Exercise 11.1

(8) Verify for the Cobb-Douglas production function $P(L,k) = 1.01 L^{0.75} k^{0.25}$

discussed in example 2 that the production will be doubled if both the amount of labour and the amount of capital are doubled. Determined whether this is also true for the general production function

$$P(L,k) = bL^{\alpha} k^{1-\alpha}$$

Soln:

Here given function is,

$$P(L,k) = 1.01 L^{0.75} k^{0.25}$$

Let, $L = 2L$, $k = 2k$,

$$P(2L, 2k) = 1.01 (2L)^{0.75} (2k)^{0.25}$$
$$= 1.01\ 2^{0.75} \cdot L^{0.75} \cdot 2^{0.25} \cdot k^{0.25}$$
$$= 1.01\ 2\ L^{0.75} \cdot k^{0.25}$$
$$= 2 [1.01 L^{0.75} \cdot k^{0.25})$$
$$= 2 P(L,k)$$ proved

Also, general production function is,

$$P(L,k) = b L^{\alpha} k^{1-\alpha}$$

Let $L = 2k$, $k = 2k$

$$P(2L, 2k) = b (2L)^{\alpha} (2k)^{1-\alpha}$$
$$= b\ 2^{\alpha} L^{\alpha}\ 2^{1-\alpha} k^{1-\alpha}$$
$$= b\ 2\ L^{\alpha} k^{1-\alpha}$$
$$= 2 (b L^{\alpha} k^{1-\alpha})$$
$$= 2 P(L,k)$$

$\therefore P(2L, 2k) = 2P(L,k)$ proved

Scanned with CamScanner

[5] Find and sketch the domain of the function $f(x,y) = \ln(9 - x^2 - 9y^2)$. What is the range of f?

Soln Here given function is,

$$f(x,y) = \ln(9 - x^2 - 9y^2)$$

Since ln function is defined only for positive value, hence domain must be,

$$\cancel{-(x^2+9y^2) < 9}$$

$$9 - x^2 - 9y^2 \geq 0$$

For range, put $f(x,y) = z$,

$$z = \ln(9 - x^2 - 9y^2)$$

If $y = 0$

$$z = \ln(9 - x^2)$$

$$\Rightarrow e^z = 9 - x^2$$

$$\Rightarrow x^2 = e^z - 9$$

$$\Rightarrow x = \sqrt{e^z - 9}$$

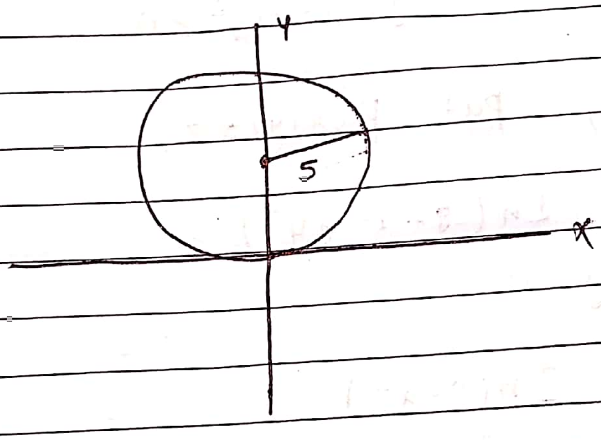
range, $e^z - 9 \geq 0$, Hence range of given function is all positive value which satisfied the condition $e^z - 9 \geq 0$.

[6] Find and sketch the domain of the Function $F(x,y) = \sqrt{y} + \sqrt{25-x^2-y^2}$.

Soln Here given Function,

$F(x,y) = \sqrt{y} + \sqrt{25-x^2-y^2}$

domain of function is $y \geq 0$ and $25-x^2-y^2 \geq 0$



[8] Let $g(x,y,z) = x^3y^2z\sqrt{10-x-y-z}$

(a) Evaluate $g(1,2,3)$

(b) Find and describe the domain of g.

Soln Here given Function is,

$g(x,y,z) = x^3y^2z\sqrt{10-x-y-z}$

(a) $g(1,2,3) = 1\cdot4\cdot3\sqrt{10-1-2-3}$

$= 12\sqrt{10-6}$

$= 12\times2 = 24$

(b) For domain,

$g(x,y,z) = x^3y^2z\sqrt{10-x-y-z}$

is defined, when,

$10-x-y-z \geq 0$

Hence domain of given function is $10-x-y-z \geq 0$.

(9) Draw a contour map of the function showing several level curve.

