \cos \frac{5\pi t}{l\sqrt{LC}}$$

which is the required solution.

Example 18.24. *A telephone line 3000 km. long has a resistance of 4 ohms/km. and a capacitance of 5×10^{-7} farad/km. Initially both the ends are grounded so that the line is uncharged. At time $t = 0$, a constant e.m.f. E is applied to one end, while the other end is left grounded. Assuming the inductance and leakance to be negligible, show that the steady state current of the grounded end at the end of 1 sec. is 5.3%.*

Solution. Since $L = 0$, $G = 0$, we use the telegraph equation

$$\frac{\partial^2 v}{\partial x^2} = RC\frac{\partial v}{\partial t}$$

Let $v = X(x)T(t)$ be its solution so that

$$TX'' = RCXT' \quad \text{or} \quad \frac{X''}{X} = RC\frac{T'}{T} = -k^2 \text{ (say)}$$

$$\therefore \qquad X'' + k^2X = 0 \text{ and } T' + (k^2/RC)T = 0$$

Solving these equations, we get

$$X = c_1 \cos kx + c_2 \sin kx,\ T = c_3 e^{-k^2t/RC}$$

giving
$$v = (c_1 \cos kx + c_2 \sin kx)\,c_3 e^{-k^2t/RC} \qquad ...(i)$$

When $t = 0$; $v = 0$ at $x = 0$ and $v = 0$ at $x = l$

$$\therefore \qquad 0 = c_1c_3 \,;\, 0 = (c_1 \cos kl + c_2 \sin kl)c_3$$

i.e.,
$$c_1c_3 = 0 \text{ and } kl = n\pi \ (n \text{ an integer})$$

Putting these in (i) and adding a linear term, we have

$$v = a_0x + b_0 + \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\, e^{-n^2\pi^2t/RCl^2} \qquad ...(ii)$$

The end conditions of the problem are

$$v = 0 \text{ at } x = 0 \text{ and } v = E \text{ at } x = l$$

Using these, (ii) gives $b_0 = 0$ and $a_0 = E/l$

Then (ii) becomes
$$v = \frac{E}{l}x + \sum b_n \sin\frac{n\pi x}{l}\, e^{-n^2\pi^2t/RCl^2}$$

Also $v = 0$ when $t = 0$, we get $-Ex/l = \sum b_n \sin n\pi x/l$

This requires the expansion of $(-Ex/l)$ as a half-range sine series in $(0, l)$.

$$\therefore \qquad b_n = \frac{2}{l}\int_0^l \left(\frac{-Ex}{l}\right)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[\left(\frac{-Ex}{l}\right)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - \left(\frac{-E}{l}\right)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_0^l = \frac{2}{l}\left(\frac{El}{n\pi}\cos n\pi\right) = \frac{2E}{n\pi}(-1)^n.$$

Thus
$$v = \frac{Ex}{l} + \frac{2E}{\pi}\sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\, e^{-n^2\pi^2t/RCl^2} \qquad ...(iii)$$

Also when $L = 0$, $\dfrac{-\partial v}{\partial x} = Ri$

i.e.,
$$i = -\frac{1}{R}\frac{\partial v}{\partial x} = -\frac{E}{lR} - \frac{2E}{lR}\sum_{n=1}^{\infty} (-1)^n \cos\frac{n\pi x}{l}\, e^{-n^2\pi^2t/RCl^2}$$

At the grounded end $(x = 0)$, the current is

$$i = -\frac{E}{lR} - \frac{2E}{lR}\sum_{n=1}^{\infty} (-1)^n e^{-n^2\pi^2t/RCl^2}$$

When $t = 1$ sec,
$$i = -\frac{E}{lR}\left(1 - 2e^{\pi^2/RCl^2} + 2e^{-4\pi^2/RCl^2} - ...\right) \qquad ...(iv)$$

Since
$$\frac{\pi^2}{RCl^2} = \frac{(3.14)^2}{4(5\times10^{-7})(3000)^2} = 0.548$$

$$\therefore \qquad e^{-\pi^2/RCl^2} = e^{-0.548} = 0.578$$

When $t \to \infty$, $i \to -E/lR$

Hence from (*iv*), we have

$$i = -\frac{E}{lR}\{1 - 2(0.578) + 2(0.578)^4 - 2(0.578)^9 + ...\}$$
$$= -\frac{E}{lR}\{1 - 1.156 + 0.223 - 0.014 + ...\}$$
$$= i_\infty(0.053) = 5.3\% \text{ of } i_\infty.$$

Example 18.25. *A transmission line 1000 kilometers long is initially under steady-state conditions with potential 1300 volts at the sending end (x = 0) and 1200 volts at the receiving end (x = 1000). The terminal end of the line is suddenly grounded, but the potential at the source is kept at 1300 volts. Assuming the inductance and leakance to be negligible, find the potential v(x, t).* (*Andhra, 2000*)

Solution. The equation of the telegraph line is

$$\frac{\partial^2 v}{\partial x^2} = RC\frac{\partial v}{\partial t} \quad \text{or} \quad \frac{\partial v}{\partial t} = \frac{1}{RC}\frac{\partial^2 v}{\partial x^2} \qquad ...(i)$$

v_s = initial steady voltage satisfying $\frac{\partial^2 v}{\partial x^2} = 0 = 1300 - x/10 = v(x, 0)$ ...(*ii*)

v_s' = steady voltage (after grounding the terminal end) when steady conditions are ultimately reached = $1300 - 1.3x$

$\therefore \quad v(x, t) = v_s' + v_t(x, t)$ where $v_t(x, t)$ is the transient part

$$= 1300 - 1.3x + \sum_{n=1}^{\infty} b_n e^{-(n^2\pi^2 t)/(l^2 RC)} \sin\frac{n\pi x}{l} \qquad \text{[By } (viii), \text{ p. 614]} \qquad ...(iii)$$

where l = 1000 kilometers.

Putting $t = 0$, we have from (*ii*) and (*iii*)

$$1300 - 0.1x = v(x, 0) = 1300 - 1.3x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

i.e.

$$1.2x = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \text{ where } b_n = \frac{2}{l}\int_0^l 1.2 \sin\frac{n\pi x}{l}\,dx = \frac{2400}{\pi}\cdot\frac{(-1)^{n+1}}{n}$$

Hence

$$v(x, t) = 1300 - 1.3x + \frac{2400}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} e^{-(n^2\pi^2 t)/(l^2 RC)} \sin\frac{n\pi x}{1000}.$$

PROBLEMS 18.7

1. Find the current i and voltage e in a line of length l, t seconds after the ends are suddenly grounded, given that $i(x, 0) = i_0$, $e(x, 0) = e_0 \sin(\pi x/l)$.
   Also R and G are negligible.
2. Show that a transmission line with negligible resistance and leakage propagates waves of current and potential with a velocity equal to $l/\sqrt{(LC)}$, where L is the self-inductance and C is the capacitance.
3. A steady voltage distribution of 20 volts at the sending end and 12 volts at the receiving end is maintained in a telephone wire of length l. At time $t = 0$, the receiving end is grounded. Find the voltage and current t sec later. Neglect leakance and inductance.
4. Obtain the solution of the *radio equation*
   $$\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2}$$
   appropriate to the case when a periodic e.m.f. $V_0 \cos pt$ is applied at the end $x = 0$ of the line.

18.11 LAPLACE'S EQUATION IN THREE DIMENSIONS

We have seen that the three dimensional heat flow equation in steady state reduces to

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad \text{...(1)}$$

which is the *Laplace's equation in three dimensions.*

Laplace's equation also arises in the study of gravitational potential at (x, y, z) of a particle of mass m situated at (ξ, η, ζ) given by

$$\frac{Gm}{r} \text{ where } r = \sqrt{[(x-\zeta)^2 + (y-\eta)^2 + (z-\zeta)^2]}$$

This function is called the *potential of the gravitational field* and satisfies the Laplace's equation.

If a mass of density ρ at (ξ, η, ζ) is distributed throughout a region R, then the gravitational potential u at an external point (x, y, z) is given by

$$u(x, y, z) = G \iiint_R \frac{\rho}{r} \, d\xi \, d\eta \, d\zeta \qquad \text{...(2)}$$

Since $\nabla^2(1/r) = 0$ and ρ is independent of x, y and z, we get

$$\nabla^2 u = \iiint_R \rho \, \nabla^2 (1/r) \, d\xi \, d\eta \, d\zeta = 0.$$

This shows that the gravitational potential defined by (2) also obeys Laplace's equation.

Thus Laplace's equation (1) is one of the most important equations arising in connection with numerous problems of physics and engineering. *The theory of its solutions is called the* **potential theory** *and its solutions are called the* **harmonic functions.**

In most of the problems leading to Laplace's equation, it is required to solve the equation subject to certain boundary conditions. A proper choice of coordinate system makes the solution of the problem simpler. Now we proceed to take up the solutions of (1) in its other forms.

18.12 SOLUTIONS OF THREE DIMENSIONAL LAPLACE'S EQUATION

(1) **Cartesian form of $\nabla^2 u = 0$** is $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$ ...(1)

Let $u = X(x)Y(y)Z(z)$ ...(2)

be a solution of (1). Substituting (2) in (1) and dividing by XYZ, we obtain

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z} \cdot \frac{d^2Z}{dz^2} = 0 \qquad \text{...(3)}$$

which is of the form $F_1(x) + F_2(y) + F_3(z) = 0$.

As x, y, z are independent, this will hold good only if F_1, F_2, F_3 are constants. Assuming these constants to be $k^2, l^2, -(k^2 + l^2)$ respectively, (3) leads to the equations

$$\frac{d^2X}{dx^2} - k^2X = 0, \; \frac{d^2Y}{dy^2} - l^2Y = 0, \; \frac{d^2Z}{dz^2} + (k^2 + l^2)Z = 0$$

Their solutions are $X = c_1e^{kx} + c_2e^{-kx}, \; Y = c_3e^{ly} + c_4e^{-ly}$

$$Z = c_5 \cos \sqrt{(k^2 + l^2)}z + c_6 \sin \sqrt{(k^2 + l^2)}z$$

Thus a possible solution of (1) is

$$u = (c_1e^{kx} + c_2e^{-kx})(c_3e^{ly} + c_4e^{-ly}) \{c_5 \cos \sqrt{(k^2 + l^2)}z + c_6 \sin \sqrt{(k^2 + l^2)}z\}.$$

Since the three constants could have been taken as $-k^2, -l^2$ and $k^2 + l^2$, an alternative solution of (1) will be

$$u = (c_1 \cos kx + c_2 \sin kx)(c_3 \cos ly + c_4 \sin ly)\{c_5e^{\sqrt{(k^2+l^2)}z} + c_6e^{-\sqrt{(k^2+l^2)}z}\}.$$

(2) **Cylindrical form of $\nabla^2 u = 0$** is $\dfrac{\partial^2 u}{\partial \rho^2} + \dfrac{1}{\rho}\dfrac{\partial u}{\partial \rho} + \dfrac{1}{\rho^2}\dfrac{\partial^2 u}{\partial \phi^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$ ...(1)

Let $u = R(\rho)\, H(\phi)\, Z(z)$ [(*iv*) p. 359]

be a solution of (1). Substituting it in (1), and dividing by RHZ, we get

$$\frac{1}{R}\left(\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho}\right) + \frac{1}{\rho^2 H}\frac{d^2H}{d\phi^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0 \quad \text{...(2)}$$

Assuming that $\dfrac{d^2H}{d\phi^2} = -n^2H$ and $\dfrac{d^2Z}{dz^2} = k^2Z$, ..(3)

(2) reduces to $\dfrac{1}{R}\left(\dfrac{d^2R}{d\rho^2} + \dfrac{1}{\rho}\dfrac{dR}{d\rho}\right) - \dfrac{n^2}{\rho^2} + k^2 = 0$

or $$\rho^2\frac{d^2R}{d\rho^2} + \rho\frac{dR}{d\rho} + (k^2\rho^2 - n^2)R = 0.$$

This is Bessel's equation [§ 16.10 (1)] and its solution is $R = c_1J_n(k\rho) + c_2Y_n(k\rho)$.

Also the solutions of equations (3) are

$$H = c_3 \cos n\phi + c_4 \sin n\phi,\ Z = c_5e^{kz} + c_6e^{-kz}$$

Thus a solution of (1) is

$$u = [c_1J_n(k\rho) + c_2Y_n(k\rho)]\,[c_3 \cos n\phi + c_4 \sin n\phi]\,[c_5\, e^{kz} + c_6e^{-kz}]$$

which is known as a *cylindrical harmonic.* (*Assam, 1999*)

(3) Spherical form of $\nabla^2 u = 0$ is

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = 0 \quad \text{...(1)} \quad [(iv)\ \text{p. 361}]$$

Let $u = R(r)\, G(\theta)\, H(\phi)$ be a solution of (1).

Then $$\frac{1}{R}\left(r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr}\right) + \frac{1}{G}\left(\frac{d^2G}{d\theta^2} + \cot\theta\frac{dG}{d\theta}\right) + \frac{1}{H\sin^2\theta}\frac{d^2H}{d\phi^2} = 0$$

Putting $\dfrac{1}{R}\left(r^2\dfrac{d^2R}{dr^2} + 2r\dfrac{dR}{dr}\right) = n(n+1)$ ...(2) and $\dfrac{1}{H}\dfrac{d^2H}{d\phi^2} = -m^2$, ...(3)

the above equation takes the form

$$\frac{d^2G}{d\theta^2} + \cot\theta\frac{dG}{d\theta} + [n(n+1) - m^2\operatorname{cosec}^2\theta]G = 0 \quad \text{...(4)}$$

Now differentiating the *Legendre's equation* (§ 16.13)

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0,$$

m times with respect to x and writing $u = d^my/dx^m$, we get

$$(1-x^2)u'' - 2(m+1)xu' + (n-m)(n+m+1)u = 0 \quad \text{...(5)}$$

Now putting $G = (1-x^2)^{m/2}\, u$ in (5), we get

$$(1-x^2)\frac{d^2G}{dx^2} - 2x\frac{dG}{dx} + \left[n(n+1) - \frac{m^2}{1-x^2}\right]G = 0 \quad \text{...(6)}$$

Now putting $x = \cos\theta$ in (6), it reduces to (4) and its solution is

$$G = c_1P_n^m(\cos\theta) + c_2Q_n^m(\cos\theta)$$

The solution of (3) is $H = c_3 \cos m\phi + c_4 \sin m\phi$

To solve (2), write $R = r^k$, so that $k(k-1) + 2k = n(n+1)$ which gives $k = n$ or $-(n+1)$

Thus $$R = c_5r^n + c_6r^{-n-1}$$

Hence the general solution of (1) is

$$u = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\{c_1P_n^m(\cos\theta) + c_2Q_n^m(\cos\theta)\}(c_3\cos m\phi + c_4\sin m\phi) \times (c_5r^n + c_6r^{-n-1})$$

Any solution of (1) is known as a *spherical harmonic.*

Example 18.26. *Find the potential in the interior of a sphere of unit radius when the potential on the surface is $f(\theta) = \cos^2\theta$.*

Solution. Take the origin at the centre of the given sphere S. Since the potential is independent of ϕ on S, so also is the potential at any point. Therefore, the Laplace's equation in spherical co-ordinates reduces to

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = 0 \qquad ...(i)$$

Putting $u(r, \theta) = R(r)\, G(\theta)$ in (i) and proceeding as in § 18.12 (3), we obtain the equations

$$\frac{\partial^2 G}{d\theta^2} + \cot\theta \frac{dG}{d\theta} + n(n+1)G = 0 \qquad ...(ii)$$

and

$$\frac{1}{R}\left(r^2 \frac{d^2R}{dr^2} + 2r\frac{dR}{dr}\right) = n(n+1) \qquad ...(iii)$$

Putting $\cot\theta = v$, (ii) takes the form

$$(1 - v^2)\frac{d^2G}{dv^2} - 2v\frac{dG}{dv} + n(n+1)G = 0$$

which is Legendre's equation. Its solutions are

$$G = P_n(v) = P_n(\cos\theta) \text{ for } n = 0, 1, 2, ...$$

The solutions of (iii) are $R_n(r) = r^n$, $\overline{R_n}(r) = 1/r^{n+1}$.

Hence the equation (i) has the following two sets of solutions

$$u_n(r, \theta) = c_n r^n P_n(\cos\theta) \text{ and } \bar{u}_n(r, \theta) = c_n P_n(\cos\theta)/r^{n+1}, \text{ where } n = 0, 1, 2, ...$$

For points inside S, we have the general equation $u(r, \theta) = \sum_{n=0}^{\infty} c_n r^n P_n(\cos\theta)$ ...(iv)

On the boundary of S, $u(1, \theta) = f(\theta)$ $\therefore$ $f(\theta) = \sum_{n=0}^{\infty} c_n P_n(\cos\theta)$

which is Fourier-Legendre expansion of $f(\theta)$. Hence by (5) p. 560, we have

$$c_n = \left(n + \frac{1}{2}\right)\int_{-1}^{1} f(\theta)\, P_n(x)\, dx \text{ where } x = \cos\theta.$$

$$= \left(n + \frac{1}{2}\right)\int_{-1}^{1} x^2 P_n(x) dx \qquad [\because f(\theta) = \cos^2\theta]$$

$$= \left(n + \frac{1}{2}\right)\int_{-1}^{1}\left[\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right]P_n(x)\, dx \qquad [\because P_2(x) = \tfrac{1}{2}(3x^2 - 1)]$$

Using the orthogonality of Legendre polynomials, we get

$$c_n = 0, \text{ except for } n = 0, 2. \text{ Hence}$$

$$c_0 = \frac{1}{2}\cdot\frac{1}{3}\int_{-1}^{1} P_0^2(x)\, dx = \frac{1}{3},\ c_2 = \frac{5}{2}\cdot\frac{2}{3}\int_{-1}^{0} P_2^2(x)\, dx = \frac{2}{3}.$$

Substituting in (iv), we get $u(r, \theta) = \frac{1}{3} + \frac{2}{3}r^2 P_2(\cos\theta)$ or $u(r, \theta) = \frac{1}{3} + r^2(\cos^2\theta - \frac{1}{3})$.

PROBLEMS 18.8

1. Show that a solution of Laplace's equation in cylindrical co-ordinates, which remains finite at $r = 0$, may be expressed in the form

$$u = \sum_{n=0}^{\infty} J_n(kr)\,\{e^{kz}(A_n\cos n\theta + B_n \sin n\theta) + e^{-kz}(C_n \cos n\theta + D_n \sin n\theta)\}.$$

2. The potential on the surface of a unit sphere is $f(\theta) = \cos 2\theta$. Show that the potential at all points of space is given by

$$u(r, \theta) = 2r^2(\cos^2\theta - 1/3) - \frac{1}{3} \text{ for } r < 1,$$

and
$$u(r, \theta) = 2r^{-3}(\cos^2\theta - 1/3) - r^{-1}/3 \text{ for } r > 1.$$

3. Show that in spherical polar coordinates (r, θ, ϕ), Laplace's equation possesses solutions of the form

$$(Ar^n + B/r^{n+1})P_n(\mu)e^{\pm im\phi},$$

where $\mu = \cos\theta$, A, B, m, n are constants and $P_n(\mu)$ satisfies Legendre's equation

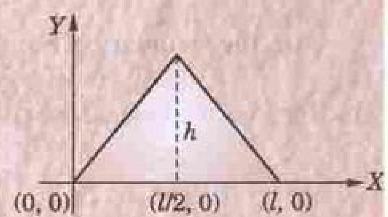
$$(1-\mu^2)\frac{d^2P_n}{d\mu^2} - 2\mu\frac{dP_n}{d\mu} + \left\{n(n+1) - \frac{m^2}{1-\mu^2}\right\}P_n = 0.$$

18.13 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 18.9

Fill up the blanks in each of the following questions :

1. The radio equations for the potential and current are ....
2. The partial differential equation representing variable heat flow in three dimensions, is ....
3. Temperature gradient is defined as ....
4. The differential equation $f_{xx} + 2f_{xy} + 4f_{yy} = 0$ is classified as ....
5. The partial differential equation of the transverse vibrations of a string is ....
6. The steady state temperature of a rod of length l whose ends are kept at 30° and 40° is ....
7. The equation $u_t = c^2u_{xx}$ is classified as ....
8. The two dimensional steady state heat flow equation in polar coordinates is ....
9. The mathematical function of the initial form of the string given by the following graph is ....



10. When a vibrating string fastened to two points l apart, has an initial velocity u_0, its initial conditions are ....
11. In two dimensional heat flow, the temperature along the normal to the xy-plane is ....
12. If a square plate has its faces and the edge $y = 0$ insulated, its edges $x = 0$ and $x = a$ are kept at zero temperature and the fourth edge is kept at temperature u, then the boundary conditions for this problem are ....
13. If the ends $x = 0$ and $x = l$ are insulated in one dimensional heat flow problems, then the boundary conditions are ....
14. D'Alembert's solution of the wave equation is ....
15. The partial differential equation of 2-dimensional heat flow in ....
16. A rod 50 cm long with insulated sides has its end A and B kept at 20° and 70°C respectively. The steady state temperature distribution of the rod is .... *(Anna, 2008)*
17. The three possible solutions of Laplace equation in polar coordinates are ....
18. Solution of $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$, given $u(0, y) = 8e^{-3y}$, is ....
19. Solution of $\frac{\partial z}{\partial x} + 4z = \frac{\partial z}{\partial t}$, given $z(x, 0) = 4e^{-3x}$, is ....
20. In the equation $\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}$, α^2 represents ....
21. The telegraph equations for potential and current are ......
22. The general solution of one-dimensional heat flow equation when both ends of the bar are kept at zero temperature, is of the form ....
23. The three possible solutions of Laplace equation $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ are ....

CHAPTER 19

Complex Numbers and Functions

1. Complex Numbers. 2. Argand's diagram. 3. Geometric representation of $z_1 \pm z_2$; z_1z_2 and z_1/z_2. 4. De Moivre's theorem. 5. Roots of a complex number. 6. To expand sin $n\theta$, cos $n\theta$ and tan $n\theta$ in powers of sin θ, cos θ and tan θ respectively; Addition formulae for any number of angles; To expand $\sin^m \theta$, $\cos^n \theta$ and $\sin^m \theta \cos^n \theta$ in a series of sines or cosines of multiples of θ. 7. Complex function: Definition. 8. Exponential function of a complex variable. 9. Circular functions of a complex variable. 10. Hyperbolic functions. 11. Inverse hyperbolic functions. 12. Real and imaginary parts of circular and hyperbolic functions. 13. Logarithmic functions of a complex variable. 14. Summation of series – '$C + iS$' method. 15. Approximations and Limits. 16. Objective Type of Questions.

19.1 COMPLEX NUMBERS

Definition. *A number of the form $x + iy$, where x and y are real numbers and $i = \sqrt{(-1)}$, is called a* **complex number.**

x is called the *real part* of $x + iy$ and is written as $R(x + iy)$ and y is called the *imaginary part* and is written as $I(x + iy)$.

A pair of complex numbers $x + iy$ and $x - iy$ are said to be conjugate of each other.

Properties : (1) *If $x_1 + iy_1 = x_2 + iy_2$, then $x_1 - iy_1 = x_2 - iy_2$.*

(2) *Two complex numbers $x_1 + iy_1 =$ and $x_2 + iy_2$ are said to be equal when*

$$R(x_1 + iy_1) = R(x_2 + iy_2), \text{ i.e., } x_1 = x_2$$

and $$I(x_1 + iy_1) = I(x_2 + iy_2), \text{ i.e., } y_1 = y_2.$$

(3) *Sum, difference, product and quotient of any two complex numbers is itself a complex number.*

If $x_1 + iy_1$ and $x_2 + iy_2$ be two given complex numbers, then

(*i*) their sum $= (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$

(*ii*) their difference $= (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2)$

(*i*) their product $= (x_1 + iy_1) + (x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$

and (*iv*) their quotient $= \dfrac{x_1 + iy_1}{x_2 + iy_2} = \dfrac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \dfrac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\,\dfrac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}$

(4) *Every complex number $x + iy$ can always be expressed in the form r (cos θ + i sin θ).*

Put $$R(x + iy), \text{ i.e., } x = r\cos\theta \quad ...(i)$$

and $$I(x + iy), \text{ i.e., } y = r\sin\theta \quad ...(ii)$$

Squaring and adding, we get $x^2 + y^2 = r^2$ *i.e.* $r = \sqrt{(x^2 + y^2)}$ (taking positive square root only)

Dividing (*ii*) by (*i*), we get $y/x = \tan\theta$ *i.e.* $\theta = \tan^{-1}(y/x)$.

Thus $x + iy = r(\cos\theta + i\sin\theta)$ where $r = \sqrt{(x^2 + y^2)}$ and $\theta = \tan^{-1}(y/x)$.

Definitions. *The number* $r = +\sqrt{(x^2 + y^2)}$ *is called the* **modulus** *of* $x + iy$ *and is written as mod* $(x + iy)$ *or* $|x + iy|$.

The angle θ *is called the amplitude or argument of* $x + iy$ *and is written as amp* $(x + iy)$ *or arg* $(x + iy)$.

Evidently the amplitude θ has an infinite number of values. The value of θ which lies between $-\pi$ and π is called the *principal value of the amplitude*. Unless otherwise specified, we shall take amp (z) to mean the principal value.

Note. *$\cos\theta + i\sin\theta$ is briefly written as cis θ (pronounced as 'sis θ')*

(5) *If the conjugate of* $z = x + iy$ be $\bar{z}$, *then*

(*i*) $R(z) = \frac{1}{2}(z + \bar{z}),\ I(z) = \frac{1}{2i}(z - \bar{z})$ (*ii*) $|z| = \sqrt{R^2(z) + I^2(z)} = |\bar{z}|$

(*iii*) $z\bar{z} = |z|^2$ (*iv*) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(*v*) $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$ (*vi*) $\overline{(z_1 / z_2)} = \bar{z}_1 / \bar{z}_2$, where $\bar{z}_2 \neq 0$.

Example 19.1. *Reduce* $1 - \cos\alpha + i\sin\alpha$ *to the modulus amplitude form.*

Solution. Put $1 - \cos\alpha = r\cos\theta$ and $\sin\alpha = r\sin\theta$

$\therefore$ $$r = (1 - \cos\alpha)^2 + \sin^2\alpha = 2 - 2\cos\alpha = 4\sin^2\alpha/2$$

i.e., $$r = 2\sin\alpha/2$$

and $$\tan\theta = \frac{\sin\alpha}{1 - \cos\alpha} = \frac{2\sin\alpha/2\cos\alpha/2}{2\sin^2\alpha/2} = \cot\alpha/2$$

$$= \tan\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) \quad \therefore \theta = \frac{\pi - \alpha}{2}.$$

Thus $1 - \cos\alpha + i\sin\alpha = 2\sin\frac{\alpha}{2}\left[\cos\frac{\pi - \alpha}{2} + i\sin\frac{\pi - \alpha}{2}\right]$.

Example 19.2. *Find the complex number* z *if arg* $(z + 1) = \pi/6$ *and arg* $(z - 1) = 2\pi/3$. *(Mumbai, 2009)*

Solution. Let $z = x + iy$ so that $z + 1 = (x + 1) + iy$ and $(z - 1) = (x - 1) + iy$

Since $\quad arg\,(z + 1) = \pi/6, \quad \therefore \tan^{-1}\left(\frac{y}{x + 1}\right) = 30°$

i.e., $$\frac{y}{x + 1} = \tan 30° = 1/\sqrt{3}, \text{ or } \sqrt{3}y = x + 1 \qquad \ldots(i)$$

Also since $\quad \arg\,(z - 1) = 2\pi/3, \quad \therefore \tan^{-1}\left(\frac{y}{x + 1}\right) = 120°$

i.e., $$\frac{y}{x - 1} = \tan 120° = -\sqrt{3}, \text{ or } y = -\sqrt{3}x + \sqrt{3} \text{ or } \sqrt{3}y = -3x + 3 \qquad \ldots(ii)$$

Subtracting (*ii*) from (*i*), we get $4x - 2 = 0$ *i.e.,* $x = 1/2$

From (*i*), $\quad \sqrt{3}\,y = 1/2 + 1$, *i.e.,* $\quad y = \sqrt{3}/2$

Hence $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$.

Example 19.3. *Find the real values of* x, y *so that* $-3 + ix^2y$ *and* $x^2 + y + 4i$ *may represent complex conjugate numbers.*

Solution. If $z = -3 + ix^2y$, then $\bar{z} = x^2 + y + 4i$

so that $$z = (x^2 + y) - 4i$$

$\therefore$ $$-3 + ix^2y = x^2 + y - 4i$$

Equating real and imaginary parts from both sides, we get

$$-3 = x^2 + y, x^2y = -4$$

Eliminating x, $(y + 3)y = -4$

or $y^2 + 3y - 4 = 0$ *i.e.*, $y = 1$ or -4

When $y = 1$, $x^2 = -3 - 1$ or $x = +2i$ which is not feasible

When $y = -4$, $x^2 = 1$ or $x = \pm 1$

Hence $x = 1$, $y - 4$ or $x = -1, y = -4$.

19.2 (1) GEOMETRIC REPRESENTATION OF IMAGINARY NUMBERS

Let all the real numbers be represented along $X'OX$, the positive real numbers being along OX and negative ones along OX'. Let OA be equal to one unit of measurement (Fig. 19.1).

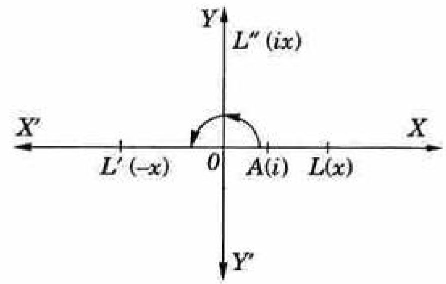


Fig. 19.1

Take a point L on OX such that $OL = x\ (OA)$.

Then L on OX represents the positive real number x and $i.ix = i^2x = -x$ is represented by a point L' on OX' distant OL from O.

From this we infer that the multiplication of the real number x by i twice amounts to the rotation of OL through two right angles to the position OL'.

Thus it naturally follows that the multiplication of a real number by i is equivalent to the rotation of OL through one right angle to the position OL''.

Hence, if Y′OY be a line perpendicular to the real axis X′OX, then all imaginary numbers are represented by points on Y′OY, called the **imaginary axis**, *the positive ones along OY and negative ones along OY′.**

Obs. Geometric interpretation of i*. From the above, it is clear that i is an operation which when multiplied to any real number makes it imaginary and rotates its direction through a right angle on the complex plane.

(2) Geometric representation of complex numbers†

Consider two lines $X'OX$, $Y'OY$ at right angles to each other.

Let all the real numbers be represented by points on the line $X'OX$ (called the *real axis*), positive real numbers being along OX and negative ones along OX'. Let the point L on OX represent the real number x (Fig. 19.2).

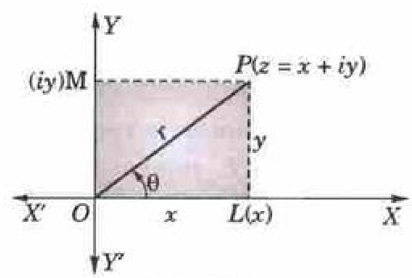


Fig. 19.2

Since the multiplication of a real number by i is equivalent to the rotation of its direction through a right angle. Therefore, let all the imaginary numbers be represented by points on the line $Y'OY$ (called the *imaginary axis*), the positive ones along OY and negative ones along OY'. Let the point M on OY represent the imaginary number iy.

Complete the rectangle $OLPM$. *Then the point whose cartesian coordinates are (x, y) uniquely represents the complex number z = x + iy on the complex plane z.* The diagram in which this representation is carried out is called the **Argand's diagram**.

If (r, θ) be the polar coordinates of P, then r is the modulus of z and θ is its amplitude.

Obs. Since a complex number has magnitude and direction, therefore, it can be represented like a vector. *Hereafter we shall often refer to the complex number $z = x + iy$ as*

(i) the point z whose co-ordinates are (x, y) or (ii) the vector z from O to P(x, y).

Example 19.4. *The centre of a regular hexagon is at the origin and one vertex is given by $\sqrt{3} + i$ on the Argand diagram. Determine the other vertices.*

---

* The first mathematician to propose a geometric representation of imaginary number i was *Kuhn of Denzig* (1750–51).

† The geometric representation of complex numbers came into mathematics through the memoire of *Jean Robert Argand*, Paris 1806.

Solution. Let $\vec{OA} = \sqrt{3} + i$ so that

$OA = 2$ and $\angle XOA = \tan^{-1} 1/\sqrt{3} = 30°$. (Fig. 19.3)

Being a regular hexagon, $OB = OC = 2$

$$\angle XOB = 30° + 60° = 90°$$

and $$\angle XOC = 30° + 120° = 150°$$

$\therefore$ $$\vec{OB} = 2(\cos 90° + i \sin 90°) = 2i$$

$$\vec{OC} = 2(\cos 150° + i \sin 150°) = -\sqrt{3} + i$$

Since $\vec{AD}, \vec{BE}, \vec{CF}$ are bisected at O,

$\therefore$ $$\vec{OD} = -\vec{OA} = -\sqrt{3} - i$$

$$\vec{OE} = -\vec{OB} = -2i \text{ and } \vec{OF} = -\vec{OC} = \sqrt{3} - i.$$

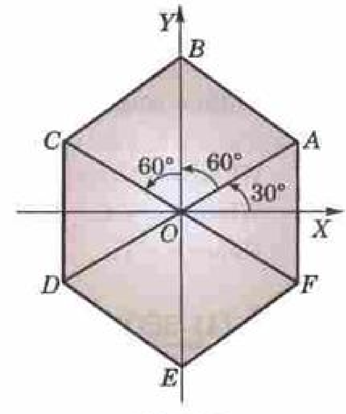


Fig. 19.3

19.3 (1) GEOMETRIC REPRESENTATION OF $z_1 + z_2$

Let P_1, P_2 represent the complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. (Fig. 19.4)

Complete the parallelogram OP_1PP_2. Draw P_1L, P_2M and $PN \perp$ s to OX. Also draw $P_1K \perp PN$.

Since $$ON = OL + LN = OL + OM = x_1 + x_2 \quad [\because LN = P_1K = OM]$$

and $$NP = NK + KP = LP_1 + MP_2 = y_1 + y_2.$$

The coordinates of P are $(x_1 + x_2, y_1 + y_2)$ and it represents the complex number

$$z = x_1 + x_2 + i(y_1 + y_2) = (x_1 + iy_1) + (x_2 + iy_2) = z_1 + z_2.$$

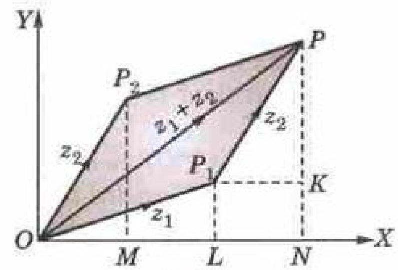


Fig. 19.4

Thus the point P which is the extremity of the diagonal of the parallelogram having OP_1 and OP_2 as adjacent sides, represents the sum of the complex numbers $P_1(z_1)$ and $P_2(z_2)$ such that

$$|z_1 + z_2| = OP \text{ and amp } (z_1 + z_2) = \angle XOP.$$

Obs. Vectorially, we have $\vec{OP_1} + \vec{P_1P} = \vec{OP}$.

(2) Geometric representation of $z_1 - z_2$

Let P_1, P_2 represent the complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ (Fig. 19.5). Then the subtraction of z_2 from z_1 may be taken as addition of z_1 to $-z_2$.

Produce P_2O backwards to R such that $OR = OP_2$. Then the coordinates of R are evidently $(-x_2, -y_2)$ and so it corresponds to the complex number $-x_2 - iy_2 = -z_2$.

Complete the parallelogram $ORQP_1$, then the sum of z_1 and $(-z_2)$ is represented by OQ *i.e.*, $z_1 - z_2 = \vec{OQ} = \vec{P_2P_1}$.

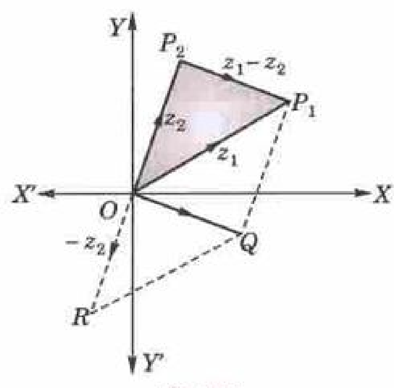


Fig. 19.5

Hence the complex number $z_1 - z_2$ is represented by the vector P_2P_1.

Obs. By means of the relation $\vec{P_2P_1} = \vec{OP_1} - \vec{OP_2}$, any vector $\vec{P_2P_1}$ may be referred to the origin.

Example 19.5. *Find the locus of $P(z)$ when*

(*i*) $|z - a| = k$;

(*ii*) *amp* $(z - a) = \alpha$, where k and α are constants. (*Gorakhpur, 1999*)

Solution. Let a, z be represented by A and P in the complex plane, O being the origin (Fig. 19.6).

Then $z - a = \vec{OP} - \vec{OA} = \vec{AP}$

(*i*) $|z - a| = k$ means that $AP = k$.

Thus the locus of P(z) is a circle whose centre is A(a) and radius k.

(*ii*) amp $(z - a)$, *i.e.*, amp $(\vec{AP}) = \alpha$, means that AP always makes a constant angle with the X-axis.

Thus the locus of P(z) is a straight line through A(a) making an $\angle\alpha$ *with OX.*

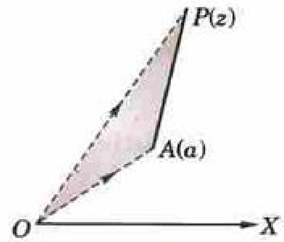


Fig. 19.6

Example 19.6. *Determine the region in the z-plane represented by*
(*i*) $1 < |z + 2i| \le 3$ (*ii*) $R(z) > 3$ (*iii*) $\pi/6 \le amp\ (z) \le \pi/3$.

Solution. (*i*) $|z + 2i| = 1$ is a circle with centre $(-2i)$ and radius 1 and $|z + 2i| = 3$ is a circle with the same centre and radius 3.

Hence $1 < |z + 2i| \le 3$ represents the region outside the circle $|z + 2i| = 1$ and inside (including circumference of) the circle $|z + 2i| = 3$ [Fig. 19.7].

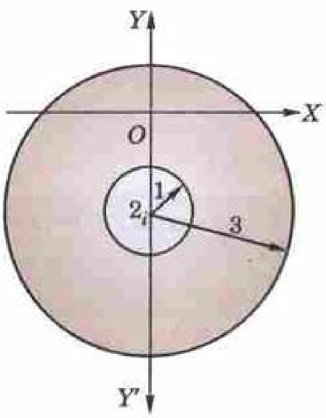


Fig. 19.7

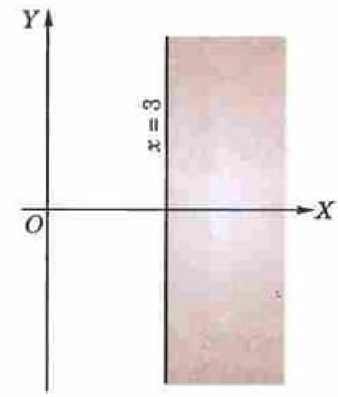


Fig. 19.8

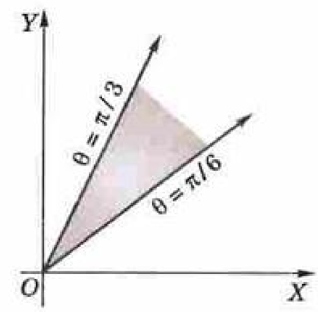


Fig. 19.9

(*ii*) $R(z) > 3$, defines all points (z) whose real part is greater than 3. Hence it represents the region of the complex plane to the right of the line $x = 3$ [Fig. 19.8].

(*iii*) If $z = r(\cos\theta + i\sin\theta)$, then amp $(z) = \theta$.

$\therefore \pi/6 \le$ amp $(z) \le \pi/3$ defines the region bounded by and including the lines $\theta = \pi/6$ and $\theta = \pi/3$. [Fig. 19.9].

Example 19.7. *If* z_1, z_2 *be any two complex numbers, prove that*
(*i*) $|z_1 + z_2| \le |z_1| + |z_2|$ [*i.e., the modulus of the sum of two complex numbers is less than or at the most equal to the sum of their moduli*].
(*ii*) $|z_1 - z_2| \ge |z_1| - |z_2|$ [*i.e., the modulus of the difference of two complex numbers is greater than or at the most equal to the difference of their moduli*].

Solution. Let P_1, P_2 represent the complex numbers z_1, z_2 (Fig. 19.10). Complete the parallelogram OP_1PP_2, so that

$$|z_1| = OP_1,\ |z_2| = OP_2 = P_1P,$$

and
$$|z_1 + z_2| = OP.$$

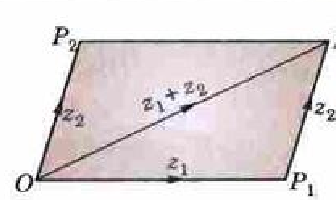


Fig. 19.10

Now from ΔOP_1P, $OP \le OP_1 + P_1P$, the sign of equality corresponding to the case when O, P_1, P are collinear.

Hence $\quad |z_1 + z_2| \le |z_1| + |z_2| \quad$ ...(*i*)

Again $\quad |z_1| = |(z_1 - z_2) + z_2| \le |z_1 - z_2| + |z_2| \quad$ [By (*i*)]

Thus $\quad |z_1 - z_2| \ge |z_1| - |z_2| \quad$ ...(*ii*)

Obs. $|z_1 + z_2 + z_3| \le |z_1| + |z_2| + |z_3|$.
In general, $|z_1 + z_2 + ... + z_n| \le |z_1| + |z_2| + ... + |z_n|$.

Example 19.8. *If* $|z_1+z_2| = |z_1-z_2|$, *prove that the difference of amplitudes of* z_1 *and* z_2 *is* $\pi/2$.
(*Mumbai, 2007*)

Solution. Let $z_1 + z_2 = r(\cos\theta + i\sin\theta)$ and $z_1 - z_2 = r(\cos\phi + i\sin\phi)$

Then $$2z_1 = r[(\cos\theta + \cos\phi) + i(\sin\theta + \sin\phi)]$$
$$= r\left\{2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2} + 2i\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}\right\}$$

or $$z_1 = r\cos\frac{\theta-\phi}{2}\left(\cos\frac{\theta+\phi}{2} + i\sin\frac{\theta+\phi}{2}\right) \text{ i.e., amp }(z_1) = \frac{\theta+\phi}{2} \quad ...(i)$$

Also $$2z_2 = r(\cos\theta - \cos\phi) + i(\sin\theta - \sin\phi)$$
$$= 2r\sin\frac{\theta-\phi}{2}\left(-\sin\frac{\theta+\phi}{2} + i\cos\frac{\theta+\phi}{2}\right)$$

or $$z_2 = r\sin\frac{\theta-\phi}{2}\left\{\cos\left(\frac{\pi}{2}+\frac{\theta+\phi}{2}\right) + i\sin\left(\frac{\pi}{2}+\frac{\theta+\phi}{2}\right)\right\}$$

i.e., $$\text{amp}(z_2) = \frac{\pi}{2} + \frac{\theta+\phi}{2} \quad ...(ii)$$

Hence [(*ii*) – (*i*)], gives amp (z_2) – amp $(z_1) = \dfrac{\pi}{2}$.

Example 19.9. *Show that the equation of the ellipse having foci at* z_1, z_2 *and major axis* $2a$, *is* $|z-z_1| + |z-z_2| = 2a$.
Also find its eccentricity.

Solution. Let $P(z)$ be any point on the given ellipse (Fig. 19.11) having foci at $S(z_1)$ and $S'(z_2)$ so that $SP = |z-z_1|$ and $S'P = |z-z_2|$.

We know that $SP + S'P = AA' (= 2a)$

i.e., $$|z-z_1| + |z-z_2| = 2a$$

which is the desired equation of the ellipse.

Also we know that $SS' = 2ae$, e being the eccentricity.

or $$\left|\vec{OS'} - \vec{OS}\right| = 2ae \quad \text{or} \quad |z_2 - z_1| = 2ae$$

or $$|z_1 - z_2| = 2ae \text{ whence } e = |z_1 - z_2|/2a.$$

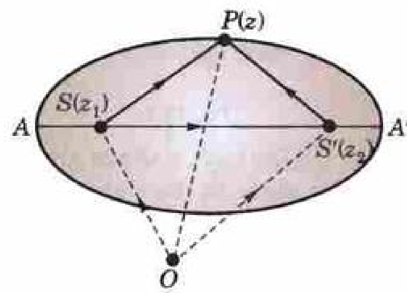


Fig. 19.11

(3) Geometric Representation of z_1z_2. Let P_1, P_2 represent the complex numbers

$$z_1 = x_1 + iy_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$

and $$z_2 = x_2 + iy_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

Measure off $OA = 1$ along OX (Fig. 19.12). Construct ΔOP_2P on OP_2 directly similar to ΔOAP_1,

so that $$OP/OP_1 = OP_2/OA \text{ i.e., } OP = OP_1 \cdot OP_2 = r_1r_2$$

and $$\angle AOP = \angle AOP_2 + \angle P_2OP = \angle AOP_2 + \angle AOP_1 = \theta_2 + \theta_1$$

$\therefore$ P represents the number

$$r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)].$$

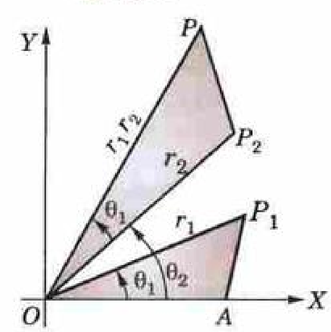


Fig. 19.12

Hence *the product of two complex numbers* z_1, z_2 *is represented by the point P, such that* (*i*) $|\mathbf{z_1z_2}| = |\mathbf{z_1}| \cdot |\mathbf{z_2}|$.

(*ii*) **amp $(\mathbf{z_1z_2})$ = amp $(\mathbf{z_1})$ + amp $(\mathbf{z_2})$.**

Cor. The effect of multiplication of any complex number z by $\cos\theta + i\sin\theta$ is to rotate its direction through an angle θ, for the modulus of $\cos\theta + i\sin\theta$ is unity.

(4) Geometric representation of z_1/z_2.

Let P_1, P_2 represent the complex numbers

$$z_1 = x_1 + iy_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$$

and
$$z_2 = x_2 + iy_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$$

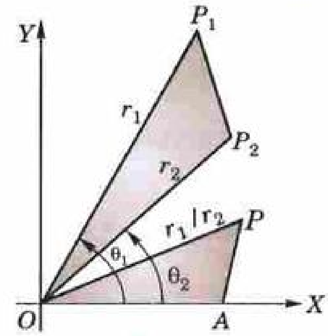


Fig. 19.13

Measure off $OA = 1$, construct triangle OAP on OA directly similar to the triangle OP_2P_1 (Fig. 19.13), so that

$$\frac{OP}{OA} = \frac{OP_1}{OP_2} \qquad i.e., \qquad OP = \frac{OP_1}{OP_2} = \frac{r_1}{r_2}$$

and
$$\angle XOP = \angle P_2OP_1 = \angle AOP_1 - \angle AOP_2 = \theta_1 - \theta_2.$$

$\therefore$ P represents the number

$$(r_1/r_2)\,[\cos (\theta_1 - \theta_2) + i \sin (\theta_1 - \theta_2)].$$

Hence the complex number z_1/z_2 is represented by the point P, such that

(*i*) $|\mathbf{z_1/z_2}| = |\mathbf{z_1}|/|\mathbf{z_2}|$

(*ii*) **amp** $(\mathbf{z_1/z_2})$ = **amp** $(\mathbf{z_1})$ – **amp** $(\mathbf{z_2})$.

Note. *If $P_1(z_1)$, $P_2(z_2)$ and $P_3(z_3)$ be any three points, then*

$$\text{amp}\left(\frac{z_3 - z_2}{z_1 - z_2}\right) = \angle P_1P_2P_3.$$

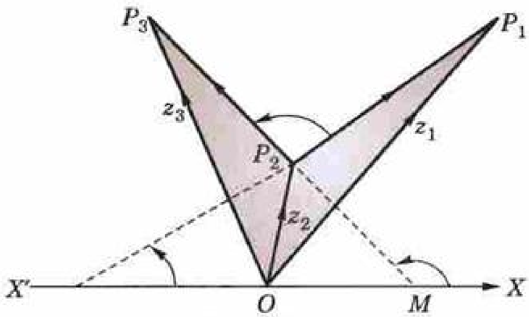


Fig. 19.14

Join O, the origin, to P_1, P_2, and P_3. Then from the figure 19.14, we have

$$\overrightarrow{P_2P_1} = z_1 - z_2 \qquad \text{and} \qquad \overrightarrow{P_2P_3} = z_3 - z_2$$

$$\therefore \qquad \text{amp}\left(\frac{z_3 - z_2}{z_1 - z_2}\right) = \text{amp}\left[\frac{\overrightarrow{P_2P_3}}{\overrightarrow{P_2P_1}}\right]$$

$$= \text{amp}\,(\overrightarrow{P_2P_3}) - \text{amp}\,(\overrightarrow{P_2P_1}) = \beta - \alpha = \angle P_1P_2P_3.$$

Example 19.10. *Find the locus of the point z, when*

(*i*) $\left|\dfrac{z-a}{z-b}\right| = k$ (*ii*) *amp* $\left(\dfrac{z-a}{z-b}\right) = \alpha$ *where k and α are constants.*

Solution. Let $A(a)$ and $B(b)$ be any two fixed points on the complex plane and let $P(z)$ be any variable point (Fig. 19.15).

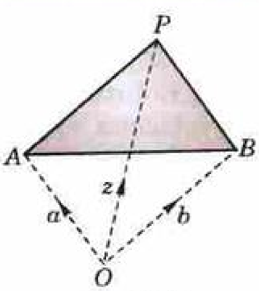


Fig. 19.15

(*i*) Since $|z - a| = AP$ and $|z - b| = BP$.

$\therefore$ The point P moves so that $\left|\dfrac{z-a}{z-b}\right| = \left|\dfrac{z-a}{z-b}\right| = \dfrac{AP}{BP} = k$

i.e., P moves so that its distances from two fixed points are in a constant ratio, which is obviously the *Appollonius* circle.

When $k = 1$, $BP = AP$ *i.e.*, P moves so that its distance from two fixed points are always equal and thus the locus of P is the right bisector of AB.

Hence the locus of P(z) is a circle (unless $k = 1$, when the locus is the right bisector of AB).

Obs. For different values of k, the equation represents family of non-intersecting coaxial circles having A and B as its limiting points.

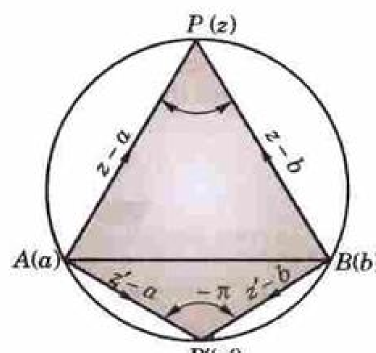


Fig. 19.16

(*ii*) From the figure 19.16, we have amp $\left(\dfrac{z-a}{z-b}\right) = \angle APB = \alpha$.

Hence the locus of P(z) is the arc APB of the circle which passes through the fixed points A and B.

If, however, P' (z') be a point on the lower arc AB of this circle, then

amp $\left(\dfrac{z'-a}{z'-b}\right) = \angle BP'A = \alpha - \pi$, which shows that the locus of P' is the arc $AP'B$ of the same circle.

Obs. For different values of α from $-\pi$ to π, the equation represents a family of intersecting coaxial circles having AB as their common radical axis.

Example 19.11. *If z_1, z_2 be two complex numbers, show that*

$$(z_1 + z_2)^2 + (z_1 - z_2)^2 = 2[|z_1|^2 + |z_2|^2].$$

Solution. Let $z_1 = r_1 \text{ cis } \theta_1$ and $z_2 = r_2 \text{ cis } \theta_2$ so that

$$|z_1 + z_2|^2 = (r_1 \cos\theta_1 + r_2 \cos\theta_2)^2 + (r_1 \sin\theta_1 + r_2 \sin\theta_2)^2$$
$$= r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_2 - \theta_1)$$

and
$$|z_1 - z_2|^2 = (r_1 \cos\theta_1 - r_2 \cos\theta_2)^2 + (r_1 \sin\theta_1 - r_2 \sin\theta_2)^2$$
$$= r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1)$$

$$\therefore \quad |z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(r_1^2 + r_2^2) = 2\left\{|z_1|^2 + |z_2|^2\right\}.$$

Example 19.12. *If z_1, z_2, z_3 be the vertices of an isosceles triangle, right angled at z_2, prove that*

$$z_1^2 + z_3^2 + 2z_2^2 = 2z_2(z_1 + z_3).$$

Solution. Let $A(z_1)$, $B(z_2)$, $C(z_3)$ be the vertices of ΔABC such that

$$AB = BC \text{ and } \angle ABC = \pi/2. \text{ (Fig. 19.17)}$$

Then $|z_1 - z_2| = |z_3 - z_2| = r$ (say).

If amp $(z_1 - z_2) = \theta$ then amp $(z_3 - z_2) = \pi/2 + \theta$

$$\therefore \quad z_1 - z_2 = r(\cos\theta + i\sin\theta),$$

and
$$z_3 - z_2 = r\left[\cos\left(\frac{\pi}{2} + \theta\right) + i\sin\left(\frac{\pi}{2} + \theta\right)\right] = r(-\sin\theta + i\cos\theta)$$

i.e.,
$$z_3 - z_2 = ir(\cos\theta + i\sin\theta) = i(z_1 - z_2)$$

or
$$(z_3 - z_2)^2 = -(z_1 - z_2)^2 \text{ or } z_1^2 + z_3^2 + 2z_2^2 = 2z_3(z_1 + z_2).$$

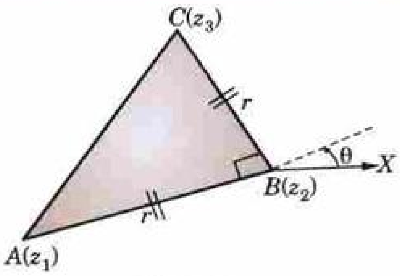


Fig. 19.17

Example 19.13. *If z_1, z_2, z_3 be the vertices of an equilateral triangle, prove that*

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.$$

Solution. Since ΔABC is equilateral, therefore, BC when rotated through $60°$ coincides with BA (Fig. 19.18). But to turn the direction of a complex number through an $\angle\,\theta$, we multiply it by $\cos\theta + i\sin\theta$.

$$\therefore \quad \vec{BC}\,(\cos\pi/3 + i\sin\pi/3) = \vec{BA}$$

i.e.,
$$(z_3 - z_2)\left(\frac{1 + i\sqrt{3}}{2}\right) = z_1 - z_2$$

or
$$i\sqrt{3}\,(z_3 - z_2) = 2z_1 - z_2 - z_3$$

Squaring,
$$-3(z_3 - z_2)^2 = (2z_1 - z_2 - z_3)^2$$

or
$$4(z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1) = 0$$

whence follows the required condition.

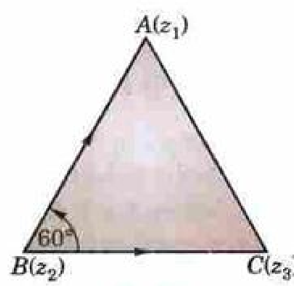


Fig. 19.18

PROBLEMS 19.1

1. Express the following in the modulus-amplitude form:

   (*i*) $1 + \sin\alpha + i\cos\alpha$ (*ii*) $\dfrac{1}{(2+1)^2} - \dfrac{1}{(2-1)^2}$. (*V.T.U., 2011 S*)

2. If $\dfrac{1}{x + iy} + \dfrac{1}{u + iv} = 1$; x, y, u, v being real quantities, express v in terms of x and y.

3. If x and y are real, solve the equation $\frac{iy}{ix+1} - \frac{3y+4i}{3x+y} = 0$.
4. If $\alpha - i\beta = \frac{1}{a - ib}$, prove that $(\alpha^2 + \beta^2)(a^2 + b^2) = 1$. *(Mumbai, 2008 S)*
5. Find what curve $z\bar{z} + (1 + i)z + (1 - i)\bar{z} = 0$ represents?
6. In an Argand diagram, show that $9 + i, 4 + 13i, -8 + 8i$ and $-3 - 4i$ form a square.
7. If $|z_1| = |z_2|$ and amp (z_1) + amp $(z_2) = 0$, then show that z_1 and z_2 are conjugate complex numbers.
8. A rectangle is constructed in the complex plane and its sides parallel to the axes and its centre is situated at the origin. If one of the vertices of the rectangle is $1 + i\sqrt{3}$, find the complex numbers representing the other three vertices of the rectangle. Find also the area of the rectangle.
9. An equilateral triangle constructed in the complex plane has its one vertex at the point $1 + i\sqrt{3}$. Find the complex numbers representing the other two vertices, O the origin being its circumcentre.
10. The centre of a regular hexagon is at the origin and one vertex is given by $1 + i$ on the Argand diagram. Find the remaining vertices.
11. What domain of the z-plane is represented by
    (*i*) $2 \le |z + 3| < 4$ (*ii*) $I(z) > 2$
    (*iii*) $\pi/3 <$ amp $(z) < \pi/2$ (*iv*) $|z + 2| + |z - 2| < 4$.
12. If $|z^2 - 1| = |z|^2 + 1$, prove that z lies on the imaginary axis. *(Mumbai, 2007)*
13. What are the loci given by (*i*) $|z - 1| + |z + 1| = 3$ (*ii*) $|z - 3| = k\,|z + 1|$ for $k = 1$ and 2.
14. Find the locus of z given by :
    (*i*) $|z| = |z - 2|$. (*ii*) $|3z - 1| = |z - 3|$.
15. Find the locus of z :
    (*i*) when $\frac{z+i}{z+2}$ is real, (*ii*) when $\frac{z-i}{z-2}$ is purely imaginary. *(Osmania, 2003 S)*

19.4 DE MOIVRE'S THEOREM*

Statement : *If n be (i) an integer, positive or negative* $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$;
(ii) a fraction, positive or negative, one of the values of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$.

Proof. Case I. *When n is a positive integer.*

By actual multiplication

$$\text{cis}\,\theta_1\,\text{cis}\,\theta_2 = (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)$$
$$= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2), \textit{ i.e.}, \text{cis}(\theta_1 + \theta_2)$$

Similarly $\text{cis}\,\theta_1\,\text{cis}\,\theta_2\,\text{cis}\,\theta_3 = \text{cis}(\theta_1 + \theta_2)\,\text{cis}\,\theta_3 = \text{cis}(\theta_1 + \theta_2 + \theta_3)$

Proceeding in this way,

$$\text{cis}\,\theta_1\,\text{cis}\,\theta_2\,\text{cis}\,\theta_3 \ldots \text{cis}\,\theta_n = \text{cis}(\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n)$$

Now putting $\theta_1 = \theta_2 = \theta_3 = \ldots = \theta_n = \theta$, we obtain $(\text{cis}\,\theta)^n = \text{cis}\,n\theta$.

Case II. *When n is a negative integer.*

Let $n = -m$, where m is a + ve integer.

$$\therefore \quad (\text{cis}\,\theta)^n = (\text{cis}\,\theta)^{-m} = \frac{1}{(\text{cis}\,\theta)^m} = \frac{1}{\text{cis}\,m\theta} \qquad \text{(By case I)}$$

$$= \frac{\cos m\theta - i\sin m\theta}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)}$$

[Multiplying the num. and denom. by $(\cos m\theta - i\sin m\theta)$]

*One of the remarkable theorems in mathematics; called after the name of its discoverer *Abraham De Moivre* (1667–1754), a French Mathematician.

$$= \frac{\cos m\theta - i \sin m\theta}{\cos^2 m\theta + \sin^2 m\theta} = \cos m\theta - i \sin m\theta$$

$$= \cos(-m\theta) + i \sin(-m\theta) = \text{cis}(-m\theta) = \text{cis}\, n\theta \qquad [\because \ -m = n]$$

Case III. *When n is a fraction, positive or negative.*

Let $n = p/q$, where q is a + ve integer and p is any integer + ve or – ve

Now $(\text{cis}\, \theta/q)^q = \text{cis}\,(q\,.\,\theta/q) = \text{cis}\, \theta$

$\therefore$ Taking qth root of both sides cis (θ/q) is one of the q values of $(\text{cis}\, \theta)^{1/q}$, *i.e.*, one of the values of $(\text{cis}\, \theta)^{1/q} = \text{cis}\, \theta/p$

Raise both sides to power p, then one of the values of $(\text{cis}\, \theta)^{p/q} = (\text{cis}\, \theta/q)^p = \text{cis}\,(p/q)\,\theta$ *i.e.*, one of the values of $(\text{cis}\, \theta)^n = \text{cis}\, n\theta$. (By case I and II)

Thus *the theorem is completely established for all rational values of n.*

Cor. **1.** $\text{cis}\, \theta_1 \,.\, \text{cis}\, \theta_2 \ldots \text{cis}\, \theta_n = \text{cis}\,(\theta_1 + \theta_2 + \ldots + \theta_n)$

2. $(\cos\theta - i \sin\theta)^n = \cos n\theta - i \sin n\theta = (\cos\theta + i\sin\theta)^{-n}$

3. $(\text{cis}\, m\theta)^n = \text{cis}\, mn\theta = (\text{cis}\, n\theta)^m$.

Example 19.14. *Simplify* $\dfrac{(\cos 3\theta + i \sin 3\theta)^4 (\cos 4\theta - i \sin 4\theta)^5}{(\cos 4\theta + i \sin 4\theta)^3 (\cos 5\theta + i \sin 5\theta)^{-4}}$.

Solution. We have, $(\cos 3\theta + i \sin 3\theta)^4 = \cos 12\theta + i \sin 12\theta = (\cos\theta + i\sin\theta)^{12}$

$(\cos 4\theta - i \sin 4\theta)^5 = \cos 20\theta - i \sin 20\theta = (\cos\theta + i\sin\theta)^{-20}$

$(\cos 4\theta + i \sin 4\theta)^3 = \cos 12\theta + i \sin 12\theta = (\cos\theta + i\sin\theta)^{12}$

$(\cos 5\theta + i \sin 5\theta)^{-4} = \cos 20\theta - i \sin 20\theta = (\cos\theta + i\sin\theta)^{-20}$

$$\therefore \quad \text{The given expression} = \frac{(\cos\theta + i\sin\theta)^{12}\,(\cos\theta + i\sin\theta)^{-20}}{(\cos\theta + i\sin\theta)^{12}\,(\cos\theta + i\sin\theta)^{-20}} = 1.$$

Example 19.15. *Prove that*

$$(1 + \cos\theta + i \sin\theta)^n + (1 + \cos\theta - i\sin\theta)^n = 2^{n+1}\cos^n(\theta/2)\,.(\cos n\theta/2).$$

Solution. Put $1 + \cos\theta = r\cos\alpha$, $\sin\theta = r\sin\alpha$.

$$\therefore \qquad r^2 = (1+\cos\theta)^2 + \sin^2\theta = 2 + 2\cos\theta = 4\cos^2\theta/2 \quad i.e., \quad r = 2\cos\theta/2$$

and $$\tan\alpha = \frac{\sin\theta}{1+\cos\theta} = \frac{2\sin\theta/2\,.\cos\theta/2}{2\cos^2\theta/2} = \tan\theta/2 \quad i.e., \quad \alpha = \theta/2.$$

$$\therefore \qquad \text{L.H.S.} = [r(\cos\alpha + i\sin\alpha)]^n + [r(\cos\alpha - i\sin\alpha)]^n$$

$$= r^n[(\cos\alpha + i\sin\alpha)^n + (\cos\alpha - i\sin\alpha)^n] = r^n(\cos n\alpha + i\sin n\alpha + \cos n\alpha - i\sin n\alpha)$$

$$= r^n\,.\,2\cos n\alpha \qquad \text{[Substituting the values of } r \text{ and } \alpha]$$

$$= 2^{n+1}\cos^n(\theta/2)\cos(n\theta/2).$$

Example 19.16. *If* $2\cos\theta = x + \dfrac{1}{x}$, *prove that*

(i) $2\cos r\theta = x^r + 1/x^r$, *(ii)* $\dfrac{x^{2n}+1}{x^{2n-1}+x} = \dfrac{\cos n\theta}{\cos(n-1)\theta}$ *(Madras, 2000 S)*

Solution. Since $x + 1/x = 2\cos\theta$ $\quad\therefore\quad$ $x^2 - 2x\cos\theta + 1 = 0$

whence $$x = \frac{2\cos\theta \pm \sqrt{(4\cos^2\theta - 4)}}{2} = \cos\theta \pm i\sin\theta.$$

(i) Taking the + ve sign, $x^r = (\cos\theta + i\sin\theta)^r = \cos r\theta + i\sin r\theta$ *(S.V.T.U., 2009)*

and $$x^{-r} = (\cos\theta + i\sin\theta)^{-r} = \cos r\theta - i\sin r\theta$$

Adding $x^r + 1/x^r = 2\cos r\theta$. Similarly with the – ve sign, the same result follows.

(*ii*)
$$\frac{x^{2n}+1}{x^{2n-1}+x}=\frac{(\cos\theta+i\sin\theta)^{2n}+1}{(\cos\theta+i\sin\theta)^{2n-1}+\cos\theta+i\sin\theta}$$
$$=\frac{\cos 2n\theta+i\sin 2n\theta+1}{\cos(2n-1)\theta+i\sin(2n-1)\theta+\cos\theta+i\sin\theta}$$
$$=\frac{(1+\cos 2n\theta)+i\sin 2n\theta}{(\cos\overline{2n-1}\,\theta+\cos\theta)+i(\sin\overline{2n-1}\,\theta+\sin\theta)}$$
$$=\frac{2\cos^2 n\theta+2i\sin n\theta\cos\theta}{2\cos n\theta\cos\overline{n-1}\,\theta+2i\sin n\theta\cos\overline{n-1}\,\theta}$$
$$=\frac{\cos n\theta\,(2\cos n\theta+2i\sin n\theta)}{\cos\overline{n-1}\,\theta\,(2\cos n\theta+2i\sin n\theta)}=\frac{\cos n\theta}{\cos\overline{n-1}\,\theta}.$$

Example 19.17. *If* $\sin\alpha+\sin\beta+\sin\gamma=\cos\alpha+\cos\beta+\cos\gamma=0$,
prove that (*i*) $\sin 2\alpha+\sin 2\beta+\sin 2\gamma=0$
(*ii*) $\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(\alpha+\beta+\gamma)$
(*iii*) $\sin 4\alpha+\sin 4\beta+\sin 4\gamma=2\Sigma\sin 2(\alpha+\beta)$
(*iv*) $\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=0$. (*Mumbai, 2009*)

Solution. Let $a=\text{cis}\,\alpha,\ b=\text{cis}\,\beta$ and $c=\text{cis}\,\gamma$.

Then $a+b+c=(\cos\alpha+\cos\beta+\cos\gamma)+i(\sin\alpha+\sin\beta+\sin\gamma)=0$ ...(1)

(*i*)
$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=(\cos\alpha+i\sin\alpha)^{-1}+(\cos\beta+i\sin\beta)^{-1}+(\cos\gamma+i\sin\gamma)^{-1}$$
$$=\sum\frac{\cos\alpha-i\sin\alpha}{\cos\alpha-i\sin\alpha}\cdot\frac{1}{\cos\alpha+i\sin\alpha}=\sum(\cos\alpha-i\sin\alpha)$$
$$=(\cos\alpha+\cos\beta+\cos\gamma)-i(\sin\alpha+\sin\beta+\sin\gamma)=0 \qquad \text{(Given)}$$

or $bc+ca+ab=0$ ...(2)

$\therefore\quad a^2+b^2+c^2=(a+b+c)^2-2(bc+ca+ab)=0$ [By (1) & (2) ...(3)]

or $(cis\,\alpha)^2+(cis\,\beta)^2+(cis\,\gamma)^2=cis\,2\alpha+cis\,2\beta+cis\,2\gamma=0$

Equating imaginary parts from both sides, we get
$$\sin 2\alpha+\sin 2\beta+\sin 2\gamma=0$$

(*ii*) Since $a+b+c=0$, $\quad\therefore\quad a^3+b^3+c^3=3abc$

or $(cis\,\alpha)^3+(cis\,\beta)^3+(cis\,\gamma)^3=3\,cis\,\alpha\,cis\,\beta\,cis\,\gamma$

or $cis\,3\alpha+cis\,3\beta+cis\,3\gamma=3\,cis\,(\alpha+\beta+\gamma)$

Equating imaginary parts from both sides, we get
$$\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(\alpha+\beta+\gamma)$$

(*iii*) From (1), $\quad a+b=-c$ or $(a+b)^2=c^2$ or $a^2+b^2-c^2=-2ab$

Again squaring, $\quad a^4+b^4+c^4+2a^2b^2-2b^2c^2-2c^2a^2=4a^2b^2$

i.e., $a^4+b^4+c^4=2(a^2b^2+b^2c^2+c^2a^2)$

or $(cis\,\alpha)^4+(cis\,\beta)^4+(cis\,\gamma)^4=2\sum(\cos\alpha)^2\,(cis\,\beta)^2$

or $cis\,4\alpha+cis\,4\beta+cis\,4\gamma=2\sum cis\,2\alpha\,cis\,2\beta=2\sum cis\,2(\alpha+\beta)$

Equating imaginary parts from both sides, we get
$$\sin 4\alpha+\sin 4\beta+\sin 4\gamma=2\sum\sin 2(\alpha+\beta)$$

(*iv*) From (2), $ab+bc+ca=0$

or $cis\,\alpha\,cis\,\beta+cis\,\beta\,cis\,\gamma+cis\,\gamma\,cis\,\alpha=0$

or $cis\,(\alpha+\beta)+cis\,(\beta+\gamma)+cis\,(\gamma+\alpha)=0$

Equating imaginary parts from both sides, we get
$$\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=0$$

PROBLEMS 19.2

1. Prove that (i) $\dfrac{(\cos 5\theta - i\sin 5\theta)^2 (\cos 7\theta + i\sin 7\theta)^{-3}}{(\cos 4\theta - i\sin 4\theta)^9 (\cos\theta + i\sin\theta)^5} = 1$

(ii) $\dfrac{(\cos\alpha + i\sin\alpha)^4}{(\sin\beta + i\cos\beta)^5} = \sin(4\alpha + 5\beta) - i\cos(4\alpha + 5\beta)$. (iii) $\left(\dfrac{\cos\theta + i\sin\theta)^4}{\sin\theta + i\cos\theta}\right) = \cos 8\theta + i\sin 8\theta$.

2. If $p = \text{cis}\,\theta$ and $q = \text{cis}\,\phi$, show that

(i) $\dfrac{p-q}{p+q} = i\tan\dfrac{\theta-\phi}{2}$ (*Mumbai, 2008*) (ii) $\dfrac{(p+q)(pq-1)}{(p-q)(pq+1)} = \dfrac{\sin\theta + \sin\phi}{\sin\theta - \sin\phi}$. (*Kurukshetra, 2005*)

3. If $a = \text{cis}\,2\alpha$, $b = \text{cis}\,2\beta$, $c = \text{cis}\,2\gamma$ and $d = \text{cis}\,2\delta$, prove that

(i) $\sqrt{\dfrac{ab}{c}} + \sqrt{\dfrac{c}{ab}} = 2\cos(\alpha + \beta - \gamma)$ (ii) $\sqrt{\dfrac{ab}{cd}} + \sqrt{\dfrac{cd}{ab}} = 2\cos(\alpha + \beta - \gamma - \delta)$.

4. If $x_r = \text{cis}\,(\pi/2^r)$, show that $\underset{n\to\infty}{\text{Lt}}\; x_1 x_2 x_3 \dots x_n = -1$. (*S.V.T.U., 2009 ; Mumbai, 2007*)

5. Find the general value of θ which satisfies the equation

$$(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta)\dots(\cos n\theta + i\sin n\theta) = 1.$$

6. Prove that (i) $(a + ib)^{m/n} + (a - ib)^{m/n} = 2(a^2 + b^2)^{m/2n}\cos\left(\dfrac{m}{n}\tan^{-1}\dfrac{b}{a}\right)$.

(ii) $(1 + i)^n + (1 - i)^n = 2^{n/2+1}\cos n\pi/4$.

7. Simplify $[\cos\alpha - \cos\beta + i(\sin\alpha - \sin\beta)]^n + [\cos\alpha - \cos\beta - i(\sin\alpha - \sin\beta)]^n$

8. Prove that (i) $(1 + \sin\theta + i\cos\theta)^n + (1 + \sin\theta - i\cos\theta)^n = 2^{n+1}\cos^n\left(\dfrac{\pi}{4} - \dfrac{\theta}{2}\right)\cos\left(\dfrac{n\pi}{4} - \dfrac{n\theta}{2}\right)$.

(ii) $\left[\dfrac{1 + \sin\alpha + i\cos\alpha}{1 + \sin\alpha - i\cos\alpha}\right]^n = \cos\left(\dfrac{n\pi}{2} - n\alpha\right) + i\sin\left(\dfrac{n\pi}{2} - n\alpha\right)$. (*S.V.T.U., 2006*)

9. If $2\cos\theta = x + 1/x$ and $2\cos\phi = y + 1/y$, show that one of the values of

(i) $x^m y^n + \dfrac{1}{x^m y^n}$ is $2\cos(m\theta + n\phi)$. (*S.V.T.U., 2007*)

(ii) $\dfrac{x^m}{y^n} + \dfrac{y^n}{x^m}$ is $2\cos(m\theta - n\phi)$. (*Nagpur, 2009*)

10. If α, β be the roots of $x^2 - 2x + 4 = 0$, prove that $\alpha^n + \beta^n = 2^{n+1}\cos n\pi/3$. (*Delhi, 2002*)

11. If α, β are the roots of the equation $z^2\sin^2\theta - z\sin\theta + 1 = 0$, then prove that

(i) $\alpha^n + \beta^n = 2\cos n\theta\,\text{cosec}^n\,\theta$ (ii) $\alpha^n\beta^n = \text{cosec}^{2n}\,\theta$. (*Mumbai, 2009*)

12. If $x^2 - 2x\cos\theta + 1 = 0$, show that $x^{2n} - 2x^n\cos n\theta + 1 = 0$.

13. If $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$, $z = \cos\gamma + i\sin\gamma$ and $x + y + z = 0$, then prove that

$$x^{-1} + y^{-1} + z^{-1} = 0.$$

14. If $\sin\theta + \sin\phi + \sin\psi = 0 = \cos\theta + \cos\phi + \cos\psi$, prove that

(i) $\cos 2\theta + \cos 2\phi + \cos 2\psi = 0$ (*Mumbai, 2009*)

(ii) $\cos 3\theta + \cos 3\phi + \cos 3\psi = 3\cos(\theta + \phi + \psi)$

(iii) $\cos 4\theta + \cos 4\phi + \cos 4\psi = 2\sum\cos 2(\phi + \psi)$.

15. If $\cos\alpha + \cos\beta + \cos\gamma = 0$ and $\sin\alpha + \sin\beta + \sin\gamma = 0$, prove that

(i) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 3/2$

(ii) $\cos(\alpha + \beta) + \cos(\beta + \gamma) + \cos(\gamma + \alpha) = 0$ (*Mumbai, 2009 ; S.V.T.U., 2008*)

16. If $\sin\alpha + 2\sin\beta + 3\sin\gamma = 0$, $\cos\alpha + 2\cos\beta + 3\cos\gamma = 0$, prove that $\sin 3\alpha + 8\sin 3\beta + 27\sin 3\gamma = 18\sin(\alpha + \beta + \gamma)$ and $\cos 3\alpha + 8\cos 3\beta + 27\sin 3\gamma = 18\cos(\alpha + \beta + \gamma)$.

19.5 ROOTS OF A COMPLEX NUMBER

There are q and only q distinct values of $(\cos\theta + i\sin\theta)^{1/q}$, *q being an integer.*

Since $\cos\theta = \cos(2n\pi + \theta)$ and $\sin\theta = \sin(2n\pi + \theta)$, where n is any integer.

$\therefore$ $$\text{cis}\,\theta = \text{cis}\,(2n\pi + \theta).$$

By De Moivre's theorem one of the values of

$$(\text{cis } \theta)^{1/q} = [\text{cis } (2n\pi + \theta)]^{1/q} = \text{cis } (2n\pi + \theta)/q \qquad ...(1)$$

Giving n the values 0, 1, 2, 3, ..., $(q - 1)$ successively, we get the following q values of $(\text{cis } \theta)^{1/q}$;

$$\left.\begin{array}{ll} \text{cis } \theta / q & (\text{for } n = 0) \\ \text{cis } (2\pi + \theta)/q & (\text{for } n = 1) \\ \text{cis } (4\pi + \theta)/q & (\text{for } n = 2) \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ \text{cis } [2(q-1)\pi + \theta]/q & (\text{for } n = q - 1) \end{array}\right\} \qquad ...(2)$$

Putting $n = q$ in (1), we get a value of $(\text{cis } \theta)^{1/q} = \text{cis } (2\pi + \theta/q) = \text{cis } \theta/q$, which is the same as the value of $n = 0$.

Similarly for $n = q + 1$, we get a value of $(\text{cis } \theta)^{1/q}$ to be cis $(2\pi + \theta)/q$, which is the same as the value for $n = 1$ and so on.

Thus, the values of $(\text{cis } \theta)^{1/q}$ for $n = q, q + 1, q + 2$ etc. are the mere repetition of the q values obtained in (2).

Moreover, the q values given by (2) are clearly distinct from each other, for no two of the angles involved therein are equal or differ by a multiple of 2π.

Hence $(\text{cis } \theta)^{1/q}$ has q and only q distinct values given by (2).

Obs. $(\text{cis } \theta)^{p/q}$ *where p/q is a rational fraction in its lowest terms, has also q and only q distinct values*; which are obtained by putting $n = 0, 1, 2, ..., q - 1$ successively in cis $p\ (2n\pi + \theta)/q$.

Note that $(\text{cis } \theta)^{6/15}$ has only 5 distinct values and not 15; because 6/15 in its lowest terms = 2/5

$\therefore$ *In order to find the distinct values of* $(\text{cis } \theta)^{p/q}$ *always see that* p/q *is in its lowest terms.*

Note. The above discussion can usefully be employed for extracting any assigned root of a given quantity. We have only to express it in the form $r (\cos \theta + i \sin \theta)$ and proceed as above.

Example 19.18. *Find the cube roots of unity and show that they form an equilateral triangle in the Argand diagram.*

Solution. If x be a cube root of unity, then

$$x = (1)^{1/3} = (\cos 0 + i \sin 0)^{1/3} = (\text{cis } 0)^{1/3} = (\text{cis } 2n\pi)^{1/3} = \text{cis } 2n\pi/3$$

where $n = 0, 1, 2$.

$\therefore$ the three vlaues of x are cis $0 = 1$,

$$\text{cis } 2\pi/3 = \cos 120° + i \sin 120° = -\frac{1}{2} + i\frac{\sqrt{3}}{2},$$

and $$\text{cis } 4\pi/3 = \cos 240° + i \sin 240° = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

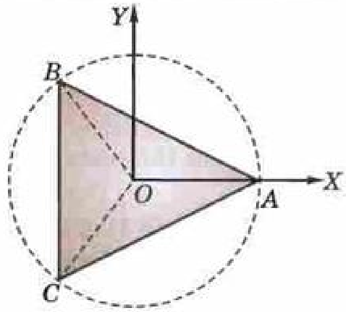


Fig. 19.19

These three cube roots are represented by the points A, B, C on the Argand diagram such that $OA = OB = OC$ and $\angle AOB = 120°$, $\angle AOC = 240°$ (Fig. 19.19).

$\therefore$ these points lie on a circle with centre O and unit radius such that $\angle AOB = \angle BOC = \angle COA = 120°$ *i.e.*, $AB = BC = CA$.

Hence A, B, C form an equilateral triangle.

Example 19.19. *Find all the values of* $\left(\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)^{3/4}$.

Also show that the continued product of these values is 1. (*Nagpur, 2009*)

Solution. Put $1/2 = r \cos \theta$ and $\sqrt{3}/2 = r \sin \theta$ so that $r = 1$ and $\theta = \pi/3$

$$\therefore \quad (1/2 + \sqrt{3}i/2)^{3/4} = [(\cos \pi/3 + i \sin \pi/3)^3]^{1/4} = (\text{cis } \pi)^{1/4}$$

$$= [\text{cis } (2n + 1)\pi]^{1/4} = \text{cis } (2n + 1)\, \pi/4 \text{ where } n = 0, 1, 2, 3.$$

Hence the required values are cis $\pi/4$, cis $3\pi/4$, cis $5\pi/4$ and cis $7\pi/4$.

$\therefore$ their continued product $= \text{cis}\left(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4}\right) = \text{cis } 4\pi = 1.$

Example 19.20. *Use De Moivre's theorem to solve the equation.* *(P.T.U., 2005)*

$$x^4 - x^3 + x^2 - x + 1 = 0.$$

Solution. '$x^4 - x^3 + x^2 - x + 1$' is a G.P. with common ratio $(-x)$, therefore

$$\frac{1-(-x)^5}{1-(-x)} = 0, \quad x \neq -1 \quad \text{or} \quad x^5 + 1 = 0$$

i.e., $$x^5 = -1 = \text{cis } \pi = \text{cis } (2n+1)\pi$$

$\therefore$ $$x = [\text{cis } (2n+1)\pi]^{1/5} = \text{cis } (2n+1)\pi/5, \text{ where } n = 0, 1, 2, 3, 4$$

Hence the values are cis $\pi/5$, cis $3\pi/5$, cis π, cis $7\pi/5$, cis $9\pi/5$

or $$\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}, \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}, -1, \cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5}, \cos\frac{\pi}{5} - i\sin\frac{\pi}{5}$$

Rejecting the value -1 which corresponds to the factor $x + 1$, the required roots are :

$$\cos \pi/5 \pm i \sin \pi/5, \cos 3\pi/5 \pm i \sin 3\pi/5.$$

Example 19.21. *Show that the roots of the equation $(x-1)^n = x^n$, n being a positive integer are $\frac{1}{2}(1 + i \cot r\pi/n)$, where r has the values 1, 2, 3, ..., n − 1.*

Solution. Given equation is $\left(\frac{x-1}{x}\right)^n = 1$ or $1 - \frac{1}{x} = (1)^{1/n}$

or $$\frac{1}{x} = 1 - (1)^{1/n} = 1 - \text{cis } \frac{2r\pi}{n}, r = 0, 1, 2, \ldots (n-1). \qquad [\because \ 1 = \text{cis } 2\pi r]$$

or $$= \left(1 - \cos\frac{2r\pi}{n}\right) - i\sin\frac{2r\pi}{n} = 2\sin^2\frac{r\pi}{n} - 2i\sin\frac{r\pi}{n}\cos\frac{r\pi}{n}$$

$\therefore$ $$x = \frac{1}{2\sin\frac{r\pi}{n}} \cdot \frac{1}{\left(\sin\frac{r\pi}{n} - i\cos\frac{r\pi}{n}\right)} = \frac{\sin\frac{r\pi}{n} + i\cos\frac{r\pi}{n}}{2\sin\frac{r\pi}{n}}$$

$$= \frac{1}{2}\left(1 + i\cot\frac{r\pi}{n}\right), r = 1, 2, \ldots (n-1). \qquad [\because \ \cot 0 \to \infty]$$

Hence the roots of the given equation are $\frac{1}{2}(1 + i \cot r\pi/n)$ where $r = 1, 2, 3, \ldots (n-1)$.

Example 19.22. *Find the 7th roots of unity and prove that the sum of their nth powers always vanishes unless n be a multiple number of 7, n being an integer, and then the sum is 7.*

(Mumbai, 2008 ; Kurukshetra, 2005)

Solution. We have $\quad (1)^{1/7} = (\cos 2r\pi + i \sin 2r\pi)^{1/7} = \text{cis } \frac{2r\pi}{7} = \left(\text{cis } \frac{2\pi}{7}\right)^r$

Putting $r = 0, 1, 2, 3, 4, 5, 6$, we find that 7th roots of unity are $1, \rho, \rho^2, \rho^3, \rho^4, \rho^5, \rho^6$ where $\rho = \cos 2\pi/7$.

$\therefore$ sum S of the nth powers of these roots $= 1 + \rho^n + \rho^{2n} + \ldots + \rho^{6n}$ ...(*i*)

$$= \frac{1-\rho^{7n}}{1-\rho^n}, \text{ being a G.P. with common ratio } \rho$$

When n is not a multiple of 7, $\rho^{7n} = (\rho^7)^n = (\text{cis } 2\pi)^n = 1$.

i.e., $$1 - \rho^{7n} = 0 \text{ and } 1 - \rho^n \neq 0, \text{ as } n \text{ is not a multiple of 7.}$$

Thus $S = 0$.

When n is a multiple of 7 $= 7p$ (say)

From (*i*), $\quad S = 1 + (\rho^7)^p + (\rho^7)^{2p} + \ldots + (\rho^7)^{6p} = 1 + 1 + 1 + 1 + 1 + 1 + 1 = 7.$

Example 19.23. *Find the equation whose roots are 2 cos π/7, 2 cos 3π/7, 2 cos 5π/7.*

Solution. Let $y = \cos\theta + i \sin\theta$, where $\theta = \pi/7, 3\pi/7, \ldots, 13\pi/7$.

Then $y^7 = (\cos\theta + i\sin\theta)^7 = \cos 7\theta + i\sin 7\theta = -1$ or $y^7 + 1 = 0$

or $(y+1)(y^6 - y^5 + y^4 - y^3 + y^2 - y + 1) = 0$

Leaving the factor $y + 1$ which corresponds to $\theta = \pi$,

We get $y^6 - y^5 + y^4 - y^3 + y^2 - y + 1 = 0$ ...(*i*)

Its roots are $y = \text{cis}\,\theta$ where $\theta = \pi/7, 3\pi/7, 5\pi/7, 9\pi/7, 11\pi/7, 13\pi/7$.

Dividing (*i*) by y^3, $(y^3 + 1/y^3) - (y^2 + 1/y^2) + (y + 1/y) - 1 = 0$

or $\{(y + 1/y)^3 - 3(y + 1/y)\} - \{(y + 1/y)^2 - 2\} - (y + 1/y) - 1 = 0$

or $x^3 - x^2 - 2x + 1 = 0$ ...(*ii*)

where $x = y + 1/y = 2\cos\theta$.

Now since $\cos 13\pi/7 = \cos\pi/7$, $\cos 11\pi/7 = \cos 3\pi/7$, $\cos 9\pi/7 = \cos 5\pi/7$

Hence the roots of (*ii*) are $2\cos\dfrac{\pi}{7}, 2\cos\dfrac{3\pi}{7}, 2\cos\dfrac{5\pi}{7}$.

PROBLEMS 19.3

1. Find all the values of
   (*i*) $(1 + i)^{1/4}$ (*ii*) $(-1 + i)^{2/5}$
   (*iii*) $(-1 + i\sqrt{3})^{3/2}$ (*iv*) $(1 + i\sqrt{3})^{1/3} + (1 - i\sqrt{3})^{1/3}$.
2. If w is a complex cube root of unity, prove that $1 + w + w^2 = 0$.
3. Find all the values of $(-1)^{1/6}$.
4. Mark by points on the Argand diagram, all the values of $(1 + i\sqrt{3})^{1/5}$ and verify that they form a pentagon.
5. Use De Moivre's theorem to solve the following equations :
   (*i*) $x^5 + 1 = 0$ (*ii*) $x^7 + x^4 + x^3 + 1 = 0$
   (*iii*) $x^9 + x^5 - x^4 - 1 = 0$ (*Madras, 2000*) (*iv*) $(x-1)^5 + x^5 = 0$.
6. Find the roots common to the equations $x^4 + 1 = 0$ and $x^6 - i = 0$.
7. Solve the equation $x^{12} - 1 = 0$ and find which of its roots satisfy the equation $x^4 + x^2 + 1 = 0$.
8. Show that the roots of $(x + 1)^7 = (x - 1)^7$ are given by $\pm i \cot r\pi/7$, $r = 1, 2, 3$. (*Mumbai, 2008*)
9. Prove that the nth roots of unity form a geometric progression. (*Mumbai, 2007*)
   Also show that the sum of these n roots is zero and their product is $(-1)^{n-1}$.
10. Find the equation whose roots are $2\cos 2\pi/7$, $2\cos 4\pi/7$, $2\cos 6\pi/7$.

19.6 (1) TO EXPAND sin nθ, cos nθ AND tan nθ IN POWERS OF sin θ, cos θ AND tan θ RESPECTIVELY (n BEING A POSITIVE INTEGER)

We have $\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$ (By De Moivre's theorem)

$= \cos^n\theta + {}^nC_1\cos^{n-1}\theta\,(i\sin\theta) + {}^nC_2\cos^{n-2}\theta\,(i\sin\theta)^2 + {}^nC_3\cos^{n-3}\theta\,(i\sin\theta)^3 + \ldots$

(By Binomial theorem)

$= (\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + \ldots) + i\,({}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + \ldots)$

Equating real and imaginary parts from both sides, we get

$\cos n\theta = \cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta - \ldots$ ...(1)

$\sin n\theta = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_5\cos^{n-5}\theta\sin^5\theta - \ldots$ ...(2)

Replacing every $\sin^2\theta$ by $1 - \cos^2\theta$ in (1) and every $\cos^2\theta$ by $1 - \sin^2\theta$ in (2), we get the desired expansions of $\cos n\theta$ and $\sin n\theta$.

Dividing (2) by (1),

$$\tan n\theta = \frac{{}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_5\cos^{n-5}\theta\sin^5\theta - \ldots}{\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta - \ldots}$$

and dividing numerator and denominator by $\cos^n\theta$, we get

$$\tan n\theta = \frac{{}^nC_1\tan\theta - {}^nC_3\tan^3\theta + {}^nC_5\tan^5\theta - \ldots}{1 - {}^nC_2\tan^2\theta + {}^nC_4\tan^4\theta - \ldots}.$$

Example 19.24. *Express* $\cos 6\theta$ *in terms of* $\cos\theta$. *(Madras, 2002)*

Solution. We know that $\cos n\theta = \cos^n\theta - {}^nC_2 \cos^{n-2}\theta \sin^2\theta + {}^nC_4 \cos^{n-4}\theta \sin^4\theta - \ldots$

Put $n = 6$, then $\cos 6\theta = \cos^6\theta - {}^6C_2 \cos^4\theta \sin^2\theta + {}^6C_4 \cos^2\theta \sin^4\theta - {}^6C_6 \sin^6\theta$

$$= \cos^6\theta - 15\cos^4\theta(1-\cos^2\theta) + 15\cos^2\theta(1-\cos^2\theta)^2 - (1-\cos^2\theta)^3$$
$$= 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1.$$

(2) Addition formulae for any number of angles

We have, $\cos(\theta_1+\theta_2+\ldots+\theta_n) + i\sin(\theta_1+\theta_2+\ldots+\theta_n)$

$$= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ldots(\cos\theta_n + i\sin\theta_n)$$

Now $\cos\theta_1 + i\sin\theta_1 = \cos\theta_1(1+i\tan\theta_1)$, $\cos\theta_2 + i\sin\theta_2 = \cos\theta_2(1+i\tan\theta_2)$ and so on.

$\therefore$ $\cos(\theta_1+\theta_2+\ldots+\theta_n) + i\sin(\theta_1+\theta_2+\ldots+\theta_n)$

$$= \cos\theta_1\cos\theta_2\ldots\cos\theta_n(1+i\tan\theta_1)(1+i\tan\theta_2)\ldots(1+i\tan\theta_n)$$
$$= \cos\theta_1\cos\theta_2\ldots\cos\theta_n[1+i(\tan\theta_1+\tan\theta_2+\ldots+\tan\theta_n)$$
$$+ i^2(\tan\theta_1\tan\theta_2+\tan\theta_2\tan\theta_3+\ldots) + i^3(\tan\theta_1\tan\theta_2\tan\theta_3+\ldots)+\ldots+\ldots]$$
$$= \cos\theta_1\cos\theta_2\ldots\cos\theta_n(1+is_1-s_2-is_3+s_4+\ldots)$$

where $s_1 = \tan\theta_1+\tan\theta_2+\ldots+\tan\theta_n$, $s_2 = \Sigma\tan\theta_1\tan\theta_2$, $s_3 = \Sigma\tan\theta_1\tan\theta_2\tan\theta_3$ etc.

Equating real and imaginary parts, we have

$$\cos(\theta_1+\theta_2+\ldots+\theta_n) = \cos\theta_1\cos\theta_2\ldots\cos\theta_n(1-s_2+s_4-\ldots)$$
$$\sin(\theta_1+\theta_2+\ldots+\theta_n) = \cos\theta_1\cos\theta_2\ldots\cos\theta_n(s_1-s_3+s_5-\ldots)$$

and by division, we get $\mathbf{\tan(\theta_1+\theta_2+\ldots+\theta_n) = \dfrac{s_1-s_3+s_5-\ldots}{1-s_2+s_4-s_6+\ldots}}$.

Example 19.25. *If* $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi/2$, *show that* $xy+yz+zx = 1$. *(P.T.U., 2003)*

Solution. Let $\tan^{-1}x = \alpha$, $\tan^{-1}y = \beta$, $\tan^{-1}z = \gamma$ so that $x = \tan\alpha$, $y \tan\beta$, $z = \tan\gamma$

We know that $$\tan(\alpha+\beta+\gamma) = \frac{\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma}{1-\tan\alpha\tan\beta-\tan\beta\tan\gamma-\tan\gamma\tan\alpha}$$

$\therefore$ $$\tan\pi/2 = \frac{x+y+z-xyz}{1-xy-yz-zx} \quad \text{or} \quad 1-xy-yz-zx = 0$$

Hence $xy+yz+zx = 1$.

Example 19.26. *If* $\theta_1, \theta_2, \theta_3$ *be three values of* θ *which satisfy the equation* $\tan 2\theta = \lambda\tan(\theta+\alpha)$ *and such that no two of them differ by a multiple of* π, *show that* $\theta_1+\theta_2+\theta_3+\alpha$ *is a multiple of* π.

Solution. Given equation can be written as $\dfrac{2t}{1-t^2} = \lambda\dfrac{t+\tan\alpha}{1-t\,.\tan\alpha}$ where $t = \tan\theta$

or $$\lambda t^3 + (\lambda-2)\tan\alpha\,.\,t^2 + (2-\lambda)t - \lambda\tan\alpha = 0$$

$\therefore$ $\tan\theta_1$, $\tan\theta_2$, $\tan\theta_3$, being its roots, we have

$$s_1 = \Sigma\tan\theta_1 = -\frac{\lambda-2}{\lambda}\tan\alpha \qquad \text{[By § 1.3]}$$

$$s_2 = \Sigma\tan\theta_1\tan\theta_2 = \frac{2-\lambda}{\lambda} \quad \text{and} \quad s_3 = \tan\alpha$$

$\therefore$ $$\tan(\theta_1+\theta_2+\theta_3) = \frac{s_1-s_3}{1-s_2} = \frac{(-1+2/\lambda)\tan\alpha-\tan\alpha}{1-(2/\lambda-1)}$$
$$= -\tan\alpha = \tan(n\pi-\alpha)$$

Thus $\theta_1+\theta_2+\theta_3 = n\pi-\alpha$, whence follows the result.

(3) To expand $\sin^m\theta$, $\cos^n\theta$ or $\sin^m\theta\cos^n\theta$ in a series of sines or cosines of multiples of θ

If $z = \cos\theta + i\sin\theta$ then $1/z = \cos\theta - i\sin\theta$.

By De Moivre's theorem, $z^p = \cos p\theta + i\sin p\theta$ and $1/z^p = \cos p\theta - i\sin p\theta$

$\therefore$ $z + 1/z = 2\cos\theta$, $z - 1/z = 2i\sin\theta$; $z^p + 1/z^p = 2\cos p\theta$, $z^p - 1/z^p = 2i\sin p\theta$

These results are used to expand the powers of sin θ or cos θ or their products in a series of sines or cosines of multiples of θ.

Example 19.27. *Expand $\cos^8 \theta$ in a series of cosines of multiples of θ.*

Solution. Let $z = \cos\theta + i\sin\theta$, so that $z + 1/z = 2\cos\theta$ and $z^p + 1/z^p = 2\cos p\theta$.

