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D. D. U. Gorakhpur University

GORAKHPUR - 273 009, INDIA

e-mail : nationalacademyofmathematics@gmail.com

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WEIBULL INVERSE EXPONENTIAL DISTRIBUTION: THEORY AND APPLICATIONS

By

Ramesh Kumar Joshi and Shankar Kumar Shrestha

Abstract

In this study, we have established a new distribution by using the Weibull generating family with baseline distribution as inverse exponential distribution called Weibull inverse exponential distribution. Some mathematical and statistical properties of the proposed distribution are discussed, including the shapes of the probability density function (PDF), cumulative density function (CDF) and hazard rate function (HRF), quantile function and the skewness, and kurtosis. We have discussed the maximum likelihood estimation method to estimate the parameters of the purposed distribution and also constructed the asymptotic confidence intervals based on maximum likelihood. All the computations have been performed in R software. The application of the proposed model has been illustrated considering two real data sets and investigated the goodness of fit attained by the observed model via different test statistics and graphical methods. We have found that the proposed distribution provided a better fit and more flexible in comparison with some other lifetime distributions.

Keywords : Weibull inverse exponential, MLE, LSE and CVME.
2010 AMS Subject Classification : 62F15, 65C05.

1. Introduction

The Weibull distribution is one of the most popular and widely used model for survival analysis and reliability theory. However, a weakness of this model as far as survival analysis is concerned is the monotonic behavior of its hazard rate function (HRF). In actual life applications, empirical hazard rate plot often display non-monotonic shapes such as a bathtub, inverted bathtub (unimodal) and others. So, there is a genuine need to investigate for some generalizations or modifications of the Weibull distribution that can give more flexibility in survival analysis and modeling. Let X be a random variable follows the Weibull distribution with scale parameter $\theta > 0$ and shape parameter $\lambda > 0$ and its CDF and PDF can be expressed as,

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$$F(x;\lambda,\theta)=1-\exp\left(-\theta x^{\lambda}\right)\ ;\ x>0,\ \text{and}$$

$$f(x;\lambda,\theta)=\lambda\theta x^{\lambda-1}\exp\left(-\theta x^{\lambda}\right);\ x>0.$$

Recently, several families of univariate continuous distributions have been seen in the statistical literature, allowing the creation of new flexible models with desirable properties in terms of data analysis. In particular, the generating families considering the Weibull distribution as main generator demonstrate a great capability in this regard, (Pham & Lai, 2007). There are some well-known Weibull generating families. The Weibull-G family of distributions has introduced by (Tahir et al., 2016) whose CDF and PDF are

$$\begin{aligned}F(x;\alpha,\beta,\psi)&=1-\int\limits_{0}^{-\log[G(x;\psi)]}\alpha\beta t^{\beta-1}\exp\left(-\alpha t^{\beta}\right)\\&=\exp\left(-\alpha\{-\ln G(x;\psi)\}^{\beta}\right),\end{aligned}$$

and

$$f(x;\alpha,\beta,\psi)=\alpha\beta\frac{g(x;\psi)}{G(x;\psi)}\{-\ln G(x;\psi)\}^{\beta-1}\exp\left(-\alpha\{-\ln G(x;\psi)\}^{\beta}\right).$$

Tahir et al. (2016) has taken six special models to study the capability of Weibull-G family namely Weibull-Uniform, Weibull-Weibull, Weibull-Rayleigh, Weibull-logistic, Weibull-log-logistic and Weibull-Burr XII distribution and found that the shape of density can be symmetrical, left skewed, right skewed, bathtub, J, reverse-J shaped and constant, increasing, decreasing, bathtub, inverted bathtub, J, reverse-J and S-shaped hazard rate. The Weibull-Pareto distribution has been introduced by (Alzaatreh et al., 2013, 2013a) by using the T-X family of distribution having CDF

$$G(x)=\int\limits_{0}^{-\log[1-F(x)]}r(t)dt,$$

where

$$r(t)=(c/\lambda)(t/\lambda)^{c-1}e^{-(t/\lambda)^{c}};\ t\geqslant 0.$$

Oguntunde et al. (2015) has defined the Weibull- exponential distribution using the Weibull generalized family of distribution introduced by (Bourguignon et al.,