(a) $f(x,y) = (y-2x)^2$

Soln To Draw a Contour map we need Constant functional value,

Hence

Case I $F(x,y)=0 \Rightarrow (y-2x)^2 = y = 2x \rightarrow L_1$

Case II $f(x,y) = 1 \Rightarrow (y-2x)^2 = 1 \Rightarrow y-2x = 1$

$\Rightarrow y = 2x+1 - L_2$

Case II $F(x,y) = 2 \Rightarrow (y-2x)^2 = 2$

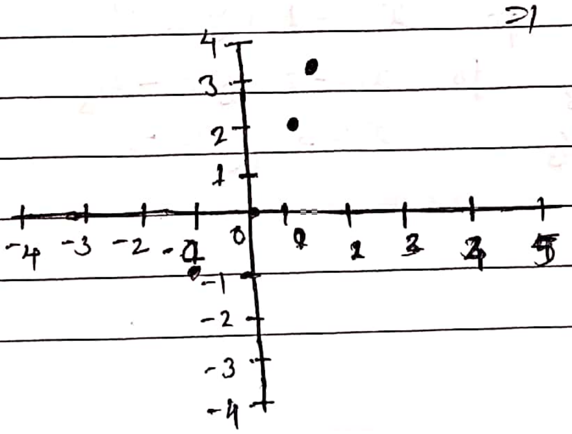
$\Rightarrow y-2x = \sqrt{2}$

$\Rightarrow y = 2x+\sqrt{2}$

Case III $F(x,y)=3 \Rightarrow (y-2x)^2 = 3$

$\Rightarrow y-2x = \sqrt{3}$

$\Rightarrow y = 2x+\sqrt{3}$



(20) $f(x,y) = x^3 - y$

Soln

Let $c = f(x,y) = x^3 - y$

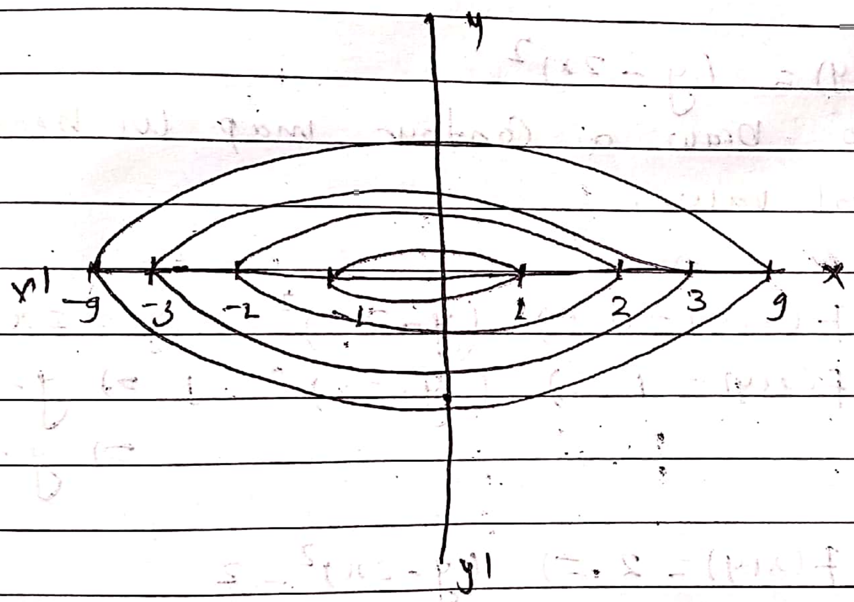
For $c=0$, $x^3 - y = 0 \Rightarrow x^3 = y$

For $c=1$, $x^3 - y = 1 \Rightarrow x^3 = y+1 \Rightarrow y = x^3 - 1$

For $c=2$, $x^3 - y = 2 \Rightarrow x^3 = y+2 \Rightarrow y = x^3 - 2$

For $c=3$, $x^2 + 9y^2 = 3 \Rightarrow \frac{x^2}{3} + 3y^2 = 1$

represents an ellipse with centre $(0,0)$ & axes $(3, 1/3)$



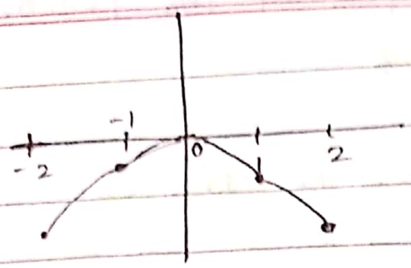
[27] Sketch both a Contour map and a graph of the Function and Compared them,

(a) $f(x,y) = x^2 + 9y^2 = 0$

For $x=0, y=0$ For $x=-1, y=-1/3$

For $x=1, y=-1/3$ for $x=-2, y=-2/3$

For $x=2$ $y=-2/3$ $x=3$ $y=-1$



Let $c = f(x,y) = x^2 + 9y^2$

For $c = 0$, $x^2 + 9y^2 = 0 \Rightarrow \frac{x^2}{9} + \frac{y^2}{1} = 0$ [e1]

represents an ellipse with centre (0,0) with axes (9,1)