$\therefore$ $(2\cos\theta)^8 = (z + 1/z)^8$

$$= z^8 + {}^8C_1 z^7 \cdot \frac{1}{z} + {}^8C_2 z^6 \cdot \frac{1}{z^2} + {}^8C_3 z^5 \cdot \frac{1}{z^3} + {}^8C_4 z^4 \cdot \frac{1}{z^4} + {}^8C_5 z^3 \cdot \frac{1}{z^5} + {}^8C_6 z^2 \cdot \frac{1}{z^6} + {}^8C_7 z \cdot \frac{1}{z^7} + \frac{1}{z^8}$$

$$= (z^8 + 1/z^8) + {}^8C_1(z^6 + 1/z^6) + {}^8C_2(z^4 + 1/z^4) + {}^8C_3(z^2 + 1/z^2) + {}^8C_4$$

$$= (2\cos 8\theta) + 8(2\cos 6\theta) + 28(2\cos 4\theta) + 56(2\cos 2\theta) + 70.$$

Hence $\cos^8\theta = \dfrac{1}{128}[\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35]$.

Example 19.28. *Expand $\sin^7\theta\cos^3\theta$ in a series of sines of multiples of θ.*

Solution. Let $z = \cos\theta + i\sin\theta$

so that $z + 1/z = 2\cos\theta,\ z - 1/z = 2i\sin\theta$ and $z^p - 1/z^p = 2i\sin p\theta$.

$\therefore$ $(2i\sin\theta)^7(2\cos\theta)^3 = (z - 1/z)^7(z + 1/z)^3$

$$= (z - 1/z)^4[(z - 1/z)(z + 1/z)]^3 = (z - 1/z)^4(z^2 - 1/z^2)^3$$

$$= \left(z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4}\right)\left(z^6 - 3z^2 + \frac{3}{z^2} - \frac{1}{z^6}\right)$$

$$= \left(z^{10} - \frac{1}{z^{10}}\right) - 4\left(z^8 - \frac{1}{z^8}\right) + 3\left(z^6 - \frac{1}{z^6}\right) + 8\left(z^4 - \frac{1}{z^4}\right) - 14\left(z^2 - \frac{1}{z^2}\right)$$

$$= 2i\sin 10\theta - 4(2i\sin 8\theta) + 3(2i\sin 6\theta) + 8(2i\sin 4\theta) - 14(2i\sin 2\theta)$$

Since $i^7 = -i$,

$\therefore$ $\sin^7\theta\cos^3\theta = -\dfrac{1}{2^9}[\sin 10\theta - 4\sin 8\theta + 3\sin 6\theta + 8\sin 4\theta - 14\sin 2\theta]$.

Obs. The expansion of $\sin^m\theta\cos^n\theta$ is a series of sines or cosines of multiples of θ according as m is odd or even.

PROBLEMS 19.4

1. Express sin 6θ/sin θ as a polynomial in cos θ?

Prove that (2–5) :

2. $\sin 7\theta/\sin\theta = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta$.
3. $\dfrac{1+\cos 7\theta}{1+\cos\theta} = (x^3 - x^2 - 2x + 1)^2$, where $x = 2\cos\theta$. *(Madras, 2002)*
4. $2(1 + \cos 8\theta) = (x^4 - 4x^2 + 2)^2$ where $x = 2\cos\theta$.
5. $\tan 5\theta = \dfrac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$ where $t = \tan\theta$.
6. If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$, show that $x + y + z = xyz$.
7. If α, β, γ be the roots of the equation $x^3 + px^2 + qx + p = 0$, prove that $\tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma = n\pi$ radians except in one particular case.

Prove that (8-12) :

8. $\cos^7\theta = \dfrac{1}{16}(\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta)$. *(Madras, 2003 S)*
9. $\cos^6\theta - \sin^6\theta = \dfrac{1}{16}(\cos 6\theta + 15\cos 2\theta)$. *(Mumbai, 2007)*
10. $\sin^8\theta = 2^{-7}(\cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 35)$.
11. $32\sin^4\theta\cos^2\theta = \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$.
12. $\sin^5\theta\cos^2\theta = \dfrac{1}{64}(\sin 7\theta - 3\sin 5\theta + \sin 3\theta + 5\sin\theta)$. *(Madras, 2003)*

13. Expand $\cos^5\theta \sin^7\theta$ in a series of sines of multiples of θ ? (*Madras, 2002*)
14. If $\cos^5\theta = A\cos\theta + B\cos 3\theta + C\cos 5\theta$, find $\sin^5\theta$ in terms of A, B, C.
15. If $\sin^4\theta \cos^3\theta = A_1\cos\theta + A_3\cos 3\theta + A_5\cos 5\theta + A_7\cos 7\theta$, prove that

$$A_1 + 9A_3 + 25A_5 + 49A_7 = 0.$$

19.7 COMPLEX FUNCTION

Definition. *If for each value of the complex variable z $(= x + iy)$ in a given region R, we have one or more values of w $(= u + iv)$, then w is said to be a* **complex function** *of z* and we write $w = u(x, y) + iv(x, y) = f(z)$ where u, v are real functions of x and y.

If to each value of z, there corresponds one and only one value of w, then w is said to be a *single-valued function of z* otherwise a *multi-valued function*. For example, $w = 1/z$ is a single-valued function and $w = \sqrt{z}$ is a multi-valued function of z. The former is defined at all points of the z-plane except at $z = 0$ and the latter assumes two values for each value of z except at $z = 0$.

19.8 EXPONENTIAL FUNCTION OF A COMPLEX VARIABLE

(1) Definition. When x is real, we are already familiar with the exponential function

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots \infty.$$

Similarly, *we define the exponential function of the complex variable $z = x + iy$, as*

$$\mathbf{e^z \text{ or } exp(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!} + \ldots \infty} \qquad \ldots(i)$$

(2) Properties :

I. Exponential form of $z = re^{i\theta}$

Putting $x = 0$ in (i), we get

$$e^{iy} = 1 + \frac{iy}{1!} + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \ldots \infty$$

$$= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \ldots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \ldots\right) = \cos y + i\sin y$$

Thus $\qquad e^z = e^x \cdot e^{iy} = e^x(\cos y + i\sin y)$

Also $\qquad x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$. Thus, $z = re^{i\theta}$

II. e^z is periodic function having imaginary period $2\pi i$, $[\because \; e^{z+2n\pi i} = e^z \cdot e^{2n\pi i} = e^z]$.

III. e^z is not zero for any value of z.

Since $\qquad e^z = e^{x+iy} = re^{i\theta}$ or $e^x \cdot e^{iy} = re^{i\theta}$

$\therefore \qquad r = e^x > 0, y = \theta, |e^{iy}| = 1,$

Thus $\qquad |e^z| = |e^x| \cdot |e^{iy}| = e^x \neq 0.$

IV. $\qquad e^{\bar{z}} = \overline{e^z}$

Since $\qquad e^{\bar{z}} = e^{x-iy} = e^x \cdot e^{-iy} = e^x(\cos y - i\sin y)$

$$= \overline{e^x(\cos y + i\sin y)} = \overline{e^z}$$

19.9 CIRCULAR FUNCTIONS OF A COMPLEX VARIABLE

(1) Definitions:

Since $\qquad e^{iy} = \cos y + i\sin y$ and $e^{-iy} = \cos y - i\sin y$.

$\therefore$ the circular functions of real angles can be written as

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i}, \cos y = \frac{e^{iy} + e^{-iy}}{2} \text{ and so on.}$$

It is, therefore, natural to *define the circular functions of the complex variable z by the equations :*

$$\mathbf{\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \cos z = \frac{e^{iz} + e^{-iz}}{2}, \tan z = \frac{\sin z}{\cos z}}$$

with cosec z, sec z and cot z as their respective reciprocals.

(2) Properties :

I. Circular functions are periodic : $\sin z$, $\cos z$ are periodic functions having real period 2π while $\tan z$, $\cot z$ have period π. [$a \sin (z + 2n\pi) = \sin z$, $\tan (z + n\pi) = \tan z$ etc.]

II. Even and odd functions : cos z, sec z are even functions while sin z, cosec z are odd functions. [$\because$ $\cos z = \dfrac{e^{-iz} + e^{iz}}{2} = \cos z$, and $\sin (-z) = \dfrac{e^{-iz} - e^{iz}}{2i} = \dfrac{e^{iz} - e^{-iz}}{2i} = -\sin z$]

III. Zeros of sin z are given by $z = \pm 2n\pi$ *and zeros of cos z are given by* $z = \pm \frac{1}{2}(2n + 1)\pi$, $n = 0, 1, 2, ...$

IV. All the formulae for real circular functions are valid for complex circular functions

e.g., $\sin^2 z + \cos^2 z = 1$, $\sin (z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$.

(3) Euler's theorem $e^{iz} = \cos z + i \sin z$.

By definition $\cos z + i \sin z = \dfrac{e^{iz} + e^{-iz}}{2} + i\dfrac{e^{iz} - e^{-iz}}{2i} = e^{iz}$ where $z = x + iy$.

Also we have shown that $e^{iy} = \cos y + i \sin y$, where y is real.

Thus $e^{i\theta} = \mathbf{\cos \theta + i \sin \theta}$, *where θ is real or complex. This is called the Euler's theorem.**

Cor. *De Moivre's theorem for complex numbers*

Whether θ is real or complex, we have

$$(\cos \theta + i \sin \theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i \sin n\theta$$

Thus De Moivre's theorem is true for all θ (real or complex).

Example 19.29. *Prove that (i)* $[\sin (\alpha + \theta) - e^{i\alpha} \sin \theta]^n = \sin^n \alpha\, e^{-in\theta}$

(ii) $\sin (\alpha - n\theta) + e^{-i\alpha} \sin n\theta = e^{-in\theta} \sin \alpha$.

Solution. (*i*) L.H.S. $= [\sin \alpha \cos \theta + \cos \alpha \sin \theta - (\cos \alpha + i \sin \alpha) \sin \theta]^n$

$= (\sin \alpha \cos \theta - i \sin \alpha \sin \theta)^n$

$= \sin^n \alpha (\cos \theta - i \sin \theta)^n = \sin^n \alpha (e^{-i\theta})^n = \sin^n \alpha e^{-in\theta}$

(*ii*) L.H.S. $= \sin \alpha \cos n\theta - \cos \alpha \sin n\theta + (\cos \alpha - i \sin \alpha) \sin n\theta$

$= \sin \alpha \cos n\theta - i \sin \alpha \sin n\theta$

$= \sin \alpha (\cos n\theta - i \sin n\theta) = \sin \alpha \,.\, e^{-in\theta}$.

Example 19.30. *Given* $\dfrac{1}{\rho} = \dfrac{1}{L\rho i} + C\rho i + \dfrac{1}{R}$, *where L, ρ, R are real, express ρ in the form* $Ae^{i\theta}$ *giving the values of A and θ.*

Solution.
$$\frac{1}{\rho} = \frac{R + L\rho^2 CR(-1) + L\rho i}{L\rho Ri} = \frac{(R - L\rho^2 CR) + iLR}{L\rho Ri}$$

or
$$\rho = L\rho \frac{Ri}{(R - L\rho^2 CR) + iL\rho} \times \frac{(R - L\rho^2 CR) - iL\rho}{(R - L\rho^2 CR) - iL\rho}$$

$$= \frac{L^2\rho^2 R + iL\rho R\,(R - L\rho^2 CR)}{(R - L\rho^2 CR)^2 + (L\rho)^2} = A(\cos \theta + i \sin \theta), \text{ say}$$

*See footnote p. 205.

Equating real and imaginary parts, we have

$$A\cos\theta = \frac{L^2\rho^2 R}{(R - L\rho^2 CR)^2 + (L\rho)^2} \quad ...(i)$$

$$A\sin\theta = \frac{L\rho R(R - L\rho^2 CR)}{(R - L\rho^2 CR)^2 + (L\rho)^2} \quad ...(ii)$$

Squaring and adding (*i*) and (*ii*),

$$A^2 = \frac{(L^2\rho^2 R)^2 + (L\rho R)^2 (R - L\rho^2 CR)^2}{[(R - L\rho^2 CR)^2 + (L\rho)^2]^2} \quad \text{or} \quad A = \frac{L\rho R}{\sqrt{[(R - L\rho^2 CR)^2 + (L\rho)]^2}} \quad ...(iii)$$

Dividing (*ii*) by (*i*),

$$\tan\theta = \frac{R - L\rho^2 CR}{L\rho} \quad \text{or} \quad \theta = \tan^{-1}\left\{\frac{R(1 - LC\rho^2)}{L\rho}\right\} \quad ...(iv)$$

Hence $P = A(\cos\theta + i\sin\theta) = Ae^{i\theta}$

where A and θ are given by (*iii*) and (*iv*).

19.10 HYPERBOLIC FUNCTIONS

(1) Definitions: *If x be real or complex,*

(*i*) $\dfrac{e^x - e^{-x}}{2}$ *is defined as* **hyperbolic sine of x** and is written as **sinh x.**

(*ii*) $\dfrac{e^x + e^{-x}}{2}$ *is defined as* **hyperbolic cosine of x** *and is written as* **cosh x.**

Thus $\sinh x = \dfrac{e^x - e^{-x}}{2}$ and $\cosh x = \dfrac{e^x + e^{-x}}{2}$

Also we define,

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}; \ \coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\text{sech}\, x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}; \ \text{cosech}\, x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

(2) Properties

I. Periodic functions : sinh z and cosh z are periodic functions having imaginary period $2\pi i$.

[$\because$ $\sinh(z + 2\pi i) = \sinh z$; $\cosh(z + 2\pi i) = \cosh z$]

II. Even and odd functions : cosh z is an even function while sinh z is an odd function

III. $\sinh 0 = 0$, $\cosh 0 = 1$, $\tanh 0 = 0$.

IV. **Relations between hyperbolic and circular functions.**

Since for all values of θ, $\quad \sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$ and $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

$\therefore$ Putting $\theta = ix$, we have

$$\sin ix = \frac{e^{-x} - e^x}{2i} = -\frac{e^x - e^{-x}}{2i} \quad [\because \ e^{i\theta} = e^{i \cdot ix} = e^{-x}]$$

$$= i^2 \frac{e^x - e^{-x}}{2i} = i \cdot \frac{e^x - e^{-x}}{2} = i \sinh x$$

and, therefore,

$$\cos ix = \frac{e^{-x} + e^x}{2} = \cosh x$$

Thus

$$\mathbf{\sin ix = i \sinh x} \quad ...(i)$$

$$\mathbf{\cos ix = \cosh x} \quad ...(ii)$$

and $\therefore$

$$\mathbf{\tan ix = i \tanh x} \quad ...(iii)$$

Cor.

$$\mathbf{\sinh ix = i \sin x} \quad ...(iv)$$

$$\mathbf{\cosh ix = \cos x} \quad ...(v)$$

$$\mathbf{\tanh ix = i \tan x} \quad ...(vi)$$

V. Formulae of hyperbolic functions

(a) Fundamental formulae

(1) $\cosh^2 x - \sinh^2 x = 1$ (2) $\text{sech}^2 x + \tanh^2 x = 1$ (3) $\coth^2 x - \text{cosech}^2 x = 1.$

(b) Addition formulae

(4) $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$ (5) $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

(6) $\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$

(c) Functions of 2x.

(7) $\sinh 2x = 2 \sinh x \cosh x$

(8) $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$

(9) $\tanh 2x = \dfrac{2\tanh x}{1 + \tanh^2 x}$

(d) Functions of 3x

(10) $\sinh 3x = 3\sinh x + 4\sinh^3 x$

(11) $\cosh 3x = 4\cosh^3 x - 3\cosh x$ (12) $\tanh 3x = \dfrac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$

(e) (13) $\sinh x + \sinh y = 2\sinh\dfrac{x+y}{2}\cosh\dfrac{x-y}{2}$ (14) $\sinh x - \sinh y = 2\cosh\dfrac{x+y}{2}\sinh\dfrac{x-y}{2}$

(15) $\cosh x + \cosh y = 2\cosh\dfrac{x+y}{2}\cosh\dfrac{x-y}{2}$ (16) $\cosh x - \cosh y = 2\sinh\dfrac{x+y}{2}\sinh\dfrac{x-y}{2}$.

Proofs. (1) Since, for all values of θ, we have $\cos^2\theta + \sin^2\theta = 1$.

$\therefore$ putting $\theta = ix$, we get $\cos^2 ix + \sin^2 ix = 1$ or $\cosh^2 x - \sinh^2 x = 1$

Otherwise : $\cosh^2 x - \sinh^2 x = \left(\dfrac{e^x + e^{-x}}{2}\right)^2 - \left(\dfrac{e^x - e^{-x}}{2}\right)^2 = \dfrac{1}{4}[e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2] = 1.$

Similarly we can establish the formulae (2) and (3).

(4) $\sinh(x + y) = (1/i)\sin i(x + y) = -i[\sin ix \cos iy + \cos ix \sin iy]$

$$= -i[i\sinh x \,.\, \cosh y + \cosh x \,.\, i\sinh y] = \sinh x \cosh y + \cosh x \sinh y.$$

Otherwise : $\sinh x \cosh y + \cosh x \sinh y$

$$= \frac{e^x - e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} + \frac{e^x + e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2} = \frac{e^{x+y} - e^{-(x+y)}}{2} = \sinh(x + y)$$

Similarly we can establish the formulae (5) and (6).

(12) $$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

Putting $A = ix$, $\tan 3ix = \dfrac{3\tan ix - \tan^3 ix}{1 - 3\tan^2 ix}$ or $i\tanh 3x = \dfrac{3(i\tanh x) - (i\tanh x)^3}{1 - 3(i\tanh x)^2}$

$$\tanh 3x = \frac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$$

Similarly, we can establish the formulae (7) to (11).

(16) $\cos C - \cos D = -2\sin\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

Putting $C = ix$, and $D = iy$, $\cos ix - \cos iy = -2\sin i\,\dfrac{x+y}{2}\sin i\,\dfrac{x-y}{2}$

or $$\cosh x - \cosh y = -2\left(i\sinh\frac{x+y}{2}\right)\left(i\sinh\frac{x-y}{2}\right) = 2\sinh\frac{x+y}{2}\sinh\frac{x-y}{2}$$

Similarly, we can establish the formulae (13) to (15).

19.11 INVERSE HYPERBOLIC FUNCTIONS

(1) Definitions: If $\sinh u = z$, then u is called the hyperbolic sine inverse of z and is written as $u = \sinh^{-1} z$. Similarly we define $\cosh^{-1} z$, $\tanh^{-1} z$, etc.

The inverse hyperbolic functions like other inverse functions are many-valued, but we shall consider only their principal values.

(2) *To show that (i)* $\sinh^{-1} z = \log [z + \sqrt{(z^2 + 1)}]$ *(Mumbai, 2009)*

(ii) $\cosh^{-1} z = \log [z + \sqrt{(z^2 - 1)}]$, *(iii)* $\tanh^{-1} z = \dfrac{1}{2} \log \dfrac{1+z}{1-z}$.

(*i*) Let $\sinh^{-1} z = u$, then $z = \sinh u = \dfrac{1}{2}(e^u - e^{-u})$

or $$2z = e^u - 1/e^u \quad \text{or} \quad e^{2u} - 2ze^u - 1 = 0$$

This being a quadratic in e^u, we have

$$e^u = \frac{2z \pm \sqrt{(4z^2 + 4)}}{2} = z \pm \sqrt{(z^2 + 1)}$$

$\therefore$ Taking the positive sign only, we have

$$e^u = z + \sqrt{(z^2 + 1)} \quad \text{or} \quad u = \log [z + \sqrt{(z^2 + 1)}]$$

Similarly we can establish (*ii*)

(*iii*) Let $\tanh^{-1} z = u$, then $z = \tanh u$

i.e., $$z = \frac{e^u - e^{-u}}{e^u + e^{-u}}.$$

Applying componendo and dividendo, we get $\dfrac{1+z}{1-z} = e^u/e^{-u} = e^{2u}$

or $$2u = \log \left(\frac{1+z}{1-z}\right) \text{ whence follows the result.}$$ *(P.T.U., 2005)*

Example 19.31. *If $u = \log \tan (\pi/4 + \theta/2)$, prove that*

(i) $\tanh u/2 = \tan \theta/2$ *(Mumbai, 2008 ; P.T.U., 2006 ; Madras, 2003)*

(ii) $\theta = -i \log \tan\left(\dfrac{\pi}{4} + \dfrac{iu}{2}\right)$. *(Kurukshetra, 2006)*

Solution. We have $$e^u = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) \quad \text{or} \quad \frac{e^{u/2}}{e^{-u/2}} = \frac{1 + \tan \theta/2}{1 - \tan \theta/2}$$

By componendo and dividendo, we get

$$\frac{e^{u/2} - e^{-u/2}}{e^{u/2} + e^{-u/2}} = \tan \theta/2 \quad i.e., \quad \tanh \frac{u}{2} = \tan \frac{\theta}{2} \qquad ...(i)$$

or $$\frac{1}{i} \tan \frac{iu}{2} = \frac{1}{i} \tanh \frac{i\theta}{2} \quad \text{or} \quad \frac{i\theta}{2} = \tanh^{-1}\left(\tan \frac{iu}{2}\right) = \frac{1}{2} \log \frac{1 + \tan iu/2}{1 - \tan iu/2}$$

or $$\theta = \frac{1}{i} \log \tan\left(\frac{\pi}{4} + \frac{iu}{2}\right) = -i \log \tan\left(\frac{\pi}{4} + \frac{iu}{2}\right). \qquad ...(ii)$$

Example 19.32. *Show that $\tanh^{-1}(\cos \theta) = \cosh^{-1}(\operatorname{cosec} \theta)$.* *(Kurukshetra, 2005)*

Solution. Let $\tanh^{-1}(\cos \theta) = \phi$ so that $\cos \theta = \tanh \phi$

or $$\tanh^2 \phi = \cos^2 \theta \quad \text{or} \quad 1 - \operatorname{sech}^2 \phi = \cos^2 \theta$$

or $$\operatorname{sech}^2 \phi = 1 - \cos^2 \theta = \sin^2 \theta \quad \text{or} \quad \operatorname{sech} \phi = \sin \theta$$

or $$\cosh \phi = \operatorname{cosec} \theta \quad \text{or} \quad \phi = \cosh^{-1}(\operatorname{cosec} \theta).$$

Example 19.33. *Find tanh x, if 5 sinh x – cosh x = 5.* *(Mumbai, 2004)*

Solution. We have $5(\sinh x - 1) = \cosh x$

or $$25(\sinh x - 1)^2 = \cosh^2 x = 1 + \sinh^2 x$$

or $$24\sinh^2 x - 50\sinh x + 24 = 0 \quad \text{or} \quad 12\sinh^2 x - 25\sinh x + 12 = 0$$

or $$(3\sinh x - 4)(4\sinh x - 3) = 0 \quad \text{whence } \sinh x = 4/3 \quad \text{or} \quad 3/4.$$

$\therefore$ $$\cosh x = \sqrt{(1 + \sinh^2 x)} = 5/3 \quad \text{or} \quad -5/4$$ [$\because$ $\cosh x = 5/4$ doesn't satisfy (*i*)]

Hence $$\tanh x = \frac{4}{5} \quad \text{or} \quad -\frac{3}{5}.$$

PROBLEMS 19.5

1. Separate into real and imaginary parts
   (*i*) $\exp(z^2)$ where $z = x + iy$ (*ii*) $\exp(5 + i\pi/2)$ (*iii*) $\exp(5 + 3i)^2$.
2. From the definitions of $\sin z$ and $\cos z$, prove that
   (*i*) $\cos 2z = 2\cos^2 z - 1$ (*ii*) $\dfrac{\sin 2z}{1 - \cos 2z} = \cot z$ (*iii*) $\sin 3z = 3\sin z - 4\sin^3 z$.
3. Prove that $[\sin(\alpha - \theta) + e^{-i\alpha}\sin\theta]^n = \sin^{n-1}\alpha\,\{\sin(\alpha - n\theta) + e^{-i\alpha}\sin n\theta\}$
4. If $z = e^{i\theta}$, show that $\dfrac{z^2 - 1}{z^2 + 1} = i\tan\theta$.
5. Eliminate z from p cosech $z + q$ sech $z + r = 0$, p' cosech $z + q'$ sech $z + r' = 0$.
6. If $y = \log\tan x$, show that $\sinh ny = \dfrac{1}{2}(\tan^n x - \cot^n x)$.
7. If $\tan y = \tan\alpha\tanh\beta$ and $\tan z = \cot\alpha\tanh\beta$, prove that $\tan(y + z) = \sinh 2\beta\,\text{cosec}\,2\alpha$.
8. Prove that
   (*i*) $\cosh(\alpha + \beta) - \cosh(\alpha - \beta) = 2\sinh\alpha\sinh\beta$
   (*ii*) $\sinh(\alpha + \beta)\cosh(\alpha - \beta) = \dfrac{1}{2}(\sinh 2\alpha + \sinh 2\beta)$.
9. Prove that (*i*) $(\cosh\theta \pm \sinh\theta)^n = \cosh n\theta + \sinh n\theta$; (*ii*) $\left(\dfrac{1 + \tanh\theta}{1 - \tanh\theta}\right)^3 = \cosh 6\theta + \sinh 6\theta$.
10. Express $\cosh^7\theta$ in terms of hyperbolic cosines of multiples of θ.
11. If $\sin\theta = \tanh x$, prove that $\tan\theta = \sinh x$.
12. If $\tan x/2 = \tanh u/2$, prove that
    (*i*) $\tan x = \sinh u$ and $\cos x\cosh u = 1$; (*ii*) $u = \log_e\tan(\pi/4 + x/2)$.
13. If $\cosh x = \sec\theta$, prove that
    (*i*) $\tanh^2 x/2 = \tan^2\theta/2$ (*ii*) $x = \log_e\tan(\pi/4 + \theta/2)$.
14. Show that $\tan^{-1} z = \dfrac{i}{2}\log\dfrac{i + z}{i - z}$.
15. Prove that
    (*i*) $\sinh^{-1} x = \cosh^{-1}\sqrt{(1 + x^2)} = \tanh^{-1}\dfrac{x}{\sqrt{(1 - x^2)}} = \dfrac{1}{2}\,\text{cosech}^{-1}\dfrac{1}{2x\sqrt{(1 + x^2)}}$
    (*ii*) $\tanh^{-1} x = \sinh^{-1}\dfrac{x}{\sqrt{(1 - x^2)}}$.
16. Show that
    (*i*) $\sinh^{-1}(\tan\theta) = \log\tan(\pi/4 + \theta/2)$ (*ii*) $\text{sech}^{-1}(\sin\theta) = \log\cot\theta/2$.
17. Solve the equation $7\cosh x + 8\sinh x = 1$ for real values of x. *(Mumbai, 2008)*
18. Find $\tanh x$ if $\sinh x - \cosh x = 5$.

19.12 REAL AND IMAGINARY PARTS OF CIRCULAR AND HYPERBOLIC FUNCTIONS

(1) *To separate the real and imaginary parts of*

(i) sin (x + iy); (ii) cos (x + iy); (iii) tan (x + iy); (iv) cot (x + iy); (v) sec (x + iy); (vi) cosec (x + iy).

Proofs. (*i*) $\sin(x+iy) = \sin x \cos iy + \cos x \sin iy = \sin x \cosh y + i \cos x \sinh y$.

Similarly, $\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y$

(*iii*) Let $\alpha + i\beta = \tan(x+iy)$ then $\alpha - i\beta = \tan(x-iy)$

Adding, $2\alpha = \tan(x+iy) + \tan(x-iy)$

i.e., $$\alpha = \frac{\sin(x+iy+x-iy)}{2\cos(x+iy)\cos(x-iy)} = \frac{\sin 2x}{\cos 2x + \cos 2iy} = \frac{\sin 2x}{\cos 2x + \cosh 2y}$$

Subtracting, $2i\beta = \tan(x+iy) - \tan(x-iy)$

i.e., $$i\beta = \frac{\sin 2iy}{2\cos(x+iy)\cos(x-iy)} = \frac{i \sinh 2y}{\cos 2x + \cosh 2y}$$

$\therefore$ $$\beta = \frac{\sinh 2y}{\cos 2x + \cosh 2y}$$

Similarly, $\cot(x+iy) = \dfrac{\sin 2x - i \sinh 2y}{\cosh 2y - \cos 2x}$.

(*v*) Let $\alpha + i\beta = \sec(x+iy)$ then $\alpha - i\beta = \sec(x-iy)$

Adding, $2\alpha = \sec(x+iy) + \sec(x-iy)$

i.e., $$\alpha = \frac{\cos(x-iy)+\cos(x+iy)}{2\cos(x+iy)\cos(x-iy)} = \frac{2\cos x \cos iy}{\cos 2x + \cos 2iy} = \frac{2\cos x \cosh y}{\cos 2x + \cosh 2y}$$

Subtracting, $2i\beta = \sec(x+iy) - \sec(x-iy)$

i.e., $$i\beta = \frac{\cos(x-iy)-\cos(x+iy)}{2\cos(x+iy)\cos(x-iy)} = \frac{2\sin x \sin iy}{\cos 2x + \cos 2iy} = \frac{2i \sin x \sinh y}{\cos 2x + \cosh 2y}$$

$\therefore$ $$\beta = \frac{2\sin x \sinh y}{\cos 2x + \cosh 2y}$$

Similarly, $\operatorname{cosec}(x+iy) = 2\,\dfrac{\sin x \cosh y - i \cos x \sinh y}{\cosh 2y - \cos 2x}$.

(2) *To separate the real and imaginary parts of*

(i) sinh (x + iy); (ii) cosh (x + iy); (iii) tanh (x + iy).

Proofs. (*i*) $\sinh(x+iy) = (1/i)\sin i(x+iy) = (1/i)\sin(ix-y)$

$= (1/i)[\sin ix \cos y - \cos ix \sin y] = (1/i)[i \sinh x \cos y - \cosh x \sin y]$

$= \sinh x \cos y + i \cosh x \sin y$

Similarly, $\cosh(x+iy) = \cosh x \cos y + i \sinh x \sin y$.

(*iii*) If $\alpha + i\beta = \tanh(x+iy) = (1/i)\tan(ix-y)$

then $\alpha - i\beta = \tanh(x-iy) = (1/i)\tan(ix+y)$

Adding, $2\alpha = (1/i)[\tan(ix-y) + \tan(ix+y)]$

$$\alpha = \frac{\sin(ix-y+ix+y)}{i\,.\,2\cos(ix-y)\cos(ix+y)} = \frac{(1/i)\sin 2ix}{\cos 2ix + \cos 2y} = \frac{\sinh 2x}{\cosh 2x + \cos 2y}.$$

Subtracting, $2i\beta = (1/i)[\tan(ix-y) - \tan(ix+y)]$

i.e., $$i\beta = -\frac{\sin[(ix+y)-(ix-y)]}{i\,.\,2\cos(ix+y)\cos(ix-y)}$$

$\therefore$ $$\beta = \frac{\sin 2y}{\cos 2ix + \cos 2y} = \frac{\sin 2y}{\cosh 2x + \cos 2y}.$$

Example 19.34. *If cosh (u + iv) = x + iy, prove that*

$$\frac{x^2}{\cosh^2 u} + \frac{y^2}{\sinh^2 u} = 1 \quad \text{(P.T.U., 2009 S)} \qquad \frac{x^2}{\cos^2 v} - \frac{y^2}{\sin^2 v} = 1. \quad \text{(Madras, 2000)}$$

Solution. Since $x + iy = \cosh(u + iv) = \cos(iu - v)$

$$= \cos iu \cos v + \sin iu \sin v = \cosh u \cos v + i \sinh u \sin v.$$

$\therefore$ equating real and imaginary parts, we get $x = \cosh u \cos v$; $y = \sinh u \sin v$

i.e., $$\frac{x}{\cosh u} = \cos v \quad \text{and} \quad \frac{y}{\sinh u} = \sin v$$

Squaring and adding, we get the first result.

Again $$\frac{x}{\cos v} = \cosh u \quad \text{and} \quad \frac{v}{\sin v} = \sinh u.$$

$\therefore$ squaring and subtracting, we get the second result.

Example 19.35. *If* $\tan(\theta + i\phi) = e^{i\alpha}$, *show that*

$\theta = (n + 1/2)\pi/2$ *and* $\phi = \frac{1}{2} \log \tan(\pi/4 + \alpha/2)$. *(S.V.T.U., 2007 ; Rohtak, 2005)*

Solution. Since $\tan(\theta + i\phi) = \cos\alpha + i \sin\alpha \quad \therefore \quad \tan(\theta - i\phi) = \cos\alpha - i \sin\alpha$

$\therefore$ $$\tan 2\theta = \tan[(\theta + i\phi) + (\theta - i\phi)]$$

$$= \frac{\tan(\theta + i\phi) + \tan(\theta - i\phi)}{1 - \tan(\theta + i\phi)\tan(\theta - i\phi)} = \frac{2\cos\alpha}{1 - (\cos^2\alpha + \sin^2\alpha)} = \frac{2\cos\alpha}{0} \to \infty$$

i.e., $$2\theta = n\pi + \pi/2 \quad \text{or} \quad \theta = (n + 1/2)\pi/2$$

Also $$\tan 2i\phi = \tan[(\theta + i\phi) - (\theta - i\phi)] = \frac{\tan(\theta + i\phi) - \tan(\theta - i\phi)}{1 + \tan(\theta + i\phi)\tan(\theta - i\phi)}$$

or $$i \tanh 2\phi = \frac{2i \sin\alpha}{1 + (\cos^2\alpha + \sin^2\alpha)} = i \sin\alpha \quad \text{or} \quad \frac{e^{2\phi} - e^{-2\phi}}{e^{2\phi} + e^{-2\phi}} = \frac{\sin\alpha}{1}$$

By componendo and dividendo, we get

$$\frac{e^{2\phi}}{e^{-2\phi}} = \frac{1 + \sin\alpha}{1 - \sin\alpha} = \frac{\cos^2\alpha/2 + \sin^2\alpha/2 + 2\sin\alpha/2\cos\alpha/2}{\cos^2\alpha/2 + \sin^2\alpha/2 - 2\sin\alpha/2\cos\alpha/2}$$

or $$e^{4\phi} = \frac{(\cos\alpha/2 + \sin\alpha/2)^2}{(\cos\alpha/2 - \sin\alpha/2)^2} = \left(\frac{1 + \tan\alpha/2}{1 - \tan\alpha/2}\right)^2$$

or $$e^{2\phi} = \frac{1 + \tan\alpha/2}{1 - \tan\alpha/2} = \tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right). \quad \text{Hence } \phi = \frac{1}{2}\log\tan(\pi/4 + \alpha/2).$$

Example 19.36. *Separate* $\tan^{-1}(x + iy)$ *into real and imaginary parts.* *(S.V.T.U., 2009)*

Solution. Let $\alpha + i\beta = \tan^{-1}(x + iy)$. Then $\alpha - i\beta = \tan^{-1}(x - iy)$

Adding, $$2\alpha = \tan^{-1}(x + iy) + \tan^{-1}(x - iy)^* = \tan^{-1}\frac{(x + iy) + (x - iy)}{1 - (x + iy)(x - iy)}$$

$\therefore$ $$\alpha = \frac{1}{2}\tan^{-1}\frac{2x}{1 - x^2 - y^2}$$

Subtracting, $$2i\beta = \tan^{-1}(x + iy) - \tan^{-1}(x - iy) = \tan^{-1}\frac{(x + iy) - (x - iy)}{1 + (x + iy)(x - iy)}$$

$$= \tan^{-1} i\frac{2y}{1 + x^2 + y^2} = i\tanh^{-1}\frac{2y}{1 + x^2 + y^2} \qquad [\because \tan^{-1} iz = i\tanh^{-1} z]$$

$\therefore$ $$\beta = \frac{1}{2}\tanh^{-1}\frac{2y}{1 + x^2 + y^2}.$$

Example 19.37. *Separate* $\sin^{-1}(\cos\theta + i\sin\theta)$ *into real and imaginary parts, where* θ *is a positive acute angle.*

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* $\tan^{-1} A \pm \tan^{-1} B = \tan^{-1}\dfrac{A \pm B}{1 \mp AB}$

Solution. Let $\sin^{-1}(\cos\theta + i\sin\theta) = x + iy$

Then $$\cos\theta + i\sin\theta = \sin(x+iy) = \sin x\cosh y + i\cos x\sinh y$$

$\therefore$ $$\cos\theta = \sin x\cosh y \quad ...(i) \qquad \text{and} \qquad \sin\theta = \cos x\sinh y \quad ...(ii)$$

Squaring and adding, we have

$$1 = \sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y = \sin^2 x(1+\sinh^2 y) + \cos^2 x\sinh^2 y$$
$$= \sin^2 x + \sinh^2 y(\sin^2 x + \cos^2 x)$$

or $$1 - \sin^2 x = \sinh^2 y, \quad i.e. \quad \cos^2 x = \sinh^2 y.$$

Hence from (*ii*), we have $\sin^2\theta = \cos^4 x$, *i.e.*, $\cos^2 x = \sin\theta$ because θ being a positive acute angle, $\sin\theta$ is positive.

As x is to be between $-\pi/2$ and $\pi/2$, therefore, we have

$$\cos x = +\sqrt{(\sin\theta)} \quad \text{or} \quad x = \cos^{-1}\sqrt{(\sin\theta)}$$

The relation (*ii*), then, gives $\sinh y = \sqrt{(\sin\theta)}$ so that $y = \log[\sqrt{(\sin\theta)} + \sqrt{(1+\sin\theta)}]$.

PROBLEMS 19.6

1. If $\sin(A+iB) = x+iy$, prove that

   (*i*) $\dfrac{x^2}{\cosh^2 B} + \dfrac{y^2}{\sinh^2 B} = 1$ (*ii*) $\dfrac{x^2}{\sin^2 A} - \dfrac{y^2}{\cos^2 A} = 1.$ (*P.T.U., 2010*)

2. If $\cos(\alpha+i\beta) = r(\cos\theta + i\sin\theta)$, prove that (*i*) $e^{2\beta} = \dfrac{\sin(\alpha-\theta)}{\sin(\alpha+\theta)}$. (*Kurukshetra, 2005 ; Madras, 2003*)

   (*ii*) $\beta = \dfrac{1}{2}\log\dfrac{\sin(\alpha-\theta)}{\sin(\alpha+\theta)}$. (*V.T.U., 2006*)

3. If $\cos(\theta+i\phi) = \cos\alpha + i\sin\alpha$, prove that

   (*i*) $\sin^2\theta = \pm\sin\alpha$ (*Madras, 2003*) (*ii*) $\cos 2\theta + \cosh 2\phi = 2$.

4. If $\tan(A+iB) = x+iy$, prove that

   (*i*) $x^2+y^2+2x\cot 2A = 1$. (*ii*) $x^2+y^2-2y\coth 2B+1=0$. (*iii*) $x\sinh 2B = y\sin 2A$.

5. If $\tan(\theta+i\phi) = \tan\alpha + i\sec\alpha$, prove that $e^{2\phi} = \pm\cot\alpha/2$ and $2\theta = \left(n+\dfrac{1}{2}\right)\pi+\alpha$. (*Nagpur, 2009 ; S.V.T.U., 2008*)

6. If $\tan(x+iy) = \sin(u+iv)$, prove that $\dfrac{\sin 2x}{\sinh 2y} = \dfrac{\tan u}{\tan v}$. (*S.V.T.U., 2006*)

7. If $\operatorname{cosec}(\pi/4+ix) = u+iv$, prove that $(u^2+v^2) = 2(u^2-v^2)$. (*Mumbai, 2009*)

8. If $x = 2\cos\alpha\cosh\beta$, $y = 2\sin\alpha\sinh\beta$, prove that $\sec(\alpha+i\beta)+\sec(\alpha-i\beta) = \dfrac{4x}{x^2+y^2}$.

9. If $a+ib = \tanh(v+i\pi/4)$, prove that $a^2+b^2=1$.

10. Reduce $\tan^{-1}(\cos\theta+i\sin\theta)$ to the form $a+ib$. (*Mumbai, 2009*)

    Hence show that $\tan^{-1}(e^{i\theta}) = \dfrac{n\pi}{2}+\dfrac{\pi}{4}-\dfrac{i}{2}\log\tan\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right)$.

11. Separate $\cos^{-1}(\cos\theta+i\sin\theta)$ into real and imaginary parts, where θ is a positive acute angle.

12. If $\sin^{-1}(u+iv) = \alpha+i\beta$, prove that $\sin^2\alpha$ and $\cosh^2\beta$ are the roots of the equation

    $$x^2 - x(1+u^2+v^2)+u^2 = 0.$$

13. If $\cos^{-1}(x+iy) = \alpha+i\beta$, show that

    (*i*) $x^2\sec^2\alpha - y^2\operatorname{cosec}^2\alpha = 1$, (*ii*) $x^2\operatorname{sech}^2\beta + y^2\operatorname{cosech}^2\beta = 1$.

14. Prove that (*i*) $\sin^{-1}(ix) = 2n\pi + i\log(\sqrt{1+x^2}+x)$ (*ii*) $\sin^{-1}(\operatorname{cosec}\theta) = \pi/2 + i\log\cot\theta/2$.

19.13 LOGARITHMIC FUNCTION OF A COMPLEX VARIABLE

(1) *Definition. If $z(=x+iy)$ and $w(=u+iv)$ be so related that $e^w = z$, then w is said to be a logarithm of z to the base e and is written as $w = \log_e z$.* ...(*i*)

Also $$e^{w+2in\pi} = e^w \,.\, e^{2in\pi} = z \qquad [\because \; e^{2in\pi} = 1]$$

$\therefore$ $$\log z = w + 2in\pi \qquad ...(ii)$$

i.e., the logarithm of a complex number has an infinite number of values and is, therefore, a multi-valued function.

The general value of the logarithm of z is written as $Log\ z$ (beginning with capital L) so as to distinguish it from its principal value which is written as $\log z$. This principal value is obtained by taking $n = 0$ in $Log\ z$.

Thus from (*i*) and (*ii*), $Log\,(x + iy) = 2in\pi + \log\,(x + iy)$.

Obs. 1. If $y = 0$, then $Log\ x = 2in\pi + \log x$.

This shows that the logarithm of a real quantity is also multi-valued. Its principal value is real while all other values are imaginary.

2. *We know that the logarithm of a negative quantity has no real value. But we can now evaluate this.*

e.g. $\log_e(-2) = \log_e 2(-1) = \log_e 2 + \log_e(-1) = \log_e 2 + i\pi$ $\quad [\because\ -1 = \cos\pi + i\sin\pi = e^{i\pi}]$

$= 0.6931 + i\,(3.1416)$.

(2) *Real and imaginary parts of* $Log\,(x + iy)$.

$$Log\,(x + iy) = 2in\pi + \log\,(x + iy)$$

$$= 2in\pi + \log\,[r\,(\cos\theta + i\sin\theta)] \qquad \left\{\begin{array}{l}\text{Put } x = r\cos\theta,\ y = r\sin\theta \text{ so that} \\ r = \sqrt{(x^2 + y^2)} \text{ and } \theta = \tan^{-1}(y/x)\end{array}\right.$$

$$= 2in\pi + \log\,(re^{i\theta})$$

$$= 2in\pi + \log r + i\theta = \log\sqrt{(x^2 + y^2)} + i\,\{2n\pi + \tan^{-1}(y/x)\}$$

(3) *Real and imaginary parts of* $(\alpha + i\beta)^{x+iy}$

$$(\alpha + i\beta)^{x+iy} = e^{(x+iy)\,Log\,(\alpha+i\beta)} = e^{(x+iy)\,[2in\pi + \log(\alpha+i\beta)]} \qquad \left\{\begin{array}{l}\text{Put } \alpha = r\cos\theta,\ \beta = r\sin\theta \text{ so that} \\ r = \sqrt{(\alpha^2 + \beta^2)} \text{ and } \theta = \tan^{-1}\beta/\alpha\end{array}\right.$$

$$= e^{(x+iy)\,[2in\pi + \log re^{i\theta}]} = e^{(x+iy)\,[\log r + i\,(2n\pi+\theta)]}$$

$$= e^{A+iB} = e^A\,(\cos B + i\sin B).$$

where $A = x\log r - y\,(2n\pi + \theta)$ and $B = y\log r + x\,(2n\pi + \theta)$.

$\therefore$ the required real part $= e^A\cos B$ and the imaginary part $= e^A\sin B$.

Example 19.38. *Find the general value of* $\log(-i)$.

Solution. $Log\,(-i) = 2in\pi + \log\,[0 + i(-1)]$ $\qquad \left\{\begin{array}{l}\text{Put } 0 = r\cos\theta,\ -1 = r\sin\theta 2 \\ \text{so that } r = 1 \text{ and } \theta = -\pi/2\end{array}\right.$

$$= 2in\pi + \log\,[r\,(\cos\theta + i\sin\theta)] = 2in\pi + \log\,(re^{i\theta})$$

$$= 2in\pi + \log r + i\theta = 2in\pi + \log 1 + i\,(-\pi/2) = i\left(2n - \frac{1}{2}\right)\pi.$$

Example 19.39. *Prove that* (*i*) $i^i = e^{-(4n+1)\pi/2}$ *and* $Log\ i^i = -\left(2n + \frac{1}{2}\right)\pi$.

(*ii*) $(\sqrt{i})^{\sqrt{i}} = e^{-\alpha}\ cis\ \alpha$ *where* $\alpha = \pi/4\sqrt{2}$. *(Mumbai, 2008)*

Solution. (*i*) By definition, we have

$$i^i = e^{i\,Log\,i} = e^{i\,(2in\pi + \log i)} = e^{-2n\pi + i\log[\exp(i\pi/2)]} \qquad [\because\ i = \text{cis}\,\pi/2 = \exp(i\pi/2)]$$

$$= e^{-2n\pi + i(i\pi/2)} = e^{-(2n+1/2)\pi}$$

Taking logarithms, we get (*ii*)

(*ii*) $$(\sqrt{i})^{\sqrt{i}} = e^{\sqrt{i}\log\sqrt{i}}$$

Now $$\sqrt{i}\,\log\sqrt{i} = \frac{1}{2}\left(\cos\frac{\pi}{2} + i\sin\pi/2\right)^{1/2}\log\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$= \frac{1}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\log\,(e^{i\pi/2}) = \frac{1}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\frac{i\pi}{2}$$

$$= \frac{i\pi}{4}\left(\frac{i}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = -\frac{\pi}{4\sqrt{2}} + i\,\frac{\pi}{4\sqrt{2}}$$

Hence $\qquad (\sqrt{i})^{\sqrt{i}} = e^{-\alpha + i\alpha}$ where $\alpha = \pi/4\sqrt{2}$

$$= e^{-\alpha} \,.\, e^{i\alpha} = e^{-\alpha} (\cos \alpha + i \sin \alpha).$$

Example 19.40. *If $(a + ib)^p = m^{x+iy}$, prove that one of the values of y/x is*
$$2 \tan^{-1} (b/a) \div \log (a^2 + b^2).$$

Solution. Taking logarithms, $(a + ib)^p = m^{x + iy}$ gives $p \log (a + ib) = (x + iy) \log m$

or
$$p\left(\frac{1}{2}\log (a^2 + b^2) + i \tan^{-1} \frac{b}{a}\right) = x \log m + iy \log m$$

Equating real and imaginary parts from both sides, we get

$$\frac{p}{2} \log (a^2 + b^2) = x \log m \quad ...(i), \qquad p \tan^{-1} \frac{b}{a} = y \log m \quad ...(ii)$$

Division of (*ii*) by (*i*) gives
$$y/x = 2 \tan^{-1} (b/a)/\log (a^2 + b^2).$$

Example 19.41. *If $i^{i^{i\cdots\infty}} = A + iB$, prove that $\tan \pi A/2 = B/A$ and $A^2 + B^2 = e^{-\pi/B}$.* (S.V.T.U., 2006 S)

Solution. $\qquad i^{i^{i\cdots\infty}} = A + iB \quad$ i.e. $\quad i^{A + iB} = A + iB$

or
$$A + iB = e^{(A+iB)\log i} = e^{(A+iB)\log(\cos \pi/2 + i \sin \pi/2)}$$
$$= \exp [(A + iB) \log (e^{i\pi/2})] = e^{(A + iB)(i\pi/2)}$$
$$= e^{-B\pi/2} \,.\, e^{i\pi A/2} = e^{-B\pi/2}\left(\cos\frac{\pi A}{2} + i \sin \frac{\pi A}{2}\right)$$

Equating real and imaginary parts, we get

$$A = e^{-B\pi/2} \cos \frac{\pi A}{2} \quad ...(i) \qquad B = e^{-B\pi/2} \sin \frac{\pi A}{2} \quad ...(ii)$$

Division of (*ii*) by (*i*) gives $B/A = \tan \pi A/2$

Squaring and adding (*i*) and (*ii*), $A^2 + B^2 = e^{-B\pi}$.

Example 19.42. *Prove that $\log\left(\dfrac{a+ib}{a-ib}\right) = 2i \tan^{-1}\left(\dfrac{b}{a}\right)$. Hence evaluate $\cos\left[i \log\left(\dfrac{a+ib}{a-ib}\right)\right]$.* (P.T.U., 2006)

Solution. Putting $a = r \cos \theta$, $b = r \sin \theta$ so that $\theta = \tan^{-1} b/a$, we have

$$\log\left(\frac{a+ib}{a-ib}\right) = \log\frac{r(\cos\theta + i\sin\theta)}{r(\cos\theta - i \sin\theta)} = \log (e^{i\theta} \div e^{-i\theta})$$
$$= \log e^{2i\theta} = 2i\theta = 2i \tan^{-1} b/a.$$

Thus
$$\cos\left[i \log\left(\frac{a+ib}{a-ib}\right)\right] = \cos [i\,(2i\theta)] = \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-(b/a)^2}{1+(b/a)^2} = \frac{a^2-b^2}{a^2+b^2}.$$

Example 19.43. *Separate into real and imaginary parts $\log \sin (x + iy)$.*

Solution. $\qquad \log \sin (x + iy) = \log (\sin x \cos iy + \cos x \sin iy)$

$$= \log (\sin x \cosh y + i \cos x \sinh y) = \log r (\cos \theta + i \sin \theta),$$

where
$$r \cos\theta = \sin x \cosh y \text{ and } r \sin \theta = \cos x \sinh y,$$

so that
$$r = \sqrt{(\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y)}$$
$$= \sqrt{\frac{1-\cos 2x}{2} \cdot \frac{1+\cosh 2y}{2} + \frac{1+\cos 2x}{2} \cdot \frac{\cosh 2y - 1}{2}} = \sqrt{\left[\frac{1}{2}(\cosh 2y - \cos 2x)\right]}$$

and
$$\theta = \tan^{-1} (\cot x \tanh y).$$

Thus $\log \sin (x + iy) = \log (re^{i\theta}) = \log r + i\theta$

$$= \frac{1}{2}\log\left[\frac{1}{2}(\cosh 2y - \cos 2x)\right] + i \tan^{-1} (\cot x \tanh y).$$

Example 19.44. *Find all the roots of the equation*

(*i*) *sin z = cosh 4* (*ii*) *sinh z = i.*

Solution. (*i*) $\sin z = \cosh 4 = \cos 4i = \sin (\pi/2 - 4i)$

$\therefore \quad z = n\pi + (-1)^n (\pi/2 - 4i)$ $\left\{\because \text{ If } \sin\theta = \sin\alpha \text{ then } \theta = n\pi + (-1)^n \alpha\right.$

(*ii*) $i = \sinh z = \dfrac{e^z - e^{-z}}{2}$

or $e^{2z} - 2ie^z - 1 = 0$, *i.e.* $(e^z - i)^2 = 0$ *i.e.*, $e^z = i$

or $z = Log\, i = 2in\pi + \log i = 2in\pi + \log e^{i\pi/2} = 2in\pi + i\pi/2 = i\left(2n + \dfrac{1}{2}\right)\pi.$

PROBLEMS 19.7

1. Find the general value of
   (*i*) $\log (6 + 8i)$ (*Rohtak, 2006*) (*ii*) $\log (-1)$. (*J.N.T.U., 2003*)
2. Show that (*i*) $\log (1 + i \tan \alpha) = \log (\sec \alpha) + i\alpha$, where α is an acute angle.
   (*ii*) $\text{Log}_e \dfrac{3-i}{3+i} = 2i\left(n\pi - \tan^{-1}\dfrac{1}{3}\right)$.
3. If $(a_1 + ib_1)(a_2 + ib_2)......(a_n + ib_n) = A + iB$, prove that
   (*i*) $(a_1^2 + b_1^2)(a_2^2 + b_2^2)......(a_n^2 + b_n^2) = A^2 + B^2$
   (*ii*) $\tan^{-1}\dfrac{b_1}{a_1} + \tan^{-1}\dfrac{b_2}{a_2} + ... + \tan^{-1}\dfrac{b_n}{a_n} = \tan^{-1}\dfrac{B}{A}$.
4. Find the modulus and argument of (*i*) $(1 - i)^{1+i}$. (*P.T.U., 2010*) (*ii*) $i^{\log(1+i)}$
5. If $i^{\alpha + i\beta} = \alpha + i\beta$, prove that $\alpha^2 + \beta^2 = e^{-(4n+1)\pi\beta}$. (*Kurukshetra, 2005*)
6. Prove that $\log\left\{\dfrac{\sin(x+iy)}{\sin(x-iy)}\right\} = 2i\tan^{-1}(\cot x \tanh y)$. (*Mumbai, 2007*)
7. Prove that $\tan\left[i\log\left(\dfrac{a-ib}{a+ib}\right)\right] = \dfrac{2ab}{a^2 - b^2}$.
8. If $\tan \log (x + iy) = a + ib$ where $a^2 + b^2 \neq 1$, show that $\tan \log (x^2 + y^2) = \dfrac{2a}{1 - a^2 - b^2}$.
9. If $\sin^{-1}(x + iy) = \log (A + iB)$, show that $\dfrac{x^2}{\sin^2 u} - \dfrac{y^2}{\cos^2 u} = 1$, where $A^2 + B^2 = e^{2u}$.
10. Separate into real and imaginary parts $\log \cos (x + iy)$.
11. Find all the roots of the equation, (*i*) $\cos z = 2$, (*ii*) $\tanh z + 2 = 0$.

19.14 SUMMATION OF SERIES – 'C + iS' METHOD

This is the most general method and is applied to find the sum of a series of the form

$$a_0 \sin \alpha + a_1 \sin (\alpha + \beta) + a_2 \sin (\alpha + 2\beta) + ...$$

or

$$a_0 \cos \alpha + a_1 \cos (\alpha + \beta) + a_2 \cos (\alpha + 2\beta) + ...$$

Procedure. (*i*) Put the given series = S (or C) according as it is a series of sines (or cosines). Then write C (or S) = a similar series of cosines (or sines).

e.g., If $S = a_0 \sin \alpha + a_1 \sin (\alpha + \beta) + a_2 \sin (\alpha + 2\beta) + ...$

then $C = a_0 \cos \alpha + a_1 \cos (\alpha + \beta) + a_2 \cos (\alpha + 2\beta)...$

(*ii*) Multiply the series of sines by i and add to the series of cosines, so that

$$C + iS = a_0 [\cos \alpha + i \sin \alpha] + a_1 [\cos (\alpha + \beta) + i \sin (\alpha + \beta)] + ...$$
$$= a_0 e^{i\alpha} + a_1 e^{i(\alpha+\beta)} + a_2 e^{i(\alpha + 2\beta)} + ...$$

(*iii*) Sum up this last series using any of the following standard series :

(1) Exponential series *i.e.,* $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \infty = e^x$

(2) Sine, cosine, sinh or cosh series

i.e., $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \infty = \sin x, \quad 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \infty = \cos x$

$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \infty = \sinh x, \quad 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \infty = \cosh x$

(3) Logarithmic series

i.e., $x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \infty = \log (1 + x), \quad -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots \infty\right) = \log (1 - x)$

(4) Gregory's series

i.e., $x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots \infty = \tan^{-1} x, \quad x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots \infty = \tanh^{-1} x = \frac{1}{2} \log \frac{1+x}{1-x}$

(5) Binomial series

i.e., $1 + nx + \frac{n(n-1)}{1.2} x^2 + \frac{n(n-1)(n-2)}{1.2.3} x^3 + \ldots \infty = (1 + x)^n$

$1 - nx + \frac{n(n+1)}{2!} x^2 - \frac{n(n+1)(n+2)}{3!} x^3 + \ldots \infty = (1 + x)^{-n}$

$1 + nx + \frac{n(n+1)}{2!} x^2 + \frac{n(n+1)(n+2)}{3!} x^3 + \ldots \infty = (1 - x)^{-n}$

(6) Geometric series

i.e., $a + ar + ar^2 + \ldots$ to n terms $= a \, \frac{1 - r^n}{1 - r}$, $a + ar + ar^2 + \ldots \infty = \frac{a}{1-r}, |r| < 1.$

(*iv*) Finally express the sum thus obtained in the form $A + iB$ so that by equating the real and imaginary parts, we get $C = A$ and $S = B$.

Series depending on exponential series

Example 19.45. *Sum the series* $\sin \alpha + x \sin (\alpha + \beta) + \frac{x^2}{2!} \sin (\alpha + 2\beta) + \ldots \infty.$

Solution. Let $S = \sin \alpha + x \sin (\alpha + \beta) + \frac{x^2}{2!} \sin (\alpha + 2\beta) + \ldots \infty$

and $C = \cos \alpha + x \cos (\alpha + \beta) + \frac{x^2}{2!} \cos (\alpha + 2\beta) + \ldots \infty$

$\therefore \quad C + iS = [\cos \alpha + i \sin \alpha] + x [\cos (\alpha + \beta) + i \sin (\alpha + \beta)]$

$+ \frac{x^2}{2!} [\cos (\alpha + 2\beta) + i \sin (\alpha + 2\beta)] + \ldots\ldots \infty$

$$= e^{i\alpha} + xe^{i(\alpha + \beta)} + \frac{x^2}{2!} \cdot e^{i(\alpha + 2\beta)} + \ldots \infty = e^{i\alpha} \left[1 + \frac{xe^{i\beta}}{1!} + \frac{x^2 e^{2i\beta}}{2!} + \ldots \infty\right]$$

$$= e^{i\alpha} \cdot e^{xe^{i\beta}} = e^{i\alpha} e^{x(\cos \beta + i \sin \beta)} = e^{x \cos \beta + i (\alpha + x \sin \beta)} = e^{x \cos \beta} e^{i(\alpha + x \sin \beta)}$$

$$= e^{x \cos \beta} [\cos (\alpha + x \sin \beta) + i \sin (\alpha + x \sin \beta)]$$

Equating imaginary parts from both sides, we have $S = e^{x \cos \beta} \sin (\alpha + x \sin \beta)$.

Series depending on logarithmic series

Example 19.46. *Sum the series*

$$\sin^2 \theta - \frac{1}{2} \sin 2\theta \sin^2 \theta + \frac{1}{3} \sin 3\theta \sin^3 \theta - \frac{1}{4} \sin 4\theta \sin^4 \theta + \ldots \infty.$$

(*P.T.U., 2010 ; V.T.U., 2006 S*)

Solution. Let $S = \sin\theta \,.\, \sin\theta - \frac{1}{2}\sin 2\theta \,.\, \sin^2\theta + \frac{1}{3}\sin 3\theta \,.\, \sin^3\theta - \ldots \infty$

and $C = \cos\theta \,.\, \sin\theta - \frac{1}{2}\cos 2\theta \,.\, \sin^2\theta + \frac{1}{3}\cos 3\theta \,.\, \sin^3\theta - \ldots \infty$

$$\therefore \quad C + iS = e^{i\theta}\sin\theta - \frac{e^{2i\theta}\sin^2\theta}{2} + \frac{e^{3i\theta}\sin^3\theta}{3} - \ldots \infty$$
$$= \log(1 + e^{i\theta}\sin\theta) = \log[1 + (\cos\theta + i\sin\theta)\sin\theta]$$
$$= \log[1 + \cos\theta\sin\theta + i\sin^2\theta] \text{ [Put } 1 + \cos\theta\sin\theta = r\cos\alpha \,;\, \sin^2\theta = r\sin\alpha] \quad \ldots(i)$$
$$= \log r(\cos\alpha + i\sin\alpha) = \log re^{i\alpha} = \log r + i\alpha$$

Equating imaginary parts, we have $S = \alpha = \tan^{-1}\left(\frac{\sin^2\theta}{1 + \cos\theta\sin\theta}\right)$. [from (*i*)]

Series depending on binomial series

Example 19.47. *Find the sum to infinity of the series*

$$1 - \frac{1}{2}\cos\theta + \frac{1.3}{2.4}\cos 2\theta - \frac{1.3.5}{2.4.6}\cos 3\theta + \ldots (-\pi < \theta < \pi).$$ (*S.V.T.U., 2009*)

Solution. Let $C = 1 - \frac{1}{2}\cos\theta + \frac{1.3}{2.4}\cos 2\theta - \frac{1.3.5}{2.4.6}\cos 3\theta + \ldots \infty$

and $S = 0 - \frac{1}{2}\sin\theta + \frac{1.3}{2.4}\sin 2\theta - \frac{1.3.5}{2.4.6}\sin 3\theta + \ldots \infty$

$$\therefore \quad C + iS = 1 - \frac{1}{2}e^{i\theta} + \frac{1.3}{2.4}e^{2i\theta} - \frac{1.3.5}{2.4.6}e^{3i\theta} - \ldots$$
$$= 1 + \left(-\frac{1}{2}\right)e^{i\theta} + \frac{-\frac{1}{2}\left(-\frac{1}{2} - 1\right)}{1.2}e^{2i\theta} + \frac{-\frac{1}{2}\left(-\frac{1}{2} - 1\right)\left(-\frac{1}{2} - 2\right)}{1.2.3}e^{3i\theta} + \ldots$$
$$= (1 + e^{i\theta})^{-1/2} = (1 + \cos\theta + i\sin\theta)^{-1/2} = \left(2\cos^2\frac{\theta}{2} + i\,.\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{-1/2}$$
$$= \left(2\cos\frac{\theta}{2}\right)^{-1/2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^{-1/2} = \left(2\cos\frac{\theta}{2}\right)^{-1/2}\left(\cos\frac{\theta}{4} - i\sin\frac{\theta}{4}\right).$$

Equating real parts, we have $C = (2\cos\theta/2)^{-1/2}\cos\theta/4$.

PROBLEMS 19.8

Sum the following series :

1. $\cos\theta + \sin\theta\cos 2\theta + \frac{\sin^2\theta}{1.2}\cos 3\theta + \ldots \infty.$ (*P.T.U., 2005*)
2. $\sin\alpha - \frac{\sin(\alpha + 2\beta)}{2!} + \frac{\sin(\alpha + 4\beta)}{4!} - \ldots \infty.$
3. $x\sin\theta - \frac{1}{2}x^2\sin 2\theta + \frac{1}{3}x^3\sin 3\theta - \ldots \infty.$ (*Kurukshetra, 2005*)
4. $\cos\theta - \frac{1}{2}\cos 2\theta + \frac{1}{3}\cos 3\theta \ldots \infty.$ (*S.V.T.U., 2006*)
5. $e^{\alpha}\cos\beta - \frac{e^{3\alpha}}{3}\cos 3\beta + \frac{e^{5\alpha}}{5}\cos 5\beta - \ldots \infty.$
6. $c\sin\alpha + \frac{c^3}{3}\sin 3\alpha + \frac{c^5}{5}\sin 5\alpha + \ldots \infty.$
7. $1 - \frac{1}{2}\cos 2\theta + \frac{1.3}{2.4}\cos 4\theta - \frac{1.3.5}{2.4.6}\cos 6\theta + \ldots \infty.$ (*Kurukshetra, 2006*)
8. $n\sin\alpha + \frac{n(n+1)}{1.2}\sin 2\alpha + \frac{n(n+1)(n+2)}{1.2.3}\sin 3\alpha + \ldots \infty.$
9. $\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \ldots \sin(\alpha + \overline{n-1}\,\beta)$ (*P.T.U., 2009 S*)
10. $\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \ldots$ to n terms. (*Kurukshetra, 2006*)
11. $\sin\alpha\cos\alpha + \sin^2\alpha\cos 2\alpha + \sin^3\alpha\cos 3\alpha + \ldots \infty.$
12. $1 + x\cos\theta + x^2\cos 2\theta + \ldots + x^{n-1}\cos(n-1)\theta.$

19.15 APPROXIMATIONS AND LIMITS

Example 19.48. *If* $\dfrac{\sin\theta}{\theta} = \dfrac{599}{600}$, *find an approximate value of* θ *in radians.*

Solution. Since $\dfrac{\sin\theta}{\theta} = 1 - \dfrac{1}{600}$ which is nearly equal to 1. $\therefore$ θ must be very small.

We know that $\sin\theta = \theta - \dfrac{\theta^3}{3!} + \dfrac{\theta^5}{5!} + \ldots$

$$\therefore \qquad \frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{6} + \frac{\theta^4}{5!}$$

Omitting θ^4 and higher powers, we have

$$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{6} = 1 - \frac{1}{600} \quad \text{or} \quad \theta^2 = \frac{1}{100}. \text{ Hence } \theta = 0.1 \text{ radians.}$$

Example 19.49. *Solve approximately* $\sin\left(\dfrac{\pi}{6} + \theta\right) = 0.51.$

Solution. Since 0.51 is nearly equal to 1/2, which is the value of $\sin \pi/6$, so θ must be very small.

$$\therefore \qquad \sin\left(\frac{\pi}{6} + \theta\right) = \sin\frac{\pi}{6}\cos\theta + \cos\frac{\pi}{6}\sin\theta = \frac{1}{2}\left(1 - \frac{\theta^2}{2!} + \ldots\right) + \frac{\sqrt{3}}{2}\left(\theta - \frac{\theta^3}{3!} + \ldots\right)$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2}\theta, \text{ omitting } \theta^2 \text{ and higher powers of } \theta.$$

Hence the given equation becomes,

$$\frac{1}{2} + \frac{\sqrt{3}}{2}\theta = 0.51 \quad \text{or} \quad \theta = \frac{1}{50\sqrt{3}}$$

or

$$\theta = \frac{1}{50\sqrt{3}} \text{ radian} = \frac{\sqrt{3}}{150} \times 57.29 \text{ degrees nearly} = 39.7'.$$

PROBLEMS 19.9

1. Given $\dfrac{\sin\theta}{\theta} = \dfrac{5045}{5046}$, show that θ is $1°58'$ nearly.
2. If $\dfrac{\sin\theta}{\theta} = \dfrac{2165}{2166}$, find an approximate value of θ in radians. (*Madras, 2003*)
3. If $\cos\theta = \dfrac{1681}{1682}$, find θ approximately.
4. Solve approximately the equation $\cos\left(\dfrac{\pi}{3} + \theta\right) = 0.49$.

19.16 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 19.10

Choose the correct answer or fill up the blanks in each of the following problems :

1. If $x + iy = \sqrt{2} + 3i$, then $x^2 + y$ is

   (*a*) 7 (*b*) 5 (*c*) 13 (*d*) $\sqrt{2} + 3$

2. The real part of $(\sin x + i\cos x)^5$ is

   (*a*) $-\cos 5x$ (*b*) $-\sin 5x$ (*c*) $\sin 5x$ (*d*) $\cos 5x$.

3. The number $(i)^i$ is
(a) a purely imaginary number (b) an irrational number
(c) a rational number (d) an integer.
4. The relation $|3-z|+|3+z|=5$ represents
(a) a circle (b) a parabola (c) an ellipse (d) a hyperbola.
5. z is a complex number with $|z|=1$ and arg $(z)=3\pi/4$. The value of z is
(a) $(1+i)/\sqrt{2}$ (b) $(-1+i)/\sqrt{2}$ (c) $(1-i)/\sqrt{2}$ (d) $(-1-i)/\sqrt{2}$.
6. If $f(z)=e^{2z}$, then the imaginary part of $f(z)$ is
(a) $e^y \sin x$ (b) $e^x \cos y$ (c) $e^{2x} \cos 2y$ (d) $e^{2x} \sin 2y$.
7. Expansion of $\sin^m \theta \cos^n \theta$ is a series of sines of multiples of θ when m is ......
8. Expansion of $\cos 6\theta$ in terms of $\cos \theta$ is ......
9. If $f(z)=3\bar{z}$, then the value of $f(z)$ at $z=2+4i$ is ......
10. If $x=\cos\theta+i\sin\theta$, then $x^n-1/x^n=$ ......
11. Imaginary part of $(2+i3)/(3-i4)$ is ......
12. Real part of $\cosh(x+iy)$ is ......
13. If $\dfrac{\sin\theta}{\theta}=\dfrac{2165}{2166}$, then $\theta=$ ...... approximately.
14. If $\tan x/2=\tanh y/2$, then $\cos x \cosh y=$ ......
15. Imaginary part of $\sin \bar{z}$ is ......
16. Modulus of $(\sqrt{i})^{\sqrt{i}}=$ ....
17. If $\sin\alpha+\sin\beta+\sin\gamma=0=\cos\alpha+\cos\beta+\cos\gamma$, then $\cos 3\alpha+\cos 3\beta+\cos 3\gamma=3\cos$ (.............)
18. $\log(-1)=$ ....
19. $(i)^i$ is purely real or imaginary
20. If $\sin\theta=\tanh\phi$, then $\tan\theta=$ ....
21. Imaginary part of $\tan(\theta+i\phi)=$ ....
22. $\cos 5\alpha=$ (.........) $\cos^5\alpha+$ (.....) $\cos^3\alpha+$ (.....) $\cos\alpha$.
23. Cube roots of unity form .... triangle.
24. If $|z_1+z_2|=|z_1-z_2|$ then amp $(z_1)-$ amp (z_2) is ....
25. If $-3+ix^2y$ and x^2+y+4i represent conjugate complex number then $x=$ .... and $y=$ ....
26. If $\left|\dfrac{z-a}{z-b}\right|=k\ (\neq 1)$, then the locus of z is ....
27. $(-i)^{-i}$ is purely real. (True or False)
28. The statements Re $z>0$ and $|z-1|<|z+1|$ are equivalent. *(Mumbai, 2007)* (True or False)
29. Hyperbolic functions are periodic. (True or False)
30. nth roots of unity form a G.P. (True or False)
31. $\sin ix=-i\sinh x$. *(Mumbai, 2008)* (True or False)
32. If the sum and product of two complex numbers are real, then the two numbers must be either real or conjugate. *(Mumbai, 2008)* (True or False)
33. The modulus of the sum of two complex numbers $\geq$ to the sum of their moduli. (True or False)

CHAPTER 20

Calculus of Complex Functions

20.1 INTRODUCTION

In the previous chapter, we have dealt with some elementary complex functions—the exponential, logarithmic, circular and hyperbolic functions, evaluated at specific complex values. These functions are useful in the study of fluid mechanics, thermodynamics and electric fields. It, therefore, seems desirable to study the calculus of such functions.

20.2 (1) LIMIT OF A COMPLEX FUNCTION

A function $w = f(z)$ is said to tend to **limit** *l as z approaches a point z_0, if for every real $\in$, we can find a positive real δ such that*

$$| f(z) - l | < \in \quad \text{for} \quad | z - z_0 | < \delta$$

i.e., for every $z \neq z_0$ in the δ–disc (dotted) of z-plane, $f(z)$ has a value lying in the $\in$-disc of w-plane (Fig. 20.1). In symbols, we write $\underset{z \to z_0}{\text{Lt}} f(z) = l$.

This definition of limit though similar to that in ordinary calculus, is quite different for in real calculus x approaches x_0 only along the line whereas here *z approaches z_0 from any direction in the z-plane.*

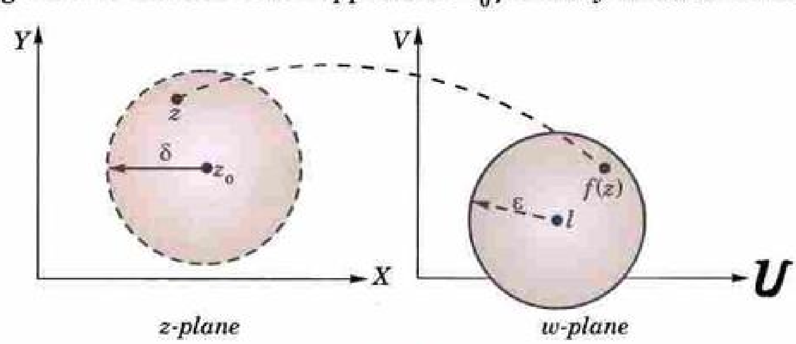

Fig. 20.1

(2) Continuity of f(z). *A function $w = f(z)$ is said to be* **continuous** *at $z = z_0$, if*

$$\underset{z \to z_0}{Lt} f(z) = f(z_0).$$

Further $f(z)$ is said to be continuous in any region R of the z-plane, if it is continuous at every point of that region.

Also if $w = f(z) = u(x, y) + iv(x, y)$ is continuous at $z = z_0$, then $u(x, y)$ and $v(x, y)$ are also continuous at $z = z_0$, *i.e.*, at $x = x_0$ and $y = y_0$. Conversely if $u(x, y)$ and $v(x, y)$ are continuous at (x_0, y_0), then $f(z)$ will be continuous at $z = z_0$. [cf. § 5.1 (3)].

20.3 (1) DERIVATIVE OF f(z)

Let $w = f(z)$ be a single-valued function of the variable $z = x + iy$. Then *the derivative of $w = f(z)$ is defined to be*

$$\frac{dw}{dz} = f'(z) = \underset{\delta z \to 0}{Lt} \frac{f(z + \delta z) - f(z)}{\delta z},$$

provided the limit exists and has the same value for all the different ways in which δz approaches zero.

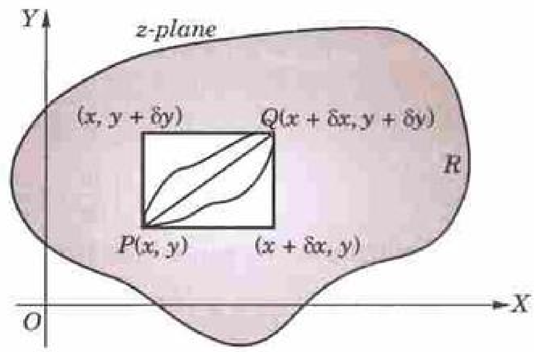

Fig. 20.2

Suppose $P(z)$ is fixed and $Q(z + \delta z)$ is a neighbouring point (Fig. 20.2). The point Q may approach P along any straight or curved path in the given region, *i.e.*, δz may tend to zero in any manner and dw/dz may not exist. It, therefore, becomes a fundamental problem to determine the necessary and sufficient conditions for dw/dz to exist. The fact is settled by the following theorem.

(2) Theorem. *The necessary and sufficient conditions for the derivative of the function $w = u(x, y) + iv(x, y) = f(z)$ to exist for all values of z in a region R, are*

(*i*) $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ *are continuous functions of x and y in R;*

(*ii*) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$.

The relations (ii) are known as **Cauchy-Riemann* equations** or briefly *C-R* equations.

(*a*) *Condition is necessary.*

If $f(z)$ possesses a unique derivative at $P(z)$, then

$$f'(z) = \underset{\delta z \to 0}{\text{Lt}} \frac{f(z + \delta z) - f(z)}{\delta z}$$

$$= \underset{\substack{\delta x \to 0 \\ \delta y \to 0}}{\text{Lt}} \frac{\{u(x + \delta x, y + \delta y) + iv(x + \delta x, y + \delta y)\} - \{u(x, y) + iv(x, y)\}}{\delta x + i\delta y}$$

Since δz can approach zero in any manner, we can first assume δz to be wholly real and then wholly imaginary. When δz is wholly real, then $\delta y = 0$ and $\delta z = \delta x$.

$$\therefore \quad f'(z) = \underset{\delta x \to 0}{\text{Lt}} \left(\frac{u(x + \delta x, y) - u(x, y)}{\delta x} + i \frac{v(x + \delta x, y) - v(x, y)}{\delta x} \right) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad ...(1)$$

When δz is wholly imaginary, then $\delta x = 0$ and $\delta z = i\delta y$.

$$\therefore \quad f'(z) = \underset{\delta y \to 0}{\text{Lt}} \left(\frac{u(x, y + \delta y) - u(x, y)}{i\delta y} + i \frac{v(x, y + \delta y) - v(x, y)}{i\delta y} \right) = \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\delta y} - i \frac{\partial u}{\partial y} \quad ...(2)$$

Now the existence of $f'(z)$ requires the equality of (1) and (2).

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* Named after *Cauchy* (p. 144) and the German mathematician *Bernhard Riemann* (1826—1866) who along with *Weierstrass* (p. 390) laid the foundations of complex analysis. Riemann introduced the concept of integration and made basic contributions to number theory and mathematical analysis. He developed the Riemannian geometry which formed the mathematical base for Einstein's relativity theory.

$$\therefore \qquad \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

On equating the real and imaginary parts from both sides, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad ...(3)$$

Thus the necessary conditions for the existence of the derivative of $f(z)$ is that the C-R equations should be satisfied. (V.T.U., 2011 S)

(*b*) *Condition is sufficient.* Suppose $f(z)$ is a single-valued function possessing partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ at each point of the region and the C-R equations (3) are satisfied.

Then by Taylor's theorem for functions of two variables (p. 220)

$$f(z + \delta z) = u(x + \delta x, y + \delta y) + iv(x + \delta x, y + \delta y)$$

$$= u(x, y) + \left(\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y\right) + ... + i\left[v(x, y) + \left(\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y\right) + ...\right]$$

$$= f(z) + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

[Omitting terms beyond the first powers of δx and δy]

or
$$f(z + \delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y.$$

Now using the C-R equation (3), replace $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ by $-\frac{\partial v}{\partial x}$ and $\frac{\partial u}{\partial x}$ respectively.

Then
$$f(z + \delta z) - f(z) = \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\delta x + \left[-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right]\delta y = \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\delta x + \left[i\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\right]i\delta y$$

$$= \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right](\delta x + i\delta y) = \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\delta z$$

$$\therefore \qquad f'(z) = \underset{\delta z \to 0}{\text{Lt}} \frac{f(z + \delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{or} \quad \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

which by (1) or (2) proves the sufficiency of conditions.

20.4 ANALYTIC FUNCTIONS

A function $f(z)$ which is single-valued and possesses a unique derivative with respect to z at all points of a region R, is called an **analytic function** *of z in that region. An analytic function is also called a regular function* or an *holomorphic function.*

A function which is analytic everywhere in the complex plane, is known as an **entire function.** As derivative of a polynomial exists at every point, a polynomial of any degree is an *entire function.*

A point at which an analytic function ceases to possess a derivative is called a **singular point** *of the function.*

Thus if u and v are real single-valued functions of x and y such that $\delta u/\delta x$, $\delta u/\delta y$, $\delta v/\delta x$, $\delta v/\delta y$ are continuous throughout a region R, then the Cauchy-Riemann equations

$$\frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \frac{\partial \mathbf{v}}{\partial \mathbf{y}} \text{ and } \frac{\partial \mathbf{u}}{\partial \mathbf{y}} = -\frac{\partial \mathbf{v}}{\partial \mathbf{x}} \qquad ...(1)$$

are both necessary and sufficient conditions for the function $f(z) = u + iv$ to be analytic in R. The derivative of $f(z)$ is then, given by (1) of p. 664 or (2) of p. 665.

The real and imaginary parts of an analytic function are called *conjugate functions.* The relation between two conjugate functions is given by C-R equation (1).

Example 20.1. *If $w = \log z$, find dw/dz and determine where w is non-analytic.*

(U.P.T.U., 2005 ; J.N.T.U., 2005)

Solution. We have $w = u + iv = \log(x + iy) = \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1} y/x$ [By **(2)**, p. 665]

so that $u = \frac{1}{2}\log(x^2 + y^2),\ v = \tan^{-1} y/x.$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{x}{x^2+y^2} = \frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2} = -\frac{\partial v}{\partial x}.$$

Since the Cauchy-Riemann equations are satisfied and the partial derivatives are continuous except at (0, 0). Hence w is analytic everywhere except at $z = 0$.

$$\therefore \quad \frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{1}{x+iy} = \frac{1}{z}\ (z \neq 0).$$

Obs. The definition of the derivative of a function of complex variable is identical in form to that of the derivative of a function of real variable. Hence the rules of differentiation for complex functions are the same as those of real calculus. *Thus* **if, a complex function is once known to be analytic, it can be differentiated just in the ordinary way.**

Example 20.2. *If $f(z)$ is an analytic function with constant modulus, show that $f(z)$ is constant.*
(U.P.T.U., 2008 ; Mumbai, 2005 S ; Madras 2003 ; Bhopal, 2002 S)

Solution. If $f(z) = u + iv$ is an analytic function, then

$| f(z) | = \sqrt{(u^2 + v^2)}$ is constant $= c$ (say) or $u^2 + v^2 = c^2$ ...(*i*)

Differentiating (*i*) partially w.r.t. x and y, we get

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0\ ; \qquad 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$$

or $$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \quad ...(ii) \qquad u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0 \quad ...(iii)$$

Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y};\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ by C-R equations,

$\therefore$ (*iii*) becomes $-u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} = 0$ ...(*iv*)

Squaring and adding (*ii*) and (*iv*), we obtain

$$u^2\left(\frac{\partial u}{\partial x}\right)^2 + v^2\left(\frac{\partial v}{\partial x}\right)^2 + u^2\left(\frac{\partial v}{\partial x}\right)^2 + v^2\left(\frac{\partial u}{\partial x}\right)^2 = 0$$

or $$(u^2+v^2)\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] = 0 \text{ or } \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = 0 \quad [\because\ u^2 + v^2 = c^2 \neq 0] \quad ...(v)$$

Now $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\therefore \quad | f'(z) |^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = 0 \qquad \text{[By } (v)]$$

or $f'(z) = 0$. or $f(z) =$ constant.

Example 20.3. *Show that the function $f(z) = \sqrt{|xy|}$ is not analytic at the origin even though C.R. equations are satisfied thereof.* *(A.M.I.E.T.E., 2005 S ; Osmania, 2003)*

Solution. If $f(z) = \sqrt{|xy|} = u(x, y) + iv(x, y)$, then $u(x, y) = \sqrt{|xy|}$, $v(x, y) = 0$

At the origin, we have

$$\frac{\partial u}{\partial x} = \underset{x \to 0}{\text{Lt}} \frac{u(x,0) - u(0,0)}{x} = \underset{x \to 0}{\text{Lt}} \frac{0-0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \underset{y \to 0}{\text{Lt}} \frac{u(0,y) - u(0,0)}{y} = \underset{x \to 0}{\text{Lt}} \frac{0-0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \underset{x \to 0}{\text{Lt}} \frac{v(x,0) - v(0,0)}{x} = \underset{x \to 0}{\text{Lt}} \frac{0-0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \underset{y \to 0}{\text{Lt}} \frac{v(0, y) - v(0, 0)}{y} = \underset{y \to 0}{\text{Lt}} \frac{0 - 0}{y} = 0$$

$\therefore$
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

i.e., C.R. equations are satisfied at the origin.

However
$$f'(0) = \underset{z \to 0}{\text{Lt}} \frac{f(z) - f(0)}{z - 0} = \underset{z \to 0}{\text{Lt}} \frac{\sqrt{|xy|} - 0}{x + iy}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\sqrt{|mx^2|} - 0}{x(1 + im)}, \text{ when } z \to 0 \text{ along the line } y = mx$$

$$= \frac{\sqrt{|m|}}{1 + im} \text{ which is not unique.}$$

$\therefore$ $f'(0)$ does not exist. Hence $f(z)$ is not analytic at the origin.

Example 20.4. *Prove that the function $f(z)$ defined by*

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} (z \neq 0), f(0) = 0$$

is continuous and the Cauchy-Riemann equations are satisfied at the origin, yet $f'(0)$ does not exist.

(S.V.T.U., 2009 ; V.T.U., 2001)

Solution.
$$\underset{z \to 0}{\text{Lt}} f(z) = \underset{\substack{x \to 0 \\ y \to 0}}{\text{Lt}} \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} = \underset{y \to 0}{\text{Lt}} \frac{-y^3(1-i)}{y^2} = \underset{y \to 0}{\text{Lt}} [-y(1-i)] = 0$$

$$\underset{z \to 0}{\text{Lt}} f(z) = \underset{\substack{y \to 0 \\ x \to 0}}{\text{Lt}} \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} = \underset{x \to 0}{\text{Lt}} \frac{x^3(1+i)}{x^2} = \underset{x \to 0}{\text{Lt}} [x(1+i)] = 0$$

Also $f(0) = 0$ (given).

Thus $\underset{z \to 0}{\text{Lt}} f(z) = f(0)$ when $x \to 0$ first and then $y \to 0$ and also vice-versa. Now let both x and y tend to zero simultaneously along the path $y = mx$. Then

$$\underset{z \to 0}{\text{Lt}} f(z) = \underset{\substack{y \to mx \\ x \to 0}}{\text{Lt}} \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{x^3(1+i) - m^3x^3(1-i)}{(1+m^2)x^2} = \underset{x \to 0}{\text{Lt}} \frac{x[1 + i - m^3(1-i)]}{1 + m^2} = 0$$

Hence $\underset{z \to 0}{\text{Lt}} f(z) = f(0)$, in whatever manner $z \to 0$. $\therefore$ $f(z)$ is continuous at the origin.

Now
$$f(z) = \frac{x^3 - y^3}{x^2 + y^2} + i \frac{x^3 + y^3}{x^2 + y^2} = u(x, y) + iv(x, y).$$

Also $u(0, 0) = 0$, and $v(0, 0) = 0$ $\quad [\because f(0) = 0]$

$\therefore$
$$\left(\frac{\partial u}{\partial x}\right)_{0,0} = \underset{x \to 0}{\text{Lt}} \frac{u(x, 0) - u(0, 0)}{x} = \underset{x \to 0}{\text{Lt}} \frac{x}{x} = 1$$

$$\left(\frac{\partial u}{\partial y}\right)_{0,0} = \underset{y \to 0}{\text{Lt}} \frac{u(0, y) - u(0, 0)}{y} = \underset{y \to 0}{\text{Lt}} \frac{-y}{y} = -1$$

$$\left(\frac{\partial v}{\partial x}\right)_{0,0} = \underset{x \to 0}{\text{Lt}} \frac{v(x, 0) - v(0, 0)}{x} = \underset{x \to 0}{\text{Lt}} \frac{x}{x} = 1$$

and
$$\left(\frac{\partial v}{\partial y}\right)_{0,0} = \underset{y \to 0}{\text{Lt}} \frac{v(0, y) - v(0, 0)}{y} = \underset{y \to 0}{\text{Lt}} \frac{y}{y} = 1.$$

Hence at (0, 0), $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

Thus the *C-R* equations are satisfied at the origin.

But $$f'(0) = \underset{z\to 0}{\text{Lt}} \frac{f(z)-f(0)}{z} = \underset{z\to 0}{\text{Lt}} \frac{f(z)}{z} = \underset{z\to 0}{\text{Lt}} \frac{x^3 - y^3 + i(x^3+y^3)}{(x^2+y^2)(x+iy)}.$$

If $z \to 0$ along the path $y = mx$, then $f'(0) = \dfrac{1-m^3+i(1+m^3)}{(1+m^2)(1+im)}$

which assumes different values as m varies. So $f'(z)$ is not unique at (0, 0) *i.e.*, $f'(0)$ does not exist. Thus $f(z)$ is not analytic at the origin even though it is continuous and satisfies the *C-R* equations thereat.

Example 20.5. *Show that polar form of Cauchy-Riemann equations are*

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$ (*U.P.T.U., 2008 ; V.T.U., 2006*)

Deduce that $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0.$ (*Bhopal, 2009 ; Kurukshetra, 2005*)

Solution. If (r, θ) be the coordinates of a point whose cartesian coordinates are (x, y), then $z = x + iy = re^{i\theta}$.

$\therefore$ $$u + iv = f(z) = f(re^{i\theta})$$

where u and v are now expressed in terms of r and θ.

Differentiating it partially w.r.t. r and θ, we have

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta}) \,.\, e^{i\theta}$$

and $$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta}) \,.\, ire^{i\theta} = ir\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

Equating real and imaginary parts, we get

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad ...(i) \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \quad ...(ii)$$

Differentiating (*i*) partially w.r.t. r, we get

$$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial \theta \partial r} \quad ...(iii)$$

Differentiating (*ii*) partially w.r.t. θ, we have

$$\frac{\partial^2 u}{\partial \theta^2} = -r\frac{\partial^2 v}{\partial r \partial \theta} \quad ...(iv)$$

Thus using (*i*), (*ii*) and (*iv*)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial \theta \partial r} + \frac{1}{r}\left(\frac{1}{r}\frac{\partial v}{\partial \theta}\right) + \frac{1}{r^2}\left(-r\frac{\partial^2 v}{\partial r \partial \theta}\right) = 0 \qquad \left[\because \frac{\partial^2 v}{\partial \theta \partial r} = \frac{\partial^2 v}{\partial r \partial \theta}\right]$$

20.5 (1) HARMONIC FUNCTIONS

If $f(z) = u + iv$ be an analytic function in some region of the z-plane, then the Cauchy-Riemann equations are satisfied.

i.e., $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad ...(1) \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \quad ...(2)$$

Differentiating (1) with respect to x and (2) with respect to y, we obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad ...(3) \qquad \text{and} \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}. \quad ...(4)$$

Adding (3) and (4) and assuming that $\dfrac{\partial^2 v}{\partial x \partial y} = \dfrac{\partial^2 v}{\partial y \partial x}$, we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \qquad ...(5)$$

Similarly, by differentiating (1) with respect to y and (2) with respect to x and subtracting, we obtain

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \qquad ...(6)$$

Thus both the functions u and v satisfy the Laplace's equation in two variables. For this reason, they are known as **harmonic functions** and their theory is called **potential theory.** *(Rohtak, 2005)*

(2) Orthogonal system. Consider the two families of curves

$$u(x, y) = c_1 \quad ...(7) \qquad \text{and} \qquad v(x, y) = c_2 \qquad ...(8)$$

Differentiating (7), we get $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} \cdot \dfrac{dy}{dx} = 0$

or
$$\frac{dy}{dx} = -\frac{\partial u/\partial x}{\partial u/\partial y} = \frac{\partial v/\partial y}{\partial v/\partial x} = m_1 \text{ (say)} \qquad \text{[By (1) and (2)]}$$

Similarly (8) gives $\dfrac{dy}{dx} = -\dfrac{\partial v/\partial x}{\partial v/\partial y} = m_2$ (say)

$\therefore \quad m_1 m_2 = -1$, *i.e.*, (7) and (8) form an orthogonal system.

Hence *every analytic function $f(z) = u + iv$ defines two families of curves $u(x, y) = c_1$ and $v(x, y) = c_2$, which form an orthogonal system.* *(U.P.T.U., 2009)*

20.6 APPLICATIONS TO FLOW PROBLEMS

As the real and imaginary parts of an analytic function are the solutions of the Laplace's equation in two variables, the conjugate functions provide solutions to a number of field and flow problems.

As an illustration, consider the irrotational motion of an incompressible fluid in two dimensions. Assuming the flow to be in planes parallel to the xy-plane, the velocity $\mathbf{V}$ of a fluid particle can be expressed as

$$\mathbf{V} = v_x \mathbf{I} + v_y \mathbf{J} \qquad ...(1)$$

Since the motion is irrotational, therefore, by § 6.18 (1), there exist a scalar function $\phi(x, y)$ such that

$$\mathbf{V} = \nabla\phi(x, y) = \frac{\partial \phi}{\partial x}\mathbf{I} + \frac{\partial \phi}{\partial y}\mathbf{J} \qquad ...(2)$$

[The function $\phi(x, y)$ is called the *velocity potential* and the curves $\phi(x, y) = c$ are known as *equipotential lines*.]

Thus from (1) and (2), $v_x = \dfrac{\partial \phi}{\partial x}$ and $v_y = \dfrac{\partial \phi}{\partial y}$ ...(3)

Also the fluid being incompressible div $\mathbf{V} = 0$ [by § 8.7 (1)] *i.e.*, $\dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} = 0.$

Substituting the values of v_x and v_y from (3), we get $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = 0$

which shows that the velocity potential ϕ is *harmonic*. It follows that there must exist a conjugate harmonic function $\psi(x, y)$ such that $w(z) = \phi(x, y) + i\psi(x, y)$ ...(4)
is analytic.

Also the slope at any point of the curve $\psi(x, y) = c'$ is given by

$$\frac{dy}{dx} = -\frac{\partial \psi/\partial x}{\partial \psi/\partial y} = \frac{\partial \phi/\partial y}{\partial \phi/\partial x} \qquad \text{[By } C\text{-}R \text{ equations]}$$

$$= v_y/v_x \qquad \text{[By (3)]}$$

This shows that the velocity of the fluid particle is along the tangent to the curve $\psi(x, y) = c'$, *i.e.* the particle moves along this curve. Such curves are known as *stream lines* and $\psi(x, y)$ is called the *stream function.* Also the equipotential lines $\phi(x, y) = c$ and the stream lines $\psi(x, y) = c'$ cut orthogonally.

From (4), $$\frac{\partial w}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y}$$ [By *C-R* equations]

$$= v_x - iv_y$$ [By (3)]

$\therefore$ The magnitude of the fluid velocity = $\sqrt{(v_x^2 + v_y^2)} = | dw/dz |$.

Thus the flow pattern is fully represented by the function $w(z)$ which is known as the **complex potential.**

Similarly the complex potential $w(z)$ can be taken to represent any other type of 2-dimensional steady flow. In electrostatics and gravitational fields, the curves $\phi(x, y) = c$ and $\psi(x, y) = c'$ are *equipotential lines* and *lines of force*. In heat flow problems, the curves $\phi(x, y) = c$ and $\psi(x, y) = c'$ are known as *isothermals* and *heat flow lines* respectively.

Given $\phi(x, y)$, we can find $\psi(x, y)$ and *vice-versa*.

Example 20.6. *If $w = \phi + i\psi$ respresents the complex potential for an electric field and $\psi = x^2 - y^2 + \dfrac{x}{x^2 + y^2}$, determine the function ϕ.* (V.T.U., 2011 ; Mumbai, 2008 ; Bhopal, 2002 S)

Solution. It is readily verified that ψ satisfies the Laplace's equation.

$\therefore$ ϕ and ψ must satisfy the Cauchy-Riemann equations :

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad ...(i) \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \quad ...(ii)$$

$\therefore$ by (*i*), $$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial y}\left[x^2 - y^2 + \frac{x}{x^2 + y^2}\right] = -2y - \frac{2xy}{(x^2 + y^2)^2}$$

Integrating w.r.t. x, we get $\phi = -2xy + \dfrac{y}{x^2 + y^2} + \eta(y)$ where $\eta(y)$ is an arbitrary function of y.

$\therefore$ (*ii*) gives $-2x + \dfrac{x^2 - y^2}{(x^2 + y^2)^2} + \eta'(y) = -2x + \dfrac{x^2 - y^2}{(x^2 + y^2)^2}$

whence $\eta'(y) = 0$, *i.e.*, $\eta(y) = c$, an arbitrary constant.

Thus $$\phi = -2xy + \frac{y}{x^2 + y^2} + c$$

Otherwise (Milne-Thomson's method*) :

We have $$\frac{dw}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} + i\frac{\partial \psi}{\partial x} = \left[-2y - \frac{2xy}{(x^2 + y^2)^2}\right] + i\left[2x + \frac{y^2 - x^2}{(x^2 + y^2)^2}\right]$$

By Milne-Thomson's method, we express dw/dz in terms of z, on replacing x by z and y by 0.

$$\therefore \quad \frac{dw}{dz} = i\left(2z - \frac{1}{z^2}\right)$$

Integrating w.r.t. z, we get $w = i\,(z^2 + 1/z) + A$ where A is a complex constant.

---

* Since $z = x + iy$ and $\bar{z} = x - iy$, we have

$$x = \frac{1}{2}(z + \bar{z}), \qquad y = \frac{1}{2i}(z + \bar{z})$$

$$\therefore \quad f(z) = \phi\,(x, y) + i\psi(x, y) \quad ...(1)$$

$$= \phi\left[\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right] + i\psi\left[\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right]$$

Now considering this as a formal identity in the two independent variables z, $\bar{z}$ and putting $\bar{z} = z$, we get

$$f(z) = \phi(z, 0) + i\psi(z, 0) \quad ...(2)$$

$\therefore$ (2) is the same as (1), if we replace x by z and y by 0.