2014) having CDF

$$G(x) = \int_0^{\frac{F(x)}{1-F(x)}} \alpha\beta t^{\beta-1} e^{-\alpha t^{\beta}} dt,$$

where $F(x)$ is the CDF of any baseline distribution. Another well-known Weibull-G (W-G) Family of Distributions was introduced by (Bourguignon et al., 2014). The CDF and PDF of W-G family are respectively as

$$F(x;\alpha,\beta,\psi) = 1 - \exp\left(-\alpha\left\{\frac{G(x;\psi)}{1-G(x;\psi)}\right\}^{\beta}\right); \alpha,\beta > 0, \tag{1.1}$$

and

$$f(x;\alpha,\beta,\psi) = \alpha\beta g(x;\psi)\frac{\{G(x;\psi)\}^{\beta-1}}{\{1-G(x;\psi)\}^{\beta+1}} \exp\left(-\alpha\left\{\frac{G(x;\psi)}{1-G(x;\psi)}\right\}^{\beta}\right), \tag{1.2}$$

here $G(x;\psi)$ and $g(x;\psi)$ are the CDF and PDF of any baseline distribution respectively and ψ is a parameter space of baseline distribution. This (1.1) and (1.2) were used by (Bourguignon et al., 2014) to define the CDF and PDF of Weibull-Uniform distribution, Weibull-Weibull distribution, Weibull-Burr XII distribution and Weibull-Normal distribution. These distributions has shown that the shape of density can be symmetrical, left skewed, right skewed, bathtub, J, reverse J shaped and constant, increasing, decreasing, bathtub, inverted bathtub, J, reverse J and S shaped hazard rate. With these motivations, the authors are focused to increase the flexibility of the one parameter inverse exponential distribution using the W-G family of distributions created by (Bourguignon et al., 2014). Our preference is based on the fact that this form of W-G family is a more flexible family of distributions.

The main aim of this paper is to create a more flexible distribution by adding just one extra parameter to the inverse exponential distribution using (1.1) and (1.2) to achieve a better fit to real data. We investigate the properties of the WIE distribution and illustrate its applicability. The contents of the proposed study are organized as follows. The Weibull inverse exponential distribution is introduced and various mathematical and statistical properties are discussed in Section 2. To estimate the model parameters, we have employed the maximum likelihood estimation (MLE) method and presented the asymptotic confidence interval for MLEs and Fisher information matrix in Section 3. In Section 4 two real data sets has been analyzed to explore the applications and suitability of the proposed distribution, also AIC, BIC, CAIC and HQIC are calculated to assess the goodness of fit of the WIE model. Finally, Section 5 ends up with some general concluding remarks.

2. The Weibull Inverse exponential (WIE) distribution

Using (1.1) and (1.2) we are introduced a new distribution where the baseline distribution is the inverse exponential distribution. The Inverse Exponential (IE) distribution has been introduced by (Keller & Kamath, 1982) and it has been studied and discussed as a lifetime model. If a random variable $X \sim IE(\lambda)$ then the variable $U = \frac{1}{X}$ will have an inverse exponential distribution and its CDF and PDF can be written as,

$$G(x) = e^{-\lambda/x};\ \lambda > 0, x > 0, \tag{2.1}$$

and

$$g(x) = \frac{\lambda}{x^2} e^{-\lambda/x};\ \lambda > 0, x > 0. \tag{2.2}$$

By substituting equations (2.1) and (2.2) in (1.1) and (1.2) we get the CDF and PDF of the three parameters Weibull inverse exponential distribution and can be expressed as follows

$$F(x;\alpha,\beta,\lambda) = 1 - \exp\left\{-\alpha\left(e^{\lambda/x} - 1\right)^{-\beta}\right\};\ (\alpha,\beta,\lambda) > 0,\ x > 0, \tag{2.3}$$