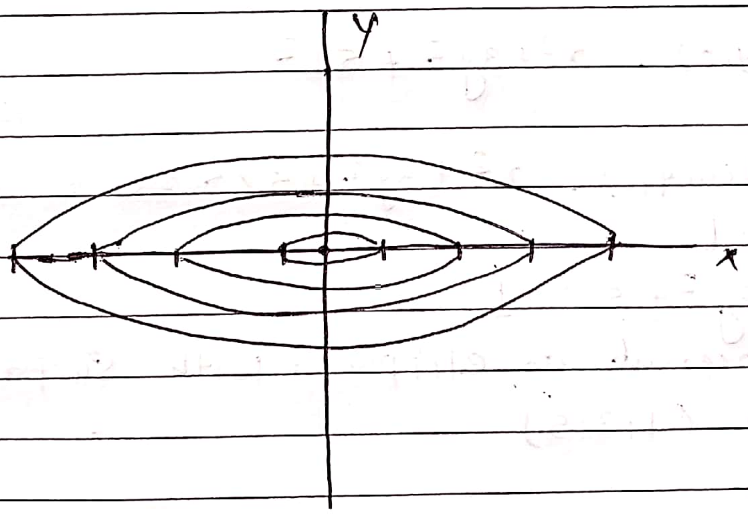
For $c = 1$, $x^2 + 9y^2 = 1 \Rightarrow \frac{x^2}{9} + y^2 = \frac{1}{9}$ [e2]

$\Rightarrow x^2 + 9y^2 = 1 \Rightarrow \frac{x^2}{1} + \frac{y^2}{1/9} = 1$

represents an ellipse with $c = (0,0)$ & axes $(1, 1/9)$

For $c = 2$, $x^2 + 9y^2 = 2 \Rightarrow \frac{x^2}{9} + \frac{y^2}{2/9} = 1$ (e3)

represents an ellipse with $c = (0,0)$ & axes $(9, 2/9)$



Describe the level surface of the Function,

[41] $F(x,y,z) = x+3y+5z$

Soln:

Let $F(x,y,z) = x+3y+5z$ is the equation of parallel planes with normal vector of plane $(1,3,5)$

when $c=1$,

$F(x,y,z) = x+3y+5z = 1$

when $c=2$,

$F(x,y,z) = x+3y+5z = 2$

when $c=3$,

$F(x,y,z) = x+3y+5z = 3$

when $c=4$,

$F(x,y,z) = x+3y+5z = 4$

Hence, we see that for any value of $c \in \mathbb{R}$ we obtain planes parallel with each other.

(42) $F(x,y,z) = x^2+3y^2+5z^2$

Soln:

let $F(x,y,z) = x^2+3y^2+5z^2 = c$

when $c=1$

$x^2+3y^2+5z^2 = 1$

$\rightarrow$ represent an ellipsoid with surface normal vector $(1,3,5)$

when $c=2$, $x^2+3y^2+5z^2=2$

represents an ellipsoid with SNV $(1,3,5)$

when $c=3$, $x^2+3y^2+5z^2 = 3$

represents an ellipsoid with SNV $(1,3,5)$

Ie. level surface of given function is a family of ellipsoids for $c>0$ & origin for $c=0$

Date
Page

Exercise 11.2

Find the limit, if it exists, or show that the limit does not exist.

[5] $\lim_{(x,y)\to(1,3)} (5x^3 - x^2y^2)$

Soln

Since, $5x^3 - x^2y^2$ is continious function, it is continious everywhere so we can find limit by direct substitution.

$\lim_{(x,y)\to(1,3)} (5x^3 - x^2y^2)$

$= 5\cdot(1)^3 - (1)^2(3)^2$

$= 5 - 9$

$= -4$

[8] $\lim_{(x,y)\to(0,0)} \dfrac{x^2+\sin^2 y}{2x^2+y^2}$

Soln

Let approach (0,0) first axis,

$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,0)\to(0,0)} \dfrac{x^2}{2x^2} = \frac{1}{2}$

Also for second axis,

$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(0,y)\to(0,0)} \dfrac{x^2+\sin^2 y}{2x^2+y^2}$

$= \dfrac{0}{1} = 0.$

Here, both are not equal so limit does not exist.

[Q3] Find $h(x,y) = g(F(x,y))$ and the set on which h is Continious,

(Q3) $g(t) = t^2 + \sqrt{t}$, $F(x,y) = 2x+3y-6$.

$$h(x,y) = h(\underbrace{x, f(x)}_{g(x)})$$

Here, $g(F(x,y)) = (2x+3y-6)^2 + \sqrt{(2x+3y-6)}$

Function $g(F(x,y))$ exists when

$$2x+3y-6 \geq 0$$
$$\Rightarrow 2x+3y \geq 6$$

Hence, Function $g(F(x,y))$ is Continious everywhere when $2x+3y \geq 6$.