Thus **to express any function in terms of z, replace x by z and y by 0.** *This provides an elegant method of finding $f(z)$ when its real part or the imaginary part is given. It is due to Milne-Thomson.*

Hence $$\phi = R\left[i\left(z^2 + \frac{1}{z}\right) + A\right] = -2xy + \frac{y}{x^2 + y^2} + c.$$

Example 20.7. *Find the analytic function, whose real part is* $\sin 2x/(\cosh 2y - \cos 2x)$.

(*J.N.T.U., 2005 ; Anna, 2003*)

Solution. Let $f(z) = u + iv$, where $u = \sin 2x/(\cosh 2y - \cos 2x)$

$$\therefore \quad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \text{[By } C\text{-}R \text{ equations]}$$

$$= \frac{(\cosh 2y - \cos 2x)\, 2\cos 2x - \sin 2x\,(2\sin 2x)}{(\cosh 2y - \cos 2x)^2} - i\,\frac{\sin 2x(-2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2\cos 2x\cosh 2y - 2}{(\cosh 2y - \cos 2x)^2} + i\,\frac{2\sin 2x\sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

By Milne-Thomson's method, we express $f'(z)$ in terms of z by putting $x = z$ and $y = 0$.

$$\therefore \quad f'(z) = \frac{2\cos 2z - 2}{(1 - \cos 2z)^2} + i(0) = \frac{-2}{1 - \cos 2z} = \frac{-2}{2\sin^2 z} = -\operatorname{cosec}^2 z$$

Integrating w.r.t. z, we get $f(z) = \cot z + ic$, taking the constant of integration as imaginary since u does not contain any constant.

Example 20.8. *Determine the analytic function* $f(z) = u + iv$, *if* $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$ *and* $f(\pi/2) = 0$.

(*A.M.I.E.T.E., 2005 ; Osmania, 2003*)

Solution. We have $$u - v = \frac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$$

$$\therefore \quad \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = \frac{(\sin x - \cos x)\cosh y + 1 - e^{-y}\sin x}{2(\cos x - \cosh y)^2} \qquad \ldots(i)$$

and $$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = \frac{(\cos x - \cosh y)\,e^{-y} + (\cos x + \sin x - e^{-y})\sinh y}{2(\cos x - \cosh y)^2}$$

or $$-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} = \frac{(\sin x + \cos x)\sinh y + e^{-y}(\cos x - \cosh y - \sinh y)}{2(\cos x - \cosh y)^2} \qquad \ldots(ii)$$

Subtracting (*ii*) from (*i*), we get

$$2\frac{\partial u}{\partial x} = \frac{(\sin x - \cos x)\cosh y - (\sin x + \cos x)\sinh y + 1 - e^{-y}(\sin x + \cos x - \cosh y - \sinh y)}{2(\cos x - \cosh y)^2}$$

Adding (*i*) and (*ii*), we have

$$-2\frac{\partial v}{\partial x} = \frac{(\sin x - \cos x)\cosh y + (\sin x + \cos x)\sinh y + 1 + e^{-y}(-\sin x + \cos x - \cosh y - \sinh y)}{2(\cos x - \cosh y)^2}$$

Thus $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{1 - \cos z}{2(1 - \cos z)^2} \qquad [\text{Putting } x = z \text{ and } y = 0]$$

$$= \frac{1}{2(1 - \cos z)} = \frac{1}{4\sin^2 z/2} = \frac{1}{4}\operatorname{cosec}^2\frac{z}{2} \quad \text{or} \quad f(z) = -\frac{1}{2}\cot\frac{z}{2} + c$$

Since $f(\pi/2) = 0$, $\quad 0 = -\dfrac{1}{2}\cot \pi/4 + c$, whence $c = \dfrac{1}{2}$

Hence $$f(z) = \frac{1}{2}\left(1 - \cot\frac{z}{2}\right).$$

Example 20.9. *Find the conjugate harmonic of $v(r, \theta) = r^2 \cos 2\theta - r \cos \theta + 2$. Show that v is harmonic.* (*Marathwada, 2008*)

Solution. Let $f(z) = u + v$. Using C-R equations in polar coordinates (Ex. 20.5),

$$r \frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} = -2r^2 \sin 2\theta + r \sin \theta \quad ...(i)$$

$$-\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r} = 2r \cos 2\theta - \cos \theta \quad ...(ii)$$

$\therefore$ (*i*) gives, $\dfrac{\partial u}{\partial r} = -2r \sin 2\theta + \sin \theta$

Integrating w.r.t., r

$$u = -r^2 \sin 2\theta + r \sin \theta + \phi(\theta)$$ where $\phi(\theta)$ is an arbitrary function.

$$\therefore \quad \frac{\partial u}{\partial \theta} = -2r^2 \cos 2\theta + r \cos \theta + \phi'(\theta) \quad ...(iii)$$

From (*ii*) and (*iii*), we get

$$-2r^2 \cos 2\theta + r \cos \theta = \frac{\partial u}{\partial \theta} = -2r^2 \cos 2\theta + r \cos \theta + \phi'(\theta)$$

$$\therefore \quad \phi'(\theta) = 0 \quad \text{or} \quad \phi(\theta) = c$$

Thus $u = -r^2 \sin 2\theta + r \sin \theta + c$ is the conjugate harmonic of v.

Now v will be harmonic if it satisfies the Laplace equation $\dfrac{\partial^2 v}{\partial r^2} + \dfrac{1}{r} + \dfrac{\partial v}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 v}{\partial \theta^2} = 0$

From (*i*), $\dfrac{\partial^2 v}{\partial \theta^2} = -4r^2 \cos 2\theta + r \cos \theta$. From (*ii*), $\dfrac{\partial^2 v}{\partial r^2} = 2 \cos 2\theta$

$$\therefore \quad \frac{\partial t^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 2 \cos 2\theta + \frac{1}{r}(2r \cos 2\theta - \cos \theta) + \frac{1}{r^2}(-4r^2 \cos 2\theta + r \cos \theta)$$

$$= 4 \cos 2\theta - \frac{1}{r}\cos \theta - 4 \cos 2\theta + \frac{1}{r}\cos \theta = 0$$

Hence v is harmonic.

Example 20.10. (*a*) *Find the orthogonal trajectories of the family of curves*

$$x^4 + y^4 - 6x^2y^2 = \textit{constant}$$

(*b*) *Show that the curves $r^n = \alpha \sec n\theta$ and $r^n = \beta \operatorname{cosec} n\theta$ cut orthogonally.*

(*Mumbai, 2005 ; J.N.T.U., 2003*)

Solution. (*a*) Take $u(x, y) = x^4 + y^4 - 6x^2y^2$. Then the family of curves $v(x, y)$ = constant will be the required trajectories if $f(z) = u + iv$ is analytic.

Now $\dfrac{\partial u}{\partial x} = 4x^3 - 12xy^2, \; \dfrac{\partial u}{\partial y} = 4y^3 - 12x^2y$

$$\therefore \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 4x^3 - 12xy^2$$

Integrating, $v = 4x^3y - 4xy^3 + c(x)$

Differentiating partially w.r.t. x

$$12x^2y - 4y^3 + \frac{dc(x)}{dx} = \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -4y^3 + 12x^2y$$

$$\therefore \quad \frac{dc(x)}{dx} = 0 \quad \text{or} \quad c = \text{constant}$$

Thus the required orthogonal trajectories are v = constant or $x^3y - xy^3$ = constant.

(*b*) Writing $u(r, \theta) = r^n \cos n\theta = \alpha$ and $v(r, \theta) = r^n \sin n\theta = \beta$,

we have $u(r, \theta) + iv(r, \theta) = \alpha + i\beta = r^n (\cos n\theta + i \sin n\theta) = r^n . e^{in\theta} = (re^{i\theta})^n = z^n$

This is an analytic function.

Thus $f(z) = u + iv$, gives the curves $u = \alpha$ and $v = \beta$ which cut orthogonally.

Example 20.11. *Two concentric circular cylinders of radii r_1, r_2 $(r_1 < r_2)$ are kept at potentials ϕ_1 and ϕ_2 respectively. Using complex function $w = a \log z + c$, prove that the capacitance per unit length of the capacitor formed by them is $2\pi\lambda / \log (r_2/r_1)$ where λ is the dielectric constant of the medium.*

Solution. We have $\quad \phi + i\psi = a \log (re^{i\theta}) + c \quad$ where $z = x + iy = re^{i\theta}$

$$\therefore \qquad \phi = a \log r + c, \quad \text{and} \quad \psi = a\theta$$

so that
$$\phi_1 = a \log r_1 + c, \quad \phi_2 = a \log r_2 + c$$

Thus the potential difference $= \phi_2 - \phi_1 = a (\log r_2 - \log r_1)$

Also the total charge (or flux) $= \int_0^{2\pi} d\psi = \int_0^{2\pi} a\, d\theta = 2\pi a.$

The capacitance being the charge required to maintain a unit potential difference ; the capacitance without dielectric

$$= \frac{\text{charge}}{\text{potential difference}} = \frac{2\pi a}{a(\log r_2 - \log r_1)} = \frac{2\pi}{\log (r_2/r_1)}.$$

A medium of dielectric constant λ increases the potential difference to λ times that in vacuum for the same charge. Thus the capacitance with dielectric $= 2\pi\lambda/\log (r_2/r_1)$.

Example 20.12. *If $f(z)$ is a regular function of z, prove that*

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) | f(z) |^2 = 4 \,| f'(z) |^2. \qquad \textit{(J.N.T.U., 2006 ; Kottayam, 2005)}$$

or
$$\nabla^2 \,| f(z) |^2 = 4 \,| f'(z) |^2 \qquad \textit{(Madras, 2006)}$$

Solution. Let $f(z) = u(x, y) + iv(x, y)$ so that $| f(z) |^2 = u^2 + v^2 = \phi(x, y)$, (say).

$$\therefore \qquad \frac{\partial \phi}{\partial x} = 2u \frac{\partial u}{\partial x} + 2v \frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial^2 \phi}{\partial x^2} = 2\left\{u \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + v \frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2\right\}$$

Similarly,
$$\frac{\partial^2 \phi}{\partial y^2} = 2\left\{u \frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2 + v \frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2\right\}$$

Adding, we have

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2\left\{u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)\right\} + 2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right\} \qquad ...(i)$$

Since u, v have to satisfy Cauchy-Riemann equations and the Laplace's equation.

$$\therefore \qquad \left(\frac{\partial u}{\partial x}\right)^2 = \left(\frac{\partial v}{\partial y}\right)^2 ; \left(\frac{\partial u}{\partial y}\right)^2 = \left(-\frac{\partial v}{\partial x}\right)^2 \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}.$$

Thus (i) takes the form $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = 4\left\{\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial v}{\partial x}\right)^2\right\}$

Hence
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) | f(z) |^2 = 4 \,| f'(z) |^2 \quad \text{or} \quad \nabla^2 \,| f(z) |^2 = 4 \,| f'(z) |^2.$$

PROBLEMS 20.1

1. If $f(z) = \begin{cases} x^3 y(y - ix)/(x^6 + y^2), & z \neq 0 \\ 0 & , z = 0 \end{cases}$ prove that $|f(z) - f(0)|/z \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ along the curve $y = ax^3$.

2. Show that (*a*) $f(z) = xy + iy$ is everywhere continuous but is not analytic. (*Osmania, 2003 S*)
(*b*) $f(z) = z + 2\bar{z}$ is not analytic anywhere in the complex plane. (*J.N.T.U., 2003*)

3. If $f(z) = u + iv$ is analytic, then show that $|f'(z)|^2 = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$. (*Mumbai, 2007*)

4. Find the constants a, b, c, d and e if $f(z) = (ax^4 + bx^2y^2 + cy^4 + dx^2 - 2y^2) + i(4x^3y - exy^3 + 4xy)$ is analytic. (*Mumbai, 2008*)

5. Show that z^n is analytic. Hence find its derivative. (*V.T.U., 2010 S*)

6. Determine which of the following functions are analytic :
(*i*) $2xy + i(x^2 - y^2)$ (*ii*) $(x - iy)/(x^2 + y^2)$ (*iii*) $\cosh z$.

7. (*a*) Determine p such that the function $f(z) = \frac{1}{2}\log_e(x^2 + y^2) + i\tan^{-1}(px/y)$ be an analytic function. (*Mumbai, 2007 ; J.N.T.U., 2003*)
(*b*) Show that $u = \frac{1}{2}\log(x^2 + y^2)$ is harmonic and find its harmonic conjugate function. (*U.P.T.U., 2010*)

8. Show that each of the following functions is not analytic at any point :
(*i*) $\bar{z}$ (*J.N.T.U., 2003*) (*ii*) $|z|^2$.

9. Show that $u + iv = (x - iy)/(x - iy + a)$ where $a \neq 0$, is not an analytic function of $z = x + iy$ whereas $u - iv$ is such a function.

10. Show that $f(z) = \begin{cases} xy^2(x + iy) \div (x^2 + y^4), & z \neq 0 \\ 0, & z = 0 \end{cases}$
is not analytic at $z = 0$, although C-R equations are satisfied at the origin. (*J.N.T.U., 2003*)

11. Verify if $f(z) = \dfrac{xy^2(x + iy)}{x^2 + y^4}, z \neq 0; f(0) = 0$ is analytic or not. (*U.P.T.U., 2008*)

12. Examine the nature of the function $f(z) = \dfrac{x^2y^5(x + iy)}{x^4 + y^{10}}, z \neq 0 ; f(0) = 0$. (*Rohtak, 2004*)

13. For the function $f(z)$ defined by $f(z)^2 = (\bar{z})^2/z, z \neq 0; f(0) = 0$, show that the C-R equations are satisfied at (0, 0), but $f(z)$ is not differentiable at (0, 0). (*P.T.U., 2010*)

14. Determine the analytic function whose real part is
(*i*) $x^3 - 3xy^2 + 3x^2 - 3y^2$ (*Bhopal, 2009*) (*ii*) $\cos x \cosh y$ (*Rohtak, 2004*)
(*iii*) $y/(x^2 + y^2)$ (*iv*) $y + e^x \cos y$ (*S.V.T.U., 2008 ; V.T.U., 2006*)
(*v*) $e^{-x}(x \sin y - y \cos y)$ (*U.P.T.U., 2008*)
(*vi*) $e^{2x}(x \cos 2y - y \sin 2y)$ (*V.T.U., 2008 S ; Mumbai, 2005 ; Kottayam, 2005*)
(*vii*) $x \sin x \cosh y - y \cos x \sinh y$ (*V.T.U., 2006*)
(*viii*) $e^x[(x^2 - y^2)\cos y - 2xy \sin y]$. (*V.T.U., 2010 S ; Rohtak, 2005*)

15. Find the regular function whose imaginary part is
(*i*) $(x - y)/(x^2 + y^2)$ (*ii*) $-\sin x \sinh y$ (*iii*) $e^x \sin y$
(*iv*) $e^{-x}(x \sin y - y \cos y)$ (*v*) $e^{-x}(x \cos y + y \sin y)$ (*U.P.T.U., 2009*) (*vi*) $\dfrac{2 \sin x \sin y}{\cos 2x + \cosh 2y}$. (*Mumbai, 2006*)

16. Find the analytic function $z = u + iv$, if
(*i*) $u - v = (x - y)(x^2 + 4xy + y^2)$ (*Mumbai, 2008 ; V.T.U., 2007 ; W.B.T.U., 2005*)
(*ii*) $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - e^y - e^{-y}}$ when $f\left(\dfrac{\pi}{2}\right) = 0$ (*Mumbai, 2007*)
(*iii*) $u + v = \dfrac{2 \sin 2x}{e^{2y} - e^{-2y} - 2\cos 2x}$. (*P.T.U., 2002*)

17. An electrostatic field in the xy-plane is given by the potential function $\phi = 3x^2y - y^3$, find the stream function.

18. If the potential function is $\log(x^2 + y^2)$, find the flux function and the complex potential function.

19. Prove that $u = x^2 - y^2$ and $v = \dfrac{y}{x^2 + y^2}$ are harmonic functions of (x, y) but are not harmonic conjugates. (*U.P.T.U., 2004 S*)

20. Show that the function $u = e^{-2xy} \sin (x^2 - y^2)$ is harmonic. Find the conjugate function v and express $u + iv$ as an analytic function of z. *(Bhopal, 2007)*
21. For $w = \exp (z^2)$, find u and v, and prove that the curves $u(x, y) = c_1$ and $v(x, y) = c_2$ where c_1 and c_2 are constants, cut orthogonally. *(J.N.T.U., 2003)*
22. Find the orthogonal trajectories of the family of curves
(*i*) $x^3y - xy^3 = c$ *(Mumbai, 2007)* (*ii*) $e^x \cos y - xy = c$ *(Mumbai, 2008)* (*iii*) $r^2 \cos 2\theta = c$.
23. In a two dimensional fluid flow, the stream function ψ is given, find the velocity potential ϕ :
(*i*) $\psi = -y/(x^2 + y^2)$ (*ii*) $\psi = \tan^{-1} (y/x)$.
24. Find the analytic function $f(z) = u + iv$, given
(*i*) $u = a (1 + \cos \theta)$ (*ii*) $v = (r - 1/r) \sin \theta, r \neq 0$.
25. If $f(z)$ is an analytic function of z, show that
$$\left\{\frac{\partial}{\partial x} | f(z) |\right\}^2 + \left\{\frac{\partial}{\partial y} | f(z) |\right\}^2 = | f'(z) |^2.$$ *(U.P.T.U., 2009 ; V.T.U., 2008 S ; P.T.U., 2005)*
26. If $f(z)$ is an analytic function of z, prove that
(*i*) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \log | f(z) | = 0$ *(Madras, 2000 S)* (*ii*) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) | R f(z) |^2 = 2 | f'(z) |^2$
(*iii*) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) | f(z) |^p = p^2 | f'(z) |^2 | f(z) |^{p-2}$. *(Kerala, 2005)*
27. Prove that $\psi = \log [(x - 1)^2 + (y - 2)^2]$ is harmonic in every region which does not include the point (1, 2). Find a function ϕ such that $\phi + i\psi$ is an analytic function of the complex variable $z = x + iy$. Express $\phi + i\psi$ as a function of z.

20.7 GEOMETRICAL REPRESENTATION OF w = f(z)

To find the geometrical representation of a function of a complex variable, it requires a departure from the usual practice of cartesian plotting, where we associate a curve to a real function $y = f(x)$.

In the complex domain, the function $w = f(z)$

i.e., $$u + iv = f(x + iy) \quad ...(1)$$

involves four real variables x, y, u, v. Hence a four dimensional region is required to plot (1) in the cartesian fashion. As it is not possible to have 4-dimensional graph papers, we make use of two complex planes, one for the variable $z = x + iy$, and the other for the variable $w = u + iv$. If the point z describes some curve C in the z-plane, the point w will move along a corresponding curve C' in the w-plane, since to each point (x, y), there corresponds a point (u, v) (Fig. 20.3). We then, say that *a curve C in the z-plane is mapped into the corresponding curve C′ in the w-plane by the function w = f (z) which defines a* **mapping** *or* **transformation** *of the z-plane into the w-plane.*

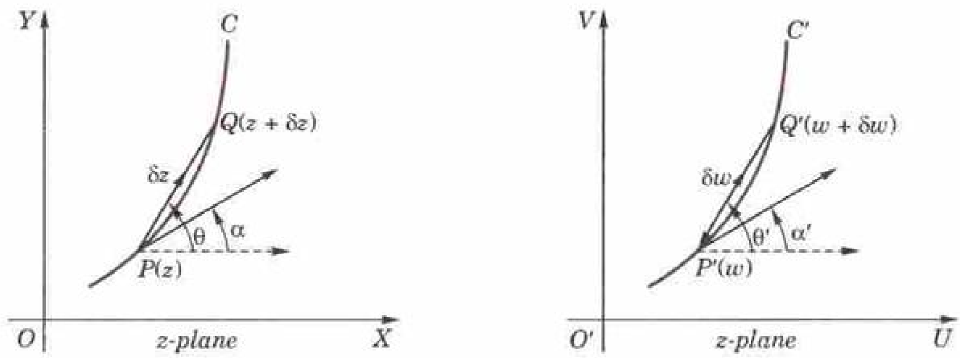

Fig. 20.3

20.8 SOME STANDARD TRANSFORMATIONS

(1) Translation. $w = z + c$, where c is a complex constant.

If $z = x + iy$, $c = c_1 + ic_2$ and $w = u + iv$, then the transformation becomes $u + iv = x + iy + c_1 + ic_2$ whence $u = x + c_1$ and $v = y + c_2$, *i.e.* the point $P(x, y)$ in the z-plane is mapped onto the point $P'(x + c_1, y + c_2)$ in the

w-plane. Every point in the z-plane is mapped onto w-plane in the same way. Thus if the w-plane is superposed on the z-plane, figure is shifted through a distance given by the vector c. Accordingly, this transformation maps a figure in the z-plane into a figure in the w-plane of the same shape and size.

In particular, this transformation changes circles into circles.

(2) Magnification and rotation. w = cz, where c is a complex constant.

If $c = \rho e^{i\alpha}$, $z = re^{i\theta}$ and $w = Re^{i\phi}$, then

$$Re^{i\phi} = \rho e^{i\alpha} \cdot re^{i\theta} = \rho re^{i(\theta+\alpha)}$$

whence $R = \rho r$ and $\phi = \theta + \alpha$, *i.e.* the point $P(r, \theta)$ in the z-plane is mapped onto the point $P'(\rho r, \theta + \alpha)$ in the w-plane. Hence the transformation consists of magnification (or contraction) of the radius vector of P by $\rho = |c|$ and its rotation through an $\angle\alpha = \text{amp}(c)$. Accordingly any figure in the z-plane is transformed into a geometrically similar figure in the w-plane. In particular, this transformation maps circles into circles.

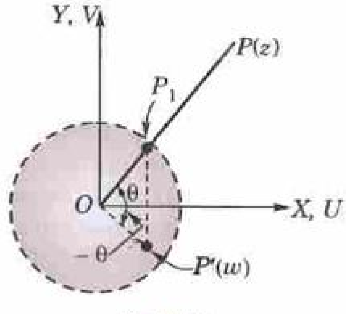

Fig. 20.4

(3) Inversion and reflection. w = 1/z.

Here it is convenient to think the w-plane as superposed on z-plane (Fig. 20.4).

If $z = re^{i\theta}$ and $w = Re^{i\phi}$, then $Re^{i\phi} = \dfrac{1}{r} e^{-i\theta}$

whence $R = 1/r$ and $\phi = -\theta$. Thus, if P be (r, θ) and P_1 be $(1/r, \theta)$, *i.e.* P_1 is the inverse* of P w.r.t. the unit circle with centre O, then the reflection P' of P_1 in the real axis represents $w = 1/z$.

Hence this transformation is an inversion of z w.r.t. the unit circle $|z| = 1$ followed by reflection of the inverse into the real axis.

Obs. 1. Clearly the function $w = 1/z$ maps the interior of the unit circle $|z| = 1$ onto the exterior of the unit circle $|w| = 1$ and the exterior of $|z| = 1$ onto the interior of $|w| = 1$. In particular, the origin $z = 0$ corresponds to the improper point $w = \infty$, called the *point at infinity* and the image of the improper point $z = \infty$ is the origin $w = 0$.

2. *This transformation maps a circle onto a circle or to a straight line if the former goes through the origin.*

To prove this, we write $z = 1/w$ as $x + iy = \dfrac{1}{u+iv} = \dfrac{u-iv}{u^2+v^2}$

so that
$$x = \frac{u}{u^2+v^2} \text{ and } y = \frac{-v}{u^2+v^2}. \qquad ...(1)$$

Now the general equation of any circle in the z-plane is
$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad ...(2)$$

which on substituting from (1), becomes $\dfrac{u^2}{(u^2+v^2)^2} + \dfrac{v^2}{(u^2+v^2)^2} + 2g\dfrac{u}{u^2+v^2} + 2f\dfrac{-v}{u^2+v^2} + c = 0$

or
$$c(u^2 + v^2) + 2gu - 2fv + 1 = 0 \qquad ...(3)$$

This is the equation of a circle in the w-plane. If $c = 0$, the circle (2) passes through the origin and its image, *i.e.*, (3) reduces to a straight line. Hence the result.

Regarding a straight line as the limiting form of a circle with infinite radius, we conclude that the *transformation $w = 1/z$ always maps a circle into a circle.*

(4) Bilinear transformation. The transformation

$$w = \frac{az+b}{cz+d} \qquad ...(1)$$

where a, b, c and d are complex constants and $ad - bc \neq 0$ is known as the **bilinear transformation.**** The condition $ad - bc \neq 0$ ensures that $dw/dz \neq 0$, *i.e.*, the transformation is conformal. If $ad - bc = 0$ every point of the z-plane is a *critical point*.

The inverse mapping of (1) is

$$z = \frac{-dw+b}{cw-a} \qquad ...(2)$$

which is also a bilinear transformation.

* The inverse of a point A w.r.t. a circle with centre O and radius k is defined as the point B on the line OA such that $OA.OB = k^2$.

** First studied by *Möbius* (p. 337). Hence, sometimes called *Möbius transformation*

Obs. 1. From (1), we see that each point in the z-plane except $z = -d/c$, corresponds a unique point in the w-plane. Similarly, (2) shows that each point in the w-plane except $w = a/c$, maps into a unique point in the z-plane. Including the images of the two exceptional points as the infinite points in the two planes, it follows that *there is one to one correspondence between all points in the two planes.*

Obs. 2. Invariant points of bilinear transformation. If z maps into itself in the w-plane (*i.e.*, $w = z$), then (1) gives

$$z = \frac{az+b}{cz+d} \quad \text{or} \quad cz^2 + (d-a)z - b = 0$$

The roots of this equation (say : z_1, z_2) *are defined as the invariant or fixed points of the bilinear transformation (1).*

If however, the two roots are equal, the bilinear transformation is said to be *parabolic.*

Obs. 3. Dividing the numerator and denominator of the right side of (1) by one of the four constants, it is clear that (1) has only three essential arbitrary constants. Hence *three conditions are required to determine a bilinear transformation.* For instance, three distinct points z_1, z_2, z_3 can be mapped into any three specified points w_1, w_2, w_3.

Two important properties :

I. A bilinear transformation maps circles into circles.

By actual division, (1) can be written as $w = \dfrac{a}{c} + \dfrac{bc-ad}{c^2} \cdot \dfrac{1}{z+d/c}$

which is a combination of the transformations

$$w_1 = z + d/c,\ w_2 = 1/w_1,\ w_3 = \frac{bc-ad}{c^2} w_2,\ w = \frac{a}{c} + w_3 .$$

By these transformations, we successively pass from z-plane to w_1-plane, from w_1-plane to w_2-plane, from w_2-plane to w_3-plane and finally from w_3-plane to w-plane. Now each of these transformations is one or other of the standard transformations $w = z + c$, $w = cz$, $w = 1/z$ and under each of these a circle always maps onto a circle. Hence the bilinear transformation maps circles into circles.

II. A bilinear transformation preserves cross-ratio† *of four points.*

Let the points z_1, z_2, z_3, z_4 of the z-plane map onto the points w_1, w_2, w_3, w_4 of the w-plane respectively under the bilinear transformation (1). If these points are finite, then from (1), we have

$$w_j - w_k = \frac{az_j+b}{cz_j+d} - \frac{az_k+b}{cz_k+d} = \frac{ad-bc}{(cz_j+d)(cz_k+d)}(z_j - z_k).$$

Using this relation for $j, k = 1, 2, 3, 4$, we get $\dfrac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)} = \dfrac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$

Thus the cross-ratio of four points is invariant under bilinear transformation.

This property is very useful in finding a bilinear transformation. If one of the points, say : $z_1 \to \infty$, the quotient of those two differences which contain z_1, is replaced by 1 *i.e.*,

$$\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)} = \frac{z_3-z_4}{z_3-z_2}.$$

Example 20.13. *Find the bilinear transformation which maps the points $z = 1, i, -1$ onto the points $w = i, 0, -i$.*

Hence find (a) the image of $|z| < 1$, *(Mumbai, 2006 ; Delhi, 2002)*

(b) the invariant points of this transformation. *(U.P.T.U., 2008 ; V.T.U., 2000)*

Solution. Let the points $z_1 = 1, z_2 = i, z_3 = -1$ and $z_4 = z$ map onto the points $w_1 = i, w_2 = 0, w_3 = -i$ and $w_4 = w$. Since the cross-ratio remains unchanged under a bilinear transformation.

$$\therefore \quad \frac{(1-i)(-1-z)}{(1-z)(-1-i)} = \frac{(i-0)(-i-w)}{(i-w)(-i-0)} \quad \text{or} \quad \frac{w+i}{w-i} = \frac{(z+1)(1-i)}{(z-1)(1+i)}$$

By componendo dividendo, we get $\dfrac{2w}{2i} = \dfrac{(z+1)(1-i)+(z-1)(1+i)}{(z+1)(1-i)-(z-1)(1+i)}$

† **Def.** *If t_1, t_2, t_3, t_4 be any four numbers, then* $\dfrac{(t_1-t_2)(t_3-t_4)}{(t_1-t_4)(t_3-t_2)}$ *is said to be their cross-ratio and is denoted* (t_1, t_2, t_3, t_4).

$\therefore \qquad w = \dfrac{1 + iz}{1 - iz}$ ...(i)

which is the required bilinear transformation.

(a) Rewriting (i) as $z = i\,\dfrac{1 - w}{1 + w}$

$\therefore \qquad \left|\dfrac{i(1 - w)}{1 + w}\right| = | z | < 1 \quad \text{or} \quad | i |\, | 1 - w | < | 1 + w |$

or $\qquad | 1 - u - iv | < | 1 + u + iv |$ $\qquad [\because \; | i | = 1]$

or $\qquad (1 - u)^2 + v^2 < (1 + u)^2 + v^2$ which reduces to $u > 0$.

Hence the interior of the circle $x^2 + y^2 = 1$ in the z-plane is mapped onto the entire half of the w-plane to the right of the imaginary axis.

(b) To find the invariant points of the transformation, we put $w = z$ in (i).

$\therefore \qquad z = \dfrac{1 + iz}{1 - iz} \quad \text{or} \quad iz^2 + (i - 1)z + 1 = 0$

or $\qquad z = \dfrac{1 - i \pm \sqrt{[(i-1)^2 - 4i]}}{2i} = -\tfrac{1}{2}\{1 + i \pm \sqrt{(6i)}\}$

which are the required invariant points.

Example 20.14. *Show that $w = \dfrac{i - z}{i + z}$ maps the real axis of z-plane into the circle $| w | = 1$ and the half plane $y > 0$ into the interior of the unit circle $| w | = 1$ in the w-plane.* (Mumbai, 2007)

Solution. Since $\qquad w = (i - z)/(i + z)$,

$\therefore \qquad | w | = 1$ becomes $| (i - z)/(i + z) | = 1 \quad \text{or} \quad | i - z | = | i + z |$

i.e., $\qquad | i - x - iy | = | i + x + iy | \quad \text{or} \quad | -x + i(1 - y) | = | x + i(1 + y) |$

$\therefore \qquad \sqrt{[x^2 + (1 - y)^2]} = \sqrt{(x^2 + (1 + y)^2]} \text{ or } (1 - y)^2 = (1 + y)^2$

$\therefore \qquad 4y = 0 \quad \text{or} \quad y = 0$ which is the real axis.

Hence the real axis of the z-plane is mapped to the circle $| w | = 1$

Now for the interior of the circle $| w | = 1$

$| w | < 1 \quad i.e., \quad | i - z | < | i + z | \quad \text{or} \quad (1 - y)^2 < (1 + y)^2$

$\therefore \qquad -4y < 0 \quad i.e., \quad y > 0$

Hence the half plane $y > 0$ is mapped into the interior of the circle $| w | = 1$.

PROBLEMS 20.2

1. Find the invariant points of the transformation $w = (z - 1)/(z + 1)$. (Madras, 2003)
2. Find the transformation which maps the points $-1, i, 1$ of the z-plane onto $1, i, -1$ of the w-plane respectively. Also find its *invariant points*. (V.T.U., 2011)
3. Find the bilinear transformation which maps $1, i, -1$ to $2, i, -2$ respectively. Find the fixed and critical points of the transformation. (S.V.T.U., 2008 ; Mumbai, 2007 ; V.T.U., 2006)
4. Determine the bilinear transformation that maps the points $1 - 2i$, $2 + i$, $2 + 3i$ respectively into $2 + 2i$, $1 + 3i$, 4. (J.N.T.U., 2003 ; Coimbatore, 1999)
5. Find the bilinear transformation which maps
   (i) the points $z = 1, i, -1$ into the points $w = 0, 1, \infty$ (V.T.U., 2008 ; Mumbai, 2007)
   (ii) the points $z = 0, 1, i$ into the points $w = 1 + i, -i, 2 - i$ (V.T.U., 2010 S)
   (iii) $R(z) > 0$ into interior of unit circle so that $z = \infty, i, 0$ map into $w = -1, -i, 1$.
6. Under the transformation $w = \dfrac{z - 1}{z + 1}$, show that the map of the straight line $x = y$ is a circle and find its centre and radius. (Marathwada, 2008)

7. Show that the bilinear transformation $w = (2z + 3)/(z - 4)$ maps the circle $x^2 + y^2 - 4x = 0$ into the line $4u + 3 = 0$. *(Mumbai, 2007 ; J.N.T.U., 2003 ; Bhopal, 2002)*
8. Show that the condition for transformation $w = (az + b)/(cz + d)$ to make the circle $|w| = 1$ correspond to a straight line in the z-plane is $|a| = |c|$.
9. Show that the transformation $w = i(1 - z)/(1 + z)$ maps the circle $|z| = 1$ into the real axis of the w-plane and the interior of the circle $|z| < 1$ into the upper half of the w-plane. *(Osmania, 2003 S ; V.T.U., 2001)*
10. If z_0 is the upper half of the z-plane, show that the bilinear transformation $w = e^{i\alpha}\left(\frac{z - z_0}{z - \bar{z}_0}\right)$ maps the upper half of the z-plane into the interior of the unit circle at the origin in the w-plane.

20.9 (1) CONFORMAL TRANSFORMATION

Suppose two curves C, C_1 in the z-plane intersect at the point P and the corresponding curves C' and C_1' in the w-plane intersect at P' (Fig. 20.5). *If the angle of intersection of the curves at P is the same as the angle of intersection of the curves at P' in magnitude and sense, then the transformation is said to be* **conformal.**

(2) Theorem. *The transformation effected by an analytic function $w = f(z)$ is conformal at every point of the z-plane where $f'(z) \neq 0$.*

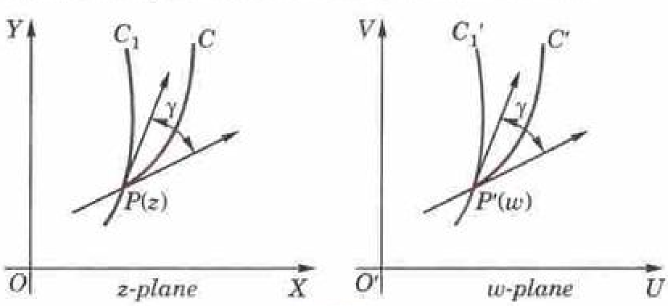

Fig. 20.5

Let $P(z)$ be a point in the region R of the z-plane and $P'(w)$ the corresponding point in the region R' of the w-plane (Fig. 20.3). Suppose z moves on a curve C and w moves on the corresponding curve C'. Let $Q(z + \delta z)$ be a neighbouring point on C and $Q'(w + \delta w)$ be the corresponding point on C' so that $\vec{PQ} = \delta z$ and $\vec{P'Q'} = \delta w$. Then δz is a complex number whose modulus r is the length PQ and amplitude θ is the angle which PQ makes with the x-axis.

$$\therefore \qquad \delta z = re^{i\theta}$$

Similarly, if the modulus and amplitude of δw be r' and θ', then $\delta w = r'e^{i\theta'}$.

Hence $$\frac{\delta w}{\delta z} = \frac{r'}{r} e^{i(\theta' - \theta)}$$

Now if the tangent at P to the curve C makes an $\angle\alpha$ with the x-axis and the tangent at P' to C' makes an $\angle\alpha'$ with the u-axis, then as $\delta z \to 0$, $\theta \to \alpha$ and $\theta' \to \alpha'$.

$$\therefore \qquad f'(z) = \frac{dw}{dz} = \left(\text{Lt}\, \frac{r'}{r}\right) . e^{i(\alpha' - \alpha)} \qquad \text{...(1)}$$

If ρ is the modulus and ϕ the amplitude of the function $f(z)$ which is supposed to be non-zero, then

$$f'(z) = \rho e^{i\phi} \qquad \text{...(2)}$$

$\therefore$ from (1) and (2), we have $$\rho = \text{Lt}\, \frac{r'}{r} \qquad \text{...(3)}$$

and $$\phi = \alpha' - \alpha. \qquad \text{...(4)}$$

Now let C_1 be another curve through P in the z-plane and C_1' the corresponding curve through P' in the w-plane. If the tangent at P to C_1 makes an $\angle\beta$ with the x-axis and tangent at P' to C_1' makes an $\angle\beta'$ with the u-axis, then as in (4),

$$\psi = \beta' - \beta \qquad \text{...(5)}$$

Equating (4) and (5), $\quad \alpha' - \alpha = \beta' - \beta \quad$ or $\quad \beta - \alpha = \beta' - \alpha' = \gamma$ (Fig. 20.5)

Thus the angle between the curves before and after the mapping is preserved in magnitude and direction. Hence the mapping by the analytic function $w = f(z)$ is conformal at each point where $f'(z) \neq 0$.

Obs. 1. *A point at which $f'(z) = 0$ is called a* **critical point** *of the transformation.*

Obs. 2. The relation (4), *i.e.*, $\alpha' = \alpha + \phi$ shows that the tangent at P to the curve C is rotated through an $\angle\phi = \text{amp}\, (f'(z))$ under the given transformation.

Obs. 3. The relation (3) shows that in the transformation, elements of arc passing through P in any direction are changed in the ratio $\rho : 1$, where $\rho = |f'(z)|$, *i.e., an infinitesimal length in the z-plane is magnified by the factor* $|f'(z)|$. Consequently *the infinitesimal areas are magnified by the factor* $|f'(z)|^2$ *in a conformal transformation.*

If $w = f(z)$ is analytic then u and v must satisfy C-R equations.

$$\therefore \quad J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{vmatrix} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left|\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right|^2 = |f'(z)|^2$$

Hence *in a conformal transformation, infinitesimal areas are magnified by the factor* $J\left(\frac{u,v}{x,y}\right)$.

Also the condition of a conformal mapping is $J\left(\frac{u,v}{x,y}\right) \neq 0$.

Obs. 4. *The angle preserving property of the conformal transformation has many important physical applications.* For instance, consider the flow of an incompressible fluid in a plane with velocity potential $\phi(x, y)$ and stream function $\psi(x, y)$. We know that ϕ and ψ are real and imaginary parts of some analytic function $w = f(z)$. As ϕ = constant and ψ = constant represent a system of orthogonal curves; these are transformed by the function $w = f(z)$ into a set of orthogonal lines in the w-plane and *vice-versa*.

Thus, *the conjugate functions ϕ and ψ when subjected to conformal transformation remain conjugate functions, i.e., the solutions of Laplace's equation remain solutions of the Laplace's equation after the transformation.* This is the main reason for the great importance of the conformal transformation in applications.

20.10 SPECIAL CONFORMAL TRANSFORMATIONS

(1) *Transformation* $w = z^2$.

We have $\quad u + iv = (x + iy)^2 = x^2 - y^2 + 2ixy.$

$\therefore \quad u = x^2 - y^2$ and $v = 2xy$ ...(1)

If u is constant (say, a), then $x^2 - y^2 = a$ which is a rectangular hyperbola. Similarly, if v is constant (say, b), then $xy = b/2$ which also represents a rectangular hyperbola.

Hence a pair of lines $u = a$, $v = b$ parallel to the axes in the w-plane, map into pair of orthogonal rectangular hyperbolae in the z-plane as shown in Fig. 12.7 (p. 455).

Again, if x is constant (say, c), then $y = v/2c$ and $y^2 = c^2 - u$. Elimination of y from these equations gives $v^2 = 4c^2(c^2 - u)$, which represents a parabola. Similarly, if y is a constant (say, d), then elimination of x from the equations (1) gives $v^2 = 4d^2(d^2 + u)$ which is also a parabola.

Hence the pair of lines $x = c$ and $y = d$ parallel to the axes in the z-plane map into orthogonal parabolas in the w-plane as shown in Fig. 20.6.

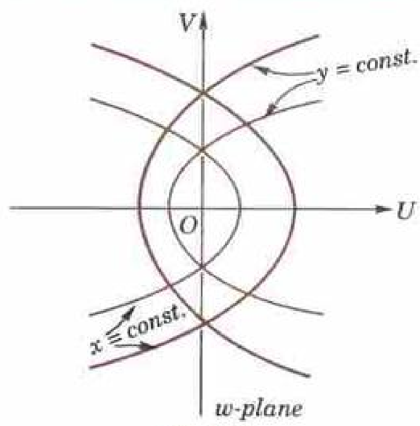

Fig. 20.6

Also since $\dfrac{dw}{dz} = 2z = 0$ for $z = 0$, therefore, it is a *critical point* of the mapping.

Taking $z = re^{i\theta}$ and $w = Re^{i\phi}$ then in polar form $w = z^2$ becomes $Re^{i\phi} = r^2e^{2i\theta}$.

This shows that upper half of the z-plane $0 < \theta < \pi$ transforms into the entire w-plane $0 \leq \phi < 2\pi$. The same is true of the lower half. **(P.T.U., 2003)**

Obs. 1. Taking the axes to represent two walls, a single quadrant could be used to represent fluid flow at a corner wall. This transformation can also represent the electrostatic field in the vicinity of a corner conductor.

Obs. 2. *For the tansformation* $w = z^n$, n being a positive integer, we have $dw/dz = 0$ at $z = 0$.

Also $\quad Re^{i\phi} = (re^{i\theta})^n = r^ne^{in\theta}$

$\therefore \quad R = r^n$ and $\phi = n\theta$, when $0 < \theta < \pi/n$, correspondingly $0 < \phi < \pi$.

Hence $w = z^n$ gives a conformal mapping of the z-plane everywhere except at the origin and that is fans out a sector of z-plane of central angle π/n to cover the upper half of the w-plane.

(2) *Joukowski's transformation** $w = z + 1/z$.

Since $\dfrac{dw}{dz} = \dfrac{(z+1)(z-1)}{z^2}$, the mapping is conformal except at the points $z = 1$ and $z = -1$ which correspond to the points $w = 2$ and $w = -2$ of the w-plane.

---

* Named after the Russian mathematician *Nikolai Jegorovich Joukowshi* (1847-1921).

Changing to polar coordinates,

$$w = u + iv = r(\cos\theta + i\sin\theta) + \frac{1}{r(\cos\theta + i\sin\theta)}$$

$$= r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$

$\therefore$ $$u = (r + 1/r)\cos\theta \text{ and } v = (r - 1/r)\sin\theta$$

Elimination of θ gives $$\frac{u^2}{(r+1/r)^2} + \frac{v^2}{(r-1/r)^2} = 1 \qquad ...(1)$$

while the elimination of r gives $$\frac{u^2}{4\cos^2\theta} - \frac{v^2}{4\sin^2\theta} = 1 \qquad ...(2)$$

From (1), it follows that the circles r = constant of z-plane transform into a family of ellipses of the w-plane (Fig. 20.7). These ellipses are confocal for $(r + 1/r)^2 - (r - 1/r)^2 = 4$, *i.e.*, a constant.

In particular, the unit circle ($r = 1$) in the z-plane flattens out to become the segment $u = -2$ to $u = 2$ of the real axis in w-plane. Thus the exterior of the unit circle in the z-plane maps into the entire w-plane.

(*A.M.I.E.T.E., 2005 S*)

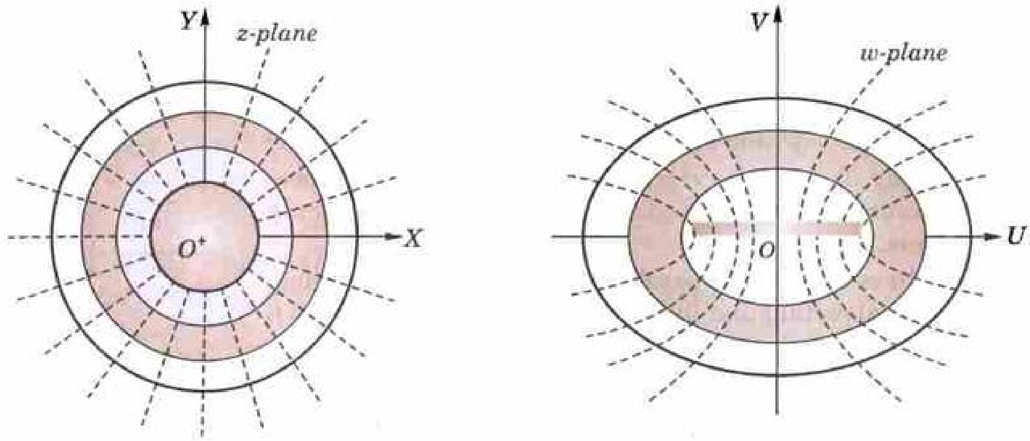

Fig. 20.7

From (2), it is clear that the radial lines θ = constant of the z-plane transform into a family of hyperbolae which are also confocal (Fig. 20.7).

Obs. 1. $v = \left(r - \frac{1}{r}\right)\sin\theta = 0$ gives $r = \pm 1$ or $\theta = 0, \pi$, *i.e.*, this streamline consists of the unit circle $r = 1$ and the x-axis ($\theta = 0$ to $\theta = \pi$). For large z, the flow is nearly uniform and parallel to the x-axis. This can be interpreted as a flow around a circular cylinder of unit radius having two **stagnation points*** at $A(z = 1)$ and $B(z = -1)$. (Fig. 20.8)

[$\because$ $dw/dz = 0$ at $z = \pm 1$]

Obs. 2. This transformation is also used to map the exterior of the profile of an aeroplane wing on the exterior of a nearly circular region. These airfoils are known as *Joukowski airfoils*.

(3) *Transformation* $w = e^z$.

Writing $z = x + iy$ and $w = \rho e^{i\phi}$, we have $\rho e^{i\phi} = e^{x+iy} = e^x \cdot e^{iy}$

whence $\rho = e^x$ ...(1) and $\phi = y$ ...(2)

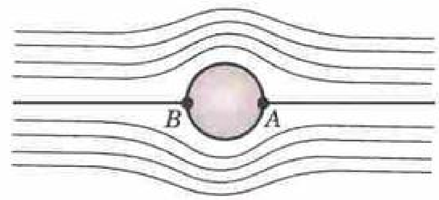

Fig. 20.8

From (1), it is clear that the lines parallel to y-axis (x = const.) map into circles (ρ = const.) in the w-plane, their radii being less than or greater than 1 according as x is less than or greater than 0 (Fig. 20.9). (*V.T.U., 2011*)

Similarly, it follows from (2) that the lines parallel to the x-axis (y = const.) map into the radial lines (ϕ = const.) of the w-plane. Thus any horizontal strip of height 2π in the z-plane will cover once the entire w-plane.

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* *Stagnation points* are those at which the fluid velocity is zero.

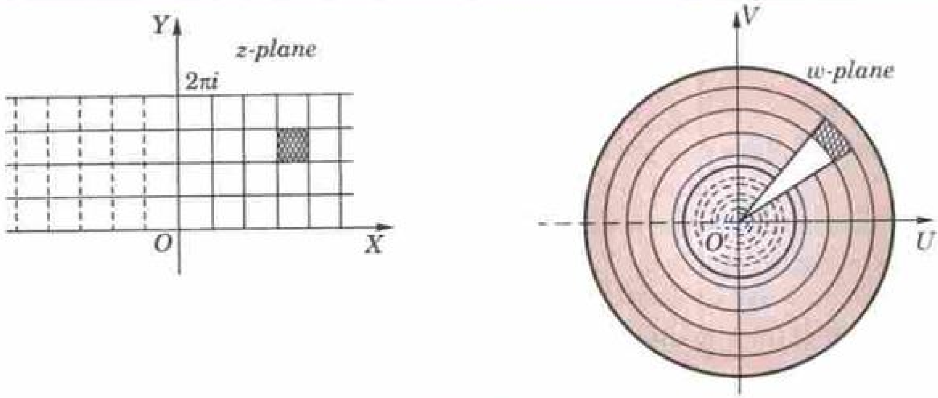

Fig. 20.9

The rectangular region $a_1 \le x \le a_2$, $b_1 \le y \le b_2$ in the z-plane (shown shaded) transforms into the region $e^{a_1} \le \rho \le e^{a_2}$, $b_1 \le \phi \le b_2$ in the w-plane bounded by circles and rays (shown shaded).

(P.T.U., 2005 ; Kerala, 2005)

Obs. This transformation can be used to obtain the circulation of a liquid around a cylindrical obstacle, the electrostatic field due to a charged circular cylinder etc.

(4) *Transformation* $w = \cosh z$.

We have $u + iv = \cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y$ [By (2) (*ii*), p. 662]

so that $u = \cosh x \cos y$ and $v = \sinh x \sin y$.

Elimination of x from these equations gives

$$\frac{u^2}{\cos^2 y} - \frac{v^2}{\sin^2 y} = 1 \qquad ...(1)$$

while elimination of y gives

$$\frac{u^2}{\cosh^2 x} + \frac{v^2}{\sinh^2 z} = 1 \qquad ...(2)$$

(1) shows that the lines parallel to x-axis (*i.e.*, y = const.) in the z-plane map into hyperbolae in the w-plane.

(2) shows that the lines parallel to the y-axis (*i.e.*, x = const.) in the z-plane map into ellipse in the w-plane (Fig. 20.10). The rectangular region $a_1, \le x \le a_2$, $b_1 \le y \le b_2$ in the z-plane (shown shaded) transforms into the shaded region in the w-plane bounded by the corresponding hyperbolae and ellipses. *(Kerala M. Tech., 2005)*

Obs. This transformation can be used.
(*i*) to obtain the circulation of liquid around an elliptic cylinder;
(*ii*) to determine the electrostatic field due to a charged cylinder;
(*iii*) to determine the potential between two confocal elliptic (or hyperbolic) cylinders.

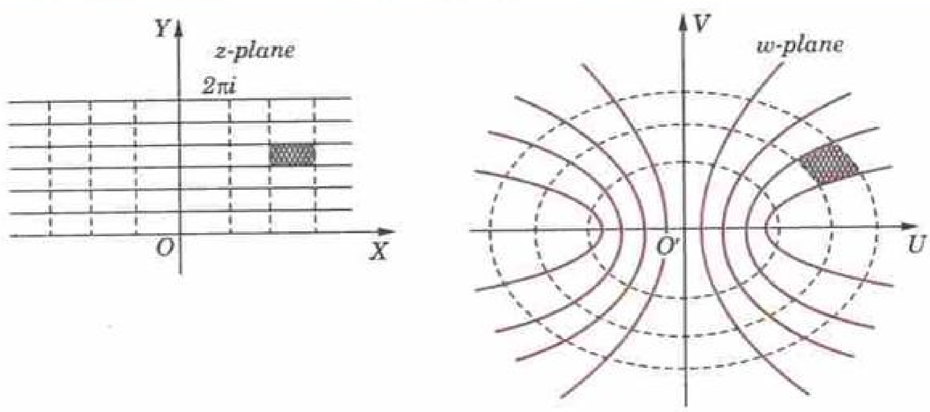

Fig. 20.10

Example 20.15. *Show that under the transformation $w = (z - i)/(z + i)$, real axis in the z-plane is mapped into the circle $|w| = 1$. Which portion of the z-plane corresponds to the interior of the circle?* (J.N.T.U., 2003)

Solution. We have $$|w| = \left|\frac{z-i}{z+i}\right| = \frac{|z-i|}{|z+i|} = \frac{|x+i(y-1|}{|x+i(y+1)|}$$
$$= \sqrt{\{x^2+(y-1)^2\}}/\sqrt{\{x^2+(y+1)^2\}}$$

Now the real axis in z-plane *i.e.*, $y = 0$, transforms into
$$|w| = \sqrt{(x^2+1)}/\sqrt{\{x^2+1)} = 1.$$

Hence the real axis in the z-plane is mapped into the circle $|w| = 1$.

The interior of the circle, *i.e.*, $|w| < 1$, gives
$$\{x^2+(y-1)^2\}/\{x^2+(y+1)^2\} < 1 \text{ i.e., } -4y < 0 \text{ or } y > 0.$$

Thus the upper half of the z-plane corresponds to the interior of the circle $|w| = 1$.

PROBLEMS 20.3

1. Determine the region of the w-plane into which the following regions are mapped by the transformation $w = z^2$:
   (*i*) first quadrant of z-plane (*J.N.T.U., 2000*)
   (*ii*) region bounded by $x = 1$, $y = 1$, $x + y = 1$ (*Kottayam, 2005 ; V.T.U., 2000 S*)
   (*iii*) the region $1 \le x \le 2$ and $1 \le y \le 2$ (*Osmania, 2003 ; V.T.U., 2000*)
   (*iv*) circle $|z - 1| = 2$.
2. Find the transformation which maps the triangular region $0 \le \arg z \le \pi/3$ into the unit circle $w \le 1$.
3. Discuss the transformation $w = \sqrt{z}$. Is it conformal at the origin? (*Delhi, 2002*)
4. Under the transformation $w = 1/z$, find the image of
   (*a*) the circle $|z - 2i| = 2$ (*Bhopal, 2009 ; Kerala M.Tech., 2005*)
   (*b*) the straight line $y - x + 1 = 0$ (*P.T.U., 2007*)
   (*c*) the hyperbola $x^2 - y^2 = 1$. (*Mumbai, 2005 ; J.N.T.U., 2005*)
5. Show that under the transformation $w = 1/z$, (*a*) circle $x^2 + y^2 - 6x = 0$ is transformed into a straight line in the w-plane.
   (*b*) the circle $(x-3)^2 + y^2 = 2$ is transformed into a circle with centre $(3/7, 0)$ and radius $\sqrt{2}/17$. (*Mumbai, 2007*)
6. Show that the transformation $w = 1/z$ transforms all circles and straight lines into the circles and straight lines in the w-plane. Which circles in the z-plane become straight lines in the w-plane, and which straight lines are transformed into other straight lines? (*Anna, 2003*)
7. Show that the transformation $w = z + 1/z$ converts the straight line $\arg z = \alpha$ ($|\alpha| < \alpha/2$) into a branch of hyperbola of eccentricity $\sec \alpha$. (*Mumbai, 2005 S*)
8. Show that the transformation $w = z + (a^2 - b^2)/4z$ transforms the circle of radius $\frac{1}{2}(a + b)$, centre at the origin, in the z-plane into ellipse of semi-axes a, b in the w-plane.
9. Show that the transformation $w = z + a^2/z$ transforms circles with origin at the cent`re in the z-plane into co-axial concentric, confocal ellipses in the w-plane. (*Kurukshetra, 2005 ; J.N.T.U., 2005*)
10. Show that the function $w = A(z + a^2/z)$ may be used to represent the flow of a perfect incompressible fluid past a circular cylinder. Also find the stagnation points.
11. Show that by the relation $u + iv = \cos(x + iy)$, the infinite strip bounded by $x = c$, $x = d$, where c and d lie between 0 and $\pi/2$, is mapped into the region between the two branches of the hyperbola lying in $u > 0$. (*Osmania, 2002*)
12. Prove that the transformation $w = \sin z$, maps the families of lines x = constant and y = constant into two families of confocal central conics. (*J.N.T.U., 2003*)
13. Discuss the transformation $w = e^z$, and show that it transforms the region between the real azis and a line parallel to real axis at $y = \pi$, into the upper half of the w-plane.
14. Discuss fully the transformation $w = c \cosh z$, where c is a real number. What physical problem can we study with the help of this transformation?

20.11 SCHWARZ-CHRISTOFFEL TRANSFORMATION

This transformation maps the interior of a polygon of the w-plane into the upper half of the z-plane and the boundary of the polygon into the real axis. The formula of this transformation is

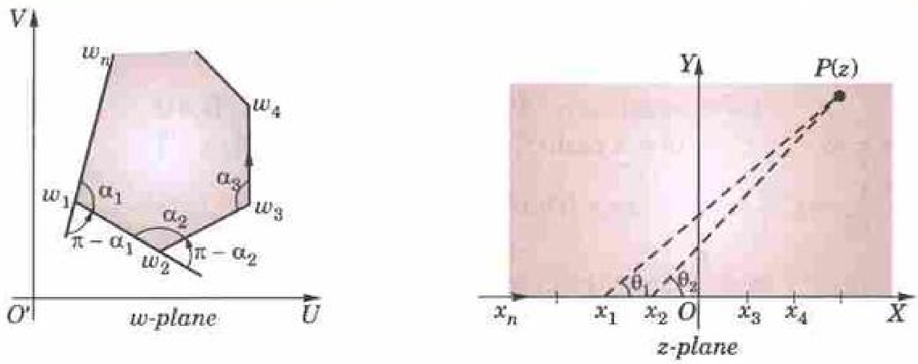

Fig. 20.11

$$\frac{dw}{dz} = A(z - x_1)^{\frac{\alpha_1}{\pi} - 1}(z - x_2)^{\frac{\alpha_2}{\pi} - 1} \dots (z - x_n)^{\frac{\alpha_n}{\pi} - 1} \quad \dots(1)$$

or $$w = A \int (z - x_1)^{\frac{\alpha_1}{\pi} - 1}(z - x_2)^{\frac{\alpha_2}{\pi} - 1} \dots (z - x_n)^{\frac{\alpha_n}{\pi} - 1} dz + B \quad \dots(2)$$

where $\alpha_1, \alpha_2, \dots, \alpha_n$ are the interior angles of the polygon having vertices $w_1, w_2, \dots, w_n$ which map into the points $x_1, x_2, \dots, x_n$ on the real-axis of the z-plane (Fig. 20.11). Also A and B are complex constants which determines the size and position of the polygon.

Proof. We have from (1),

$$\text{amp}\left(\frac{dw}{dz}\right) = \text{amp}\,[(A) + \left(\frac{\alpha_1}{\pi} - 1\right) \text{amp}\,(z - x_1) + \left(\frac{\alpha_2}{\pi} - 1\right) \text{amp}\,(z - x_2)$$

$$\dots + \left(\frac{\alpha_n}{\pi} - 1\right) \text{amp}\,(z - x_n) \quad \dots(3)$$

As z moves along the real axis from the left towards x_1, suppose that w moves along the side $w_n w_1$ of the polygon towards w_1. As z crosses x_1 from left to right, $\theta_1 = \text{amp}\,(z - x_1)$ changes from π to 0 while all other terms of (3) remain unaffected. Hence only $\left(\frac{\alpha_1}{\pi} - 1\right)$ amp $(z - x_1)$ decreases by $\left(\frac{\alpha_1}{\pi} - 1\right) \pi = \alpha_1 - \pi$, *i.e.* increases by $\pi - \alpha_1$ in the anti-clockwise direction. In other words, amp (dw/dz) increases by $\pi - \alpha_1$. Thus the direction of w_1 turns through the angle $\pi - \alpha_1$ and w now moves along the side $w_1 w_2$ of the polygon.

Similarly when z passes through x_2, $\theta_1 = \text{amp}\,(z - x_1)$ and $\theta_2 = \text{amp}\,(z - x_2)$ change from π to 0 while all other terms remain unchanged. Hence the side $w_1 w_2$ turns through the angle $\pi - \alpha_2$. Proceeding in this way, we see that as z moves along x-axis, w traces the polygon $w_1\, w_2\, w_3 \dots w_n$ and conversely.

Example 20.16. *Find the transformation which maps the semi-infinite strip in the w-plane (Fig. 20.12) into the upper half of the z-plane* *(V.T.U., M.E. 2006 ; Osmania, 2003)*

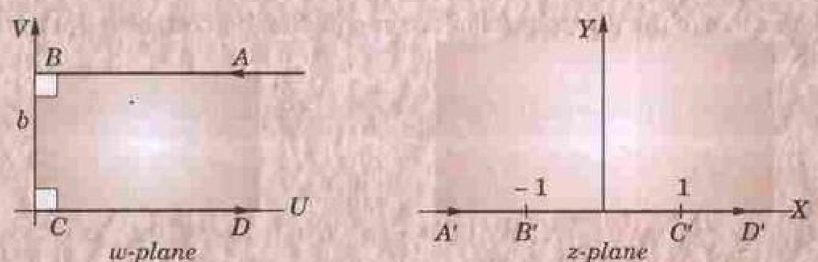

Fig. 20.12

Solution. Consider $ABCD$ as the limiting case of a triangle with two vertices B and C and the third vertex A or D at infinity. Let the vertices B and C map into the points $B'(-1)$ and $C'(1)$ of the z-plane. Since the interior angles at B and C are $\pi/2$, we have by the Schwarz-Christoffel transformation,

$$\frac{dw}{dz} = A(z+1)^{\frac{\pi/2}{\pi} - 1}(z - 1)^{\frac{\pi/2}{\pi} - 1} = A/\sqrt{(z^2 - 1)}$$

$$\therefore \qquad w = A\int \frac{dz}{\sqrt{(z^2-1)}} + B = A\cosh^{-1} z + B$$

When $\qquad z = 1, w = 0. \quad \therefore \quad 0 = A\cosh^{-1}(1) + B$, *i.e.*, $B = 0$.

When $z = -1, w = ib. \quad \therefore \quad ib = A\cosh^{-1}(-1) + 0$, *i.e.*, $\cosh(ib/A) = -1$

or
$$\cos\frac{b}{A} = -1 = \cos\pi. \text{ Thus } A = \frac{b}{\pi}.$$

Hence
$$w = \frac{b}{\pi}\cosh^{-1} z \text{ or } z = \cosh\frac{\pi w}{b}.$$

PROBLEMS 20.4

1. Find the transformation which maps the semi-infinite strip of width π bounded by the lines $v = 0, v = \pi$ and $u = 0$ into the upper half of the z plane.
2. Show how you will use Schwarz-Christoffel transformation to map the semi-infinite strip enclosed by the real axis and the lines $u = \pm 1$ of the w-plane into the upper half of the z-plane.
3. Find the mapping function which maps semi-infinite strip in the z-plane $-\pi/2 \le x \le \pi/2, y \ge 0$ into half w-plane for which $v \ge 0$, such that the points $(-\pi/2, 0), (\pi/2, 0)$ in the π-plane are mapped into the points $(-1, 0), (1, 0)$ respectively in w-plane.
4. Find the transformation which will map the interior of the infinite strip bounded by the lines $v = 0, v = \pi$ onto the upper half of the z-plane.

20.12 COMPLEX INTEGRATION

We have already discussed the concept of the line integral as applied to vector fields in § 8.11. Now we shall consider the line integral of a complex function.

Fig. 20.13

Consider a continuous function $f(z)$ of the complex variable $z = x + iy$ defined at all points of a curve C having end points A and B. Divide C into n parts at the points

$$A = P_0(z_0), P_1(z_1), \ldots, P_i(z_i), \ldots P_n(z_n) = B.$$

Let $\delta z_i = z_i - z_{i-1}$ and ζ_i be any point on the arc $P_{i-1}P_i$. The limit of the sum $\sum_{i=1}^{n} f(\zeta_i)\,\delta z_i$ as $n \to \infty$ *in such a way that the length of the chord* δz_i *approaches zero, is called the* **line integral of f (z)** *taken along the path C, i.e.,*

$$\int_C f(z)\, dz.$$

Writing $f(z) = u(x, y) + iv(x, y)$ and noting that $dz = dx + idy$,

$$\int_C f(z)\,dz = \int_C (udx - vdy) + i\int_C (vdx + udy)$$

which shows that the evaluation of the line integral of a complex function can be reduced to the evaluation of two line integrals of real functions.

Obs. *The value of the integral is independent of the path of integration when the integrand is analytic.*

Example 20.17. *Prove that*

(*i*) $\int_C \frac{dz}{z-a} = 2\pi i.$ $\qquad$ (*ii*) $\int_C (z-a)^n\,dz = 0$ [*n, any integer* $\neq -1$]

where C is the circle $|z - a| = r$. (*U.P.T.U.*, 2003)

Solution. The parametric equation of C is $z - a = re^{i\theta}$, where θ varies from 0 to 2π as z describes C once in the positive (anti-clockwise) sense. (Fig. 20.14)

(*i*)
$$\int_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{1}{re^{i\theta}}\,.\,ire^{i\theta}\,d\theta \qquad [\because dz = ire^{i\theta}\,d\theta]$$

$$= i\int_0^{2\pi} d\theta = 2\pi i$$

(ii) $\int_C (z-a)^n\, dz = \int_0^{2\pi} r^n e^{ni\theta} \,.\, ire^{i\theta}\, d\theta$

$$= ir^{n+1} \int_0^{2\pi} e^{(n+1)\theta i}\, d\theta = \frac{r^{n+1}}{n+1} \left| e^{(n+1)\theta i} \right|_0^{2\pi}, \text{ provided } n \neq -1$$

$$= \frac{r^{n+1}}{n+1} [e^{2(n+1)\pi i} - 1] = 0 \qquad [\because \; e^{2(n+1)\pi i} = 1]$$

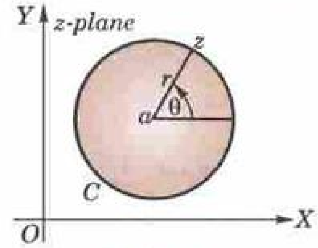

Fig. 20.14

Example 20.18. *Evaluate* $\int_0^{2+i} (\bar{z})^2\, dz$, *along (i) the line* $y = x/2$, *(Bhopal, 2007 ; U.P.T.U., 2002)*
(ii) the real axis to 2 and then vertically to 2 + i. *(S.V.T.U., 2009 ; P.T.U., 2008 S ; Mumbai, 2006)*

Solution. (i) Along the line OA, $x = 2y$, $z = (2+i)y$, $\bar{z} = (2-i)y$ and $dz = (2+i)\,dy$ (Fig. 20.15)

$$\therefore \qquad I = \int_0^{2+i} (\bar{z})^2\, dz = \int_0^1 (2-i)^2 y^2 \,.\, (2+i)\, dy$$

$$= 5(2-i) \left| \frac{y^3}{3} \right|_0^1 = \frac{5}{3}(2-i)$$

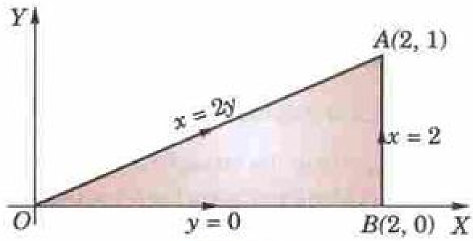

Fig. 20.15

(ii) $$I = \int_{OB} (\bar{z})^2\, dz + \int_{BA} (\bar{z})^2\, dz.$$

Now along OB, $z = x$, $\bar{z} = x$, $dz = dx$;
and along BA, $z = 2 + iy$, $\bar{z} = 2 - iy$, $dz = idy$

$$\therefore \qquad I = \int_0^2 x^2 dx + \int_0^1 (2-iy)^2 \,.\, idy = \left| \frac{x^3}{3} \right|_0^2 + \int_0^1 [\,4y + (4-y^2)i\,]\, dy$$

$$= \frac{8}{3} + 4 \,.\, \frac{1}{2} + \left(4.1 - \frac{1}{3}\right) i = \frac{1}{3}(14 + 11i).$$

Example 20.19. *Evaluate* $\int_C (z^2 + 3z + 2)\, dz$ *where C is the arc of the cycloid* $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ *between the points (0, 0) and* $(\pi a, 2a)$. *(Rohtak, 2004)*

Solution. $f(z) = z^2 + 3z + 2$ is analytic in the z-plane being a polynomial. As such, the line integral of $f(z)$ between O and A is independent of the path (Fig. 20.16). We therefore, take the path from O to L and L to A so that

$$\int_C f(z)\, dz = \int_{OL} f(z)\, dz + \int_{LA} f(z)\, dz \qquad ...(i)$$

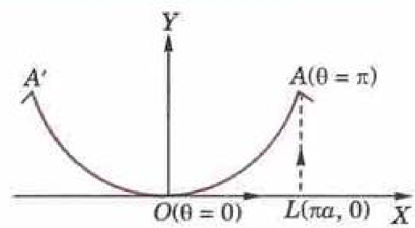

Fig. 20.16

$$\therefore \quad \int_{OL} f(z)\, dz = \int_0^{\pi a} (x^2 + 3x + 2)\, dx$$

[$\because$ along OL, $y = 0$, $x = 0$ at O, $x = \pi a$ at L

$$= \left| \frac{x^3}{3} + \frac{3x^2}{2} + 2x \right|_0^{\pi a} = \frac{\pi a}{6}(2\pi^2 a^2 + 9\pi a + 12) \qquad ...(ii)$$

and $$\int_{LA} f(z)\, dz = \int_0^{2a} [(\pi a + iy)^2 + 3(\pi a + iy) + 2]\, idy$$

[$\because$ along LA, $x = \pi a$, $z = \pi a + iy$, $dz = idy$ and y varies from 0 (at L) to $2a$ (at A)

$$= L \left| \frac{(\pi a + iy)^3}{3i} + 3\frac{(\pi a + iy)^2}{2i} + 2y \right|_0^{2\pi} = \frac{a^3}{3}(\pi + 2i)^3 + \frac{3a^2}{2}(\pi + 2i)^2 + 4ia \qquad ...(iii)$$

$\therefore$ substituting from (ii) and (iii) in (i), we get

$$\int_C f(z)\, dz = \frac{\pi a}{6}(2\pi^2 a^2 + 9\pi a + 12) + \frac{a^3}{3}(\pi + 2i)^3 + \frac{3a^2}{2}(\pi + 2i)^2 + 4ia$$

PROBLEMS 20.5

1. Evaluate $\int_0^{1+i}(x^2 - iy)\,dz$ along the paths (a) $y = x$ and (b) $y = x^2$. (U.P.T.U., 2010)

2. Evaluate $\int_{1-i}^{2+i}(2x + iy + 1)\,dz$, along the two paths: (U.P.T.U., 2010)
   (i) $x = t + 1, y = 2t^2 - 1$ (ii) the straight line joining $1 - i$ and $2 + i$. (U.P.T.U., 2006)

3. Evaluate $\int_{1-i}^{2+3i}(z^2 + z)\,dz$ along the line joining the points $(1, -1)$ and $(2, 3)$. (V.T.U., 2004)

4. Show that for every path between the limits, $\int_{-2}^{-2+i}(2 + z)^2\,dz = -i/3$. (Delhi, 2002)

5. Show that $\oint_C (z + 1)\,dz = 0$, where C is the boundary of the square whose vertices are at the points $z = 0$, $z = 1$, $z = 1 + i$ and $z = i$. (Rohtak, 2006)

6. Evaluate $\int_C |z|\,dz$, where C is the contour
   (i) straight line from $z = -i$ to $z = i$.
   (ii) left half of the unit circle $|z| = 1$ from $z = -i$ to $z = i$.
   (iii) circle given by $|z + 1| = 1$ described in the clockwise sense.

7. Find the value of $\int_0^{1+i}(x - y + ix^2)\,dz$
   (i) along the straight line from $z = 0$ to $z = 1 + i$
   (ii) along real axis from $z = 0$ to $z = 1$ and then along a line parallel to the imaginary axis from $z = 1$ to $z = 1 + i$. (U.P.T.U., 2003)

8. Prove that $\int dz/z = -\pi i$ or πi, according as C is the semi-circular arc $|z| = 1$ above or below the real axis. (Rohtak, 2005)

9. Evaluate $\int_C (z - z^2)\,dz$, where C is the upper half of the circle $|z| = 1$.
   What is the value of this integral if C is the lower half of the above circle ?

20.13 CAUCHY'S THEOREM

If $f(z)$ is an analytic function and $f'(z)$ is continuous at each point within and on a closed curve C, then $\oint_C f(z)\,dz = 0$.

Writing $f(z) = u(x, y) + iv(x, y)$ and noting that $dz = dx + idy$

$$\oint_C f(z)dz = \oint_C (udx - vdy) + i \oint_C (vdx + udy) \qquad ...(1)$$

Since $f'(z)$ is continuous, therefore, $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ are also continuous in the region D enclosed by C. Hence the Green's theorem (p. 376) can be applied to (1), giving

$$\oint_C f(z)dz = -\iint_D \left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] dxdy + i \iint_D \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] dxdy \qquad ...(2)$$

Now $f(z)$ being analytic, u and v necessarily satisfy the Cauchy-Riemann equations and thus the integrands of the two double integrals in (2) vanish identically.

Hence $$\oint_C f(z)dz = 0.$$

Obs. 1. The Cauchy-Riemann equations are precisely the conditions for the two real integrals in (1) to be independent of the path. *Hence the line integral of a function $f(z)$ which is analytic in the region D, is independent of the path joining any two points of D.*

Obs. 2. Extension of Cauchy's theorem. *If $f(z)$ is analytic in the region D between two simple closed curves C and C_1, then* $\oint_C f(z)dz = \oint_{C_1} f(z)dz$.

To prove this, we need to introduce the cross-cut AB. Then $\oint f(z)dz = 0$ where the path is as indicated by arrows in Fig. 20.17, *i.e.*, along AB—along C_1 in clockwise sense and along BA—along C in anti-clockwise sense.

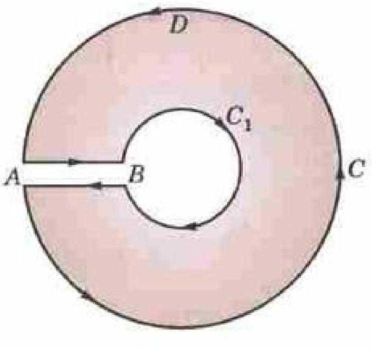

Fig. 20.17(*a*)

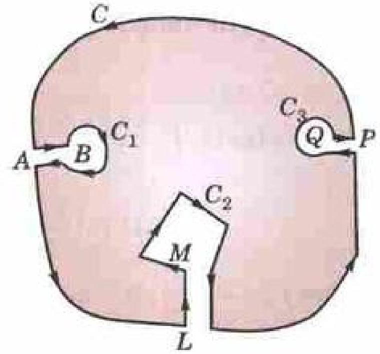

Fig. 20.17(*b*)

i.e.,
$$\int_{AB} f(z)dz + \int_{C_1} f(z)dz + \int_{AB} f(z)dz + \int_C f(z)dz = 0$$

But, since the integrals along AB and along BA cancel, it follows that
$$\int_C f(z)dz + \int_{C_1} f(z)dz = 0$$

Reversing the direction of the integral around C_1 and transposing, we get
$$\int_C f(z)dz + \int_{C_1} f(z)dz \text{ each integration being taken in the anti-clockwise sense.}$$

If $C_1, C_2, C_3, \ldots$ *be any number of closed curves within C (Fig. 20.17(b)), then*
$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \oint_{C_3} f(z)dz + \ldots$$

20.14 CAUCHY'S INTEGRAL FORMULA

If f (z) is analytic within and on a closed curve and if a is any point within C, then
$$f(a) = \frac{1}{2\pi i} = \oint_C \frac{f(z)dz}{z-a}$$

Consider the function $f(z)/(z-a)$ which is analytic at all points within C except at $z = a$. With the point a as centre and radius r, draw a small circle C_1 lying entirely within C.

Now $f(z)/(z-a)$ being analytic in the region enclosed by C and C_1, we have by Cauchy's theorem,
$$\oint_C \frac{f(z)}{z-a}dz = \oint_{C_1} \frac{f(z)}{z-a}dz \qquad \left\{\begin{array}{l}\text{For any point on } C_1, \\ z-a = re^{i\theta} \text{ and } dz = ire^{i\theta}\, d\theta\end{array}\right.$$
$$= \oint_C \frac{f(a+re^{i\theta})}{re^{i\theta}} . ire^{i\theta}\, d\theta = i \oint_{C_1} f(a+re^{i\theta})d\theta \qquad \text{...(1)}$$

In the limiting form, as the circle C_1 shrinks to the point a, *i.e.*, as $r \to 0$, the integral (1) will approach to
$$\oint_C f(a)d\theta = if(a)\int_0^{2\pi} d\theta = 2\pi i f(a) \oint f(z)dz. \text{ Thus } \oint_C \frac{f(z)}{z-a}dz = 2\pi i f(a)$$

i.e.,
$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz \qquad \text{...(2)}$$

which is the desired *Cauchy's integral formula.* (*V.T.U., 2011 S*)

Cor. Differentiating both sides of (2) w.r.t. a,
$$f'(a) = \frac{1}{2\pi i}\oint_C \frac{\partial}{\partial a}\left[\frac{f(z)}{z-a}\right]dz = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2}dz \qquad \text{...(3)}$$

Similarly,
$$f''(a) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^3}\oint_C dz \qquad \text{...(4)}$$

and in general, $$f^n(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz. \qquad ...(5)$$

Thus it follows from the results (2) to (5) that *if a function f (z) is known to be analytic on the simple closed curve C then the values of the function and all its derivatives can be found at any point of C.* Incidently, we have established a remarkable fact that **an analytic function possesses derivatives of all orders and these are themselves all analytic.**

Example 20.20. *Evaluate* $\int_C \frac{z^2 - z + 1}{z - 1} dx$, *where C is the circle*

(*i*) $|z| = 1$, (*ii*) $|z| = \frac{1}{2}$. (*S.V.T.U., 2007*)

Solution. (*i*) Here $f(z) = z^2 - z + 1$ and $a = 1$.

Since $f(z)$ is analytic within and on circle $C : |z| = 1$ and $a = 1$ lies on C.

$\therefore$ by Cauchy's integral formula $\frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} = f(a) = 1$ *i.e.*, $\int_C \frac{z^2 - z + 1}{z - 1} dz = 2\pi i.$

(*ii*) In this case, $a = 1$ lies outside the circle $C : |z| = 1/2$. So $(z^2 - z + 1)/(z - 1)$ is analytic everywhere within C.

$\therefore$ by Cauchy's theorem $\int_C \frac{z^2 - z + 1}{z - 1} dz = 0.$

Example 20.21. *Evaluate, using Cauchy's integral formula:*

(*i*) $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$ *where C is the circle* $|z| = 3$ (*U.P.T.U., 2010*)

(*ii*) $\oint_C \frac{\cos \pi z}{z^2 - 1} dz$ *around a rectangle with vertices* $2 \pm i, -2 \pm 1$

(*iii*) $\oint_C \frac{e^{tz}}{z^2 + 1} dz$ *where C is the circle* $|z| = 3$. (*U.P.T.U., 2009*)

Solution. (*i*) $f(z) = \sin \pi z^2 + \cos \pi z^2$ is analytic within the circle $|z| = 3$ and the two singular points $z = 1$ and $z = 2$ lie inside this circle.

$$\therefore \quad \oint_C \frac{f(z)dz}{(z-1)(z-2)} = \oint_C (\sin \pi z^2 + \cos \pi z^2) = \left(\frac{1}{z-2} - \frac{1}{z-1}\right)$$

$$= \oint_C \frac{(\sin \pi z^2 + \cos \pi z^2)}{z-2} dz - \oint_C \frac{(\sin \pi z^2 + \cos \pi z^2)}{z-1} dz$$

$$= 2\pi i\, [\sin \pi(2)^2 + \cos \pi(2)^2] - 2\pi i [\sin \pi(1)^2 + \cos \pi(1)^2]$$

[By Cauchy's integral formula]

$$= 2\pi i\,(0 + 1) - 2\pi i\,(0 - 1) = 4\pi i$$

(*ii*) $f(z) = \cos \pi z$ is analytic in the region bounded by the given rectangle and the two singular points $z = 1$ and $z = -1$ lie inside this rectangle. (Fig. 20.18)

$$\therefore \quad \oint_C \frac{\cos \pi z}{z^2 - 1} dz = \frac{1}{2} \oint_C \left(\frac{1}{z-1} - \frac{1}{z+1}\right) \cos \pi z \, dz$$

$$= \frac{1}{2} \oint_C \frac{\cos \pi z}{z-1} dz - \oint_C \frac{\cos \pi z}{z+1} dz$$

$$= \frac{1}{2} \{2\pi i \cos \pi (1)\} - \frac{1}{2} \{2\pi i \cos \pi (-1)\} = 0.$$

[By Cauchy's integral formula]

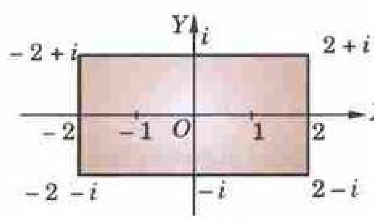

Fig. 20.18

(*iii*) $f(z) = e^{tz}$ is analytic within the circle $|z| = 3$.

The singular points are given by $z^2 + 1 = 0$ *i.e.*, $z = i$ and $z = -i$ which lie within this circle.

$$\therefore \quad \oint_C \frac{e^{tz}}{z^2+1}\,dz = \oint_C \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)e^{tz}\,dz = \frac{1}{2i}\left\{\oint_C \frac{e^{tz}}{z-i}\,dz - \oint_C \frac{e^{tz}}{z+i}\,dz\right\}$$

$$= \frac{1}{2i}\{2\pi i e^{t(i)} - 2\pi i e^{t(-i)}\} \qquad \text{[By Cauchy's integral formula]}$$

$$= 2\pi i\left(\frac{e^{it} - e^{-it}}{2i}\right) = 2\pi i \sin t.$$

Example 20.22. *Evaluate*

(*i*) $\oint_C \frac{\sin^2 z}{(z-\pi/6)^3}\,dz$, *where C is the circle* $|z| = 1$ (*Rohtak, 2005*)

(*ii*) $\oint_C \frac{e^{2z}}{(z+i)^4}\,dz$, *where C is the circle* $|z| = 3$ (*U.P.T.U., 2008*)

(*iii*) $\oint_C \frac{e^z}{(z^2+\pi^2)^2}\,dz$, *where C is* $|z| = 4$. (*U.P.T.U., 2008; J.N.T.U., 2000*)

Solution. (*i*) $f(z) = \sin^2 z$ is analytic inside the circle C: $|z| = 1$ and the point $a = \pi/6$ (= 0.5 approx.) lies within C.

$\therefore$ by Cauchy's integral formula $f''(a) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^3}dz$,

we get
$$\oint_C \frac{\sin^2 z}{(z-\pi/6)^3}\,dz = \pi i\left[\frac{d^2}{dz^2}(\sin^2 z)\right]_{z=\pi/6}$$

$$= \pi i(2\cos 2z)_{z=\pi/6} = 2\pi i \cos \pi/3 = \pi i.$$

(*ii*) $f(z) = e^{2z}$ is analytic within the circle C: $|z| = 3$. Also $z = -1$ lies inside C.

$\therefore$ By Cauchy's integral formula: $f'''(a) = \frac{3!}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^4}$

we get
$$\int_C \frac{e^{2z}}{(z+1)^4}\,dz = \frac{2\pi i}{6}\left|\frac{d^3(e^{2z})}{dz^3}\right|_{z=-1} = \frac{\pi i}{3}[8e^{2z}]_{z=-1} = \frac{8\pi i}{3}e^{-2}$$

(*iii*) $\frac{e^z}{(z^2+\pi^2)^2} = \frac{e^z}{(z+\pi i)^2(z-\pi i)^2}$ is not analytic at $z = \pm\pi i$.

However both $z = \pm \pi i$ lie within the circle $|z| = 4$.

Now
$$\frac{1}{(z+\pi i)^2(z-\pi i)^2} = \frac{A}{z+\pi i} + \frac{B}{(z+\pi i)^2} + \frac{C}{z-\pi i} + \frac{D}{(z-\pi i)^2}$$

where $A = 7/2\pi^3 i$, $C = -7/2\pi^3 i$, $B = D = -1/4\pi^2$

$$\therefore \quad \int_C \frac{e^z}{(z^2+\pi^2)^2}\,dz = \frac{7}{2\pi^3 i}\left\{\int_C \frac{e^z}{z+\pi i}\,dz - \int_C \frac{e^z}{z-\pi i}\,dz\right\} - \frac{1}{4\pi^2}\left\{\int_C \frac{e^z}{(z+\pi i)^2}\,dz + \int \frac{e^z}{(z-\pi i)^2}\,dz\right\}$$

$$= \frac{7}{2\pi^3 i}[2\pi i f(-\pi i) - 2\pi i f(\pi i)] - \frac{1}{4\pi^2}[2\pi i f'(-\pi i) + 2\pi i f'(\pi i)] \qquad \text{where } f(z) = e^z.$$

$$= \frac{7}{\pi^2}(e^{-\pi i} - e^{\pi i}) - \frac{i}{2\pi}(e^{-\pi i} + e^{\pi i}) = -\frac{14i}{\pi^2}\left(\frac{e^{\pi i} - e^{-\pi i}}{2i}\right) - \frac{i}{\pi}\left(\frac{e^{\pi i} + e^{-\pi i}}{2}\right) \qquad [\S\ 19.9]$$

$$= -\frac{14i}{\pi^2}\sin\pi - \frac{i}{\pi}\cos\pi = \frac{i}{\pi}.$$

Example 20.23. *Verify Cauchy's theorem by integrating e^{iz} along the boundary of the triangle with the verticles at the points $1 + i$, $-1 + i$ and $-1 - i$.* (U.P.T.U., 2006)

Solution. The boundary of the given triangle consists of three lines AB, BC, CA. (Fig. 29.19).

$$\oint_{ABC} e^{iz}dz = \int_{AB} e^{iz}dz + \int_{BC} e^{iz}dz + \int_{CA} e^{iz}dz$$

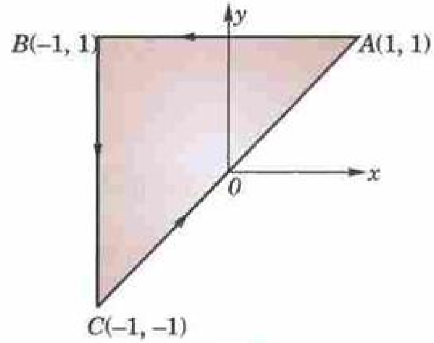

Fig. 20.19

Now $\displaystyle\int_{AB} e^{iz}dz = \int_{1}^{-1} e^{i(x+i)}dx$ $\quad\left|\begin{array}{l}\because \text{Along } AB : y = 1 \\ \therefore z = x + i \text{ and } dz = dx\end{array}\right.$

$$= \int_{1}^{-1} e^{(ix-1)}\,dx = \left|\frac{e^{ix-1}}{i}\right|_{1}^{-1} = \frac{e^{-i-1} - e^{i-1}}{i}$$

$$\int_{BC} e^{iz}dz = \int_{1}^{-1} e^{i(-1+1iy)}\,idy$$

$\left|\begin{array}{l}\because \text{Along } BC : x = -1 \\ \therefore z = -1 + iy,\ dz = idy\end{array}\right.$

$$= i\int_{1}^{-1} e^{-i-y}\,dy = i\left|\frac{e^{-i-y}}{-1}\right|_{1}^{-1} = \frac{e^{-i+1} - e^{-i-1}}{i}$$

$$\int_{CA} e^{iz}dz = \int_{-1}^{1} e^{i(1+i)x}\,(1+i)dx$$

$\left|\begin{array}{l}\because \text{Along } CA : y = x \\ \therefore z = (1 + i)x,\ dz = (1 + i)\,dx\end{array}\right.$

$$= (1+i)\,\frac{e^{(i-1)} - e^{-(i-1)}}{i(1+i)} = \frac{e^{i-1} - e^{-i+1}}{i}$$

Thus from (i) $\displaystyle\oint_{ABC} e^{iz}\,dz = \frac{e^{-i-1} - e^{i-1}}{i} + \frac{e^{-i+1} - e^{-i-1}}{i} + \frac{e^{i-1} - e^{-i+1}}{i} = 0$ ...(ii)

Also since $f(x) = e^{iz}$ is analytic everywhere,

$\therefore$ by Cauchy's theorem $\oint f(z) = 0$ ...(iii)

Hence from (ii) and (iii), $\displaystyle\oint_{ABC}$ Cauchy's theorem is verified.

Example 20.24. *If $F(\zeta) = \displaystyle\oint_C \frac{4z^2 + z + 5}{z - \zeta}\,dz$, where C is the ellipse $(x/2)^2 + (y/3)^2 = 1$, find the value of (a) $F(3.5)$; (b) $F(i)$, $F'(-1)$ and $F''(-i)$.* (Bhopal, 2009 ; Marathwada, 2008 ; Mumbai, 2006)

Solution. (a) $$F(3.5) = \oint_C \frac{z^3 + z + 1}{z^2 - 7z + 2}dz$$

Since $\zeta = 3.5$ is the only singular point of $(4z^2 + z + 5)/(z - 3.5)$ and it lies outside the ellipse C, therefore, $(4z^2 + z + 5)/(z - 3.5)$ is analytic everywhere within C.

Hence by Cauchy's theorem,

$$\oint_C \frac{4z^2 + z + 5}{z - 3.5}\,dz = 0, \text{ i.e., } F(3.5) = 0.$$

(b) Since $f(z) = 4z^2 + z + 5$ is analytic within C and $\zeta = i, -1$ and $-i$ all lie within C, therefore, by Cauchy's integral formula

$$f(\zeta) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - \zeta}\,dz$$

i.e., $$\oint_C \frac{4z^2+z+5}{z-\zeta}dz = 2\pi i(4\zeta^2+\zeta+5)$$

i.e., $$F(\zeta) = 2\pi i(4\zeta^2+\zeta+5)$$

$\therefore$ $$F'(\zeta) = 2\pi i\,(8\zeta+1) \text{ and } F''(\zeta) = 16\pi i$$

Thus $$F(i) = 2\pi i(-4+i+5) = 2\pi(i-1)$$

$$F'(-1) = 2\pi i[8(-1)+1] = -14\pi i \text{ and } F''(-i) = 16\pi i.$$

20.15 (1) CONVERSE OF CAUCHY'S THEOREM: MORERA'S THEOREM*

If $f(z)$ is continuous in a region D and $\oint_C f(z)dz = 0$ around every simple closed curve C in D, then $f(z)$ is analytic in D.

Since $\oint f(z)dz = 0$, then the line integral of $f(z)$ from a fixed point z_0 to a variable point z must be independent of the path and hence must be a function of z only. Thus

$$\int_{z_0}^{z} f(z)dz = \phi(z), \text{ (say)},$$

Let $\phi(z) = U + iV$ and $f(z) = u + iv$

Then $$U + iV = \int_{(x_0, y_0)}^{(x,y)} (u+iv)(dx+idy) = \int_{(x_0, y_0)}^{(x,y)} (udx - vdy) + i\int_{(x_0, y_0)}^{(x,y)} (vdx+udy)$$

$\therefore$ $$U = \int_{(x_0, y_0)}^{(x,y)} (udx - vdy),\ V = \int_{(x_0, y_0)}^{(x,y)} (vdx+udy)$$

Differentiating under the integral sign,

$$\frac{\partial U}{\partial x} = u, \frac{\partial U}{\partial y} = -v, \frac{\partial V}{\partial x} = v, \frac{\partial V}{\partial y} = u \quad \therefore \quad \frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}, \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

Thus U and V satisfy C-R equations.

Also, since $f(z)$ is given to be continuous, u and v and therefore, $\partial U/\partial x$, $\partial U/\partial y$, $\partial V/\partial x$, $\partial V/\partial y$, are also continuous.

$\therefore$ $\phi(z)$ is an analytic function and

$$\phi'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = u + iv = f(z).$$

Thus, $f(z)$ is the derivative of an analytic function $\phi(z)$. Hence $f(z)$ is analytic by § 20.14 Cor.

(2) Cauchy's inequality†. *If $f(z)$ is analytic within and on the circle C: $|z-a| = r$, then*

$$|f^n(a)| \le \frac{Mn!}{r^n} \quad \text{...(I)}$$

where M is the maximum value of $|f(z)|$ on C.

From (5) of § 20.14, we get

$$|f^n(a)| = \frac{n!}{2\pi}\left|\oint_C \frac{f(z)dz}{(z-a)^{n+1}}\right|$$

$$\le \frac{n!}{2\pi}\cdot\frac{M}{r^{n+1}}\oint_C |z| \qquad [\because\ |f(z)| < M]$$

$$= \frac{n!M}{2\pi r^{n+1}}\oint_C ds = \frac{Mn!}{2\pi r^{n+1}}2\pi r = \frac{Mn!}{r^n} \qquad (U.P.T.U., 2005)$$

(3) Liouville's theorem$. *If $f(z)$ is analytic and bounded for all z in the entire complex plane, then $f(z)$ is a constant.* (U.P.T.U., 2008)

* Named after the Italian mathematician, *Giacinto Morera* (1856–1909) who worked in Turin.

† See footnote p. 144

$ See footnote p. 573.

Taking $n = 1$ and replacing a by z, (I) gives

$$|f'(z)| \leq M/r$$

As $r \to \infty$, it gives $f'(z) = 0$ *i.e.*, $f(z)$ is constant for all z.

(4) Poisson's integral formulae. *If $f(z)$ is analytic within and on the circle C: $|z| = \rho$ and $z = re^{i\theta}$ is any point within C, then*

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2 - r^2}{\rho^2 - 2r\rho\cos(\theta - \phi) + r^2} f(re^{i\phi})d\phi$$

By Cauchy's integral formula, $f(z) = \dfrac{1}{2\pi i}\oint_C \dfrac{f(w)}{w - z}\,dw$ ...(1)

As the inverse of the point x w.r.t. C lies outside C and is given by $\rho^2/\bar{z}$. [See footnote p. 685]

$\therefore$ by Cauchy's theorem,

$$0 = \frac{1}{2\pi i}\int \frac{f(w)}{w - \rho^2/\bar{z}}\,dw \qquad ...(2)$$

Subtracting (2) from (1), $f(z) = \dfrac{1}{2\pi i}\displaystyle\int\left(\frac{1}{w - z} - \frac{1}{w - \rho^2/\bar{z}}\right) f(w)\,dw$

$$= \frac{1}{2\pi i}\oint_C \frac{z\bar{z} - \rho^2}{\bar{z}w^2 - (z\bar{z} + \rho^2)\,w + z\rho^2)} f(w)\,dw \qquad ...(3)$$

Taking $w = \rho e^{i\phi}$ and noting that $\bar{z} = re^{-i\theta}$, (3) gives

$$f(re^{i\theta}) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{(r^2 - \rho^2)\,f(\rho e^{i\phi})\,.\,\rho ie^{i\phi}\,d\phi}{re^{-i\theta}\,.\,\rho^2 e^{2i\phi} - (r^2 + \rho^2)\,\rho e^{i\phi} + re^{i\theta}\rho^2}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(\rho^2 - r^2)\,f(\rho e^{i\phi})d\phi}{\rho^2 + r^2 - r\rho[e^{i(\theta-\phi)} + e^{-i(\theta-\phi)}]} = \frac{1}{2\pi}\int_0^{2\pi} \frac{(\rho^2 - r^2)\,f(\rho e^{i\phi})d\phi}{\rho^2 - 2r\rho\cos(\theta - \phi) + r^2} \qquad ...(4)$$

This is called *Poisson's integral formula** for a circle. It expresses the values of a harmonic function within a circle in terms of its values on the boundary.

Writing $f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$ and $f(\rho e^{i\phi}) = u(\rho, \phi) + iv(\rho, \phi)$ in (4) and equating real and imaginary parts from both sides, we get the formulae:

$$u(r, \theta) = \int_0^{2\pi} \frac{(\rho^2 - r^2)u(\rho, \phi)d\phi}{\rho^2 - 2r\rho\cos(\theta - \phi) + r^2} \qquad ...(5)$$

and

$$v(r, \theta) = \int_0^{2\pi} \frac{(\rho^2 - r^2)v(\rho, \phi)d\phi}{\rho^2 - 2r\rho\cos(\theta - \phi) + r^2} \qquad ...(6)$$

PROBLEMS 20.6

1. Evaluate $\oint_C (z - a)^{-1}dz$, where C is a simple closed curve and the point $z = a$ is (*i*) outside C, (*ii*) inside C.
2. Evaluate $\oint_C \dfrac{dz}{(z - a)^n}$, $n = 2, 3, 4, \ldots$, where C is a closed curve containing the point $z = a$.
3. Evaluate (*i*) $\oint_C \dfrac{e^z}{z^2 + 1}dz$, where C is the circle $|z| = 1/2$. (*P.T.U., 2010*)

   (*ii*) $\oint_C \dfrac{e^{3iz}}{(z + \pi)^3}dz$, where C is the circle $|z - \pi| = 3$. (*U.P.T.U., 2007*)

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* Named after the French mathematician *Simeon Denis Poisson* (1781–1840) who was a professor in Paris and made contributions to partial differential equations, potential theory and probability.