and

$$f(x;\alpha,\beta,\lambda) = \alpha\beta\lambda\, x^{-2} e^{\lambda/x}(e^{\lambda/x} - 1)^{-(\beta+1)} \exp\left\{-\alpha\left(e^{\lambda/x} - 1\right)^{-\beta}\right\}. \tag{2.4}$$

respectively. Figure 1 demonstrates the graph for PDF and hazard function for WIE distribution for different values of parameters. From Fig. 1 (left panel), the density function of the WIE distribution can bear different shapes according to the values of the parameters. Figure 1 (right panel) demonstrates the increasing, decreasing, increasing - decreasing and constant graph of the hazard function. This proves that WIE distribution is more flexible than Rayleigh distribution.

Survival function: The survival function $R(t)$, which is the probability of an item surviving up to time t, is defined by $R(t) = 1 - F(t)$. The survival /reliability function of a Weibull inverse exponential distribution is given by

$$R(t) = \exp\left\{-\alpha\left(e^{\lambda/x} - 1\right)^{-\beta}\right\};\ (\alpha,\beta,\lambda) > 0,\ x > 0. \tag{2.5}$$

The hazard rate function (HRF): Let t be survival time of a component or item and the probability that it will not survive for an additional time Δt

then, hazard rate function is,

$$h(t) = \frac{f(t)}{R(t)},$$

where $R(t)$ is a reliability function. Hence let, $X \sim WIE(\alpha, \beta, \lambda)$ then its hazard rate function is

$$h(x) = \alpha\beta\lambda\, x^{-2} e^{\lambda/x}(e^{\lambda/x} - 1)^{-(\beta+1)};\ (\alpha, \beta, \lambda) > 0,\ x > 0. \tag{2.6}$$

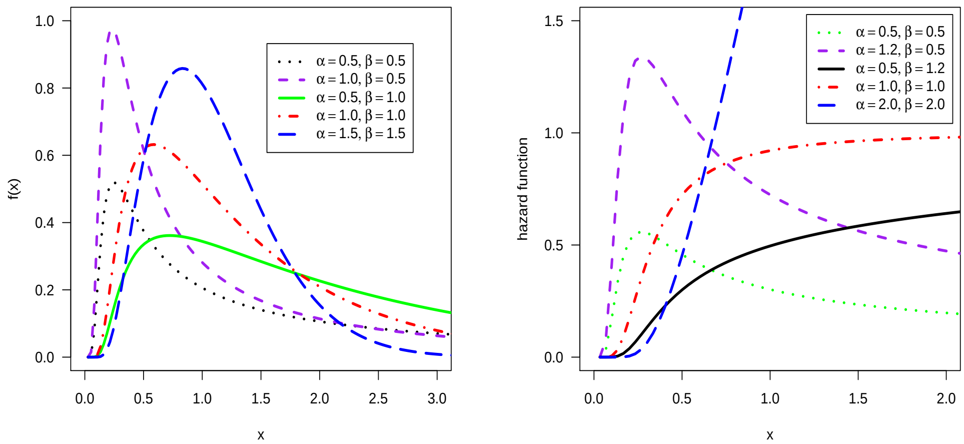


Figure 1. Plots of the probability density function(left panel) and hazard function (right panel), for λ=1 and different values of α.

Quantile function: The value of the p_{th} quantile can be obtained by solving the following equation,

$$Q(p) = F^{-1}(p),$$

and we get quantile function by inverting (2.3) as

$$Q(p) = \lambda\left[\ln\left\{\left(-\frac{1}{\alpha}\ln(1-p)\right)^{-1/\beta}\right\} - 1\right]^{-1};0 < p < 1. \tag{2.7}$$

For the generation of the random numbers of the WIE distribution, we suppose simulating values of random variable X with the CDF (2.3). Let U denote a uniform random variable in (0,1), then the simulated values of X can be obtained by

$$x = \lambda\left[\ln\left\{\left(-\frac{1}{\alpha}\ln(1-u)\right)^{-1/\beta}\right\} - 1\right]^{-1};0 < u < 1. \tag{2.8}$$