[24] $g(t) = t + \ln t$, $f(x,y) = \dfrac{1-xy}{1+x^2y^2}$

Soln Here given function is,

~~Let~~

$$g(t) = t + \ln t, \quad f(x,y) = \frac{1-xy}{1+x^2y^2}$$

Hence, $g(f(x,y)) = \dfrac{1-xy}{1+x^2y^2} + \ln\left(\dfrac{1-xy}{1+x^2y^2}\right)$

$g(f(x,y))$ is defined when $x^2y^2 < 1$

Hence, the function $g(f(x,y))$ is continious everywhere when $xy < 1$

Scanned with CamScanner

Determine the set of points at which the function is continious,

(27) $f(x,y) = \arctan(x+\sqrt{y})$

Soln Here given function is

$$f(x,y) = \arctan(x+\sqrt{y})$$

Since $\arctan(x+\sqrt{y})$ is defined for every $y>0$ values. So the function $f(x,y) = \arctan(x+\sqrt{y})$ is continious everywhere when $y\geq 0$.

Hence set is $(0,\infty)$

(29) $G(x,y) = \ln(x^2+y^2-4)$

Soln

Given function is

$G(x,y) = \ln(x^2+y^2-4)$

Since logarithmic function is continous, Hence $G(x,y)$ is continious for every value of x, y.

In which

$$x^2+y^2 \geq 4$$

Hence set is, $[4,\infty]$

(31) $f(x,y,z) = \dfrac{\sqrt{y}}{x^2-y^2+z^2}$

Soln given function is,

$$f(x,y,z) = \frac{\sqrt{y}}{x^2-y^2+z^2}$$

Since function,

$f(x,y,z)$ is rational function we know that every rational function is continious. Hence the function is continious every where in R^3 except where $y < 0$.

Exercise 11.3

partial derivatives,

11, 15, 17, 18, 39, 43, 45, 40, 49, 51, 53, 57, 59, 71, 78,

[11] If $f(x,y) = 16 - 4x^2 - y^2$, find $f_x(1,2)$ and $f_y(1,2)$ and interpret these numbers as slopes. Illustrate with either hand-draws sketches.

Soln given function is,

$$f(x,y) = 16 - 4x^2 - y^2$$

partial derivatives w.r.t x is,

$$f_x(x,y) = -8x$$

partial derivatives w.r.t y is,

$$f_y(x,y) = -2y$$

now, $f_x(1,2) = -8$

$f_y(1,2) = -4$

The graph of f is paraboloid $z = 4 - x^2 - 2y^2$

Find the partial derivatives of the function.

[15] $f(x,y) = y^5 - 3xy$

Soln Here given function is,

$$f(x,y) = y^5 - 3xy \quad -(*)$$

Differenciate $f(x,y)$ w.r.t x keeping y constant

$$f_x(x,y) = -3y$$

Again, Differenciate $f(x,y)$ w.r.t y keeping x constant

$$f_y(x,y) = 5y^4 - 3x$$

(17) $f(x,t) = e^{-t}\cos \pi x$

Soln Here given function is,

$$f(x,t) = e^{-t}\cos \pi x \quad -(*)$$

Differenciate $(*)$ w.r.t t

$$f_t(x,t) = e^{-t}\cos \pi x\,(-1) = -e^{-t}\cos \pi x$$

Differenciate $(*)$ w.r.t x

$$f_x(x,t) = e^{-t} \cdot -\sin(\pi x)\cdot \pi$$
$$= -\pi e^{-t}\sin(\pi x)$$

(18) $F(x,t) = \sqrt{x} \ln t$

Soln given Function is,

$F(x,t) = \sqrt{x} \ln t$ — (*)

Differenciate (*) w.r.t x we get,

$F_x(x,t) = \frac{1}{2}(x)^{-1/2} \ln t$

$= \frac{1}{2\sqrt{x}} \ln t$

Differenciate (*) w.r.t t we get

$F_t(x,t) = \sqrt{x} \cdot \frac{1}{t}$

(39) Find Indicated partial derivatives

(a) $f(x,y) = \ln(x + \sqrt{x^2+y^2})$; $F_x(3,4)$

Soln Here given Function is,

$F(x,y) = \ln(x + \sqrt{x^2+y^2})$ — (*)

Differenciate (*) w.r.t x partially,

$$F_x(x,y) = \frac{\partial \ln(x+\sqrt{x^2+y^2})}{\partial(x+\sqrt{x^2+y^2})} \cdot \frac{\partial(x+\sqrt{x^2+y^2})}{\partial(x)}$$

$$= \frac{1}{(x+\sqrt{x^2+y^2})} \cdot \left(1 + \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2x\right)$$

$$= \frac{1 + \frac{x}{\sqrt{x^2+y^2}}}{x+\sqrt{x^2+y^2}}$$

Now,

$$f_x(3,4) = \frac{1+\frac{3}{\sqrt{9+16}}}{3+\sqrt{9+16}}$$

$$= \frac{1+\frac{3}{5}}{3+5}$$

$$= \frac{\frac{3+5}{5}}{3+5}$$

$$= \frac{8}{8\times 5} = \frac{1}{5}.$$

(43) Use the defination of partial derivatives as limit(4) to find $f_x(x,y)$ and $f_y(x,y)$