4. Use Cauchy's integral formula to calculate:

(i) $\oint_C \frac{3z-5}{z^2+2z}dz$, where C is $|z| = 1$. (P.T.U., 2005 S) (ii) $\oint_C \frac{z^2+1}{z(2z+1)}dz$, where C is $|z| = 1$.

(iii) $\oint_C \frac{\sin \pi z + \cos \pi z}{(z-1)(z-2)}\, dz$ where C is $|z| = 4$. (U.P.T.U., 2008)

5. Evaluate (a) $\oint_C \frac{z^3 - 2z + 1}{(z-i)^2}$ where C is $|z| = 2$.

(b) $\oint_C \frac{e^{-z}}{(z-1)(z-2)^2}dz$ where C is $|z| = 3$. (Rohtak, 2003)

6. Evaluate, using Cauchy's integral formulae:

(i) $\oint_C \frac{z}{z^2-3z+2}dz$, where C is $|z-2| = \frac{1}{2}$. (U.P.T.U., 2009 ; Hissar, 2007 ; Madras, 2000)

(ii) $\oint_C \frac{e^z dz}{(z+1)^2}$, where C is $|z-1| = 3$. (Bhopal, 2009)

(iii) $\oint_C \frac{\log z}{(z-1)^3}dz$ where C is $|z-1| = \frac{1}{2}$. (J.N.T.U., 2003)

7. Evaluate $f(2)$ and $f(3)$ where $f(a) = \oint_C \frac{2z^2 - z - 2}{z-a}dz$ and C is the circle $|z| = 2.5$.

8. If $\phi(\zeta) = \oint_C \frac{3z^2+7z+1}{z-\zeta}\, dz$, where C is the circle $|z| = 2$ find the values of

(i) $\phi(3)$, (ii) $\phi'(1-i)$ (iii) $\phi''(1-i)$. (Mumbai, 2006)

9. Evaluate $\oint_C \frac{z^3+z+1}{z^2-7z+2}\, dz$, where C is the ellipse $4x^2 + 9y^2 = 1$. (Rohtak, 2006)

10. Verify Cauchy's theorem for the integral of z^3 taken over the boundary of the (i) rectangle with vertices $-1, 1, 1+i, -1+i$; (ii) triangle with vertices (1, 2), (1, 4), (3, 2). (V.T.U., 2003)

20.16 (1) SERIES OF COMPLEX TERMS

Let $$(a_1 + ib_1) + (a_2 + ib_2) + \ldots + (a_n + ib_n) + \ldots \quad \ldots(1)$$
be an infinite series of complex terms ; a's and b's being real numbers. *If the series Σa_n and Σb_n converge to the sums A and B, then series (1) is said to* **converge** *to the sum* $A + iB$. Also if (1) is a convergent series, then $\underset{n\to\infty}{\text{Lt}} (a_n + ib_n) = 0$.

The series (1) is said to be **absolutely convergent** *if the series*
$$|a_1 + ib_1| + |a_2 + ib_2| + \ldots + |a_n + ib_n| + \ldots$$
is convergent. Since $|a_n|$ and $|b_n|$ are both $\leq |a_n + ib_n|$, it follows *that an absolutely convergent series is convergent.*

Next let the series of functions $u_1(z) + u_2(z) + \ldots + u_n(z) + \ldots\ldots$ ...(2)
converge to the sum $S(z)$ and $S_n(z)$ be the sum of its first n terms. Then *the series (2) is said to be* **uniformly convergent** *in a region R, if corresponding to any positive number ε, there exists a positive number N, depending on ε, but not on z, such that for every z in R.*
$$|S(z) - S_n(z)| < \varepsilon \text{ for } n > N. \quad [cf.\ Def.\ p.\ 389]$$

As in the case of real series (p. 390) **Weirstrass's M-test** holds for series of complex terms. So *the series (2) is uniformly convergent in a region R if there is a convergent series of positive constants ΣM_n such that $|u_n(z)| \leq M_n$ for all z in R.*

Also a *uniformly convergent series of continuous complex functions is itself continuous and can be integrated term by term.*

Obs. *If a power series $\Sigma a_n z^n$ converges for $z = z_1$, then it converges absolutely for $|z| < |z_1|$.*

Since $\Sigma a_n z_1^n$ converges, therefore, $\underset{n \to \infty}{\text{Lt}}\ a_1 z_1^n = 0$ and so we can find a number k such that $|a_n z_1^n| < k$ for all n. Then $\Sigma a_n z^n = \Sigma |a_n z_1^n| \,.\, |z/z_1|^n < \Sigma k t^n$ where $t = |z/z_1|$.

But the series Σt^n converges for $t < 1$. Hence the series $\Sigma a_n z^n$ converges absolutely for $|z| < |z_1|$, *i.e.*, if a circle with centre at the origin and radius $|z|$ be drawn, then the given series converges absolutely at all points inside the circle.

Such a circle $|z| = R$ within which series $\Sigma a_n z^n$ converges, is called the *circle of convergence and R is called the radius of convergence.*

A power series is uniformly convergent in any region which lies entirely within its circle of convergence.

(2) Taylor's series*. *If $f(z)$ is analytic inside a circle C with centre at a, then for z inside C,*

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \ldots + \frac{f^n(a)}{n!}(z-a)^n + \ldots \qquad \ldots(i)$$

Proof. Let z be any point inside C. Draw a circle C_1 with centre at a enclosing z (Fig. 20.20). Let t be a point on C_1. We have

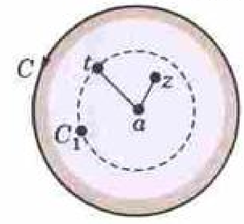

Fig. 20.20

$$\frac{1}{t-z} = \frac{1}{t-a-(z-a)} = \frac{1}{t-a}\left(1 - \frac{z-a}{t-a}\right)^{-1}$$

$$= \frac{1}{t-a}\left[1 + \frac{z-a}{t-a} + \left(\frac{z-a}{t-a}\right) + \ldots + \left(\frac{z-a}{t-a}\right)^n + \ldots\right] \qquad \ldots(ii)$$

As $|z-a| < |t-a|$, *i.e.* $|(z-a)/(t-a)| < 1$, this series converges uniformly. So, multiplying both sides of (*ii*) by $f(t)$, we can integrate over C_1.

$$\therefore \quad \oint_{C_1} \frac{f(t)}{t-z} dz = \oint_{C_1} \frac{f(t)}{t-a} dz + (z-a)\oint_{C_1} \frac{f(t)}{(t-a)^2} dt + \ldots + (z-a)^n \,.\, \oint_{C_1} \frac{f(t)}{(t-a)^{n+1}} dt + \ldots \qquad \ldots(iii)$$

Since $f(t)$ is analytic on and inside C_1, therefore, applying the formulae (2) to (5) of p. 697-698 (*iii*), we get (*i*) which is known as *Taylor's series.*

Obs. Another remarkable fact is that *complex analytic functions can always be represented by power series of the form* (*i*).

(3) Laurent's series†. *If $f(z)$ is analytic in the ring-shaped region R bounded by two concentric circles C and C_1 of radii r and r_1 $(r > r_1)$ and with centre at a, then for all z in R*

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \ldots$$

where $a_n = \dfrac{1}{2\pi i}\displaystyle\oint \frac{f(t)}{(t-a)^{n+1}}\, dt,$

Γ being any curve in R, encircling C_1 (as in Fig. 20.21).

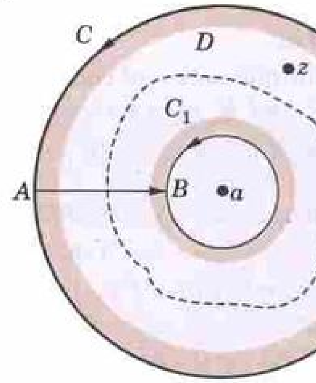

Fig. 20.21

Proof. Introduce cross-out AB, then $f(z)$ is analytic in the region D bounded by AB, C_1 described clockwise, BA and C described anti-clockwise (see Fig. 20.17). Then if z be any point in D, we have

$$f(z) = \frac{1}{2\pi i}\left[\int_{AB} \frac{f(t)}{t-z} dt + \oint_{C_1} \frac{f(t)}{t-z} dt + \int_{BA} \frac{f(t)}{t-z} dt + \oint_{C_1} \frac{f(t)}{t-z} dt\right]$$

$$= \frac{1}{2\pi i}\left[\oint_{C} \frac{f(t)}{t-z} dt - \oint_{C_1} \frac{f(t)}{t-z} dt\right] \qquad \ldots(i)$$

where both C and C_1 are described anti-clockwise in (*i*) and integrals along AB and BA cancel (Fig. 20.21).

For the first integral in (*i*), expanding $1/(t-z)$ as in § 20.16 (2), we get

$$\frac{1}{2\pi i}\oint_{C} \frac{f(t)}{t-z} dt = \sum_{n=1}^{\infty} \frac{(z-a)^n}{2\pi i} \oint_{C} \frac{f(t)}{(t-a)^{n+1}} dt$$

* See footnote p. 145.

† Named after the French engineer and mathematician *Pierre Alphonse Laurent* (1813–1854) who published this theorem in 1843.

$$= \Sigma a_n (z-a)^n \text{ where } a_n = \frac{1}{2\pi i} \oint_C \frac{f(t)}{(t-a)^{n+1}} dt \qquad ...(ii)$$

For the second integral in (i), let t lie on C_1. Then we write

$$\frac{1}{t-z} = \frac{1}{(t-a)-(z-a)} = -\frac{1}{z-a}\left(1-\frac{t-a}{z-a}\right)^{-1}$$

$$= -\frac{1}{z-a}\left[1+\frac{t-a}{z-a}+\left(\frac{t-a}{z-a}\right)^2+...+\left(\frac{t-a}{z-a}\right)^{n-1}+...\right]$$

As $|t-a| < |z-a|$, *i.e.*, $|(t-a)/(z-a)| < 1$, this series converges uniformly. So multiplying both sides by $f(t)$ and integrating over C_1, we get

$$-\frac{1}{2\pi i}\oint_C \frac{f(t)}{t-z} dt = \sum_{n=1}^{\infty} \frac{1}{(z-a)^n} \cdot \frac{1}{2\pi i} \oint_C (t-a)^{n-1} f(t) dt = \sum_{n=1}^{\infty} a_{-n}(z-a)^{-n} \qquad ...(iii)$$

where $$a_{-n} = \frac{1}{2\pi i}\oint_{C_1} \frac{f(t)}{(t-a)^{-n+1}} dt$$

Substituting from (ii) and (iii) in (i), we get

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} a_{-n}(z-a)^{-n}. \qquad ...(iv)$$

Now $f(t)/(t-a)^{n+1}$ being analytic in the region between C and Γ, we can take the integral giving a_n over Γ. Similarly we can take the integral giving a_{-n} over Γ. Hence (iv) can be written as

$$f(z) = \sum_{-\infty}^{\infty} a_n (z-a)^n \text{ where } a_n = \frac{1}{2\pi i}\oint_\Gamma \frac{f(t)}{(t-a)^{n+1}} dt$$

which is known as *Laurent's series.*

Obs. 1. *As $f(z)$ is not given to be analytic inside Γ,* $a_n = \frac{1}{2\pi i}\oint_\Gamma \frac{f(t)}{(t-a)^{n+1}} dt \neq \frac{f^n(a)}{n!}$

However, *if $f(z)$ is analytic inside Γ*, then $a_{-n} = 0$; $a_n = \frac{1}{2\pi i}\oint_\Gamma \frac{f(t)}{(t-a)^{n+1}} dt = \frac{f^n(a)}{n!}$

and *Laurent's series reduces to Taylor's series.*

Obs. 2. *To obtain Taylor's or Laurent's series, simply expand f(z) by binomial theorem* instead of finding a_n by complex integration which is quite complicated.

Obs. 3. *Laurent series of a given analytic function f(z) in its annulus of convergence is unique.* There may be different Laurent series of $f(z)$ in two annuli with the same centre.

Example 20.25. *Show that the series $z(1-z) + z^2(1-z) + z^3(1-z) + ... \infty$ converges for $|z| < 1$. Determine whether it converges absolutely or not.*

Solution. Let the sum of the first n terms of the series be s_n, so that

$$s_n = z - z^2 + z^2 - z^3 + z^3 - z^4 + ... + z^n - z^{n+1} = z - z^{n+1}$$

For $|z| < 1$, $z^{n+1} \to 0$ as $n \to \infty$.

$\therefore$ $\qquad \underset{n\to\infty}{\text{Lt}} (s_n) = z$, *i.e.*, the given series converges for $|z| < 1$.

$$|s_n(z)| = |z(1-z)| + |z^2(1-z)| + ... + |z^n(1-z)|$$
$$= |1-z| \{|z| + |z|^2 + |z|^3 + ... + |z|^n\}$$

For $|z| < 1$, $\qquad \underset{n\to\infty}{\text{Lt}} |s_n(z)| = |1-z| \frac{|z|}{1-|z|}$ [G.P.]

Hence the given series converges absolutely.

Example 20.26. *Expand sin z in a Taylor's series about z = 0 and determine the region of convergence.* (P.T.U., 2009 S)

Solution. Given $f(z) = \sin z, f'(z) = \cos \theta, f''(z) = -\sin z, f'''(z) = -\cos z, \ldots$

$\therefore$ $f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = 1$

By Taylor's series about $z = 0$, we have

$$f(z) = f(0) + \frac{(z-0)}{1!}f'(0) + \frac{(z-0)^2}{2!}f''(0) + \frac{(z-0)^3}{3!}f'''(0) + \ldots$$

i.e.,
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots + (-1)^{n-1}\frac{z^{2n}-1}{(2n-1)!} + \ldots$$

Hence
$$\sin z = \sum_{n=1}^{\infty} a_n (z-0)^{2n-1} \text{ where } a_n = \frac{(-1)^{n-1}}{(2n-1)!}$$

Since
$$\rho = \underset{n\to\infty}{\text{Lt}} \left|\frac{a_{n+1}}{a_n}\right| = \underset{n\to\infty}{\text{Lt}} \left|\frac{(2n-1)!}{(2n+1)!}\right| = 0$$

Thus the radius of convergence of $f(z) = 1/\rho = \infty$

i.e., the region of convergence of $f(z)$ is all reals.

Example 20.27. *Find Taylor's expansion of*

(*i*) $f(z) = \dfrac{1}{(z+1)^2}$ *about the point* $z = -i$. (V.T.U., 2009 S)

(*ii*) $f(z) = \dfrac{2z^3+1}{z^2+z}$ *about the point* $z = i$. (P.T.U., 2003)

Solution. (*i*) To expand $f(z)$ about $z = -i$, *i.e.*, in powers of $z + i$, put $z + i = t$. Then

$$f(z) = \frac{1}{(t-i+1)^2} = (1-i)^{-2}\,[1 + t/(1-i)]^{-2} = \frac{i}{2}\left[1 - \frac{2t}{1-i} + \frac{3t^2}{(1-i)^2} - \frac{4t^3}{(1-i)^3} + \ldots\right]$$

[Expanding by Binomial theorem]

$$= \frac{i}{2}\left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{(n+1)(z+i)^n}{(1-i)^n}\right]$$

(*ii*)
$$f(z) = \frac{2z^3+1}{z(z+1)} = 2z - 2 + \frac{2z+1}{z(z+1)} = (2i-2) + 2(z-i) + \frac{1}{z} + \frac{1}{z+1} \qquad \ldots(i)$$

[By partial fractions]

To expand $1/z$ and $1/(z+1)$ about $z = i$, put $z - i = t$, so that

$$\frac{1}{z} = \frac{1}{(t+i)} = \frac{1}{i}\left(1 + \frac{t}{i}\right)^{-1}$$ [Expanding by Binomial theorem]

$$= \frac{1}{i}\left[1 - \frac{t}{i} + \frac{t^2}{i^2} - \frac{t^3}{i^3} + \frac{t^4}{i^4} - \ldots \infty\right] = \frac{1}{i} + \frac{t}{1} + \frac{t^2}{i^3} - \frac{t^3}{i^4} + \frac{t^4}{i^5} - \ldots \infty$$

$$= -i + (z-i) + \sum_{n=2}^{\infty} (-1)^n \frac{(z-i)^n}{i^{n+1}} \qquad \ldots(ii)$$

and
$$\frac{1}{z+1} = \frac{1}{t+i+1} = \frac{1}{1+i}\left(1 + \frac{t}{1+i}\right)^{-1}$$ [Expanding by Binomial theorem]

$$= \frac{1}{1+i}\left[1 - \frac{t}{1+i} + \frac{t^2}{(1+t)^2} - \frac{t^3}{(1+i)^3} + \frac{t^4}{(1+i)^4} - \ldots \infty\right]$$

$$= \frac{1-i}{2} - \frac{t}{2i} + \left[\frac{t^2}{(1+i)^3} - \frac{t^3}{(1+i)^4} + \frac{t^4}{(1+i)^5} - \ldots \infty\right] = \frac{1}{2} - \frac{i}{2} - \frac{z-i}{2i} + \sum_{n=2}^{\infty} (-1)^n \frac{(z-i)^n}{(1+i)^{n+1}} \qquad \ldots(iii)$$

Substituting from (*ii*) and (*iii*) in (*i*), we get

$$f(z) = \left(2i - 2 - i + \frac{1}{2} - \frac{i}{2}\right) + \left(2 + 1 - \frac{1}{2i}\right)(z - i) + \sum_{n=2}^{\infty} (-1)^n \left\{\frac{1}{i^{n+1}} + \frac{1}{(1+i)^{n+1}}\right\}(z-i)^n$$

$$= \left(\frac{i}{2} - \frac{3}{2}\right) + \left(3 + \frac{i}{2}\right)(z - i) + \sum_{n=2}^{\infty} (-1)^n \left\{\frac{1}{i^{n+1}} + \frac{1}{(1+i)^{n+1}}\right\}(z-i)^n.$$

Example 20.28. *Expand $f(z) = 1/[(z-1)(z-2)]$ in the region:*
(a) $|z| < 1$, (b) $1 < |z| < 2$, (c) $|z| > 2$, (d) $0 < |z - 1| < 1$.
(U.P.T.U., 2010 ; V.T.U., 2010 ; Bhopal, 2009)

Solution. (*a*) By partial fractions $\dfrac{1}{(z-1)(z-2)} = \dfrac{1}{z-2} - \dfrac{1}{z-1}$ ...(*i*)

$$= -\frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1} + (1-z)^{-1} \quad ...(ii)$$

For $|z| < 1$, both $|z/2|$ and $|z|$ are less than 1. Hence (*ii*) gives on expansion

$$f(z) = -\frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \ldots\right) + (1 + z + z^2 + z^3 + \ldots)$$

$$= \frac{1}{2} + \frac{3}{4}z + \frac{7}{8}z^2 + \frac{15}{16}z^3 + \ldots \text{ which is a } \textit{Taylor's series.}$$

(*b*) *For* $1 < |z| < 2$, we write (*i*) as

$$f(z) = -\frac{1}{2}\frac{1}{(1 - z/2)} - \frac{1}{z(1 - z^{-1})} \quad ...(iii)$$

and notice that both $|z/2|$ and $|z^{-1}|$ are less than 1. Hence (*iii*) gives on expansion

$$f(z) = -\frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \ldots\right) - \frac{1}{z}(1 + z^{-1} + z^{-2} + z^{-3} + \ldots)$$

$$= \ldots - z^{-4} - z^{-3} - z^{-2} - z^{-1} - \frac{1}{2} - \frac{1}{4}z - \frac{1}{8}z^2 - \frac{1}{16}z^3 - \ldots$$

which is a *Laurent's series.*

(*c*) *For* $|z| > 2$, we write (*i*) as

$$f(z) = \frac{1}{z(1 - 2z^{-1})} - \frac{1}{z(1 - z^{-1})}$$

$$= z^{-1}(1 + 2z^{-1} + 4z^{-2} + 8z^{-3} + \ldots) - z^{-1}(1 + z^{-1} + z^{-2} + z^{-3} + \ldots)$$

$$= \ldots + 7z^{-4} + 3z^{-3} + z^{-2} + \ldots$$

(*d*) *For* $0 < |z - 1| < 1$, we write (*i*) as

$$f(z) = \frac{1}{(z-1)-1} - \frac{1}{z-1}$$

$$= -(z-1)^{-1} - [1 - (z-1)]^{-1}$$

$$= -(z-1)^{-1} - [1 + (z-1) + (z-1)^2 + (z-1)^3 + \ldots].$$

Example 20.29. *Find the Laurents' expansion of $f(z) = \dfrac{7z-2}{(z+1)z(z-2)}$ in the region $1 < z + 1 < 3$.*
(S.V.T.U., 2009 ; Anna, 2003 ; V.T.U., 2003)

Solution. Writing $z + 1 = u$, we have

$$f(z) = \frac{7(u-1)-2}{u(u-1)(u-1-2)} = \frac{7u-9}{u(u-1)(u-3)}$$

$$= -\frac{3}{u} + \frac{1}{u-1} + \frac{2}{u-3} \quad \text{(splitting into partial fraction)}$$

$$= -\frac{3}{u} + \frac{1}{u(1-1/u)} - \frac{2}{3(1-u/3)} = -\frac{3}{u} + \frac{1}{u}\left(1 - \frac{1}{u}\right)^{-1} - \frac{2}{3}\left(1 - \frac{u}{3}\right)^{-1}$$

Since $1 < u < 3$ or $1/u < 1$ and $u/3 < 1$, expanding by Binomial theorem,

$$f(z) = \frac{-3}{u} + \frac{1}{u}\left(1 + \frac{1}{u} + \frac{1}{u^2} + \frac{1}{u^3} + \dots \infty\right) - \frac{2}{3}\left(1 + \frac{u}{3} + \frac{u^2}{3^2} + \frac{u^3}{3^3} + \dots \infty\right)$$

$$= -\frac{2}{u} + \frac{1}{u^2} + \frac{1}{u^3} + \dots \infty - \frac{2}{3}\left(1 + \frac{u}{3} + \frac{u^2}{3^2} + \frac{u^3}{3^3} + \dots \infty\right)$$

Hence $$f(z) = -\frac{2}{z+1} + \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} + \dots \infty - \frac{2}{3}\left[1 + \frac{z+1}{3} + \frac{(z+1)^2}{3^2} + \frac{(z+1)^3}{3^3} + \dots \infty\right]$$

which is valid in the region $1 < z + 1 < 3$.

20.17 (1) ZEROS OF AN ANALYTIC FUNCTION

Def. *A zero of an analytic function $f(z)$ is that value of z for which $f(z) = 0$.*

If $f(z)$ is analytic in the neighbourhood of a point $z = a$, then by Taylor's theorem

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_n(z-a)^n + \dots \qquad \text{where } a_n = \frac{f^n(a)}{n!}.$$

If $a_0 = a_1 = a_2 = \dots = a_{m-1} = 0$ but $a_m \neq 0$, then $f(z)$ is said to have a *zero of order* m at $z = a$.

When $m = 1$, the zero is said to be *simple*. In the neighbourhood of zero ($z = a$) of order m,

$$f(z) = a_m(z-a)^m + a_{m+1}(z-a)^{m+1} + \dots \infty$$

$$= (z-a)^m \phi(z) \text{ where } \phi(z) = a_m + a_{m+1}(z-a) + \dots$$

Then $\phi(z)$ is analytic and non-zero in the neighbourhood of $z = a$.

(2) Singularities of an analytic function

We have already defined a *singular point of a function as the point at which the function ceases to be analytic.*

(*i*) **Isolated singularity.** *If $z = a$ is a singularity of $f(z)$ such that $f(z)$ is analytic at each point in its neighourhood* (*i.e.*, there exists a circle with centre a which has no other singularity), *then $z = a$ is called an* **isolated singularity.**

In such a case, $f(z)$ can be expanded in a Laurent's series around $z = a$, giving

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \dots \qquad \dots(1)$$

For example, $f(z) = \cot(\pi/z)$ is not analytic where $\tan(\pi/z) = 0$ *i.e.* at the points $\pi/z = 4\pi$ or $z = 1/n$ ($n = 1, 2, 3, \dots$).

Thus $z = 1, 1/2, 1/3, \dots$ are all *isolated singularities* as there is no other singularity in their neighbourhood.

But when n is large, $z = 0$ is such a singularity that there are infinite number of other singularities in its neighbourhood. Thus $z = 0$ is the *non-isolated singularity* of $f(z)$.

(*ii*) **Removable singularity.** If all the negative powers of $(z - a)$ in (1) are zero, then $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$.

Here the singularity can be removed by defining $f(z)$ at $z = a$ in such a way that it becomes analytic at $z = a$. Such a singularity is called a *removable singularity.*

Thus *if* $\underset{z\to a}{Lt} f(z)$ *exists finitely, then $z = a$ is a removable singularity.*

(*iii*) **Poles.** *If all the negative powers of $(z - a)$ in (i) after the nth are missing, then the singularity at $z = a$ is called a* **pole of order n.**

A pole of first order is called a **simple pole.**

(*iv*) **Essential singularity.** *If the number of negative powers of $(z - a)$ in (1) is infinite, then $z = a$ is called an essential singularity.* In this case, $\underset{z\to a}{\text{Lt}} f(z)$ does not exist.

Example 20.30. *Find the nature and location of singularities of the following functions:*

(*i*) $\dfrac{z - \sin z}{z^2}$ (*ii*) $(z + 1)\sin\dfrac{1}{z-2}$ (*iii*) $\dfrac{1}{\cos z - \sin z}$.

Solution. (*i*) Here $z = 0$ is a singularity.

Also $$\frac{z - \sin z}{z^2} = \frac{1}{z^2}\left\{z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots\right)\right\} = \frac{z}{3!} - \frac{z^3}{5!} + \frac{z^5}{7!} - \dots$$

Since there are no negative powers of z in the expansion, $z = 0$ *is a removable singularity.*

(*ii*) $(z + 1) \sin \dfrac{1}{z-2} = (t + 2 + 1) \sin \dfrac{1}{t}$ where $t = z - 2$

$$= (t + 3)\left\{\frac{1}{t} - \frac{1}{3!t^3} + \frac{1}{5!t^5} - \dots\right\} = \left(1 - \frac{1}{3!t^2} + \frac{1}{5!t^4} - \dots\right) + \left(\frac{3}{t} - \frac{1}{2t^3} + \frac{3}{5!t^5} - \dots\right)$$

$$= 1 + \frac{3}{t} - \frac{1}{6t^2} - \frac{1}{2t^3} + \frac{1}{120t^4} - \dots = 1 + \frac{3}{z-2} - \frac{1}{6(z-2)^2} - \frac{1}{2(z-2)^3} + \dots$$

Since there are infinite number of terms in the negative powers of $(z - 2)$, $z = 2$ *is an essential singularity.*

(*iii*) Poles of $f(z) = \dfrac{1}{\cos z - \sin z}$ are given by equating the denominator to zero, *i.e.*, by $\cos z - \sin z = 0$ or $\tan z = 1$ or $z = \pi/4$. Clearly $z = \pi/4$ is a *simple pole of* $f(z)$.

Example 20.31. *What type of singularity have the following functions :*

(*i*) $\dfrac{1}{1-e^z}$ (*ii*) $\dfrac{e^{2z}}{(z-1)^4}$ (*iii*) $\dfrac{e^{1/z}}{z^2}$. (*U.P.T.U., 2009*)

Solution. (*i*) Poles of $f(z) = 1/(1 - e^z)$ are found by equating to zero $1 - e^z = 0$ or $e^z = 1 = e^{2n\pi i}$

$\therefore$ $$z = 2n\pi i \ (n = 0, \pm 1, \pm 2, \dots)$$

Clearly $f(z)$ *has a simple pole at* $z = 2\pi i$.

(*ii*) $\dfrac{e^{2z}}{(z-1)^4} = \dfrac{e^{2(t+1)}}{t^4} = \dfrac{e^2}{t^4}\cdot e^{2t}$ where $t = z - 1$

$$= \frac{e^2}{t^4}\left\{1 + \frac{2t}{1!} + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} + \frac{(2t)^5}{5!} + \dots\right\} = e^2\left\{\frac{1}{t^4} + \frac{2}{t^3} + \frac{2}{t^2} + \frac{4}{3t} + \frac{2}{3} + \frac{4t}{15} + \dots\right\}$$

$$= e^2\left\{\frac{1}{(z-1)^4} + \frac{2}{(z-1)^3} + \frac{2}{(z-1)^2} + \frac{4}{3(z-1)} + \frac{2}{3} + \frac{4}{15}(z-1) + \dots\right\}$$

Since there are finite (4) number of terms containing negative powers of $(z - 1)$,

$\therefore$ $z = 1$ *is a pole of 4th order.*

(*iii*) $$f(z) = \frac{e^{1/z}}{z^2} = \frac{1}{z^2}\left\{1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots\right\} = z^{-2} + z^{-3} + \frac{z^{-4}}{2} + \dots \infty$$

Since there are infinite number of terms in the negative powers of z, therefore $f(z)$ has *an essential singularity* at $z = 0$.

PROBLEMS 20.7

1. Obtain the expansion of $(z - 1)/z^2$ in a Taylor's series in powers of $(z - 1)$ and determine the region of convergence.
2. Find the first three times of the Taylor's series expansion of $f(z) = 1/(z^2 + 4)$ about $z = -i$. Also find the region of convergence. (*U.P.T.U., 2006*)
3. Expand in Taylor's series (*i*) $(z - 1)/(z + 1)$ about the point $z = 1$. (*Andhra, 2000*)

   (*ii*) $\cos z$ about the point $z = \pi/2$. (*Marathwada, 2008*) (*iii*) $\dfrac{1}{z^2-z-6}$ about (*a*) $z = -1$ (*b*) $z = 1$ (*P.T.U., 2009*)
4. Expand the following functions in Laurent's series :

   (*i*) $f(z) = \dfrac{1}{z-z^2}$ for $1 < |z + 1| < 2$. (*Madras, 2006*) (*ii*) $f(z) = \dfrac{1}{(z-1)(z+3)}$ for $1 < |z| < 3$. (*J.N.T.U., 2006*)

(*iii*) $f(z) = z/[(z-1)(z-3)]$ for $|z-1| < 2$. (*V.T.U., 2007*)

5. Find the Laurent's expansion of (*i*) $\dfrac{e^z}{(z-1)^2}$, about $z = 1$. (*Rohtak, 2006*)

   (*ii*) $e^{2z}/(z-1)^3$ about the singularity $z = 1$.

6. Expand the following functions in Laurent series.

   (*i*) $(z-1)/z^2$ for $|z-1| > 1$ (*ii*) $\dfrac{1-\cos z}{z^3}$, about $z = 0$. (*Rohtak, 2004*)

7. Find the Laurent's series expansion of

   (*i*) $\dfrac{z^2-1}{z^2+5z+6}$ about $z = 0$ in the region $2 < |z| < 3$ (*V.T.U., 2011 S ; Osmania, 2003*)

   (*ii*) $\dfrac{z^2-6z-1}{(z-1)(z-3)(z+2)}$ in the region $3 < |z+2| < 5$

   (*iii*) $\dfrac{7z^2-9z-18}{z^3-9z}$ in the region (*a*) $|z| > 3$ (*b*) $0 < |z-3| < 3$. (*V.T.U., 2010 S*)

8. Find the Laurent's expansion of $1/[(z^2+1)(z^2+2)]$ for (*a*) $0 < |z| < 1$; (*b*) $1 < |z| < \sqrt{2}$; (*c*) $|z| > 2$.

   Find the nature and location of the singularities of the following functions : (*P.T.U., 2005*)

9. $\dfrac{1}{z(2-z)}$.
10. $\sin(1/z)$. (*U.P.T.U., 2009*)
11. $\tan\left(\dfrac{1}{z}\right)$. (*P.T.U., 2006*)
12. $\dfrac{z^2-1}{(z-1)^3}$. (*Osmania, 2003*)
13. $\dfrac{e^z}{(z-1)^4}$.
14. $\dfrac{\cot \pi z}{(z-a)^2}$. (*U.P.T.U., 2008*)

20.18 (1) RESIDUES

The coefficient of $(z-a)^{-1}$ *in the expansion of* $f(z)$ *around an isolated singularity is called the* **residue** *of* $f(z)$ *at that point.* Thus in the Laurent's series expansion of $f(z)$ around $z = a$ *i.e.*, $f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \dots$, the residue of $f(z)$ at $z = a$ is a_{-1}.

$$\therefore \qquad Res\, f(a) = \frac{1}{2\pi i}\oint_C f(z)$$

i.e.,
$$\oint_C f(z)\,dz = 2\pi i \operatorname{Res} f(a). \qquad \dots(1)$$

(2) Residue Theorem

If $f(z)$ *is analytic in a closed curve C except at a finite number of singular points within C, then* $\oint_C f(z)\,dz = 2\pi i \times$ *(sum of the residues at the singular points within C).*

Let us surround each of the singular points $a_1, a_2, \dots, a_n$ by a small circle such that it encloses no other singular point (Fig. 20.22). Then these circles $C_1, C_2, \dots, C_n$ together with C, form a multiply connected region in which $f(z)$ is analytic.

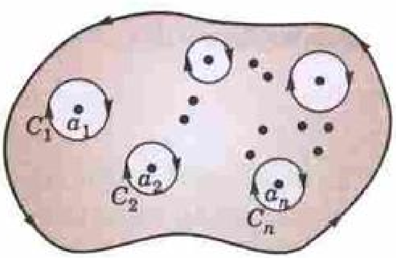

Fig. 20.22

$\therefore$ applying Cauchy's theorem, we have

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \dots + \oint_{C_n} f(z)dz \qquad \text{[by (1)}$$

$$= 2\pi i\,[\operatorname{Res} f(a_1) + \operatorname{Res} f(a_2) + \dots + \operatorname{Res} f(a_n)] \text{ which is the desired result.}$$

20.19 CALCULATION OF RESIDUES

(1) *If $f(z)$ has a simple pole at $z = a$, then*

$$\textbf{Res f(a)} = \underset{z \to a}{\textbf{Lt}} \; [(\mathbf{z - a})\mathbf{f(z)}]. \quad ...(1)$$

Laurent's series in this case is

$$f(z) = c_0 + c_1(z-a) + c_2(z-a)^2 \ldots + c_{-1}(z-a)^{-1}$$

Multiplying throughout by $z - a$, we have

$$(z-a)f(z) = c_0(z-a) + c_1(z-a)^2 + \ldots + c_{-1}.$$

Taking limits as $z \to a$, we get

$$\underset{z \to a}{\text{Lt}} [(z-a)f(z)] = c_{-1} = \text{Res } f(a).$$

(2) *Another formula for Res $f(a)$:*

Let $f(z) = \phi(z)/\psi(z)$, where $\psi(z) = (z-a)\,F(z)$, $F(a) \neq 0$.

Then $$\underset{z \to a}{\text{Lt}} [(z-a)\phi(z)/\psi(z)]$$

$$= \underset{z \to a}{\text{Lt}} \frac{(z-a)[\phi(a) + (z-a)\phi'(a) + \ldots]}{\psi(a) + (z-a)\psi'(a) + \ldots}$$

$$= \underset{z \to a}{\text{Lt}} \frac{\phi(a) + (z-a)\phi'(a) + \ldots}{\psi'(a) + (z-a)\psi''(a) + \ldots}, \quad \text{since } \psi(a) = 0$$

Thus $$\textbf{Res f(a)} = \frac{\boldsymbol{\phi}(\mathbf{a})}{\boldsymbol{\psi}'(\mathbf{a})}.$$

(3) *If $f(z)$ has a pole of order n at $z = a$, then*

$$\textbf{Res f(a)} = \frac{\mathbf{1}}{(\mathbf{n-1})!}\left\{\frac{\mathbf{d^{n-1}}}{\mathbf{dz^{n-1}}}[(\mathbf{z-a})^{\mathbf{n}}\mathbf{f(z)}]\right\}_{\mathbf{z=a}}$$

Here $$f(z) = c_0 + c_1(z-a) + c_2(z-a)^2 + \ldots + c_{-1}(z-a)^{-1} + \ldots + c_{-n}(z-a)^{-n}.$$

Multiplying throughout by $(z-a)^n$, we get

$$(z-a)^n f(z) = c_0(z-a)^n + c_1(z-a)^{n+1} + c_2(z-a)^{n+2} + \ldots + c_{-1}(z-a)^{n-1} + c_{-2}(z-a)^{n-2} + \ldots + c_{-n}.$$

Differentiating both sides w.r.t. z, $n - 1$ times and putting $z = a$, we get

$$\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]\right\}_{z=a} = (n-1)!\, c_{-1} \text{ whence follows the result.}$$

Obs. In many cases, the residue of a pole $(z = a)$ can be found, by putting $z = a + t$ in $f(z)$ and expanding it in powers of t where $|\,t\,|$ is quite small.

Example 20.32. *Find the sum of the residues of $f(z) = \dfrac{\sin z}{z \cos z}$ at its poles inside the circle $|z| = 2$.*

(Rohtak, 2004)

Solution. $f(z)$ has simple poles at $z = 0, \pm \pi/2, \pm 3\pi/2, - \ldots$

Only the poles $z = 0$ and $z = \pm \pi/2$ lies inside $|z| = 2$.

$$\therefore \quad \text{Res } f(0) = \underset{z \to 0}{\text{Lt}} [z \, . \, f(z)] = \underset{z \to 0}{\text{Lt}} \left(\frac{\sin z}{\cos z}\right) = 0.$$

$$\text{Res } f(\pi/2) = \underset{z \to \pi/2}{\text{Lt}} \left[\left(z - \frac{\pi}{2}\right) f(z)\right] = \underset{z \to \pi/2}{\text{Lt}} \left\{\frac{(z - \pi/2)\sin z}{z \cos z}\right\}$$

$$= \underset{z \to \pi/2}{\text{Lt}} \frac{(z - \pi/2)\cos z + \sin z}{\cos z - z \sin z} \qquad \left[\text{Being } \frac{0}{0} \text{ form}\right]$$

$$= \frac{1}{-\pi/2} = -\frac{2}{\pi}$$

and $\quad \text{Res } f(-\pi/2) = \underset{z \to -\pi/2}{\text{Lt}} \left\{ \dfrac{(z + \pi/2) \sin z}{z \cos z} \right\} = \underset{z \to -\pi/2}{\text{Lt}} \dfrac{(z + \pi/2) \cos z + \sin z}{\cos z - z \sin z} = \dfrac{-1}{-\pi/2} = \dfrac{2}{\pi}$

Hence sum of residues $= 0 - \dfrac{2}{\pi} + \dfrac{2}{\pi} = 0.$

Example 20.33. *Determine the poles of the function*

$f(z) = z^2/(z-1)^2(z+2)$ *and the residue at each pole.* (S.V.T.U., 2008 ; J.N.T.U., 2005)

Hence evaluate $\oint_C f(z)dz$, *where C is the circle* $|z| = 2.5$.

Solution. Since $\underset{z \to -2}{\text{Lt}} \{(z+2)f(z)\} = \underset{z \to -2}{\text{Lt}} \dfrac{z^2}{(z-1)^2} = \dfrac{4}{9}$.

which is finite and non-zero, the function has a simple pole at $z = -2$ and Res $f(-2) = 4/9$.

Also since $\underset{z \to 1}{\text{Lt}} [(z-1)^2 f(z)]$ is finite and non-zero, $f(z)$ has a pole of order two at $z = 1$.

$$\therefore \quad \text{Res } f(1) = \frac{1}{1!}\left[\frac{d}{dz}\{(z-1)^2 f(z)\}\right]_{z=1} = \left[\frac{d}{dz}\left(\frac{z^2}{z+2}\right)\right]_{z=1} = \left[\frac{z^2+4z}{(z+2)^2}\right]_{z=1} = \frac{5}{9}.$$

[*Otherwise writing* $z = 1 + t$,

$$f(z) = \frac{(1+t)^2}{t^2(3+t)} = \frac{1}{3t^2}(1+t)^2(1+t/3)^{-1} = \frac{1}{3t^2}(1+t)^2\left(1 - \frac{t}{3} + \frac{t^2}{9} - \ldots\right)$$

$$= \frac{1}{3t^2}\left(1 + \frac{5}{3}t + \frac{4}{9}t^2 - \ldots\right) = \frac{1}{3t^2} + \frac{5}{9t} + \frac{4}{27} - \ldots \quad \ldots(i)$$

$$\therefore \quad \text{Res } f(1) = \text{coefficient of } \frac{1}{t} \text{ in } (i) = \frac{5}{9}.]$$

Clearly $f(z)$ is analytic on $|z| = 2.5$ and at all points inside except the poles $z = -2$ and $z = 1$. Hence by residue theorem

$$\oint_C f(z)dz = 2\pi i \,[\text{Res } f(-2) + \text{Res } f(1)] = 2\pi i \left[\frac{4}{9} + \frac{5}{9}\right] = 2\pi i.$$

Example 20.34. *Find the residue of* $f(z) = \dfrac{z^3}{(z-1)^4(z-2)(z-3)}$ *at its poles and hence evaluate* $\oint_C f(z)dz$ *where C is the circle* $|z| = 2.5$. (U.P.T.U., 2003)

Solution. The poles of $f(z)$ are given by $(z-1)^4(z-2)(z-3) = 0$.

$\therefore$ $z = 1$ is a pole of order 4, while $z = 2$ and $z = 3$ are simple poles.

$$\text{Res } f(1) = \frac{1}{3!}\frac{d^3}{dz^3}\left\{(z-1)^4 \cdot \frac{z^3}{(z-1)^4(z-2)(z-3)}\right\}_{z=1} = \frac{1}{6}\frac{d^3}{dz^3}\left\{\frac{z^3}{(z-2)(z-3)}\right\}_{z=1}$$

$\therefore$ $z = 1$ is a pole of order 4, while $z = 2$ and $z = 3$ are simple poles.

$$\text{Res } f(1) = \frac{1}{3!}\frac{d^3}{dz^3}\left\{(z-1)^4 \cdot \frac{z^3}{(z-1)^4(z-2)(z-3)}\right\}_{z=1} = \frac{1}{6}\frac{d^3}{dz^3}\left\{\frac{z^3}{(z-2)(z-3)}\right\}_{z=1}$$

$$= \frac{1}{6}\frac{d^3}{dz^3}\left[z + 5 - \frac{8}{z-2} + \frac{27}{z-3}\right] = \frac{1}{6}\left[-8 \cdot \frac{(-1)^3\, 3!}{(z-2)^4} + \frac{27 \cdot (-1)^3\, 3!}{(z-2)^4}\right]_{z=1}$$

$$= -\left[-8 + \frac{27}{16}\right] = \frac{101}{16}.$$

$$\text{Res } f(2) = \underset{z \to 2}{\text{Lt}} \left\{ (z-2) \cdot \frac{z^3}{(z-1)^4 (z-2)(z-3)} \right\} = \underset{z \to 2}{\text{Lt}} \left\{ \frac{z^3}{(z-1)^4 (z-3)} \right\} = \frac{8}{(1)^4(-1)} = -8$$

$$\text{Res } f(3) = \underset{z \to 3}{\text{Lt}} \left\{ (z-3) \cdot \frac{z^3}{(z-1)^4 (z-2)(z-3)} \right\} = \frac{27}{(2)^4 \cdot 1} = \frac{27}{16}$$

Now $\oint_C f(z) dz = 2\pi i \, [\text{Res } f(1) + \text{Res } f(2)]$ [$\because$ Pole $z = 3$ is outside C]

$$= 2\pi i \left(\frac{101}{16} - 8 \right) = \frac{-27\pi i}{8}.$$

Example 20.35. *Evaluate*

$\oint_C \frac{z-3}{z^2 + 2z + 5} dz$, *where C is the circle*

(*i*) $|z| = 1$, (*ii*) $|z + 1 - i| = 2$, (*iii*) $|z + 1 + i| = 2$. (*J.N.T.U., 2003*)

Solution. The poles of $f(z) = \dfrac{z-3}{z^2 + 2z + 5}$ are given by $z^2 + 2z + 5 = 0$

i.e., by $z = \dfrac{-2 \pm \sqrt{(4 - 20)}}{2} = -1 \pm 2i.$

(*i*) Both the poles $z = -1 + 2i$ and $z = -1 - 2i$ lie outside the circle $|z| = 1$. Therefore, $f(z)$ is analytic everywhere within C.

Hence by Cauchy's theorem, $\oint_C \dfrac{z-3}{z^2 + 2z + 5} dz = 0.$

(*ii*) Here only one pole $z = -1 + 2i$ lies inside the circle $C : |z + 1 - i| = 2$. Therefore, $f(z)$ is analytic within C except at this pole.

$$\therefore \quad \text{Res } f(-1 + 2i) = \underset{z \to -1+2i}{\text{Lt}} [\{z - (-1 + 2i)\} f(z)] = \underset{z \to -1+2i}{\text{Lt}} \frac{(z + 1 - 2i)(z-3)}{z^2 + 2z + 5}$$

$$= \underset{z \to -1+2i}{\text{Lt}} \frac{z-3}{z + 1 + 2i} = \frac{-4 + 2i}{4i} = i + 1/2.$$

Hence by residue theorem $\oint_C f(z) dz = 2\pi i \text{ Res } f(-1 + 2i) = 2\pi i (i + 1/2) = \pi(i - 2).$

(*iii*) Here only the pole $z = -1 - 2i$ lies inside the circle $C : |z + 1 + i| = 2$. Therefore, $f(z)$ is analytic within C except at this pole.

$$\therefore \quad \text{Res } f(-1 - 2i) = \underset{z \to -1-2i}{\text{Lt}} \frac{(z + 1 + 2i)(z-3)}{z^2 + 2z + 5}$$

$$= \underset{z \to -1-2i}{\text{Lt}} \frac{z-3}{z + 1 - 2i} = \frac{-4 - 2i}{-4i} = \tfrac{1}{2} - i$$

Hence by residue theorem, $\oint_C f(z) dz = 2\pi i \text{ Res } f(-1 - 2i) = 2\pi i \left(\tfrac{1}{2} - i\right) = \pi(2 + i).$

Example 20.36. *Evaluate* $\oint_C \dfrac{e^z}{\cos \pi z} dz$, *where C is the unit circle* $|z| = 1$. (*Rohtak, 2006*)

Solution. $f(z) = e^x/\cos \pi z$ has simple poles at $z = \pm 1/2, \pm 3/2, \pm 5/2, \ldots$

Out of these only the poles at $z = 1/2$ and $z = -1/2$ lie inside the given circle $|z| = 1$.

$$\therefore \quad \text{Res } f(1/2) = \underset{z \to 1/2}{\text{Lt}} \left[\left(z - \frac{1}{2} \right) f(z) \right] = \underset{z \to 1/2}{\text{Lt}} \left\{ \frac{\left(z - \frac{1}{2} \right) e^z}{\cos \pi z} \right\} \quad \left[\frac{0}{0} \text{ form} \right]$$

$$= \underset{z \to 1/2}{\text{Lt}} \frac{e^z + \left(z - \frac{1}{2}\right)e^z}{-\pi \sin \pi z} = \frac{e^{1/2}}{-\pi}$$

and
$$\text{Res } f(-1/2) = \underset{z \to -1/2}{\text{Lt}} \left\{ \frac{\left(z + \frac{1}{2}\right)e^z}{\cos \pi z} \right\} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \underset{z \to -1/2}{\text{Lt}} \frac{e^z + \left(z + \frac{1}{2}\right)e^z}{-\pi \sin \pi z} = \frac{e^{-1/2}}{\pi}$$

Hence
$$\oint_C \frac{e^z}{\cos \pi z} dz = 2\pi i \left(\text{Res } f\left(\frac{1}{2}\right) + \text{Res } f\left(-\frac{1}{2}\right) \right\}$$

$$= 2\pi i \left(-\frac{e^{1/2}}{\pi} + \frac{e^{-1/2}}{\pi} \right) = -4i \left(\frac{e^{1/2} - e^{-1/2}}{2} \right) = -4i \sinh \frac{1}{2}.$$

Example 20.37. *Evaluate* $\oint_C \tan z \, dz$ *where C is the circle* $|z| = 2$. *(V.T.U., 2010 S)*

Solution. The poles of $f(z) = \sin z/\cos z$ are given by $\cos z = 0$ *i.e.* $z = \pm \pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$
Of these poles, $z = \pi/2$, and $-\pi/2$ only are within the given circle.

$$\therefore \quad \text{Res } f(\pi/2) = \underset{z \to \pi/2}{\text{Lt}} \frac{\sin z}{\frac{d}{dz}(\cos z)} = \underset{z \to \pi/2}{\text{Lt}} \left(\frac{\sin z}{-\sin z} \right) = -1 \qquad \text{[By § 20.19 (2)]}$$

Similarly
$$\text{Res } f(-\pi/2) = \underset{z \to -\pi/2}{\text{Lt}} \frac{\sin z}{\frac{d}{dz}(\cos z)} = -1.$$

Hence by residue theorem,

$$\oint_C f(z) dz = 2\pi i \{\text{Res } f(\pi/2) + \text{Res } f(-\pi/2)\} = 2\pi i (-1 - 1) = -4\pi i.$$

Example 20.38. *Evaluate* $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z-2)} dz$, *where C is the circle* $|z| = 3$.
(V.T.U., 2010 ; Anna, 2003 S ; U.P.T.U., 2002)

Solution.
$$f(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z-2)}$$

is analytic within the circle $|z| = 3$ excepting the poles $z = 1$ and $z = 2$.

Since $z = 1$ is a pole of order 2.

$$\therefore \quad \text{Res } f(1) = \frac{1}{1!} \left[\frac{d}{dz} \{(z-1)^2 f(z)\} \right]_{z=1} = \left[\frac{d}{dz} \left(\frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)} \right) \right]_{z=1}$$

$$= \left[\frac{(z-2)(2\pi z \cos \pi z^2 - 2\pi z \sin \pi z^2) - (\sin \pi z^2 + \cos \pi z^2)}{(z-2)^2} \right]_{z=1}$$

$$= (-1)(-2\pi) - (-1) = 2\pi + 1$$

Also
$$\text{Res } f(2) = \underset{z \to 2}{\text{Lt}} [(z-2) f(z)] = \underset{z \to 2}{\text{Lt}} \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2} = 1$$

Hence by residue theorem,

$$\oint_C f(z) \, dz = 2\pi i \, [\text{Res } f(1) + \text{Res } f(2)] = 2\pi i \, (2\pi + 1 + 1) = 4\pi(\pi + 1)i.$$

PROBLEMS 20.8

1. Expand $f(z) = 1/[z^2 (z - i)]$ as a Laurent's series about i and hence find the residue thereat.
2. Find the residue of (*i*) $ze^z/(z-1)^3$ at its pole. (*J.N.T.U., 2003*)
   (*ii*) $z^2/(z^2 + a^2)$ at $z = ai$. (*P.T.U., 2009 S*)
3. Determine the poles of the following functions and the residue at each pole :
   (*i*) $\dfrac{z^2+1}{z^2-2z}$ (*ii*) $\dfrac{z^2-2z}{(z+1)^2(z^2+1)}$ (*J.N.T.U., 2005*) (*iii*) $\dfrac{2z+4}{(z+1)(z^2+1)}$ (*J.N.T.U., 2006*)
4. Find the residues of the following functions at each pole.
   (*i*) $(1-e^{2z})/z^4$ (*ii*) $ze^{iz}/(z^2+1)$ (*P.T.U., 2010*) (*iii*) $\cot z$.
5. $\oint_C \dfrac{z^2+4}{(z-2)(z+3)}\, dz$, where C is (*i*) $|z+1| = 2$ (*ii*) $|z-2| = 2$. (*Mumbai, 2006*)
6. Evaluate the following integrals :
   (*i*) $\oint_C \dfrac{e^{2z}\, dz}{(z+2)(z+4)(z+7)}$ for C as circle $|z| = 3$. (*V.T.U., 2009*)
   (*ii*) $\oint_C \dfrac{4z^2-4z+1}{(z-2)(4+z^2)}\, dz$, $C : |z| = 1$
   (*iii*) $\oint_C \dfrac{3z^2+z+1}{(z^2-1)(z+3)}\, dz$, $C : |z| = 2$. (*U.P.T.U., 2004*)
7. Evaluate
   (*i*) $\int_C \dfrac{2z+1}{(2z-1)^2}\, dz$, where C is $|z| = 1$ (*ii*) $\oint_C \dfrac{z+4}{z^2+2z+5}\, dz$, where C is $|z+1-i| = 2$
   (*iii*) $\int_C \dfrac{z^2-2z}{(z+1)^2(z^2+4)}\, dz$, where C is the circle $|z| = 10$. (*U.P.T.U., 2009*)
8. Evaluate :
   (*i*) $\oint_C \dfrac{z\, dz}{(z-1)(z-2)^2}$, $C : |z-2| = \frac{1}{2}$. (*Madras, 2006*)
   (*ii*) $\oint_C \dfrac{3z^2+2}{(z-1)(z^2+9)}\, dz$, $C : |z-2| = 2$. (*Rohtak, 2005*)
   (*iii*) $\oint_C \dfrac{dz}{(z^2+4)^2}$, $C : |z-i| = 2$. (*Hissar, 2007 ; Anna, 2003 S ; Osmania, 2003*)
9. Evaluate :
   (*i*) $\oint_C \dfrac{e^{-z}}{z^2}\, dz$, $C : |z| = 1$. (*ii*) $\oint_C z^2 e^{1/z}$, $C : |z| = 1$.
   (*iii*) $\oint_C \dfrac{e^z\, dz}{z^2+4}$, $C : |z-i| = 2$. (*V.T.U., 2006*) (*iv*) $\oint_C \dfrac{e^{2z}\, dz}{(z+1)^4}$, $C : |z| = 2$.
10. Evaluate the following integrals : (*i*) $\oint_C \dfrac{\sin^6 z}{(z-\pi/6)^3}\, dz$, $C : |z| = 1$
    (*ii*) $\oint_C \dfrac{z \sec z}{(1-z)^2}\, dz$, $C : |z| = 3$ (*iii*) $\oint_C \dfrac{z \cos z}{(z-\pi/2)^3}\, dz$, $C : |z-1| = 1$. (*V.T.U., 2007*)
11. Evaluate $\oint_C \dfrac{dz}{\sinh 2z}$ where C is the circle $|z| = 2$. (*Marathwada, 2008*)
12. Obtain Laurent's expansion for the function $f(z) = 1/z^2 \sinh z$ and evaluate
    $\oint_C \dfrac{z}{z^2 \sinh z}$, where C is the circle $|z-1| = 2$. (*J.N.T.U., 2005*)

20.20 EVALUATION OF REAL DEFINITE INTEGRALS

Many important definite integrals can be evaluated by applying the Residue theorem to properly chosen integrals. The contours chosen will consist of straight lines and circular arcs.

(*a*) **Integration around the unit circle.** An integral of the type $\int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta$, where the integrand is a rational function of $\sin\theta$ and $\cos\theta$ can be evaluated by writing $e^{i\theta} = z$.

Since $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$ and $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$, then integral takes the form $\int_C f(z)\, dz$, where $f(z)$ is a rational function of z and C is a unit circle $|z| = 1$.

Hence *the integral is equal to* $2\pi i$ *times the sum of the residues at those poles of* $f(z)$ *which are within* C.

Example 20.39. *Show that*

$$\int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{1 - 2a\cos\theta + a^2} = \frac{2\pi a^2}{1 - a^2}, \ (a^2 < 1).$$

(Bhopal, 2009 ; Rohtak, 2003)

Solution. Putting $z = e^{i\theta}$, $d\theta = dz/iz$, $\cos\theta = \frac{1}{2}(z + 1/z)$ and $\cos 2\theta = \frac{1}{2}(e^{2i\theta} + e^{-2i\theta}) = \frac{1}{2}(z^2 + 1/z^2)$

$\therefore$ the given integral

$$I = \int_C \frac{\frac{1}{2}(z^2 + 1/z^2)}{1 - a(z + 1/z) + a^2} \cdot \frac{dz}{iz} = \frac{1}{2i}\int_C \frac{(z^4 + 1)\, dz}{z^2(z - az^2 - a + a^2 z)}$$

$$= \frac{1}{2i}\int_C \frac{(z^4 + 1)\, dz}{z^2(z - a)(1 - az)} = \int_C f(z)\, dz \qquad \text{where } C \text{ is the unit circle } |z| = 1.$$

Now $f(z)$ has simple poles at $z = a$, $1/a$ and the second order pole at $z = 0$, of which the poles at $z = 0$ and $z = a$ lie within the unit circle.

$\therefore$
$$\text{Res } f(a) = \underset{z \to a}{\text{Lt}}\, [(z - a) f(z)] = \frac{1}{2i} \underset{z \to a}{\text{Lt}} \left[\frac{z^4 + 1}{z^2(1 - az)}\right] = \frac{a^4 + 1}{2ia^2(1 - a^2)}$$

and
$$\text{Res } f(0) = \underset{z \to 0}{\text{Lt}}\, \frac{d}{dz}[z^2 f(z)] = \frac{1}{2i} \underset{z \to 0}{\text{Lt}}\, \frac{d}{dz}\left[\frac{z^4 + 1}{z - az^2 - a + a^2 z}\right]$$

$$= \frac{1}{2i} \underset{z \to 0}{\text{Lt}}\, \frac{(z - az^2 - a + a^2 z)(4z^3) - (z^4 + 1)(1 - 2az + a^2)}{(z - az^2 - a + a^2 z)^2} = -\frac{1 + a^2}{2ia^2}$$

Hence
$$I = 2\pi i\, [\text{Res } f(a) + \text{Res } f(0)] = 2\pi i \left[\frac{a^4 + 1}{2ia^2(1 - a^2)} - \frac{1 + a^2}{2ia^2}\right] = \frac{2\pi a^2}{1 - a^2}.$$

Example 20.40. *By integrating around a unit circle, evaluate* $\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta}\, d\theta$.

(S.V.T.U, 2009 ; U.P.T.U., 2009 ; Madras, 2003)

Solution. Putting $z = e^{i\theta}$, $d\theta = dz/iz$, $\cos\theta = \frac{1}{2}(z + 1/z)$

and
$$\cos 3\theta = \frac{1}{2}(e^{3i\theta} + e^{-3i\theta}) = \frac{1}{2}(z^3 + 1/z^3).$$

$\therefore$ the given integral
$$I = \int_C \frac{\frac{1}{2}(z^3 + 1/z^3)}{5 - 2(z + 1/z)} \cdot \frac{dz}{iz}$$

$$= -\frac{1}{2i}\int_C \frac{z^6 + 1}{z^3(2z^2 - 5z + 2)}\, dz = -\frac{1}{2i}\int_C \frac{(z^6 + 1)\, dz}{z^3(2z - 1)(z - 2)}$$

$$= -\frac{1}{2i}\int f(z)\,dz, \quad \text{where } C \text{ is the unit circle } |z| = 1.$$

Now $f(z)$ has a pole of order 3 at $z = 0$ and simple poles at $z = \frac{1}{2}$ and $z = 2$. Of these only $z = 0$ and $z = 1/2$ lie within the unit circle.

$$\therefore \quad \text{Res } f(1/2) = \underset{z \to 1/2}{\text{Lt}} \frac{(z-1/2)(z^6+1)}{(2z-1)(z-2)} = \underset{z \to 1/2}{\text{Lt}} \left\{\frac{z^6+1}{2z^3(z-2)}\right\} = -\frac{65}{24}$$

$$\text{Res } f(0) = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-0)^n f(z)]\right\}_{z=0} \quad \text{where } n = 3$$

$$= \frac{1}{2}\left\{\frac{d^2}{dz^2}\left(\frac{z^6+1}{2z^2-5z+2}\right)\right\}_{z=0} = \frac{d}{dz}\left[\frac{(2z^2-5z+2)\,6z^5-(z^6+1)(4z-5)}{2(2z^2-5z+2)^2}\right] \text{ at } z=0$$

$$= \left\{\frac{d}{dz}\left[\frac{8z^7-25z^6+12z^5-4z+5}{2(2z^2-5z+2)^2}\right]\right\}_{z=0}$$

$$= \left[\frac{(2z^2-5z+2)^2(56z^6-150z^5+60z^4-4)-(8z^7-25z^6+12z^5-4z+5)2(2z^2-5z+2)(4z-5)}{2(2z^2-5z+2)^4}\right]_{z=0}$$

$$= \frac{4(-4)-5(-20)}{2\times 16} = \frac{84}{32} = \frac{21}{8}$$

Hence $$I = \frac{-1}{2i}\{2\pi i\,[\text{Res } f(1/2) + \text{Res } f(0)]\} = -\pi\left[-\frac{65}{24}+\frac{21}{8}\right] = -\pi\left(-\frac{1}{12}\right) = \frac{\pi}{12}.$$

(*b*) **Integration around a small semi-circle.** To evaluate $\int_{-\infty}^{\infty} f(x)\,dx$, we consider $\int_C f(z)$, where C is the contour consisting of the semi-circle $C_R : |z| = R$, together with the diameter that closes it.

Supposing that $f(z)$ has no singular point on the real axis, we have, by the Residue theorem,

$$\int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx = 2\pi i \sum \text{Res } f(a).$$

Finally making R tend to ∞, we find the value of $\int_{-\infty}^{\infty} f(x)\,dx$, provided $\int_{C_R} f(z)\,dz \to 0$ as $R \to \infty$.

Example 20.41. *Evaluate* $\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)}$. (*U.P.T.U., 2008*)

Solution. Consider $$\int_C \frac{z^2\,dz}{(z^2+1)(z^2+4)} = \int_C f(z)\,dz \quad \text{...(i)}$$

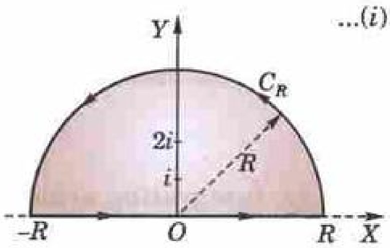

Fig. 20.23

where C is the contour consisting of the semi-circle C_R of radius R together with the part of the real axis from $-R$ to R as shown in Fig. 20.23.

The integrand has simple poles at $z = \pm i$, $z = \pm 2i$ of which $z = i$, $2i$ only lie inside C.

$\therefore$ by the Residue theorem,

$$\int_C f(z)\,dz = 2\pi i\,[\text{Res } f(i) + \text{Res } f(2i)]$$

$$= 2\pi i\,[\underset{z \to i}{\text{Lt}} (z-i)f(z) + \underset{z \to 2i}{\text{Lt}} (z-2i)f(z)]$$

$$= 2\pi i\left[\frac{i^2}{2i(i^2+4)} + \frac{4i^2}{(4i^2+1)(4i)}\right] = 2\pi i\left(\frac{i}{6}-\frac{i}{3}\right) = \frac{\pi}{3} \quad \text{...(ii)}$$

Also $$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz \qquad ...(iii)$$

Now let $R \to \infty$, so as to show that the second integral in (*iii*) vanishes. For any point on C_R as $|z| \to \infty$

$$f(z) = \frac{1}{z^2} \cdot \frac{1}{(1+z^{-2})(1+4z^{-2})}$$

decreases as $1/z^2$ and tends to zero whereas the length of C_R increases with z.

Consequently, $\underset{|z|\to\infty}{\text{Lt}} \int_{C_R} f(z)\,dz = 0.$

Hence from (*i*), (*ii*) and (*iii*), we get $\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)} = \frac{\pi}{3}$.

Example 20.42. *Evaluate* $\int_0^{\infty} \frac{\cos ax}{x^2+1}\,dx$. *(U.P.T.U., 2006 ; Delhi, 2002)*

Solution. Consider $\int_C \frac{e^{iaz}}{z^2+1}\,dz = \int_C f(z)\,dz$

where C is the contour consisting of the semi-circle C_R of radius R together with the part of the real axis from $-R$ to R as shown in Fig. 20.23.

The integrand has simple poles at $z = i$ amd $z = -i$, of which $z = i$ only lies inside C.

$\therefore$ by Residue theorem, $\int_C f(z)\,dz = 2\pi i \operatorname{Res} f(i) = 2\pi i \underset{z\to i}{\text{Lt}} [(z-i) f(z)]$

$$= 2\pi i \underset{z\to i}{\text{Lt}} \frac{(z-i)\,e^{iaz}}{z^2+1} = 2\pi i \underset{z\to i}{\text{Lt}} \frac{e^{iaz}}{z+i} = 2\pi i \frac{e^{-a}}{2i} = \pi e^{-a} \qquad ...(i)$$

Also $$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz \qquad ...(ii)$$

Now $|z| = R$ on C_R and $|z^2+1| \geq R^2 - 1$.

Also $|e^{iaz}| = |e^{ia(x+iy)}| = |e^{iax} \cdot e^{-ay}| = e^{-ay} < 1$ $\qquad [\because\ y > 0]$

$$\therefore \quad \left|\frac{e^{iaz}}{z^2+1}\right| = |e^{iaz}| \cdot \frac{1}{|z^2+1|} < 1 \cdot \frac{1}{R^2-1}$$

Thus $$\int_{C_R} f(z)\,dz = \left|\int_{C_R} \frac{e^{iaz}}{z^2+1}\,dz\right| < \int_{C_R} \frac{1}{R^2-1}|dz| < \frac{\pi R}{R^2-1} \text{ which} \to \text{to 0 as } R \to \infty. \qquad ...(iii)$$

Hence from (*i*), (*ii*) and (*iii*), we get

$$\pi e^{-a} = \int_{-\infty}^{\infty} f(x)\,dx + 0 \quad \text{or} \quad \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}\,dx = \pi e^{-a}$$

Equating real parts from both sides, we obtain

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\,dx = \pi e^{-a}$$

Since $\cos ax/(x^2+1)$ is an even function of x, we have

$$2\int_0^{\infty} \frac{\cos ax}{x^2+1}\,dx = \pi e^{-a} \quad \text{or} \quad \int_0^{\infty} \frac{\cos ax}{x^2+1}\,dx = \frac{\pi}{2} e^{-a}.$$

(*c*) **Integration around rectangular contours**

Example 20.43. *Evaluate* $\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}\,dx$.

Solution. Consider $\int_C \frac{e^{az}}{e^z+1}\,dz = \int_C f(z)\,dz$ where C is the rectangle $ABCD$ with vertices at $(R, 0)$, $(R, 2\pi)$, $(-R, 2\pi)$ and $(-R, 0)$, R being positive (Fig. 20.24).

$f(z)$ has finite poles given by

$$e^z = -1 = e^{(2n+1)\pi i}$$

or $$z = (2n+1)\pi i, \text{ where } n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

The only pole inside the rectangle is $z = \pi i$.

$\therefore$ by Residue theorem,

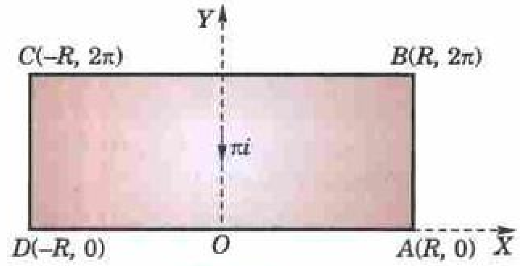

Fig. 20.24

$$\int_C f(z)dz = 2\pi i \operatorname{Res} f(\pi i)$$

$$= 2\pi i \left[e^{az} \Big/ \frac{d}{dz}(e^z + 1) \right]_{z=\pi i}$$

$$= 2\pi i e^{a\pi i}/e^{\pi i} = -2\pi i e^{a\pi i} \qquad [\because \; e^{\pi i} = -1] \qquad \ldots(i)$$

Also $$\int_C f(z)dz = \int_{AB} f(z)dz + \int_{BC} f(z)dz + \int_{CD} f(z)dz + \int_{DA} f(z)dz$$

$$= \int_0^{2\pi} f(R+iy)idy + \int_R^{-R} f(x+2\pi i)dx + \int_{2\pi}^0 f(-R+iy)idy + \int_{-R}^R f(x)dx$$

[$\because$ $z = R + iy$ along AB, $z = x + 2\pi i$ along BC, $z = -R + iy$ along CD and $z = x$ along DA.]

or $$\int_C f(z)dz = i\int_0^{2\pi} \frac{e^{a(R+iy)}}{e^{R+iy}+1}dy - \int_{-R}^R \frac{e^{a(x+2\pi i)}}{e^{x+2\pi i}+1}dx - i\int_0^{2\pi} \frac{e^{a(-R+iy)}}{e^{-R+iy}+1}dy + \int_{-R}^R \frac{e^{ax}}{e^x+1}dx \qquad \ldots(ii)$$

Now for any two complex numbers z_1, z_2

$$|z_1| \geq |z_2|, \text{ we have } |z_1 + z_2| \geq |z_1| - |z_2|$$

so that $|e^{R+iy}+1| \geq e^R - 1$. Also $|e^{a(R+iy)}| = e^{aR}$

$\therefore$ for the integrand of first integral in (ii), we have

$$\left| \frac{e^{a(R+iy)}}{e^{R+iy}+1} \right| \leq \frac{e^{aR}}{e^R - 1} \text{ which} \to 0 \text{ as } R \to \infty. \qquad [\because \; a > 1]$$

Similarly, for the integrand of the third integral in (ii), we get

$$\left| \frac{e^{a(-R+iy)}}{e^{-R+iy}+1} \right| \leq \frac{e^{-aR}}{1 - e^{-R}} \text{ which also} \to 0 \text{ as } R \to \infty. \qquad [\because \; a < 0]$$

Hence as $R \to \infty$, since the first and third integrals in (ii) approach zero, we get

$$\int_C f(z)dz = -e^{2a\pi i} \int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}dx + \int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}dx = \left(1 - e^{2a\pi i}\right) \int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}dx \qquad \ldots(iii)$$

Thus from (i) and (iii), we obtain $(1 - e^{2a\pi i}) \displaystyle\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}dx = -2\pi i e^{a\pi i}$

$\therefore$ equating real parts, we get $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}dx = \frac{\pi}{\sin a\pi}$.

Example 20.44. *Show that* $\displaystyle\int_0^{\infty} e^{-x^2} \cos 2mx \, dx = \frac{1}{2}\sqrt{\pi e^{-m^2}}$.

Solution. Integrate $f(z) = e^{-z^2}$ along the rectangle ABCDA having vertices $A(-l)$, $B(l)$, $C(l + im)$, $D(-l + im)$ (Fig. 20.25). $f(z)$ has no poles inside this contour. As such

$$\int_{AB} f(z)dz + \int_{BC} f(z)dz + \int_{CD} f(z)dz + \int_{DA} f(z)dz = 0 \qquad \ldots(i)$$

On $AB : z = x$, on $BC : z = l + iy$, on $CD : z = x + im$ and on $DA : z = -l + iy$.

Therefore, (i) becomes

$$\int_{-l}^{l} e^{-x^2} dx + \int_0^m e^{-(l+iy)^2} idy + \int_l^{-l} e^{-(x+im)^2} dx + \int_m^0 e^{-(-l+iy)^2} dy = 0$$

or $$\int_{-l}^{l} e^{-x^2} dx - \int_{-l}^{l} e^{-x^2 - 2imx + m^2} dx + \int_0^m e^{-l^2 - 2ily + y^2} . idy$$

$$- \int_0^m e^{-l^2 + 2ily + y^2} . idy = 0 \qquad \ldots(ii)$$

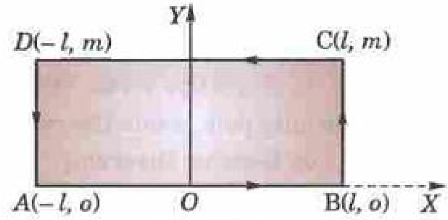

Fig. 20.25

Now let $l \to \infty$. Then the last two integrals

$$= ie^{-l^2} \int_0^m e^{y^2} (e^{-2ily} - e^{2ily})\, dy = 2e^{-l^2} \int_0^m e^{y^2} \sin 2ly\, dy \to 0$$

$[\because$ As $l \to \infty$, $e^{-l^2} \to 0$ and $\sin 2ly$ is finite$]$

Hence (*ii*) reduces to

$$\int_{-\infty}^{\infty} e^{-x^2} dx - e^{m^2} \int_{-\infty}^{\infty} e^{-x^2} (\cos 2mx - i \sin 2mx) dx = 0$$

Equating real parts, we get

$$e^{m^2} \int_{-\infty}^{\infty} e^{-x^2} \cos 2mx\, dx = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$ [See p. 289]

or
$$\int_0^{\infty} e^{-x^2} \cos 2mx\, dx = \frac{1}{2}\sqrt{\pi e^{-m^2}}$$

(*d*) **Indenting the contours having poles on the real axis.** So far we have considered such cases in which there is no pole on the real axis. When the integrand has a simple pole on the real axis, we delete it from the region by indenting the contour (*i.e.*, by drawing a small semi-circle having the pole for the centre). The method will be clear from the following example.

Example 20.45. *Evaluate* $\int_0^{\infty} \frac{\sin mx}{x} dx$, *when* $m > 0$. *(U.P.T.U., 2007)*

Solution. Consider the integral $\int_C \frac{e^{miz}}{z} dz = \int_C f(z) dz$ where C consists of

(*i*) the real axis from r to R,
(*ii*) the upper half of the circle $C_R : |z| = R$,
(*iii*) the real axis $-R$ to $-r$,
(*iv*) the upper half of the circle $C_r : |z| = r$ (Fig. 20.26).

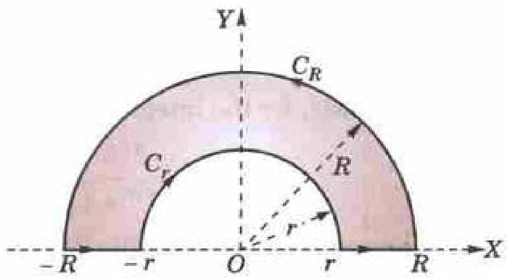

Fig. 20.26

Since $f(z)$ has no singularity inside C (its only singular point being a simple pole at $z = 0$ which has been deleted by drawing C_r), we have by Cauchy's theorem :

$$\int_r^R f(x) dx + \int_{C_R} f(z) dz + \int_{-R}^{-r} f(x) dx + \int_{C_r} f(z) dz = 0 \quad ...(i)$$

Now
$$\int_{C_R} f(z) dz = \int_0^{\pi} \frac{e^{imR(\cos\theta + i \sin\theta)}}{Re^{i\theta}} . Rie^{i\theta} d\theta$$ $[\because \; z = Re^{i\theta}]$

$$= i \int_0^{\pi} e^{imR(\cos\theta + i\sin\theta)} d\theta$$

Since $|e^{imR(\cos\theta + i\sin\theta)}| = |e^{-mR\sin\theta + imR\cos\theta}| = e^{-mR\sin\theta}$

$\therefore$
$$\left|\int_{C_R} f(z) dz\right| \le \int_0^{\pi} e^{-mR\sin\theta} d\theta = 2\int_0^{\pi/2} e^{-mR\sin\theta} d\theta$$

$$= 2\int_0^{\pi/2} e^{-2mR\theta/\pi} d\theta$$ $[\because$ for $0 \le \theta \le \pi/2$, $\sin\theta/\theta \ge 2/\pi]$

$$= \frac{\pi}{mR}\left(1 - e^{-mR}\right) \text{ which } \to 0 \text{ as } R \to \infty,$$

Also
$$\int_{C_r} f(z) dz = i\int_{\pi}^{0} e^{imr(\cos\theta + i\sin\theta)} d\theta \to i\int_{\pi}^{0} d\theta \;\; i.e., -i\pi \text{ as } r \to 0.$$

Hence as $r \to 0$ and $R \to \infty$, we get from (*i*) $\int_0^{\infty} f(x) dx + 0 + \int_{-\infty}^{0} f(x) dx - i\pi = 0$

or
$$\int_{-\infty}^{\infty} f(x) dx = i\pi \;\; i.e., \int_{-\infty}^{\infty} \frac{e^{imx}}{x} dx = i\pi \quad ...(ii)$$

Equating imaginary parts from both sides,

$$\int_{-\infty}^{\infty} \frac{\sin mx}{x} dx = \pi. \text{ Hence } \int_0^{\infty} \frac{\sin mx}{x} dx = \frac{\pi}{2}.$$

Obs. Equating real parts from both sides of (*ii*), we get

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x} dx = 0.$$

Example 20.46. *Show that* $\int_0^\infty \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi}, 0 < p < 1.$

Solution. Integrate $f(z) = \frac{z^{p-1}}{1+z}$ along the contour consisting of the circles α and γ of radii a and R and the lines AB and FG along x-axis (Fig. 20.27). There is a simple pole at $z = -1$ which is within the contour.

$$\therefore \qquad \text{Res}\, f(-1) = \underset{z \to -1}{\text{Lt}} (1+z) \cdot \frac{z^{p-1}}{1+z} = \underset{z \to -1}{\text{Lt}}\, z^{p-1} = (-1)^{p-1} = e^{i\pi(p-1)}$$

Thus
$$\int_{AB} f(z)dz + \int_\gamma f(z)dz + \int_{FG} f(z)dz + \int_\alpha f(z)dz = 2\pi i\, e^{i\pi(p-1)} \qquad \text{...}(i)$$

On $AB : z = x$ and on $FG : z = xe^{2\pi i}$

$$\therefore \qquad \int_{AB} f(z)dz + \int_{FG} f(z)dz = \int_a^R \frac{x^{p-1}}{1+x} dx + \int_R^a \frac{(xe^{2\pi i})^{p-1}}{1+xe^{2\pi i}} dx\, e^{2\pi i}$$

$$= \int_a^R \frac{x^{p-1}}{1+x} [1 - e^{2\pi i(p-1)}]\, dx$$

On the circle $\gamma : z = Re^{i\theta}$. So

$$\int_\gamma f(z)dz = \int_0^{2\pi} \frac{(Re^{i\theta})^{p-1}}{1+Re^{i\theta}} Re^{i\theta}\, i\, d\theta$$

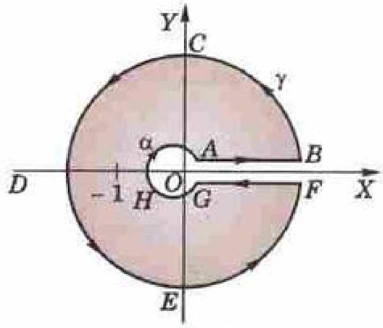

Fig. 20.27

For large R, the integrand is of the order $\frac{R^{p-1} \cdot R}{1+R}$ i.e.
R^{p-1} which tends to zero as $R \to \infty$. $(\because \; p < 1)$

Hence $\int_\gamma f(z)dz \to 0$ as $R \to \infty$

On the circle $\alpha : z = ae^{i\theta}$. So

$$\int_\alpha f(z)dz = \int_{2\pi}^{0} \frac{(ae^{i\theta})^{p-1}}{1+ae^{i\theta}} ae^{i\theta}\, id\theta$$

For small α, the integrand is of the order a^p which tends to zero as $a \to 0$. $(\because \; p > 0)$

Thus on taking limits as $a \to 0$ and $R \to \infty$, (*i*) gives

$$\int_0^\infty \frac{x^{p-1}}{1+x} \{1 - e^{2\pi i(p-1)}\} dx = 2\pi i\, e^{i\pi(p-1)}$$

or
$$\int_0^\infty \frac{x^{p-1}}{1+x} dx = \frac{2\pi i\, e^{i\pi(p-1)}}{1 - e^{2\pi i(p-1)}} = \frac{2\pi i e^{ip\pi}(-1)}{1 - e^{2ip\pi}\,(1)} = \frac{2i \cdot \pi}{e^{ip\pi} - e^{-ip\pi}} = \frac{\pi}{\sin p\pi}.$$

Example 20.47. *Prove that* $\int_0^\infty \sin x^2\, dx = \int_0^\infty \cos x^2\, dx = \frac{1}{2}\sqrt{\left(\frac{\pi}{2}\right)}$. *(Osmania, 2003)*

Solution. Consider $\int_C e^{-z^2} dz$ where C consists of the real axis from O to A, part of circle AB of radius R and the line $\theta = \frac{\pi}{4}$. (Fig. 20.28).

e^{-z^2} has no singularity within C.

$$\therefore \qquad \int_{OA} e^{-z^2} dz + \int_{AB} e^{-z^2} dz + \int_{BO} e^{-z^2} dz = 0 \qquad \text{...}(i)$$

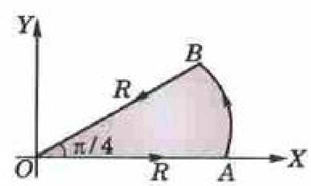

Fig. 20.28

On $OA: z = x$, $\therefore$ $\int_{OA} e^{-z^2} dz = \int_0^R e^{-x^2} dx \to \sqrt{\pi/2}$ as $R \to \infty$ [See p. 289]

On $AB: z = Re^{i\theta}$,

$\therefore$ $$\int_{AB} e^{-z^2} dz = \int_0^{\pi/4} e^{-R^2(\cos 2\theta + i \sin 2\theta)} . Re^{i\theta} . id\theta \to 0 \quad \text{as} \quad R \to \infty.$$

[$\because$ integrand $\to 0$ as $R \to \infty$]

On $BO: z = re^{i\pi/4}$ and $z^2 = r^2 e^{i\pi/2} = ir^2$

$\therefore$ $$\int_{BO} e^{-z^2} dz = \int_R^0 e^{-ir^2} . e^{i\pi/4} dr = -\int_0^R e^{-ix^2} \frac{1+i}{\sqrt{2}} dx$$

$$\to -\int_0^\infty (\cos x^2 - i \sin x^2) \frac{1+i}{\sqrt{2}} dx \qquad \text{when } R \to \infty$$

Substituting these in (i), we get

$$\frac{1}{2}\sqrt{\pi} + 0 - \int_0^\infty (\cos x^2 - i \sin x^2) \left(\frac{1+i}{\sqrt{2}}\right) dx = 0$$

Equating real and imaginary parts, we obtain

$$\int_0^\infty (\cos x^2 + \sin x^2) dx = \frac{1}{2}\sqrt{(2\pi)} \quad \text{and} \quad \int_0^\infty (\cos x^2 - \sin x^2) dx = 0$$

Hence $$\int_0^\infty \sin x^2 \, dx = \int_0^\infty \cos x^2 \, dx = \frac{1}{2}\sqrt{\left(\frac{\pi}{2}\right)}.$$

PROBLEMS 20.9

Apply the calculus of residues, to prove that

1. $\int_0^{2\pi} \frac{d\theta}{1 - 2p \sin\theta + p^2} = \frac{2\pi}{1-p^2} (0 < p < 1)$. (*Hissar, 2007 ; Mumbai, 2006 ; Kerala, 2005*)
2. $\int_0^{2\pi} \frac{d\theta}{1 - 2r \cos\theta + r^2} = \frac{\pi}{1-r^2}$. (*J.N.T.U., 2006 ; Madras, 2006 ; Anna, 2003*)
3. $\int_0^{2\pi} \frac{d\theta}{a + \cos\theta} = \frac{2\pi}{\sqrt{(a^2-1)}} (a > 1)$. (*P.T.U., 2010*)
4. $\int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta} d\theta = \frac{\pi}{6}$. (*U.P.T.U., 2010*)
5. $\int_0^{2\pi} \frac{\sin^2\theta \, d\theta}{a + b\cos\theta} = \frac{2\pi}{b^2}[a - \sqrt{(a^2 - b^2)}], (0 < b < a)$. (*J.N.T.U., 2003*)
6. $\int_0^{2\pi} \frac{a d\theta}{a^2 + \sin^2\theta} = \frac{2\pi}{\sqrt{(1+a^2)}}, (a > 0)$. (*S.V.T.U., 2009*)
7. $\int_0^{2\pi} \frac{d\theta}{(5 - 3\cos\theta)^2} = \frac{5\pi}{32}$.
8. $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a+b} (a, b > 0)$. (*P.T.U., 2007 ; Mumbai, 2006 ; Anna, 2003*)
9. $\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx = \frac{5\pi}{12}$. (*A.M.I.E.T.E., 2003 ; Delhi, 2002*)
10. $\int_{-\infty}^{\infty} \frac{dx}{x^4 + 1} = \frac{\pi}{\sqrt{2}}$. (*J.N.T.U., 2006*)
11. $\int_0^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{4}$. (*Madras, 2006 ; Kerala, 2005*)
12. $\int_0^{\infty} \frac{dx}{x^6 + 1} = \frac{\pi}{3}$. (*Kerala, 2005*)
13. $\int_0^{\infty} \frac{x^2}{x^6 + 1} dx = \frac{\pi}{6}$. (*Rohtak, 2006*)
14. $\int_{-\infty}^{\infty} \frac{\cos mx}{e^x + e^{-x}} dx = \frac{\pi}{2} \text{sech} \frac{m\pi}{2}$.
15. $\int_0^{\infty} \frac{1 - \cos x}{x^2} dx = \pi/2$. (*P.T.U., 2005*)
16. $\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$. (*Kerala, 2005*)
17. $\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 4x + 5} dx = \frac{-\pi \sin 2}{e}$.
18. $\int_{-\infty}^{\infty} \frac{\cos x \, dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2 - b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right) (a > b > 0)$.
19. $\int_0^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \frac{\pi}{2} e^{-a}$.

20.12. OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 20.10

Select the correct answer or fill up the blanks in each of the following questions :

1. The only function that is analytic from the following is
   (i) $f(z) = \sin z$ (ii) $f(z) = \bar{z}$ (iii) $f(z) = \text{Im}(z)$ (iv) $R(iz)$.
2. If $f(z) = u(x, y) + iv(x, y)$ is analytic, then $f'(z) =$
   (i) $\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}$ (ii) $\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ (iii) $\frac{\partial v}{\partial y} - i\frac{\partial v}{\partial x}$.
3. If $2x - x^2 + ay^2$ is to be harmonic, then a should be
   (a) 1 (b) 2 (c) 3 (d) 0.
4. The analytic function which maps the angular region $0 \le \theta \le \pi/4$ onto the upper half plane is
   (i) z^2 (ii) $4z$ (iii) z^4 (iv) 2θ.
5. An angular domain in the complex plane is defined by $0 < \text{amp}(z) < \pi/4$. The mapping which maps this region onto the left half plane is
   (i) $w = z^4$ (ii) $w = iz^4$ (iii) $w = -z^4$ (iv) $w = -iz^4$.
6. The mapping $w = z^2 - 2z - 3$ is
   (i) conformal within $|z| = 1$ (ii) not conformal at $z = 1$
   (iii) not conformal at $z = -1$ and $z = 3$ (iv) conformal everywhere.
7. If $z = re^{i\theta}$, then the image of θ = constant under the mapping $w(z) = Re^{i\phi} = iz^3$ is
   (i) $\phi = 3\theta$ (ii) $\phi = 3\theta + \pi/2$ (iii) $\phi = 3\theta - \pi/2$ (iv) $\phi = \theta^3$.
8. The fixed points of the mapping $w = (5z + 4)/(z + 5)$ are
   (i) 2, 2 (ii) 2, -2 (iii) $-2, -2$ (iv) $-4/5$, 5.
9. The value of $\int_C (4x^3\,dx + 3y^2z^2\,dy + 2y^3 z\,dz)$ where C is any path joining $A(-1, 1, 0)$ to $B(1, 2, 1)$ is
   (i) 0 (ii) 1 (iii) 8 (iv) -8.
10. The value of $\int_C \frac{3z^2 + 7z + 1}{z + 1}\,dz$ where C is $|z| = 1/2$ is
    (i) $2\pi i$ (ii) 0 (iii) πi (iv) $\pi i/2$.
11. The value of $\int_C \frac{3z + 4}{z(2z + 1)}\,dz$ where C is the circle $|z| = 1$ is
    (i) $2\pi i$ (ii) $3\pi i$ (iii) 4 (iv) -4.
12. The residue of a function can be found if the pole is an isolated singularity :
    (i) True (ii) False (iii) Partially false (iv) none of these.
13. The value of $\int_C \frac{z\,dz}{\sin z}$ where $C : |z| = 4$ is
    (i) $2\pi i$ (ii) 0 (iii) $-2\pi i$ (iv) $4\pi i$.
14. The value of $\int_C \tanh z\,dz$, where $C : |z| = 3$, is
    (i) 0 (ii) πi (iii) $2\pi i$ (iv) $4\pi i$.
15. The harmonic conjugate of the function $u(x, y) = 2x(1 - y)$ is ...... (U.P.T.U., 2009)
16. Harmonic conjugate of $x^3 - 3xy^2$ is ....
17. The curves $u(x, y) = c$ and $v(x, y) = c'$ are orthogonal if ......
18. The value of $\int_0^{1+i} z^2\,dz$ along the line $x = y$ is ......
19. Residue of $\frac{\cos z}{z}$ at $z = 0$ is ......
20. The critical point of the transformation $w^2 = (z - a)(z - b)$ is ......
21. Image of $|z + 1| = 1$ under the mapping $w = 1/z$ is......
22. The poles of $f(z) = (z^3 - 1)/(z^3 + 1)$ are $z =$ ......
23. $w = \log z$ is analytic everywhere except at $z =$ ......
24. If $f(z) = -\frac{1}{z - 1} - 2[1 + (z - 1) + (z - 1)^2 + ...]$, then the residue of $f(z)$ at $z = 1$ is ......
25. If $|z| < 1$ then Taylor's series expansion of $\log(1 + z)$ about $z = 0$ is ......

26. The value of $\int_C \frac{4z^2+z+5}{z-4}dz$ where C is $9x^2+4y^2=36$, is ......

27. The value of $\int z^4 e^{1/z}\,dz$, where C is $|z|=1$, is

(i) πi (ii) $\pi i/12$ (iii) $\pi i/60$ (iv) $-\pi i/60$.

28. If $f(z)$ has a pole of order three at $z=a$ Res $[f(a)]=$ ......

29. The value of $\int_C \frac{e^z\,dz}{(z-3)^2}$, C being $|z|=2$, is ......

30. The CR equations for $f(z)=u(x,y)+iv(x,y)$ to be analytic are ......

31. If $f(z)$ is analytic in a simply connected domain D and C is any simple closed path then $\int_C f(z)\,dz=$ ......

32. The harmonic conjugate of $e^x\cos y$ is ......

33. The value of $\oint_C \cos z\,dz$ where C is the circle $|z|=1$, is ......

34. The singularity of $f(z)=z/(z-2)^3$ is ......

35. The function $f(z)=\bar{z}$ is analytic at ......

36. C-R equations for a function to be analytic, in polar form, are ......

37. If C is the circle $|z-a|=r$, $\int_C (z-a)^n\,dz$ [n, any integer $\neq -1$] $=$ ......

38. A simply connected region is that ......

39. A holomorphic function is that ......

40. The poles of the function $f(z)=\frac{z^2}{(z-1)^2(z+2)}$ are at $z=$ ......

41. The cross-ratio of four points z_1, z_2, z_3, z_4 is ......

42. The value of $\int_C |z|\,dz$, where C is the contour represented by the straight line from $z=-i$ to $z=i$, is ......

43. Taylor's series expansion of $\left(\frac{1}{z-2}-\frac{1}{z-1}\right)$ in the region $|z|<1$, is ......

44. The invariant points of the transformation $w=(1+z)/(1-z)$ are $z=$ ......

45. The residue at $z=0$ of $\frac{1+e^z}{z\cos z+\sin z}$ is ......

46. The transformation $w=Cz$ consists of ......

47. The residue of $f(z)$ at a pole is ......

48. The value of $\int_C \frac{1}{z-1}dz$, C being $|z|=2$, is ......

49. If C is $|z|=1/2$, $\int_C \frac{z^2-z+1}{z-1}dz=$ ......

50. Singular points of $\frac{\cos \pi z}{(z-1)(z-2)}$ are ......

51. Taylor series expansion of $\frac{1}{z-2}$ in $|z|<1$ is ......

52. $\underset{z\to\infty}{\text{Lt}} \frac{iz^3+iz-1}{(2z+3)(z-1)^2}=$ .... (P.T.U., 2007)

53. The poles of $\frac{(z-1)^2}{z(z-2)^2}$ are at $z=$ ......

54. Cauchy's integral theorem states that ......

55. The critical points of the transformation $w=z+1/z$ are ......

56. $\int_c \frac{dz}{2z-3}$, where $|z|=1$, is ....

57. The zeroes and singularities of $\frac{z^2+1}{1-z^2}$ are ......

58. Residue of $\tan z$ at $z=\pi/2$ is ......

59. Singularity of $e^{z^{-1}}$ at $z=0$ is of the type ....

60. Res $(e^{1/z})_{z=0}=$ ......

61. Taylor's series expansion of $\sin z$ about $z=\pi/4$ is ......

62. Image of $|z|=2$ under $w=z+3+2i$ is ....

63. The poles of $\cot z$ are ....

64. If a is simple pole, then Res $[\phi(z)/\psi(z)]_{z=a}=$ ....

65. Bilinear transformation always transforms circles into ....

66. If $f(z)$ and $\overline{f(z)}$ are analytic functions, then $f(z)$ is constant. (True or False) (*Mumbai, 2006*)

67. The function $u(x,y)=2xy+3xy^2-2y^3$ is a harmonic functions. (True or False) (*P.T.U., 2009 S*)

68. The function $e^x\cos y$ is harmonic. (True or False)

69. $\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$, if $z = a$ is a point within C. (True or False)

70. The transformation affected by an analytic function $w = f(z)$ is conformal at every point of the z-plane where $f'(z) = 0$. (True or False)

71. The function $\bar{z}$ is not analytic at any point. (True or False)

72. Under the transformation $w = 1/z$, circle $x^2 + y^2 - 6x = 0$ transforms into a straight line in the w-plane. (True or False)

73. If $w = f(z)$ is analytic, then $\frac{dw}{dz} = -i\frac{\partial w}{\partial y}$. (True or False)

74. An analytic function with constant imaginary part is constant. (True or False)

75. If $u + iv$ is analytic, then $v - iu$ is also analytic. (True or False)

76. $f(z) = I_m(z)$ is not analytic. (True or False)

77. The cross-ratio of four points is not invariant under bilinear transformation. (True or False)

78. $z = 0$ is not a critical point of the mapping $w = z^2$. (True or False)

79. $f(z) = \text{Re}(z^2)$ is analytic. (True or False)

80. An analytic function with constant modulus is constant. (True or False)

81. The function $|\bar{z}|^2$ is not analytic at any point. (True or False)

82. If $f(z) = z^2$, then the family of curves $x^2 - y^2 = C_1$, and $xy = C_2$ are orthogonal. (True or False)

CHAPTER 21

Laplace Transforms

1. Introduction. 2. Definition ; Conditions for existence. 3. Transforms of elementary functions. 4. Properties of Laplace transforms. 5. Transforms of Periodic functions. 6. Transforms of Special functions. 7. Transforms of derivatives. 8. Transforms of integrals. 9. Multiplication by t^n. 10. Division by t. 11. Evaluation of integrals by Laplace transforms. 12. Inverse transforms. 13. Other methods of finding inverse transforms. 14. Convolution theorem. 15. Application to differential equations. 16. Simultaneous linear equations with constant co-efficients. 17. Unit step function. 18. Unit impulse function. 19. Objective Type of Questions.

21.1 INTRODUCTION

The knowledge of Laplace transforms has in recent years become an essential part of mathematical background required of engineers and scientists. This is because the transform methods provide an easy and effective means for the solution of many problems arising in engineering.

This subject originated from the operational methods applied by the English engineer Oliver Heaviside (1850–1925), to problems in electrical engineering. Unfortunately, Heaviside's treatment was unsystematic and lacked rigour, which was placed on sound mathematical footing by Bromwich and Carson during 1916–17. It was found that Heaviside's operational calculus is best introduced by means of a particular type of definite integrals called Laplace transforms.*

The method of Laplace transforms has the advantage of directly giving the solution of differential equations with given boundary values without the necessity of first finding the general solution and then evaluating from it the arbitrary constants. Moreover, the ready tables of Laplace transforms reduce the problem of solving differential equations to mere algebraic manipulation.

21.2 (1) DEFINITION

Let $f(t)$ be a function of t defined for all positive values of t. Then the **Laplace transforms** *of $f(t)$, denoted by $L\{f(t)\}$ is defined by*

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt \qquad ...(1)$$

provided that the integral exists. s is a parameter which may be a real or complex number.