The median of WIE distribution can be obtained as

$$Median = \lambda\left[\ln\left\{\left(\frac{0.301}{\alpha}\right)^{-1/\beta}\right\} - 1\right]^{-1}. \tag{2.9}$$

Skewness and Kurtosis: The skewness and kurtosis measures are used in statistical analyses to characterize a distribution or a data set. The Bowley's skewness measure based on quartiles is given by

$$S_k = \frac{Q\,(0.75) + Q\,(0.25) - 2Q\,(0.5)}{Q\,(0.75) - Q\,(0.25)}, \tag{2.10}$$

and the Moors's kurtosis measure based on octiles (Moors (1988)) is given by

$$K_u = \frac{Q\,(0.875) - Q\,(0.625) + Q\,(0.375) - Q\,(0.125)}{Q\,(0.75) - Q\,(0.25)}, \tag{2.11}$$

where the $Q\,(.)$ is the quantile function.The skewness and kurtosis measures based on quantiles like Bowley's skewness and Moors's kurtosis have a number of advantages compared to the classical measures of skewness and kurtosis, e.g. they are less sensitive to outliers and they exist for the distributions even without defined the moments.

3. Maximum Likelihood Estimation

In this section, we have illustrated the maximum likelihood estimators (MLE's) of the WIE(α,β,λ) distribution. Let $\underline{x} = (x_1,\ldots,x_n)$ be the observed values of size n from WIE(α,β,λ) then the likelihood function for the parameter vector $\Phi = (\alpha,\beta,\lambda)^T$ can be written as,

$$L(\Phi) = \alpha\beta\lambda\prod_{i=1}^{n}\frac{1}{x_i^2}e^{\lambda/x_i}(e^{\lambda/x_i}-1)^{-(\beta+1)}\exp\left\{-\alpha\left(e^{\lambda/x_i}-1\right)^{-\beta}\right\}.$$

It is easy to deal with log-likelihood function as,

$$\ln L(\Phi) = n\ln(\alpha\beta\lambda) - 2\sum_{i=1}^{n}\ln x_i + \frac{\lambda}{x_i} - (\beta+1)\sum_{i=1}^{n}\ln(e^{\lambda/x_i}-1) - \alpha\sum_{i=1}^{n}(e^{\lambda/x_i}-1)^{-\beta}. \tag{3.1}$$

The elements of the score function $A\,(\Phi) = (A_\alpha, A_\beta, A_\lambda)$ are obtained as

$$
\begin{aligned}
A_\alpha &= \frac{n}{\alpha} - \sum_{i=1}^{n} \left(e^{\lambda/x_i} - 1\right)^{-\beta}, \\
A_\beta &= \frac{n}{\beta} - \sum_{i=1}^{n} \ln\left(e^{\lambda/x_i} - 1\right) + \alpha \sum_{i=1}^{n} \left(e^{\lambda/x_i} - 1\right)^{-\beta} \ln\left(e^{\lambda/x_i} - 1\right), \\
A_\lambda &= \frac{n}{\lambda} + \sum_{i=1}^{n} \frac{1}{x_i} - (\beta+1) \sum_{i=1}^{n} \frac{e^{\lambda/x_i}}{x_i(e^{\lambda/x_i} - 1)} + \alpha\beta \sum_{i=1}^{n} \frac{e^{\lambda/x_i}(e^{\lambda/x_i} - 1)^{-(1+\beta)}}{x_i}.
\end{aligned}
\tag{3.2}
$$