(43) $f(x,y) = xy^2 - x^3y$

Soln By defination partial derivatives of f with respect to x at (x,y) is given by

$$f_x(x,y) = g'(x)$$

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

Hence, $f_x(x,y) = y^2 - 3x^2y = g(x)$

$$g'(x) = y^2 - 3x^2y$$

Also,

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

$$\therefore f_y(x,y) = 2xy - x^3$$

[45] Use Impeicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$
$x^2+y^2+z^2 = 3xyz$

Soln given Function is,

$$x^2+y^2+z^2 = 3xyz$$

$$\therefore x^2+y^2+z^2 - 3xyz = 0$$

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial f}{\partial z}} \quad \text{--- (A)} \quad \& \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

$$\frac{\partial F}{\partial x} = 2x - 3yz \qquad \frac{\partial F}{\partial y} = 2y - 3xz$$

$$\frac{\partial F}{\partial z} = 2z - 3xy$$

Substitute value in (A)

$$\frac{\partial z}{\partial x} = -\frac{(2x-3yz)}{(2z-3xy)}$$

$$\frac{\partial z}{\partial z} = -\frac{(2y-3xz)}{(2z-3xy)}$$

Scanned with CamScanner

(40) Find Indicated partial derivatives,
$f(x,y) = \arctan(y/x) \quad f_x(2,3)$

Soln :- $f(x,y) = \arctan(y/x)$

we know that,

$$\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$$

so, $$\frac{\partial}{\partial x}(\arctan(y/x)) = \frac{1}{1+(y/x)^2} \cdot \frac{\partial (y/x)}{\partial (x)}$$

$$= \frac{1}{1+(y/x)^2} \cdot \cancel{y}$$

$$= \frac{y}{1+(y/x)^2}$$

$\therefore$ $$f_x(2,3) = \frac{3}{1+(3/2)^2} = \frac{3}{1+\frac{3^2}{2^2}} = \frac{3 \cdot 2^2}{2^2+3^2}$$

$$= \frac{3\times4}{4+9} = \frac{12}{13}$$

(49) find $\partial z/\partial x$ and $\partial z/\partial y$

(a) $z = f(x) + g(y)$

given Function is,
$z = f(x) + g(y)$

$$\frac{\partial z}{\partial x} = F'(x)$$

$$\frac{\partial z}{\partial y} = g'(y)$$

(b) $z = F(x+y)$

Soln given $z = f(x+y)$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} F(x+y) = F_x(x+y)$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} F(x+y) = F_y(x+y)$$

[51] Find all the second partial derivatives,

(a) $F(x,y) = x^3y^5 + 2x^4y$

given Function is,

$$f(x,y) = x^3y^5 + 2x^4y \quad -(1)$$

Differenciating (1) w.r.t x partially,

$$F_x(x,y) = 3x^2y^5 + 8x^3y \quad -(2)$$

again differenciating w.r.t x partially,

$$F_{xx}(x,y) = \frac{\partial(3x^2y^5 + 8x^3y)}{\partial x}$$

$$= 6xy^5 + 24x^2y.$$

Scanned with CamScanner

Date
Page

Differenciating ① w.r.t y partially,

$$F_y(x,y) = \frac{\partial}{\partial y}(x^3y^5 + 2x^4y) = 5x^3y^4 + 2x^4$$

again Differenciating w.r.t y partially,

$$F_{yy}(x,y) = \frac{\partial}{\partial y}(5x^3y^4 + 2x^4) = 20x^3y^3$$

[53] $w = \sqrt{u^2+v^2}$

Soln given Function is,

$$w = \sqrt{u^2+v^2} \quad — ⊛$$

Differenciating ⊛ w.r.t u partially,

$$\frac{\partial w}{\partial u} = \frac{\partial}{\partial u}\sqrt{u^2+v^2} = \frac{1}{2}(u^2+v^2)^{-1/2} \cdot 2u$$

$$= \frac{u}{\sqrt{u^2+v^2}}$$

again differenciating w.r.t u. partially,

$$\frac{\partial^2 w}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{u}{\sqrt{u^2+v^2}}\right)$$

$$= \frac{u \cdot \frac{\partial}{\partial u}(\sqrt{u^2+v^2}) - \sqrt{u^2+v^2} \cdot \frac{\partial}{\partial u} u}{(\sqrt{u^2+v^2})^2}$$

$$= \frac{\frac{u \cdot u}{\sqrt{u^2+v^2}} - \sqrt{u^2+v^2}}{u^2+v^2}$$

$$= \frac{u^2 - u^2 - v^2 / \sqrt{u^2+v^2}}{u^2+v^2} = \frac{\frac{-v^2}{\sqrt{u^2+v^2}}}{u^2+v^2} = \frac{-v^2}{(u^2+v^2)\sqrt{u^2+v^2}}$$