$L\{f(t)\}$ being clearly a function of s is briefly written as $\bar{f}(s)$ *i.e.*, $L\{f(t)\} = \bar{f}(s)$,

which can also be written as $f(t) = L^{-1}\{\bar{f}(s)\}$.

Then $f(t)$ is called the **inverse Laplace transform** of $\bar{f}(s)$. The symbol L, which transforms $f(t)$ into $\bar{f}(s)$, is called the *Laplace transformation operator.*

**Pierre de Laplace* (1749–1827) (See footnote p. 18) used such transforms, much earlier in 1799, while developing the theory of probability.

(2) Conditions for the existence

The Laplace transform of $f(t)$ i.e., $\int_0^\infty e^{-st} f(t)\,dt$ exists for $s > a$, if

(i) $f(t)$ is continuous *(iii)* $\underset{t \to \infty}{\text{Lt}} \{e^{-at} f(t)\}$ *is finite.*

It should however, be noted that the above conditions are sufficient and not necessary.

For example, $L(1/\sqrt{t})$ exists, though $1/\sqrt{t}$ is infinite at $t = 0$.

21.3 TRANSFORMS OF ELEMENTARY FUNCTIONS

The direct application of the definition gives the following formulae :

(1) $\mathbf{L(1) = \dfrac{1}{s}}$ $(s > 0)$

(2) $\mathbf{L(t^n) = \dfrac{n!}{s^{n+1}}}$, when $n = 0, 1, 2, 3, \ldots$ $\left[\text{Otherwise } \dfrac{\Gamma(n+1)}{s^{n+1}}\right]$

(3) $\mathbf{L(e^{at}) = \dfrac{1}{s-a}}$ $(s > a)$

(4) $\mathbf{L(\sin at) = \dfrac{a}{s^2+a^2}}$ $(s > 0)$

(5) $\mathbf{L(\cos at) = \dfrac{s}{s^2+a^2}}$ $(s > 0)$

(6) $\mathbf{L(\sinh at) = \dfrac{a}{s^2-a^2}}$ $(s > |a|)$

(7) $\mathbf{L(\cosh at) = \dfrac{s}{s^2-a^2}}$ $(s > |a|)$

Proofs. **(1)** $$L(1) = \int_0^\infty e^{-st} . 1\, dt = \left| -\frac{e^{-st}}{s} \right|_0^\infty = \frac{1}{s} \text{ if } s > 0.$$

(2) $$L(t^n) = \int_0^\infty e^{-st} . t^n\, dt = \int_0^\infty e^{-p} . \left(\frac{p}{s}\right)^n \frac{dp}{s}, \text{ on putting } st = p$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-p} . p^n\, dp = \frac{\Gamma(n+1)}{s^{n+1}}, \text{ if } n > -1 \text{ and } s > 0. \text{ [Page 302]}$$

In particular $L(t^{-1/2}) = \dfrac{\Gamma(1/2)}{s^{1/2}} = \sqrt{\dfrac{\pi}{s}}$; $L(t^{1/2}) = \dfrac{\Gamma(3/2)}{s^{3/2}} = \dfrac{\sqrt{\pi}}{2 s^{3/2}}$

In n be a positive integer, $\Gamma(n + 1) = n!$ [(*v*) p. 302],

therefore, $L(t^n) = n!/s^{n+1}$.

(3) $$L(e^{at}) = \int_0^\infty e^{-st} . e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\, dt = \left| \frac{e^{-(s-a)t}}{-(s-a)} \right|_0^\infty = \frac{1}{s-a}, \text{ if } s > a.$$

(4) $$L(\sin at) = \int_0^\infty e^{-st} \sin at\, dt = \left| \frac{e^{-st}}{s^2+a^2}(-s \sin at - a \cos at) \right|_0^\infty = \frac{a}{s^2+a^2}$$

Similarly, the reader should prove **(5)** himself.

(6) $$L(\sinh at) = \int_0^\infty e^{-st} \sinh at\, dt = \int_0^\infty e^{-st} \left(\frac{e^{at} - e^{-at}}{2}\right) dt$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t}\, dt - \int_0^\infty e^{-(s+a)t}\, dt\right] = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{a}{s^2-a^2} \text{ for } s > |a|.$$

Similarly, the reader should prove **(7)** himself.

21.4 PROPERTIES OF LAPLACE TRANSFORMS

I. Linearity property. *If a, b, c be any constants and f, g, h any functions of t, then*

$$\mathbf{L\,[af(t) + bg(t) - ch(t)] = aL\,\{f(t)\} + bL\,\{g(t)\} - cL\,\{h(t)\}}$$

For by definition,

$$\text{L.H.S.} = \int_0^\infty e^{-st}\,[af(t) + bg(t) - ch(t)]\,dt$$

$$= a\int_0^\infty e^{-st}\,f(t)\,dt + b\int_0^\infty e^{-st}\,g(t)\,dt - c\int_0^\infty e^{-st}\,h(t)\,dt = aL\{f(t)\} + bL\{g(t)\} - cL\{h(t)\}$$

This result can easily be generalised.

Because of the above property of L, it is called a *linear operator.*

II. First shifting property. *If* $L\{f(t)\} = \bar{f}(s)$, then

$$\mathbf{L\,\{e^{at}f(t)\} = \bar{f}(s - a)}.$$

By definition, $L\{e^{at}f(t)\} = \int_0^\infty e^{-st}\,e^{at}\,f(t)\,dt = \int_0^\infty e^{-(s-a)t}\,f(t)\,dt$

$$= \int_0^\infty e^{-rt}\,f(t)\,dt, \text{ where } r = s - a = \bar{f}(r) = \bar{f}(s - a).$$

Thus, if we know the transform $\bar{f}(s)$ *of* $f(t)$, *we can write the transform of* $e^{at}f(t)$ *simply replacing* s *by* $s - a$ *to get* $\bar{f}(s - a)$.

Application of this property leads us to the following useful results :

(1) $\mathbf{L\,(e^{at}) = \dfrac{1}{s - a}}$ $\left[\because\ L(1) = \dfrac{1}{s}\right.$

(2) $\mathbf{L\,(e^{at}t^n) = \dfrac{n!}{(s - a)^{n+1}}}$ $\left[\because\ L(t^n) = \dfrac{n!}{s^{n+1}}\right.$

(3) $\mathbf{L\,(e^{at}\sin bt) = \dfrac{b}{(s - a)^2 + b^2}}$ $\left[\because\ L(\sin bt) = \dfrac{b}{s^2 + b^2}\right.$

(4) $\mathbf{L\,(e^{at}\cos bt) = \dfrac{s - a}{(s - a)^2 + b^2}}$ $\left[\because\ L(\cos bt) = \dfrac{s}{s^2 + b^2}\right.$

(5) $\mathbf{L\,(e^{at}\sinh bt) = \dfrac{b}{(s - a)^2 - b^2}}$ $\left[\because\ L(\sinh bt) = \dfrac{b}{s^2 - b^2}\right.$

(6) $\mathbf{L\,(e^{at}\cosh bt) = \dfrac{s - a}{(s - a)^2 - b^2}}$ $\left[\because\ L(\cosh bt) = \dfrac{s}{s^2 - b^2}\right.$

where in each case $s > a$.

Example 21.1. *Find the Laplace transforms of*

(i) $\sin 2t \sin 3t$ *(ii)* $\cos^2 2t$ *(iii)* $\sin^3 2t$.

Solution. (*i*) Since $\sin 2t \sin 3t = \frac{1}{2}[\cos t - \cos 5t]$

$$\therefore\quad L(\sin 2t \sin 3t) = \tfrac{1}{2}[L(\cos t) - L(\cos 5t)] = \frac{1}{2}\left[\frac{s}{s^2 + 1^2} - \frac{s}{s^2 + 5^2}\right] = \frac{12s}{(s^2 + 1)(s^2 + 25)}$$

(*ii*) Since $\cos^2 2t = \dfrac{1}{2}(1 + \cos 4t)$

$$\therefore\quad L(\cos^2 2t) = \frac{1}{2}[L(1) + L\cos 4t] = \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2 + 16}\right)$$

(*iii*) Since $\sin 6t = 3 \sin 2t - 4 \sin^3 2t$

or $\sin^3 2t = \frac{3}{4} \sin 2t - \frac{1}{4} \sin 6t$

$\therefore$ $L(\sin^3 2t) = \frac{3}{4} L(\sin 2t) - \frac{1}{4} L(\sin 6t)$

$$= \frac{3}{4} \cdot \frac{2}{s^2 + 2^2} - \frac{1}{4} \cdot \frac{2}{s^2 + 6^2} = \frac{48}{(s^2 + 4)(s^2 + 36)}.$$

Example 21.2. *Find the Laplace transform of*

(*i*) $e^{-3t}(2 \cos 5t - 3 \sin 5t)$. (*ii*) $e^{2t} \cos^2 t$ (V.T.U., 2006) (*iii*) $\sqrt{t}e^{3t}$. (P.T.U., 2009)

Solution. (*i*) $L\{e^{-3t}(2 \cos 5t - 3 \sin 5t)\} = 2L(e^{-3t} \cos 5t) - 3L(e^{-3t} \sin 5t)$

$$= 2 \cdot \frac{s+3}{(s+3)^2 + 5^2} - 3 \cdot \frac{5}{(s+3)^2 + 5^2} = \frac{2s - 9}{s^2 + 6s + 34}.$$

(*ii*) Since $L(\cos^2 t) = \frac{1}{2} L(1 + \cos 2t) = \frac{1}{2}\left\{\frac{1}{s} + \frac{s}{s^2 + 4}\right\}$

$\therefore$ by shifting property, we get

$$L(e^{2t} \cos^2 t) = \frac{1}{2}\left\{\frac{1}{s-2} + \frac{s-2}{(s-2)^2 + 4}\right\}.$$

(*iii*) Since $L(\sqrt{t}) = \frac{\Gamma(3/2)}{s^{3/2}} = \frac{(1/2) \, . \, \Gamma\pi}{s^{3/2}}$

$\therefore$ by shifting property, we obtain $L(e^{3t}\sqrt{t}) = \frac{\sqrt{\pi}}{2} \frac{1}{(s-3)^{3/2}}$.

Example 21.3. *If* $L f(t) = \bar{f}(s)$, *show that*

$$L[(\sinh at) f(t)] = \frac{1}{2}[\bar{f}(s-a) - \bar{f}(s+a)]$$

$$L[(\cosh at) f(t)] = \frac{1}{2}[\bar{f}(s-a) + \bar{f}(s+a)]$$

Hence evaluate (*i*) $\sinh 2t \sin 3t$ (*ii*) $\cosh 3t \cos 2t$.

Solution. We have $L[(\sinh at) f(t)] = L\left[\frac{1}{2}(e^{at} - e^{-at}) f(t)\right] = \frac{1}{2}[L\{e^{at} f(t) - L\{e^{-at} f(t)\}]$

$$= \frac{1}{2}[\bar{f}(s-a) - \bar{f}(s+a)], \text{ by shifting property.}$$

Similarly, $L[(\cosh at) f(t)] = \frac{1}{2}[L\{e^{at} f(t)\} + \{e^{-at} f(t)\}]$

$$= \frac{1}{2}[\bar{f}(s-a) + \bar{f}(s+a)], \text{ by shifting property.}$$

(*i*) Since $L(\sin 3t) = \frac{3}{s^2 + 3^2}$, the first result gives

$$L(\sinh 2t \sin 3t) = \frac{1}{2}\left\{\frac{3}{(s-2)^2 + 3^2} - \frac{3}{(s+2)^2 + 3^2}\right\} = \frac{12s}{s^4 + 10s^2 + 169}$$

(*ii*) Since $L(\cos 2t) = \frac{s}{s^2 + 2^2}$, the second result gives

$$L(\cosh 3t \cos 2t) = \frac{1}{2}\left\{\frac{s-3}{(s-3)^2 + 2^2} + \frac{s+3}{(s+3)^2 + 2^2}\right\} = \frac{2s(s^2 - 5)}{s^4 - 10s^2 + 169}.$$

Example 21.4. *Show that*

(i) $L(t \sin at) = \dfrac{2as}{(s^2+a^2)^2}$ (Bhopal, 2001) (ii) $L(t \cos at) = \dfrac{s^2-a^2}{(s^2+a^2)^2}$.

Solution. Since $L(t) = 1/s^2$. $\therefore$ $L(te^{iat}) = \dfrac{1}{(s-ia)^2} = \dfrac{(s+ia)^2}{[(s-ia)(s+ia)]^2}$

or $$L[t(\cos at + i \sin at)] = \frac{(s^2-a)^2 + i(2as)}{(s^2+a^2)^2}$$

Equating the real and imaginary parts from both sides, we get the desired results.

Example 21.5. *Find the Laplace transform of $f(t)$ defined as*

(i) $f(t) = t/\tau$, when $0 < t < \tau$
$= 1$, when $t > \tau$. (Kerala, 2005)

(ii) $f(t) = \begin{cases} 1, & 0 < t \le 1 \\ t, & 1 < t \le 2 \\ 0, & t > 2 \end{cases}$ (J.N.T.U., 2006 ; W.B.T.U., 2005)

Solution. (i) $$Lf(t) = \int_0^\tau e^{-st} \cdot \frac{t}{\tau}\, dt + \int_\tau^\infty e^{-st} \cdot 1\, dt = \frac{1}{\tau}\left[\left| t \cdot \frac{e^{-st}}{-s}\right|_0^\tau - \int_0^\tau 1 \cdot \frac{e^{-st}}{-s}\, dt\right] + \left|\frac{e^{-st}}{-s}\right|_\tau^\infty$$

$$= \frac{1}{\tau}\left[\frac{\tau e^{-s\tau} - 0}{-s} - \left|\frac{e^{-st}}{s^2}\right|_0^\tau\right] + \frac{0 - e^{-s\tau}}{-s} = \frac{-e^{-s\tau}}{s} - \frac{e^{-s\tau}-1}{\tau s^2} + \frac{e^{-s\tau}}{s} = \frac{1-e^{-s\tau}}{\tau s^2}.$$

(ii) $$L\{f(t)\} = \int_0^1 e^{-st} \cdot 1\, dt + \int_1^2 e^{-st} \cdot t\, dt + \int_2^\infty e^{-st} \cdot (0)\, dt$$

$$= \left|\frac{e^{-st}}{-s}\right|_0^1 + \left| t \cdot \frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right|_1^2 = \frac{1-e^{-s}}{s} + \left\{\left(-\frac{2e^{-2s}}{s} - \frac{e^{-2s}}{s^2}\right) - \left(\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2}\right)\right\}$$

$$= \frac{1}{s} - \frac{2e^{-2s}}{s} + \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2}.$$

Example 21.6. *Find the Laplace transform of (i) $\left(\sqrt{t} - \dfrac{1}{\sqrt{t}}\right)^3$.* (Kurukshetra, 2005)

(ii) $\dfrac{\cos\sqrt{t}}{\sqrt{t}}$ (Mumbai, 2009)

Solution. (i) Since $(\sqrt{t} - 1/\sqrt{t})^3 = t^{3/2} - 3t^{1/2} + 3t^{-1/2} - t^{-3/2}$

$\therefore$ $$L(\sqrt{t} - 1/\sqrt{t}) = L(t^{3/2}) - 3L(t^{1/2}) + 3L(t^{-1/2}) - L(t^{-3/2})$$

$$= \frac{\Gamma(3/2+1)}{s^{3/2+1}} - 3\frac{\Gamma(1/2+1)}{s^{1/2+1}} + 3\frac{\Gamma(-1/2+1)}{s^{-1/2+1}} - \frac{\Gamma(-3/2+1)}{s^{-3/2+1}}$$

$$= \frac{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)}{s^{5/2}} - 3\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{3/2}} + 3\frac{\Gamma\left(\frac{1}{2}\right)}{s^{1/2}} - \frac{\Gamma\left(-\frac{1}{2}\right)}{s^{-1/2}}$$

$$= \frac{3}{4}\frac{\sqrt{\pi}}{s^{5/2}} - \frac{3}{2}\frac{\sqrt{\pi}}{s^{3/2}} + \frac{3\sqrt{\pi}}{s^{1/2}} + \frac{2\sqrt{\pi}}{s^{-1/2}} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}\right]$$

$$= \frac{\sqrt{\pi}}{4}\left(\frac{3}{s^{5/2}} - \frac{6}{s^{3/2}} + \frac{12}{s^{1/2}} + \frac{8}{s^{-1/2}}\right).$$

(ii) We know that $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots \infty$

$$\therefore \quad \cos\sqrt{t} = 1 - \frac{t}{2!} + \frac{t^2}{4!} - \frac{t^3}{6!} + \dots$$

and
$$\frac{\cos\sqrt{t}}{\sqrt{t}} = t^{-1/2} - \frac{t^{1/2}}{2!} + \frac{t^{3/2}}{4!} - \frac{t^{5/2}}{6!} + \dots$$

and
$$L\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right) = \frac{\Gamma(1/2)}{s^{1/2}} - \frac{1}{2!}\frac{\Gamma(3/2)}{s^{3/2}} + \frac{1}{4!}\frac{\Gamma(5/2)}{s^{5/2}} - \frac{1}{6!}\frac{\Gamma(7/2)}{s^{7/2}} + \dots$$
$$= \frac{\Gamma 1/2}{\sqrt{s}} - \frac{1}{2}\cdot\frac{1/2\,\Gamma 1/2}{s^{3/2}} + \frac{1}{4!}\frac{3/2\,.\,1/2\,.\,\Gamma(1/2)}{s^{5/2}} - \frac{1}{6!}\frac{5/2\,.\,3/2\,.\,1/2\,.\,\Gamma(1/2)}{s^{7/2}} + \dots$$
$$= \sqrt{\left(\frac{\pi}{2}\right)}\left[1 - \frac{1}{(4s)} + \frac{1}{2!}\frac{1}{(4s)^2} - \frac{1}{3!}\frac{1}{(4s)^3}\dots\right] = \sqrt{\left(\frac{\pi}{s}\right)}e^{-1/4s}.$$

Example 21.7. *Find the Laplace transform of the function*
(i) $f(t) \mid t-1 \mid + \mid t+1 \mid, t \geq 0$ *(S.V.T.U., 2009)*
(ii) $f(t) = [t]$, *where [ ] stands for the greatest integer function.* *(P.T.U., 2010)*

Solution. (i) Given function is equivalent to

$$f(t) = \begin{cases} 2, & 0 \le t < 1 \\ 2t, & t \ge 1 \end{cases}$$

$$\therefore \quad L f(t) = \int_0^1 e^{-st}(2)\,dt + \int_1^\infty e^{-st}(2t)\,dt = 2\left[\left|\frac{e^{-st}}{-s}\right|_0^1 + 2\left|\frac{t.e^{-st}}{-s}\right|_1^\infty - \left|\frac{e^{-st}}{(-s)^2}\right|_1^\infty\right]$$
$$= 2\left(\frac{e^{-s}}{-s} + \frac{1}{s}\right) + 2\left(\frac{0-e^{-s}}{-s} - \frac{0-e^{-s}}{s^2}\right) = \frac{2}{s}\left(1 + \frac{e^{-s}}{s}\right)$$

(ii) Given function is equivalent to

$$[t] = 0 \text{ in } (0,1) + 1 \text{ in } (1,2) + 2 \text{ in } (2,3) + 3 \text{ in } (3,4) + \dots$$

$$\therefore \quad L[f(t)] = \int_0^\infty e^{-st}[f(t)]dt = \int_0^\infty e^{-st}\,[t]dt$$
$$= \int_0^1 e^{-st}(0)dt + \int_1^2 e^{-st}(1)dt + \int_2^3 e^{-st}(2)dt + \int_3^4 e^{-st}(3)dt + \dots \infty$$
$$= 0 + \left|\frac{e^{-st}}{-s}\right|_1^2 + 2\left|\frac{e^{-st}}{-s}\right|_2^3 + 3\left|\frac{e^{-st}}{-s}\right|_3^4 + \dots \infty$$
$$= -\frac{1}{s}[(e^{-2s} - e^{-s}) + 2(e^{-3s} - e^{-2s}) + 3(e^{-4s} - e^{-3s}) + \dots \infty]$$
$$= \frac{1}{s}(e^{-s} + e^{-2s} + e^{-3s} + \dots \infty) = \frac{1}{s}\left(\frac{e^{-s}}{1-e^{-s}}\right) = \frac{1}{s(e^s - 1)}.$$

III. Change of scale property. *If* $L\{f(t)\} = \bar{f}(s)$, *then* $L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$

$$L\{f(at)\} = \int_0^\infty e^{-st} f(at)\,dt = \int_0^\infty e^{-su/a} f(u)\,.\,du/a$$

| Put $at = u$
| $dt = du/a$

$$= \frac{1}{a}\int_0^\infty e^{-su/a} f(u)du = \frac{1}{a}\bar{f}(s/a).$$

Example 21.8. *Find* $L\left\{\frac{\sin at}{t}\right\}$, *given that* $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left\{\frac{1}{s}\right\}$.

Solution. By the above property,

$$L\left\{\frac{\sin at}{at}\right\} = \frac{1}{a}\tan^{-1}\left\{\frac{1}{(s/a)}\right\} = \frac{1}{a}\tan^{-1}\left(\frac{a}{s}\right) \text{ i.e., } L\left\{\frac{\sin at}{t}\right\} = \tan^{-1}\left\{\frac{a}{s}\right\}.$$

PROBLEMS 21.1

Find the Laplace transforms of

1. $e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t$. (*J.N.T.U., 2003*)
2. $1 + 2\sqrt{t} + 3/\sqrt{t}$.
3. $3\cosh 5t - 4\sinh 5t$. (*Nagarjuna, 2006*)
4. $\cos(at + b)$.
5. $(\sin t - \cos t)^2$.
6. $\sin 2t \cos 3t$. (*Kottayam, 2005*)
7. $\sin\sqrt{t}$.
8. $\sin^5 t$. (*Mumbai, 2007*)
9. $\cos^3 2t$.
10. $e^{-at}\sinh bt$
11. $e^{2t}(3t^5 - \cos 4t)$. (*P.T.U., 2007*)
12. $e^{-3t}\sin 5t \sin 3t$. (*V.T.U., 2006*)
13. $e^{-t}\sin^2 t$. (*Mumbai, 2009*)
14. $e^{2t}\sin^4 t$ (*Mumbai, 2007*)
15. $\cosh at \sin at$. (*Delhi, 2002*)
16. $\sinh 3t \cos^2 t$ (*Madras, 2000*)
17. $t^2 e^{2t}$. (*V.T.U., 2008 S*)
18. $(1 + te^{-t})^3$
19. $t\sqrt{(1 + \sin t)}$ (*Mumbai, 2007*)
20. $f(t) = \begin{cases} 4, & 0 \le t <= 1 \\ 3, & t > 1 \end{cases}$ (*U.P.T.U., 2009*)
21. $f(t) = \begin{cases} \sin t, 0 < t < \pi \\ 0, t > \pi. \end{cases}$ (*Madras, 2000 S*)
22. $f(x) = \begin{cases} \sin(x - \pi/3), & x > \pi/3 \\ 0 & ; x < \pi/3 \end{cases}$ (*Rajasthan, 2006*)
23. $f(t) = \begin{cases} \cos(t - 2\pi/3), & t > 2\pi/3 \\ 0, & t < 2\pi/3 \end{cases}$
24. $f(t) = \begin{cases} t^2, 0 < t < 2 \\ t - 1, 2 < t < 3 \\ 7, t > 3. \end{cases}$ (*Mumbai, 2007*)
25. If $L[f(t)] = \frac{1}{s(s^2+1)}$, find $L[e^{-t}f(2t)]$.

21.5 TRANSFORMS OF PERIODIC FUNCTIONS

If $f(t)$ is a periodic function with period T, i.e., $f(t + T) = f(t)$, then

$$L\{f(t)\} = \frac{\int_0^T e^{-st} f(t)dt}{1 - e^{-sT}}$$

We have $\quad L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \int_0^T e^{-st} f(t)dt + \int_T^{2T} e^{-st} f(t)dt + \int_{2T}^{3T} e^{-st} f(t)dt + \ldots$

In the second integral put $t = u + T$, in the third integral put $t = u + 2T$, and so on. Then

$$L\{f(t)\} = \int_0^T e^{-st} f(t)dt + \int_0^T e^{-s(u+T)} f(u+T)du + \int_0^T e^{-s(u+2T)} f(u+2T)du + \ldots$$

$$= \int_0^T e^{-st} f(t)dt + e^{-sT}\int_0^T e^{-su} f(u)du + e^{-2sT}\int_0^T e^{-su} f(u)du + \ldots$$

$$[\because \; f(u) = f(u+T) = f(u+2T) \text{ etc.}]$$

$$= \int_0^T e^{-st} f(t)dt + e^{-sT}\int_0^T e^{-st} f(t)dt + e^{-2sT}\int_0^T e^{-st} f(t)dt + \ldots$$

$$= (1 + e^{-sT} + e^{-2sT} + \ldots \infty)\int_0^T e^{-st} f(t)dt$$

$$= \frac{1}{1 - e^{-sT}}\int_0^T e^{-st} f(t)dt.$$ (*V.T.U., 2008 ; Mumbai, 2006*)

Example 21.9. *Find the Laplace transform of the function*

$$f(t) = \sin \omega t, \quad 0 < t < \pi/\omega$$
$$= 0, \quad \pi/\omega < t < 2\pi/\omega.$$

(Kurukshetra, 2005 ; Madras, 2003)

Solution. Since $f(t)$ is a periodic function with period $2\pi/\omega$.

$$\therefore \quad L\{f(t)\} = \frac{1}{1-e^{-2\pi s/\omega}} \int_0^{2\pi/\omega} e^{-st} f(t) dt$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left[\int_0^{\pi/\omega} e^{-st} \sin \omega t \, dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st} . 0 \, dt \right]$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left| \frac{e^{-st}(-s \sin \omega t - \omega \cos \omega t)}{s^2+\omega^2} \right|_0^{\pi/\omega} = \frac{\omega e^{-\pi s/\omega} + \omega}{(1-e^{-2\pi s/\omega})(s^2+\omega^2)} = \frac{\omega}{(1-e^{-\pi s/\omega})(s^2+\omega^2)}.$$

Example 21.10. *Draw the graph of the periodic function*

$$f(t) = \begin{cases} t, & 0 < t < \pi \\ \pi - t, & \pi < t < 2\pi. \end{cases}$$

and find its Laplace transform. *(U.P.T.U., 2003)*

Solution. Here the period of $f(t) = 2\pi$ and its graph is as in Fig. 21.1.

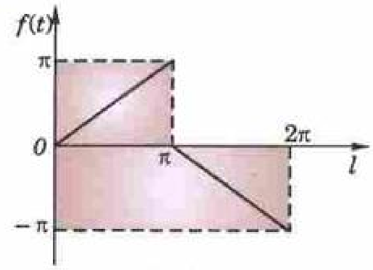

Fig. 21.1

$$\therefore \quad L f(t) = \frac{1}{1-e^{-2\pi s}} \left\{ \int_0^{\pi} e^{-st} t \, dt + \int_{\pi}^{2\pi} e^{-st} (\pi - t) dt \right\}$$

$$= \frac{1}{1-e^{-2\pi s}} \left\{ \left| t\left(\frac{e^{-st}}{-s}\right) - 1.\left(\frac{e^{-st}}{s^2}\right) \right|_0^{\pi} + \left| (\pi - t)\left(\frac{e^{-st}}{-s}\right) - (-1)\left(\frac{e^{-st}}{s^2}\right) \right|_{\pi}^{2\pi} \right\}$$

$$= \frac{1}{1-e^{-2\pi s}} \left\{ \frac{-\pi e^{-\pi s}}{s} - \frac{e^{-\pi s}}{s^2} + \frac{1}{s^2} + \frac{\pi e^{-2\pi s}}{s} + \frac{e^{-2\pi s}}{s^2} - \frac{e^{-\pi s}}{s^2} \right\}$$

$$= \frac{1}{1-e^{-2\pi s}} \left\{ \frac{\pi}{s}\left(e^{-2\pi s} - e^{-\pi s}\right) + \frac{1}{s^2}\left(1 + e^{-2\pi s} - 2e^{-\pi s}\right) \right\}.$$

21.6 TRANSFORMS OF SPECIAL FUNCTIONS

(1) Transform of Bessel functions $J_0(x)$ and $J_1(x)$.

We know that $$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 . 4^2} - \frac{x^6}{2^2 . 4^2 . 6^2} + \ldots$$ [§ 16.7 (1), p. 553]

$$\therefore \quad L\{J_0(x)\} = \frac{1}{s} - \frac{1}{2^2}\frac{2!}{s^3} + \frac{1}{2^2 . 4^2} \cdot \frac{4!}{s^5} - \frac{1}{2^2 . 4^2 . 6^2} \cdot \frac{6!}{s^7} + \ldots$$

$$= \frac{1}{s}\left\{ 1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1 . 3}{2 . 4}\left(\frac{1}{s^4}\right) - \frac{1 . 3 . 5}{2 . 4 . 6}\left(\frac{1}{s^6}\right) + \ldots \right\}$$

$$= \frac{1}{s}\left(1 + \frac{1}{s^2}\right)^{-1/2} = \frac{1}{\sqrt{(s^2+1)}} \qquad \ldots(1)$$

Also since $$J_0'(x) = -J_1(x).$$ [Problem 4(*i*), p. 557]

$$\therefore \quad L\{J_1(x)\} = -L\{J_0'(x)\} = -[sL\{J_0(x)\} - 1] = 1 - \frac{s}{\sqrt{(s^2+1)}} \qquad \ldots(2)$$

(2) Transform of Error function

We know that $$erf(\sqrt{x}) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{x}} e^{-t^2} dt$$ (§ 7.18, p. 312)

$$= \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{x}} \left(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \dots\right) dt = \frac{2}{\sqrt{\pi}} \left(x^{1/2} - \frac{x^{3/2}}{3} + \frac{x^{5/2}}{5.2!} - \frac{x^{7/2}}{7.3!} + \dots\right)$$

$$\therefore \quad L\{erf(\sqrt{x})\} = \frac{2}{\sqrt{\pi}} \left\{\frac{\Gamma(3/2)}{s^{3/2}} - \frac{\Gamma(5/2)}{3s^{5/2}} + \frac{\Gamma(7/2)}{5.2!\,s^{7/2}} - \frac{\Gamma(9/2)}{7.3!\,s^{9/2}} + \dots\right\}$$

$$= \frac{1}{s^{3/2}} - \frac{1}{2}\frac{1}{s^{5/2}} + \frac{1.3}{2.4}\cdot\frac{1}{s^{7/2}} - \frac{1.3.5}{2.4.6}\cdot\frac{1}{s^{9/2}} + \dots$$

$$= \frac{1}{s^{3/2}} \left\{1 - \frac{1}{2}\cdot\frac{1}{s} + \frac{1.3}{2.4}\cdot\frac{1}{s^2} - \frac{1.3.5}{2.4.6}\cdot\frac{1}{s^3} + \dots\right\}$$

$$= \frac{1}{s^{3/2}}\left[1 + \frac{1}{s}\right]^{-1/2} = \frac{1}{s\sqrt{(s+1)}}. \qquad (Mumbai, 2009) \quad ...(3)$$

(3) Transform of Laguerre's polynomials $L_n(x)$

We know that $L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x})$ (§ 16.18, p. 571)

$$L[L_n(t)] = \int_0^\infty e^{-st} e^t \frac{d^n}{dt^n}(t^n e^{-t}) dt = \int_0^\infty e^{-(s-1)t} \frac{d^n}{dt^n}(e^{-t} t^n) dt$$

$$= \left| e^{-(s-1)t} \frac{d^{n-1}}{dt^{n-1}}(e^{-t}t^n)\right|_0^\infty + \int_0^\infty e^{-(s-1)t\,(s-1)}\,(s-1)\frac{d^{n-1}}{dt^{n-1}}(e^{-t}t^n)dt$$

$$= (s-1)\int_0^\infty e^{-(s-1)t}\,\frac{d^{n-1}}{dt^{n-1}}(e^{-t}t^n)dt. \qquad \text{(Integrating by parts)}$$

$$= (s-1)^n \int_0^\infty e^{-(s-1)t}\,.\,e^{-t}\,.\,t^n\,dt = (s-1)^n \int_0^\infty e^{-st}\,.\,t^n\,dt$$

$$= (s-1)^n L(t^n) = (s-1)^n\,.\,\frac{n!}{s^{n+1}}$$

Hence $$L[L_n(x)] = \frac{n!(s-1)^n}{s^n + 1} \quad (s > 1).$$

Example 21.11. *Evaluate (i) $L\{e^{-at} J_0(at)\}$ (ii) $L(erf\, 2\sqrt{t})$.* (*Mumbai, 2006*)

Solution. (*i*) We know that $L\{J_0(at)\} = \frac{1}{\sqrt{(s^2+a^2)}}$

By shifting property, we get

$$L\{e^{-at} J_0(at)\} = \frac{1}{\sqrt{[(s+a)^2 + a^2]}} = \frac{1}{\sqrt{(s^2 + 2sa + 2a^2)}}$$

(*ii*) We know that $L(erf\sqrt{t}) = \frac{1}{s(s+1)}$

$$\therefore \quad L(erf\ 2\sqrt{t}) = L[erf\sqrt{(4t)}] = \frac{1}{4}\cdot\frac{1}{\frac{s}{4}\sqrt{\left(\frac{s}{4}+1\right)}} = \frac{2}{s\sqrt{(s+4)}}.$$

PROBLEMS 21.2

1. Find the Laplace transform of the *saw-toothed wave* of period T, given $f(t) = t/T$ for $0 < t < T$. (*V.T.U., 2007*)
2. Find the Laplace transform of the *full-wave rectifier*

   $f(t) = E \sin wt$, $0 < t < \pi/w$, having period π/w.

3. Find the Laplace transform of the *square-wave* (or *meander*) function of period a defined as

$$f(t) = k, \quad \text{when } 0 < t < a$$
$$= -k, \quad \text{when } a < t < 2a.$$

(V.T.U., 2011)

4. Find the Laplace transform of the *triangular wave* of period $2a$ given by

$$f(t) = t, \quad 0 < t < a$$
$$= 2a - t, \quad a < t < 2a.$$

(Nagarjuna, 2008 ; V.T.U., 2008 S ; U.P.T.U., 2002)

Find the Laplace transform of the following functions :

5. $J_0(ax)$. 6. $e^{-at} J_0(bt)$. 7. $e^{2t} \, erf(\sqrt{t})$.

21.7 TRANSFORMS OF DERIVATIVES

(1) *If $f'(t)$ be continuous and $L\{f(t)\} = f(s)$, then $L\{f'(t)\} = s\bar{f}(s) - f(0)$.*

$$L\{f(t)\} = \int_0^\infty e^{-st} f'(t)dt \qquad \text{[Integrate by parts]}$$

$$= \left| e^{-st} f(t) \right|_0^\infty - \int_0^\infty (-s)e^{-st} . f(t)dt.$$

Now assuming $f(t)$ to be such that $\underset{t\to\infty}{\text{Lt}} \; e^{-st} f(t) = 0$. When this condition is satisfied, $f(t)$ is said to be *exponential order s*.

Thus, $$L\{f'(t)\} = f(0) + s\int_0^\infty e^{-st} f(t)dt$$

whence follows the desired result.

(2) *If $f'(t)$ and its first $(n-1)$ derivatives be continuous, then*

$$\mathbf{L\{f^n(t)\} = s^n \,\bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{n-1}(0).}$$

Using the general rule of integration by parts (Footnote p. 398).

$$L\{f^n(t)\} = \int_0^\infty e^{-st} f^n(t)dt$$

$$= |e^{-st} f^{n-1}(t) - (-s)e^{-st} f^{n-2}(t) + (-s)^2 e^{-st} f^{n-3}(t) - \ldots$$

$$+ (-1)^{n-1}(-s)^{n-1} e^{-st} . f(t)|_0^\infty + (-1)^n(-s)^n \int_0^\infty e^{-st} f(t)dt$$

$$= -f^{n-1}(0) - sf^{n-2}(0) - s^2 f^{n-3}(0) - \ldots - s^{n-1} f(0) + s^n \int_0^\infty e^{-st} f(t)dt$$

Assuming that $\underset{t\to\infty}{\text{Lt}} \; e^{-st} f^m(t) = 0$ for $m = 0, 1, 2, \ldots, n-1$.

This proves the required result.

21.8 TRANSFORMS OF INTEGRALS

If $L\{f(t)\} = \bar{f}(s)$, then $\mathbf{L\left\{\int_0^t f(u)du\right\} = \frac{1}{s}\bar{f}(s).}$

Let $$\phi(t) = \int_0^t f(u)\,du, \text{ then } \phi'(t) = f(t) \text{ and } \phi(0) = 0$$

$\therefore$ $$L\{\phi'(t)\} = s\bar{\phi}(s) - \phi(0) \qquad \text{[By § 21.7 (1)]}$$

or $$\bar{\phi}(s) = \frac{1}{s} L\{\phi'(t)\} \quad i.e., \quad L\left(\int_0^t f(u)du\right) = \frac{1}{s}\bar{f}(s).$$

21.9 MULTIPLICATION BY t^n

If $L\{f(t)\} = \bar{f}(s)$, then

$$\mathbf{L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[\bar{f}(s)]}, \textit{ where } n = 1, 2, 3 \ldots$$

We have $\int_0^\infty e^{-st} f(t)\,dt = \bar{f}(s)$.

Differentiating both sides with respect to s, $\frac{d}{ds}\left\{\int_0^\infty e^{-st} f(t)\,dt\right\} = \frac{d}{ds}\{\bar{f}(s)\}$

or By Leibnitz's rule for differentiation under the integral sign (p. 233).

$$\int_0^\infty \frac{\partial}{\partial s}(e^{-st}) f(t)\,dt = \frac{d}{ds}\{\bar{f}(s)\}$$

or $$\int_0^\infty \{-te^{-st} f(t)\}\,dt = \frac{d}{ds}[\bar{f}(s)] \quad \text{or} \quad \int_0^\infty e^{-st}[tf(t)]\,dt = -\frac{d}{ds}[\bar{f}(s)]$$

which proves the theorem for $n = 1$.

Now assume the theorem to be true for $n = m$ (say), so that

$$\int_0^\infty e^{-st}[t^m f(t)]\,dt = (-1)^m \frac{d^m}{ds^m}[\bar{f}(s)]$$

Then $$\frac{d}{ds}\left[\int_0^\infty e^{-st} t^m f(t)\,dt\right] = (-1)^m \frac{d^{m+1}}{ds^{m+1}}[\bar{f}(s)]$$

or By Leibnitz's rule, $$\int_0^\infty (-te^{-st}) \cdot t^m f(t)\,dt = (-1)^m \frac{d^{m+1}}{ds^{m+1}}[\bar{f}(s)]$$

or $$\int_0^\infty e^{-st}[t^{m+1} f(t)]\,dt = (-1)^{m+1} \frac{d^{m+1}}{ds^{m+1}}[\bar{f}(s)].$$

This shows that, if the theorem, is true for $n = m$, it is also true for $n = m + 1$. But it is true for $n = 1$. Hence it is true for $n = 1 + 1 = 2$, and $n = 2 + 1 = 3$ and so on.

Thus the theorem is true for all positive integral values of n. *(U.P.T.U., 2005)*

Example 21.12. *Find the Laplace transforms of*

(*i*) *t cos at* *(Raipur, 2005)* (*ii*) t^2 *sin at* *(S.V.T.U., 2007)*

(*iii*) $t^3 e^{-3t}$ *(Kottayam, 2005)* (*iv*) te^{-t} *sin 3t.* *(Kurukshetra, 2005)*

Solution. (*i*) Since $L(\cos at) = s/(s^2 + a^2)$

$$\therefore \quad L(t\cos at) = -\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right) = -\frac{s^2+a^2-s\,.\,2s}{(s^2+a^2)^2}$$

$$= \frac{s^2-a^2}{(s^2+a^2)^2}$$ [*cf.* Example 21.4]

(*ii*) Since $\sin at = \frac{a}{s^2+a^2}$,

$$\therefore \quad L(t^2 \sin at) = (-1)^2 \frac{d^2}{ds^2}\left(\frac{a}{s^2+a^2}\right) = \frac{d}{ds}\left\{\frac{-2as}{(s^2+a^2)^2}\right\} = \frac{2a(3s^2-a^2)}{(s^2+a^2)^3}.$$

(*iii*) Since $L(e^{-3t}) = 1/(s+3)$,

$$\therefore \quad L(t^3 e^{-3t}) = (-1)^3 \frac{d^3}{ds^3}\left(\frac{1}{s+3}\right) = -\frac{(-1)^3\,.\,3\,!}{(s+3)^{3+1}} = 6/(s+3)^4.$$

(*iv*) Since $L(\sin 3t) = \frac{3}{s^2+3^2}$, therefore $L(t\sin 3t) = -\frac{d}{ds}\left(\frac{3}{s^2+3^2}\right) = \frac{6s}{(s^2+9)^2}$

Now using the shifting property (§ 21.4 II), we get

$$L(e^{-t}\, t\sin 3t) = \frac{6(s+1)}{[(s+1)^2+9]^2} = \frac{6(s+1)}{(s^2+2s+10)^2}.$$

Example 21.13. *Evaluate (i) $L\{t\,J_0(at)\}$ (ii) $L\{t\,J_1(t)\}$ (iii) $L\{t\,erf\,2\sqrt{t}\}$.*

Solution. (*i*) Since $L\{J_0(at)\} = \dfrac{1}{\sqrt{(s^2+a^2)}}$

$$\therefore \qquad L\{tJ_0(at)\} = -\frac{d}{ds}\{L\,J_0(at)\} = -\frac{d}{ds}\frac{1}{\sqrt{(s^2+a^2)}} = \frac{s}{(s^2+a^2)^{3/2}}$$

(*ii*) Since $\qquad L\{J_1(t)\} = 1 - \dfrac{s}{\sqrt{(s^2+1)}}$

$$\therefore \qquad L\{t\,J_1(t)\} = -\frac{d}{ds}\{L\,J_0(t)\} = -\frac{d}{ds}\left\{1 - \frac{s}{(\sqrt{s^2+1})}\right\} = \frac{1}{(s^2+1)^{3/2}}$$

(*iii*) Since $\qquad L(erf\sqrt{t}) = \dfrac{1}{s\sqrt{(s+1)}}$

$$\therefore \qquad L(erf\,2\sqrt{t}) = L\,erf\sqrt{4(t)} = \frac{1}{4}\cdot\frac{1}{\frac{s}{4}\sqrt{\left(\frac{s}{4}+1\right)}} = \frac{2}{s\sqrt{(s+4)}}$$

Thus $$L(t\,erf\,2\sqrt{t}) = -\frac{d}{ds}\left\{\frac{2}{s\sqrt{(s+4)}}\right\} = -\frac{d}{ds}\left\{\frac{2}{\sqrt{(s^3+4s^2)}}\right\} = \frac{3s+8}{s^2(s+4)^{3/2}}$$

21.10 DIVISION BY t

If $L\{f(t)\} = \bar{f}(s)$, then $\mathbf{L}\left\{\frac{1}{t}\mathbf{f(t)}\right\} = \int_s^\infty \bar{\mathbf{f}}(\mathbf{s})\,\mathbf{ds}$ *provided the integral exists.*

We have $\bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt$

Integrating both sides with respect to s from s to ∞.

$$\int_s^\infty \bar{f}(s)\,ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)\,dt\right] ds = \int_0^\infty \int_s^\infty f(t)\,e^{-st}\,ds\,dt$$

[Changing the order of integration]

$$= \int_0^\infty f(t)\left[\int_s^\infty e^{-st}\,ds\right] dt \qquad [\because\ t \text{ is independent of } s]$$

$$= \int_0^\infty f(t)\left|\frac{e^{-st}}{-t}\right|_s^\infty dt = \int_0^\infty e^{-st}\cdot\frac{f(t)}{t}\,dt = L\left\{\frac{1}{t}f(t)\right\}.$$

Example 21.14. *Find the Laplace transform of (i) $(1-e^t)/t$* (*Madras, 2000*)
(*ii*) $\dfrac{\cos at - \cos bt}{t} + t\sin at.$ (*V.T.U., 2010*)

Solution. (*i*) Since $\qquad L(1-e^t) = L(1) - L(e^t) = \dfrac{1}{s} - \dfrac{1}{s-1}$

$$\therefore \qquad L\left(\frac{1-e^t}{t}\right) = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1}\right) ds = |\log s - \log(s-1)|_0^\infty$$

$$= \left|\log\left(\frac{s}{s-1}\right)\right|_s^\infty = -\log\left[\frac{1}{1-1/s}\right] = \log\left(\frac{s-1}{s}\right)$$

(*ii*) Since $\qquad L(\cos at - \cos bt) = \dfrac{s}{s^2+a^2} - \dfrac{s}{s^2+b^2}$ and $L(\sin at) = \dfrac{a}{s^2+a^2}$

$$\therefore \quad L\left(\frac{\cos at - \cos bt}{t}\right) + L(t \sin at) = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds - \frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)$$

$$= \left| \frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)\right|_s^\infty - a\,\frac{-2s}{(s^2+a^2)^2}$$

$$= \frac{1}{2}\operatorname*{Lt}_{s\to\infty} \log\frac{s^2+a^2}{s^2+b^2} - \frac{1}{2}\log\frac{s^2+a^2}{s^2+b^2} + \frac{2as}{(s^2+a^2)^2}$$

$$= \frac{1}{2}\log\left(\frac{1+0}{1+0}\right) - \frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right) + \frac{2as}{(s^2+a^2)^2} = \log\left(\frac{s^2+b^2}{s^2+a^2}\right)^{1/2} + \frac{2as}{(s^2+a^2)^2}$$

$[\because \quad \log 1 = 0]$

Example 21.15. *Evaluate (i)* $L\left\{e^{-t}\int_0^t \frac{\sin t}{t}\,dt\right\}$ *(Madras, 2006)*

(ii) $L\left\{t\int_0^t \frac{e^{-t}\sin t}{t}\,dt\right\}$ *(P.T.U., 2005)* *(iii)* $L\left\{\int_0^t\int_0^t\int_0^t (t\sin t)\,dt\,dt\,dt\right\}$. *(Mumbai, 2006)*

Solution. (*i*) We know that $L(\sin t) = \dfrac{1}{s^2+1}$

$$L\left(\frac{\sin t}{t}\right) = \int_0^\infty \frac{1}{s^2+1}\,ds = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$$

$$\therefore \quad L\left\{\int_0^t \frac{\sin t}{t}\,dt\right\} = \frac{1}{s}\cot^{-1} s$$

Thus by shifting property, $L\left\{e^{-t}\left(\int_0^t \frac{\sin t}{t}\,dt\right)\right\} = \dfrac{1}{s+1}\cot^{-1}(s+1)$.

(*ii*) Since $\quad L\left(\dfrac{\sin t}{t}\right) = \cot^{-1} s$

$$\therefore \quad L\left(e^{-t}\cdot\frac{\sin t}{t}\right) = \cot^{-1}(s+1)$$

and $\quad L\left\{\displaystyle\int_0^t e^{-t}\frac{\sin t}{t}\,dt\right\} = \dfrac{1}{s}\cot^{-1}(s+1)$

Hence $\quad L\left\{t\,.\displaystyle\int_0^t e^{-t}\frac{\sin t}{t}\,dt\right\} = -\dfrac{d}{ds}\left\{\dfrac{\cot^{-1}(s+1)}{s}\right\}$

$$= -\frac{s\,.\left[\dfrac{-1}{1+(s+1)^2}\right] - \cot^{-1}(s+1)}{s^2} = \frac{s + (s^2+2s+2)\cot^{-1}(s+1)}{s^2(s^2+2s+2)}$$

(*iii*) Since $\quad L(\sin t) = \dfrac{1}{s^2+1}$

$$\therefore \quad L(t\sin t) = -\frac{d}{ds}\frac{1}{(s^2+1)} = \frac{2s}{(s^2+1)^2}$$

Thus $\quad L\left\{\displaystyle\int_0^t\int_0^t\int_0^t (t\sin t)\,dt\,.\,dt\,.\,dt\right\} = \dfrac{1}{s^3}L(t\sin t) = \dfrac{1}{s^3}\cdot\dfrac{2s}{(s^2+1)^2} = \dfrac{2}{s^2(s^2+1)^2}$

21.11 EVALUATION OF INTEGRALS BY LAPLACE TRANSFORMS

Example 21.16. *Evaluate* (*i*) $\int_0^\infty te^{-3t} \sin t \, dt$ (V.T.U., 2007)

(*ii*) $\int_0^\infty \frac{\sin mt}{t} dt$ (*iii*) $\int_0^\infty e^{-t}\left(\frac{\cos at - \cos bt}{t}\right) dt$ (Mumbai, 2009)

(*iv*) $L\left\{\int_0^t \frac{e^{-t}\sin t}{t} dt\right\}$.

Solution. (*i*) $$\int_0^\infty te^{-3t} \sin t \, dt = \int_0^\infty e^{-st}(t \sin t) \, dt \quad \text{where } s = 3$$

$$= L(t \sin t), \text{ by definition.}$$

$$= (-1) \frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2} = \frac{2\times 3}{(3^2+1)^2} = \frac{3}{50}.$$

(*ii*) Since $L(\sin mt) = m/(s^2 + m^2) = f(s)$, say.

$\therefore$ Using § 21.10, $L\left(\frac{\sin mt}{t}\right) = \int_s^\infty f(s)\, ds = \int_0^\infty \frac{m\, ds}{s^2+m^2} = \left|\tan^{-1}\frac{s}{m}\right|_s^\infty$

or by Def., $$\int_0^\infty e^{-st} \frac{\sin mt}{t} dt = \frac{\pi}{2} - \tan^{-1}\frac{s}{m}$$

Now $\underset{s\to 0}{\text{Lt}} \tan^{-1}(s/m) = 0$ if $m > 0$ or π if $m < 0$.

Thus taking limits as $s \to 0$, we get

$$\int_0^\infty \frac{\sin mt}{t} dt = \frac{\pi}{2} \text{ if } m > 0 \quad \text{or} \quad -\pi/2 \text{ if } m < 0$$

(*iii*) We know that $L(\cos at) = \frac{s}{s^2+a^2}$ and $L(\cos bt) = \frac{s}{s^2+b^2}$

$$\therefore \quad L\,\frac{\cos at - \cos bt}{t} = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

$$= \frac{1}{2}\left\{\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\}_s^\infty = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

This implies that $\int_0^\infty e^{-st}\left(\frac{\cos at - \cos bt}{t}\right) dt = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$

Taking $s = 1$, we get $\int_0^\infty \left(e^{-t}\,\frac{\cos at - \cos bt}{t}\right) dt = \frac{1}{2}\log\left(\frac{1+b^2}{1+a^2}\right)$

(*iv*) Since $$L\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{ds}{s^2+1} = \tan^{-1} s = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s.$$

$$\therefore \quad L\left\{e^t\left(\frac{\sin t}{t}\right)\right\} = \cot^{-1}(s-1), \text{ by shifting property (§ 21.4 } II).$$

Thus $$L\left[\int_0^t \left\{e^t\left(\frac{\sin t}{t}\right)\right\} dt\right] = \frac{1}{s}\cot^{-1}(s-1), \text{ by § 21.8.}$$

PROBLEMS 21.3

1. Find $L\left(\int_0^t e^{-t}\cos t\,dt\right)$.
2. Given $L\{2\sqrt{(t/\pi)}\} = 1/s^{3/2}$, show that $L\{1/\sqrt{(\pi t)}\} = 1/\sqrt{s}$. (U.P.T.U., 2005 ; Madras, 2003)
3. Given $L[\sin(\sqrt{t})] \dfrac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4\pi}$, prove that $L\left[\dfrac{\cos(\sqrt{t})}{\sqrt{t}}\right] = \sqrt{\left(\dfrac{\pi}{s}\right)} e^{-1/4\pi}$. (Mumbai, 2009)

Find the Laplace transforms of the following functions :

4. $t\sin^2 t$ (Nagarjuna, 2008)
5. $\sin 2t - 2t\cos 2t$. (Anna, 2003)
6. $t^2\cos at$.
7. $t\sinh at$.
8. $te^{2t}\sin 3t$. (Madras, 2003)
9. $te^{-2t}\sin 4t$. (V.T.U., 2008)
10. $t^2e^{-3t}\sin 2t$. (Madras, 2000 S)
11. $(e^{-at} - e^{-bt})/t$. (Anna, 2005 S)
12. $(\sin t)/t$. (P.T.U., 2010)
13. $\dfrac{(\sin t \sin 5t)}{t}$. (Mumbai, 2008)
14. $(e^{at} - \cos bt)/t$. (U.P.T.U., 2003)
15. $(e^{-t}\sin t)/t$. (V.T.U., 2009 S)
16. $(1 - \cos 3t)/t$. (V.T.U., 2006)
17. $(1 - \cos t)/t^2$. (Hazaribag, 2008)
18. $2^t + \dfrac{\cos 2t - \cos 3t}{t} + t\sin t$. (V.T.U., 2004)
19. Evaluate (i) $\int_0^\infty \dfrac{\cos 6t - \cos 4t}{t}\,dT$ (Mumbai, 2008 ; P.T.U., 2006)

    (ii) $\int_0^\infty \dfrac{e^{-\sqrt{2}t}\sinh t\sin t}{t}\,dt$ (Mumbai, 2005) (iii) $\int_0^\infty te^{-2t}\sin 3t\,dt$ (V.T.U., 2008)

    (iv) $\int_0^\infty te^{-t}\sin^4 t\,dt$.
20. Prove that (i) $\int_0^\infty \dfrac{e^{-at} - e^{-bt}}{t}\,dt = \log\dfrac{b}{a}$. (S.V.T.U., 2009 ; Mumbai, 2007 ; J.N.T.U., 2006)

    (ii) $\int_0^\infty \dfrac{e^{-2t}\sinh t}{t}\,dt = \dfrac{1}{2}\log 3$ (Mumbai, 2008) (iii) $\int_0^\infty \dfrac{e^{-t}\sin t}{t}\,dt = \dfrac{\pi}{4}$. (V.T.U., 2009 S)

    (iv) $\int_0^\infty \dfrac{e^{-t}\sin^2 t}{t}\,dt = \dfrac{1}{4}\log 5$. (Kurukshetra, 2006)
21. Evaluate (i) $L\left(\int_0^t \dfrac{\sin t}{t}\,dt\right)$ (J.N.T.U., 2005)

    (ii) $L\left(\int_0^t e^{-t}\cos t\,dt\right)$ (iii) $L\int_0^t \dfrac{e^t\sin t}{t}\,dt$. (P.T.U., 2009 S ; S.V.T.U., 2009 ; Bhopal, 2008)
22. Show that (i) $L[tJ_0(at)] = \dfrac{s}{(s^2+a^2)^{3/2}}$ (ii) $\int_0^\infty te^{-3t}J_0(4t)\,dt = 3/125$.

21.12 INVERSE TRANSFORMS — METHOD OF PARTIAL FRACTIONS

Having found the Laplace transforms of a few functions, let us now determine the inverse transforms of given functions of s. We have seen that $L\{f(t)\}$ in each case, is a rational algebraic function. Hence *to find the inverse transforms, we first express the given function of s into partial fractions which will, then, be recognizable as one of the following standard forms* :

(1) $\mathbf{L^{-1}\left[\dfrac{1}{s}\right] = 1.}$ **(2)** $\mathbf{L^{-1}\left[\dfrac{1}{s-a}\right] = e^{at}.}$

(3) $L^{-1}\left[\frac{1}{s^n}\right]=\frac{t^{n-1}}{(n-1)!}$, $n = 1, 2, 3, \ldots$ (4) $L^{-1}\left[\frac{1}{(s-a)^n}\right]=\frac{e^{at}\,t^{n-1}}{(n-1)!}$.

(5) $L^{-1}\left(\frac{1}{s^2+a^2}\right)=\frac{1}{a}\sin at$. (6) $L^{-1}\left(\frac{s}{s^2+a^2}\right)=\cos at$.

(7) $L^{-1}\left(\frac{1}{s^2-a^2}\right)=\frac{1}{a}\sinh at$. (8) $L^{-1}\left(\frac{s}{s^2-a^2}\right)=\cosh at$.

(9) $L^{-1}\left[\frac{1}{(s-a)^2+b^2}\right]=\frac{1}{b}e^{at}\sin bt$. (10) $L^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right]=e^{at}\cos bt$.

(11) $L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right)=\frac{1}{2a}t\sin at$. (12) $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]=\frac{1}{2a^3}(\sin at - at\cos at)$.

The reader is strongly advised to commit these results to memory. The results (1) to (10) follow at once from their corresponding results in § 21.3 and 21.4. As illustrations, we shall prove (11) and (12). Example 21.4 gives

$$L(t\sin at)=\frac{2as}{(s^2+a^2)^2} \text{ and } L(t\cos at)=\frac{s^2-a^2}{(s^2+a^2)^2}$$

$\therefore$ $$t\sin at = 2a\,L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right], \text{ whence follows (11).}$$

Also $$t\cos at = L^{-1}\left[\frac{s^2-a^2}{(s^2+a^2)^2}\right]=L^{-1}\left[\frac{(s^2+a^2)-2a^2}{(s^2+a^2)^2}\right]$$

$$=L^{-1}\left[\frac{1}{s^2+a^2}\right]-2a^2\,L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$$

$$=\frac{1}{a}\sin at - 2a^2\,L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] \text{ whence follows (12).}$$

Obs. *Go through the note on the 'partial fractions' given in para 10 of 'useful information' in Appendix I.*

Example 21.17. *Find the inverse transforms of*

(*i*) $\frac{s^2-3s+4}{s^3}$ (*ii*) $\frac{s+2}{s^2-4s+13}$ (*V.T.U., 2008*)

Solution. (*i*) $L^{-1}\left(\frac{s^2-3s+4}{s^3}\right)=L^{-1}\left(\frac{1}{s}\right)-3L^{-1}\left(\frac{1}{s^2}\right)+4L^{-1}\left(\frac{1}{s^3}\right)=1-3t+4\,.\,t^2/2!=1-3t+2t^2.$

(*ii*) $$L^{-1}\left(\frac{s+2}{s^2-4s+13}\right)=L^{-1}\left[\frac{s+2}{(s-2)^2+9}\right]=L^{-1}\left[\frac{s-2+4}{(s-2)^2+3^2}\right]$$

$$=L^{-1}\left[\frac{s-2}{(s-2)^2+3^2}\right]+4L^{-1}\left[\frac{1}{(s-2)^2+3^2}\right]=e^{2t}\cos 3t+\tfrac{4}{3}e^{2t}\sin 3t.$$

Example 21.18. *Find the inverse transforms of*

(*i*) $\frac{2s^2-6s+5}{s^3-6s^2+11s-6}$ (*V.T.U., 2007 ; U.P.T.U., 2004*)

(*ii*) $\frac{4s+5}{(s-1)^2(s+2)}$ (*Kurukshetra, 2005*)

Solution. (*i*) Here the denominator = $(s-1)(s-2)(s-3)$.

So let $$\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$$

Then $A = [2 \,.\, 1^2 - 6 \,.\, 1 + 5]/(1-2)(1-3) = \frac{1}{2}$

$B = [2 \,.\, 2^2 - 6 \,.\, 2 + 5]/(2-1)(2-3) = -1$

and $C = [2 \,.\, 3^2 - 6 \,.\, 3 + 5]/(3-1)(3-2) = \frac{5}{2}$.

$$\therefore \quad L^{-1}\left(\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right) = \frac{1}{2}L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left(\frac{1}{s-2}\right) + \frac{5}{2}L^{-1}\left(\frac{1}{s-3}\right)$$

$$= \tfrac{1}{2}e^{t} - e^{2t} + \tfrac{5}{2}e^{3t}.$$

(*ii*) Let $$\frac{4s+5}{(s-1)^2\,(s+2)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{4(-2)+5}{(-2-1)^2(s+2)}$$

Multiplying both sides by $(s-1)^2(s+2)$, $4s+5 = A(s-1)(s+2) + B(s+2) - \frac{1}{3}(s-1)^2$

Putting $s = 1$, $9 = 3B$, $\therefore \; B = 3$.

Equating the coefficients of s^2 from both sides,

$$0 = A - \tfrac{1}{3}, \quad \therefore \quad A = \tfrac{1}{3}.$$

$$\therefore \quad L^{-1}\left[\frac{4s+5}{(s-1)^2\,(s+2)}\right] = \frac{1}{3}L^{-1}\left(\frac{1}{s-1}\right) + 3L^{-1}\left[\frac{1}{(s-1)^2}\right] - \frac{1}{3}L^{-1}\left(\frac{1}{s+2}\right)$$

$$= \tfrac{1}{3}e^{t} + 3te^{t} - \tfrac{1}{3}e^{-2t}.$$

Example 21.19. *Find the inverse transforms of*

(*i*) $\dfrac{5s+3}{(s-1)(s^2+2s+5)}$ *(Rohtak, 2009 ; U.P.T.U., 2005)*

(*ii*) $\dfrac{s}{s^4+4a^4}$. *(Mumbai, 2008)*

Solution. (*i*) Let $$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{5(1)+3}{(s-1)(1^2+2\,.\,1+5)} + \frac{As+B}{s^2+2s+5}$$

Multiplying both sides by $(s-1)(s^2+2s+5)$,

$$5s+3 = 1\,.\,(s^2+2s+5) + (As+B)(s-1).$$

Equating the coefficients of s^2 from both sides,

$$0 = 1 + A, \quad \therefore \quad A = -1.$$

Putting $s = 0$, $3 = 5 - B$, $\therefore \; B = 2$.

$$\therefore \quad L^{-1}\left[\frac{5s+3}{(s-1)(s^2+2s+5)}\right] = L^{-1}\left(\frac{1}{s-1}\right) + L^{-1}\left(\frac{-s+2}{s^2+2s+5}\right)$$

$$= L^{-1}\left(\frac{1}{s-1}\right) + L^{-1}\left[\frac{-(s+1)+3}{(s+1)^2+4}\right] = L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left[\frac{s+1}{(s+1)^2+2^2}\right] + 3L^{-1}\left[\frac{1}{(s+1)^2+2^2}\right]$$

$$= e^{t} - e^{-t}\cos 2t + \tfrac{3}{2}\,e^{-t}\sin 2t.$$

(*ii*) Since $s^4 + 4a^4 = (s^2+2a^2)^2 - (2as)^2 = (s^2+2as+2a^2)(s^2-2as+2a^2)$

$$\therefore \quad \text{Let} \quad \frac{s}{s^4+4a^4} = \frac{As+B}{s^2+2as+2a^2} + \frac{Cs+D}{s^2-2as+2a^2}$$

Multiplying both sides by s^4+4a^4,

$$s = (As+B)(s^2-2as+2a^2) + (Cs+D)(s^2+2as+2a^2)$$

Equating coefficients of s^3, $0 = A + C$ ...(i)

Equating coefficients of s^2, $0 = -2aA + B + 2aC + D$ ...(ii)

Equating coefficients of s, $1 = 2a^2A - 2aB + 2a^2C + 2aD$ ...(iii)

Putting $s = 0$, $\quad 0 = 2a^2B + 2a^2D$ ...(iv)

From (iv), $B + D = 0$ ...(v)

$\therefore$ (ii) becomes $-A + C = 0$, and by (i), we get $A = C = 0$.

Then (iii) reduces to $D - B = 1/2a$ and by (v), $B = -1/4a$, $D = 1/4a$.

$$\therefore \quad L^{-1}\left(\frac{s}{s^4 + 4a^4}\right) = -\frac{1}{4a} L^{-1}\left(\frac{1}{s^2 + 2as + 2a^2}\right) + \frac{1}{4a} L^{-1}\left(\frac{1}{s^2 - 2as + 2a^2}\right)$$

$$= -\frac{1}{4a} L^{-1}\left[\frac{1}{(s + a)^2 + a^2}\right] + \frac{1}{4a} L^{-1}\left[\frac{1}{(s - a)^2 + a^2}\right]$$

$$= -\frac{1}{4a} \cdot \frac{1}{a} e^{-at} \sin at + \frac{1}{4a} \cdot \frac{1}{a} e^{at} \sin at = \frac{1}{2a^2} \sin at \left(\frac{e^{at} - e^{-at}}{2}\right) = \frac{1}{2a^2} \sin at \sinh at.$$

PROBLEMS 21.4

Find the inverse Laplace transforms of :

1. $\dfrac{2s - 5}{4s^2 + 25} + \dfrac{4s - 18}{9 - s^2}$.
2. $\dfrac{1}{s^2 - 5s + 6}$. (S.V.T.U., 2008)
3. $\dfrac{s}{(2s - 1)(3s - 1)}$. (V.T.U., 2010)
4. $\dfrac{3s}{s^2 + 2s - 8}$.
5. $\dfrac{3s + 2}{s^2 - s - 2}$. (V.T.U., 2010 S)
6. $\dfrac{1}{s(s^2 - 1)}$. (Nagarjuna, 2008)
7. $\dfrac{1 - 7s}{(s - 3)(s - 1)(s + 2)}$. (B.P.T.U., 2005 S)
8. $\dfrac{s^2 - 10s + 13}{(s - 7)(s^2 - 5s + 6)}$.
9. $\dfrac{2p^2 - 6p + 5}{p^3 - 6p^2 + 11p - 6}$. (U.P.T.U., 2004)
10. $\dfrac{s}{(s^2 - 1)^2}$. (Kurukshetra, 2005)
11. $\dfrac{1 + 2s}{(s + 2)^2(s - 1)^2}$.
12. $\dfrac{s}{(s - 3)(s^2 + 4)}$.
13. $\dfrac{s}{(s + 1)^2 (s^2 + 1)}$.
14. $\dfrac{s^3}{s^4 - a^4}$. (Kurukshetra, 2005)
15. $\dfrac{1}{s^3 - a^3}$.
16. $\dfrac{s^2 + 6}{(s^2 + 1)(s^2 + 4)}$.
17. $\dfrac{2s - 3}{s^2 + 4s + 13}$.
18. $\dfrac{s^2 + s}{(s^2 + 1)(s^2 + 2s + 2)}$.
19. $\dfrac{s^2 + 2s + 3}{(s^2 + 2s + 2)(s^2 + 2s + 5)}$ (Mumbai, 2008)
20. $\dfrac{s}{s^4 + s^2 + 1}$. (Raipur, 2005)
21. $\dfrac{a(s^2 - 2a^2)}{s^4 + 4a^4}$. (Mumbai, 2009)

21.13 OTHER METHODS OF FINDING INVERSE TRANSFORMS

We have seen that the most effective method of finding the inverse transforms is by means of partial fractions. However, various other methods are available which depend on the following *important inversion formulae.*

I. Shifting property for inverse Laplace transforms.

If $L^{-1}[\bar{f}(s)] = f(t)$, *then*

$$L^{-1}[\bar{f}'(s-a)] = e^{at} f(t) = e^{at} L^{-1}[\bar{f}(s)].$$

II. *If* $L^{-1}\{\bar{f}(s)\} = f(t)$ *and* $f(0) = 0$, *then*

$$L^{-1}[s\,\bar{f}(s)] = \frac{d}{dt}\{f(t)\}$$

In general, $L^{-1}\{s^n \bar{f}(s)\} = \dfrac{d^n}{dt^n}\{f(t)\}$ provided $f(0) = f'(0) = \ldots = f^{n-1}(0) = 0$.

The above formulae at once follow from the results of § 21.7 (Transforms of derivatives).

III. *If* $L^{-1}\{\bar{f}(s)\} = f(t)$, *then*

$$L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(t)\,dt$$

This result follows from § 21.8 (Transforms of integrals)

Also $$L^{-1}\left\{\frac{\bar{f}(s)}{s^2}\right\} = \int_0^t \left\{\int_0^t f(t)\,dt\right\} dt$$

$$L^{-1}\left\{\frac{\bar{f}(s)}{s^3}\right\} = \int_0^t \left\{\int_0^t \left(\int_0^t f(t)\,dt\right) dt\right\} dt \text{ and so on.}$$

IV. *If* $L^{-1}\{\bar{f}(s)\} = f(t)$, *then*

$$t\,f(t) = L^{-1}\left\{-\frac{d}{ds}[\bar{f}(s)]\right\}$$

This result follows from $L\{t\,f(t)\} = -\dfrac{d}{ds}\{\bar{f}(s)\}$ (§ 21.9)

V. The formula of § 21.10, *i.e.*,

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)\,ds$$

is useful in finding $f(t)$ when $f(s)$ is given, provided the inverse transform of $\int_s^\infty \bar{f}(s)\,ds$ can be conveniently calculated.

Example 21.20. *Find the inverse Laplace transforms of the following :*

(*i*) $\dfrac{s^2}{(s-2)^3}$ (*ii*) $\dfrac{s+3}{s^2-4s+13}$ (*iii*) $\dfrac{(s+2)^2}{(s^2+4s+8)^2}$ (*Mumbai, 2005*)

Solution. (*i*) Since $s^2 = (s-2)^2 + 4(s-2) + 4$

$$\therefore \quad \frac{s^2}{(s-2)^3} = \frac{1}{s-2} + \frac{4}{(s-2)^2} + \frac{4}{(s-2)^3}$$

$$\therefore \quad L^{-1}\left\{\frac{s^2}{(s-2)^3}\right\} = L^{-1}\left\{\frac{1}{s-2}\right\} + 4L^{-1}\left\{\frac{1}{(s-2)^2}\right\} + 4L^{-1}\left\{\frac{1}{(s-2)^3}\right\}$$

$$= e^{2t} + 4e^{2t}\,t + 2e^{2t}\,t^2.$$ [using shifting property]

(*ii*) $$\frac{s+3}{s^2-4s+13} = \frac{s-2}{(s-2)^2+3^2} + \frac{5}{(s-2)^2+3^2}$$

$$\therefore \quad L^{-1}\left\{\frac{s+3}{s^2-4s+13}\right\} = L^{-1}\left\{\frac{s-2}{(s-2)^2+3^2}\right\} + \frac{5}{3}L^{-1}\left\{\frac{3}{(s-2^2)+3^2}\right\}$$

$$= e^{2t}\cos 3t + \frac{5}{3}\, e^{2t}\sin 3t. \qquad \text{[Using shifting property]}$$

(iii) $$L^{-1}\frac{(s+2)^2}{(s^2+4s+8)^2} = L^{-1}\frac{(s+2)^2}{(s^2+4s+4+4)^2} = L^{-1}\frac{(s+2)^2}{[(s+2)^2+4]^2}$$

$$= e^{-2t}L^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\} = e^{-2t}L^{-1}\left\{\frac{s^2+4-4}{(s^2+4)^2}\right\}$$

$$= e^{-2t}L^{-1}\left\{\frac{1}{s^2+4} - \frac{4}{(s^2+4)^2}\right\} = \frac{e^{-2t}\sin 2t}{2} - 4e^{-2t}L^{-1}\left\{\frac{1}{(s^2+4)^2}\right\}$$

$$= \frac{e^{-2t}\sin 2t}{2} - 4e^{-2t}\left\{\frac{1}{4}\left(\frac{\sin 2t}{4} - \frac{t\cos 2t}{2}\right)\right\}$$

$$= e^{-2t}\left\{\frac{\sin 2t}{2} - \frac{\sin 2t}{4} + \frac{t\cos 2t}{2}\right\} = e^{-2t}\left\{\frac{\sin 2t}{4} + \frac{t\cos 2t}{2}\right\}.$$

Example 21.21. *Find the inverse transform of (i) $1/s\,(s^2+a^2)$ (ii) $1/s\,(s+a)^3$.* *(P.T.U., 2003)*

Solution. (*i*) Since $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at.$

therefore, by formula III above,

$$L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} = \int_0^t \frac{1}{a}\sin at\,dt = \frac{1}{a^2}[-\cos at]_0^t = (1-\cos at)/a^2$$

(*ii*) $$L^{-1}\left\{\frac{1}{s(s+a)^3}\right\} = L^{-1}\left\{\frac{1}{[(s+a)-a](s+a)^3}\right\} = e^{-at}L^{-1}\left\{\frac{1}{(s-a)s^3}\right\}$$

Now $$L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} \quad \therefore \quad L^{-1}\left\{\frac{1}{(s-a)s}\right\} = \int_0^t e^{at}\,dt = \frac{e^{at}}{a} - \frac{1}{a}, \text{ by III above}$$

$\therefore$ $$L^{-1}\left\{\frac{1}{(s-a)s^2}\right\} = \frac{1}{a}\int_0^t (e^{at}-1)\,dt = \frac{1}{a^2}(e^{at}-at-1)$$

$$L^{-1}\left\{\frac{1}{(s-a)s^3}\right\} = \frac{1}{a^2}\int_0^t (e^{at}-at-1)\,dt = \frac{1}{a^3}\left(e^{at} - \frac{a^2}{2}t^2 - at - 1\right)$$

Hence $$L^{-1}\left\{\frac{1}{s(s+a)^3}\right\} = e^{-at}\cdot\frac{1}{a^3}\left(e^{at} - \frac{a^2t^2}{2} - at - 1\right) = \frac{1}{a^3}\left(1 - e^{-at} - ate^{-at} - \frac{a^2}{2}t^2e^{-at}\right).$$

Example 21.22. *Find the inverse Laplace transforms of :*

(*i*) $\frac{s}{(s^2+a^2)^2}$ *(S.V.T.U., 2009)* (*ii*) $\frac{s^2}{(s^2+a^2)^2}$ *(Hazaribag, 2009)* (*iii*) $\frac{1}{(s^2+a^2)^2}$.

Solution. (*i*) If $f(t) = L^{-1}\frac{s}{(s^2+a^2)^2}$, then by formula V above,

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \frac{s}{(s^2+a^2)^2}\,ds = \frac{1}{2}\int_s^\infty \frac{2s}{(s^2+a^2)^2}\,ds = -\frac{1}{2}\left(\frac{1}{s^2+a^2}\right)_s^\infty = \frac{1}{2}\cdot\frac{1}{s^2+a^2}$$

$\therefore$ $$\frac{f(t)}{t} = \frac{1}{2}L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{\sin at}{2a}$$

Hence, $f(t) = \dfrac{1}{2a}\, t \sin at$.

Otherwise : Let $\quad f(t) = L^{-1}\left(\dfrac{1}{s^2+a^2}\right) = \dfrac{\sin at}{a}$ so that $\bar{f}(s) = \dfrac{1}{s^2+a^2}$

Then by (IV) above, $\quad t\,f(t) = L^{-1}\left\{-\dfrac{d}{ds}[\bar{f}(s)]\right\} = L^{-1}\left\{-\dfrac{d}{ds}\left(\dfrac{1}{s^2+a^2}\right)\right\}$

or
$$\frac{t \sin at}{a} = L^{-1}\left\{\frac{2s}{(s^2+a^2)^2}\right\}. \text{ Hence } L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{2a}\, t \sin at.$$

(*ii*) In (*i*), we have proved that

$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{2a}\, t \sin at = f(t), \text{ say}$$

Since $f(0) = 0$, we get from formula II above, that

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\} = L^{-1}\left\{s \cdot \frac{s}{(s^2+a^2)^2}\right\} = \frac{d}{dt}\{f(t)\}$$

$$= \frac{d}{dt}\left(\frac{1}{2a} t \sin at\right) = \frac{1}{2a}(\sin at + at \cos at)$$

(*iii*) In (*i*), we have shown that

$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{2a}(t \sin at) = f(t), \text{ say}$$

By formula III above, we have

$$L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\frac{1}{s} \cdot \frac{s}{(s^2+a^2)^2}\right\} = \int_0^t f(t)\, dt = \int_0^t \frac{t \sin at}{2a}\, dt$$

$$= \frac{1}{2a}\left\{\left| t \cdot \frac{-\cos at}{a}\right|_0^t - \int_0^t 1 \cdot \left(\frac{-\cos at}{a}\right) dt\right\}$$

$$= \frac{1}{2a}\left\{\frac{-t\cos at}{a} + \frac{\sin at}{a^2}\right\} = \frac{1}{2a^3}(\sin at - at \cos at).$$

Example 21.23. *Find the inverse Laplace transforms of*

(*i*) $\dfrac{s+2}{s^2(s+1)(s-2)}$ (*V.T.U., 2003*) (*ii*) $\dfrac{s+2}{(s^2+4s+5)^2}$. (*S.V.T.U., 2009 ; P.T.U., 2005*)

Solution. (*i*) $L^{-1}\left\{\dfrac{s+2}{(s+1)(s-2)}\right\} = \dfrac{4}{3} L^{-1}\left(\dfrac{1}{s-2}\right) - \dfrac{1}{3} L^{-1}\left(\dfrac{1}{s+1}\right) = \dfrac{4}{3}e^{2t} - \dfrac{1}{3}e^{-t}$

By III above, $L^{-1}\left\{\dfrac{s+2}{s(s+1)(s-2)}\right\} = \displaystyle\int_0^t L^{-1}\left(\dfrac{s+2}{(s+1)(s-2)}\right) dt$

$$= \int_0^t \left(\frac{4}{3}e^{2t} - \frac{1}{3}e^{-t}\right) dt = \frac{2}{3}e^{2t} + \frac{1}{3}e^{-t} - 1$$

Again by III above, $L^{-1}\dfrac{s+2}{s^2(s+1)(s-2)} = \displaystyle\int_0^t L^{-1}\left\{\dfrac{s+2}{s(s+1)(s-2)}\right\} dt$

$$= \int_0^t \left(\frac{2}{3}e^{2t} + \frac{1}{3}e^{-t} - 1\right) dt = \frac{1}{3}(e^{2t} - e^{-t} - t).$$

(ii) $$L^{-1}\left(\frac{1}{s^2+4s+5}\right)=L^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}=e^{-2t}\sin t$$

By II above, $L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+4s+5}\right)\right\}=(-1)^1 t\,.\,e^{-2t}\sin t$

i.e., $$L^{-1}\left\{\frac{-(2s+4)}{(s^2+4s+5)^2}\right\}=-t\,.\,e^{-2t}\sin t$$

or $$L^{-1}\left\{\frac{s+2}{(s^2+4s+5)^2}\right\}=\frac{1}{2}t\,.\,e^{-2t}\sin t.$$

Example 21.24. *Find the inverse Laplace transforms of the following :*

(i) $\log\frac{s+1}{s-1}$ *(S.V.T.U., 2009 ; Bhopal, 2008)* *(ii)* $\log\frac{s^2+1}{s(s+1)}$ *(S.V.T.U., 2009 ; V.T.U., 2008)*

(iii) $\cot^{-1}\left(\frac{s}{2}\right)$ *(iv)* $\tan^{-1}\left(\frac{2}{s^2}\right)$. *(V.T.U., 2011 ; Mumbai, 2005 S)*

Solution. (*i*) If $f(t)=L^{-1}\log\frac{s+1}{s-1}$, then by IV above,

$$tf(t)=L^{-1}\left\{-\frac{d}{ds}\log\left(\frac{s+1}{s-1}\right)\right\}=-L^{-1}\left\{\frac{d}{ds}\log(s+1)\right\}+L^{-1}\left\{\frac{d}{ds}\log(s-1)\right\}$$

$$=-L^{-1}\left(\frac{1}{s+1}\right)+L^{-1}\left(\frac{1}{s-1}\right)=-e^{-t}+e^{t}=2\sinh t$$

Thus $f(t)=(2\sinh t)/t$.

(*ii*) If $f(t)=L^{-1}\log\frac{s^2+1}{s(s+1)}$, then by IV above,

$$tf(t)=L^{-1}\left\{-\frac{d}{ds}\log\left(\frac{s^2+1}{s(s+1)}\right)\right\}=-L^{-1}\left\{\frac{d}{ds}\log(s^2+1)\right\}+L^{-1}\left\{\frac{d}{ds}\log s\right\}$$

$$+L^{-1}\left\{\frac{d}{ds}\log(s+1)\right\}$$

$$=-L^{-1}\left(\frac{2s}{s^2+1}\right)+L^{-1}\left(\frac{1}{s}\right)+L^{-1}\left(\frac{1}{s+1}\right)=-2\cos t+1+e^{-t}$$

Thus $f(t)=\frac{1}{t}(1+e^{-t}-2\cos t)$.

(*iii*) If $f(t)=L^{-1}\cot^{-1}\left(\frac{s}{2}\right)$, then by IV above,

$$tf(t)=L^{-1}\left\{-\frac{d}{ds}\cot^{-1}\left(\frac{s}{2}\right)\right\}=L^{-1}\left(\frac{2}{s^2+2^2}\right)=\sin 2t$$

Thus $f(t)=(\sin 2t)/t$.

(*iv*) If $f(t)=L^{-1}\left(\tan^{-1}\frac{2}{s^2}\right)$, then by IV above,

$$tf(t)=L^{-1}\left\{-\frac{d}{ds}\tan^{-1}\left(\frac{2}{s^2}\right)\right\}=L^{-1}\left\{\frac{4s}{s^4+4}\right\}$$

$$= L^{-1}\left\{\frac{4s}{(s^2+2)^2-(2s)^2}\right\} = L^{-1}\left\{\frac{4s}{(s^2+2+2s)(s^2+2-2s)}\right\}$$

$$= L^{-1}\left\{\frac{1}{s^2-2s+2}-\frac{1}{s^2+2s+2}\right\} = L^{-1}\left\{\frac{1}{(s-1)^2+1}-\frac{1}{(s+1)^2+1}\right\}$$

$$= e^t \sin t - e^{-t}\sin t = 2\sinh t \sin t.$$

21.14 CONVOLUTION THEOREM

If $\quad L^{-1}\{\bar{f}(s)\} = f(t)$, *and* $L^{-1}\{\bar{g}(s)\} = g(t)$,

then $\quad L^{-1}\{\bar{f}(s)\,\bar{g}(s)\} = \int_0^t f(u)\,g(t-u)\,du = F * G$

[*$F * G$ is called the* **convolution** *or* **falting** *of F and G.*]

Let $\quad \phi(t) = \int_0^t f(u)\,g(t-u)\,du$

$$L\{\phi(t)\} = \int_0^\infty e^{-st}\left\{\int_0^t f(u)\,g(t-u)\,du\right\}dt = \int_0^\infty \int_0^t e^{-st} f(u)\,g(t-u)\,du\,dt \qquad ...(1)$$

The domain of integration for this double integral is the entire area lying between the lines $u = 0$ and $u = t$ (Fig. 21.2).

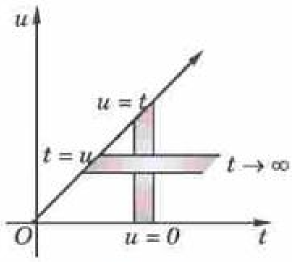

Fig. 21.2

On changing the order of integration, we get

$$L\{\phi(t)\} = \int_0^\infty \int_u^\infty e^{-st} f(u)\,g(t-u)\,dt\,du$$

$$= \int_0^\infty e^{-su} f(u)\left\{\int_0^\infty e^{-s(t-u)}\,g(t-u)\,dt\right\}du$$

$$= \int_0^\infty e^{-su} f(u)\left\{\int_0^\infty e^{-sv}\,g(v)\,dv\right\}du \text{ on putting } t-u=v$$

$$= \int_0^\infty e^{-su} f(u)\,\bar{g}(s)\,du = \int_0^\infty e^{-su} f(u)\,du\,.\,\bar{g}(s)$$

$= \bar{f}(s)\,.\,\bar{g}(s)$ whence follows the desired result.

Example 21.25. *Apply Convolution theorem to evaluate*

(*i*) $L^{-1}\dfrac{s}{(s^2+a^2)^2}$. (*V.T.U., 2010*)

(*ii*) $L^{-1}\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$. (*V.T.U., 2011 S ; Bhopal, 2008 ; Mumbai, 2007*)

Solution. (*i*) Since $f(t) = L^{-1}\left(\dfrac{s}{s^2+a^2}\right) = \cos at$ and $g(t) = L^{-1}\left(\dfrac{s}{s^2+a^2}\right) = \dfrac{1}{a}\sin at$

$\therefore$ by Convolution theorem, we get

$$L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right] = \int_0^t \cos au\,\frac{\sin a(t-u)}{a}\,du \qquad \left[\begin{array}{l}\because\ f(u) = \cos au \\ g(t-u) = \dfrac{1}{a}\sin a(t-u)\end{array}\right.$$

$$= \frac{1}{2a}\int_0^t [\sin at - \sin(2au - at)]dt = \frac{1}{2a}\left|u\sin at + \frac{1}{2a}\cos(2au-at)\right|_0^t = \frac{1}{2a}t\sin at$$

Hence $\quad L^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\} = \dfrac{1}{2a}t\sin at.$

(*ii*) Since $f(t) = L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$ and $g(t) = L^{-1}\left(\frac{s}{s^2+b^2}\right) = \cos bt$,

$\therefore$ by Convolution theorem, we get

$$L^{-1}\left(\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}\right) = \int_0^t \cos au \cos b(t-u)\,du \qquad [\because f(u) = \cos au, g(t-u) = \cos b(t-u)]$$

$$= \frac{1}{2}\int_0^t \{\cos[(a-b)u + bt] + \cos[(a+b)u - bt]\}\,du$$

$$= \frac{1}{2}\left|\frac{\sin[(a-b)u+bt]}{a-b} + \frac{\sin[(a+b)u-bt]}{a+b}\right|_0^t$$

$$= \frac{1}{2}\left\{\frac{\sin at - \sin bt}{a-b} + \frac{\sin at + \sin bt}{a+b}\right\} = \frac{a\sin at - b\sin bt}{a^2-b^2}.$$

Example 21.26. *Evaluate* (*i*) $L^{-1}\dfrac{1}{(s^2+1)(s^2+9)}$ (*Mumbai, 2005 S*)

(*ii*) $L^{-1}\dfrac{s}{(s^2+1)(s^2+4)(s^2+9)}$. (*Madras, 2006*)

Solution. (*i*) Since $L^{-1}\left(\frac{1}{s^2+1}\right) = \sin t$, $L^{-1}\left(\frac{1}{s^2+9}\right) = \frac{\sin 3t}{3}$

$\therefore$ by Convolution theorem, we get

$$L^{-1}\left(\frac{1}{s^2+1}\cdot\frac{1}{s^2+9}\right) = \int_0^t \sin u \,.\, \frac{\sin 3(t-u)}{3}\,du$$

$$= \frac{1}{6}\int_0^t [\cos(4u-3t) - \cos(3t-2u)]\,du = \frac{1}{6}\left|\frac{\sin(4u-3t)}{4} - \frac{\sin(3t-2u)}{-2}\right|_0^t$$

$$= \frac{1}{6}\left\{\frac{1}{4}(\sin t + \sin 3t) + \frac{1}{2}(\sin t - \sin 3t)\right\} = \frac{1}{8}\left(\sin t - \frac{1}{3}\sin 3t\right)$$

(*ii*) Since $L^{-1}\left(\frac{s}{s^2+4}\right) = \cos 2t$ and $L^{-1}\left\{\frac{1}{(s^2+1)(s^2+9)}\right\} = \frac{1}{8}\left[\sin t - \frac{1}{3}\sin 3t\right]$ [By (*i*)]

$\therefore$ by Convolution theorem, we get

$$L^{-1}\frac{s}{(s^2+1)(s^2+4)(s^2+9)} = L^{-1}\left\{\frac{1}{(s^2+1)(s^2+9)}\cdot\frac{s}{s^2+4}\right\}$$

$$= \int_0^t \frac{1}{8}\left(\sin u - \frac{1}{3}\sin 3u\right).\cos 2(t-u)\,du$$

$$= \frac{1}{8}\int_0^t \left[\sin u \cos 2(t-u) - \frac{1}{3}\sin 3u \cos 2(t-u)\right]du$$

$$= \frac{1}{8}\int_0^t \left[\frac{1}{2}\{\sin(2t-u) - \sin(3u-2t)\} - \frac{1}{6}\{\sin(u+2t) - \sin(5u-2t)\}\right]du$$

$$= \frac{1}{16}\left[\left|\frac{-\cos(2t-u)}{-1} + \frac{\cos(3u-2t)}{3}\right|_0^t\right] - \frac{1}{48}\left[\left|-\cos(u+2t) + \frac{\cos(5u-2t)}{5}\right|_0^t\right]$$

$$= \frac{1}{12}\cos t - \frac{1}{10}\cos 2t + \frac{1}{60}\cos 3t.$$

PROBLEMS 21.5

Find the inverse transforms of :

1. $\dfrac{1}{s^2(s+5)}$. *(Madras, 2003 S)*
2. $\dfrac{1}{s(s+2)^3}$.
3. $\dfrac{s}{a^2s^2+b^2}$. *(Madras, 2000 S)*
4. $\dfrac{1}{s^2(s^2+a^2)}$.
5. $\dfrac{1}{s^3(s^2+1)}$.
6. $\dfrac{s+2}{(s^2+4s+8)^2}$. *(Mumbai, 2006)*
7. $\dfrac{2as}{(s^2+a^2)^2}$.
8. $\dfrac{s^2}{(s+a)^3}$.
9. $\log\left(\dfrac{1+s}{s}\right)$.
10. $\log\left(\dfrac{s+a}{s+b}\right)$. *(Anna, 2003 ; U.P.T.U., 2003)*
11. $\log\left\{\dfrac{s+1}{(s+2)(s+3)}\right\}$.
12. $\dfrac{1}{2}\log\left(\dfrac{s^2+b^2}{s^2+a^2}\right)$. *(Mumbai, 2008 ; V.T.U., 2008)*
13. $\log\left(1-\dfrac{a^2}{s^2}\right)$.
14. $\log\dfrac{s^2+1}{(s-1)^2}$. *(Madras, 2000 S)*
15. $\tan^{-1}\left(\dfrac{2}{s}\right)$ *(Mumbai, 2007 ; P.T.U., 2005)*
16. $\cot^{-1}(s)$. *(V.T.U., 2005)*
17. $s\log\dfrac{s-1}{s+1}$ *(Madras, 1999)*

Using Convolution theorem, evaluate :

18. $L^{-1}\left\{\dfrac{1}{(s+a)(s+b)}\right\}$.
19. $L^{-1}\dfrac{1}{(s^2+a^2)^2}$.
20. $L^{-1}\dfrac{1}{s^2(s^2+a^2)}$.
21. $L^{-1}\left\{\dfrac{1}{s^2(s+1)^2}\right\}$.
22. $L^{-1}\left\{\dfrac{1}{(s-2)(s+2)^2}\right\}$. *(Mumbai, 2009)*
23. $L^{-1}\left\{\dfrac{s}{(s+2)(s^2+9)}\right\}$. *(V.T.U., 2008 S)*
24. $\dfrac{1}{s^3(s^2+1)}$. *(V.T.U., 2007 ; U.P.T.U., 2005)*
25. $\dfrac{1}{(s^2+4s+13)^2}$. *(Mumbai, 2008)*
26. Show that (*i*) $L^{-1}\left(\dfrac{1}{s}\sin\dfrac{1}{s}\right)=t-\dfrac{t^3}{(3!)^2}+\dfrac{t^5}{(5!)^2}-\dfrac{t^7}{(7!)^2}+\ldots$

    (*ii*) $L^{-1}\left(\dfrac{1}{s}\cos\dfrac{1}{s}\right)=1-\dfrac{t^2}{(2!)^2}+\dfrac{t^4}{(4!)^2}-\dfrac{t^6}{(6!)^2}+\ldots$

21.15 (1) APPLICATION TO DIFFERENTIAL EQUATIONS

The Laplace transform method of solving differential equations yields particular solutions without the necessity of first finding the general solution and then evaluating the arbitrary constants. This method is, in general, shorter than our earlier methods and is specially useful for solving linear differential equations with constant coefficients.

(2) Working procedure *to solve a linear differential equation with constant coefficients by transform method :*

1. Take the Laplace transform of both sides of the differential equation using the formula of § 21.7, and the given initial conditions.

2. Transpose the terms with minus signs to the right.

3. Divide by the coefficient of $\bar{y}$, getting $\bar{y}$ as a known function of s.

4. Resolve this function of s into partial fractions and take the inverse transform of both sides. This gives y as a function of t which is the desired solution satisfying the given conditions.

Example 21.27. *Solve by the method of transforms, the equation*

$$y''' + 2y'' - y' - 2y = 0 \text{ given } y(0) = y'(0) = 0 \text{ and } y''(0) = 6.$$ (V.T.U., 2011 S)

Solution. Taking the Laplace transform of both sides, we get

$$[s^3 \bar{y} - s^2 y(0) - sy'(0) - y''(0)] + 2[s^2 \bar{y} - sy(0) - y'(0)] - [s\bar{y} - y(0)] - 2\bar{y} = 0$$

Using the given conditions, it reduces to

$$(s^3 + 2s^2 - s - 2)\bar{y} = 6$$

$$\therefore \qquad \bar{y} = \frac{6}{(s-1)(s+1)(s+2)} = \frac{6}{(s-1)(6)} + \frac{6}{(-2)(s+1)} + \frac{6}{3(s+2)}$$

On inversion, we get $\quad y = L^{-1}\dfrac{1}{(s-1)} - 3L^{-1}\dfrac{1}{(s+2)} + 2L^{-1}\left(\dfrac{1}{s+2}\right)$

or $\quad y = e^t - 3e^{-t} + 2e^{-2t}$ which is the desired result.

Example 21.28. *Use transform method to solve*

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t \text{ with } x = 2, \frac{dx}{dt} = -1 \text{ at } t = 0.$$ (Anna, 2005 S)

Solution. Taking the Laplace transforms of both sides, we get

$$[s^2\bar{x} - sx(0) - x'(0)] - 2\{s\bar{x} - x(0)\} + \bar{x} = \frac{1}{s-1}$$

Using the given conditions, it reduces to

$$(s^2 - 2s + 1)\bar{x} = \frac{1}{s-1} + 2s - 5 = \frac{2s^2 - 7s + 6}{s-1}$$

$$\therefore \qquad \bar{x} = \frac{2s^2 - 7s + 6}{(s-1)^3} = \frac{2}{s-1} - \frac{3}{(s-1)^2} + \frac{1}{(s-1)^3}$$ on breaking into partial fractions.

On inversion, we obtain $x = 2L^{-1}\left(\dfrac{1}{s-1}\right) - 3L^{-1}\dfrac{1}{(s-1)^2} + L^{-1}\dfrac{1}{(s-1)^3}$

$$= 2e^t - \frac{3e^t \cdot t}{1!} + \frac{e^t \cdot t^2}{2!} = 2e^t - 3te^t + \tfrac{1}{2}t^2e^t.$$

Example 21.29. *Solve* $(D^2 + n^2)x = a\sin(nt + \alpha)$, $x = Dx = 0$ *at* $t = 0$.

Solution. Taking the Laplace transforms of both sides, we get

$$[s^2\bar{x} - sx(0) - x'(0)] + n^2\bar{x} = aL\{\sin nt \cdot \cos\alpha + \cos nt \cdot \sin\alpha\}$$

On using the given conditions,

$$(s^2 + n^2)\bar{x} = a\cos\alpha \cdot \frac{n}{s^2+n^2} + a\sin\alpha \cdot \frac{s}{s^2+n^2}$$

$$\therefore \qquad \bar{x} = an\cos\alpha \cdot \frac{1}{(s^2+n^2)^2} + a\sin\alpha \cdot \frac{s}{(s^2+n^2)^2}$$

On inversion, we obtain

$$x = an\cos\alpha \cdot \frac{1}{2n^3}(\sin nt - nt\cos nt) + a\sin\alpha \cdot \frac{t}{2n}\sin nt$$ [By (11) and (12) p. 741]

$$= a\{\sin nt\cos\alpha - nt\cos(nt + \alpha)\}/2n^2.$$

Example 21.30. *Solve* $(D^3 - 3D^2 + 3D - 1)y = t^2e^t$ *given that* $y(0) = 1$, $y'(0) = 0$, $y''(0) = -2$.

(S.V.T.U., 2009)

Solution. Taking the Laplace transforms of both sides, we get

$$[s^3\bar{y} - s^2y(0) - sy'(0) - y''(0)] - 3\,[s^2\bar{y} - sy(0) - y'(0)] + 3\,[s\,\bar{y} - y(0)] - \bar{y} = \frac{2}{(s-1)^3}$$

Using the given conditions, it reduces to

$$\bar{y} = \frac{s^2 - 3s + 1}{(s-1)^3} + \frac{2}{(s-1)^6} = \frac{(s-1)^2 - (s-1) - 1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$= \frac{1}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

On inversion, we obtain $y = L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\frac{1}{(s-1)^2} - L^{-1}\frac{1}{(s-1)^3} + 2L^{-1}\frac{1}{(s-1)^6}$

$$= e^t\left(1 - t - \frac{1}{2}t^2 + \frac{1}{60}t^5\right).$$

Example 21.31. *Solve* $\frac{d^2x}{dt^2} + 9x = \cos 2t$, *if* $x(0) = 1$, $x(\pi/2) = -1$. *(Bhopal, 2008 ; U.P.T.U., 2006)*

Solution. Since $x'(0)$ is not given, we assume $x'(0) = a$.