Equating A_α, A_β and A_λ to zero and solving these non-linear equations simultaneously gives the MLE $\hat{\Phi} = \left(\hat{\alpha}, \hat{\beta}, \hat{\lambda}\right)$ of $\Phi = (\alpha, \beta, \lambda)^T$. These equations cannot be solved analytically and by using the computer software R, Mathematica, Matlab, or any other programs and Newton-Raphson's iteration method, one can solve these equations. Let us denote the parameter vector by $\Phi = (\alpha, \beta, \lambda)^T$ and the corresponding MLE of Θ as $\hat{\Phi} = \left(\hat{\alpha}, \hat{\beta}, \hat{\lambda}\right)$, then the asymptotic normality results in, $\left(\hat{\Phi} - \Phi\right) \to N_3\left[0, \left(I\left(\Phi\right)\right)^{-1}\right]$ where $I\left(\Phi\right)$ is the Fisher's information matrix. The observed fisher information matrix $O\left(\hat{\Phi}\right)$ as an estimate of the information matrix $I\left(\Phi\right)$ given by

$$
O\left(\hat{\Phi}\right) = -\begin{pmatrix} A_{\alpha\alpha} & A_{\alpha\beta} & A_{\alpha\lambda} \\ A_{\alpha\beta} & A_{\beta\beta} & A_{\beta\lambda} \\ A_{\alpha\lambda} & A_{\beta\lambda} & A_{\lambda\lambda} \end{pmatrix}_{|(\hat{\alpha},\hat{\beta},\hat{\lambda})} = -H(\Phi)_{|(\Phi=\hat{\Phi})},
$$

here H is the Hessian matrix. The Newton-Raphson algorithm to maximize the likelihood produces the observed information matrix. Therefore, the variance-covariance matrix is given by

$$
\left[-H(\Phi)_{|(\Phi=\hat{\Phi})}\right]^{-1} = \begin{pmatrix} \mathrm{var}(\hat{\alpha}) & \mathrm{cov}(\hat{\alpha}, \hat{\beta}) & \mathrm{cov}(\hat{\alpha}, \hat{\lambda}) \\ \mathrm{cov}(\hat{\alpha}, \hat{\beta}) & \mathrm{var}(\hat{\beta}) & \mathrm{cov}(\hat{\lambda}, \hat{\beta}) \\ \mathrm{cov}(\hat{\alpha}, \hat{\lambda}) & \mathrm{cov}(\hat{\lambda}, \hat{\beta}) & \mathrm{var}(\hat{\lambda}) \end{pmatrix}
$$

Further differentiating (3.2) we get,

$$A_{\alpha\alpha} = -\frac{n}{\alpha^2}\,,$$

$$A_{\beta\beta} = -\frac{n}{\beta^2} - \sum_{i=1}^{n}\left\{\ln(e^{\lambda/x_i}-1)\right\}^2 (e^{\lambda/x_i}-1)^{-\beta}\,,$$

$$A_{\lambda\lambda} = -\frac{n}{\lambda^2} - \sum_{i=1}^{n}\frac{e^{\lambda/x_i}}{x_i^2(e^{\lambda/x_i}-1)^2}\left\{1+\beta-\alpha\beta x_i(e^{\lambda/x_i}-1)^{-\beta}\right\},$$

$$A_{\alpha\beta} = -\sum_{i=1}^{n}(e^{\lambda/x_i}-1)^{-\beta}\ln(e^{\lambda/x_i}-1)\,,$$

$$A_{\alpha\lambda} = \beta\sum_{i=1}^{n}(1/x_i)\,e^{\lambda/x_i}(e^{\lambda/x_i}-1)^{-(1+\beta)}\,,$$

$$A_{\beta\lambda} = -\sum_{i=1}^{n}\frac{e^{\lambda/x_i}}{x_i(e^{\lambda/x_i}-1)}.$$

Hence from the asymptotic normality of MLEs, approximate $100(1-\gamma)\%$ confidence intervals for α, β and λ can be constructed as,

$$\hat{\alpha} \pm z_{\gamma/2}\sqrt{\mathrm{var}(\hat{\alpha})},\ \hat{\beta} \pm z_{\gamma/2}\sqrt{\mathrm{var}(\hat{\beta})} \text{ and } \hat{\lambda} \pm z_{\gamma/2}\sqrt{\mathrm{var}(\hat{\lambda})},$$

where $z_{\gamma/2}$ is the upper percentile of standard normal variate.