Date
Page

Differenciating (x) w.r.t v

$$\frac{\partial w}{\partial v} = \frac{\partial}{\partial v}\sqrt{u^2+v^2} = \frac{1}{2}(u^2+v^2)^{-1/2}\cdot 2v = \frac{v}{\sqrt{u^2+v^2}}$$

again differenciating w.r.t v

$$\frac{\partial^2 w}{\partial v^2} = \frac{\partial}{\partial v}\frac{v}{\sqrt{u^2+v^2}}$$

$$= \frac{v\cdot\frac{\partial}{\partial v}\sqrt{u^2+v^2} - \sqrt{u^2+v^2}\cdot\frac{\partial}{\partial v}\cdot v}{\left(\sqrt{u^2+v^2}\right)^2}$$

$$= \frac{v\cdot\frac{1}{2}(u^2+v^2)^{-1/2}\cdot 2v - \sqrt{u^2+v^2}}{u^2+v^2}$$

$$= \frac{\frac{v^2}{\sqrt{u^2+v^2}} - \sqrt{u^2+v^2}}{u^2+v^2} = \frac{\frac{v^2-u^2-v^2}{\sqrt{u^2+v^2}}}{u^2+v^2}$$

$$= \frac{-u^2}{(u^2+v^2)\sqrt{u^2+v^2}}$$

(57) Varify that the Conclusion of Clairaut's Theorem holds that is $u_{xy} = u_{yx}$

(a) $u = xe^{xy}$

Soln given function is,

$$u = xe^{xy} \quad \text{--- (i)}$$

Differentiating (i) w.r.t x partially,

$u_x = xe^{xy}\cdot y + e^{xy}$

$u_{xx} = y(xe^{xy}\cdot y + e^{xy}) + e^{xy}\cdot y$

$= xy^2e^{xy} + ye^{xy} + ye^{xy} = xy^2e^{xy} + 2ye^{xy}$

Scanned with CamScanner

$u_x = xe^{xy}.y + e^{xy}$

$u_{xy} = x(e^{xy}.xy + e^{xy}) + e^{xy}.x$

$= x^2ye^{xy} + xe^{xy} + xe^{xy}$

$= x^2ye^{xy} + 2xe^{xy}$ ← (*)

Differenciating (i) w.r.t y

$u_y = xe^{xy}.x$

$= x^2 e^{xy}$

$u_{yx} = 2xe^{xy} + x^2e^{xy}.y$

$= x^2ye^{xy} + 2xe^{xy}$ — (* *)

From (*) and (* *)
we can say,

$u_{xy} = u_{yx}$. proved

(59) Find the Indicated partial derivatives (s)

10) $f(x,y) = 3xy^4 + x^3y^2$; f_{xxy}, f_{yyy}

Soln given function is,

$f(x,y) = 3xy^4 + x^3y^2$

$f_x(x,y) = 3y^4 + 3x^2y^2$

$f_{xx}(x,y) = 6xy^2$

$f_{xxy}(x,y) = 12xy$

$F_y(x,y) = 12xy^3 + 2x^3y$

$F_{yy}(x,y) = 48xy^2 + 2x^3$

$F_{yyy}(x,y) = 96xy$

[71] Verify that the equation $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$ is a Solution of three-dimensional Laplace eqn $u_{xx} + u_{yy} + u_{zz} = 0$

Soln given function is,

$$u = \frac{1}{\sqrt{x^2+y^2+z^2}} = (x^2+y^2+z^2)^{-\frac{1}{2}}$$

$$u_x = -\frac{1}{2}(x^2+y^2+z^2)^{-2} \cdot 2x$$
$$= -x(x^2+y^2+z^2)^{-2}$$

$$u_{xx} = -2(x^2+y^2+z^2) + 2 \cdot 2x(x^2+y^2+z^2)^{-3} \cdot 2x$$
$$= -2(x^2+y^2+z^2) + 8x^2(x^2+y^2+z^2)^{-3}$$

$$u_y = -1(x^2+y^2+z^2) \cdot 2y$$
$$= -2y(x^2+y^2+z^2)^{-2}$$

$$u_{yy} = -2(x^2+y^2+z^2) - 2y \cdot (-2)(x^2+y^2+z^2) \cdot (2y)$$
$$= -2(x^2+y^2+z^2) + 8y^2(x^2+y^2+z^2)^{-3}$$