Taking the Laplace transforms of both sides of the equation, we have

$$L(x'') + 9L(x) = L(\cos 2t) \text{ i.e., } [s^2\bar{x} - s\,x(0) - x'(0)] + 9\,\bar{x} = \frac{s}{s^2+4}$$

$$(s^2 + 9)\,\bar{x} = s + a + \frac{s}{s^2+4} \quad \text{or} \quad \bar{x} = \frac{s+a}{s^2+9} + \frac{s}{(s^2+4)(s^2+9)}$$

or
$$\bar{x} = \frac{a}{s^2+9} + \frac{1}{5}\cdot\frac{s}{s^2+4} + \frac{4}{5}\cdot\frac{s}{s^2+9}.$$

On inversion, we get $\quad x = \frac{a}{3}\sin 3t + \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t$

When $t = \pi/2$, $-1 = -\frac{a}{3} - \frac{1}{5}$ or $\frac{a}{3} = \frac{4}{5}$ $\quad \left[\because\ x\left(\frac{\pi}{2}\right) = -1\right]$

Hence the solution is $x = \frac{1}{5}(\cos 2t + 4\sin 3t + 4\cos 3t)$.

Obs. *Laplace transform method can also be used for solving ordinary differential equations with variable coefficients of the form* $t^m y^{(n)}(t)$ *because* $L[t^m y^{(n)}(t)] = (-1)^m \frac{d^m}{ds^m}[L\,y^{(n)}(t)]$.

Example 21.32. *Solve* $ty'' + 2y' + ty = \cos t$ *given that* $y(0) = 1$. *(S.V.T.U., 2009)*

Solution. Taking Laplace transform of both sides of the equation and noting that

$$L[t\,\{f(t)\}] = -\frac{d}{ds}[L\,\{f(t)\}], \text{ we get}$$

$$-\frac{d}{ds}[s^2\bar{y} - sy(0) - y'(0)] + 2[s\,\bar{y} - y(0)] - \frac{d}{ds}(\bar{y}) = \frac{s}{s^2+1}$$

or
$$-\left(s^2\frac{d\bar{y}}{ds} + 2s\bar{y}\right) + y(0) + 0 + 2s\,\bar{y} - 2y(0) - \frac{d}{ds}(\bar{y}) = \frac{s}{s^2+1}$$

or
$$(s^2+1)\frac{d\bar{y}}{ds} + 1 = -\frac{s}{s^2+1} \quad \text{or} \quad \frac{d\bar{y}}{ds} = -\frac{1}{s^2+1} - \frac{s}{(s^2+1)^2}$$

On inversion and noting that $L^{-1}\{\bar{f}'(s)\} = -t f(t)$, we get

$$-ty = -\sin t - \tfrac{1}{2}t\sin t \qquad \text{[See § 21.12 (11)]}$$

or $$y = \frac{1}{2}\left(1 + \frac{2}{t}\right)\sin t$$ which is the desired solution.

Example 21.33. *Solve* $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + xy = 0,\ y(0) = 2,\ y'(0) = 0.$

Solution. Taking Laplace transform of both sides of the equation, we get

$$L(xy'') + L(y') + L(xy) = 0$$

or $$-\frac{d}{ds}[s^2\bar{y} - sy(0) - y'(0)] + [s\bar{y} - y(0)] - \frac{d\bar{y}}{ds} = 0 \quad \text{or} \quad (s^2+1)\frac{d\bar{y}}{ds} + s\bar{y} = 0$$

Separating the variables, $\displaystyle\int\frac{d\bar{y}}{\bar{y}} + \int\frac{s\,ds}{s^2+1} = c$

or $$\log\bar{y} + \tfrac{1}{2}\log(s^2+1) = \log c' \quad \text{or} \quad \bar{y} = \frac{c'}{\sqrt{(s^2+1)}}$$

Inversion gives $y = c' J_0(x)$

To find c', we have $y(0) = c' J_0(0)$, *i.e.*, $c' = 2$

Hence $y = 2J_0(x)$.

Example 21.34. *An alternating e.m.f. E sin ωt is applied to an inductance L and a capacitance C in series. Show by transform method, that the current in the circuit is* $\dfrac{E\omega}{(p^2-\omega^2)L}(\cos\omega t - \cos pt)$, *where* $p^2 = 1/LC$.

Solution. If i be a current and q the charge at time t in the circuit, then its differential equation is

$$L\frac{di}{dt} + \frac{q}{c} = E\sin\omega t \qquad [\because \ R = 0]$$

Taking Laplace transform of both sides, we get

$$L[s\bar{i}(s) - i(0)] + \frac{1}{C}L(q) = E\cdot\frac{\omega}{s^2+\omega^2}$$

Since $i = 0$ and $q = 0$ at $t = 0$

$$\therefore \quad Ls\bar{i}(s) + \frac{1}{C}L(q) = \frac{E\omega}{s^2+\omega^2} \qquad ...(i)$$

Also taking Laplace transform of $i = dq/dt$, we get

$$\bar{i}(s) = L(dq/dt) = sL(q) - q(0)$$

i.e. $$L(q) = \bar{i}(s)/s \qquad [\because \ q(0) = 0]$$

$\therefore$ (*i*) becomes $Ls\bar{i}(s) + \dfrac{1}{C}[\bar{i}(s)/s] = \dfrac{E\omega}{s^2+\omega^2}$

or $$\left(Ls + \frac{1}{Cs}\right)\bar{i}(s) = \frac{E\omega}{s+\omega^2} \quad \text{or} \quad \bar{i}(s) = \frac{E\omega s}{L(s^2+1/LC)(s^2+\omega^2)}$$

or $$\bar{i}(s) = \frac{E\omega}{L}\cdot\frac{s}{(s^2+p^2)(s^2+\omega^2)} \quad \text{where } p^2 = 1/LC$$

$$\bar{i}(s) = \frac{E\omega}{L(p^2-\omega^2)}\left\{\frac{s}{s^2+\omega^2} - \frac{s}{s^2+p^2}\right\}$$

Now taking inverse Laplace transform of both sides, we get

$$i(t) = \frac{E\omega}{L(p^2 - \omega^2)} L^{-1}\left\{\frac{s}{s^2 + \omega^2} - \frac{s}{(s^2 + p^2)}\right\}$$

or
$$i(t) = \frac{E\omega}{L(p^2 - \omega^2)} (\cos \omega t - \cos pt).$$

PROBLEMS 21.6

Solve the following equations by the transform method :

1. $y'' + 4y' + 3y = e^{-t}$, $y(0) = y'(0) = 1$. (*V.T.U., 2008 S ; Kurukshetra, 2005*)
2. $(D^2 - 1)x = a \cosh t$, $x(0) = x'(0) = 0$.
3. $y'' + y = t$, $y(0) = 1$, $y'(0) = 0$. (*Mumbai, 2009*)
4. $y'' - 3y' + 2y = e^{3t}$, when $y(0) = 1$ and $y'(0) = 0$. (*V.T.U., 2010*)
5. $(D^2 - 3D + 2)y = 4e^{2t}$ with $y(0) = -3$, $y(0) = 5$. (*Mumbai, 2008*)
6. $y'' + 25y = 10 \cos 5t$ given that $y(0) = 2$, $y'(0) = 0$. (*S.V.T.U., 2008*)
7. $(D^2 + \omega^2)y = \cos \omega t$, $t > 0$, given that $y = 0$ and $Dy = 0$ at $t = 0$.
8. $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} - 3y = \sin t$, $y = \dfrac{dy}{dt} = 0$ when $t = 0$. (*Kurukshetra, 2005 ; Madras, 2003*)
9. $\dfrac{d^4y}{dt^4} - k^4y = 0$, where $y(0) = 1$, $y'(0) = y''(0) = y'''(0) = 0$.
10. $y''''(t) + 2y''(t) + y(t) = \sin t$, when $y(0) = y'(0) = y''(0) = y'''(0) = 0$.
11. $\dfrac{d^2y}{dt^2} + 5\dfrac{dy}{dt} + 5y = e^{-t} \sin t$, where $y(0) = 0$ and $y'(0) = 1$. (*P.T.U., 2010*)
12. $y'' + 2y' + 5y = 5y = 5(t - 2)$, $y(0) = 0$, $y'(0) = 0$. (*P.T.U., 2005 S*)
13. $\dfrac{d^3y}{dt^3} - 3\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} - y = t^2 e^{2t}$, where $y = 1$, $\dfrac{dy}{dt} = 0$, $\dfrac{d^2y}{dt^2} = -2$ at $t = 0$.
14. $(D^2 + 1)x = t \cos 2t$, $x = Dx = 0$ at $t = 0$. (*Raipur, 2005 ; U.P.T.U., 2005*)
15. $ty'' + 2y' + ty = \sin t$, when $y(0) = 1$.
16. $ty'' + (1 - 2t)y' - 2y = 0$, when $y(0) = 1$, $y'(0) = 2$. (*P.T.U., 2002*)
17. $y'' + 2ty' - y = t$, when $y(0) = 0$, $y'(0) = 1$. (*U.P.T.U., 2003*)
18. $ty'' + y' + 4ty = 0$ when $y(0) = 3$, $y'(0) = 0$.
19. A voltage Ee^{-at} is applied at $t = 0$ to a circuit of inductance L and resistance R. Show (by the transform method) that the current at time t is $\dfrac{E}{R - aL}(e^{-at} - e^{-Rt/L})$. (*V.T.U., 2000*)
20. Workout example 12.17, p. 465 by the transform method.
21. Obtain the equation for the forced oscillation of a mass m attached to the lower end of an elastic spring whose upper end is fixed and whose stiffness is k, when the driving force is $F_0 \sin at$. Solve this equation (using the Laplace transforms) when $a^2 \neq k/m$, given that initial velocity and displacement (from equilibrium position) are zero.

$\left[\textbf{Hint :} \text{ The equation of motion is } \dfrac{d^2x}{dt^2} + \dfrac{k}{m}x = \dfrac{F_0}{m} \sin at \text{ and } x = \dfrac{dx}{dt} = 0 \text{ when } t = 0.\right]$

21.16 SIMULTANEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

The Laplace transform method can also be applied with advantage to the solution of simultaneous linear differential equations.

Example 21.35. *Solve the simultaneous equations* $\dfrac{dx}{dt} + 5x - 2y = t$, $\dfrac{dy}{dt} + 2x + y = 0$ *being given* $x = y = 0$ *when* $t = 0$. [Ex. 13.38]

Solution. Taking the Laplace transforms of the given equations, we get

$$[s\bar{x} - x(0)] + 5\bar{x} - 2\bar{y} = 1/s^2 \quad i.e., \quad (s+5)\,\bar{x} - 2\bar{y} = 1/s^2 \qquad ...(i)\ [\because\ x(0)=0]$$

and $$s\bar{y} - y(0) + 2\bar{x} + \bar{y} = 0 \quad i.e., \quad 2\bar{x} + (s+1)\,\bar{y} = 0 \qquad ...(ii)\ [\because\ y(0)=0]$$

Solving (*i*) and (*ii*) for $\bar{x}$, we get

$$\bar{x} = \begin{vmatrix} 1/s^2 & -2 \\ 0 & s+1 \end{vmatrix} \div \begin{vmatrix} s+5 & -2 \\ 2 & s+1 \end{vmatrix} = \frac{s+1}{s^2(s+3)^2} = \frac{1}{27s} + \frac{1}{9s^2} - \frac{1}{27(s+3)} - \frac{2}{9(s+3)^2}$$

Substituting the value of $\bar{x}$ in (*ii*), we get

$$\bar{y} = -\frac{2}{s^2(s+3)^2} = \frac{4}{27s} - \frac{2}{9s^2} - \frac{4}{27(s+3)} - \frac{2}{9(s+3)^2}$$

On inversion, we get

$$x = \frac{1}{27} + \frac{t}{9} - \frac{1}{27}e^{-3t} - \frac{2}{9}te^{-3t},\ y = \frac{4}{27} - \frac{2t}{9} - \frac{4}{27}e^{-3t} - \frac{2}{9}te^{-3t}.$$

Example 21.36. *The coordinates (x, y) of a particle moving along a plane curve at any time t, are given by $dy/dt + 2x = \sin 2t$, $dx/dt - 2y = \cos 2t$, $(t > 0)$. If at $t = 0$, $x = 1$ and $y = 0$, show by transforms, that the particle moves along the curve $4x^2 + 4xy + 5y^2 = 4$.* (U.P.T.U., 2003)

Solution. Taking the Laplace transforms of the given equations and noting that $y(0) = 0$, $x(0) = 1$,

we get $$[s\bar{y} - y(0)] + 2\bar{x} = \frac{2}{s^2+2^2} \quad \text{or} \quad 2\bar{x} + s\bar{y} = \frac{2}{s^2+4} \qquad ...(i)$$

and $$[s\bar{x} - x(0)] - 2\bar{y} = \frac{s}{s^2+2^2} \quad \text{or} \quad s\bar{x} - 2\bar{y} = \frac{s}{s^2+4} + 1 \qquad ...(ii)$$

Multiplying (*i*) by s and (*ii*) by 2 and subtracting, we get

$$(s^2+4)\,\bar{y} = -2 \quad \text{or} \quad \bar{y} = -2/(s^2+4)$$

On inversion, $$y = -2L^{-1}\left[\frac{1}{s^2+4}\right] = -\sin 2t$$

From the given first equation,

$$2x = \sin 2t - dy/dt = \sin 2t - \frac{d}{dt}(-\sin 2t)$$

or $$2x = \sin 2t + 2\cos 2t \quad \text{or} \quad 4x^2 = (\sin 2t + 2\cos 2t)^2 \qquad ...(iii)$$

Also $$4xy = (\sin 2t + 2\cos 2t)(-2\sin 2t) = -2(\sin^2 2t + 2\sin 2t\cos 2t) \qquad ...(iv)$$

and $$5y^2 = 5\sin^2 2t. \qquad ...(v)$$

Adding (*iii*), (*iv*), and (*v*), we obtain

$$4x^2 + 4xy + 5y^2 = \sin^2 2t + 4\sin 2t\cos 2t + 4\cos^2 2t - 2\sin^2 2t$$
$$- 4\sin 2t\cos 2t + 5\sin^2 2t = 4\sin^2 2t + 4\cos^2 2t = 4.$$

Example 21.37. *The small oscillations of a certain system with two degrees of freedom are given by the equations : $D^2x + 3x - 2y = 0$, $D^2x + D^2y - 3x + 5y = 0$ where $D = d/dt$. If $x = 0$, $y = 0$, $x = 3$, $y = 2$ when $t = 0$, find x and y when $t = 1/2$.* [Example 13.41]

Solution. Taking the Laplace transform of both the equations, we get

$$[s^2\bar{x} - sx(0) - x'(0)] + 3\bar{x} - 2\bar{y} = 0 \quad i.e., \quad (s^2+3)\bar{x} - 2\bar{y} = 3 \qquad ...(i)$$

and $$[s^2\bar{x} - sx(0) - x'(0)] + [s^2\bar{y} - sy(0) - y'(0)] - 3\bar{x} + 5\bar{y} = 0\ i.e., (s^2-3)\bar{x} + (s^2+5)\bar{y} = 5 \qquad ...(ii)$$

Solving (*i*) and (*ii*) for $\bar{x}$ and $\bar{y}$, we get

$$\bar{x} = \begin{vmatrix} 3 & -2 \\ 5 & s^2+5 \end{vmatrix} \div \begin{vmatrix} s^2+3 & -2 \\ s^2-3 & s^2+5 \end{vmatrix} = \frac{3s^2+25}{(s^2+1)(s^2+9)}$$
$$= \frac{11}{4}\cdot\frac{1}{s^2+1} + \frac{1}{4}\cdot\frac{1}{s^2+9}$$

and $$\bar{y} = \begin{vmatrix} s^2+3 & 3 \\ s^2-3 & 5 \end{vmatrix} \div \begin{vmatrix} s^2+3 & -2 \\ s^2-3 & s^2+5 \end{vmatrix} = \frac{2s^2+24}{(s^2+1)(s^2+9)} = \frac{11}{4}\cdot\frac{1}{s^2+1} + \frac{3}{4}\cdot\frac{1}{s^2+9}.$$

On inversion, we get $x = \frac{11}{4}\sin t + \frac{1}{12}\sin 3t$; $y = \frac{11}{4}\sin t - \frac{1}{4}\sin 3t$

which are the same as the solution in (*vii*) on p. 499.

Obs. The student should compare the earlier solutions of the above examples with those given now and appreciate the superiority of the transform method over others.

PROBLEMS 21.7

Solve the following simultaneous equations (by using Laplace transforms) :

1. $\frac{dx}{dt} - y = e^t$, $\frac{dy}{dt} + x = \sin t$, given $x(0) = 1$, $y(0) = 0$. (*U.P.T.U., 2006 ; Delhi, 2002*)
2. $\frac{dx}{dt} + y = \sin t$, $\frac{dy}{dt} + x = \cos t$, given that $x = 2$ and $y = 0$ when $t = 0$. (*Kerala, 2005 ; U.P.T.U., 2004*)
3. $\frac{d^2x}{dt^2} - x = y$, $\frac{d^2y}{dt^2} + y = -x$, given that at $t = 0$; $x = 2$, $y = -1$, $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$. (*P.T.U., 2009 S*)
4. $3\frac{dx}{dt} + \frac{dy}{dt} + 2x = 1$, $\frac{dx}{dt} + 4\frac{dy}{dt} + 3y = 0$; given $x = 0$, $y = 0$ when $t = 0$. (*Madras, 2003 S*)
5. $(D-2)x - (D+1)y = 6e^{3t}$; $(2D-3)x + (D-3)y = 6e^{3t}$ given $x = 3$, $y = 0$ when $t = 0$.
6. The currents i_1 and i_2 in mesh are given by the differential equations ; $di_1/dt - \omega i_2 = a\cos pt$, $di_2/dt + \omega i_1 = a\sin pt$. Find the currents i_1 and i_2 by Laplace transform, if $i_1 = i_2 = 0$ at $t = 0$.

21.17 (1) UNIT STEP FUNCTION

At times, we come across such fractions of which the inverse transform cannot be determined from the formulae so far derived. In order to cover such cases, we introduce the *unit step function* (or *Heaviside's unit function**).

Def. *The unit step function $u(t-a)$ is defined as follows :*

$$u(t-a) = \begin{cases} 0 \text{ for } t < a \\ 1 \text{ for } t \geq a \end{cases}$$

where, a is always positive (Fig. 21.3). It is also denoted as $H(t-a)$.

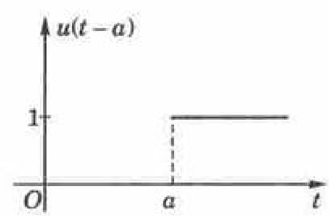

Fig. 21.3

(2) Transform of unit function.

$$L\{u(t-a)\} = \int_0^\infty e^{-st}\, u(t-a)\, dt = \int_0^a e^{-st} . 0\, dt + \int_a^\infty e^{-st} . 1\, dt = 0 + \left| \frac{e^{-st}}{-s} \right|_a^\infty$$

Thus $\quad L\{u(t-a)\} = e^{-as}/s.$

The product $f(t)\, u(t-a) = \begin{cases} 0 & \text{for } t < a \\ f(t) & \text{for } t \geq a. \end{cases}$

The function $f(t-a) . u(t-a)$ represents the graph of $f(t)$ shifted through a distance a to the right and is of special importance.

Second shifting property. If $L\{f(t)\} = \bar{f}(s)$, *then*

$$\mathbf{L\{f(t-a).\, u(t-a)\} = e^{-as}\, \bar{f}(s)}$$

$$L\{f(t-a) . u(t-a)\} = \int_0^\infty e^{-st} f(t-a)\, u(t-a)\, dt$$

*Named after the British Electrical Engineer Oliver Heaviside (1850–1925).

$$= \int_0^a e^{-st} f(t-a)(0)\,dt + \int_a^\infty e^{-st} f(t-a)\,dt \qquad [\text{Put } t-a=u]$$

$$= \int_0^\infty e^{-s(u+a)} f(u)\,du = e^{-sa}\int_0^\infty e^{-su} f(u)\,du = e^{-as}\,\bar{f}(s).$$

Example 21.38. *Express the following function (Fig. 21.4) in terms of unit step function and find its Laplace transform.* *(U.P.T.U., 2002)*

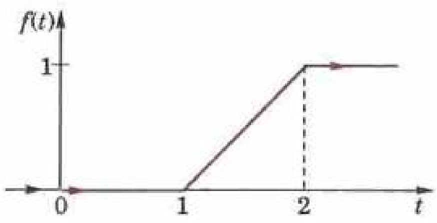

Fig. 21.4

Solution. We have $f(t) = \begin{cases} 0, & 0 < t < 1 \\ t-1, & 1 < t < 2 \\ 1, & t > 2 \end{cases}$

or $$f(t) = (t-1)[u(t-1) - u(t-2)] + u(t-2)$$
$$= (t-1)\,u(t-1) - (t-2)\,u(t-2)$$

By second shifting property,
$$L[f(t-a)\,u(t-a)] = e^{-as} L[f(t)].$$

Also $$L[f(t)] = L(t) = 1/s^2.$$

$\therefore$ $$L[(t-1)\,u(t-1)] = e^{-s}\cdot\frac{1}{s^2} \text{ and } L[(t-2)\,u(t-2)] = e^{-2s}\cdot\frac{1}{s^2}$$

Hence $$L[f(t)] = L[(t-1)\,u(t-1) - (t-2)\,u(t-2)] = \frac{e^{-s} - e^{-2s}}{s^2}.$$

Example 21.39. *Using unit step function, find the Laplace transform of*

$$f(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ \sin 2t, & \pi \le t < 2\pi. \\ \sin 3t, & t \ge 2\pi \end{cases}$$ *(V.T.U., 2004)*

Solution. $$f(t) = \sin t\,[u(t-0) - u(t-\pi)] + \sin 2t\,[u(t-\pi) - u(t-2\pi)] + \sin 3t\,.\,u(t-2\pi)$$
$$= \sin t + (\sin 2t - \sin t)\,u(t-\pi) + (\sin 3t - \sin 2t)\,u(t-2\pi)$$

Since $L[f(t-a)\,u(t-a)] = e^{-as}\,\bar{f}(s)$ and $L(\sin at) = \dfrac{a}{s^2+a^2}$,

$$L[f(t)] = L(\sin t) + L[(\sin 2t - \sin t)\,.\,u(t-\pi)] + L[(\sin 3t - \sin 2t)\,.\,u(t-2\pi)]$$

$$= \frac{1}{s^2+1} + e^{-\pi s}\left(\frac{2}{s^2+4} - \frac{1}{s^2+1}\right) + e^{-2\pi s}\left(\frac{3}{s^2+9} - \frac{2}{s^2+4}\right).$$

Example 21.40. (*i*) *Express the function (Fig. 21.5) in terms of unit step function and find its Laplace transform.* *(P.T.U., 2005 S)*

(*ii*) *Obtain the Laplace transform of* $e^{-t}[1 - u(t-2)]$.

Solution. (*i*) We have $f(t) = \begin{cases} t-1, & 1 < t < 2 \\ 3-t, & 2 < t < 3. \end{cases}$

or $$f(t) = (t-1)\{u(t-1) - u(t-2)\} + (3-t)\{u(t-2) - u(t-3)\}$$
$$= (t-1)\,u(t-1) - 2(t-2)\,u(t-2) + (t-3)\,u(t-3)$$

Since $L\{f(t-a)\,u(t-a)\} = e^{-as}\,\bar{f}(s)$ ...(λ)

$\therefore$ $$L[f(t)] = e^{-s}\cdot\frac{1}{s^2} - 2e^{-2s}\cdot\frac{1}{s^2} + e^{-3s}\cdot\frac{1}{s^2} = \frac{e^{-s}(1-e^{-s})^2}{s^2} \qquad [\because\ f(t) = t]$$

(*ii*) $$L\{e^{-t}[1 - u(t-2)]\} = L(e^{-t}) - L\{e^{-t}\,u(t-2)\} = \frac{1}{s+1} - e^{-2}\,L\{e^{-(t-2)}\,u(t-2)\}$$

Taking $f(t) = e^{-t}$, $\bar{f}(s) = \dfrac{1}{s+1}$ and using (λ) above,

$$L\{e^{-(t-2)}u(t-2)\} = e^{-2s} \cdot \frac{1}{s+1}$$

Hence $Le^{-t}\{1 - u(t-2)\} = \{1 - e^{-2(s+1)}\}/(s+1)$.

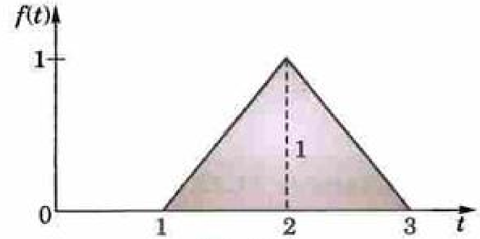

Fig. 21.4

Example 21.41. *Using Laplace transform, evaluate* $\int_0^\infty e^{-t}(1 + 2t - t^2 + t^3)H(t-1)\,dt$.

(Mumbai, 2007)

Solution. We have $L\{(1 + 2t - t^2 + t^3)H(t-1)\}$

$$= e^{-s}L[1 + 2(t+1) - (t+1)^2 + (t+1)^3] = e^{-s}L(3 + 3t + 2t^2 + t^3)$$

$$= e^{-s}\left(3 \cdot \frac{1}{s} + 3 \cdot \frac{1}{s^2} + 2 \cdot \frac{2!}{s^3} + \frac{3!}{s^4}\right) = e^{-s}\left(\frac{3}{s} + \frac{3}{s^2} + \frac{4}{s^3} + \frac{6}{s^4}\right)$$

By definition, this implies that

$$\int_0^\infty e^{-st}(1 + 2t - t^2 + t^3)H(t-1)\,dt = e^{-s}\left(\frac{3}{s} + \frac{3}{s^2} + \frac{4}{s^3} + \frac{6}{s^4}\right)$$

Taking $s = 1$, we obtain

$$\int_0^\infty e^{-t}(1 + 2t - t^2 + t^3)H(t-1)\,dt = e^{-1}(3 + 3 + 4 + 6) = 16/e.$$

Example 21.42. *Evaluate (i)* $L^{-1}\left\{\dfrac{e^{-s} - 3e^{-3s}}{s^2}\right\}$ *(U.P.T.U., 2002)*

(ii) $L^{-1}\left\{\dfrac{se^{-as}}{s^2 - w^2}\right\}, a > 0.$

Solution.

$$L^{-1}\left\{e^{-s} \cdot \frac{1}{s^2}\right\} = \begin{Bmatrix} t-1, & t > 1 \\ 0, & t < 1 \end{Bmatrix} = (t-1)\,u(t-1)$$

$$L^{-1}\left\{e^{-3s} \cdot \frac{1}{s^2}\right\} = \begin{Bmatrix} t-3, & t > 3 \\ 0, & t < 3 \end{Bmatrix} = (t-3)\,u(t-3)$$

$$\therefore \quad L^{-1}\left\{\frac{e^{-s} - 3e^{-3s}}{s^2}\right\} = L^{-1}\left(\frac{e^{-s}}{s^2}\right) - 3L^{-1}\left(\frac{e^{-3s}}{s^2}\right) = (t-1)\,u(t-1) - 3(t-3)\,u(t-3)$$

(*ii*) We know that $L^{-1}\left(\dfrac{s}{s^2 - w^2}\right) = \cosh wt$

$$\therefore \quad L^{-1}\left(\frac{se^{-as}}{s^2 - w^2}\right) = \begin{cases} \cosh w(t-a), & t > a \\ 0, & t < a \end{cases}$$

$= \cosh w(t-a)\,u(t-a)$, by second shifting property.

Example 21.43. *Find the inverse Laplace transform of :*

(*i*) $\dfrac{se^{-s/2} + \pi e^{-s}}{s^2 + \pi^2}$ *(V.T.U., 2000)* (*ii*) $\dfrac{e^{-cs}}{s^2(s+a)}$ $(c > 0)$. *(Kurukshetra, 2005)*

Solution. (*i*) Since $L^{-1}\dfrac{s}{s^2 + \pi^2} = \cos \pi t$, $L^{-1}\left(\dfrac{\pi}{s^2 + \pi^2}\right) = \sin \pi t$

and $$L^{-1}[e^{-as}\bar{f}(s)] = f(t-a)\,.\,u(t-a) \quad ...(\lambda)$$

$$\therefore \quad L^{-1}\left\{\frac{se^{-s/2}+\pi e^{-s}}{s^2+\pi^2}\right\} = L^{-1}\left\{e^{-s/2}\,.\,\frac{s}{s^2+\pi^2}\right\} + L^{-1}\left\{e^{-s}\,.\,\frac{\pi}{s^2+\pi^2}\right\}$$

$$= \cos\pi(t-1/2)\,.\,u(t-1/2) + \sin\pi(t-1)\,.\,u(t-1)$$

$$= \sin\pi t\,.\,u(t-1/2) - \sin\pi t\,.\,u(t-1) = \{u(t-1/2)-u(t-1)\}\sin\pi t$$

(ii) $$L^{-1}\left\{\frac{e^{-cs}}{s^2(s+a)}\right\} = L^{-1}\left\{e^{-cs}\left(-\frac{1}{a^2}\,.\,\frac{1}{s}+\frac{1}{a}\,.\,\frac{1}{s^2}+\frac{1}{a^2}\,.\,\frac{1}{s+a}\right)\right\}$$

Using (λ) above, we have

$$L^{-1}\left\{\frac{e^{-cs}}{s^2(s+a)}\right\} = -\frac{1}{a^2}\{1\,.\,u(t-c)\} + \frac{1}{a}\{(t-c)\,.\,u(t-c)\} + \frac{1}{a^2}\{e^{-a(t-c)}\,.\,u(t-c)\}$$

$$= \frac{1}{a^2}\{a(t-c)-1+e^{-a(t-c)}\}\,u(t-c).$$

Example 21.44. *A particle of mass m can oscillate about the position of equilibrium under the effect of a restoring force mk^2 times the displacement. It started from rest by a constant force F which acts for time T and then ceases. Find the amplitude of the subsequent oscillation.*

Solution. The constant force F acting from $t = 0$ to $t = T$ can be expressed as

$$F[1-u(t-T)],\ 0 < t < T$$

$\therefore$ equation of motion of the particle is

$$m\frac{d^2x}{dt^2} = F[1-u(t-T)] - mk^2x \quad \text{or} \quad \frac{d^2x}{dt^2} + k^2x = \frac{F}{m}[1-u(t-T)]$$

Taking Laplace transform of both sides, we get

$$(s^2+k^2)\,\bar{x} = \frac{F}{ms}(1-e^{-sT}) \qquad [\because\ x = 0,\ \dot{x} = 0 \text{ at } t = 0]$$

or $$\bar{x} = \frac{F}{m}\,.\,\frac{1-e^{-sT}}{s(s^2+k^2)} = \frac{F}{m}(1-e^{-sT})\,.\,\frac{1}{k^2}\left(\frac{1}{s}-\frac{s}{s^2+k^2}\right)$$

$$= \frac{F}{mk^2}\left\{(1-e^{-sT})\frac{1}{s} - (1-e^{-sT})\,.\,\frac{s}{s^2+k^2}\right\}$$

Taking inverse Laplace transform, we obtain

$$x = \frac{F}{mk^2}[(1-\cos kt) - \{1-\cos k(t-T)\}]\,u(t-T)$$

i.e., $$x = \frac{F}{mk^2}(1-\cos kt) \text{ for } 0 < t < T$$

and $$x = \frac{F}{mk^2}(1-\cos kt) - \{1-\cos k(t-T)\} \text{ for } t > T$$

$$= \frac{F}{mk^2}\{\cos k(t-T) - \cos kT\} \text{ for } t > T$$

or $$x = \frac{2F}{mk^2}\sin\frac{kT}{2}\,.\,\sin k(t-T/2) \text{ for } t > T$$

Hence the amplitude of subsequent oscillation (*i.e.*, for $t > T$) $= \dfrac{2F}{mk^2}\sin\dfrac{kT}{2}$.

Example 21.45. *In an electrical circuit with e.m.f. E(t), resistance R and inductance L, the current i builds up at the rate given by*

$$L\,di/dt + Ri = E(t). \quad ...(i)$$

If the switch is connected at t = 0 and disconnected at t = a, find the current i at any instant.

Solution. We have $i = 0$ at $t = 0$ and $E(t) = \begin{cases} E \text{ for } 0 < t < a \\ 0 \text{ for } t > a \end{cases}$

$\therefore$ taking the Laplace transform of both sides, (i) becomes

$$(Ls + R)\bar{i} = \int_0^{\infty} e^{-st}\; E(t)\, dt = \int_0^{a} e^{-st}\; E dt = \frac{E}{s}(1 - e^{-as})$$

or
$$\bar{i} = \frac{E}{s(Ls + R)} - \frac{Ee^{-as}}{s(Ls + R)}$$

On inversion, we get
$$i = L^{-1}\left\{\frac{E}{s(Ls + R)}\right\} - L^{-1}\left\{\frac{Ee^{-as}}{s(Ls + R)}\right\} \quad ...(ii)$$

Now
$$L^{-1}\left\{\frac{E}{s(Ls + R)}\right\} = \frac{E}{R}\left\{L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s + R/L}\right)\right\} = \frac{E}{R}\,(1 - e^{-Rt/L})$$

and
$$L^{-1}\left\{\frac{Ee^{-as}}{s(Ls + R)}\right\} = \frac{E}{R}\,\{1 - e^{-R(t-a)/L}\}\,u\,(t - a)$$ [By the second shifting property]

Thus (ii) becomes
$$i = \frac{E}{R}\{1 - e^{-Rt/L}\} - \frac{E}{R}\{1 - e^{-R(t-a)/L}\}\,u\,(t - a)$$

Hence
$$i = \frac{E}{R}\{1 - e^{-Rt/L}\} \text{ for } 0 < t < a$$

and
$$i = \frac{E}{R}[\{1 - e^{-Rt/L}\} - \{1 - e^{-R(t-a)/L}\}] = \frac{E}{R}e^{-Rt/L}\,(e^{-Ra/L} - 1) \text{ for } t > a.$$

Example 21.46. *Calculate the maximum deflection of an encastre beam 1 ft. long carrying a uniformly distributed load w lb./ft. on its central half length.*

Solution. Taking the origin at the end A, we have

$$EI\,\frac{d^4y}{dx^4} = w(x)$$

where $w(x) = w\{u(x - l/4) - u(x - 3l/4)\}$

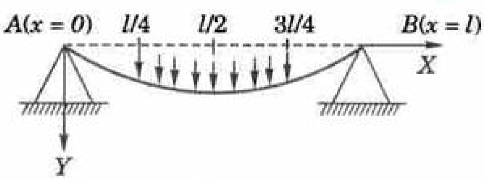

Fig. 21.6

Taking the Laplace transform of both sides, (Fig. 21.6), we get

$$EI[s^4\,\bar{y} - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)]$$
$$= w\left(\frac{e^{-ls/4}}{s} - \frac{e^{-3ls/4}}{s}\right)$$

Using the conditions $y(0) = y'(0) = 0$ and taking $y''(0) = c_1$ and $y'''(0) = c_2$, we have

$$EI\,\bar{y} = w\left(\frac{e^{-ls/4}}{s^5} - \frac{e^{-3ls/4}}{s^5}\right) + \frac{c_1}{s^3} + \frac{c_2}{s^4}$$

On inversion, we get $EIy = \frac{w}{24}\,[(x - l/4)^4\,u(x - l/4) - (x - 3l/4)^4\,u(x - 3l/4)] + \frac{1}{2}c_1x^2 + \frac{1}{6}c_2x^3$ ...(i)

For $x > 3l/4$,
$$EIy = \frac{w}{24}\,[(x - l/4)^2 - (x - 3l/4)^2] + \tfrac{1}{2}c_1x^2 + \tfrac{1}{6}c_2x^3$$

and
$$EIy' = \frac{w}{6}\,[(x - l/4)^3 - (x - 3l/4)^3] + c_1x + \tfrac{1}{2}c_2x^2$$

Using the conditions $y(l) = 0$ and $y'(l) = 0$, we get $0 = \frac{w}{24}\left\{\left(\frac{3l}{4}\right)^4 - \left(\frac{l}{4}\right)^4\right\} + \frac{1}{2}c_1l^2 + \frac{1}{6}c_2l^3$

and
$$0 = \frac{w}{6}\left\{\left(\frac{3l}{4}\right)^3 - \left(\frac{l}{4}\right)^3\right\} + c_1l + \tfrac{1}{2}c_2l^2$$

whence $c_1 = 11\,wl^2/192$; $c_2 = -\,wl/4$.

Thus for $l/4 < x < 3l/4$, (*i*) gives $EIy = \frac{w}{24}\left(x + \frac{1}{4}\right)^4 + \frac{11wl^2}{384}x^2 - \frac{wl}{24}x^3$

Hence the maximum deflection $= y(l/2) = \frac{13wl^4}{6144EI}$.

21.18 (1) UNIT IMPULSE FUNCTION

The idea of a very large force acting for a very short time is of frequent occurrence in mechanics. To deal with such and similar ideas, we introduce the *unit impulse function* (also called *Dirac delta function**).

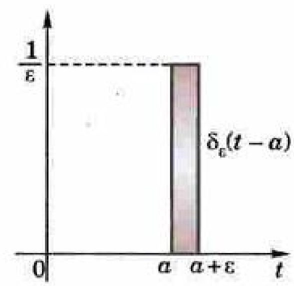

Fig. 21.7

Thus unit impulse function is considered as the limiting form of the function (Fig. 21.7) :

$$\delta_\varepsilon(t-a) = 1/\varepsilon, \quad a \le t \le a + \varepsilon$$
$$= 0, \quad \text{otherwise}$$

as $\varepsilon \to 0$. It is clear from Fig. 21.7 that as $\varepsilon \to 0$, the height of the strip increases indefinitely and the width decreases in such a way that its area is always unity.

Thus the unit impulse function $\delta(t-a)$ is defined as follows :

$$\delta(t-a) = \infty \text{ for } t = a \; ; = 0 \text{ for } t \ne a,$$

such that $$\int_0^\infty \delta(t-a)\, dt = 1. \qquad (a \ge 0)$$

As an illustration, a load w_0 acting at the point $x = a$ of a beam may be considered as the limiting case of uniform loading w_0/ε per unit length over the portion of the beam between $x = a$ and $x = a + \varepsilon$. Thus

$$w(x) = w_0/\varepsilon \qquad a < x < a + \varepsilon,$$
$$= 0, \qquad \text{otherwise}$$

i.e., $$w(x) = w_0\delta(x + a).$$

(2) Transform of unit impulse function. If $f(t)$ be a function of t continuous at $t = a$, then

$$\int_0^\infty f(t)\,\delta_\varepsilon(t-a)\,.\,dt = \int_0^{a+\varepsilon} f(t)\,.\,\frac{1}{\varepsilon}\,dt$$

$$= (a + \varepsilon - a)\, f(\eta)\,.\,\frac{1}{\varepsilon} = f(\eta), \qquad \text{where } a < \eta < a + \varepsilon.$$

by Mean value theorem for integrals.

As $\varepsilon \to 0$, we get $\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$.

In particular, when $f(t) = e^{-st}$, we have $\mathbf{L\{\delta(t-a)\} = e^{-as}}$.

Example 21.47. *Evaluate* (*i*) $\int_0^\infty \sin 2t\, \delta(t - \pi/4)\, dt$ (*ii*) $L\frac{1}{t}\,\delta(t-a)$.

Solution. (*i*) We know that $\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$

$$\therefore \quad \int_0^\infty \sin 2t\, \delta(t - \pi/4)\, dt = \sin(2\,.\,\pi/4) = 1$$

(*ii*) We know that $L\{\delta(t-a)\} = e^{-as}$

$$\therefore \quad L\left[\frac{1}{t}\,\delta(t-a)\right] = \int_s^\infty L[\delta(t-a)]\,ds = \int_s^\infty e^{-as}\,ds$$

$$= \left|\frac{e^{-as}}{-a}\right|_s^\infty = \frac{1}{a}\,e^{-as}.$$

* After the English physicist *Paul Dirac* (1902-84) who was awarded the Nobel prize in 1933 for his work in Quantum mechanics.

Example 21.48. *An impulsive voltage $E\delta(t)$ is applied to a circuit consisting of L, R, C in series with zero initial conditions. If i be the current at any subsequent time t, find the limit of i as $t \to 0$?*

Solution. The equation of the circuit governing the current i is

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i\,dt = E\delta(t) \qquad \text{where } i = 0, \text{ when } t = 0.$$

Taking Laplace transform of both sides, we get

$$L[s\bar{i} - i(0)] + R\bar{i} + \frac{1}{C}\frac{1}{s}\bar{i} = E \qquad \text{[Using § 21.7 and 21.8]}$$

or $$\left(s^2 + \frac{R}{L}s + \frac{1}{CL}\right)\bar{i} = \frac{E}{L}s \quad \text{or} \quad (s^2 + 2as + a^2 + b^2)\bar{i} = (E/L)\,s$$

where $R/L = 2a$ and $1/CL = a^2 + b^2$

or $$\bar{i} = \frac{E}{L}\frac{(s+a)-a}{(s+a)^2+b^2} = \frac{E}{L}\left\{\frac{s+a}{(s+a)^2+b^2} - a\frac{1}{(s+a)^2+b^2}\right\}$$

On inversion, we get

$$i = \frac{E}{L}\left\{e^{-at}\cos bt - \frac{a}{b}e^{-at}\sin bt\right\}$$

Taking limits as $t \to 0$, $i \to E/L$

Although the current $i = 0$ initially, yet a large current will develop instantaneously due to impulsive voltage applied at $t = 0$. In fact, we have determined the limit of this current which is E/L.

Example 21.49. *A beam is simply supported at its end $x = 0$ and is clamped at the other end $x = l$. It carries a load w at $x = l/4$. Find the resulting deflection at any point.*

Solution. The differential equation for deflection is

$$\frac{d^4y}{dx^4} = \frac{w}{EI}\delta(x - l/4)$$

Taking the Laplace transform, we have $s^4\bar{y} - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) = \frac{w}{EI}e^{-ls/4}$

Using the conditions $y(0) = 0$, $y''(0) = 0$ and taking $y'(0) = c_1$ and $y'''(0) = c_2$, we get

$$\bar{y} = \frac{c_1}{s^2} + \frac{c_2}{s^4} + \frac{w}{EI}\frac{e^{-ls/4}}{s^4}.$$

On inversion, it gives $$y = c_1x + c_2\frac{x^3}{3!} + \frac{w}{EI}\frac{(x - l/4)^3}{3!}\,u(x - l/4)$$

$$\left.\begin{array}{ll} i.e., \quad y = c_1x + \frac{1}{6}c_2x^3, & 0 < x < l/4 \\ \text{and} \quad y = c_1x + \frac{1}{6}c_2x^3 + \frac{\omega}{6EI}(x - l/4)^3, & l/4 < x < l \end{array}\right\} \quad ...(i)$$

Using the conditions $y(l) = 0$ and $y'(l) = 0$, we get

$$0 = c_1l + \tfrac{1}{6}c_2l^3 + 9wl^3/128EI \quad \text{and} \quad 0 = c_1 + \tfrac{1}{2}c_2l^2 + 9wl^2/32EI$$

whence $$c_1 = 9wl^2/256\,EI, \qquad c_2 = -81w/128EI.$$

Substituting the values of c_1 and c_2 in (i), we get the deflection at any point.

PROBLEMS 21.8

1. Represent $f(t) = \sin 2t$, $2\pi < t < 4\pi$ and 0 otherwise, in terms of the unit step function and hence find its Laplace transform. *(Mumbai, 2005)*
2. Sketch the graph of the following functions and express them in terms of unit step function. Hence find their Laplace transforms :

(i) $f(t) = 2t$ for $0 < t < \pi$, $f(t) = 1$ for $t > \pi$ (Assam, 1999)

(ii) $f(t) = t^2$ for $0 < t \le 2$, $f(t) = 0$ for $t > 2$

(iii) $f(t) = \cos(wt + \phi)$ for $0 < t < T$, $f(t) = 0$ for $t > T$.

3. Express the following functions in terms of unit step function and hence find its Laplace transform.

(i) $f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ 1, & \pi < t < 2\pi \\ \sin t, & t > 2\pi \end{cases}$ (V.T.U., 2007) (ii) $f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ \cos 2t, & \pi < t < 2\pi \\ \cos 3t, & t > \pi \end{cases}$

(Mumbai, 2008 ; V.T.U., 2003 S)

(iii) $f(t) = \begin{cases} t^2, & 0 < t < 2 \\ 4t, & 2 < t < 4 \\ 8, & t > 4 \end{cases}$ (V.T.U., 2011)

4. Evaluate (i) $L\{e^{t-1} u(t-1)\}$ (ii) $L\{(t-1)^2 u(t-1)\}$

(iii) $L(1 + 2t - 3t^2 + 4t^3) H(t-2)$ (Mumbai, 2007) (iv) $L\{t^2 u(t-1) + \delta(t-1)\}$.

5. Evaluate $\int_0^\infty e^{-t}(1 + 3t + t^2) u(t-2)\, dt$.

6. Find the inverse Laplace transforms of :

(i) $\dfrac{e^{-\pi s}}{s^2 + 1}$ (ii) $\dfrac{e^{-2s}}{s^2 + 8s + 25}$ (Mumbai, 2006)

(iii) $\dfrac{e^{-s}}{(s+1)^3}$ (P.T.U., 2010) (iv) $\dfrac{3}{s} - 4\dfrac{e^{-s}}{s^2} + 4\dfrac{e^{-3s}}{s^2}$. (P.T.U., 2002 S)

7. Solve using Laplace transforms $\dfrac{d^2y}{dt^2} + 4y = f(t)$ with conditions

$$y(0) = 0, y'(0) = 1 \text{ and } f(t) = \begin{cases} 1 \text{ when } 0 < t < 1 \\ 0 \text{ when } t > 1 \end{cases}$$ (Mumbai, 2007)

8. Using Laplace transforms, solve $x''(t) + x(t) = u(t)$, $x(0) = 1$, $x'(0) = 0$

where $u(t) = \begin{cases} 3, & 0 \le t \le 4 \\ 2t - 5, & t > 4. \end{cases}$

9. A beam has its ends clamped at $x = 0$ and $x = l$. A concentrated load W acts vertically downwards at the point $x = l/3$. Find the resulting deflection.

$\Bigg[$ **Hint.** The differential equation and the boundary conditions are $\dfrac{d^4y}{dx^2} = \dfrac{W}{EI}\, \delta(x - l/3)$ and

$y(0) = y'(0) = 0, y(l) = y'(l) = 0.$ $\Bigg]$

10. A cantilever beam is clamped at the end $x = 0$ and is free at the end $x = l$. It carries a uniform load w per unit length from $x = 0$ to $x = l/2$. Calculate the deflection y at any point. (Kurukshetra, 2006)

[**Hint.** The differential equation and boundary conditions are

$$\frac{d^4y}{dx^4} = \frac{W(x)}{EI} \ (0 < x < l) \text{ where } \quad W(x) = \begin{cases} W_0, & 0 < x < l/2 \\ 0, & x > l/2 \end{cases}$$

and $y(0) = y'(0) = 0, y''(0) = y'''(0) = 0$]

11. An impulse I (kg-sec) is applied to a mass m attached to a spring having a spring constant k. The system is damped with damping constant μ. Derive expressions for displacement and velocity of the mass, assuming initial conditions $x(0) = x'(0) = 0$.

$\Bigg[$ **Hint.** The equation of motion is $m\dfrac{d^2x}{dt^2} = I\,\delta(x) - kx - \mu\dfrac{dx}{dt}$. $\Bigg]$

21.19 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 21.9

Fill up the blanks or choose the correct answer in each of the following problems :

1. Laplace transform of $(t \sin t)$ = ...
2. $L\{\delta(t)\}$ = ...
   (a) 0 (b) e^{-as} (c) ∞ (d) 1.
3. If $L\{f(t)\} = f(s)$, then $L\{e^{-at} f(t)\}$ is
   (a) $f(s-a)$ (b) $f(s+a)$ (c) $f(s)$ (d) none of these.
4. $L(e^{2t} \sin t)$ = ...
5. Inverse Laplace transform of $(s+2)^{-2}$ is ....
6. Inverse Laplace transform of $1/(s^2 + 4s + 13)$ = ...
7. Laplace transform of $f'(t)$ = ...
8. $L^{-1}\left[\dfrac{s}{(2s+3)^2}\right]$ = ...
9. $L(\cosh^2 2t)$ = ...
10. $L(e^t)$ = ...
11. $L(e^{-t} t^k)$ = ...
12. $\int_0^\infty e^{-2t} \cos 3t \, dt$ = ...
13. $L\{u(t-a)\}$ = ...
14. $L^{-1}(\sqrt{t})$ = ...
15. If $L\{F(t)\} = f(s)$, then $L\left\{\dfrac{d^2F(t)}{dt^2}\right\}$ = ...
16. $L(\cos^3 4t)$ = ...
17. $L\left(\dfrac{\sin at}{t}\right)$ = ...
18. $L^{-1}\left\{\dfrac{1}{(s+3)^5}\right\}$ = ...
19. $L \cos(2t+3)$ = ...
20. $L^{-1}(1/s^n)$ is possible only when n is
   (a) zero (b) –ve integer (c) +ve integer (d) negative rational.
21. If $L^{-1}[\phi(s)] = f(t)$, the $L^{-1}[e^{-as}\phi(s)]$ = ...
22. $L\{u(t+2)\}$ = ...
   (a) e^{-2s}/s^2 (b) e^{2s} (c) $\dfrac{e^{2s}}{s}$ (d) $\dfrac{e^{-2s}}{s}$ (V.T.U., 2011 S)
23. $L^{-1}\left\{\dfrac{s^2 - 3s + 4}{s^3}\right\}$ = ... (V.T.U., 2010 S)
24. If $L\{f(t)\} = \bar{f}(s)$, then $L^{-1}\left\{\dfrac{\bar{f}(s)}{s}\right\}$ = ...
25. If $f(t)$ is a periodic function with period T, then $L[f(t)]$ = ...
26. If y satisfies $y'' + 3y' + 2y = e^{-t}$ with $y(0) = y'(0) = 0$, then $L[y(t)]$ = ...
27. $L[e^{3t}(2\cos 5t + 3\sin 4t)]$ = ...
28. $L(4^t)$ = ...
29. $L^{-1}\left\{\dfrac{1}{\sqrt{(s+3)}}\right\}$ = ...
30. Laplace transform of $\sin 2t \, \delta(t-2)$ is
   (a) $e^{2s} \sin 4$ (b) $e^{-2s} \sin 2$ (c) $e^{-4s} \sin 2$ (d) $e^{-2s} \sin 4$. (V.T.U., 2009 S)
31. If $L^{-1}\left\{\dfrac{s}{(s+1)^2}\right\} = \dfrac{t \sin t}{2}$ then $L^{-1}\left\{\dfrac{8s}{(4s^2+1)^2}\right\}$ (P.T.U., 2009)
32. $L^{-1}[e^{-as} F(s)]$ =
   (a) $f(t)\,u(t)$ (b) $f(t-a)\,u(t)$ (c) $f(t-a)\,u(t-a)$ (d) None of these. (V.T.U., 2009 S)
33. $L^{-1}\left\{\dfrac{1}{(s+a)^2}\right\}$ =
   (a) e^{at} (b) e^{-at} (c) te^{-at}
   (d) te^{at} (e) $-t$.

34. Laplace transform of $t^4 e^{-at}$ is

(i) $\dfrac{4!}{(s+a)^4}$ (ii) $\dfrac{4!}{(s+a)^5}$ (iii) $\dfrac{4!}{(s-a)^4}$ (iv) $\dfrac{5!}{(s-a)^5}$.

35. Laplace transform of $te^{at} \sin(at)$, $t > 0$, is

(i) $\dfrac{s-a}{(s-a)^2+a^2}$ (ii) $\dfrac{a(s-a)}{(s-a)^2+a^2}$ (iii) $\dfrac{2a(s-a)}{[(s-a)^2+a^2]^2}$ (iv) $\dfrac{(s-a)^2}{(s-a)^2+a^2}$.

36. $L^{-1} \dfrac{s^2}{(s^2+4)^2}$ is

(i) $\dfrac{1}{4} \sin 2t + t \cos 2t$ (ii) $\dfrac{1}{4} \sin 2t + \dfrac{1}{2} \cos 2t$ (iii) $\sin 2t + \dfrac{t}{2} \cos 2t$ (iv) $\dfrac{1}{2} \sin 2t + \dfrac{t}{4} \cos 2t$.

37. $L^{-1} \dfrac{1}{s(s^2+1)}$ is

(i) $1 + \sin t$ (ii) $1 - \sin t$ (iii) $1 + \cos t$ (iv) $1 - \cos t$.

38. $L\,[u(t-a)]$ where $u(t-a)$ is a unit step function, is

(i) $\dfrac{e^{-as}}{s}$ (ii) $\dfrac{e^{as}}{s}$ (iii) $\dfrac{e^{-as}}{s^2}$ (iv) $\dfrac{e^{as}}{s^2}$. (V.T.U., 2011)

39. For a periodic function of period 2π, $\displaystyle\int_{a+2\pi}^{b+2\pi} f(x)\,dx = \ldots\ldots$ (P.T.U., 2009)

40. $L\,[\,\delta(t-a)]$ where $\delta(t-a)$ is a unit impulse function, is

(i) e^{as} (ii) e^{-as} (iii) e^{s} (iv) e^{-as}/s (V.T.U., 2010 S)

41. Laplace transform of $\sin^2 3t$ is

(i) $\dfrac{3}{s^2+36}$ (ii) $\dfrac{6}{(s^2+36)}$ (iii) $\dfrac{18}{s(s^2+36)}$ (iv) $\dfrac{18}{s^2+36}$ (V.T.U., 2010)

42. $[L\,(t^2 e^{-3t})] =$

(i) $\dfrac{1}{(s+3)^3}$ (ii) $\dfrac{2}{(s+3)^2}$ (iii) $\dfrac{3}{(s+3)^3}$ (iv) $\dfrac{2}{(s+3)^3}$ (V.T.U., 2011)

43. $\dfrac{d^2}{ds^2}\,[L\,f(t)] - L(t^2 f(t)] = 0.$ (True or False)

44. Laplace transform of $f(t)$ is defined for +ve and –ve values of t. (True or False)

45. If $L[f(t)] = \phi(s)$, then $L\,[t\,f(t)] = \dfrac{d}{ds}\,[\phi(s)]$. (True or False)

CHAPTER 22

Fourier Transforms

1. Introduction. 2. Definition. 3. Fourier integrals — Fourier sine and cosine integral – Complex forms of Fourier integral. 4. Fourier transform — Fourier sine and cosine transforms — Finite Fourier sine and cosine transforms. 5. Properties of F-transforms. 6. Convolution theorem for F-transforms. 7. Parseval's identity for F-transforms. 8. Relation between Fourier and Laplace transforms. 9. Fourier transforms of the derivatives of a function— 10. Inverse Laplace transforms by method of residues. 11. Application of transforms to boundary value problems. 12. Objective Type of Questions.

22.1 INTRODUCTION

In the previous chapter, the reader has already been acquainted with the use of Laplace transforms in the solution of ordinary differential equations. In this chapter, the well-known Fourier transforms will be introduced and their properties will be studied which will be used in the solution of partial differential equations. The choice of a particular transform to be employed for the solution of an equation depends on the boundary conditions of the problem and the ease with which the transform can be inverted. A Fourier transform when applied to a partial differential equation reduces the number of its independent variables by one.

The theory of integral transforms afford mathematical devices through which solutions of numerous boundary value problems of engineering can be obtained *e.g.*, conduction of heat, transverse vibrations of a string, transverse oscillations of an elastic beam, free and forced vibrations of a membrane, transmission lines etc. Some of these applications will be illustrated in the last section.

22.2 DEFINITION

The integral transform of a function f(x) denoted by I[f(x)], is defined by

$$\bar{f}(s) = \int_{x_1}^{x_2} f(x) K(s, x)\, dx$$

where $K(s, x)$ is called the *kernel* of the transform and is a known function of s and x. The function $f(x)$ is called the *inverse transform* of $\bar{f}(s)$.

Three simple examples of a kernel are as follows :

(*i*) *When* $K(s, x) = e^{-sx}$, *it leads to the* **Laplace transform** *of* $f(x)$, *i.e.*,

$$\bar{f}(s) = \int_0^{\infty} f(x) e^{-sx}\, dx. \qquad \text{[Chap. 21]}$$

(*ii*) *When* $K(s, x) = e^{isx}$, *we have the* **Fourier transform** *of* $f(x)$, *i.e.*,

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{isx}\, dx.$$

(*iii*) When $K(s, x) = x^{s-1}$, it gives the *Mellin transform* of $f(x)$ *i.e.*,

$$M(s) = \int_0^\infty f(x) x^{s-1} dx.$$

Other special transforms arise when the kernel is a sine or a cosine function or a Bessel's function. These lead to *Fourier sine* or *cosine transforms* and the *Hankel transform* respectively.

In order to introduce the *Fourier transforms*, we shall first derive the Fourier integral theorem.

22.3 (1) FOURIER INTEGRAL THEOREM

Consider a function $f(x)$ which satisfies the Dirichlet's conditions (Art. 10.3) in every interval $(-c, c)$ so that, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{c} + b_n \sin \frac{n\pi x}{c} \right) \quad ...(1)$$

where $\quad a_0 = \frac{1}{c} \int_{-c}^{c} f(t)\, dt,\ a_n = \frac{1}{c} \int_{-c}^{c} f(t) \cos \frac{n\pi t}{c} dt$, and $b_n = \frac{1}{c} \int_{-c}^{c} f(t) \sin \frac{n\pi t}{c} dt.$

Substituting the values of a_0, a_n and b_n in (1), it takes the form

$$f(x) = \frac{1}{2c} \int_{-c}^{c} f(t)\, dt + \frac{1}{c} \sum_{n=1}^{\infty} \int_{-c}^{c} f(t) \cos \frac{n\pi(t-x)}{c} dt \quad ...(2)$$

If we assume that $\int_{-\infty}^{\infty} |f(x)|\, dx$ converges, the first term on the right side of (2) approaches 0 as $c \to \infty$, since

$$\left| \frac{1}{2c} \int_{-c}^{c} f(t)\, dt \right| \le \frac{1}{2c} \int_{-\infty}^{\infty} |f(t)|\, dt$$

The second term on the right side of (2) tends to

$$\underset{c \to \infty}{\text{Lt}} \frac{1}{c} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} f(t) \cos \frac{n\pi(t-x)}{c} dt$$

$$= \underset{\delta\lambda \to 0}{\text{Lt}} \frac{1}{\pi} \sum_{n=1}^{\infty} \delta\lambda \int_{-\infty}^{\infty} f(t) \cos n\delta\lambda\, (t-x)\, dt, \text{ on writing } \pi/c = \delta\lambda$$

This is of the form $\underset{\delta\lambda \to 0}{\text{Lt}} \sum_{n=1}^{\infty} F(n\delta\lambda)$, *i.e.*, $\int_0^\infty F(\lambda)\, d\lambda$

Thus as $c \to \infty$, (2) becomes $f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^{\infty} f(t) \cos \lambda(t-x)\, dt d\lambda \quad ...(3)$

which is known as the **Fourier integral** of $f(x)$.

Obs. We have given a heuristic demonstration of the Fourier integral theorem which simply helps in deriving the result (3). It cannot however, be taken as a rigorous proof for that would, involve a proof of the convergence of the Fourier integral which is beyond the scope of this book. When $f(x)$ satisfies the above-mentioned conditions, equation (3) holds good at a point of continuity. If however, x is point of discontinuity, we replace $f(x)$ by $\frac{1}{2}[f(x+0) + f(x-0)]$ as in the case of Fourier series.

(2) Fourier sine and cosine integrals. Expanding $\cos \lambda(t-x)$, (3) may be written as

$$f(x) = \frac{1}{\pi} \int_0^\infty \cos \lambda x \int_{-\infty}^{\infty} f(t) \cos \lambda t\, dt\, d\lambda + \frac{1}{\pi} \int_0^\infty \sin \lambda x \int_{-\infty}^{\infty} f(t) \sin \lambda t\, dt\, d\lambda \quad ...(4)$$

If $f(x)$ is an odd function, $f(t) \cos \lambda t$ is also an odd function while $f(t) \sin \lambda t$ is even. Then the first term on the right side of (4) vanishes and, we get

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin \lambda x \int_0^\infty f(t) \sin \lambda t\, dt\, d\lambda \quad ...(5)$$

which is known as the *Fourier sine integral*.

Similarly, if $f(x)$ is an even function, (4) takes the form

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos \lambda x \int_0^\infty f(t) \cos \lambda t \, dt \, d\lambda \quad ...(6)$$

which is known as the *Fourier cosine integral.*

Obs. A function $f(x)$ defined in the interval $(0, \infty)$ is expressed either as a Fourier sine integral or as a Fourier cosine integral, merely looking upon it as an odd or even function in $(-\infty, \infty)$ on the lines of half-range Fourier series.

(3) **Complex form of Fourier integrals.** Equation (3) can be written as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty f(t) \cos \lambda(t - x) dt d\lambda \quad ..(7)$$

because $\cos \lambda(t - x)$ is an even function of λ. Also since $\sin \lambda(t - x)$ is an odd function of λ, we have

$$0 = \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty f(t) \sin \lambda(t - x) dt d\lambda \quad ...(8)$$

Now multiply (8) by i and add it to (7), so that

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty f(t)\, e^{i\lambda(t-x)} dt d\lambda \quad ...(9)$$

which is the *complex* form of the *Fourier integral.*

(4) Fourier integral representation of a function

Using (4), *a function F(x) may be represented by a Fourier integral as*

$$F(x) = \frac{1}{\pi} \int_0^\infty [A(\lambda) \cos \lambda x + B(\lambda) \sin \lambda x] \, d\lambda$$

where $$A(\lambda) = \int_{-\infty}^\infty f(t) \cos \lambda t \, dt \; ; B(\lambda) = \int_{-\infty}^\infty f(t) \sin \lambda t \, dt \quad ...(10)$$

If $f(x)$ is an odd function, then [By (5)]

$$f(x) = \frac{1}{\pi} \int_0^\infty B(\lambda) \sin \lambda x \, d\lambda \text{ where } B(\lambda) = 2 \int_0^\infty f(t) \sin \lambda t \, dt \quad ...(11)$$

If $f(x)$ is an even function, then [By (6)]

$$f(x) = \frac{1}{\pi} \int_0^\infty A(\lambda) \cos \lambda x \, d\lambda \text{ where } A(\lambda) = 2 \int_0^\infty f(t) \cos \lambda t \, dt \quad ...(12)$$

Example 22.1. *Express $f(x) = 1$ for $0 \le x \le \pi$,*

$= 0$ for $x > \pi$,

as a Fourier sine integral and hence evaluate

$$\int_0^\infty \frac{1 - \cos(\pi\lambda)}{\lambda} \sin(x\lambda) d\lambda.$$

(Kottayam, 2005 ; J.N.T.U., 2004 S)

Solution. The Fourier sine integral for $f(x) = \frac{2}{\pi} \int_0^\infty \sin(\lambda x) \, d\lambda \int_0^\infty f(t) \sin(\lambda t) \, dt$

$$= \frac{2}{\pi} \int_0^\infty \sin(\lambda x) \, d\lambda \int_0^\infty \sin(\lambda t) \, dt$$

$$= \frac{2}{\pi} \int_0^\infty \sin(\lambda x) \, d\lambda \left| \frac{-\cos(\lambda t)}{\lambda} \right|_0^\pi = \frac{2}{\pi} \int_0^\infty \frac{1 - \cos(\lambda\pi)}{\lambda} \sin(\lambda x) d\lambda$$

$$\therefore \quad \int_0^\infty \frac{1 - \cos(\lambda\pi)}{\lambda} \sin(\lambda x) d\lambda = \frac{\pi}{2} f(x) = \begin{cases} \pi/2 & \text{for } 0 \le x < \pi \\ 0 & \text{for } x > \pi \end{cases}$$

At $x = \pi$, which is a point of discontinuity of $f(x)$, the value of the above integral

$$= \frac{\pi}{2} \left[\frac{f(\pi - 0) + f(\pi + 0)}{2} \right] = \frac{\pi}{2} \cdot \frac{1 + 0}{2} = \frac{\pi}{4}.$$

22.4 (1) FOURIER TRANSFORMS

Rewriting (9) of § 22.3 as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\, ds \int_{-\infty}^{\infty} f(t)e^{ist}\, dt,$$

it follows that if
$$F(s) = \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt \qquad ...(1)$$

then
$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds \qquad ...(2)$$

The function F(s), defined by (1), is called the **Fourier transform** *of* $f(x)$. *Also the function* $f(x)$*, as given by* (2), *is called the* **inverse Fourier transform** *of* $F(s)$. Sometimes, we call (2) as an *inversion formula* corresponding to (1).

(2) Fourier sine and cosine transforms. From (5) of § 22.3, it follows that if

$$F_s(s) = \int_0^{\infty} f(x)\sin sx\, dx \qquad ...(3)$$

then
$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_s(s)\sin sx\, ds \qquad ...(4)$$

The function $F_s(s)$*, as defined by* (3), *is known as the* **Fourier sine transform** *of* $f(x)$ *in* $0 < x < \infty$. *Also the function* $f(x)$*, as given by* (4) *is called the* **inverse Fourier sine transform** *of* $F_s(s)$.

Similarly, if follows from (6) of § 22.3 that if

$$F_c(s) = \int_0^{\infty} f(x)\cos sx\, dx \qquad ...(5)$$

then
$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_c(s)\cos sx\, ds \qquad ...(6)$$

The function $F_c(s)$ *as defined by* (5) *is known as the* **Fourier cosine transform** *of* $f(x)$ *in* $0 < x < \infty$. *Also the function* $f(x)$*, as given by* (6), *is called the* **inverse Fourier cosine transform** *of* $F_c(s)$.

(3) **Finite Fourier sine and cosine transforms.** These transforms are useful for such a boundary-value problem in which at least two of the boundaries are parallel and separated by a finite distance.

The **finite Fourier sine transform** *of* $f(x)$*, in* $0 < x < c$*, is defined as*

$$F_s(n) = \int_0^{c} f(x)\sin\frac{n\pi x}{c}\, dx \qquad ...(7)$$

where n is an integer.

The function $f(x)$ *is then called the* **inverse finite Fourier sine transform** *of* $F_s(n)$ *which is given by*

$$f(x) = \frac{2}{c}\sum_{n=1}^{\infty} F_s(n)\sin\frac{n\pi x}{c} \qquad ...(8)$$

The **finite Fourier cosine transform** *of* $f(x)$*, in* $0 < x < c$*, is defined as*

$$F_c(n) = \int_0^{c} f(x)\cos\frac{n\pi x}{c}\, dx \qquad ...(9)$$

where n is an integer.

The function $f(x)$ *is then called the* **inverse finite Fourier cosine transform** *of* $F_c(n)$ *which is given by*

$$f(x) = \frac{1}{c}F_c(0) + \frac{2}{c}\sum_{n=1}^{\infty} F_c(n)\cos\frac{n\pi x}{c} \qquad ...(10)$$

Obs. The finite Fourier sine transform is useful for problems involving boundary conditions of heat distribution on two parallel boundaries, while the finite cosine transform is useful for problems in which the velocities normal to two parallel boundaries are among the boundary conditions.

22.5 PROPERTIES OF FOURIER TRANSFORMS

(1) **Linear property.** *If* $F(s)$ *and* $G(s)$ *are Fourier transforms of* $f(x)$ *and* $g(x)$ *respectively, then*

$$F[a\,f(x) + bg(x)] = a\,F(s) + bG(s)$$

where a and b are constants.

We have $F(s) = \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$ and $G(s) = \int_{-\infty}^{\infty} e^{isx} g(x)\, dx$

$$\therefore \quad F[af(x) + bg(x)] = \int_{-\infty}^{\infty} e^{isx} [af(x) + bg(x)]\, dx = a \int_{-\infty}^{\infty} e^{isx} f(x)\, dx + b \int_{-\infty}^{\infty} e^{isx} g(x) dx$$
$$= aF(s) + bG(s)$$

(2) Change of scale property. *If F(s) is the complex Fourier transform of f(x), then*

$$F\{f(ax)\} = \frac{1}{a} F\left(\frac{s}{a}\right), a \neq 0$$

We have $$F(s) = \int_{-\infty}^{\infty} e^{isx} f(x) dx \quad ...(i)$$

$$\therefore \quad F\{f(ax)\} = \int_{-\infty}^{\infty} e^{isx} f(ax) dx \quad \left| \begin{array}{l} \text{Put } ax = t \\ \text{so that } dx = dt/a \end{array} \right.$$

$$= \int_{-\infty}^{\infty} e^{ist/a} f(t) dt / a = \frac{1}{a} \int_{-\infty}^{\infty} e^{i(s/a)t} f(t) dt = \frac{1}{a} F\left(\frac{s}{a}\right) \quad \text{[By } (i)]$$

Cor. *If $F_s(s)$ and $F_c(s)$ are the Fourier sine and cosine transforms of $f(x)$ respectively, then*

$$F_s\{f(ax)\} = \frac{1}{a} F_s\left(\frac{s}{a}\right) \quad \text{and} \quad F_c\{f(ax)\} = \frac{1}{a} F_s\left(\frac{s}{a}\right).$$

(3) Shifting property. *If F(s) is the complex Fourier transform of f(x), then*

$$F\{f(x-a)\} = e^{isa} F(s)$$

We have $$F(s) = \int_{-\infty}^{\infty} e^{isx} f(x) dx \quad ...(i)$$

$$\therefore \quad F\{f(x-a)\} = \int_{-\infty}^{\infty} e^{isx} f(x-a) dx \quad \left| \begin{array}{l} \text{Put } x - a = t \\ \text{so that } dx = dt \end{array} \right.$$

$$= \int_{-\infty}^{\infty} e^{is(t+a)} f(t) dt = e^{isa} \int_{-\infty}^{\infty} e^{ist} f(t) dt = e^{isa} F(s) \quad \text{[By } (i)]$$

(4) Modulation theorem. *If F(s) is the complex Fourier transform of f(x), then*

$$F\{f(x) \cos ax\} = \frac{1}{2} [F(s+a) + F(s-a)]$$

We have $$F(s) = \int_{-\infty}^{\infty} e^{isx} f(x) dx \quad ...(i)$$

$$\therefore \quad F\{f(x) \cos ax\} = \int_{-\infty}^{\infty} e^{isx} f(x) \cos ax\, dx = \int_{-\infty}^{\infty} e^{isx} . f(x) . \frac{e^{iax} + e^{-iax}}{2} dx$$

$$= \frac{1}{2} \left[\int_{-\infty}^{\infty} e^{i(s+a)x} f(x) dx + \int_{-\infty}^{\infty} e^{i(s-a)x} f(x) dx \right] = \frac{1}{2} [F(s+a) + F(s-a)].$$

Cor. *If $F_s(s)$ and $F_c(s)$ are Fourier sine and cosine transforms of $f(x)$ respectively, then*

(*i*) $F_s\{f(x) \cos ax\} = \frac{1}{2} [F_s(s+a) + F_s(s-a)]$ (*Anna, 2008*)

(*ii*) $F_c\{f(x) \sin ax\} = \frac{1}{2} [F_s(s+a) - F_s(s-a)]$

(*iii*) $F_s\{f(x) \sin ax\} = \frac{1}{2} [F_c(s-a) - F_c(s+a)]$

Obs. This theorem is of great importance in radio and television where the harmonic carrier wave is modulated by an envelope.

Example 22.2. *Find the Fourier transform of*

$$f(x) = \begin{cases} 1 \text{ for } |x| < 1 \\ 0 \text{ for } |x| > 1 \end{cases}.$$

Hence evaluate $\int_0^{\infty} \frac{\sin x}{x} dx.$ (*V.T.U., 2010 ; S.V.T.U., 2009 ; U.P.T.U., 2008*)

Solution. The Fourier transform of $f(x)$, *i.e.*,

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{isx}\,dx = \int_{-1}^{1} (1)\,e^{isx}\,dx = \left|\frac{e^{isx}}{is}\right|_{-1}^{1} = \frac{e^{is}-e^{-is}}{is}$$

Thus $\quad F\{f(x)\} = F(s) = 2\,\dfrac{\sin s}{s}$, $s \neq 0$. For $s = 0$, we have $F(s) = 2$.

Now by the inversion formula, we get

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{-isx}ds, \text{ or } \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin s}{s}\,e^{-isx}ds = \begin{cases} 1 \text{ for } |x|<1 \\ 0 \text{ for } |x|>1 \end{cases}$$

Putting $x = 0$, we get

$$\int_{-\infty}^{\infty} \frac{\sin s}{s}\,ds = \pi \quad \therefore \quad \int_{0}^{\infty} \frac{\sin s}{s}\,ds = \frac{\pi}{2}, \text{ since the integrand is even.}$$

Example 22.3. *Find the Fourier transform of :*

$$f(x) = \begin{cases} 1-x^2, & |x| \leq 1 \\ 0 & , |x|>1 \end{cases}$$

Hence evaluate $\displaystyle\int_0^{\infty} \frac{x\cos x - \sin x}{x^3}\cos\frac{x}{2}\,dx.$ *(V.T.U., 2011 S ; Anna, 2005 S ; Mumbai, 2005 S)*

Solution. $F\{f(x)\} = \displaystyle\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx = F(s)$, say

$$= \int_{-\infty}^{-1}(0)\,e^{isx}\,dx + \int_{-1}^{1}(1-x^2)\,e^{isx}\,dx + \int_{1}^{\infty}(0)\,e^{isx}\,dx = \left|(1-x^2)\frac{e^{isx}}{is} - (2x)\frac{e^{isx}}{(is)^2} + (-2)\frac{e^{isx}}{(is)^3}\right|_{-1}^{1}$$

$$= 2\left(\frac{e^{is}+e^{-is}}{-s^2}\right) - 2\left(\frac{e^{is}-e^{-is}}{-is^3}\right) = -\frac{4}{s^3}(s\cos s - \sin s)$$

Now by inversion formula, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{-isx}\,ds$$

or
$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4}{s^3}(s\cos s - \sin s)\,e^{-isx}\,ds = \begin{cases} 1-x^2, & |x|\leq 1 \\ 0, & |x|>1 \end{cases}$$

Putting $x = 1/2$, we obtain

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4}{s^3}(s\cos s - \sin s)e^{-is/2}ds = \frac{3}{4}$$

or
$$\int_{-\infty}^{\infty}\frac{s\cos s - \sin s}{s^3}\left(\cos\frac{s}{2} - i\sin\frac{s}{2}\right)ds = -\frac{3\pi}{8}$$

or
$$\int_{-\infty}^{\infty}\frac{s\cos s - \sin s}{s^3}\,.\cos\frac{s}{2}\,ds = -\frac{3\pi}{8}$$

or
$$\int_{0}^{\infty}\frac{x\cos x - \sin x}{x^3}\,.\cos\frac{x}{2}\,dx = -\frac{3\pi}{16}, \text{ since the integral is even.}$$

Example 22.4. (*a*) *Find the Fourier transform of* $e^{-a^2x^2}$, $a<0$. *Hence deduce that* $e^{-x^2/2}$ *is self reciprocal in respect of Fourier transform.* *(Madras, 2006 ; Kottayam, 2005)*

(*b*) *Find Fourier transform of* (*i*) $e^{-2(x-3)^2}$ (*ii*) $e^{-x^2}\cos 3x$.

Solution. (*a*) $\quad F(e^{-a^2x^2}) = \displaystyle\int_{-\infty}^{\infty} e^{-a^2x^2}\,.\,e^{isx}\,dx = \int_{-\infty}^{\infty} e^{-a^2(x^2 - isx/a^2)}dx$

$$= \int_{-\infty}^{\infty} e^{-a^2(x-is/2a^2)^2}\,.\,e^{-s^2/4a^2}\,dx$$

$$= \int_{-\infty}^{\infty} e^{-t^2} \cdot e^{-s^2/4a^2}\, dt/a \qquad \text{[Putting } a(x - is/2a^2) = t,\ dx = dt/a$$

$$= \frac{e^{-s^2/4a^2}}{a} \int_{-\infty}^{\infty} e^{-t^2}\, dt = \frac{e^{-s^2/4a^2}}{a} \sqrt{\pi} \qquad \left[\because \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}\right]$$

Hence $$F(e^{-a^2x^2}) = \frac{\sqrt{\pi}}{a} e^{-s^2/4a^2}$$

Taking $a^2 = 1/2$, we have

$$F(e^{-x^2/2}) = \frac{\sqrt{\pi}}{(1/\sqrt{2})} e^{-s^2/2} = \sqrt{2\pi} e^{-s^2/2}$$

i.e., Fourier transform of $e^{-x^2/2}$ is a constant times $e^{-s^2/2}$. Also the functions $e^{-x^2/2}$ and $e^{-s^2/2}$ are the same. Hence it follows that $e^{-x^2/2}$ *is self-reciprocal under the Fourier transform.*

(*b*) Since $e^{-2x^2} = e^{-(2x)^2/2} = f(2x)$ where $f(x) = e^{-x^2/2}$

$\therefore$ by change of scale property, $F[f(2x)] = \frac{1}{2} F(s/2)$

i.e., $$F(e^{-2x^2}) = F[e^{-(2x)^2/2}] = \sqrt{2\pi} e^{-(s/2)^2/2} = \sqrt{2\pi} e^{-s^2/8}$$

By shifting property $F f(x - 3) = e^{i3s} F(3)$

$$\therefore \quad F[e^{-2(x-3)^2}] = e^{3is} \sqrt{2\pi} e^{-s^2/8} = \sqrt{2\pi} e^{(3is - s^2/8)} \qquad ...(i)$$

Also by modulation theorem,

$$F[f(x) \cos 2x] = \frac{1}{2} [F(s + a) + F(s - a)]$$

$$F(e^{-x^2} \cos 3x) = \frac{1}{2} \sqrt{2\pi} [e^{-(s+3)^2/2} + e^{-(s-3)^2/2}]. \qquad ...(ii)$$

Example 22.5. *Find the Fourier cosine transform of* e^{-x^2}. (*V.T.U., 2010; Rajasthan, 2006*)

Solution. We have $F_c(e^{-x^2}) = \int_0^{\infty} e^{-x^2} \cos sx\, dx = I$ (say)

Differentiating under the integral sign w.r.t. s,

$$\frac{dI}{ds} = -\int_0^{\infty} xe^{-x^2} \sin sx\, dx = \frac{1}{2} \int_0^{\infty} (\sin sx)(-2xe^{-x^2}) dx$$

$$= \frac{1}{2} \left\{ \left| \sin sx \cdot e^{-x^2} \right|_0^{\infty} - s \int_0^{\infty} \cos sx \cdot e^{-x^2}\, dx \right\}$$

$$= -\frac{s}{2} \int_0^{\infty} e^{-x^2} \cos sx\, dx = -\frac{s}{2} I \quad \text{or} \quad \frac{dI}{I} = -\int \frac{s}{2} ds + \log c$$

or $$\log I = -\frac{s^2}{4} + \log c = \log (ce^{-s^2/4})$$

$$\therefore \quad I = ce^{-s^2/4} \quad \text{or} \quad \int_0^{\infty} e^{-x^2} \cos sx\, dx = ce^{-s^2/4}$$

Putting $s = 0$, $$c = \int_0^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}. \ \ i.e.\ I = \frac{\sqrt{\pi}}{2} e^{-s^2/4}.$$

Hence $$F_c(e^{-x^2}) = \frac{\sqrt{\pi}}{2} e^{-s^2/4}.$$

Example 22.6. *Find the Fourier sine transform of* $e^{-|x|}$.

Hence show that $\int_0^{\infty} \frac{x \sin mx}{1 + x^2} dx = \frac{\pi e^{-m}}{2},\ m > 0.$ (*V.T.U., 2010 ; S.V.T.U., 2008 ; Kottayam, 2005*)

Solution. x being positive in the interval $(0, \infty)$, $e^{-|x|} = e^{-x}$

$\therefore$ Fourier sine transform of $f(x) = e^{-|x|}$ is given by

$$F_s\{f(x)\} = \int_0^\infty f(x) \sin sx\, dx = \int_0^\infty e^{-x} \sin sx\, dx$$

$$= \left| \frac{e^{-x}}{1+s^2}(-\sin sx - s\cos sx) \right|_0^\infty = \frac{s}{1+s^2}$$

Using Inversion formula for Fourier sine transforms, we get

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s\{f(x)\} \sin sx\, dx \quad \text{or} \quad e^{-x} = \frac{2}{\pi}\int_0^\infty \frac{s}{1+s^2} \sin sx\, ds$$

or changing x to m, $$e^{-m} = \frac{2}{\pi}\int_0^\infty \frac{s \sin ms}{1+s^2}\, ds = \frac{2}{\pi}\int_0^\infty \frac{x \sin mx}{1+m^2}\, dx$$

Hence $$\int_0^\infty \frac{x \sin mx}{1+m^2}\, dx = \frac{\pi e^{-m}}{2}.$$

Example 22.7. *Find the Fourier cosine transform of* $f(x) = \begin{cases} x, \text{ for } 0 < x < 11 \\ 2 - x, \text{ for } 1 < x < 2 \\ 0, \text{ for } x > 2 \end{cases}$ *(J.N.T.U., 2006)*

Solution. Fourier cosine transform of $f(x)$ *i.e.*, $F_c[f(x)]$

$$= \int_0^\infty f_c(x) \cos sx\, dx = \int_0^1 x \cos sx\, dx + \int_1^2 (2-x)\cos sx\, dx + \int_2^\infty 0\,.\, dx$$

$$= \left| x\frac{\sin sx}{s} - \left(\frac{-\cos sx}{s^2}\right)\right|_0^1 + \left|(2-x)\frac{\sin sx}{s} - (-1)\frac{-\cos sx}{s^2}\right|_1^2$$

$$= \left(\frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{1}{s^2}\right) + \left(-\frac{\cos 2s}{s^2} - \frac{\sin s}{s} + \frac{\cos s}{s^2}\right)$$

$$= \frac{2\cos s}{s^2} - \frac{\cos 2s}{s^2} - \frac{1}{s^2}.$$

Example 22.8. *Find the Fourier sine transform of* e^{-ax}/x. *(V.T.U., 2010 S ; P.T.U., 2006 ; Rohtak, 2005)*

Solution. Let $f(x) = e^{-ax}/x$, then its Fourier sine transform

i.e. $$F_s\{f(x)\} = \int_0^\infty f(x) \sin sx\, dx = \int_0^\infty \frac{e^{-ax}}{x} \sin sx\, dx = F(s), \text{ say}$$

Differentiating both sides w.r.t. s, we get

$$\frac{d}{ds}\{F(s)\} = \int_0^\infty \frac{xe^{-ax}\cos sx}{x}\, dx = \int_0^\infty e^{-ax} \cos sx\, dx = \frac{a}{s^2+a^2}$$

Integrating w.r.t. s, we obtain $F(s) = \int_0^\infty \frac{a}{s^2+a^2}\, ds = \tan^{-1}\frac{s}{a} + c$

But $F(s) = 0$, when $s = 0$; $\therefore$ $c = 0$. Hence $F(s) = \tan^{-1}(s/a)$.

Example 22.9. *Find the Fourier cosine transform of* $f(x) = 1/(1+x^2)$. *(V.T.U., 2011 S ; Anna, 2009)*
Hence derive Fourier sine transform of $\phi(x) = x/(1+x^2)$. *(V.T.U., 2009 S)*

Solution. $$F_c[f(x)] = \int_0^\infty \frac{\cos sx}{1+x^2}\, dx = I, \text{ say} \quad \text{...(i)}$$

$$\therefore \quad \frac{dI}{ds} = \int_0^\infty \frac{-x\sin sx}{1+x^2}\, dx = -\int_0^\infty \frac{x^2 \sin sx}{x(1+x^2)}\, dx \quad \text{...(ii)}$$

$$= -\int_0^\infty \frac{[(1+x^2)-1]\sin sx}{x(1+x^2)}dx = -\int_0^\infty \frac{\sin sx}{x}dx + \int_0^\infty \frac{\sin sx}{x(1+x^2)}dx$$

or $$\frac{dI}{ds} = -\frac{\pi}{2} + \int_0^\infty \frac{\sin sx}{x(1+x^2)}dx \quad ...(iii)$$

$\therefore$ $$\frac{d^2I}{ds^2} = \int_0^\infty \frac{x\cos sx}{x(1+x^2)}dx = I$$

or $$\frac{d^2I}{ds^2} - I = 0 \quad \text{or} \quad (D^2-1)I = 0, \text{ where } D = \frac{dI}{ds}$$

Its solution is $$I = c_1e^s + c_2e^{-s} \quad ...(iv)$$

$\therefore$ $$dI/ds = c_1e^s - c_2e^{-s} \quad ...(v)$$

When $s = 0$, (i) and (iv) give $c_1 + c_2 = \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$

Also when $s = 0$, (iii) and (v) give $c_1 - c_2 = -\pi/2$.

Solving these, $c_1 = 0, c_2 = \pi/2$.

Thus from (i) and (iv), we have $F_c[f(x)] = I = (\pi/2)e^{-s}$

Now $$F_s[\phi(x)] = \int_0^\infty \frac{x\sin sx}{1+x^2}dx = -\frac{dI}{ds}, \text{ from } (ii)$$

$$= (\pi/2)e^{-s}, \text{ from } (v), \text{ with } c_1 = 0, c_2 = \pi/2.$$

Example 22.10. *Find the Fourier sine and cosine transform of x^{n-1}, $n > 0$.* (Madras, 2006)

Solution. We know that $$F_s(x^{n-1}) = \int_0^\infty x^{n-1}\sin sx\,dx \quad ...(i)$$

and $$F_c(x^{n-1}) = \int_0^\infty x^{n-1}\cos sx\,dx \quad ...(ii)$$

$\therefore$ $$F_c(x^{n-1}) + i\,F_s(x^{n-1}) = \int_0^\infty (\cos sx + i\sin sx)x^{n-1}dx$$

$$= \int_0^\infty e^{isx}x^{n-1}dx = \int_0^\infty e^{-t}\left(-\frac{t}{is}\right)^{n-1}\left(-\frac{dt}{is}\right) \quad [\text{Where } isx = -t]$$

$$= \left(-\frac{1}{(i)}\right)^n \int_0^\infty e^{-t}t^{n-1}dt = \frac{(i)^{2n}}{(i)^n s^n}\Gamma(n) = \frac{(i)^n}{s^n}\Gamma(n)$$

$$= \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^n \Gamma(n)/s^n = \left(\cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2}\right)\Gamma(n)/s^n$$

Equating real and imaginary parts, we get

$$F_c(x^{n-1}) = \frac{\Gamma(n)}{s^n}\cos\frac{n\pi}{2} \quad \text{and} \quad F_s(x^{n-1}) = \frac{\Gamma(n)}{s^n}\sin\frac{n\pi}{2}.$$

Example 22.11. *(a) Show that $F_s[x\,f(x)] = -\frac{d}{ds}\{F_c(s)\}$; $F_c[x\,f(x)] = \frac{d}{ds}\{F_s(s)\}$.*

(b) Find the Fourier sine and cosine transform of xe^{-ax} (Madras, 2006)

Solution. (a) $$\frac{d}{ds}\{F_c(s)\} = \frac{d}{ds}\left\{\int_0^\infty f(x)\cos sx\,dx\right\} = \int_0^\infty f(x)(-x\sin sx)dx$$

$$= -\int_0^\infty \{x\,f(x)\}\sin sx\,dx = -F_s[x\,f(x)] \quad ...(i)$$

$$\frac{d}{ds}\{F_s(s)\} = \frac{d}{ds}\left\{\int_0^\infty f(x)\sin sx\,dx\right\} = \int_0^\infty f(x)(x\cos sx)dx$$

$$= \int_0^\infty \{x\,f(x)\}\cos sx\,dx = F_c[x\,f(x)] \quad ...(ii)$$

(b) We have $$F_s(e^{-ax}) = \int_0^\infty e^{-ax} \sin sx\, dx = \frac{e^{-ax}}{a^2+s^2} \left| -a \sin sx - s \cos sx \right|_0^\infty$$

$$= \frac{s}{a^2+s^2} \quad ...(iii)$$

and $$F_c(e^{-ax}) = \int_0^\infty e^{-ax} \cos sx\, dx = \frac{e^{-ax}}{a^2+s^2} \left| -a \cos sx + s \sin sx \right|_0^\infty$$

$$= \frac{a}{a^2+s^2} \quad ...(iv)$$

Now $$F_s(xe^{-ax}) = -\frac{d}{ds}\{F_c(e^{-ax})\} \quad \text{[by (i)]}$$

$$= -\frac{d}{ds}\left(\frac{a}{a^2+s^2}\right) = \frac{2as}{(a^2+s^2)^2} \quad \text{[by (iv)]}$$

$$F_c(xe^{-ax}) = \frac{d}{ds}\{F_s(e^{-ax})\} \quad \text{[by (ii)]}$$

$$= \frac{d}{ds}\left(\frac{s}{a^2+s^2}\right) = \frac{(a^2+s^2) - s(2s)}{(a^2+s^2)^2} = \frac{a^2-s^2}{(a^2+s^2)^2}. \quad \text{[by (iii)]}$$

Example 22.12. *If the Fourier sine transform of $f(x) = \dfrac{1-\cos n\pi}{n^2\pi^2}$ $(0 \le x \le \pi)$, find $f(x)$.* (Delhi, 2002)

Solution. We have $f(x)$ = inverse finite Fourier sine transform of $F_s(n)$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty} F_s(n) \sin\frac{n\pi x}{\pi} = \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1-\cos n\pi}{n^2\pi^2}\right\} \sin nx$$

$$= \frac{2}{\pi^3}\sum_{n=1}^{\infty}\left\{\frac{1-\cos n\pi}{n^2}\right\} \sin nx.$$

Example 22.13. *Solve the integral equation**

$$\int_0^\infty f(\theta) \cos \alpha\theta\, d\theta = \begin{cases} 1-\alpha, & 0 \le a \le 1 \\ 0, & \alpha > 1 \end{cases}$$

Hence evaluate $\displaystyle\int_0^\infty \frac{\sin^2 t}{t^2}\, dt.$ (V.T.U., 2011 S ; Kurukshetra, 2005)

Solution. We have $$\int_0^\infty f(\theta)\cos\alpha\theta\, d\theta = F_c(\alpha)$$

$\therefore$ $$F_c(\alpha) = \begin{cases} 1-\alpha, & 0 \le a \le 1 \\ 0, & \alpha > 1 \end{cases} \quad ...(i)$$

By the inversion formula, we have

$$f(\theta) = \frac{2}{\pi}\int_0^\infty F_c(\alpha)\cos\alpha\theta\, d\alpha = \frac{2}{\pi}\int_0^1 (1-\alpha)\cos\alpha\theta\, d\alpha \quad \text{[Integrating by parts]}$$

$$= \frac{2}{\pi}\left[\left|(1-\alpha)\frac{\sin\alpha\theta}{\theta}\right|_0^1 - \int_0^1 (-1)\frac{\sin\alpha\theta}{\theta}\, d\alpha\right] = \frac{2}{\pi\theta}\left|-\frac{\cos\alpha\theta}{\theta}\right|_0^1 = \frac{2(1-\cos\theta)}{\pi\theta^2}$$

Now $$F_c(\alpha) = \int_0^\infty f(\theta)\cos\alpha\theta\, d\theta = \int_0^\infty \frac{2(1-\cos\theta)}{\pi\theta^2}\cos\alpha\theta\, d\theta \quad ...(ii)$$

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* Refer to Chapter 26.

$\therefore$ From (*i*) and (*ii*), we have

$$\frac{2}{\pi}\int_0^\infty \frac{1-\cos\theta}{\theta^2}\cos\alpha\theta\, d\theta = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases}$$

Now letting $\alpha \to 0$, we get $\frac{2}{\pi}\int_0^\infty \frac{1-\cos\theta}{\theta^2}\, d\theta = 1$ (*V.T.U., 2008*)

or $$\int_0^\infty \frac{2\sin^2\theta/2}{\theta^2}\, d\theta = \pi/2$$ [Put $\theta/2 = t$, so that $d\theta = 2dt$]

$\therefore$ $$\int_0^\infty \frac{\sin^2 t}{t^2} = \pi/2.$$

PROBLEMS 22.1

1. Express the function $f(x) = \begin{cases} 1 \text{ for } |x| \le 1 \\ 0 \text{ for } |x| > 1 \end{cases}$ as a Fourier integral.

   Hence evaluate $\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\, d\lambda$. (*Kottayam, 2005*)

2. Find the Fourier integral representation for

   (*i*) $f(x) = \begin{cases} 1-x^2, \text{ for } |x| \le 1 \\ 0, \text{ for } |x| > 1 \end{cases}$ (*Mumbai, 2008*) (*ii*) $f(x) = \begin{cases} e^{ax}, \text{ for } x \le 0, a > 0 \\ e^{-ax}, \text{ for } x \ge 0\ a < 0 \end{cases}$

3. Using the Fourier integral representation, show that

   (*i*) $\int_0^\infty \frac{\omega\sin x\omega}{1+\omega^2}\, d\omega = \frac{\pi}{2}e^{-x}\ (x > 0)$ (*ii*) $\int_0^\infty \frac{\cos\omega x}{1+\omega^2}\, d\omega = \frac{\pi}{2}e^{-x}\ (x \ge 0)$ (*U.P.T.U., 2008*)

   (*iii*) $\int_0^\infty \frac{\sin\omega\cos x\omega}{\omega}\, d\omega = \frac{\pi}{2}$ when $0 \le x < 1$. (*iv*) $\int_0^\infty \frac{\sin\pi\alpha\sin\alpha\theta}{1-\alpha^2}\, d\alpha = \begin{cases} \frac{1}{2}\pi\sin\theta, & 0 \le \theta \le \pi \\ 0, & \theta > \pi \end{cases}$

4. Find the Fourier transforms of

   (*i*) $f(x) = \begin{cases} 1, |x| < a \\ 0, |x| > a \end{cases}$ (*W.B.T.U., 2005 ; Madras, 2003 ; P.T.U., 2003*)

   Hence evaluate $\int_{-\infty}^\infty \frac{\sin ax}{x}\, dx$ (*Mumbai, 2009*)

   (*ii*) $f(x) = \begin{cases} x^2, |x| < a \\ 0, |x| > a \end{cases}$ (*S.V.T.U., 2008*)

5. Find the Fourier transform of $f(x) = \begin{cases} a^2 - x^2 \text{ for } |x| \le a \\ 0 \text{ for } |x| > a \end{cases}$ (*V.T.U., 2007*)

   Hence deduce that $\int_0^\infty \frac{\sin t - t\cos t}{t^3}\, dt = \frac{\pi}{4}$. (*Anna, 2009*)

6. Given $F(e^{-x^2}) = \sqrt{\pi}e^{-s^2/4}$, find the Fourier transform of

   (*i*) $e^{-x^2/3}$ (*ii*) $e^{-4(x-3)^2}$

7. Find the Fourier sine and cosine transforms of $f(x) = \begin{cases} 1, 0 \le x < 2 \\ 0, x \ge 2 \end{cases}$ (*V.T.U., 2008*)

8. Using the Fourier sine transform of e^{-ax} $(a > 0)$, show that $\int_0^\infty \frac{x\sin kx}{a^2+x^2}\, dx = \frac{\pi}{2}e^{-ak}\ (k > 0)$.

   Hence obtain the Fourier sine transform of $x/(a^2 + x^2)$. (*Rohtak, 2006 ; Madras, 2003 S*)

9. Find the Fourier cosine transform of e^{-ax}. (*Anna, 2009*)

   Hence evaluate $\int_0^\infty \frac{\cos\lambda x}{x^2+\alpha^2}\, dx$. (*V.T.U., 2003 S*)

10. If the Fourier sine transform of $f(x)$ is e^{-as}/s, find $f(x)$. Hence obtain the inverse Fourier sine transform of $1/s$. (*Mumbai, 2009*)

11. Find the Fourier cosine transform of e^{-x^2} and hence evaluate Fourier sine transform of xe^{-x^2}.
12. Find the Fourier cosine transform of $e^{-a^2x^2}$ for any $a > 0$ and hence prove that $e^{-x^2/2}$ is self-reciprocal under Fourier cosine transform. *(Anna, 2009)*
13. Find the Fourier sine transform of (*i*) $\dfrac{1}{x(x^2+a^2)}$. *(Rohtak, 2006)*
(*ii*) $[e^{-ax}/x]$, $a > 0$ *(U.P.T.U., 2008)*
14. Obtain Fourier sine transform of

(*i*) $f(x) = \begin{cases} \sin x, & 0 < x < a \\ 0, & x > a \end{cases}$ *(Madras, 2000)* (*ii*) $f(x) = \begin{cases} 4x, & \text{for } 0 < x < 1 \\ 4 - x, & \text{for } 1 < x < 4 \\ 0, & \text{for } x > 4 \end{cases}$ *(V.T.U., 2006)*

15. Find the Fourier cosine transform of $(1 - x/\pi)^2$. *(P.T.U., 2006)*
16. Find the finite Fourier sine and cosine transforms of $f(x) = 2x$, $0 < x < 4$. *(V.T.U., 2011)*
17. Find the finite sine transform of $f(x) = \begin{cases} -x, & x < c \\ \pi - x, & x > c \end{cases}$ where $0 \le c \le \pi$.
18. Show that the inverse finite Fourier sine transform of $F_s(n) = \dfrac{1}{\pi}\left\{1 + \cos n\pi - 2\cos\dfrac{n\pi}{2}\right\}$ is

$$f(x) = \begin{cases} 1, & 0 < x < \pi/2 \\ -1, & \pi/2 < x < \pi \end{cases}$$ *(V.T.U., 2008)*

19. Solve the integral equation $\displaystyle\int_0^\infty f(x)\sin tx\,dx = \begin{cases} 1, & 0 \le t < 1, \\ 2, & 1 \le t < 2, \\ 0, & t \ge 2 \end{cases}$ *(Kottayam, 2005)*
20. Solve the integral equation $\displaystyle\int_0^\infty f(x)\cos \alpha x\,dx = e^{-\alpha}$. *(S.V.T.U., 2009 ; Rohtak, 2004)*

22.6 (1) CONVOLUTION

The convolution of two functions f (x) and g(x) over the interval (– ∞, ∞) is defined as

$$f * g = \int_{-\infty}^{\infty} f(u)\, g(x-u)du = h(x).$$

(2) **Convolution theorem for Fourier transforms.** *The Fourier transform of the convolution of f (x) and g(x) is the product of their Fourier transforms, i.e.,*

$$F\{f(x) * g(x)\} = F\{f(x)\} \,.\, F\{g(x)\}$$

We have
$$F\{f(x) * g(x)\} = F\left\{\int_{-\infty}^{\infty} f(u)\, g(x-u)du\right\}$$

$$= \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(u)\, g(x-u)du\right\} e^{isx}\,dx = \int_{-\infty}^{\infty} f(u)\left\{\int_{-\infty}^{\infty} g(x-u)\,.\,e^{isx}\,dx\right\}du$$

[Changing the order of integration]

$$= \int_{-\infty}^{\infty} f(u)\left\{\int_{-\infty}^{\infty} e^{is(x-u)}\,.\,g(x-u)\,d(x-u)\right\} e^{isu}\,du$$

$$= \int_{-\infty}^{\infty} e^{isu} f(u)\left\{\int_{-\infty}^{\infty} e^{ist}\, g(t)\,dt\right\} du \text{ where } x - u = t$$

$$= \int_{-\infty}^{\infty} e^{isu} f(u)\,du \,.\, F\{g(t)\} = \int_{-\infty}^{\infty} e^{isx} f(x)\,dx \,.\, F\{g(x)\} = F\{f(x)\}\,.\,F\{g(x)\}$$

22.7 PARSEVAL'S IDENTITY FOR FOURIER TRANSFORMS

If the Fourier transforms of f (x) and g(x) are F(s) and G(s) respectively, then

(*i*) $\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} F(s)\overline{G}(s)ds = \int_{-\infty}^{\infty} f(x)\overline{g}(x)dx$ (*ii*) $\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} |F(s)|^2\,ds = \int_{-\infty}^{\infty} |f(x)|^2\,dx$

where bar implies the complex conjugate.

(*i*) $\int_{-\infty}^{\infty} f(x)\,\overline{g}\,(dx) \int_{-\infty}^{\infty} f(x)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{G}(s)\,e^{isx}\,ds\right\}dx$ [Using the inversion formula for Fourier transform]

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{G}(s)\left\{\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx\right\}ds$$ [Changing the order of integration]

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{G}(s)F(s)ds$, by definition of F-transform.

(*ii*) Taking $g(x) = f(x)$, we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\,\overline{F}\,(s)ds = \int_{-\infty}^{\infty} f(x)\,\overline{f}(x)dx \text{ or } \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(s)|^2\,ds = \int_{-\infty}^{\infty} |f(x)|^2\,dx$$

Obs. The following Parseval's identities for Fourier cosine and sine transforms can be proved as above :

(*i*) $\frac{2}{\pi}\int_0^{\infty} F_c(s)\,G_c(s)ds = \int_0^{\infty} f(x)\,g(x)dx$ (*ii*) $\frac{2}{\pi}\int_0^{\infty} F_s(s)\,G_s(s)ds = \int_0^{\infty} f(x)\,g(x)dx$

(*iii*) $\frac{2}{\pi}\int_0^{\infty} |F_c(s)|^2\,ds = \int_0^{\infty} |f(x)|^2\,dx$ (*iv*) $\frac{2}{\pi}\int_0^{\infty} |F_s(s)|^2\,ds = \int_0^{\infty} |f(x)|^2\,dx$.

Example 22.14. *Using Parseval's identities, prove that*

(*i*) $\int_0^{\infty} \frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab(a+b)}$ *(S.V.T.U., 2009 : [illegible] 2008)*

(*ii*) $\int_0^{\infty} \frac{t^2}{(t^2+1)^2}\,dt = \frac{\pi}{4}$ (*iii*) $\int_0^{\infty} \frac{\sin at}{t(a^2+t^2)}\,dt = \frac{\pi}{2}\cdot\frac{1-e^{-a^2}}{a^2}$.

Solution. (*i*) Let $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$. Then $F_c(s) = \frac{a}{a^2+s^2}, G_c(s) = \frac{b}{b^2+s^2}$

Now using Parseval's identity for Fourier cosine transforms, *i.e.*,

$$\frac{2}{\pi}\int_0^{\infty} F_c(s)G_c(s)ds = \int_0^{\infty} f(x)\,g(x)dx \quad \text{...(1)}$$

We have
$$\frac{2}{\pi}\int_0^{\infty} \frac{ab}{(a^2+s^2)(b^2+s^2)}\,ds = \int_0^{\infty} e^{-(a+b)x}dx$$

or
$$\frac{2ab}{\pi}\int_0^{\infty} \frac{ds}{(a^2+s^2)(b^2+s^2)} = \left|\frac{e^{-(a+b)x}}{-(a+b)}\right|_0^{\infty} = \frac{1}{a+b}$$

Thus
$$\int_0^{\infty} \frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab(a+b)}$$

(*ii*) Let $f(x) = \frac{x}{x^2+1}$ so that $F_s[f(x)] = \frac{\pi}{2}e^{-s}$

Now using Parseval's identity for sine transform, *i.e.*,

$$\frac{2}{\pi}\int_0^{\infty} [F_s\{f(x)\}]^2\,ds = \int_0^{\infty} [f(x)]^2\,dx$$

or
$$\int_0^{\infty}\left(\frac{x}{x^2+1}\right)^2 dx = \frac{2}{\pi}\int_0^{\infty}\left(\frac{\pi}{2}e^{-s}\right)^2 ds = \frac{\pi}{2}\left| e^{-2s}/-2\right|_0^{\infty} = \frac{\pi}{-4}(0-1) = \frac{\pi}{4}$$

Hence
$$\int_0^{\infty} \frac{t^2}{(t^2+1)^2}\,dt = \frac{\pi}{4}$$

(*iii*) Let $f(x) = e^{-ax}$ and $g(x) = \begin{cases} 1, & 0 < x < a \\ 0, & x > a \end{cases}$. Then $F_c(s) = \frac{a}{a^2+s^2}, G_c(s) = \frac{\sin as}{s}$

Now using (1) above, we have $\frac{2}{\pi}\int_0^{\infty}\frac{a\sin as}{s(a^2+s^2)}\,ds=\int_0^{a}e^{-ax}\,.1\,dx=\frac{1-e^{-a^2}}{a}$

Thus $$\int_0^{\infty}\frac{\sin at}{t(a^2+t^2)}\,dt=\frac{\pi}{2a^2}(1-e^{-a^2}).$$

Example 22.15. *Find the Fourier transform of $f(x)$ given by $f(x)=\begin{cases}a-|x|, & for\ |x|<a\\ 0 & , for\ |x|>a\end{cases}$.*

Hence show that $\int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt=\frac{\pi}{2}$ and $\int_0^{\infty}\left(\frac{\sin t}{t}\right)^4 dt=\pi/3$. (Anna, 2008)

Solution. Fourier transform of $f(x)$ *i.e.* $F[f(x)]=\int_{-\infty}^{\infty}f(x)\,e^{isx}\,dx=\int_{-a}^{a}[a-|x|]e^{isx}\,dx$

$$=\int_{-a}^{a}[a-|x|](\cos x+i\sin sx)dx$$

$$=2\int_0^{a}(a-x)\cos sx\,dx+0 \qquad \left\{\because \begin{array}{l}[a-|x|]\cos x \text{ in an even function}\\ [a-|x|]\sin x \text{ is an odd function}\end{array}\right.$$

$$=2\left|(a-x)\frac{\sin sx}{s}-\frac{\cos sx}{s^2}\right|_0^{a}=2\,\frac{1-\cos as}{s^2}=4\,\frac{\sin^2 as/2}{s^2}$$

(*i*) By inversion formula,

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(s)\,e^{-isx}\,ds=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4\sin^2 as/2}{s^2}\,e^{-isx}\,ds$$

To evaluate $\int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt$, put $x=0$ and $a=2$ so that

$$f(0)=\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\sin^2 s}{s^2}\,ds=\frac{4}{\pi}\int_0^{\infty}\left(\frac{\sin s}{s}\right)^2 ds \qquad \left[\because \frac{\sin s}{s} \text{ is an even function}\right]$$

$$\therefore \quad \int_0^{\infty}\left(\frac{\sin s}{s}\right)^2 ds=\frac{\pi}{4}f(0)=\frac{\pi}{2}. \qquad [\because f(0)=a=2$$

(*ii*) Using Parseval's identity

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}[F(s)]^2 ds=\int_{-\infty}^{\infty}|f(x)|^2\,dx$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{4\sin^2 as/2}{s^2}\right)^2 dx=\int_{-a}^{a}|[a-|x|]|^2\,dx$$

$$\frac{16}{\pi}\int_0^{\infty}\left(\frac{\sin as/2}{s}\right)^4 ds=2\int_0^{a}(a-x)^2\,dx=2\left|\frac{(a-x)^3}{-3}\right|_0^{a}=\frac{2}{3}a^3$$

Putting $t=as/2$ and $dt=ads/2$

$$\frac{16}{\pi}\int_0^{\infty}\left(\frac{\sin t}{2t/a}\right)\frac{2}{a}\,dt=\frac{2}{3}a^3 \quad \text{or} \quad \frac{2a^3}{\pi}\int_0^{\infty}\left(\frac{\sin t}{t}\right)^4 dt=\frac{2}{3}a^3$$

Hence $$\int_0^{\infty}\left(\frac{\sin t}{t}\right)^4 dt=\frac{\pi}{3}.$$

PROBLEMS 22.2

1. Verify Convolution theorem for $f(x) = g(x) = e^{-x^2}$. (V.T.U., 2000 S)
2. Use Convolution theorem to find the inverse Fourier transform of $\frac{i}{(1+s^2)^2}$, given that $\frac{2}{(1+s^2)}$ is the Fourier transform of $e^{-|x|}$. (V.T.U., 2010 S)
3. Using Parseval's identity, show that

   (i) $\int_0^\infty \frac{dx}{(t^2+1)^2} = \frac{\pi}{4}$, (Hissar, 2007) (ii) $\int_0^\infty \frac{t^2}{(4+t^2)(9+t^2)} dt = \frac{\pi}{10}$. (Rohtak, 2003)
4. Find the Fourier transform of $f(x) = \begin{cases} 1 \text{ if } |x| < 1 \\ 0 \text{ if } |x| > 1 \end{cases}$.

   Hence deduce that $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$. (Anna, 2009)
5. Evaluate $\int_0^\infty \left(\frac{1-\cos x}{x}\right)^2 dx$.

22.8 RELATION BETWEEN FOURIER AND LAPLACE TRANSFORMS

If $$f(t) = \begin{cases} e^{-xt} g(t), & t > 0 \\ 0, & t < 0 \end{cases} \qquad ...(i)$$

then $$F\{f(t)\} = L\{g(t)\}.$$

We have $$F\{f(t)\} = \int_{-\infty}^{\infty} e^{ist} f(t) dt = \int_{-\infty}^{0} e^{ist} . 0 . dt + \int_0^\infty e^{ist} . e^{-xt} g(t) dt$$

$$= \int_0^\infty e^{(is-x)t} g(t)\, dt = \int_0^\infty e^{-pt} g(t) dt \quad \text{where } p = x - is$$

Hence *the Fourier transform of $f(t)$* [defined by (*i*)] is the *Laplace transform of $g(t)$*.

22.9 FOURIER TRANSFORMS OF THE DERIVATIVES OF A FUNCTION

The Fourier transform of the function $u(x, t)$ is given by

$$F[u(x, t)] = \int_{-\infty}^{\infty} ue^{isx} dx$$

Then the Fourier transform of $\partial^2 u/\partial x^2$, i.e.

$$F\left[\frac{\partial^2 u}{\partial x^2}\right] = \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{isx} dx = \left| e^{isx} \frac{\partial u}{\partial x} - is\, e^{isx} . u \right|_{-\infty}^{\infty} + (is)^2 \int_{-\infty}^{\infty} ue^{isx} dx,$$

on applying the general rule of integration by parts (p. 398). If u and $\frac{\partial u}{\partial x}$ tend to zero as x tends to $\pm\infty$, then

$$\mathbf{F}\left[\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}\right] = -\mathbf{s}^2 \mathbf{F}[\mathbf{u}] \qquad ...(1)$$

Similarly in the case of Fourier sine and cosine transforms, we have

$$\mathbf{F_s}\left[\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}\right] = \mathbf{s}(\mathbf{u})_{\mathbf{x}=0} - \mathbf{s}^2 \mathbf{F_s}[\mathbf{u}] \qquad ...(2)$$

and $$\mathbf{F_c}\left[\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}\right] = -\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)_{\mathbf{x}=0} - \mathbf{s}^2 \mathbf{F_c}[\mathbf{u}] \qquad ...(3)$$

In general, the *Fourier transform of the nth derivative of $f(x)$* is given by

$$\mathbf{F}\left[\frac{\mathbf{d^n f}}{\mathbf{dx^n}}\right] = (-\mathbf{is})^{\mathbf{n}}\, \mathbf{F[f(x)]} \qquad ...(4)$$

provided the first $n-1$ derivatives vanish as $x \to \pm\infty$.

For $\quad F[f^n(x)] = \int_{-\infty}^{\infty} f^n(x)\, e^{isx}\, dx$

$$= \left| e^{isx} f^{n-1} - ise^{isx} f^{n-2} + (is)^2 e^{isx} f^{n-3} - ... \right|_{-\infty}^{\infty} + (-is)^n \int_{-\infty}^{\infty} f \,.\, e^{isx}\, dx$$

by the general rule of integration by parts, whence follows (4).

22.10 INVERSE LAPLACE TRANSFORMS BY METHOD OF RESIDUES

Let the Laplace transform of $f(x)$ be $\bar{f}(s)$ so that

$$\bar{f}(s) = \int_0^{\infty} f(t)e^{-st}\, dt \qquad ...(1)$$

Multiply both sides by e^{xs} and integrate w.r.t. s within the limits $a - ir$ and $a + ir$. Then

$$\int_{a-ir}^{a+ix} e^{xs}\, \bar{f}(s)ds = \int_{a-ir}^{a+ir} e^{xs} \int_0^{\infty} f(t)e^{-st}\, dt\, ds \qquad [\text{Put } s = a - iu]$$

$$= \int_r^{-r} e^{x(a-iu)} \int_0^{\infty} f(t)e^{-(a-iu)t}\, dt\,(-idu) = ie^{ax} \int_{-r}^{r} e^{-ixu} \int_0^{\infty} [e^{-at} f(t)]e^{iut}\, dtdu$$

$$= ie^{ax} \int_{-r}^{r} e^{-ixu} \int_{-\infty}^{\infty} \phi(t)\, e^{iut}\, dt\, du$$

where $\phi(t) = \begin{cases} e^{-at} f(t) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$.

Proceeding to limits as $r \to \infty$, we get

$$\int_{a-i\infty}^{a+i\infty} e^{xs}\, \bar{f}(s)\, ds = ie^{ax}\,.\, 2\pi\phi(x), \text{ by (2) of § 22.4} = 2\pi i e^{ax}.e^{-ax} f(x) \text{ for } x > 0.$$

Hence $\quad f(x) = \int_{a-i\infty}^{a+i\infty} e^{xs}\, \bar{f}(s)ds \ (x > 0) \qquad ...(2)$

which is called the *complex inversion formula*. It provides a direct means for obtaining the inverse Laplace transform of a given function.

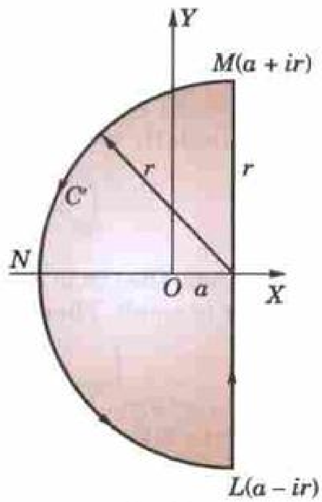

Fig. 22.1

The integration in (2) is performed along a line LM parallel to the imaginary axis in the complex plane $z = x + iy$ such that all the singularities of $\bar{f}(s)$ lie to its left* (Fig. 22.1). Let us take a contour C which is composed of the line LM and the semi-circle C' (*i.e.*, MNL). Then from (2)

$$\frac{1}{2\pi i} \int_{LM} e^{xs}\, \bar{f}(s)\, ds = \frac{1}{2\pi i} \int_C e^{xs}\, \bar{f}(s)\, ds - \frac{1}{2\pi i} \int_{C'} e^{xs}\, \bar{f}(s)\, dx$$

The integral over C' tends to zero as $r \to \infty$ (under certain conditions†). Therefore,

$$\mathbf{f(x)} = \underset{\mathbf{r \to \infty}}{\mathbf{Lt}}\ \frac{\mathbf{1}}{\mathbf{2\pi i}} \int_{\mathbf{C}} \mathbf{e^{xs}\, \bar{f}(s)\, dx}$$

$$= \textbf{sum of the residues of } e^{xs}\bar{f}(s) \textbf{ at the poles of } f(s) \qquad ...(3)$$

[By §20.18]

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* This has been so assumed simply to ensure the convergence of the integral (1).

† If positive constants A and k can be so found that $|\bar{f}(s)| < Ar^{-k}$ for every point on C', then

$$\underset{r \to \infty}{\text{Lt}}\ \frac{1}{2\pi i} \int_{C'} e^{xs} \bar{f}(s)\, ds = 0. \qquad (\textit{Jordan's Lemma})$$

Example 22.16. *Evaluate* $L^{-1}\left\{\dfrac{1}{(s-1)(s^2+1)}\right\}$ *by the method of residues.*

Solution. Since $\left|\dfrac{1}{(s-1)(s^2+1)}\right| \sim \left|\dfrac{1}{s^3}\right|$ for $|s| \to \infty$, therefore,

$$L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right] = \text{sum of Res}\left[\frac{e^{xs}}{(s-1)(s^2+1)}\right] \text{at the poles } s = 1, \pm i$$

Now
$$(\text{Res})_{s=1} = \underset{s\to 1}{\text{Lt}}\left[\frac{(s-1)\,.\,e^{xs}}{(s-1)(s^2+1)}\right] = \frac{e^x}{2} \qquad \text{[By § 20.19 (1)]}$$

$$(\text{Res})_{s=i} = \underset{s\to i}{\text{Lt}}\left[\frac{(s-i)\,.\,e^{xs}}{(s-1)(s^2+1)}\right] = \frac{e^{ix}}{(i-1)(i-1)} = -\frac{1}{2}\cdot\frac{e^{ix}}{1+i}$$

Changing i to $-i$, we get $(\text{Res})_{s=-i} = -\dfrac{1}{2}\cdot\dfrac{e^{ix}}{1-i}$

$$\therefore \qquad L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right] = \frac{e^x}{2} - \frac{1}{2}\left(\frac{e^{ix}}{1+i} + \frac{e^{-ix}}{1-i}\right) = \frac{1}{2}(e^x - \sin x - \cos x).$$

Example 22.17. *Prove that* $L^{-1}\left(\dfrac{e^{-c\sqrt{s}}}{s}\right) = 1 - erf\left(\dfrac{c}{\sqrt{2x}}\right)$.

Solution. By the complex inversion formula,

$$L^{-1}\left(\frac{e^{-c\sqrt{s}}}{s}\right) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{xs}\,.\,\frac{e^{-c\sqrt{s}}}{s}\,ds.$$

Since $s = 0$ is a branch point of the integrand, we take a contour $LMNPQST$ as shown in Fig. 22.2, so that it doesn't include any singularity. Therefore, by Cauchy's theorem (§ 20.13), we have

$$\left\{\int_{LM} + \int_{MN} + \int_{NP} + \int_{PQS} + \int_{ST} + \int_{TL}\right\} \times e^{xs}\,\frac{e^{-c\sqrt{s}}}{s}\,ds = 0 \qquad ...(i)$$

If $ON = \rho$ and $OP = \varepsilon$, then along NP, $s = Re^{i\pi}$, therefore,

$$\int_{NP} = \int_{\rho}^{\varepsilon} e^{-xR}\,\frac{e^{-ic\sqrt{R}}}{R}\,dR$$

Similarly along ST, $s = Re^{-i\pi}$, therefore,

$$\int_{ST} = \int_{\varepsilon}^{\rho} e^{-xR}\,\frac{e^{c\sqrt{R}}}{R}\,dR$$

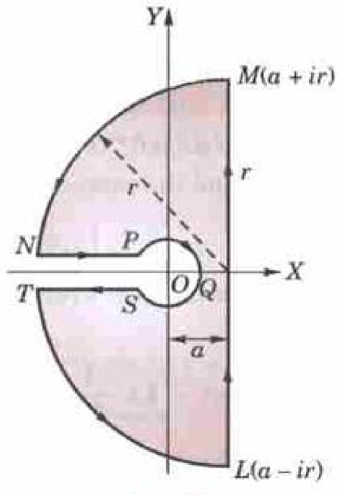

Fig. 22.2

Along the circle PQS, $s = \varepsilon\, e^{i\theta}$. Also e^{xs} and $e^{-c\sqrt{\sigma}}$ are both approximately 1 since ε is small. Therefore,

$$\int_{PQS} = \int_{\pi}^{-\pi} \frac{1}{\varepsilon e^{i\theta}}\,.\,\varepsilon e^{i\theta}\, i d\theta = -2\pi i \text{ approximately.}$$

For $c > 0$, $|\,e^{-c\sqrt{s}}/s\,| < |s|^{-1}$.

But $\int_{MN}$ and $\int_{TL}$ both tend to zero as $r \to \infty$

Thus (i) takes the form

$$\int_{a-ir}^{a+ir} \frac{e^{xs-c\sqrt{s}}}{s}\,ds + \int_{\varepsilon}^{\rho} e^{-xR}\,\frac{e^{ic\sqrt{R}} - e^{-ic\sqrt{R}}}{R}\,dR - 2\pi i = 0$$

Taking limits as $\varepsilon \to 0$ and $\rho \to \infty$, we get

$$\int_{a-i\infty}^{a+i\infty} \frac{e^{xs-c\sqrt{s}}}{s}\,ds = 2\pi i - 2i\int_0^{\infty} e^{-xR}\,\frac{\sin c\sqrt{R}}{R}\,dR$$

or $$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{xs-c\sqrt{s}}}{s}\,ds = 1-\frac{2}{\pi}\int_0^{\infty} e^{-t^2}\,\frac{\sin(ct/\sqrt{x})}{t}\,dt^*, \text{ where } R = t^2/x$$

$$= 1-\frac{2}{\pi}\cdot\frac{\pi}{2}\,erf\left(\frac{C}{2\sqrt{x}}\right) \text{ whence follows the result.}$$

PROBLEMS 22.3

Using the method of residues, evaluate the inverse Laplace transform of each of the following :

1. $\dfrac{1}{(s+1)(s-2)^2}$.
2. $\dfrac{1}{(s-2)(s^2+1)}$.
3. $\dfrac{1}{s^2(s^2-a^2)}$.
4. $\dfrac{1}{(s-1)^2(s^2+1)}$.
5. $\dfrac{1}{(s^2+1)^2}$. (*V.T.U., 2008 S*)

22.11 APPLICATION OF TRANSFORMS TO BOUNDARY VALUE PROBLEMS

In one dimensional boundary value problems, the partial differential equation can easily be transformed into an ordinary differential equation by applying a suitable transform. The required solution is then obtained by solving this equation and inverting by means of the complex inversion formula or by any other method. In two dimensional problems, it is sometimes required to apply the transforms twice and the desired solution is obtained by double inversion.

(*i*) If in a problem $u(x, t)_{x=0}$ is given then we use infinite sine transform to remove $\partial u^2/\partial x^2$ from the differential equation.

In case $[\partial u(x, t)/\partial x]_{x=0}$ is given then we employ infinite cosine transform to remove $\partial^2 u/\partial x^2$.

(*ii*) If in a problem $u(0, t)$ and $u(l, t)$ are given, then we use finite sine transform to remove $\partial^2 u/\partial x^2$ from the differential equation.

In case $(\partial u/\partial x)_{x=0}$ and $(\partial u/\partial x)_{x=l}$ are given, then we employ finite cosine transform to remove $\partial^2 u/\partial x^2$.

The method of solution is best explained through the following examples.

Heat conduction

Example 22.18. *Determine the distribution of temperature in the semi-infinite medium $x \geq 0$, when the end $x = 0$ is maintained at zero temperature and the initial distribution of temperature is $f(x)$.* (*Osmania, 2003*)

Solution. Let $u(x, t)$ be the temperature at any point x and at any time t. We have to solve the heat-flow equation (§ 18.5)

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}\,(x > 0, t > 0) \qquad ...(i)$$

subject to the initial condition $u(x, 0) = f(x)$ ...(*ii*)

and the boundary condition $u(0, t) = 0$ ...(*iii*)

Taking Fourier sine transform of (1) and denoting $F_s[u(x, t)]$ by $\bar{u}_s$, we have

$$\frac{d\bar{u}_s}{dt} = c^2\,[su(0, t) - s^2\bar{u}_s] \qquad \text{[By (2) of § 22.9]}$$

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* We know that $\int_0^{\infty} e^{-t^2}\cos 2mt\,dt = \frac{1}{2}\sqrt{\pi}e^{-m^2}$ [Example 20.44

Integrating both sides w.r.t. m from 0 to $c/2\sqrt{x}$.

$$\int_0^{\infty} e^{-t^2}\left|\frac{\sin 2mt}{2t}\right|_0^{c/2\sqrt{x}} dt = \frac{1}{2}\sqrt{\pi}\int_0^{c/2\sqrt{x}} e^{-m^2}\,dm$$

or $$\int_0^{\infty} e^{-t^2}\,\frac{\sin(ct/\sqrt{x})}{t}\,dt = \frac{\pi}{2}\,erf\left(\frac{c}{2\sqrt{x}}\right). \qquad \text{[By § 7.18(1)]}$$

or $$\frac{d\bar{u}_s}{dt} + c^2s^2\bar{u}_s = 0 \qquad \text{[By (iii)]} \quad ...(iv)$$

Also the Fourier sine transform of (*ii*) is $\bar{u}_s = \bar{f}(s)$ at $t = 0$. ...(*v*)

Solving (*iv*) and using (*v*), we get $\bar{u}_s = \bar{f}_s(s)e^{-c^2s^2t}$

Hence taking its inverse Fourier sine transform, we obtain

$$u(x, t) = \frac{2}{\pi}\int_0^\infty \bar{f}_s(s)e^{-c^2s^2t} \sin xs\, ds.$$

Example 22.19. *Solve $\partial u/\partial t = 2\partial^2 u/\partial x^2$, if $u(0, t) = 0$, $u(x, 0) = e^{-x}$ $(x > 0)$, $u(x, t)$ is bounded where $x > 0$, $t > 0$.* (*Rohtak, 2006*)

Solution. Given $\partial u/\partial t = 2\partial^2 u/\partial x^2, x > 0, t > 0$ ...(*i*)

with boundary conditions : $u(0, t) = 0$, $u(x, t)$ is bounded ...(*ii*)

and initial condition $u(x, 0) = e^{-x}, x > 0$ ...(*iii*)

Since $u(0, t)$ is given, we take Fourier sine transform of both sides of (*i*) so that

$$\int_0^\infty \frac{\partial u}{\partial t} \sin px\, dx = 2\int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin px\, dx$$

or $$\frac{d}{dt}\int_0^\infty u(x, t) \sin px\, dx = 2\left[\left|\frac{\partial u}{\partial x}\sin px\right|_0^\infty - \int_0^\infty \frac{\partial u}{\partial x}\,.\,p\cos px\, dx\right] \qquad \text{(Integrating by parts)}$$

or $$\frac{d\bar{u}_s}{dt} = -2p\int_0^\infty \frac{\partial u}{\partial x}\cos px\, dx, \text{ if } \frac{\partial u}{\partial x} \to \frac{\partial u}{\partial x} \to 0 \text{ as } x \to \infty \text{ where } \bar{u}_s(p, t) = \int_0^\infty u(x, t)\sin px\, dx$$

$$= -2p\left[|u(x, t)\cos px|_0^\infty - \int_0^\infty u(x, t) - (-p\sin px)\, dx\right] \qquad \text{[Again integrating by parts]}$$

$$= -2p\left[0 - u(0, t) + p\int_0^\infty u(x, t)\sin px\, dx\right] \qquad [\because\ u(x, t) \to 0 \text{ as } x \to \infty \text{ by } (ii)]$$

$$= 2pu(0, t) - 2p^2\bar{u}_s$$

or $$\frac{d\bar{u}_s}{dt} = -2p^2\bar{u}_s \qquad \text{[By } (ii)]$$

Integrating $\int \frac{\partial \bar{u}_s}{\bar{u}_s} - \log c = -2p^2 \int dt$ or $\log \bar{u}_s - \log c = -2p^2t$

$\therefore$ $$\bar{u}_s(p, t) = ce^{-2p^2t} \qquad ...(iv)$$

Taking Fourier sine transform of both sides of (*iii*), we get

$$\int_0^\infty u(x, 0)\sin px\, dx = \int_0^\infty e^{-x}\sin px\, dx$$

or $$\bar{u}_s(p, 0) = \left|\frac{e^{-x}}{1+p^2}(-\sin px - p\cos px)\right|_0^\infty = \frac{p}{1+p^2} \qquad ...(v)$$

Putting $t = 0$ in (*iv*) and using (*v*), we obtain $p/(1 + p^2) = c$

Thus (*iv*) becomes $\bar{u}_s(p, t) = \dfrac{p}{1+p^2}e^{-2p^2t}$

Now taking inverse Fourier sine transform, we get

$$u(x, t) = \frac{2}{\pi}\int_0^\infty \frac{pe^{-2p^2t}}{1+p^2}\sin px\, dp.$$

Example 22.20. *Solve the equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$, $(x > 0, t > 0)$ subject to the conditions*

(i) $u = 0$, when $x = 0$, $t > 0$ (ii) $u = \begin{cases} 1, & 0 < x < 1 \\ 0, & x \le 1, \text{ when } t = 0 \end{cases}$ (iii) $u(x, t)$ is bounded. (*U.P.T.U., 2003 S*)

Solution. Since $u(0, t) = 0$, we take Fourier sine transform of both sides of the given equation, we get

$$\int_0^\infty \frac{\partial u}{\partial t} \sin sx \, dx = \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin sx \, dx$$

$$\frac{\partial}{\partial t} \int_0^\infty u \sin sx \, dx = -s^2 \, \bar{u}(s) + s\, u(0) \qquad [\because u = 0, \text{ when } x = 0]$$

or $$\frac{\partial \bar{u}}{\partial t} = -s^2 \bar{u} \quad \text{or} \quad \frac{\partial \bar{u}}{\partial t} + s^2 \bar{u} = 0 \quad \text{or} \quad (D^2 + s^2)\bar{u} = 0 \text{ i.e., } D = \pm is$$

$\therefore$ Its solution is $\bar{u}(s, t) = e^{-s^2 t}$ ...(1)

Since $$\bar{u}(s, t) = \int_0^\infty u(x, t) \sin sx \, dx$$

$\therefore$ $$\bar{u}(s, 0) = \int_0^\infty u(x, 0) \sin sx \, dx = \int_0^1 1 . \sin sx \, dx \qquad [\text{By } (ii)]$$

$$= \frac{1 - \cos s}{s} \qquad ...(2)$$

From (1) and (2), $$c = \bar{u}(s, 0) = \frac{1 - \cos s}{s}$$

Thus (1) gives $$\bar{u}(s, t) = \frac{1 - \cos s}{s} e^{-s^2 t}$$

Now taking inverse Fourier sine transform, we get

$$u(x, t) = \int_0^\infty \frac{1 - \cos s}{s} e^{-s^2 t} \, ds$$

which is the desired solution.

Example 22.21. *Using finite Fourier transform, solve*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

given $u(0, t) = 0$, $u(4, t) = 0$ *and* $u(x, 0) = 2x$ *where* $0 < x < 4$, $t > 0$. *(Rajasthan, 2006)*

Solution. Since $u(0, t) = 0$, we take finite Fourier sine transform of both sides of the given equation

$$\int_0^4 \frac{\partial u}{\partial t} \sin \frac{n\pi}{4} x \, dx = \int_0^4 \frac{\partial^2 u}{\partial x^2} \sin \frac{n\pi}{4} x \, dx$$

or $$\frac{d}{dt}(\bar{u}s) = F_s\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$$= -\frac{n^2\pi^2}{16} \bar{u}_s + \frac{n\pi}{4}[u(0, t) - (-1)^n u(4, t)]$$

$$= -\frac{n^2\pi^2}{16} \bar{u}_s \qquad [\because u(0, t) = 0, u(4, t) = 0.]$$

or $$\frac{d\bar{u}_s}{\bar{u}_s} = -\frac{n^2\pi^2}{16} dt$$

Integrating both sides, $$\log \bar{u}_s = -\frac{n^2\pi^2}{16} t + c$$

or $$\bar{u}_s(x, 0) = \alpha e^{\frac{-n^2\pi^2 t}{16}} \qquad ...(i)$$

Putting $t = 0$, $$a = \bar{u}_s(x, 0) = \int_0^4 u(x, 0) \sin \frac{n\pi x}{4} dx \qquad [\because \; u(x, 0) = 2x]$$

$$= \int_0^4 2x \sin \frac{n\pi x}{4} dx = -\frac{32}{n\pi} \cos n\pi$$

Thus (*i*) gives, $\bar{u}_s(x, 0) = -\frac{32}{n\pi} \cos n\pi e^{-n^2\pi^2 t/16} = -\frac{32}{n\pi}(-1)^n e^{-n^2\pi^2 t/16}$

Now taking inverse Fourier sine transform, we get

$$u(x, 0) = \frac{2}{4} \sum_{n=1}^{\infty} \frac{32}{n\pi} (-1)^{n+1} e^{-n^2\pi^2 t/16} \sin\left(\frac{n\pi x}{4}\right)$$

$$= 16 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi} e^{-n^2\pi^2 t/16} \sin\left(\frac{n\pi x}{4}\right).$$

Example 22.22. *If the initial temperature of an infinite bar is given by*

$$\theta(x) = \begin{cases} \theta_0 \text{ for } |x| < a \\ 0 \text{ for } |x| > a, \end{cases}$$

determine the temperature at any point x and at any instant t. (*S.V.T.U., 2008 ; Rohtak, 2004*)

Solution. To determine the temperature $\theta(x, t)$ at any point at any time, we have to solve the equation

$$\frac{\partial \theta}{\partial t} = c^2 \frac{\partial^2 \theta}{\partial x^2} \quad (t > 0) \qquad ...(i)$$

subject to the initial condition $\theta(x, 0) = \begin{cases} \theta_0 \text{ for } |x| < a \\ 0 \text{ for } |x| > a \end{cases}$ ...(*ii*)

Taking Fourier transform of (*i*) and denoting $F[\theta(x, t)]$ by $\bar{\theta}$, we find

$$\frac{d\bar{\theta}}{dt} = -c^2 s^2 \bar{\theta} \qquad \text{[by (1) of § 22.9] ...(iii)}$$

Also the Fourier transform of (2) is

$$\bar{\theta}(s, 0) = \int_{-\infty}^{\infty} \theta(x, 0) e^{isx} dx = \int_{-a}^{a} \theta_0 e^{isx} dx = \theta_0 \frac{e^{isa} - e^{-isa}}{is} = 2\theta_0 \frac{\sin as}{s} \qquad ...(iv)$$

Solving (*iii*) and using (*iv*), we get $\bar{\theta} = \frac{2\theta_0 \sin as}{s} e^{-c^2 s^2 t}$

Hence taking its inverse Fourier transform, we get

$$\theta(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\theta_0 \sin as}{s} e^{-c^2 s^2 t} e^{-isx} ds = \frac{\theta_0}{\pi} \int_{-\infty}^{\infty} \frac{\sin as}{s} e^{-c^2 s^2 t} (\cos xs - i \sin xs) ds$$

$$= \frac{2\theta_0}{\pi} \int_0^{\infty} \frac{\sin as}{s} e^{-c^2 s^2 t} \cos xs \, ds \qquad \left\{\begin{array}{l}\text{The second integral vanishes as} \\ \text{its integrand is an odd function}\end{array}\right.$$

$$= \frac{\theta_0}{\pi} \int_0^{\infty} e^{-c^2 s^2 t} \frac{\sin (a + x)s + \sin (a - x)s}{s} ds$$

$$= \frac{\theta_0}{\pi} \int_0^{\infty} e^{-v^2} \left\{\sin \frac{(a + x)v}{c\sqrt{t}} + \sin \frac{(a - x)v}{c\sqrt{t}}\right\} \frac{dv}{v} \text{ where } v^2 = c^2 s^2 t$$

$$= \frac{\theta_0}{\pi} \left\{erf \frac{(a + x)}{2c\sqrt{t}} + erf \frac{(a - x)}{2c\sqrt{t}}\right\}.$$

[See footnote on p. 783]

Example 22.23. *A bar of length a is at zero temperature. At t = 0, the end x = a is suddenly raised to temperature u_0 and the end x = 0 is insulated. Find the temperature at any point x of the bar at any time t > 0, assuming that the surface of the bar is insulated.*

Solution. Here we have to solve the differential equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (0 < x < a, t > 0) \qquad ...(i)$$

subject to the conditions

$u(x, 0) = 0$ ...(*ii*) ; $u_x(0, t) = 0$ ...(*iii*) and $u(a, t) = u_0$ (*Rohtak, 2005*) ...(*iv*)

The Laplace transform of (i), if $L[u(x,t)] = \bar{u}(x,s)$, is

$$s\bar{u} - u(x,0) = c^2 \frac{d^2\bar{u}}{dx^2}$$

Using (ii), we get $$\frac{d^2\bar{u}}{dx^2} - \frac{s}{c^2}\bar{u} = 0 \qquad ...(v)$$

Similarly the Laplace transform of (iii) and (iv) are

$$\bar{u}_x(0,s) = 0 \quad ...(vi)\,; \qquad \bar{u}(a,s) = \frac{u_0}{s} \quad ...(vii)$$

Solving (v), we have $$\bar{u} = C_1 e^{x\sqrt{s}/c} + C_2 e^{-x\sqrt{s}/c}$$

Using (vi), we find $C_1 = C_2$ so that

$$\bar{u} = C_1(e^{\sqrt{s}x/c} + e^{-\sqrt{s}x/c}) = 2C_1 \cosh(\sqrt{s}x/c)$$

Now using (vii), we have $$\bar{u} = \frac{u_0 \cosh(\sqrt{s}x/c)}{s \cosh(\sqrt{s}a/c)}$$

By the inversion formula (3) § 22.10, we get

$$u(x,t) = \text{sum of the residues of } \left(\frac{e^{st} \cdot u_0 \cosh(\sqrt{s}x/c)}{s \cosh(\sqrt{s}a/c)}\right) \text{ at all the poles which occur at } s = 0$$

and $$\cosh(\sqrt{s}a/c) = 0 \text{ i.e., at } s = 0,\ \sqrt{s}a/c = \left(n - \frac{1}{2}\right)\pi i,\ n = 0, \pm 1, \pm 2, \ldots$$

or at $$s = 0,\ s\,(= s_n) = -\frac{(2n-1)^2 c^2 \pi^2}{4a^2} = 0, 1, 2, \ldots$$

Now $$(\text{Res})_{s=0} = \underset{s \to 0}{\text{Lt}} \left\{ s \cdot \frac{u_0 e^{st} \cosh(\sqrt{s}x/c)}{s \cosh(\sqrt{s}a/c)} \right\} = u_0$$

$$(\text{Res})_{s=s_n} = u_0 \underset{s \to s_n}{\text{Lt}} \left\{ (s - s_n) \cdot \frac{u_0 e^{st} \cosh(\sqrt{s}x/c)}{s \cosh(\sqrt{s}a/c)} \right\}$$

$$= u_0 \underset{s \to s_n}{\text{Lt}} \left\{ \frac{s - s_n}{\cosh(\sqrt{s}a/c)} \right\} \cdot \underset{s \to s_n}{\text{Lt}} \left\{ \frac{e^{st} \cosh(\sqrt{s}x/c)}{s} \right\} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= u_0 \underset{s \to s_n}{\text{Lt}} \frac{1}{\sinh(\sqrt{s}a/c) \cdot (a/2\sqrt{s}/c)} \cdot \underset{s \to s_n}{\text{Lt}} \left\{ \frac{e^{st} \cosh(\sqrt{s}x/c)}{s} \right\}$$

$$= \frac{4u_0(-1)^n}{(2n-1)\pi} e^{-(2n-1)^2 \pi^2 c^2 t/4a^2} \cos \frac{(2n-1)\pi x}{2a}$$

Thus we get $$u(x,t) = u_0 + \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} e^{-(2n-1)^2 \pi^2 c^2 t/4a^2} \cos \frac{(2n-1)\pi x}{2a}.$$

Vibrations of a string

Example 22.24. *An infinite string is initially at rest and that the initial displacement is $f(x)$, $(-\infty < x < \infty)$. Determine the displacement $y(x, t)$ of the string.* (Rohtak, 2000)

Solution. The equation for the vibration of the string is

$$\partial^2 y/\partial t^2 = c^2 \partial^2 y/\partial x^2 \qquad ...(i)$$

and the initial conditions are

$$(\partial y/\partial t)_{t=0} = 0\ ;\ y(x,0) = f(x) \qquad ...(ii)$$

Multiplying (i) by e^{isx} and integrating w.r.t. x from $-\infty$ to ∞, we get

$$\partial^2 Y/\partial t^2 = c^2(-s^2 Y) \qquad \text{provided } y \text{ and } \partial y/\partial x \to 0 \text{ as } x \to \infty$$

$\therefore$ a solution of $d^2Y/dt^2 + c^2s^2Y = 0$ is $Y = A_1 \cos cst + A_2 \sin cst$ ...(*iii*)

Also Fourier transforms of (*ii*) are

$$\partial y/\partial t = 0 \quad \text{and} \quad Y = F(s) \text{ when } t = 0$$

Applying these to (*iii*), we get

$$A_2 = 0 \quad \text{and} \quad A_1 = F(s)$$

Thus $\qquad Y = F(s) \cos cst$

Now taking inverse Fourier transforms, we get

$$y(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) \cos cst \,.\, e^{-isx}\, dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) \frac{e^{icst} + e^{-icst}}{2} \,.\, e^{-isx}\, dx$$

$$= \frac{1}{4\pi} \int_{-\infty}^{\infty} [F(s) e^{-is(x-ct)} + F(s) e^{-is(x+ct)}]\, ds$$

$$= \frac{1}{2} [f(x - ct) + f(x + ct)] \qquad \left[\because \ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{-isx} ds\right.$$

Example 22.25. *An infinitely long string having one end at x = 0, is initially at rest along the x-axis. The end x = 0 is given a transverse displacement f(t), t > 0. Find the displacement of any point of the string at any time.*

Solution. Let $y(x, t)$ be the transverse displacement of any point x of the string at any time t. Then we have to solve the wave equation (§ 18.4)

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad (x > 0, t > 0) \qquad ...(i)$$

subject to the conditions $y(x, 0) = 0$, $y_t(x, 0) = 0$, $y(0, t) = f(t)$ and the displacement $y(x, t)$ is bounded.

The Laplace transform of (*i*), writing $L[y(x, t)] = \bar{y}(x, s)$ is

$$s^2 \bar{y} - sy(x, 0) - \frac{\partial y(x, 0)}{\partial t} = c^2 \frac{\partial^2 \bar{y}}{\partial x^2}$$

Using the first two conditions, we have

$$\frac{\partial^2 \bar{y}}{\partial x^2} = \left[\frac{s}{c}\right]^2 \bar{y} \qquad ...(ii)$$

Similarly the Laplace transforms of the third and fourth conditions are

$\bar{y}(0, s) = \bar{f}(s)$ at $x = 0$ ...(*iii*) and $\bar{y}(x, s)$ is bounded. ...(*iv*)

Solving (*ii*), we get

$$\bar{y}(x, s) = C_1 e^{sx/c} + C_2 e^{-sx/c}$$

To satisfy condition (*iv*), we must have $C_1 = 0$

Using the condition (*iii*), we get $C_2 = \bar{f}(s)$.

$\therefore \qquad \bar{y}(x, s) = \bar{f}(s) e^{-sx/c}$

Using the complex inversion formula, we obtain

$$y = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{(t - x/c)s}\, \bar{f}(s)\, ds = f(t - x/c).$$

Example 22.26. *A tightly stretched flexible string has its ends fixed at x = 0 and x = l. At time t = 0, the string is given a shape defined by F(x) = μx(l − x), where μ is a constant and then released. Find the displacement of any point x of the string at any time t > 0.* (V.T.U., M.E., 2006)

Solution. We have to solve the wave equation $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ $\qquad (x > 0, t > 0)$

subject to the conditions $y(0, t) = 0, y(l, t) = 0$

and $y(x, 0) = \mu x(l - x), y_t(x, 0) = 0$

Now taking Laplace transform, writing $L[y(x, t)] = \bar{y}(x, s)$, we get

$$s^2\bar{y} - s\bar{y}(x, 0) - \frac{\partial y(x, 0)}{\partial t} = c^2 \frac{\partial^2 \bar{y}}{\partial x^2} \quad ...(i)$$

where $\bar{y}(0, s) = 0,\ \bar{y}(l, s) = 0$ ...(ii)

$\therefore$ (i) reduces to $\dfrac{\partial^2 \bar{y}}{\partial x^2} - \left(\dfrac{s}{c}\right)^2 \bar{y} = -\dfrac{\mu sx(l - x)}{c^2}$

Its solution is $\bar{y}(x, s) = c_1 \cosh(sx/c) + c_2 \sinh(sx/c) + \dfrac{\mu x(l - x)}{s} - \dfrac{2c^2\mu}{s^3}$

Applying the conditions (ii), we get

$$c_1 = 2c^2\mu/s^2 \quad \text{and} \quad c_2 = \frac{2c^2\mu}{s^3}\left[\frac{1 - \cosh(sl/c)}{\sinh(sl/c)}\right] - \frac{2c^2\mu}{s^3} \tanh(s/2c)$$

Thus $$\bar{y}(x, s) = \frac{2c^2\mu}{s^3}\left[\frac{\cosh\{s(2x - l)/2c\}}{\cosh(sl/2c)}\right] + \frac{\mu x(l - x)}{s} - \frac{2c^2\mu}{s^3}$$

Now using the inversion formula (3) § 22.10, we get

$y(x, t)$ = sum of the residues of

$$2c^2\mu\left[e^{st}\frac{\cosh\{s(2x - l)/2c\}}{s^3\cosh(sl/2c)}\right] \text{ at all the poles } + \mu x(l - x) - c^2\mu t^2$$

Proceeding exactly as in Example 22.23, we have,

sum of the residues of $2c^2\mu\left[\dfrac{e^{st}\cosh\{s(2x - l)/2c\}}{s^3\cosh sl/2c}\right]$ at all the poles

$$= c^2\mu\left[t^2 + \left(\frac{2x - l}{2c}\right)^2 - \left(\frac{l}{2c}\right)^2\right]$$

$$- \frac{32c^2\mu}{\pi^3}\left(\frac{l}{2c}\right)^2 \sum_{n=1}^{\infty}\left[\frac{(-1)^n}{(2n - 1)^3}\cos\left\{\frac{(2n - 1)\pi(2x - l)}{2l}\right\}\cos\left\{\frac{(2n - 1)\pi ct}{l}\right\}\right]$$

$$= c^2\mu t^2 - \mu x(l - x) + \frac{8\mu l^2}{\pi^3}\sum_{n=1}^{\infty}\left[\frac{1}{(2n - 1)^3}\sin\frac{(2n - 1)\pi x}{l}\cos\frac{(2n - 1)\pi ct}{l}\right]$$

Hence $$y(x, t) = \frac{8\mu l^2}{\pi^3}\sum_{n=1}^{\infty}\left[\frac{1}{(2n - 1)^3}\sin\frac{(2n - 1)\pi x}{l}\cos\frac{(2n - 1)\pi ct}{l}\right].$$

Transmission lines

Example 22.27. *A semi-infinite transmission line of negligible inductance and leakance per unit length has its voltage and current equal to zero. A constant voltage v_0 is applied at the sending end ($x = 0$) at $t = 0$. Find the voltage and current at any point ($x > 0$) and at any instant.*

Solution. Let $v(x, t)$ and $i(x, t)$ be the voltage and current at any point x and at any time t. If $L = 0$ and $G = 0$, then the transmission line equations [(1) and (2) of § 18.10] become

$$\frac{\partial v}{\partial x} = -Ri, \frac{\partial i}{\partial x} = -C\frac{\partial v}{\partial t} \quad i.e., \quad \frac{\partial^2 v}{\partial x^2} = RC\frac{\partial v}{\partial t} \quad ...(i)$$

The boundary conditions are $v(0, t) = v_0$ and $i(x, t)$ is finite for all x and t.

The initial conditions are $v(x, 0) = 0, i(x, 0) = 0$. ...(ii)

Laplace transforms of (*i*), are

$$\frac{d^2\bar{v}}{dx^2} = RC(s\bar{v} - 0) \quad \text{or} \quad \frac{d^2\bar{v}}{dx^2} - RCs\bar{v} = 0 \qquad ...(iii)$$

Laplace transforms of the conditions in (*ii*), are

$$\bar{v}(0, s) = \frac{v_0}{s} \text{ at } x = 0 \qquad ...(iv)$$

and $\bar{v}(x, s)$ remains finite as $x \to \infty$ ...(*v*)

$\therefore$ the solution of (*iii*) is

$$\bar{v}(x, s) = C_1 e^{\sqrt{RCs}\,x} + C_2 e^{-\sqrt{RCs}\,x}$$

To satisfy condition (*v*), we must have $C_1 = 0$.

Using the condition (*iv*), we get $C_2 = v_0/s$

Thus $$\bar{v}(x, s) = \frac{v_0}{s} e^{-\sqrt{RCs}\,x}$$

Using the inversion formula, we obtain

$$v(x, t) = v_0 L^{-1}\left\{\frac{e^{-\sqrt{RC}\,x\,.\,\sqrt{s}}}{s}\right\} = v_0\, erfc\left(x\frac{\sqrt{RC}}{2\sqrt{t}}\right) \qquad \text{[By Ex. 22.17]}$$

$$= v_0 \frac{x\sqrt{RC}}{2\sqrt{\pi}} \int_0^t u^{-3/2} e^{-(RC x^2/4u)}\, du$$

$\therefore$ since $i = -\dfrac{1}{R}\dfrac{\partial v}{\partial x}$, we obtain by differentiation,

$$i(x, t) = \frac{v_0 x}{2\sqrt{x}} \sqrt{\frac{C}{R}}\, t^{-3/2} e^{(-RC x^2/4t)}.$$

Example 22.28. *A transmission line of length l has negligible inductance and leakance. A constant voltage v_0 is applied at the sending end ($x = 0$) and is open circuited at the far end. Assuming the initial voltage and current to be zero, determine the voltage and current.*

Solution. For a transmission line with $L = G = 0$, the voltage v and current i are given by the equations

$$\frac{\partial^2 v}{\partial x^2} = RC\frac{\partial v}{\partial t} \text{ and } \frac{\partial v}{\partial x} + Ri = 0 \qquad ...(i)$$

The boundary conditions are (for $t > 0$)

$$v = v_0 \text{ at } x = 0 \text{ and } i = \frac{\partial v}{\partial x} = 0 \quad \text{at } x = l \qquad ...(ii)$$

The initial condition is $v = 0$ at $t = 0$ ($x > 0$)

Laplace transforms of (*i*) and (*ii*) are

$$\frac{\partial^2 \bar{v}}{\partial x^2} = RC(s\bar{v} - 0) \qquad ...(iii)$$

and $$\bar{v} = v_0/s \text{ at } x = 0,\ \frac{\partial \bar{v}}{\partial x} = 0 \text{ at } x = l \qquad ...(iv)$$

$\therefore$ the solution of (*iii*) is

$$\bar{v} = c_1 \cosh\sqrt{(RCs)}\,x + c_2 \sinh\sqrt{RCs}\,x$$

Applying conditions (*iv*), it gives

$$v_0/s = c_1,\ 0 = c_1 \sinh\sqrt{(RCs)}\,l + c_2 \cosh\sqrt{(RCs)}\,l$$

$$\therefore \quad \bar{v} = \frac{v_0}{s}\left[\cosh\sqrt{(RCs)}x - \frac{\sinh\sqrt{(RCs)}l}{\cosh\sqrt{(RCs)}l}\sinh\sqrt{(RCs)}x\right]$$

$$= \frac{v_0}{s}\frac{\cosh pq\sqrt{s}}{\cosh p\sqrt{s}}$$ where $p = \sqrt{(RC)}\,l$ and $q = (l - x)/l$

By the inversion formula (3) § 22.10, we get

$$v(x, t) = \text{sum of the residues of } (e^{st}\,\bar{v}) \text{ at all poles of } e^{st}\,\bar{v}\,. \quad ...(iv)$$

These poles are at $s = 0$ and $p\sqrt{s} = \pm i(2n-1)\pi/2 = \pm ipk$ (say)

Now $$Res\,(e^{st}\,\bar{v})_{s=0} = \underset{s\to 0}{\text{Lt}}\ \frac{se^{st}\,v_0\cosh pq\sqrt{s}}{s\cosh p\sqrt{s}} = v_0$$

and $$Res(e^{st}\bar{v})_{s=-k^2} = \underset{s\to -k^2}{\text{Lt}}\ \frac{(s+k^2)e^{st}\,v_0\cosh pq\sqrt{s}}{s\cosh p\sqrt{s}} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{s\to -k^2}{\text{Lt}}\ \frac{v_0\,.e^{st}\cosh pq\sqrt{s} + (s+k^2)(...)}{\cosh p\sqrt{s} + s\sinh p\sqrt{s}\,.\,\frac{1}{2}ps^{-1/2}}$$

$$= \frac{v_0\,e^{-k^2t}\cosh(ipqk) + 0}{0 + 1/2\,(ipk)\sinh(ipk)} = \frac{2v_0e^{-k^2t}\cos(pqk)}{-pk\sin pk}$$

Adding up all the residues, (iv) gives

$$v(x, t) = v_0 + \frac{4v_0}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}e^{-[(2n-1)^2\pi^2t/4RCl^2]}\cos[(2n-1)\pi(l-x)/2l]$$

$[\because\ pk = (2n-1)\pi/2,\ -\sin pk = (-1)^n,\ pqk = \frac{1}{2}(2n-1)\pi(l-x)/l,$
$k^2 = (2n-1)^2\pi^2/4RCl^2]$

Also $$i = -\frac{1}{R}\frac{\partial v}{\partial x}.$$ [By (i)]

PROBLEMS 22.4

1. Solve the differential equation using Laplace transform method, $\dfrac{\partial y}{\partial t} = 3\dfrac{\partial^2 y}{\partial x^2}$
where $y(\pi/2, t) = 0$, $(\partial y/\partial x)_{x=0} = 0$ and $y(x, 0) = 30\cos 5x$. (U.P.T.U., 2005)

2. Using suitable transforms, solve the differential equation $\dfrac{\partial^2 V}{\partial x^2} = \dfrac{\partial V}{\partial t}$, $0 \le x \le \pi$, $t \ge 0$.
where $V(0, t) = 0 = V(\pi, t)$ and $V(x, 0) = V_0$ constant.

3. The initial temperature along the length of an infinite bar is given by $u(x, 0) = \begin{cases} 2, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$. If the temperature $u(x, t)$ satisfies the equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$, $-\infty < x < \infty$, $t > 0$, find the temperature at any point of the bar at any point t. (Rohtak, 2006)

4. Use the complex form of the Fourier transform to show that
$$V = \frac{1}{2\sqrt{(\pi t)}}\int_{-\infty}^{\infty}\bar{f}(u)e^{[-(x-u)^2/4t]}\,du$$
is the solution of the boundary value problem
$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2},\ -\infty < x < \infty,\ t > 0\ ;\ V = f(x) \text{ when } t = 0.$$ (U.P.T.U., 2008)

5. A semi-infinite solid $(x > 0)$ is initially at temperature zero. At time $t = 0$, a constant temperature $\theta_0 > 0$ is applied and maintained at the face $x = 0$. Show that the temperature at any point x and at any time t, is given by $\theta(x, t) = \theta_0\,erfc(x/2c\sqrt{t})$.

6. A solid is initially at constant temperature θ_0, while the ends $x = 0$ and $x = a$ are maintained at temperature zero. Determine the temperature at any point of the solid at any later time $t > 0$.
7. An infinite string is initially at rest along the x-axis. Its one end which is at $x = 0$, is given a periodic transverse displacement $a_0 \sin \omega t$, $t > 0$. Show that the displacement of any point of the string at any time is given by
$$y(x, t) = \begin{cases} a_0 \sin \omega(t - x/c), & t > x/c \\ 0, & t < x/c, \end{cases}$$
where c is the wave velocity.
8. An infinite string has an initial transverse displacement $y(x, 0) = f(x)$, $-\infty < x < \infty$, and is initially at rest. Show that
$$y(x, t) = \frac{1}{2}[f(x + ct) + f(x - ct)].$$
9. A semi-infinite transmission line has negligible inductance and leakance per unit length. A voltage v is applied at the sending end $(x = 0)$ which is given by
$$v(0, t) = \begin{cases} v_0, & 0 < t < \tau \\ 0, & t > \tau \end{cases}$$
Show that the voltage at any point $x > 0$ at any time $t > 0$ is given by
$$v(x, t) = v_0 \, erfc\left[\frac{x}{2}\sqrt{(RC/t)}\right].$$

22.12 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 22.5

Fill in the blanks or choose the correct answer in each of the following problems :

1. Fourier cosine transform of $f(t)$ is ....
2. Fourier sine transform of $1/x$ is ....
3. Convolution theorem for Fourier transforms states that ....
4. If Fourier transform of $f(x)$ is $F(s)$, then the inversion formula is ....
5. $F[x^n f(x)] =$ ....
6. If $F\{f(x)\} = F(s)$, then $F\{f(x - a)\} =$ ....
7. Fourier sine integral representation of a function $f(x)$ is given by ....
8. If $F_c[f(ax)] = k\, F_c(s/a)$, then $k =$ ....
9. Fourier transform of second derivative of $u(x, t)$ is ....
10. If $f(x) = \begin{cases} 1, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$, then Fourier sine integral of $f(x)$ is ....
11. Fourier sine transform of $f'(x)$ in the interval $(0, l)$ is ....
12. If $F(\lambda)$ is the Fourier transform of $f(x)$, then the Fourier transform of $f(ax)$ is ....
13. Inverse finite Fourier sine transform of $F_s(p) = \dfrac{1 - \cos p\pi}{(p\pi)^2}$ for $p = 1, 2, 3, \ldots$ and $0 < x < \pi$ is ....
14. If Fourier transform of $f(x) = F(s)$, then Fourier Transform of $f(2x)$ is ....
15. Fourier cosine transform of e^{-x} is ....
16. $f(x) = 1$, $0 < x < \infty$ cannot be represented by a Fourier integral. (True or False)
17. $\int_0^\infty |f(x)|^2\, dx = \int_0^\infty |F_c(s)|^2\, dx.$ (True or False)
18. Fourier transform is a linear operation. (True or False)
19. $F_s[x f(x)] = -\dfrac{d}{ds} F_c(s)$. (True or False)
20. Kernel of Fourier transform is e^{isx}. (True or False)
21. Finite Fourier cosine transform of $f(x) = 1$ in $(0, \pi)$ is zero. (True or False)

CHAPTER 23

Z-Transforms

23.1 INTRODUCTION

The development of communication branch is based on discrete analysis. Z-transform plays the same role in discrete analysis as Laplace transform in continuous systems. As such, Z-transform has many properties similar to those of the Laplace transform (§ 21.2). The main difference is that the Z-transform operates not on functions of continuous arguments but on sequences of the discrete integer-valued arguments, *i.e.* $n = 0, \pm 1, \pm 2,$ ...... The analogy of Laplace transform to Z-transform can be carried further. For every operational rule of Laplace transforms, there is a corresponding operational rule of Z-transforms and for every application of the Laplace transform, there is a corresponding application of Z-transform. A discrete system is expressible as a difference equation (§ 30.2) and its solutions are found using Z-transforms.

23.2 DEFINITION

If the function u_n is defined for discrete values ($n = 0, 1, 2, \ldots$) and $u_n = 0$ for $n < 0$, then its Z-transform is defined to be

$$Z(u_n) = U(z) = \sum_{n=0}^{\infty} u_n z^{-n} \text{ whenever the infinite series converges.} \qquad \ldots(i)$$

The inverse Z-transform is written as $Z^{-1}[U(z)] = u_n$.

If we insert a particular complex number z into the power series (i), the resulting value of $Z(u_n)$ will be a complex number. Thus the Z-transform $U(z)$ is a complex valued function of a complex variable z.

23.3 SOME STANDARD Z-TRANSFORMS

The direct application of the definition gives the following results :

(1) $Z(a^n) = \dfrac{z}{z-a}$ **(2)** $Z(n^p) = -z \dfrac{d}{dz} Z(n^{p-1})$, *p being a +ve integer.*

Proof. **(1)** By definition, $Z(a^n) = \sum_{n=0}^{\infty} a^n z^{-n}$

$$= 1 + (a/z) + (a/z)^2 + (a/z)^3 + \ldots = \frac{1}{1-(a/z)} = \frac{z}{z-a}$$ *(Kottayam, 2005)*

(2) $$Z(n^p) = \sum_{n=0}^{\infty} n^p z^{-n} = z \sum_{n=0}^{\infty} n^{p-1} . n . z^{-(n+1)} \quad \ldots(i)$$

Changing p to $p - 1$, we get $Z(n^{p-1}) = \sum_{n=0}^{\infty} n^{p-1} . z^{-n}$

Differentiating it w.r.t. z,

$$\frac{d}{dz}[Z(n^{p-1})] = \sum_{n=0}^{\infty} n^{p-1} . (-n) z^{-(n+1)} \quad \ldots(ii)$$

Substituting (*ii*) in (*i*), we obtain $Z(n^p) = -z\frac{d}{dz}[Z(n^{p-1})]$

which is the desired *recurrence formula*.

In particular, we have the following formulae :

(3) $Z(1) = \frac{z}{z-1}$ [Taking $a = 1$ in (1) **(4)** $Z(n) = \frac{z}{(z-1)^2}$ [Taking $p = 1$ in (2)]

(5) $Z(n^2) = \frac{z^2+z}{(z-1)^3}$ *(V.T.U., 2006)* **(6)** $Z(n^3) = \frac{z^3+4z^2+z}{(z-1)^4}$

(7) $Z(n^4) = \frac{z^4+11z^3+11z^2+z}{(z-1)^5}$.

23.4 LINEARITY PROPERTY

If a, b, c be any constants and u_n, v_n, w_n be any discrete functions, then

$$Z(au_n + bv_n - cw_n) = aZ(u_n) + bZ(v_n) - cZ(w_n)$$

Proof. By definition, $Z(au_n + bv_n - cw_n) = \sum_{n=0}^{\infty} (au_n + bv_n - cw_n)z^{-n}$

$$= a\sum_{n=0}^{\infty} u_n z^{-n} + b\sum_{n=0}^{\infty} v_n z^{-n} - c\sum_{n=0}^{\infty} w_n z^{-n}$$

$$= aZ(u_n) + bZ(v_n) - cZ(w_n).$$

23.5 DAMPING RULE

If $Z(u_n) = U(z)$, then $Z(a^{-n} u_n) = U(az)$

Proof. By definition, $Z(a^{-n} u_n) = \sum_{n=0}^{\infty} a^{-n} u_n . z^{-n} = \sum_{n=0}^{\infty} u_n . (az)^{-n} = U(az)$. *(Madras, 2006)*

Cor. $\mathbf{Z(a^n u_n) = U(z/a)}$

Obs. The geometric factor a^{-n} when $|a| < 1$, damps the function u_n, hence the name *damping rule*.

23.6 SOME STANDARD RESULTS

The application of the damping rule leads to the following standard results :

(1) $Z(na^n) = \frac{az}{(z-a)^2}$ **(2)** $Z(n^2a^n) = \frac{az^2+a^2z}{(z-a)^3}$

(3) $Z(\cos n\theta) = \dfrac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$ (4) $Z(\sin n\theta) = \dfrac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$

(5) $Z(a^n \cos n\theta) = \dfrac{z(z - a\cos\theta)}{z^2 - 2az\cos\theta + a^2}$ (6) $Z(a^n \sin n\theta) = \dfrac{az\sin\theta}{z^2 - 2az\cos\theta + a^2}$.

Proofs. **(1)** We know that $Z(n) = \dfrac{z}{(z-1)^2}$. Applying damping rule, we have

$$Z(na^n) = U(a^{-1}z) = \frac{a^{-1}z}{(a^{-1}z-1)^2} = \frac{az}{(z-a)^2}.$$ (*Madras, 2000 S*)

(2) We know that $Z(n^2) = \dfrac{z^2+z}{(z-1)^3}$. Applying damping rule, we have

$$Z(n^2a^n) = U(a^{-1}z) = \frac{(a^{-1}z)^2 + a^{-1}z}{(a^{-1}z-1)^3} = \frac{a(z^2+az)}{(z-a)^3}.$$

(3) and **(4)** We know that $Z(1) = \dfrac{z}{z-1}$. Applying damping rule, we have

$$Z(e^{-in\theta}) = Z(e^{-i\theta})^n \,.\, 1 = \frac{ze^{i\theta}}{ze^{i\theta}-1} = \frac{z}{z-e^{-i\theta}} = \frac{z(z-e^{i\theta})}{(z-e^{-i\theta})(z-e^{i\theta})}$$

$$= \frac{z(z-\cos\theta) - iz\sin\theta}{z^2 - z(e^{i\theta}+e^{-i\theta})+1} = \frac{z(z-\cos\theta) - iz\sin\theta}{z^2 - 2z\cos\theta + 1}$$

Equating real and imaginary parts, we get (3) and (4). (*V.T.U., 2010 S; Anna, 2009*)

(5) We know that $Z(\cos n\theta) = \dfrac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$. By damping rule, we have

$$Z(a^n\cos n\theta) = \frac{a^{-1}z\,(a^{-1}z - \cos\theta)}{(a^{-1}z)^2 - 2(a^{-1}z)\cos\theta + 1} = \frac{z(z-a\cos\theta)}{z^2 - 2az\cos\theta + a^2}$$ (*V.T.U., 2006*)

Similarly using (4) above, we get (6).

Example 23.1. *Find the Z-transform of the following :*
(i) $3n - 4\sin n\pi/4 + 5a$ *(ii)* $(n+1)^2$ (*V.T.U., 2010*)
(iii) $\sin(3n+5)$. (*V.T.U., 2009 S ; Kottayam, 2005*)

Solution. *(i)* $Z\left(3n - 4\sin\dfrac{n\pi}{4} + 5a\right) = 3Z(n) - 4Z\left(\sin\dfrac{n\pi}{4}\right) + 5a\,Z(1)$ [By Linearty property[

$$= 3\,.\,\frac{z}{(z-1)^2} - 4\,.\,\frac{z\sin n\pi/4}{z^2 - 2z\cos\pi/4 + 1} + 5a\,.\,\frac{z}{z-1}$$ [Using formulae for $Z(1)$, $Z(n)$, $Z(\sin n\theta)$]

$$= \frac{(3-5a)z + 5az^2}{(z-1)^2} - \frac{2\sqrt{2}z}{z^2 - \sqrt{2}z + 1}$$

(ii) $$Z(n+1)^2 = Z(n^2+2n+1) = Z(n^2) + 2Z(n) + Z(1)$$

$$= \frac{z^2+z}{(z-1)^3} + 2\frac{z}{(z-1)^2} + \frac{z}{z-1} = \frac{z^2(2z+1)}{(z-1)^3}$$

(iii) $Z[\sin(3n+5)] = Z(\sin 3n\cos 5 + \cos 3n \sin 5)$

$= \cos 5 . Z(\sin 3n) + \sin 5 . Z(\cos 3n)$ (using formulae for $Z(\sin n\theta)$, $Z(\cos n\theta)$

$$= \cos 5\,.\,\frac{z\sin 3}{z^2 - 2z\cos 3 + 1} + \sin 5\,.\,\frac{z(z-\cos 3)}{z^2 - 2z\cos 3 + 1} = z\,.\,\frac{(z\sin 5 - \sin 2)}{z^2 - 2z\cos 3 + 1}.$$

Example 23.2. *Find the Z-transforms of the following*
(i) e^{an} (ii) ne^{an} (iii) n^2e^{an}.

Solution. (i) Let $u_n = 1$, $e^{an} = (e^{-a})^{-n} = k^{-n}$ where $k = e^{-a}$. By damping rule $Z(k^{-n}\, u_n) = U(kz)$,

$$\therefore \quad Z(e^{an}) = Z(k^{-n}\,.\,1) = U(kz) = \frac{kz}{kz-1} \qquad \left[\because \; U(z) = Z(1) = \frac{z}{z-1}\right]$$

$$= \frac{z}{z-1/k} = \frac{z}{z-e^a}$$

(ii) Let $\quad u_n = n,\ e^{an} = (e^{-a})^{-n} = k^{-n}$ where $k = e^{-a}$

By damping rule, $Z(e^{an}\,.\,n) = Z(k^{-n}\,.\,n) = U(k\,z)$ where $U(z) = Z(n) = \dfrac{z}{(z-1)^2}$

$$\frac{kz}{(kz-1)^2} = \frac{z}{k(z-1/k)^2} = \frac{e^a z}{(z-e^a)^2}$$

(iii) Let $\quad u_n = n^2,\ e^{an} = (e^{-a})^{-n} = k^{-n}$ where $k = e^{-a}$

By damping rule,

$$Z(e^{an}\,.\,n^2) = Z(k^{-n}\,.\,n^2) = U(kz) \quad \text{where} \quad U(z) = Z(n^2) = \frac{z^2+z}{(z-1)^3}$$

$$= \frac{(kz)^2 + kz}{(kz-1)^3} = \frac{z(z+1/k)}{(z-1/(k)^3} = \frac{ze^a(z+e^a)}{(z-e^a)^3}.$$

Example 23.3. *Find the Z-transform of* (i) *cosh nθ* (V.T.U., 2011) (ii) a^n *cosh nθ.*

Solution. (i)
$$Z(\cosh n\theta) = Z\left(\frac{e^{n\theta}+e^{-n\theta}}{2}\right)$$

$$= \frac{1}{2}\left[Z\left\{(e^{-\theta})^{-n}\,.\,1\right\} + Z\left\{(e^{\theta})^{-n}\,.\,1\right\}\right]$$

Apply damping rule to both terms, taking $u_n = 1$.

$$Z(\cosh n\theta) = \frac{1}{2}\left[\frac{ze^{-\theta}}{ze^{-\theta}-1} + \frac{ze^{\theta}}{ze^{\theta}-1}\right] \qquad \left[\because \; z(1) = \frac{z}{z-1}\right]$$

$$= \frac{1}{2}\left\{\frac{2z^2 - z(e^{\theta}+e^{-\theta})}{z^2 - z(e^{\theta}+e^{-\theta})+1}\right\} = \frac{z^2 - z\cosh\theta}{z^2 - 2z\cosh\theta + 1}$$

(ii)
$$Z(a^n \cosh n\theta) = Z[(a^{-1})^{-n}\,.\,\cosh n\theta] \qquad \text{[Apply damping rule using (i)]}$$

$$= \frac{(a^{-1}z)^2 - (a^{-1}\,z)\cosh\theta}{(a^{-1}\,z)^2 - 2(a^{-1}\,z)\cosh\theta + 1} = \frac{z(z-a\cosh\theta)}{z^2 - 2az\cosh\theta + a^2}.$$

Example 23.4. *Find the Z-transforms of*
(i) $e^t \sin 2t$ (Madras, 2003) (ii) $c^k \cos k\alpha$, $(k \geq 0)$ (U.P.T.U., 2004 S)

Solution. (i) We know that $Z(\sin 2t) = \dfrac{z\sin 2}{z^2 - 2z\cos 2 + 1}$ ...(A)

$$\therefore \quad Z(e^t \sin 2t) = Z[(e^{-1})^{-t}\,.\,\sin 2t] \qquad \text{[Apply damping rule, using (A)]}$$

$$= \frac{(e^{-1}\,z)\sin 2}{(e^{-1}z)^2 - 2(e^{-1}\,z)\cos 2 + 1} = \frac{ez\sin 2}{z^2 - 2ez\cos 2 + e^2}.$$

(ii) We know that $\quad Z(\cos k\alpha) = \dfrac{z(z-\cos\alpha)}{z^2 - 2z\cos\alpha + 1}$ ...(B)

$$\therefore \quad Z(c^k \cos k\alpha) = Z[(c^{-1})^{-k}\,.\,\cos k\alpha] \qquad \text{[Apply damping rule, using (B)]}$$

$$= \frac{(c^{-1}\,z)[c^{-1}\,z - \cos\alpha]}{(c^{-1}\,z)^2 - 2(c^{-1}\,z)\cos\alpha + 1} = \frac{z(z-c\cos\alpha)}{z^2 - 2cz\cos\alpha + c^2}.$$

Example 23.5. *Find the Z-transforms of*

(*i*) $\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)$ (*V.T.U., 2011 S*) (*ii*) $\cosh\left(\frac{n\pi}{2}+\theta\right)$. (*U.P.T.U., 2008*)

Solution. (*i*) $Z\left[\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)\right] = Z\left(\cos\frac{n\pi}{2}\cos\frac{\pi}{4}-\sin\frac{n\pi}{2}\sin\frac{\pi}{4}\right)$

$$= \cos\frac{\pi}{4}.Z\left(\cos\frac{n\pi}{2}\right)-\sin\frac{\pi}{2}.Z\left(\sin\frac{n\pi}{2}\right) \quad \text{[Using formulae for } Z(\sin n\alpha) \text{ and } Z(\cos n\alpha)\text{]}$$

$$= \frac{1}{\sqrt{2}}\left\{\frac{z(z-\cos\pi/2)}{z^2-2z\cos\pi/2+1}-\frac{z\sin\pi/2}{z^2-2z\cos\pi/2+1}\right\} = \frac{1}{\sqrt{2}}\left(\frac{z^2}{z^2+1}-\frac{z}{z^2+1}\right) = \frac{z(z-1)}{\sqrt{2}(z^2+1)}$$

(*ii*)
$$Z\left[\cosh\left(\frac{n\pi}{2}+\theta\right)\right] = Z\left[\frac{e^{n\pi/2+\theta}+e^{-(n\pi/2+\theta)}}{2}\right] = \frac{1}{2}\left[e^{\theta}Z(e^{n\pi/2})+e^{-\theta}Z(e^{-n\pi/2})\right]$$

Since, $Z(a^n) = \frac{z}{z-a}$, $\therefore$ $Z(e^{n\pi/2}) = Z(e^{\pi/2})^n = \frac{z}{z-e^{\pi/2}}$, $Z(e^{-n\pi/2}) = \frac{z}{z-e^{-\pi/2}}$

Thus
$$Z\left[\cosh\left(\frac{n\pi}{2}+\theta\right)\right] = \frac{1}{2}\left\{e^{\theta}.\frac{z}{z-e^{\pi/2}}+e^{-\theta}.\frac{z}{z-e^{-\pi/2}}\right\}$$

$$= \frac{z}{2}\left\{\frac{z(e^{\theta}+e^{-\theta})-[e^{(\pi/2-\theta)}+e^{-(\pi/2-\theta)}]}{z^2-z(e^{\pi/2}+e^{-\pi/2})+1}\right\} = \frac{z^2\cosh\theta-z\cosh\left(\frac{\pi}{2}-\theta\right)}{z^2-2z\cosh\left(\frac{\pi}{2}\right)+1}.$$

Example 23.6. *Find the Z-transform of*

(*i*) ${}^nC_p$ $(0 \le p \le n)$ (*ii*) ${}^{n+p}C_p$.

Solution. (*i*)
$$Z({}^nC_p) = \sum_{p=0}^{n}\left({}^nC_p z^{-p}\right) = 1+{}^nC_1z^{-1}+{}^nC_2z^{-2}+\ldots+{}^nC_nz^{-n} = (1+z^{-1})^n$$

(*ii*)
$$Z({}^{n+p}C_n) = \sum_{p=0}^{n}{}^{n+p}C_pz^{-p}$$
$$= 1+{}^{n+1}C_1z^{-1}+{}^{n+2}C_2z^{-2}+{}^{n+3}C_3z^{-3}+\ldots\infty$$
$$= 1+(n+1)z^{-1}+\frac{(n+2)(n+1)}{2!}z^{-2}+\frac{(n+3)(n+2)(n+1)}{3!}z^{-3}+\ldots\infty$$
$$= 1+(-n-1)(-z^{-1})+\frac{(-n-1)(-n-2)}{2!}(-z^{-1})^2$$
$$+\frac{(-n-1)(-n-2)(-n-3)}{3!}(-z^{-1})^3+\ldots\infty$$
$$= (1-z^{-1})^{-n-1}.$$

Example 23.7. *Find the Z-transform of*

(*i*) *unit impulse sequence* $\delta(n) = \begin{cases}1 & n=0\\ 0 & n\neq 0\end{cases}$ (*ii*) *unit step sequence* $u(n) = \begin{cases}0 & n<0\\ 1 & n\geq 0\end{cases}$.

Solution. (*i*) $Z[\delta(n)] = \sum_{n=0}^{\infty}\delta(n)z^{-n} = 1+0+0+\ldots = 1$

(*ii*) $Z[u(n)] = \sum_{n=0}^{\infty}u(n)z^{-n} = 1+z^{-1}+z^{-2}+z^{-3}+\ldots = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$.

23.7 (1) SHIFTING U_N TO THE RIGHT

If $Z(u_n) = U(z)$, *then* $Z(u_{n-k}) = z^{-k} U(z)$ $(k > 0)$

Proof. By definition,

$$Z(u_{n-k}) = \sum_{n=0}^{\infty} u_{n-k} z^{-n} = u_0 z^{-k} + u_1 z^{-(k+1)} + \ldots = z^{-k} \sum_{n=0}^{\infty} u_n z^{-n} = z^{-k} U(z)$$

Obs. This rule will be very useful in applications to difference equations.

(2) **Shifting u_n to the left.** If $Z(u_n) = U(z)$, then

$$Z(u_{n+k}) = z^k [U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - \ldots - u_{k-1} z^{-(k-1)}]$$

Proof. $$Z(u_{n+k}) = \sum_{n=0}^{\infty} u_{n+k} z^{-n} = z^k \sum_{n=0}^{\infty} u_{n+k} z^{-(n+k)}$$

$$= z^k \left[\sum_{n=0}^{\infty} u_n z^{-n} - \sum_{n=0}^{k-1} u_n z^{-n} \right]$$

Hence $Z(u_{n+k}) = z^k [U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - \ldots - u_{k-1} z^{-(k-1)}]$ (*J.N.T.U., 2002*)

In particular, we have the following standard results :

(1) $Z(u_{n+1}) = z[U(z) - u_0]$; (2) $Z(u_{n+2}) = z^2[U(z) - u_0 - u_1 z^{-1}]$

(3) $Z(u_{n+3}) = z^3 [U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2}]$.

Example 23.8. *Show that* $Z\left(\dfrac{1}{n!}\right) = e^{1/z}$.

Hence evaluate $Z[1/(n+1)!]$ *and* $Z[1/(n+2)!]$. (*Madras, 2006*)

Solution. We have $$Z\left(\frac{1}{n!}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} = 1 + \frac{z^{-1}}{1!} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \ldots = e^{1/z}.$$

Shifting $(1/n!)$ one unit to the left gives

$$Z\left[\frac{1}{(n+1)!}\right] = z\left[Z\left(\frac{1}{n!}\right) - 1\right] = z(e^{-1/z} - 1)$$

Similarly shifting $(1/n!)$ two units to the left gives

$$Z\left[\frac{1}{(n+2)!}\right] = z^2(e^{1/z} - 1 - z^{-1}).$$

23.8 MULTIPLICATION BY n

If $Z(u_n) = u(z)$, then $Z(nu_n) = -z \dfrac{dU(z)}{dz}$

Proof. $$Z(nu_n) = \sum_{n=0}^{\infty} n \,.\, u_n z^{-n} = -z \sum_{n=0}^{\infty} u_n(-n) z^{-n-1} = -z \sum_{n=0}^{\infty} u_n \frac{d}{dz}(z^{-n}).$$

$$= -z \sum_{n=0}^{\infty} \frac{d}{dz}(u_n z^{-n}) = -z \frac{d}{dz}\left(\sum_{n=0}^{\infty} u_n z^{-n}\right) = -z \frac{d}{dz} U(z).$$

Obs. We have, $Z(n^2 u_n) = \left(-z\dfrac{d}{dz}\right)^2 u(z)$ (*Madras, 2006*)

In general, $Z(n^m u_n) = \left(-z\dfrac{d}{dz}\right)^m u(z)$.

Example 23.9. *Find the Z-transform of (i) $n \sin n\theta$ (ii) $n^2 e^{n\theta}$.*

Solution. (*i*) We know that $Z(nu_n) = -z\dfrac{dU(z)}{dz}$ and $Z(\sin n\theta) = \dfrac{z \sin\theta}{z^2 - 2z\cos\theta + 1}$

$$\therefore \quad Z(n \sin n\theta) = -z\frac{d}{dz}[Z(\sin n\theta)] = -z\frac{d}{dz}\left(\frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}\right)$$

$$= -z\frac{\sin\theta - z^2\sin\theta}{(z^2 - 2z\cos\theta + 1)^2} = \frac{z(z^2-1)\sin\theta}{(z^2 - 2z\cos\theta + 1)^2}$$

(*ii*) We know that $Z(e^{n\theta}) = \dfrac{z}{z - e^{\theta}}$

$$\therefore \quad Z(n^2 e^{n\theta}) = \left(-z\frac{d}{dz}\right)^2 (Ze^{n\theta}) = \left(-z\frac{d}{dz}\right)\left[-z\frac{d}{dz}\left(\frac{z}{z-e^{\theta}}\right)\right]$$

$$= \left(-z\frac{d}{dz}\right)\left\{-z\frac{(z-e^{\theta})(1) - z(1)}{(z-e^{\theta})^2}\right\} = -z\frac{d}{dz}\left\{\frac{ze^{\theta}}{(z-e^{\theta})^2}\right\}$$

$$= -ze^{\theta}\left\{\frac{(z-e^{\theta})^2(1) - z[2(z-e^{\theta})]}{(z-e^{\theta})^4}\right\} = -ze^{\theta}\frac{z - e^{\theta} - 2z}{(z-e^{\theta})^3} = \frac{z(z+e^{\theta})e^{\theta}}{(z-e^{\theta})^3}.$$

23.9 TWO BASIC THEOREMS

In applications, we often need the values of u_n for $n = 0$ or as $n \to \infty$ without requiring complete knowledge of u_n. We can find this as the behaviour of u_n for small values of n is related to the behaviour of $U(z)$ as $z \to \infty$ and *vice-versa*. The precise relationship is given by the following *initial and final value theorems :*

(1) Initial value theorem. *If $Z(u_n) = U(z)$, then $u_0 = \underset{z\to\infty}{Lt}\, U(z)$*

Proof. We know that $U(z) = Z(u_n) = u_0 + u_1 z^{-1} + u_2 z^{-2} + \ldots$

Taking limits as $z \to \infty$, we get $\underset{z\to\infty}{\text{Lt}}\,[U(z)] = u_0$, as required.

Similarly additional initial values can be found successively, giving :

$$u_1 = \underset{z\to\infty}{\text{Lt}}\left\{z\,[U(z) - u_0]\right\};\; u_2 = \underset{z\to\infty}{\text{Lt}}\left\{z^2\,[U(z) - u_0 - u_1 z^{-1}]\right\} \text{ and so on.}$$

(2) Final value theorem. *If $Z(u_n) = U(z)$, then*

$$\underset{n\to\infty}{Lt}\,(u_n) = \underset{z\to 1}{Lt}\,(z-1)\,U(z)$$

Proof. By definition, $Z(u_{n+1} - u_n) = \displaystyle\sum_{n=0}^{\infty}(u_{n+1} - u_n)z^{-n}$

or $$Z(u_{n+1}) - Z(u_n) = \sum_{n=0}^{\infty}(u_{n+1} - u_n)z^{-n}$$

or $$z[U(z) - u_0] - U(z) = \sum_{n=0}^{\infty}(u_{n+1} - u_n)z^{-n}$$

or $$U(z)(z-1) - u_0 z = \sum_{n=0}^{\infty}(u_{n+1} - u_n)z^{-n}$$

Taking limits of both sides as $z \to 1$, we get

$$\underset{z\to 1}{\text{Lt}}\,[(z-1)\,U(z)] - u_0 = \sum_{n=0}^{\infty}(u_{n+1} - u_n) = \underset{n\to\infty}{\text{Lt}}\,[(u_1 - u_0) + (u_2 - u_1) + \ldots + (u_{n+1} - u_n)]$$

$$= \underset{n \to \infty}{\text{Lt}} \ [u_{n+1}] - u_0 = u_\infty - u_0$$

Hence $$u_\infty = \underset{z \to 1}{\text{Lt}} \ [(z-1)\,U(z)].$$ *(Anna, 2005 S)*

Example 23.10. *If* $U(z) = \dfrac{2z^2 + 5z + 14}{(z-1)^4}$, *evaluate* u_2 *and* u_3.

Solution. Writing $$U(z) = \frac{1}{z^2} \cdot \frac{2 + 5z^{-1} + 14z^{-2}}{(1 - z^{-1})^4}$$

By initial value theorem, $$u_0 = \underset{z \to \infty}{\text{Lt}} \ U(z) = 0$$

Similarly, $$u_1 = \underset{z \to \infty}{\text{Lt}} \ \{z\,[U(z) - u_0]\} = 0$$

Now $$u_2 = \underset{z \to \infty}{\text{Lt}} \ \{z^2\,[U(z) - u_0 - u_1 z^{-1}]\} = 2 - 0 - 0 = 2$$

and $$u_3 = \underset{z \to \infty}{\text{Lt}} \ z^3[U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2}] = \underset{z \to \infty}{\text{Lt}} \ z^3[U(z) - 0 - 0 - 2z^{-2}]$$

$$= \underset{z \to \infty}{\text{Lt}} \ z^3 \left[\frac{2z^2 + 5z + 14}{(z-1)^4} - \frac{2}{z^2}\right] = \underset{z \to \infty}{\text{Lt}} \ z^3 \left\{\frac{13z^3 + 2z^2 + 8z - 2}{z^2(z-1)^4}\right\} = 13.$$

PROBLEMS 23.1

1. Find the Z-transforms of the following sequences :

   (i) $\dfrac{a^n}{n!}$ $(n \ge 0)$ *(S.V.T.U., 2009)* (ii) $\dfrac{1}{(n+1)!}$ (iii) $(\cos\theta + i\sin\theta)^n$.

2. Using the linearity property, find the Z- transforms of the following functions :

   (i) $2n + 5\sin n\pi/4 - 3a^4$ (ii) $\dfrac{1}{2}(n-1)(n+2)$ *(S.V.T.U., 2007)*

   (iii) $(n+1)(n+2)$ *(Anna, 2008)* (iv) $(2n-1)^2$ *(V.T.U., 2011 S)*

3. Show that (i) $Z(\sinh n\theta) = \dfrac{z \sinh\theta}{z^2 - 2z\cosh\theta + 1}$ *(V.T.U., 2011)* (ii) $Z(a^n \sinh n\theta) = \dfrac{az\sinh\theta}{z^2 - 2az\cosh\theta + a^2}$.

4. Show that (i) $Z(e^{-an}\cos n\theta) = \dfrac{ze^a(ze^a - \cos\theta)}{z^2e^{2a} - 2ze^a\cos\theta + 1}$; (ii) $Z(e^{-an}\sin n\theta) = \dfrac{ze^a \sin\theta}{z^2e^{2a} - 2ze^a\cos\theta + 1}$

   Also evaluate $Z(e^{3n}\sin 2n)$. *(S.V.T.U., 2007)*

5. Using $Z(n^2) = \dfrac{z^2 + z}{(z-1)^3}$, show that $Z(n+1)^2 = \dfrac{z^3 + z^2}{(z-1)^3}$.

6. Find the Z-transforms of (i) $\sin(n+1)\theta$, (ii) $\cos\left(\dfrac{k\pi}{8} + \alpha\right)$. *(Marathwada, 2008)*

7. Find the Z-transform of $\cos n\theta$ and hence find $Z(n\cos n\theta)$. *(Anna, 2009)*

8. Find the Z-transform of $\cos(n\pi/2)$ and $a^n\cos(n\pi/2)$. *(Anna, 2008 S)*

9. Find the Z-transforms of the following

   (i) e^{-an} (ii) e^{-2n} *(V.T.U., 2010 S)* (iii) $e^{-an}n^2$.

10. Show that (i) $Z[\delta(n+1)] = 1/z$ (ii) $(1/2)^n u(n) = \dfrac{2z}{2z-1}$.

11. Show that $Z({}^{n+p}C_p) = (1 - 1/z)^{-(p+1)}$. Using the damping rule, deduce that
    $$Z({}^{n+p}C_p\,a^n) = (1 + a/z)^{-(p+1)}.$$

12. If $Z(u_n) = \dfrac{z}{z-1} + \dfrac{z}{z^2+1}$, find the Z-transform of u_{n+2}. *(S.V.T.U., 2009)*

13. If $U(z) = \dfrac{2z^2 + 3z + 12}{(z-1)^4}$, find the value of u_2 and u_3.

14. Given that $Z(u_n) = \dfrac{2z^2+3z+4}{(z-3)^3}$, $|z|>3$, show that $u_1 = 2$, $u_2 = 21$, $u_3 = 139$.

15. Show that (i) $Z\left(\dfrac{1}{n}\right) = z \log \dfrac{z}{z-1}$. (*Madras, 2003 S*) (ii) $Z\left\{\dfrac{1}{n(n+1)}\right\}$. (*Anna, 2005 S*)

16. Using $Z(n) = \dfrac{z}{(z-1)^2}$, show that $Z(n \cos n\theta) = \dfrac{(z^3+z)\cos\theta - 2z^2}{(z^2-2z\cos\theta+1)^2}$.

23.10 SOME USEFUL Z-TRANSFORMS

Sr. No.	*Sequence u_n ($n \geq 0$)*	*Z-transform $U(z) = Z(u_n)$*
1.	k	$kz/(z-1)$
2.	$-k$	$kz/(z+1)$
3.	n	$z/(z-1)^2$
4.	n^2	$(z^2+z)/(z-1)^3$
5.	n^p	$-z\, d/dz\, [Z(n^{p-1})]$, p + *ve* integer.
6.	$\delta(n) = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases}$	1
7.	$u(n) = \begin{cases} 0 & n<0 \\ 1 & n \geq 0 \end{cases}$	$z/(z-1)$
8.	a^n	$z/(z-a)$
9.	na^n	$az/(z-a)^2$
10.	n^2a^n	$(az^2+a^2z)/(z-a)^3$
11.	$\sin n\theta$	$\dfrac{z\sin\theta}{z^2-2z\cos\theta+1}$
12.	$\cos n\theta$	$\dfrac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$
13.	$a^n \sin n\theta$	$\dfrac{az\sin\theta}{z^2-2az\cos\theta+a^2}$
14.	$a^n \cos n\theta$	$\dfrac{z(z-a\cos\theta)}{z^2-2az\cos\theta+a^2}$
15.	$\sinh n\theta$	$\dfrac{z\sinh\theta}{z^2-2z\cosh\theta+1}$
16.	$\cosh n\theta$	$\dfrac{z(z-\cosh\theta)}{z^2-2z\cosh\theta+1}$
17.	$a^n \sinh n\theta$	$\dfrac{az\sinh\theta}{z^2-2az\cosh\theta+a^2}$
18.	$a^n \cosh n\theta$	$\dfrac{z(z-a\cosh\theta)}{z^2-2az\cosh\theta+a^2}$
19.	$a^n u_n$	$U(z/a)$
20.	u_{n+1}	$z[U(z)-u_0]$
	u_{n+2}	$z^2[U(z)-u_0-u_1z^{-1}]$
	u_{n+3}	$z^3[U(z)-u_0-u_1z^{-1}-u_2z^{-2}]$
21.	u_{n-k}	$z^{-k}U(z)$
22.	nu_n	$-zd/dz[U(z)]$
23.	u_0	$\underset{z\to\infty}{\text{Lt}}\ U(z)$
24.	$\underset{n\to\infty}{\text{Lt}}\ (u_n)$	$\underset{z\to 1}{\text{Lt}}\ [(z-1)\,U(z)]$

23.11 SOME USEFUL INVERSE Z-TRANSFORMS

Sr. No.	$U(z)$	Inverse Z-transform $u_n = z^{-1}[U(z)]$
1.	$\dfrac{1}{z-a}$	a^{n-1}
2.	$\dfrac{1}{z+a}$	$(-a)^{n-1}$
3.	$\dfrac{1}{(z-a)^2}$	$(n-1)a^{n-2}$
4.	$\dfrac{1}{(z-a)^3}$	$\dfrac{1}{2}(n-1)(n-2)a^{n-3}$
5.	$\dfrac{z}{z-a}$	a^n
6.	$\dfrac{z}{z+a}$	$(-a)^n$
7.	$\dfrac{z^2}{(z-a)^2}$	$(n+1)a^n$
8.	$\dfrac{z^3}{(z-a)^3}$	$\dfrac{1}{2!}(n+1)(n+2)a^n u(n)$

23.12 CONVOLUTION THEOREM

If $Z^{-1}[U(z)] = u_n$ and $Z^{-1}[V(z)] = v_n$, then

$$Z^{-1}[U(z) \,.\, V(z)] = \sum_{m=0}^{n} u_m \,.\, v_{n-m} = u_n * v_n$$

where the symbol $*$ denotes the convolution operation.