$$u_z = -2z(x^2+y^2+z^2)^{-2}$$

$$u_{zz} = -2(x^2+y^2+z^2) + 8z^2(x^2+y^2+z^2)^{-3}$$

Scanned with CamScanner

$u_{xx} + u_{yy} + u_{zz}$

$$= -2(x^2+y^2+z^2) + 8x^2(x^2+y^2+z^2)^{-3}$$
$$-2(x^2+y^2+z^2) + 8y^2(x^2+y^2+z^2)^{-3}$$
$$-2(x^2+y^2+z^2) + 8z^2(x^2+y^2+z^2)^{-3}$$

$$= -8(x^2+y^2+z^2) + 8x^2(x^2+y^2+z^2)^{-3} + 8y^2(x^2+y^2+z^2)^{-3} + 8z^2(x^2+y^2+z^2)^{-3}$$

$$= -8(x^2+y^2+z^2) + (x^2+y^2+z^2)^{-3}(8x^2+8y^2+8z^2)$$

$$u = \frac{1}{\sqrt{x^2+y^2+z^2}} = (x^2+y^2+z^2)^{-1/2}$$

$$u_x = -\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-\frac{1}{2}-1}\,\cancel{2}x = -1(x^2+y^2+z^2)^{-3/2}x$$

$$u_{xx} = -(x^2+y^2+z^2)^{-3/2} - 1\cdot -\frac{3}{\cancel{2}}(x^2+y^2+z^2)^{-5/2}\cdot \cancel{2}x\cdot x$$

$$= -(x^2+y^2+z^2)^{-3/2} + 3x^2(x^2+y^2+z^2)^{-5/2}$$

$$u_y = -1(x^2+y^2+z^2)^{-3/2}\cdot y$$

$$u_{yy} = -(x^2+y^2+z^2)^{-3/2} + 3y^2(x^2+y^2+z^2)^{-5/2}$$

$$u_z = -1(x^2+y^2+z^2)^{-3/2}\cdot z$$

$$u_{zz} = -(x^2+y^2+z^2)^{-3/2} + 3z^2(x^2+y^2+z^2)^{-5/2}$$

$$u_{xx}+u_{yy}+u_{zz} = -(x^2+y^2+z^2)^{-3/2} + 3x^2(x^2+y^2+z^2)^{-5/2}$$
$$-(x^2+y^2+z^2)^{-3/2} + 3y^2(x^2+y^2+z^2)^{-5/2}$$
$$-(x^2+y^2+z^2)^{-3/2} + 3z^2(x^2+y^2+z^2)^{-5/2}$$

$= -3(x^2+y^2+z^2)^{-3/2} + 3x^2(x^2+y^2+z^2)^{-5/2} +$
$3y^2(x^2+y^2+z^2)^{-5/2} + 3z^2(x^2+y^2+z^2)^{-5/2}$

$= -3(x^2+y^2+z^2)^{-3/2} + (x^2+y^2+z^2)^{-5/2}[3x^2+3y^2+3z^2]$

$= -3(x^2+y^2+z^2)^{-3/2} + (x^2+y^2+z^2)^{-5/2} \cdot 3(x^2+y^2+z^2)$

$= -3(x^2+y^2+z^2)^{-3/2} + 3(x^2+y^2+z^2)^{-5/2+1}$

$= -3(x^2+y^2+z^2)^{-3/2} + 3(x^2+y^2+z^2)^{-3/2}$

$= 0$

Hence, $u_{xx} + u_{yy} + u_{zz} = 0$

[72] Show that each of the following function is a solution of the wave equation $u_{tt} = a^2 u_{xx}$.

(a) $u = \sin(kx)\sin(akt)$

Soln

$u_t = \sin kx \cos(akt) \cdot ak$
$u_{tt} = -\sin kx \sin(akt) \cdot (ak)^2$
$= -(ak)^2 \sin(kx)\sin(akt)$
$= (a^2) \cdot -k^2 \sin(kx) \cdot \sin(akt)$
$u_x = \cos(kx) \cdot k \sin(akt)$
$u_{xx} = -\sin(kx) \cdot k^2 \sin(akt)$
$= -k^2 \sin(kx) \cdot \sin(akt)$