Proof. We have $U(z) = \sum_{n=0}^{\infty} u_n z^{-n}$, $V(z) = \sum_{n=0}^{\infty} v_n z^{-n}$

$$\therefore \quad U(z)\,V(z) = (u_0 + u_1 z^{-1} + u_2 z^{-2} + \ldots + u_n z^{-n} + \ldots \infty) \times (v_0 + v_1 z^{-1} + v_2 z^{-2} + \ldots + v_n z^{-n} + \ldots \infty)$$

$$= \sum_{n=0}^{\infty} (u_0 v_n + u_1 v_{n-1} + u_2 v_{n-2} + \ldots + u_n v_0) z^{-n} = Z(u_0 v_n + u_1 v_{n-1} + \ldots + u_n v_0)$$

whence follows the desired result.

Obs. The convolution theorem plays an important role in the solution of difference equations and in probability problems involving sums of two independent random variables.

Example 23.11. *Use convolution theorem to evaluate* $Z^{-1}\left\{\dfrac{z^2}{(z-a)(z-b)}\right\}$.

Solution. We know that $Z^{-1}\left\{\dfrac{z}{z-a}\right\} = a^n$ and $Z^{-1}\left\{\dfrac{z}{z-b}\right\} = b^n$

$$\therefore \quad Z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\} = Z^{-1}\left\{\frac{z}{z-a} \cdot \frac{z}{z-b}\right\} = a^n * b^n$$

$$= \sum_{m=0}^{n} a^m \,.\, b^{n-m} = b^n \sum_{m=0}^{n} \left(\frac{a}{b}\right)^m \text{ which is a G.P.}$$

$$= b^n \,.\, \frac{(a/b)^{n+1} - 1}{a/b - 1} = \frac{a^{n+1} - b^{n+1}}{a - b}.$$

23.13 CONVERGENCE OF Z-TRANSFORMS

Z-transform operation is performed on a sequence u_n which may exist in the range of integers $-\infty < n < \infty$, and we write

$$U(z) = \sum_{n=-\infty}^{\infty} u_n z^{-n} \quad ...(1)$$

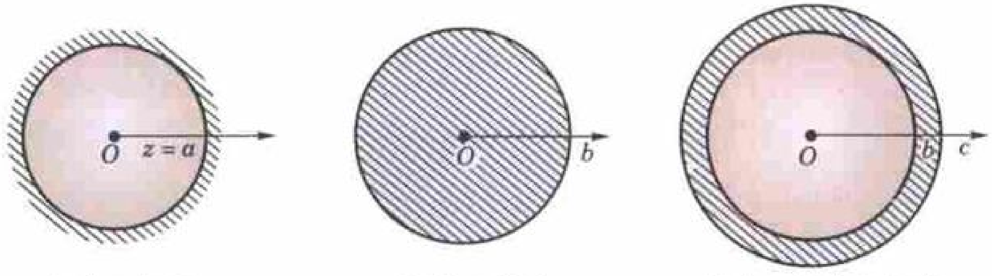
where u_n represents a number in the sequence for n = an integer. *The region of the z-plane in which (1) converges absolutely is known as the region of convergence (ROC) of U(z).*

We have so far discussed *one-sided Z-transform only for which* $n \geq 0$. Here the sequence is always *right-sided* and the *ROC* is always outside a prescribed circle say $|z| > |a|$ Fig. 23.2 (*i*)]. For a left-handed sequence, the ROC is always inside any prescribed contour $|z| < |b|$. [Fig. 23.2 (*ii*)].

23.14 TWO-SIDED Z-TRANSFORM OF u_n IS DEFINED BY

$$U(z) = \sum_{n=-\infty}^{\infty} u_n z^{-n} \quad ...(2)$$

In this case, the sequence is two-sided and the region of convergence for (2) is the *annular region* $|b| < |z| < |c|$ [Fig. 23.2 (*iii*)]. The inner circle bounds the terms in negative powers of z and the outer circle bounds the terms in positive powers of z. *The shaded annulus of convergence is necessary for the two sided sequence and its Z-transform to exist.*



$|z| > |a|$ (*i*) $\quad$ $|z| < |b|$ (*ii*) $\quad$ $|b| < |z| < |c|$ (*iii*)

Fig. 23.1

Example 23.12. *Find the Z-transfrom and region of convergence of*

(*a*) $u(n) = \begin{cases} 4^n & \text{for } n < 0 \\ 2^n & \text{for } n \geq 0 \end{cases}$ $\qquad$ (*b*) $u(n) = {}^nc_k,\ n \geq k.$

Solution. By definition $\quad Z[u(n)] = \sum_{-\infty}^{\infty} u(n) Z^{-n} = \sum_{-\infty}^{-1} 4^n z^{-n} + \sum_{n=0}^{\infty} 2^n z^{-n}$

Putting $-n = m$ in the first series, we get

$$Z[u(n)] = \sum_{1}^{\infty} 4^{-m} z^m + \sum_{n=0}^{\infty} 2^n z^{-n}$$

$$\left\{\frac{z}{4} + \frac{z^2}{4^2} + \frac{z^3}{4^3} + ...\right\} + \left\{1 + \frac{2}{z} + \frac{2^2}{z^2} + ...\right\}$$

$$= \frac{z}{4}\left\{1 + \left(\frac{z}{4}\right) + \left(\frac{z}{4}\right)^2 + ...\right\} + \left\{1 + \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 + ...\right\} \quad ...(i)$$

$$= \frac{z}{4} \cdot \frac{1}{1-(z/4)} + \frac{1}{1-(2/z)} = \frac{z}{4-z} + \frac{z}{z-2} = \frac{2z}{(4-z)(z-2)}$$

Now the two series in (*i*) being G.P. will be convergent if $| z/4 | < 1$ and $| 2/z | < 1$ *i.e.*, if $| z | < 4$ and $2 < | z |$. *i.e.* $2 < z < 4$.

Hence $Z[u(n)]$ is convergent if z lies between the annulus as shown shaded in Fig. 23.3. Hence ROC is $2 < z < 4$.

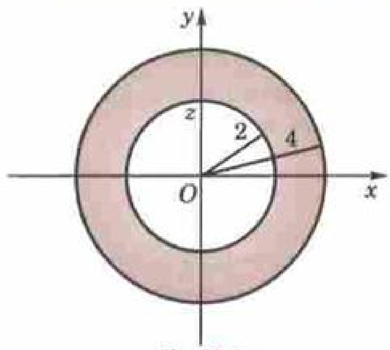


Fig. 23.2

(*b*) By definition, $Z[u(n)] = \sum_{-\infty}^{\infty} {}^nC_k z^{-n} = \sum_{n=k}^{\infty} {}^nC_k 2^n z^{-n}$

To find the sum of this series, we replace n by $k + r$

$$\therefore \quad Z[u(n)] = \sum_{r=0}^{\infty} {}^{k+r}C_k z^{-(k+r)} = z^{-k} \sum_{r=0}^{\infty} {}^{k+r}C_r z^{-r} \qquad [\because \ {}^kC_r = {}^kC_{k-r}]$$

$$= z^{-k}[1 + {}^{k+1}C_1 z^{-1} + {}^{k+1}C_2 z^{-2} + ...]$$

$$= z^{-k}(1 - 1/z)^{-(k+1)}$$

This series is convergence for $| 1/z | < 1$ *i.e.*, for $| z | > 1$.

Hence ROC is $| z | > 1$.

Example 23.13. *Find the Z-transform and the radius of convergence of*

(*a*) $f(n) = 2^n, n < 0$ (*b*) $f(n) = 5^n/n!, n \geq 0.$ (*Mumbai, 2009*)

Solution. (*a*) Assuming that $f(n) = 0$ for $n \geq 0$ we have

$$Z[f(n)] = \sum_{-\infty}^{\infty} f(n) z^{-n} = \sum_{n=-\infty}^{-1} 2^n z^{-n} = \sum_{m=1}^{\infty} 2^{-m} z^m \quad \text{where} \quad m = -n$$

$$= \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + ...\infty = \frac{z}{2}\{1 + (z/2) + (z/2)^2 + ...\infty\}$$

$$= \frac{z}{2} \cdot \frac{1}{1-(z/2)} = \frac{z}{2-z}$$

This series being a G.P. is convergent if $| z/2 | < 1$ *i.e.*, $| z | < 2$.

Hence ROC is $| z | < 2$.

(*b*) By definition, $Z[u(n)] = \sum_{n=0}^{\infty} \frac{5^n}{n!} . z^{-n} = \sum_{0}^{\infty} \frac{(5/z)^n}{n!} = 1 + \left(\frac{5}{z}\right) + \frac{1}{2!}\left(\frac{5}{z}\right)^2 + \frac{1}{3!}\left(\frac{5}{z}\right)^3 + ... \infty$

$= e^{5/z}$

The above series is convergent for all values of z.

Hence ROC is the entire z-plane.

PROBLEMS 23.2

Find the Z-transform and its ROC in each of the following sequences :

1. $u(n) = 4^n, n \geq 0.$
2. $u(n) = 2^n, n < 0.$
3. $u(n) = 4^n$, for $n < 0$ and $= 3^n$ for $n \geq 0$.
4. $u(n) = n5^n, n \geq 0.$
5. $u(n) = 2^n/n, n > 1.$
6. $u(n) = 3^n/n!, n \geq 0.$
7. $u(n) = e^{an}, n \geq 0.$

23.15 EVALUATION OF INVERSE Z-TRANSFORMS

We can obtain the inverse Z-transforms using any of the following three methods :

I. **Power series method.** This is the simplest of all the methods of finding the inverse Z-transform. If $U(z)$ is expressed as the ratio of two polynomials which cannot be factorized, we simply divide the numerator by the denominator and take the inverse Z-transform of each term in the quotient.

Example 23.14. *Find the inverse Z-transform of log (z / z + 1) by power series method.*

Solution. Putting $z = 1/t$, $U(z) = \log\left(\dfrac{1/y}{1/y+1}\right) = -\log(1+y) = -y + \dfrac{1}{2}y^2 - \dfrac{1}{3}y^3 + \ldots$

$$= -z^{-1} + \frac{1}{2}z^{-2} - \frac{1}{3}z^{-3} + \ldots$$

Thus
$$u_n = \begin{cases} 0 & \text{for } n = 0 \\ (-1)^n / n & \text{otherwise} \end{cases}.$$

Example 23.15. *Find the inverse Z-transform of $z/(z+1)^2$ by division method.*

Solution.
$$U(z) = \frac{z}{z^2 + 2z + 1} = z^{-1} - \frac{2 + z^{-1}}{z^2 + 2z + 1}, \text{ by actual division}$$

$$= z^{-1} - 2z^{-2} + \frac{3z^{-1} + 2z^{-2}}{z^2 + 2z + 1} = z^{-1} - 2z^{-2} + 3z^{-3} - \frac{4z^{-2} + 3z^{-3}}{z^2 + 2z + 1}$$

Continuing this process of division, we get an infinite series *i.e.*,

$$U(z) = \sum_{n=0}^{\infty} (-1)^{n-1}\, nz^{-n}$$

Thus $\quad u_n = (-1)^{n-1} n.$

II. **Partial fractions method.** This method is similar to that of finding the inverse Laplace transforms using partial fractions. The method consists of decomposing $U(z)/z$ into partial fractions, multiplying the resulting expansion by z and then inverting the same.

Example 23.16. *Find the inverse Z-transforms of*

(*i*) $\dfrac{2z^2 + 3z}{(z+2)(z-4)}$ (*V.T.U., 2008 S ; S.V.T.U., 2007*) (*ii*) $\dfrac{z^3 - 20z}{(z-2)^3 (z-4)}$. (*V.T.U., 2011*)

Solution. (*i*) We write $U(z) = \dfrac{2z^2 + 3z}{(z+2)(z-4)}$ as $\dfrac{U(z)}{z} = \dfrac{2z+3}{(z+2)(z-4)} = \dfrac{A}{z+2} + \dfrac{B}{z-4}$

where $A = 1/6$ and $B = 11/6$

$\therefore$
$$U(z) = \frac{1}{6}\frac{z}{z+2} + \frac{11}{6}\frac{z}{z-4}$$

On inversion, we have

$$u_n = \frac{1}{6}(-2)^n + \frac{11}{6}(4)^n \qquad \text{[Using § 23.10 (9)]}$$

(*ii*) We write
$$U(z) = \frac{z^3 - 20z}{(z-2)^3 (z-4)}$$

as
$$\frac{U(z)}{z} = \frac{z^2 - 20}{(z-2)^3 (z-4)} = \frac{A + Bz + Cz^2}{(z-2)^3} + \frac{D}{z-4}$$

Readily we get $D = 1/2$. Multiplying throughout by $(z-2)^3 (z-4)$, we get

$$z^2 - 20 = (A + Bz + Cz^2)(z-4) + D(z-2)^3.$$

Putting $z = 0, 1, -1$ successively and solving the resulting simultaneous equations, we get $A = 6$, $B = 0$, $C = 1/2$.

Thus
$$U(z) = \frac{1}{2}\cdot\frac{12z + z^3}{(z-2)^3} - \frac{z}{z-4} = \frac{1}{2}\frac{z(z-2)^2 + 4z^2 + 8z}{(z-2)^3} - \frac{z}{z-4}$$

$$= \frac{1}{2}\left\{\frac{z}{z-2} + 2\frac{2z^2 + 4z}{(z-2)^3}\right\} - \frac{z}{z-4}$$

On inversion, we get $u_n = \frac{1}{2}(2^n + 2 \,.\, n^2 2^n) - 4^n$ [Using § 23.10 (9) & (11)]

$$= 2^{n-1} + n^2 2^n - 4^n.$$

Example 23.17. *Find the inverse Z-transform of*

$2(z^2 - 5z + 6.5)/[(z-2)(z-3)^2]$, for $2 < |z| < 3$.

Solution. Splitting into partial fractions, we obtain

$$U(z) = \frac{2(z^2 - 5z + 6.5)}{(z-2)(z-3)^2} = \frac{A}{z-2} + \frac{B}{z-3} + \frac{C}{(z-3)^2} \quad \text{where } A = B = C = 1$$

$$\therefore \quad U(z) = \frac{1}{z-2} + \frac{1}{z-3} + \frac{1}{(z-3)^2}$$

$$= \frac{1}{2}\left(1 - \frac{2}{z}\right)^{-1} - \frac{1}{3}\left(1 - \frac{z}{3}\right)^{-1} + \frac{1}{9}\left(1 - \frac{z}{3}\right)^{-2} \quad \text{so that } 2/z < 1 \text{ and } z/3 < 1$$

$$= \frac{1}{z}\left(1 + \frac{2}{z} + \frac{4}{z^2} + \frac{8}{z^3} + \ldots\right) - \frac{1}{3}\left(1 + \frac{z}{3} + \frac{z^2}{9} + \frac{z^3}{27} + \ldots\right) + \frac{1}{9}\left(1 + \frac{2z}{3} + \frac{3z^2}{9} + \frac{4z^3}{27} + \ldots\right)$$

where $2 < |z| < 3$.

$$= \left(\frac{1}{2} + \frac{2}{z^2} + \frac{2^2}{z^3} + \frac{2^3}{z^4} + \ldots\right) - \left(\frac{1}{3} + \frac{z}{3^2} + \frac{z^2}{3^3} + \frac{z^3}{3^4} + \ldots\right) + \left(\frac{1}{3^2} + \frac{2z}{3^3} + \frac{3z^2}{3^4} + \frac{4z^3}{3^5} + \ldots\right)$$

$$= \sum_{n=1}^{\infty} 2^{n-1} z^{-n} - \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{n+1} z^n + \sum_{n=0}^{\infty} (n+1)\left(\frac{1}{3}\right)^{n+2} z^n$$

On inversion, we get $u_n = 2^{n-1}, n \geq 1$ and $u_n = -(n+2)3^{n-2}, n \leq 0$.

III. **Inversion integral method.** *The inverse Z-transform of $U(z)$ is given by the formula*

$$u_n = \frac{1}{2\pi i}\int_c U(z) z^{n-1}\, dz$$

= sum of residues of $U(z)\, z^{n-1}$ at the poles of $U(z)$ which are inside the contour C drawn according to the ROC given.

The following examples will illustrate the application of this formula :

Example 23.18. *Using the inversion integral method, find the inverse Z-transform of*

$$\frac{z}{(z-1)(z-2)}$$

(V.T.U., 2010 S)

Solution. Let $U(z) = \dfrac{z}{(z-1)(z-2)}$. Its poles are at $z = 1$ and $z = 2$.

Using $U(z)$ in the inversion integral, we have

$$u_n = \frac{1}{2\pi i}\int_C U(z) z^{n-1}\, dz,$$

where C is a circle large enough to enclose both the poles of $U(z)$.

$$= \text{sum of residues of } U(z)\, z^{n-1} \text{ at } z = 1 \text{ and } z = 2.$$

Now $\quad Res\,[U(z)\, z^{n-1}]_{z=1} = \underset{z \to 1}{\text{Lt}} \left\{(z-1) \,.\, \dfrac{z^n}{(z-1)(z-2)}\right\} = -1$

and $\quad Res\,[U(z)\, z^{n-1}]_{z=2} = \underset{z \to 2}{\text{Lt}} \left\{(z-2) \,.\, \dfrac{z^n}{(z-1)(z-2)}\right\} = 2^n$

Thus the required inverse Z-transform $u_n = 2^n - 1, n = 0, 1, 2, \ldots$

Example 23.19. *Find the inverse Z-transform of $2z/[(z-1)(z^2+1)]$.* (*Madras, 2000 S*)

Solution. Let $U(z) = \dfrac{2z}{(z-1)(z+i)(z-i)}$. It has three poles at $z = 1, z = \pm i$.

Using $U(z)$ in the inversion integral, we have

$$u_n = \frac{1}{2\pi i}\int_C U(z) \,.\, z^{n-1}\, dz\text{, where } C \text{ is a circle large enough to enclose the poles of } U(z).$$

$$= \text{sum of residues of } U(z) \,.\, z^{n-1} \text{ at } z = 1, z = \pm i.$$

Now $$Res\,[U(z)\, z^{n-1}]_{z=1} = \underset{z\to 1}{\text{Lt}}\left\{(z-1)\frac{2z^n}{(z-1)(z^2+1)}\right\} = 1$$

$$Res\,[U(z)\, z^{n-1}]_{z=i} = \underset{z\to i}{\text{Lt}}\left\{(z-i)\frac{2z^n}{(z-1)(z+i)(z-i)}\right\} = \frac{-(i)^n}{1+i}$$

$$Res\,[U(z)\, z^{n-1}]_{z=-i} = \underset{z\to -i}{\text{Lt}}\left\{(z+i)\frac{2z^n}{(z-1)(z+i)(z-i)}\right\} = \frac{(-i)^n}{i-1}$$

Hence $$u_n = 1 - \frac{(i)^n}{1+i} - \frac{(-i)^n}{1-i}.$$

PROBLEMS 23.3

Using convolution theorem, evaluate the inverse Z-transforms of the following :

1. $\dfrac{z^2}{(z-1)(z-3)}$.
2. $\left(\dfrac{z}{z-a}\right)^2$ (*Madras, 2003*)
3. $\left(\dfrac{z}{z-1}\right)^3$.
4. Show that (*a*) $\dfrac{1}{n!} * \dfrac{1}{n!} = \dfrac{2^n}{n!}$ (*b*) $Z^{-1}\left(\dfrac{z^2}{(z+a)(z+b)}\right) = \dfrac{(-1)}{b-a}(b^{n+1} - a^{n+1})$. (*Anna, 2009*)

Find the inverse Z-transforms of the following :

5. $\dfrac{4z}{z-a}$, $|z| > |a|$. (*Kottayam, 2005*)
6. $\dfrac{5z}{(2-z)(3z-1)}$. (*Madras, 1999*)
7. $\dfrac{z}{(z-1)^2}$.
8. $\dfrac{18z^2}{(2z-1)(4z+1)}$. (*S.V.T.U., 2009*)
9. $\dfrac{8z-z^3}{(4-z)^3}$.
10. $\dfrac{3z^2-18z+26}{(z-2)(z-3)(z-4)}$. (*Anna, 2005 S*)
11. $\dfrac{4z^2-2z}{z^3-5z^2+8z-4}$. (*V.T.U., 2011 S*)
12. $\dfrac{z^3+3z}{(z-1)^2(z^2+1)}$. (*Anna, 2009*)
13. $\dfrac{(1-e^{at})z}{(z-1)(z-e^{-at})}$.
14. Obtain $Z^{-1}[1/[(z-2)(z-3)]]$ for (*i*) $|z| < 2$; (*ii*) $2 < |z| < 3$; (*iii*) $|z| > 3$. (*Marathwada, 2008*)
15. Evaluate $Z^{-1}[(z-5)^{-3}]$ for $|z| > 5$. (*Mumbai, 2009*)

Using inversion integral, find the inverse Z-transform of the following functions :

16. $\dfrac{z+3}{(z+1)(z-2)}$.
17. $\dfrac{(2z-1)z}{2(z-1)(z+0.5)}$.
18. $\dfrac{1}{z(z-1)(z+0.5)}$. (*S.V.T.U., 2008*)
19. $\dfrac{z^2+z}{(z-1)(z^2+1)}$. (*Madras, 2003*)
20. $\dfrac{2z(z^2-1)}{(z^2+1)^2}$.

23.16 (1) APPLICATION TO DIFFERENCE EQUATIONS

Just as the Laplace transforms method is quite effective for solving linear differential equations (§ 21.15), the Z-transforms are quite useful for solving linear difference equations.

The performance of discrete systems is expressed by suitable difference equations. Also Z-transform plays an important role in the analysis and representation of discrete-time systems. To determine the frequency response of such systems, the solution of difference equations is required for which Z-transform method proves useful.

(2) **Working procedure** *to solve a linear difference equation with constant coefficients by Z-transforms :*

1. *Take the Z-transform of both sides of the difference equations using the formulae of § 26.16 and the given conditions.*

2. *Transpose all terms without U(z) to the right.*

3. *Divide by the coefficient of U(z), getting U(z) as a function of z.*

4. *Express this function in terms of the Z-transforms of known functions and take the inverse Z-transform of both sides. This gives u_n as a function of n which is the desired solution.*

Example 23.20. *Using the Z-transform, solve*

$$u_{n+2} + 4u_{n+1} + 3u_n = 3^n \text{ with } u_0 = 0, u_1 = 1. \qquad \text{(U.P.T.U., 2003)}$$

Solution. If $Z(u_n) = U(z)$, then $Z(u_{n+1}) = z[U(z) - u_0]$,

$$Z(u_{n+2}) = z^2[U(z) - u_0 - u_1 z^{-1}]$$

Also $Z(2^n) = z/(z-2)$

$\therefore$ taking the Z-transforms of both sides, we get

$$z^2[U(z) - u_0 - u_1 z^{-1}] + 4z\,[U(z) - u_0] + 3U(z) = z/(z-3)$$

Using the given conditions, it reduces to

$$U(z)\,(z^2 + 4z + 3) = z + z/(z-3)$$

$$\therefore \quad \frac{U(z)}{z} = \frac{1}{(z+1)(z+3)} + \frac{1}{(z-3)(z+1)(z+3)} = \frac{3}{8}\frac{1}{z+1} + \frac{1}{24}\frac{1}{z-3} - \frac{5}{12}\frac{1}{z+3},$$

on breaking into partial fractions.

$$U(z) = \frac{3}{8}\frac{z}{z+1} + \frac{1}{24}\frac{z}{z-3} - \frac{5}{12}\frac{z}{z+3}$$

On inversion, we obtain

$$u_n = \frac{3}{8}Z^{-1}\left(\frac{z}{z+1}\right) + \frac{1}{24}Z^{-1}\left(\frac{z}{z-3}\right) - \frac{5}{12}Z^{-1}\left(\frac{z}{z+3}\right) = \frac{3}{8}(-1)^n + \frac{1}{24}3^n - \frac{5}{12}(-3)^n.$$

Example 23.21. *Solve $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ with $y_0 = y_1 = 0$, using Z-transforms.* (V.T.U., 2011 ; Anna, 2009 ; S.V.T.U., 2009)

Solution. If $Z(y_n) = Y(z)$, then $Z(y_{n+1}) = z\,[(Y(z) - y_0]$, $Z(y_{n+2}) = z^2[Y(z) - y_0 - y_1 z^{-1}]$

Also $Z(2^n) = z/(z-2)$.

Taking Z-transforms of both sides, we get

$$z^2[Y(z) - y_0 - y_1 z^{-1}] + 6z[Y(z) - y_0] + 9Y(z) = z/(z-2)$$

Since $y_0 = 0$, and $y_1 = 0$, we have $Y(z)\,(z^2 + 6z + 9) = z/(z-2)$

or $$\frac{Y(z)}{z} = \frac{1}{(z-2)(z+3)^2} = \frac{1}{25}\left[\frac{1}{z-2} - \frac{1}{z+3} - \frac{5}{(z+3)^2}\right],$$ on splitting into partial fractions.

or $$Y(z) = \frac{1}{25}\left[\frac{z}{z-2} - \frac{z}{z+3} - 5\frac{z}{(z+3)^2}\right]$$

On taking inverse Z-transform of both sides, we obtain

$$y_n = \frac{1}{25}\left[Z^{-1}\left(\frac{z}{z-2}\right) - Z^{-1}\left(\frac{z}{z+3}\right) + \frac{5}{3}Z^{-1}\left(-\frac{3z}{(z+3)^2}\right)\right]$$

$$= \frac{1}{25}\{2^n - (-3)^n + \frac{5}{3}n(-3)^n\} \qquad \left[\because\ Z^{-1}\left\{\frac{az}{(z-a)^2}\right\} = na^n\right]$$

Example 23.22. *Find the response of the system $y_{n+2} - 5y_{n+1} + 6y_n = u_n$, with $y_0 = 0, y_1 = 1$ and $u_n = 1$ for $n = 0, 1, 2, 3, ...$ by Z-transform method.* (V.T.U., 2010)

Solution. Taking Z-transform of both sides of the given equation, we get

$$z^2(Y(z) - y_0 - y_1 z^{-1}) - 5z(Y(z) - y_0) + 6Y(z) = \frac{z}{z-1}$$

Substituting the values $y_0 = 0, y_1 = 1$, it reduces to

$$(z^2 - 5z + 6)\, Y(z) = \frac{z}{z-1} + z = \frac{z^2}{z-1}$$

or
$$\frac{Y(z)}{z} = \frac{z}{(z-1)(z-2)(z-3)}$$

$$= \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{z-3} \qquad \text{where } A = \frac{1}{2}, B = -2, C = \frac{3}{2}$$

so that
$$Y(z) = \frac{1}{2}\frac{z}{z-1} - 2\frac{z}{z-2} + \frac{3}{2}\frac{z}{z-3}$$

On inversion, we obtain
$$y_n = \frac{1}{2} - 2(2)^n + \frac{3}{2}(3)^n$$

Obs. The initial values given in the problem automatically appear in the generated sequence.

Example 23.23. *Solve the difference equation $y_n + \frac{1}{4}y_{n-1} = u_n + \frac{1}{3}u_{n-1}$ where u_n is a unit step sequence.*

Solution. Taking Z-transform of both sides of the given equation, we get

$$Y(z) + \frac{1}{4}z^{-1}\, Y(z) = 1 + \frac{1}{3}z^{-1}$$

or
$$Y(z) = \left(1 + \frac{1}{3}z^{-1}\right)\Big/\left(1 + \frac{1}{4}z^{-1}\right) = \left(z + \frac{1}{3}\right)\Big/\left(z + \frac{1}{4}\right)$$

There being only one simple pole at $z = -1/4$, consider the contour $|\, z \,| > 1/4$.

$$\therefore \quad Res\, [Y(z)\, z^{n-1}]_{z=-1/4} = \underset{z \to -1/4}{\text{Lt}} \left\{\left(z + \frac{1}{4}\right) . \left(z + \frac{1}{3}\right) z^{n-1} \Big/ \left(z + \frac{1}{4}\right)\right\}$$

$$= \underset{z \to -1/4}{\text{Lt}} \left(z + \frac{1}{3}\right) z^{n-1} = \left(-\frac{1}{4} + \frac{1}{3}\right)\left(-\frac{1}{4}\right)^{n-1} = \frac{1}{12} . \left(-\frac{1}{4}\right)^{n-1}$$

Hence by inversion integral method, we have

$$y_n = \frac{1}{12}\left(-\frac{1}{4}\right)^{n-1}.$$

Example 23.24. *Using the Z-transform, solve $u_{n+2} - 2u_{n+1} + u_n = 3n + 5$.* (S.V.T.U., 2007)

Solution. Given equation is $u_{n+2} - 2u_{n+1} + u_n = 3n + 5$.

Taking the Z-transforms of both sides, we get

$$z^2[U(z) - u_0 - u_1 z^{-1}] - 2z[U(z) - u_0] + U(z) = 3 . \frac{z}{(z-1)^2} + 5 . \frac{z}{z-1}$$

or
$$U(z)\,(z^2 - 2z + 1) = \frac{5z^2 - 2z}{(z-1)^2} + u_0(z^2 - 2z) + u_1 z$$

or
$$U(z) = \frac{5z^2 - 2z}{(z-1)^4} + u_0 \frac{z^2 - 2z}{(z-1)^2} + u_1 \frac{z}{(z-1)^2}$$

On inversion, we obtain

$$u_n = Z^{-1}\left\{\frac{5z^2-2z}{(z-1)^4}\right\} + u_0 Z^{-1}\left\{\frac{z^2-2z}{(z-1)^2}\right\} + u_1 Z^{-1}\left\{\frac{z}{(z-1)^2}\right\} \qquad ...(i)$$

Noting that $Z(1) = \dfrac{z}{z-1}$, $Z(n) = \dfrac{z}{(z-1)^2}$

$$Z(n^2) = \frac{z^2+z}{(z-1)^3},\ Z(n^3) = \frac{z^3+4z^2+z}{(z-1)^4}$$

We write $\dfrac{5z^2-2z}{(z-1)^4} \equiv A\dfrac{z^3+4z^2+z}{(z-1)^4} + B\dfrac{z^2+z}{(z-1)^3} + C\dfrac{z}{(z-1)^2} + D\dfrac{z}{z-1}$

Equating coefficients of like powers of z, we find

$$A = \frac{1}{2},\ B = 1,\ C = -\frac{3}{2},\ D = 0$$

$\therefore$
$$Z^{-1}\left\{\frac{5z^2-2z}{(z-1)^4}\right\} = \frac{1}{2}n^3 + n^2 - \frac{3}{2}n = \frac{1}{2}n(n-1)(n+3)$$

Also
$$Z^{-1}\left\{\frac{z^2-2z}{(z-1)^2}\right\} = Z^{-1}\left\{\frac{z}{z-1}\right\} - Z^{-1}\left\{\frac{z}{(z-1)^2}\right\} = 1-n$$

and
$$Z^{-1}\left\{\frac{z}{(z-1)^2}\right\} = n.$$

Substituting these values in (i) above, we get

$$u_n = \frac{1}{2}n(n-1)(n+3) + u_0(1-n) + u_1 n$$

$$= \frac{1}{2}n(n-1)(n+3) + c_0 + c_1 n. \qquad \text{where } c_0 = u_0,\ c_1 = u_1 - u_0$$

Example 23.25. *Using residue method, solve* $y_k + \dfrac{1}{9}y_{k-2} = \dfrac{1}{3^k}\cos\dfrac{k\pi}{2}$, $k \ge 0$.

Solution. Taking Z-transform of both sides of the given equation, we get

$$Z\left\{y_k + \frac{1}{9}y_{k-2}\right\} = Z\left\{\frac{1}{3^k}\cos\frac{k\pi}{2}\right\}$$

or
$$Y(z) + \frac{1}{9}z^{-2}Y(z) = \frac{z^2}{z^2+1/9} \quad \text{or} \quad \left(1+\frac{1}{9}z^{-2}\right)Y(z) = \frac{z^2}{z^2+\frac{1}{9}}$$

or
$$Y(z) = \frac{z^2}{\left(1+\frac{1}{9}z^{-2}\right)\left(z^2+\frac{1}{9}\right)} = \frac{z^4}{\left(z^2+\frac{1}{9}\right)^2}$$

There are two poles of second order at $z = i/3$ and $z = -i/3$.

$\therefore$
$$\text{Residue at } (z = i/3) = \left[\frac{d}{dz}\left\{\left(\frac{z-i}{3}\right)^2 \frac{z^{k-1}z^4}{(z^2+1/9)^2}\right\}\right]$$

$$= \left[\frac{d}{dz}\left\{\frac{z^{k+3}}{(z+i/3)^2}\right\}\right]_{z=i/3} = \left[\frac{(z+i/3)^2(k+3)z^{k+2} - z^{(k+3)}\cdot 2(z+i/3)}{(z+i/3)^4}\right]_{z=i/3}$$

$$= \left[\frac{(z+i/3)(k+3)z^{k+2} - 2z^{k+3}}{(z+i/3)^3}\right]_{z=i/3} = \left(\frac{3}{2i}\right)^3\left[(2k+6)\left(\frac{i}{3}\right)^{k+3} - 2\left(\frac{i}{3}\right)^{k+3}\right]$$

$$= \frac{1}{8}(2k+4)\left(\frac{i}{3}\right)^k = \frac{1}{4}(k+2)\left(\frac{1}{3}\right)^k\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^k = \frac{1}{4}(k+2)\left(\frac{1}{3}\right)^k\left(\cos\frac{k\pi}{2} + i\sin\frac{k\pi}{2}\right) \quad ...(i)$$

Changing i to $-i$ in (i), we have

Residue at $(z = -i/3) = \frac{1}{4}(k+2)\left(\frac{1}{3}\right)^k\left(\cos\frac{k\pi}{2} - i\sin\frac{k\pi}{2}\right)$ ...(ii)

Adding (i) and (ii), we obtain $y_k = \frac{1}{2}(k+2)\left(\frac{1}{3}\right)^k\cos\frac{k\pi}{2}$.

PROBLEMS 23.4

Solve the following difference equations using Z-transforms (1 – 8) :

1. $6y_{k+2} - y_{k+1} - y_k = 0$, given that $y(0) = y(1) = 1$. (*Kottayam, 2005*)
2. $y(n+2) + 2y(n+1) + y(n) = 0$, given that $y(0) = y(1) = 0$. (*V.T.U., 2008 S*)
3. $y_{n+2} - 4y_n = 0$ given that $y_0 = 0$, $y_1 = 2$. (*U.P.T.U., 2008*)
4. $f(n) + 3f(n-1) - 4f(n-2) = 0$, $n \geq 2$, given that $f(0) = 3$, $f(1) = -2$. (*Madras, 2003 S*)
5. $y_{(n+3)} - 3y(n+1) + 2y(n) = 0$, given that $y(0) = 4$, $y(1) = 0$ and $y(2) = 8$. (*Anna, 2005 S*)
6. $y_{n+2} - 5y_{n+1} + 6y_n = 36$, given that $y(0) = y(1) = 0$. (*Anna, 2009*)
7. $y_{n+2} - 6y_{n+1} + 9y_n = 3^n$.
8. $y_{n+2} - 4y_{n+1} + 3y_n = 5^n$.
9. $y_{n+1} + \frac{1}{4}y_n = \left(\frac{1}{4}\right)^n \quad (n \geq 0), y_0 = 0$.
10. $u_{x+2} + u_x = 5(2^x)$ given that $u_0 = 1$, $u_1 = 0$. (*Marathwada, 2008*)
11. $y_{n+2} + 4y_{n+1} + 3y_n = 2^n$ with $y_0 = 0$, $y_1 = 1$. (*Madras, 2006*)
12. $u_{k+2} - 2u_{k+1} + u_k = 2^k$ with $y_0 = 2$, $y_1 = 1$.
13. $y_{n+2} - 6y_{n+1} + 8y_n = 2^n + 6n$.
14. $y_k + \frac{1}{25}y_{k-2} = \left(\frac{1}{5}\right)^k\cos\frac{k\pi}{2}, \quad (k \geq 0)$.
15. Find the response of the system given by $y_n + 3y_{(n-1)} = u_n$ where u_n is a unit step sequence and $y_{(-1)} = 1$.
16. Find the impulse response of a system described by $y_{(n+1)} + 2y_{(n)} = \delta_n$; $y_0 = 0$.

23.1 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 23.5

Choose the correct answer or fill up the blanks in each of the following problems :

1. $Z(1) =$ ......
2. If u_n is defined for $n = 0, 1, 2, ...$ only, then $Z(u_n) =$ ....
3. Z-transform of $n =$ .... (*Anna, 2009*)
4. $Z(na^n) =$ ......
5. $Z(\sin n\theta) =$ ......
6. Z-transform of $(1/n!)$ is
7. $Z(n^2) =$ ......
8. Linear property of Z-transform states that...
9. $Z^{-1}\left(\frac{1}{z-2}\right) =$ ....
10. $Z^{-1}\left\{\frac{z}{(z+1)^2}\right\} =$ ......
11. Initial value theorem on Z-transform states that ....
12. Z-transform is linear. (True or False)
13. If $Z(u_n) = u(z)$, then $\underset{n\to\infty}{\text{Lt}}(u_n) = \underset{z\to\infty}{\text{Lt}}(z-1)u(z)$. (True or False)
14. Z-transform of the sequence $\{2^k\}$, $k \geq 0$ is $z/(z-2)$. (True or False)
15. Z-transform of $\{a^k/k!\}$, $k \geq 0 = e^{a/z}$. (True or False)
16. Z-transform of $\{^nC_r\}$, $(0 \leq r \leq n)$ is $(1+z)^n$. (True or False)
17. Z-transform of unit impulse sequence $\delta(n) = \begin{cases} 1, & n < 0 \\ 0, & n \geq 0 \end{cases}$, is $z/z - 1$. (True or False)
18. Z-transform of unit step sequence $u(n) = \begin{cases} 0, & n < 0, \\ 1, & n \geq 0, \end{cases}$ is 1. (True or False)

CHAPTER 24

Empirical Laws and Curve-fitting

24.1 INTRODUCTION

In many branches of applied mathematics, it is required to express a given data, obtained from observations, in the form of a *Law* connecting the two variables involved. Such a *Law* inferred by some scheme is known as *Empirical Law*. For example, it may be desired to obtain the law connecting the length and the temperature of a metal bar. At various temperatures, the length of the bar is measured. Then, by one of the methods explained below, a law is obtained that represents the relationship existing between temperature and length for the observed values. This relation can then be used to predict the length at an arbitrary temperature.

(2) Scatter diagram. To find a relationship between the set of paired observations x and y (say), we plot their corresponding values on the graph taking one of the variables along the x-axis and other along the y-axis *i.e.* (x_1, y_1), (x_2, y_2), (x_n, y_n). The resulting diagram showing a collection of dots is called a *scatter diagram*. A smooth curve that approximates the above set of points is known as the *approximating curve*.

(3) Curve fitting. Several equations of different types can be obtained to express the given data approximately. But the problem is to find the equation of the curve of *'best fit'* which may be most suitable for predicting the unknown values. The process of finding such an equation of 'best fit' is known as *curve-fitting*.

If there are n pairs of observed values then it is possible to fit the given data to an equation that contains n arbitrary constants for we can solve n simultaneous equations for n unknowns. If it were desired to obtain an equation representing these data but having less than n arbitrary constants, then we can have recourse to any of the four methods : *Graphical method, Method of Least squares, Method of Group averages* and *Method of Moments*. The graphical method fails to give the values of the unknowns uniquely and accurately while the other methods do. *The method of Least squares is, probably, the best to fit a unique curve to a given data.* It is widely used in applications and can be easily implemented on a computer.

24.2 GRAPHICAL METHOD

When the curve representing the given data is a **linear law** $y = mx + c$ *; we proceed as follows :*

(*i*) *Plot the given points on the graph paper* taking a suitable *scale*.

(*ii*) *Draw the straight line of best fit* such that the points are evenly distributed about the line.

(*iii*) *Taking two suitable points* (x_1, y_1) *and* (x_2, y_2) *on the line, calculate m,* the slope of the line and c, its intercept on y-axis.

When the points representing the observed values do not approximate to a straight line, a smooth curve is drawn through them. From the shape of the graph, we try to infer the law of the curve and then reduce it to the form $y = mx + c$.

24.3 LAWS REDUCIBLE TO THE LINEAR LAW

We give below some of the laws in common use, indicating the way these can be reduced to the linear form by suitable substitutions :

(1) *When the law is* $y = mx^n + c$.

Taking $x^n = X$ and $y = Y$ the above law becomes $Y = mX + c$

(2) *When the law is* $y = ax^n$.

Taking logarithms of both sides, it becomes $\log_{10} y = \log_{10} a + n \log_{10} x$

Putting $\log_{10} x = X$ and $\log_{10} y = Y$, it reduces to the form $Y = nX + c$, where $c = \log_{10} a$.

(3) *When the law is* $y = ax^n + b \log x$.

Writing it as $\frac{y}{\log x} = a \frac{x^n}{\log x} + b$ and taking $x^n/\log x = X$ and $y/\log x = Y$,
the given law becomes, $Y = aX + b$.

(4) *When the law is* $y = ae^{bx}$

Taking logarithms, it becomes $\log_{10} y = (b \log_{10} e)\, x + \log_{10} a$

Putting $x = X$ and $\log_{10} y = Y$, it takes the form $Y = mX + c$ where $m = b \log_{10} e$ and $c = \log_{10} a$.

(5) *When the law is* $xy = ax + by$.

Dividing by x, we have $y = b \frac{y}{x} + a$.

Putting $y/x = X$ and $y = Y$, it reduces to the form $Y = bX + a$.

Example 24.1. *R is the resistance to maintain a train at speed V; find a law of the type $R = a + bV^2$ connecting R and V, using the following data :*

V (miles/hour) :	*10*	*20*	*30*	*40*	*50*
R (lb/ton) :	*8*	*10*	*15*	*21*	*30*

Solution. Given law is $R = a + bV^2$ ...(*i*)

Taking $V^2 = x$ and $R = y$, (*i*) becomes

$$y = a + bx \quad ...(ii)$$

which is a *linear law*.

Table for the values of x and y is as follows :

x	100	400	900	1600	2500
y	8	10	15	21	30

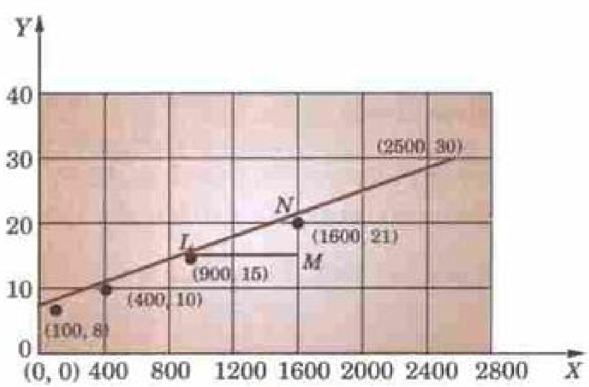


Fig. 24.1

Plot these points. Draw the straight line of best fit through these points (Fig. 24.1)

Slope of this line (= b)

$$= \frac{MN}{LM} = \frac{21 - 15}{1600 - 900} = \frac{6}{700} = 0.0085 \text{ nearly.}$$

Since L (900, 15) lies on (*ii*),

$\therefore$ $15 = a + 0.0085 \times 900$,

whence $a = 15 - 7.65 = 7.35$ nearly.

Example 24.2. *The following values of x and y are supposed to follow the law $y = ax^2 + b \log_{10} x$. Find graphically the most probable values of the constants a and b.*

x	*2.85*	*3.88*	*4.66*	*5.69*	*6.65*	*7.77*	*8.67*
y	*16.7*	*26.4*	*35.1*	*47.5*	*60.6*	*77.5*	*93.4*

Solution. Given law is $y = ax^2 + b\log_{10} x$

i.e. $$\frac{y}{\log_{10} x} = a\frac{x^2}{\log_{10} x} + b \quad ...(i)$$

Taking $x^2/\log_{10} x = X$ and $y/\log_{10} x = Y$

(*i*) becomes $Y = aX + b$ ...(*ii*)

This is a *linear law*. Table for the values of X and Y is as follows :

$X = x^2/\log_{10} x$	17.93	25.56	32.49	42.87	53.75	67.80	80.83
$Y = y/\log_{10} x$	35.59	44.83	52.50	62.90	73.65	87.04	99.56
Points	P_1	P_2	P_3	P_4	P_5	P_6	P_7

Plot these points and draw the straight line of best fit through these points (Fig. 24.2).

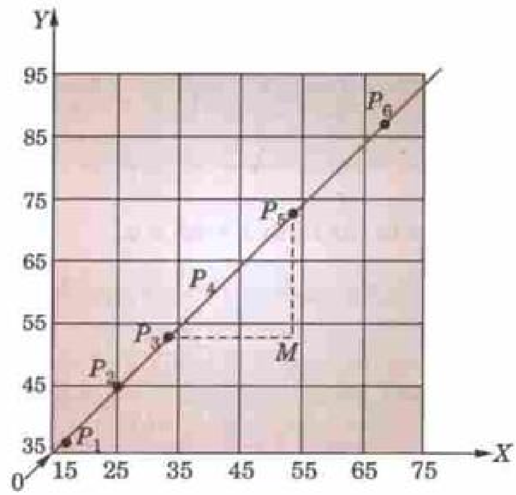


Fig. 24.2

Slope of this line (= a) = $\dfrac{MP_5}{P_3M} = \dfrac{73.65 - 52.50}{53.75 - 32.49} = \dfrac{21.15}{21.26} = 0.99$

Since P_3 lies on (*ii*), therefore, $52.50 = 0.99 \times 32.49 + b$ whence $b = 20.2$

Hence (*i*) becomes $y = (0.99)\, x^2 + (20.2)\log_{10} x$.

Example 24.3. *The values of x and y obtained in an experiment are as follows :*

x	2.30	3.10	4.00	4.92	5.91	7.20
y	33.0	39.1	50.3	67.2	85.6	125.0

Th probable law is $y = ae^{bx}$. Test graphically the accuracy of this law and if the law holds good, find the best values of the constants.

Solution. Given law is $y = ae^{bx}$ ...(*i*)

Taking logarithms to base 10, we have $\log_{10} y = \log_{10} a + (b\log_{10} e)\, x$

Putting $x = X$ and $\log_{10} y = Y$, it becomes $y = (b\log_{10} e)\, X + \log_{10} a$ ...(*ii*)

Table for the values of X and Y is as under :

$X = x$	2.30	3.10	4.00	4.92	5.91	7.20
$Y = \log_{10} y$	1.52	1.59	1.70	1.83	1.93	2.1
Points	P_1	P_2	P_3	P_4	P_5	P_6

Scale : 1 small division along x–axis = 0.1

10 small divisions along y–axis = 0.1.

Plot these points and draw the line of best fit. As these points are lying almost along a straight line, the given law is nearly accurate (Fig. 24.3).

Now slope of this line (= $b \log_{10} e$)

$$= \frac{MN}{NM} = 0.12$$

whence $$b = \frac{0.12}{\log_{10} e} = 0.12 \times 2.303 = 0.276$$

Since the point L (4, 1.71) lies on (*ii*), therefore, $1.71 = 0.12 \times 4 + \log_{10} a$ whence $a = 17$ nearly.

Hence the curve of best fit is $y = 17\, e^{0.276x}$.

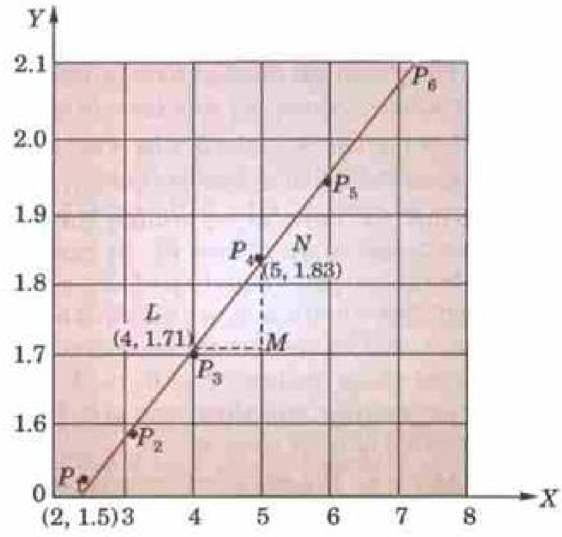


Fig. 24.3

PROBLEMS 24.1

1. If p is the pull required to lift the weight by means of a pulley block, find a linear law of the form $p = a + bw$, connecting p and w, using the following data :

w (lb) :	50	70	100	120
p (lb) :	12	15	21	25

Compute p, when w = 150 lb.

2. The resistance R of a carbon filament lamp was measured at various values of the voltage V and the following observations were made :

Voltage	V...	62	70	78	84	92
Resistance	R...	73	70.7	69.2	67.8	66.3

Assuming a law of the form $R = \frac{a}{V} + b$, find by graphical method the best value of a and b.

3. Verify if the values of x and y, related as shown in the following table, obey the law $y = a + b\sqrt{x}$. If so, find graphically the values of a and b.

x :	500	1,000	2,000	4,000	6,000
y :	0.20	0.33	0.38	0.45	0.51

4. The following values of T and l follow the law $T = al^n$. Test if this is so and find the best values of a and n.

T = 1.0	1.5	2.0	2.5
l = 25	56.2	100	1.56

5. Find the best value of a and b if $y = ax + b \log_{10} x$ is the curve which represents most closely the observed values given below :

x :	2	3	4	5	6
y :	3.74	5.99	7.47	8.92	9.86

6. Fit the curve $y = ae^{bx}$ to the following data :

x :	0	2	4
y :	5.1	10	31.1

(*Coimbatore, 1997*)

7. The following are the results of an experiment on friction of bearings. The speed being constant, corresponding values of the coefficient of friction and the temperature are shown in the table :

t :	120	110	100	90	80	70	60
μ :	0.0051	0.0059	0.0071	0.0085	0.00102	0.00124	0.00148

If μ and t are given by the law $\mu = ae^{bt}$, find the values of a and b by plotting the graph for μ and t.

24.4 PRINCIPLE OF LEAST SQUARES

The graphical method has the obvious drawback of being unable to give a unique curve of fit. *The principle of least squares, however, provides an elegant procedure for fitting a unique curve to a given data.*

Let the curve, $y = a + bx + cx^2 + ... + kx^{m-1}$ ...(1)

be fitted to the set of n data points $(x_1, y_1), (x_2, y_2)...(x_n, y_n)$.

Now we have to determine the constants $a, b, c,...k$ such that it represents the curve of best fit. In case $n = m$, on substituting the values (x_i, y_i) in (1), we get n equations from which a unique set of n constants can be found. But when $n > m$, we obtain n equations which are more than the m constants and hence cannot be solved for these constants. So we try to determine those values of $a, b, c,...k$ which satisfy all the equations as nearly as possible and thus may give the best fit. In such cases, we apply the *principle of least squares*.

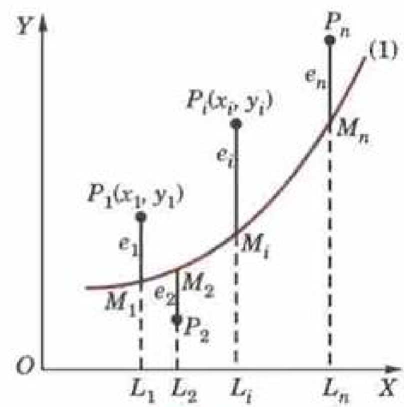


Fig. 24.4

At $x = x_i$, the *observed* (or *experimental*) *value* of the ordinate is $y_i = P_iL_i$ and the corresponding value on the fitting curve (1) is $a + bx_i + cx_i^2 + ... + kx_i^m = M_iL_i$ ($= \eta_i$, say) which is the *expected* (or *calculated*) *value* (Fig. 24.4). The difference of the observed and the expected values *i.e.* $y_i - \eta_i (= e_i)$ is called the *error* (or *residual*) at $x = x_i$. Clearly some of the errors $e_1, e_2, ... e_n$ will be positive and others negative. Thus to give equal weightage to each error, we square each of these and form their sum *i.e.* $E = e_1^2 + e_2^2 + ... e_n^2$.

The curve of best fit is that for which e's are as small as possible i.e., E, the sum of the squares of the errors is a minimum. This is known as the *principle of least squares* and was suggested by *Legendre** in 1806.

Obs. The principle of least squares does not help us to determine the form of the appropriate curve which can fit a given data. It only determines the best possible values of the constants in the equation when the form of the curve is known before hand. The selection of the curve is a matter of experience and practical considerations.

24.5 (1) METHOD OF LEAST SQUARES

For clarity, suppose it is required to fit the curve

$$y = a + bx + cx^2 \qquad ...(1)$$

to a given set of observations $(x_1, y_1), (x_2, y_2), ..., (x_5, y_5)$. For any x_i, the observed value is y_i and the expected value is $\eta_i = a + bx_i + cx_i^2$ so that the error $e_i = y_i - \eta_i$.

$\therefore$ the sum of the squares of these errors is

$$E = e_1^2 + e_2^2 + ... + e_5^2$$
$$= [y_1 - (a + bx_1 + cx_1^2)]^2 + [y_2 - (a + bx_2 + cx_2^2)]^2 + ... + [y_5 - (a + bx_5 + cx_5^2)]^2 \quad \text{[See § 5.12 (3)]}$$

For E to be minimum, we have

$$\frac{\partial E}{\partial a} = 0 = 2[y_1 - (a + bx_1 + cx_1^2)] - 2[y_2 - (a + bx_2 + cx_2^2)] - ... - 2[y_5 - (a + bx_5 + cx_5^2)] \qquad ...(2)$$

$$\frac{\partial E}{\partial b} = 0 = -2x_1[y_1 - (a + bx_1 + cx_1^2)] - 2x_2[y_2 - (a + bx_2 + cx_2^2)] - ... - 2x_5[y_5 - (a + bx_5 + cx_5^2)] \qquad ...(3)$$

$$\frac{\partial E}{\partial c} = 0 = -2x_1^2[y_1 - (a + bx_1 + cx_1^2)] - 2x_2^2[y_2 - (a + bx_2 + cx_2^2)] - ... - 2x_5^2[y_5 - (a + bx_5 + cx_5^2)] \qquad ...(4)$$

Equation (2) simplifies to

$$y_1 + y_2 + ... + y_5 = 5a + b(x_1 + x_2 + ... + x_5) + c(x_1^2 + x_2^2 + ... + x_5^2)$$

i.e.,
$$\Sigma y_i = 5a + b\Sigma x_i + c\,\Sigma x_i^2 \qquad ...(5)$$

* See footnote on p. 311.

Equation (3) becomes

$$x_1y_1 + x_2y_2 + ... + x_5y_5 = a(x_1 + x_2 + ... + x_5) + b(x_1^2 + x_2^2 + ... + x_5^2) + c(x_1^3 + x_2^3 + ... + x_5^3)$$

i.e., $$\Sigma x_i y_i = a\Sigma x_i + b\Sigma x_i^2 + c\Sigma x_i^3 \quad ...(6)$$

Similarly (4) simplifies to $\Sigma x_i^2 y_i = a\Sigma x_i^2 + b\Sigma x_i^3 + c\Sigma x_i^4$ ...(7)

The equations (5), (6) and (7) are known as *Normal equations* and can be solved as simultaneous equations in a, b, c. The values of these constants when substituted in (1) give the desired curve of best fit.

(2) Working procedure

(*a*) **To fit the straight line $y = a + bx$**

(*i*) Substitute the observed set of n values in this equation.

(*ii*) Form normal equations for each constant

i.e., $$\Sigma y = na + b\Sigma x, \ \Sigma xy = a\Sigma x + b\Sigma x^2$$

[The normal equation for the unknown a is obtained by multiplying the equations by the coefficient of a and adding. The normal equation for b is obtained by multiplying the equations by the coefficient of b (*i.e.*, x) and adding.]

(*iii*) Solve these normal equations as simultaneous equations for a and b.

(*iv*) Substitute the values of a and b in $y = a + bx$, which is the required line of best fit.

(*b*) **To fit the parabola : $y = a + bx + cx^2$**

(*i*) Form the normal equations $\Sigma y = na + b\Sigma x + c\Sigma x^2$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$$

and $$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

[The normal equation for c has been obtained by multiplying the equations by the coefficient of c (*i.e.*, x^2) and adding.]

(*ii*) Solve these as simultaneous equations for a, b, c.

(*iii*) Substitute the values of a, b, c in $y = a + bx + cx^2$, which is the required parabola of best fit.

(*c*) In general, the curve $y = a + bx + cx^2 + ... + kx^{m-1}$ can be fitted to a given data by writing m normal equations.

Example 24.4. *If P is the pull required to lift a load W by means of a pulley block, find a linear law of the form $P = mW + c$ connecting P and W, using the following data :*

P = 12	15	21	25
W = 50	70	100	120

where P and W are taken in kg-wt. Compute P when W = 150 kg. wt. (U.P.T.U., 2007 ; V.T.U., 2002)

Solution. The corresponding normal equations are

$$\left.\begin{aligned} \Sigma P &= 4c + m\Sigma W \\ \Sigma WP &= c\Sigma W + m\Sigma W^2 \end{aligned}\right\} \quad ...(i)$$

The values of ΣW etc. are calculated by means of the following table :

W	P	W²	WP
50	12	2500	600
70	15	4900	1050
100	21	10000	2100
120	25	14400	3000
Total = 340	73	31800	6750

∴ The equations (*i*) becomes $73 = 4c + 340m$ and $6750 = 340c + 31800m$

i.e., $$2c + 170m = 365 \quad ...(ii)$$

and $$34c + 3180m = 675 \quad ...(iii)$$

Multiplying (*ii*) by 17 and substracting from (*iii*), we get

$$m = 0.1879 \quad \therefore \text{ from } (ii), \quad c = 2.2785$$

Hence the line of best fit is

$$P = 2.2759 + 0.1879\ W$$

When $W = 150$ kg., $P = 2.2785 + 0.1879 \times 150 = 30.4635$ kg.

Obs. The calculations get simplified when the central values of x is zero. It is therefore, advisable to make the central value zero, if it be not so. This is illustrated by the next example.

Example 24.5. *Fit a second degree parabola to the following data :*

x	0	1	2	3	4
y	1	1.8	1.3	2.5	6.3

(P.T.U., 2006)

Solution. Let $u = x - 2$ and $v = y$ so that the parabola of fit $y = a + bx + cx^2$ becomes

$$v = A + Bu + Cu^2 \qquad ...(i)$$

The normal equations are

$$\Sigma v = 5A + B\Sigma u + C\Sigma u^2 \quad \text{or} \quad 12.9 = 5A + 10C$$
$$\Sigma uv = A\Sigma u + B\Sigma u^2 + C\Sigma u^2 \quad \text{or} \quad 11.3 = 10B$$
$$\Sigma u^2 v = A\Sigma u^2 + B\Sigma u^3 + C\Sigma u^4 \quad \text{or} \quad 33.5 = 10A + 34C$$

Solving these as simultaneous equations, we get

$$A = 1.48, \qquad B = 1.13, \qquad C = 0.55.$$

$\therefore$ (*i*) becomes, $\qquad v = 1.48 + 1.13u + 0.55u^2$

or $\qquad y = 1.48 + 1.13\ (x - 2) + 0.55\ (x - 2)^2$

Hence $y = 1.42 - 1.07x + 0.55x^2$.

Example 24.6. *Fit a second degree parabola to the following data :*

$x = 1.0$	1.5	2.0	2.5	3.0	3.5	4.0
$y = 1.1$	1.3	1.6	2.0	2.7	3.4	4.1

(V.T.U., 2009 ; Bhopal, 2008)

Solution. We shift the origin to (2.5, 0) and take 0.5 as the new unit. This amounts to changing the variable x to X, by the relation $X = 2x - 5$.

Let the parabola of fit be $y = a + bX + cX^2$. The values of ΣX etc., are calculated as below :

x	X	y	Xy	X^2	X^2y	X^3	X^4
1.0	−3	1.1	−3.3	9	9.9	−27	81
1.5	−2	1.3	−2.6	4	5.2	−8	16
2.0	−1	1.6	−1.6	1	1.6	−1	1
2.5	0	2.0	0.0	0	0.0	0	0
3.0	1	2.7	2.7	1	2.7	1	1
3.5	2	3.4	6.8	4	13.6	8	16
4.0	3	4.1	12.3	9	36.9	27	81
Total	0	16.2	14.3	28	69.9	0	196

The normal equations are

$$7a + 28c = 16.2\ ; \qquad 28b = 14.3\ ; \qquad 28a + 196c = 69.9$$

Solving these as simultaneous equations, we get

$$a = 2.07,\ b = 0.511,\ c = 0.061$$

$\therefore \qquad y = 2.07 + 0.511X + 0.061X^2$

Replacing X by $2x - 5$ in the above equation, we get

$$y = 2.07 + 0.511\ (2x - 5) + 0.061\ (2x - 5)^2$$

which simplifies to $y = 1.04 - 0.198x + 0.244x^2$. This is the required parabola of best fit.

Example 24.7. *Fit a second degree parabola to the following data :*

x	1989	1990	1991	1992	1993	1994	1995	1996	1997
y	352	356	357	358	360	361	361	360	359

(*U.P.T.U., 2009*)

Solution. Taking $u = x - 1993$ and $v = y - 357$, the equation $y = a + bx + cx^2$ becomes

$$v = A + Bu + Cu^2 \qquad ...(i)$$

x	$u = x - 1993$	y	$v = y - 357$	uv	u^2	u^2v	u^3	u^4
1989	−4	352	−5	20	16	−80	−64	256
1990	−3	360	−1	3	9	−9	−27	81
1991	−2	357	0	0	4	0	−8	16
1992	−1	358	1	−1	1	1	−1	1
1993	0	360	3	0	0	0	0	0
1994	1	361	4	4	1	4	1	1
1995	2	361	4	8	4	16	8	16
1996	3	360	3	9	9	27	27	81
1997	4	359	2	8	16	32	64	256
Total	$\Sigma u = 0$		$\Sigma v = 11$	$\Sigma uv = 51$	$\Sigma u^2 = 60$	$\Sigma u^2v = -9$	$\Sigma u^3 = 0$	$\Sigma u^4 = 708$

The normal equations are

$$\Sigma v = 9A + B\Sigma u + C\Sigma u^2 \quad \text{or} \quad 11 = 9A + 60C$$

$$\Sigma uv = A\Sigma u + B\Sigma u^2 + C\Sigma u^3 \quad \text{or} \quad 51 = 60B \quad \text{or} \quad B = \frac{17}{20}$$

$$\Sigma u^2 v = A\Sigma u^2 + B\Sigma u^3 + C\Sigma u^4 \quad \text{or} \quad -9 = 60A + 708C$$

On solving these equations, we get $A = \dfrac{694}{231}, B = \dfrac{17}{20}, C = -\dfrac{247}{924}$

$\therefore$ (*i*) becomes $\quad v = \dfrac{694}{231} + \dfrac{17}{20}u - \dfrac{247}{924}u^2$

or $\quad y - 357 = \dfrac{694}{231} + \dfrac{17}{20}(x - 1933) - \dfrac{247}{924}(x - 1933)^2$

or $\quad y = \dfrac{694}{231} - \dfrac{32861}{20} - \dfrac{247}{924}(1933)^2 + \dfrac{17}{20}x + \dfrac{247 \times 3866}{924}x - \dfrac{247}{924}x^2$

or $\quad y = 3 - 1643.05 - 998823.36 + 357 + 0.85x + 1033.44x - 0.267x^2$

Hence $\quad y = -1000106.41 + 1034.29x - 0.267x^2.$

PROBLEMS 24.2

1. By the method of least squares, find the straight line that best fits the following data :

x :	1	2	3	4	5
y :	14	27	40	55	68

(*U.P.T.U., 2008*)

2. Fit a straight line to the following data :

Year x :	1961	1971	1981	1991	2001
Production y :	8	10	12	10	16

(in thousand tons)

and find the expected production in 2006.

3. A simply supported beam carries a concentrated load P (lb) at its mid-point. Corresponding to various values of P, the maximum deflection Y (in) is measured. The data are given below :

P:	100	120	140	160	180	200
Y:	0.45	0.55	0.60	0.70	0.80	0.85

Find a law of the form $Y = a + bP$.

4. The results of measurement of electric resistance R of a copper bar at various temperatures t°C are listed below :

t :	19	25	30	36	40	45	50
R:	76	77	79	80	82	83	85

Find a relation $R = a + bt$ where a and b are constants to be determined by you.

5. Find the best possible curve of the form $y = a + bx$, using method of least squares for the data :

x :	1	3	4	6	8	9	11	14
y:	1	2	4	4	5	7	8	9

(V.T.U., 2011)

6. Fit a straight line to the following data

(a)

x:	1	2	3	4	5	6	7	8	9
y:	9	8	10	12	11	13	14	16	5

(Bhopal, 2008)

(b)

x:	6	7	7	8	8	8	9	9	10
y:	5	5	4	5	4	3	4	3	3

(J.N.T.U., 2008)

7. Find the parabola of the form $y = a + bx + cx^2$ which fits most closely with the observations :

x:	−3	−2	−1	0	1	2	3
y:	4.63	2.11	0.67	0.09	0.63	2.15	4.58

(V.T.U., 2006 ; J.N.T.U., 2000 S)

8. Fit a parabola $y = a + bx + cx^2$ to the following data :

x:	2	4	6	8	10
y:	3.07	12.85	31.47	57.38	91.29

(V.T.U., 2003 S)

9. Fit a second degree parabola to the following data :

x:	1	2	3	4	5	6	7	8	9	10
y:	124	129	140	159	228	289	315	302	263	210

(U.P.T.U., 2009)

10. The following table gives the results of the measurements of train resistances ; V is the velocity in miles per hour. R is the resistance in pounds per ton :

V:	20	40	60	80	100	120
R:	5.5	9.1	14.9	22.8	33.3	46.0

If R is related to V by the relation $R = a + bV + cV^2$, find a, b, and c. (U.P.T.U., 2002)

11. The velocity V of a liquid is known to vary with temperature according to a quadratic law $V = a + bT + cT^2$. Find the best values of a, b and c for the following table :

T:	1	2	3	4	5	6	7
V:	2.31	2.01	3.80	1.66	1.55	1.47	1.41

(U.P.T.U., MCA, 2010)

24.6 FITTING OF OTHER CURVES

(1) $y = ax^b$

Taking logarithms, $\log_{10} y = \log_{10} a + b \log_{10} x$

i.e., $Y = A + bX$ where $X = \log_{10} x$, $Y = \log_{10} y$ and $A = \log_{10} a$. (i)

$\therefore$ The normal equations for (i) are : $\Sigma Y = nA + b\Sigma X$, $\Sigma XY = A\Sigma X + b\Sigma X^2$

from which A and b can be determined. Then a can be calculated from $A = \log_{10} a$.

(2) $y = ae^{bx}$ (*Exponential curve*)

Taking logarithms, $\log_{10} y = \log_{10} a + bx \log_{10} e$

i.e., $Y = A + Bx$ where $Y = \log_{10} y$, $A = \log_{10} a$ and $B = b \log_{10} e$

Here the normal equations are : $\Sigma Y = nA + B\Sigma x$, $\Sigma xY = A\Sigma x + B\Sigma x^2$

from which A, B can be found and consequently a, b can be calculated.

(3) $xy^a = b$ (*or* $pv^\gamma = k$) (*Gas equation*)

Taking logarithms, $\log_{10} x + a \log_{10} y = \log_{10} b$ or $\log_{10} y = \dfrac{1}{a} \log_{10} b - \dfrac{1}{a} \log_{10} x$.

This is of the form $Y = A + BX$

where $X = \log_{10} x$, $Y = \log_{10} y$, $A = \frac{1}{a} \log_{10} b$, $B = -\frac{1}{a}$.

Here also the problem reduces to finding a straight line of best fit through the given data.

Example 24.8. *Find the least squares fit of the form $y = a_0 + a_1x^2$ to the following data :*

x :	−1	0	1	2
y :	2	5	3	0

(U.P.T.U., 2008)

Solution. Putting $x^2 = X$, we have $y = a_0 + a_1X$

$\therefore$ the normal equations are : $\Sigma y = 4a + a_1\Sigma X$; $\Sigma Y = a_0\Sigma X + a_1\Sigma X^2$.

The values of ΣX, ΣX^2 etc. are calculated below :

x	y	X	X^2	XY
−1	2	1	1	2
0	5	0	0	0
1	3	1	1	3
2	0	4	16	0
	$\Sigma y = 10$	$\Sigma X = 10$	$\Sigma X^2 = 18$	$\Sigma XY = 5$

$\therefore$ the normal equations become $\quad 10 = 400 + 6a_1$; $5 = 600 + 18a_1$

Solving these equations we get, $\quad a_0 = 4.167, a_1 = -1.111$.

Hence the curve of best fit is $\quad y = 4.167 - 1.111X$ *i.e.,* $y = 4.167 - 1.111x^2$.

Example 24.9. *An experiment gave the following values :*

v (ft/min) :	*350*	*400*	*500*	*600*
t (min) :	*61*	*26*	*7*	*26*

It is known that v and t are connected by the relation $v = at^b$. Find the best possible values of a and b.

Solution. We have $\quad \log_{10} v = \log_b a + b \log_{10} t$

or $\quad y = A + 6X$, where $X = \log_{10} t$, $y = \log_{10} v$, $A = \log_{10} a$

$\therefore$ the normal equations are

$$\Sigma Y = 4A + b\Sigma X \qquad ...(i)$$

$$\Sigma XY = A\Sigma X + b\Sigma X^2 \qquad ...(ii)$$

Now ΣX etc. are calculated as in the following table :

v	t	$X = \log_{10} t$	$y = \log_{10} v$	XY	X^2
350	61	1.7853	2.5441	4.542	3.187
400	26	1.4150	2.6021	3.682	2.002
500	7	0.8451	2.6990	2.281	0.714
600	2.6	0.4150	2.7782	1.153	0.172
Total ·		4.4604	10.6234	11.658	6.075

$\therefore$ Equations (*i*) and (*ii*) become

$$4A + 4.46b = 10.623 \text{ ; } 4.46A + 6.075b = 11.658$$

Solving these, $A = 2.845$, $b = -0.1697$

$\therefore$ $a = \text{antilog } A = \text{antilog } 2.845 = 699.8$.

Example 24.10. *Predict the mean radiation dose at an altitude of 3000 feet by fitting an exponential curve to the given data :*

Altitude (x) :	*50*	*450*	*780*	*1200*	*4400*	*4800*	*5300*
Dose of radiation (y) :	*28*	*30*	*32*	*36*	*51*	*58*	*69*

(S.V.T.U., 2007 ; J.N.T.U., 2003)

Solution. Let $y = ab^x$ be the exponential curve.

Then $\log_{10} y = \log_{10} a + x \log_{10} b$

or $Y = A + Bx$ where $Y = \log_{10} y, A = \log_{10} a, B = \log_{10} b$

$\therefore$ the normal equations are

$$\Sigma Y = 7A + B\,\Sigma x \quad ...(i)$$

$$\Sigma x\,Y = A\,\Sigma x + B\,\Sigma x^2 \quad ...(ii)$$

Now Σx etc. are calculated as follows :

x	y	$Y = \log_{10} y$	xY	x^2
50	28	1.447158	72.3579	2500
450	30	1.477121	664.7044	202500
780	32	1.505150	1174.0170	608400
1200	36	1.556303	1867.5636	1440000
4400	51	1.707570	7513.3080	19360000
4800	58	1.763428	8464.4544	23040000
5300	69	1.838849	9745.8997	28090000
Σ = 16980		11.295579	29502.305	72743400

$\therefore$ equations (i) and (ii) become

$$11.295579 = 7A + 16980B$$

$$29502.305 = 16980A + 72743400B$$

Solving these equations, we get $A = 1.4521015$, $B = 0.0000666289$

$$\therefore \quad \log_{10} y = Y = 1.4521015 + 0.0000666289x$$

Hence y (*at* $x = 3000$) = 44.874 *i.e.* 44.9 approx.

Example 24.11. *The pressure and volume of a gas are related by the equation $pv^{\gamma} = k$, γ and k being constants. Fit this equation to the following set of observations :*

p (kg/cm^2)	:	0.5	1.0	1.5	2.0	2.5	3.0
v (litres)	:	1.62	1.00	0.75	0.62	0.52	0.46

(V.T.U., 2011)

Solution. We have $\log_{10} p + \gamma \log_{10} v = \log_{10} k$

or $$\log_{10} v = \frac{1}{\gamma} \log_{10} k - \frac{1}{\gamma} \log_{10} p \quad \text{or} \quad Y = A + BX$$

where $X = \log_{10} p$, $Y = \log_{10} v$, $A = \dfrac{1}{\gamma} \log_{10} k$, $B = -\dfrac{1}{\gamma}$.

$\therefore$ the normal equations are

$$\Sigma Y = 6A + B\Sigma X \quad ...(i)$$

$$\Sigma XY = A\Sigma X + B\Sigma X^2 \quad ...(ii)$$

Now ΣX etc. are calculated as follows :

p	v	$X = \log_{10} p$	$Y = \log_{10} v$	XY	X^2
.5	1.62	− 0.3010	0.2095	− 0.0630	0.0906
1.0	1.00	0.0000	0.0000	− 0.0000	0.0000
1.5	0.75	0.1761	− 0.1249	− 0.0220	0.0310
2.0	0.62	0.3010	− 0.2076	− 0.0625	0.0906
2.5	0.52	0.3979	− 0.2840	− 0.1130	0.1583
3.0	0.46	0.4771	− 0.3372	− 0.1609	0.2276
Total		1.0511	− 0.7442	− 0.4214	0.5981

$\therefore$ equations (i) and (ii) become

$$6A + 1.0511B = -0.7442$$

$$1.0511A + 0.5981B = -0.4214$$

Solving these, we get $A = 0.0132$, $B = -0.7836$.

$\therefore$ $\gamma = -1/B = 1.1276$ and $k = \text{antilog}(A\gamma) = \text{antilog}(0.0168) = 1.039$.

Hence the equation of best fit is $pv^{1.276} = 1.039$.

PROBLEMS 24.3

1. If V (km/hr) and R (kg/ton) are related by a relation of the type $R = a + bV^2$, find by the method of least squares a and b with the help of the following table :

V :	10	20	30	40	50
R :	8	10	15	21	30

(*Indore, 2008*)

2. Using the method of least squares fit the curve $y = ax + bx^2$ to following observations :

x :	1	2	3	4	5
y :	1.8	5.1	8.9	14.1	19.8

3. Fit the curve $y = ax + b/x$ to the following data :

x :	1	2	3	4	5	6	7	8
y :	5.4	6.3	8.2	10.3	12.6	14.9	17.3	19.5

(*U.P.T.U., 2010*)

4. Estimate y at $x = 2.25$ by fitting the *indifference curve* of the form $xy = Ax + B$ to the following data :

x :	1	2	3	4
y :	3	1.5	6	7.5

(*J.N.T.U., 2003*)

5. Find the least square curve $y = ax + b/x$ for the following data :

x :	1	2	3	4
y :	−1.5	0.99	3.88	7.66

(*Madras, 2003*)

6. Predict y at $x = 3.75$, by fitting a *power curve* $y = ax^b$ to the given data :

x :	1	2	3	4	5	6
y :	298	4.26	5.21	6.10	6.80	7.50

(*J.N.T.U., 2003*)

7. Fit the curve of the form $y = ae^{bx}$ to the following data :

x :	77	100	185	239	285
y :	2.4	3.4	7.0	11.1	19.6

(*V.T.U., 2011 S ; J.N.T.U., 2006*)

8. Obtain the least squares fit of the form $f(t) = ae^{-3t} + be^{-2t}$ for the data :

x :	0.1	0.2	0.3	0.4
$f(t)$:	0.76	0.58	0.44	0.35

(*U.P.T.U., 2008*)

9. The voltage v across a capacitor at time t seconds is given by the following table :

t :	0	2	4	6	8
v :	150	63	28	12	5.6

Use the method of least squares to fit a curve of the form $v = ae^{kt}$ to this data.

10. Using method of least squares, fit a relation of the form $y = ab^x$ to the following data :

x :	2	3	4	5	6
y :	144	172.8	207.4	248.8	298.5

(*Tiruchirapalli, 2001*)

24.7 METHOD OF GROUP AVERAGES

Let the straight line, $y = a + bx$ ...(1)

fit the set of n observations

$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ quite closely. (Fig. 24.5)

When $x = x_1$, the *observed* (*or experimental*) *value of* $y = y_1 = L_1P_1$ and from (1),

$$y = a + bx_1 = L_1M_1,$$

which is known as the *expected* (or *calculated*) *value* of y at L_1.

Then e_1 = observed value at L_1 – expected value at L_1

$$= y_1 - (a + bx_1) = M_1P_1,$$

which is called the *error* (or *residual*) at x_1. Similarly the errors for the other observations are

$$e_2 = y_2 - (a + bx_2) = M_2P_2$$

....

$$e_n = y_n - (a + bx_n) = M_nP_n$$

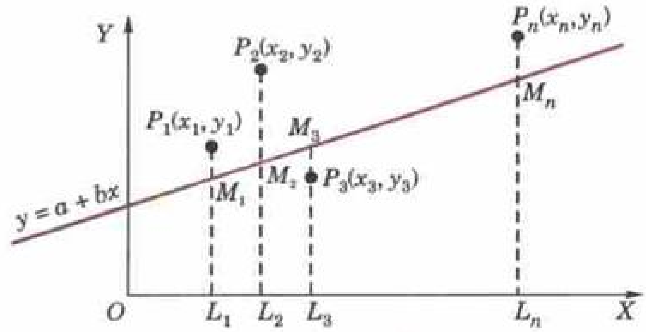


Fig. 24.5

Some of these errors may be positive and others negative.

The method of group averages is based on the assumption that the sum of the residuals is zero. To find the constants a and b is (1), we require two equations. As such we divide the data into two groups : the first containing k observations

$$(x_1, y_1), (x_2, y_2) \ldots (x_k, y_k);$$

and the second group having the remaining $n - k$ observations

$$(x_{k+1}, y_{k+1}), (x_{k+2}, y_{k+2}), \ldots, (x_n, y_n).$$

Assuming that the sum of the errors in each group is zero, we get

$$\{y_1 - (a + bx_1)\} + \{y_2 - (a + bx_2)\} + \ldots \{y_k - (a + bx_k)\} = 0$$

$$\{y_{k+1} - (a + bx_{k+1})\} + \{y_{k+2} - (a + bx_{k+2})\} + \ldots + \{y_n - (a + bx_n)\} = 0$$

On simplification, we obtain

$$\frac{y_1 + y_2 + \ldots + y_k}{k} = a + b\,\frac{x_1 + x_2 + \ldots + x_k}{k} \qquad \ldots(2)$$

$$\frac{y_{k+1} + y_{k+2} + \ldots + y_n}{n-k} = a + b\,\frac{x_{k+1} + x_{k+2} + \ldots + x_n}{n-k} \qquad \ldots(3)$$

In (2), $\frac{1}{k}(x_1 + x_2 + \ldots + x_k)$ and $\frac{1}{k}(y_1 + y_2 + \ldots + y_k)$ are simply the average values of x's and y's of the first group. Hence the equations (2) and (3) are obtained from (1) by replacing x and y by their respective averages of the two groups. Solving (2) and (3), we get a and b.

Obs. The main drawback of this method is that a different grouping of the observations will give different values of a and b. In practice, we divide the data in such a way that each group contains almost an equal number of observations.

Example 24.12. *The latent heat of vaporisation of steam r, is given in the following table at different temperatures t :*

t :	*40*	*50*	*60*	*70*	*80*	*90*	*100*	*110*
r :	*1069.1*	*1063.6*	*1058.2*	*1052.7*	*1049.3*	*1041.8*	*1036.3*	*1030.8*

For this range of temperature, a relation of the form r = a + bt is known to fit the data. Find the values of a and b by the method of group averages. (*Madras, 2003*)

Solution. Let us divide the data into two groups each containing four readings. Then we have

t	r	t	r
40	1069.1	80	1049.3
50	1063.6	90	1041.8
60	1058.2	100	1036.3
70	1052.7	110	1030.8
$\Sigma t = 220$	$\Sigma r = 4243.6$	$\Sigma t = 380$	$\Sigma r = 4158.2$

Substituting the averages of t's and r's of the two groups in the given relation, we get

$$\frac{4243.6}{4} = a + b\,\frac{220}{4} \quad i.e.,\ 1060.9 = a + 55b \qquad ...(i)$$

$$\frac{4158.2}{4} = a + b\,\frac{380}{4} \quad i.e.,\ 1039.55 = a + 95b \qquad ...(ii)$$

Solving (*i*) and (*ii*), we obtain

$$a = 1090.26,\ b = -0.534.$$

24.8 FITTING A PARABOLA

We have applied the method of averages to *linear law* involving two constants only. To fit the parabola

$$y = a + bx + cx^2 \qquad ...(1)$$

which contains three constants, to a set of observations, we proceed as follows :

Let (x_1, y_1) be a point on (1) satisfying the given data so that

$$y_1 = a + bx_1 + cx_1^2$$

Then $$y - y_1 = b\,(x - x_1) + c\,(x^2 - x_1^2)$$

or $$\frac{y - y_1}{x - x_1} = b + c\,(x + x_1)$$

Putting $x + x_1 = X$ and $(y - y_1)/(x - x_1) = Y$, it takes the linear form

$$Y = b + cX.$$

Now b and c can be found as before.

Example 24.13. *The corresponding values of x and y are given by the following table :*

x :	87.5	84.0	77.8	63.7	46.7	36.9
y :	292	283	270	235	197	181

Solution. Taking $x = 84, y = 283$ as a particular point on $y = a + bx + cx^2$,

we get $$283 = a + b\,(84) + c\,(84)^2 \qquad ...(i)$$

$\therefore$ $$y - 283 = b\,(x - 84) + c\,[x^2 - (84)^2]$$

or $$\frac{y - 283}{x - 84} = b + c\,(x + 84)$$

i.e., $$Y = b + cX \qquad ...(ii)$$

where $X = x + 84$, $Y = (y - 283)/(x - 84)$.

Now we have the following table of values :

x	y	$X = x + 84$	$Y = (y - 283)/(x - 84)$
87.5	292	171.5	2.571
84.0	283	—	—
77.8	270	161.8	2.097
		$\Sigma X = 333.3$	$\Sigma Y = 4.668$
63.7	235	147.7	2.364
46.7	197	130.7	2.306
36.9	181	120.9	2.166
		$\Sigma X = 399.3$	$\Sigma Y = 6.836$

Substituting the averages of X and Y in (*ii*), we get

$$\frac{4.668}{2} = b + c\,\frac{333.3}{2} \quad i.e.,\ 2.33 = b + 166.65\,c \qquad ...(iii)$$

$$\frac{6.836}{3} = b + c\,\frac{399.3}{3} \quad i.e.,\ 2.28 = b + 131.1\,c \qquad ...(iv)$$

(iv)–(iii) gives $c = 0.0014$

and (iii) gives $b = 2.0967$ *i.e.*, 2.1 nearly

From (i), we get $a = 96.9988$ *i.e.*, 97 nearly.

Hence the parabola of fit is

$$y = 97 + 2.1x + .0014x^2.$$

Example 24.14. *The train resistance R (lbs/ton) is measured for the following values of its velocity V (km/hr) :*

V :	20	40	60	80	100
R :	5	9	14	25	36

If R is related to V by the formula $R = a + bV^n$, find a, b, and n.

Solution. To find a, we take the following three values of v which are in G.P. :

$$v_1 = 20, \quad v_2 = 40, \quad v_3 = 80$$

Then $R_1 = 5, \quad R_2 = 9, \quad R_3 = 25$

$\therefore \quad (R_1 - a)(R_3 - a) = (R_2 - a)^2$

whence $$a = \frac{R_1R_3 - R_2^2}{R_1 + R_3 - 2R_2} = 3.67$$

Thus $R - 3.67 = bV^n$ or $\log_{10}(R - 3.67) = \log_{10} b + n \log_{10} V$

i.e., $$Y = k + nX \quad \text{...(i)}$$

where $X = \log_{10} V$, $Y = \log_{10}(R - 3.67)$, $k = \log_{10} b$.

Now we have the following table of values :

V	R	$X = \log_{10} V$	$Y = \log_{10}(R - 3.67)$
20	5	1.3010	0.1238
40	9	1.6021	0.7267
60	14	1.7782	1.0141
		$\Sigma X = 4.6813$	$\Sigma Y = 1.8646$
80	25	1.9031	1.3290
100	36	2.0000	1.5096
		$\Sigma X = 3.9031$	$\Sigma X = 2.8396$

Substituting the averages of X's and Y's in (i), we obtain

$$\frac{1.8646}{2} = k + n\,\frac{4.6813}{2} \qquad \textit{i.e.}, \ 0.6215 = k + 1.5604\,n \quad \text{...(ii)}$$

$$\frac{2.8386}{2} = k + n\,\frac{3.9031}{2} \qquad \textit{i.e.}, \ 1.4193 = k + 1.9516\,n \quad \text{...(iii)}$$

Solving (ii) and (iii), we get $n = 2.04$, $k = -2.56$ approx.

$$b = \text{antilog } k = \text{antilog } (-2.56) = 0.0028.$$

PROBLEMS 24.4

1. Fit a straight line of the form $y = a + bx$ to the following data by the method of group averages :

x :	0	5	10	15	20	25
y :	12	15	17	22	24	30

(*Tiruchirapalli, 2001*)

2. The weights of a calf taken at weekly intervals are given below :

Age :	1	2	3	4	5	6	7	8	9	10
Weight :	52.5	58.7	65.0	70.2	75.4	81.1	87.2	95.5	102.2	108.4

Find a straight line of best fit.

3. Using the method of averages, fit a parabola $y = ax^2 + bx + c$ to the following data :

x :	20	40	60	80	100	120
y :	5.5	9.1	14.9	22.8	33.3	46.0

4. While testing a centrifugal pump, the following data is obtained. It is assumed to fit the equation $y = a + bx + cx^2$, where x is the discharge in litre/sec and y, head in metres of water. Find the values of the constants a, b, c by the method of group averages.

x :	2	2.5	3	3.5	4	4.5	5	5.5	6
y :	18	17.8	17.5	17	15.8	14.8	13.3	11.7	9

5. By the method of averages, fit a curve of the form $y = ae^{bx}$ to the following data :

x :	5	15	20	30	35	40
y :	10	14	25	40	50	62

(Madras, 2002)

24.9 METHOD OF MOMENTS

Let $(x_1, y_1), (x_2, y_2), \ldots (x_n, y_n)$ be the set of n observations such that

$$x_2 - x_1 = x_3 - x_2 = \ldots = x_n - x_{n-1} = h \text{ (say)}$$

We define the *moments of the observed values* of y as follows :

m_1, the 1st moment $= h\Sigma y$

m_2, the 2nd moment $= h\Sigma xy$

m_3, the 3rd moment $= h\Sigma x^2 y$ and so on.

Let the curve fitting the given data be $y = f(x)$. Then the *moments of the calculated values* of y are

μ_1, the 1st moment $= \int y\,dx$

μ_2, the 2nd moment $= \int xy\,dx$

μ_3, the 3rd moment $= \int x^2 y\,dx$ and so on.

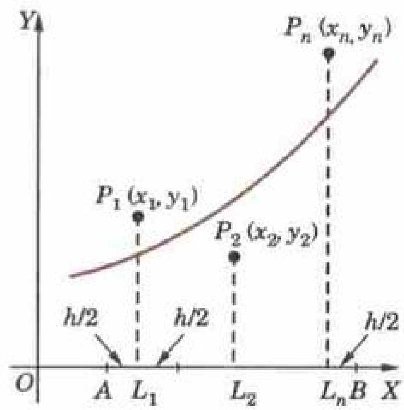


Fig. 24.6

This method is based on the assumption that the moment of the observed values of y are respectively equal to the moments of the calculated values of y i.e., $m_1 = \mu_1$, $m_2 = \mu_2$, $m_3 = \mu_3$ *etc. These equations (known as observation equations)* are used to determine the constants in $f(x)$.

m's are calculated from the tabulated values of x and y while μ's are computed as follows :

In Fig. 24.6, y_1 the ordinate of P_1 $(x = x_1)$, can be taken as the value of y at the mid-point of the interval $(x_1 - h/2, x_1 + h/2)$. Similarly, y_n, the ordinate of P_n $(x = x_n)$, can be taken as the value of y at the mid-point of the interval $(x_n - h/2, x_n + h/2)$. If A and B be the points such that

$$OA = x_1 - h/2 \text{ and } OB = x_n + h/2,$$

then
$$\mu_1 = \int y\,dx = \int_{x_1 - h/2}^{x_n + h/2} f(x)\,dx$$

$$\mu_2 = \int_{x_1 - h/2}^{x_n + h/2} xf(x)\,dx$$

and
$$\mu_3 = \int_{x_1 - h/2}^{x_n + h/2} x^2 f(x)\,dx.$$

Example 24.15. *Fit a straight line $y = a + bx$ to the following data by the method of moments :*

x :	1	2	3	4
y :	16	19	23	26

(Madras, 2001 S)

Solution. Since only two constants a and b are to be found, it is sufficient to calculate the first two moments in each case. Here $h = 1$.

$$m_1 = h\Sigma y = 1\,(16 + 19 + 23 + 26) = 84$$

$$m_2 = h\Sigma xy = 1\,(1 \times 16 + 2 \times 19 + 3 \times 22 + 4 \times 26) = 227$$

To compute the moments of calculated values of $y = a + bx$, the limits of integration will be $1 - h/2$ and $4 + h/2$ *i.e.*, 0.5 to 4.5

$$\therefore \qquad \mu_1 = 2\int_{0.5}^{4.5}(a + bx)\,dx = \left| ax + b\frac{x^2}{2} \right|_{0.5}^{4.5} = 4a + 10b$$

$$\mu_2 = \int_{0.5}^{4.5} x(a + bx)\,dx = 10a + \frac{91}{3}b.$$

Thus, the observation equations $m_r = \eta_r\,(r = 1, 2)$ are $4a + 10b = 84$; $10a + \frac{91}{3}b = 227$

Solving these, $a = 13.02$ and $b = 3.19$.

Hence the required equation is $y = 13.02 + 3.19x$.

Example 24.16. *Given the following data :*

x :	0	1	2	3	4
y :	1	5	10	22	38

find the parabola of best fit by the method of moments.

Solution. Let the parabola of best fit be $y = a + bx + cx^2$ ...(*i*)

Since three constants are to be found, we calculate the first three moments in each case. Here $h = 1$.

$$m_1 = h\Sigma y = 1\,(1 + 5 + 10 + 22 + 38) = 76$$
$$m_2 = h\Sigma xy = 1\,(0 + 5 + 20 + 66 + 152) = 243$$
$$m_3 = h\Sigma x^2y = 1\,(0 + 5 + 40 + 198 + 608) = 851$$

For computing the moments of calculated values of (*i*), the limits of integration will be $0 - h/2$ and $4 + h/2$ *i.e.*, -0.5 and 4.5.

$$\therefore \qquad \mu_1 = \int_{-0.5}^{4.5}(a + bx + cx^2)dx = 5a + 10b + 30.4c$$

$$\mu_2 = \int_{-0.5}^{4.5} x\,(a + bx + cx^2)\,dx = 10a + 30.4b + 102.5c$$

$$\mu_3 = \int_{-0.5}^{4.5} x^2\,(a + bx + cx^2)\,dx = 30.4a + 102.5b + 369.1c$$

Thus the observation equations $m_r = \mu_r\,(r = 1, 2, 3)$ are

$$5a + 10b + 30.4c = 76\ ;\ 10a + 30.4 + 102.5c = 243\ ;\ 30.4a + 102.5b + 369.1c = 851$$

Solving these equations, we get $a = 0.4$, $b = 3.15$, $c = 1.4$.

Hence the parabola of best fit is $y = 0.4 + 3.15x + 1.4x^2$.

PROBLEMS 24.5

1. Use the method of moments to fit the straight line $y = a + bx$ to the data :

x :	1	2	3	4
y :	0.17	0.18	0.23	0.32

2. Fit a straight line to the following data, using the method of moments :

x :	1	3	5	7	9
y :	1.5	2.8	4.0	4.7	6.0

(*Madras, 2001*)

3. Fit a parabola of the form $y = a + bx + cx^2$ to the data :

x :	1	2	3	4
y :	1.7	1.8	2.3	3.2

by the method of moments.

4. By using the method of moments, fit a parabola to the following data :

x :	1	2	3	4
y :	0.30	0.64	1.32	5.40

(*Madras, 2000 S*)

24.10 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 24.6

Fill up the blanks or choose the correct answer in the following problems :

1. The law $y = ax^2 + bx$ converted to linear form is ....
2. The gas equation $pv^r = k$ can be reduced to $y = a + bx$ where $a =$ .... and $b =$ ....
3. The principle of '*least squares*' states that ....
4. $y = ax^b + c$ in linear form is ....
5. To fit the straight line $y = mx + c$ to n observations, the normal equations are
   (*i*) $\Sigma y = n\,\Sigma x + \Sigma cm$, $\Sigma xy = c\,\Sigma x^2 + c\,\Sigma n$.
   (*ii*) $\Sigma y = m\,\Sigma x + nc$, $\Sigma xy = m\,\Sigma x^2 + c\,\Sigma x$.
   (*iii*) $\Sigma y = c\,\Sigma x + m\,\Sigma n$, $\Sigma xy = c\,\Sigma x^2 + m\,\Sigma x$.
6. To fit $y = ab^x$ by least square method, normal equations are ....
7. The observation equations for fitting a straight line by *method of moments* are ....
8. The *method of group averages* is based on the assumption that the sum of the residuals is ....
9. $y = ax^2 + b\log_{10} x$ reduced to linear law takes the form ....
10. Given $\begin{bmatrix} x: & 0 & 1 & 2 \\ y: & 0 & 1.1 & 2.1 \end{bmatrix}$ then the straight line of best fit is ....
11. The *method of moments* is based on the assumption that ....
12. In $y = a + bx$, $\Sigma x = 50$, $\Sigma y = 80$, $\Sigma xy = 1030$, $\Sigma x^2 = 750$ and $n = 10$, then $a =$ ...., $b =$ ....
13. $y = x/(ax + b)$ in linear form is ....
14. If $y = a + bx + cx^2$ and

x :	0	1	2	3	4
y :	1	1.8	1.3	2.5	7.3

then the first normal equation is :
(α) $15 = 5a + 10b + 29c$, (β) $15 = 5a + 10b + 31c$
(γ) $12.9 = 5a + 10b + 30c$ (δ) $34 = 5a + 10b + 27c$.

15. If $y = 2x + 5$ is the best fit for 8 pairs of values (x, y) by the method of least squares and $\Sigma y = 120$, then $\Sigma X =$
(*a*) 35 (*b*) 40 (*c*) 45 (*d*) 30.

CHAPTER 25

Statistical Methods

1. Introduction. 2. Collection and classification of data. 3. Graphical representation. 4. Comparison of frequency distributions. 5. Measures of central tendency. 6. Measures of dispersion. 7. Coefficient of variation; Relations between measures of dispersion. 8. Standard deviation of the combination of two groups. 9. Moments. 10. Skewness. 11. Kurtosis. 12. Correlation. 13. Coefficient of correlation. 14. Lines of regression. 15. Standard error of estimate. 16. Rank correlation. 17. Objective Type of Questions.

25.1 INTRODUCTION

Statistics deals with the methods for collection, classification and analysis of numerial data for drawing valid conclusions and making reasonable decisions. It has meaningful applications in production engineering, in the analysis of experimental data, etc. The importance of statistical methods in engineering is on the increase. As such we shall now introduce the student to this interesting field.

25.2 (1) COLLECTION OF DATA

The collection of data constitutes the starting point of any statistical investigation. Data may be collected for each and every unit of the whole lot (*population*), for it would ensure greater accuracy. But complete enumeration is prohibitively expensive and time consuming. As such out of a very large number of items, a few of them (a *sample*) are selected and conclusions drawn on the basis of this sample are taken to hold for the population.

(2) **Classification of data.** The data collected in the course of an inquiry is not in an easily assimilable form. As such, its proper classification is necessary for making intelligent inferences. The classification is done by dividing the raw data into a convenient number of groups according to the values of the variable and finding the frequency of the variable in each group.

Let us, for example, consider the raw data relating to marks obtained in Mechanics by a group of 64 students :

79	88	75	60	93	71	59	85
84	75	82	68	90	62	88	76
65	75	87	74	62	95	78	63
78	82	75	91	77	69	74	68
67	73	81	72	63	76	75	85
80	73	57	88	78	62	76	53
62	67	97	78	85	76	65	71
78	89	61	75	95	60	79	83