$\therefore u_{tt} = (a^2) \cdot u_{xx}$ Proved

(b) $u = \dfrac{t}{(a^2t^2-x^2)}$

$$u_t = \frac{t \cdot \frac{\partial}{\partial t}(a^2t^2-x^2) - (a^2t^2-x^2)}{(a^2t^2-x^2)^2}$$

$$= \frac{t \cdot [2a^2t] - a^2t^2 - x^2}{(a^2t^2-x^2)^2}$$

$$= \frac{2a^2t^2 - a^2t^2 - x^2}{(a^2t^2-x^2)^2} = \frac{a^2t^2-x^2}{(a^2t^2-x^2)^2}$$

$$= \frac{1}{(a^2t^2-x^2)}$$

$u_{tt} = \cancel{(-1)(a^2t^2-x^2)}$

$$= 1 \cdot a^2(2t) = \frac{2ta^2}{(a^2t^2-x^2)^2}$$

$$u_x = \frac{t(-2x)}{(a^2t^2-x^2)^2}$$

$u_{xx} = -2t$

$$= \frac{-2t \cdot (a^2t^2-x^2)^2 - (a^2t^2-x^2)^{2-1} \cdot (-2x) \cdot t}{(a^2t^2-x^2)^4}$$

$$= \frac{-2t(a^2t^2-x^2)^2 + (a^2t^2-x^2)2xt}{(a^2t^2-x^2)^4}$$

$$= \frac{-2t(a^2t^2-x^2)^2 + 2a^2t^3x - 2x^3t}{(a^2t^2-x^2)^4}$$

$$= \frac{2t[-(a^2t^2-x^2)^2 + (a^2t^2-x^2)x]}{(a^2t^2-x^2)^4}$$

$$= 2t \cdot a \frac{1}{(a^2t^2-x^2)^2}$$

$u_{tt} = a^2 \cdot u_{xx}$ proved

(c) $u = (x-at)^6 + (x+at)^6$

Soln

$u_t = 6(x-at)^5 \cdot (-a) + 6(x+at)^5 \cdot (a)$

$= -6a(x-at)^5 + 6a(x+at)^5$

$u_{tt} = -5 \times 6a(x-at)^4 \cdot (-a) + 6 \times 5a(x+at)^4 \cdot a$

$= +30a^2(x-at)^4 + 30a^2(x+at)^4$ — (*)

$u_x = 6(x-at)^5 + 6(x+at)^5$

$u_{xx} = 6 \times 5(x-at)^4 + 6 \times 5(x+at)^4$

$= 30(x-at)^4 + 30(x+at)^4$

From (*)

$u_{tt} = a^2[30(x-at)^4 + 30(x+at)^4]$

$u_{tt} = a^2 u_{xx}$ Proved

(d) $u = \sin(x-at) + \ln(x+at)$

Soln

$u_t = \cos(x-at) \cdot (-a) + \frac{1}{(x+at)} \cdot a$

$u_{tt} = -\sin(x-at) \cdot (-a) \cdot (-a) + \frac{a}{(x+at)^2} \cdot a$

$= -a^2 \sin(x-at) + \frac{a^2}{(x+at)^2}$ — (*)

$u_x = \cos(x-at) + \frac{1}{(x+at)} \cdot 1$

$u_{xx} = -\sin(x-at) + \frac{1}{(x+at)^2}$

From (*)

$u_{tt} = a^2\left[\sin(x-at) + \frac{1}{(x+at)^2}\right]$

$u_{tt} = a^2 u_{xx}$ Proved

[78] The temperature at a point (x,y) on a flat metal plate is given by $T(x,y) = 60/(1+x^2+y^2)$, where T is measured in °C and x, y in meters. Find the rate of change of temperature with respect to distance at the point $(2,1)$ in (a) the x-direction (b) the y-direction.

Soln Here given function is,

$$T(x,y) = \frac{60}{(1+x^2+y^2)}$$

$$T_x(x,y) = \frac{60 \times 2x}{(1+x^2+y^2)^2} = \frac{120x}{(1+x^2+y^2)^2}$$

$$T_y(x,y) = \frac{60 \times 2y}{(1+x^2+y^2)^2} = \frac{120y}{(1+x^2+y^2)^2}$$

$$\nabla T(x,y) = \frac{120x}{(1+x^2+y^2)^2} + \frac{120y}{(1+x^2+y^2)^2}$$

$$\nabla T(2,1) = \frac{120 \times 2}{(1+4+1)^2}\vec{i} + \frac{120 \times 1}{(1+4+1)^2}\vec{j}$$

$$= \frac{240}{36}\vec{i} + \frac{120}{36}\vec{j}$$

$$=$$

(a) the x-direction,

Unit vector along x-direction is $(1,0)$ so,

$$\nabla T(2,1)\cdot(1,0) = \left(\frac{240}{36}\vec{i} + \frac{120}{36}\vec{j}\right)\cdot(\vec{i},0)$$

$$= \frac{240}{36}$$

(b) In the y-axis

Unit vector along y-direction is, $(0, \vec{j})$

Hence,

$$\nabla T(2,1)\cdot(0,\vec{j}) = \left(\frac{240}{36}\vec{i} + \frac{120}{36}\vec{j}\right)\cdot(0,\vec{j})$$

$$= \frac{120}{36}$$

Hence rate of change alone x-direction is $\frac{240}{36}$ and rate of change alone y-direction is $\frac{120}{36}$

Scanned with CamScanner