4. Data Analysis: Application

In this section, we illustrate the applicability of WIE distribution by considering two real datasets used by different researchers.

Dataset I: The data below are from an accelerated life test of 59 conductors, (Nelson Doganaksoy, 1995). The failures can occur in microcircuits because of the movement of atoms in the conductors in the circuit; this is referred to as electro-migration. The failure times are in hours, and there are no censored observations, (Lawless, 2003).

6.545, 9.289, 7.543, 6.956, 6.492, 5.459, 8.120, 4.706, 8.687, 2.997,
8.591, 6.129, 11.038, 5.381, 6.958, 4.288, 6.522, 4.137, 7.459, 7.495,
6.573, 6.538, 5.589, 6.087, 5.807, 6.725, 8.532, 9.663, 6.369, 7.024,
8.336, 9.218, 7.945, 6.869, 6.352, 4.700, 6.948, 9.254, 5.009, 7.489,
7.398, 6.033, 10.092, 7.496, 4.531, 7.974, 8.799, 7.683, 7.224, 7.365,
6.923, 5.640, 5.434, 7.937, 6.515, 6.476, 6.071, 10.491, 5.923.

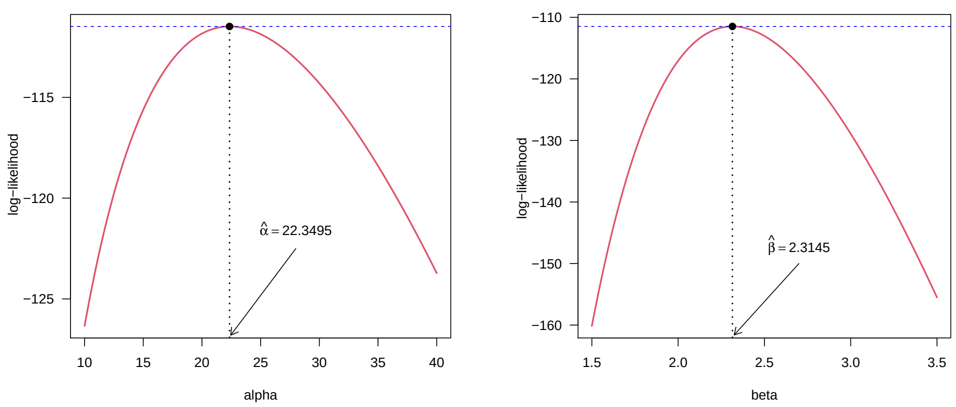
We have presented the MLEs directly by using optim() function (R Core Team, 2015) and Rizzo (2008) by maximizing the likelihood function (3.1). We have obtained $\hat{\alpha} = 22.3495$, $\hat{\beta} = 2.3145$, $\hat{\lambda} = 11.8104$ and corresponding value of log-likelihood is -111.4818. Using the method described in Section 3, we can construct the approximate confidence intervals(ACI) based on MLE's. Table 1 shows the MLE's with their standard errors(SE) and 95% confidence intervals for α, β and λ. In Table 1 we have presented the MLE's with their standard errors (SE) and 95% asymptotic confidence intervals(ACI) for α, β and λ.

Table 1

MLE, SE and 95% confidence intervals

Parameter	**MLE**	**SE**	**95% ACI**
alpha	22.3495	3.6957	(15.1059, 29.5931)
beta	2.3145	0.3905	(1.5491, 3.0799)
lambda	11.8104	1.5181	(8.8349, 14.7859)

In Figure 2 we have displayed the graph of profile log-likelihood functions of ML estimates of α, β and λ. We have noticed that ML estimates of α, β and λ exist and can be obtained uniquely. To evaluate the goodness of fit of a given distribution we generally use the PDF and CDF plot. To get the additional information we have to plot Q-Q and P-P plots. In particular, the Q-Q plot may provide information about the lack-of-fit at the tails of the distribution, whereas the P-P plot emphasizes the lack-of-fit, (Kumar & Ligges 2011). From Figure 3 we have shown that the WIE model fits the data very well.



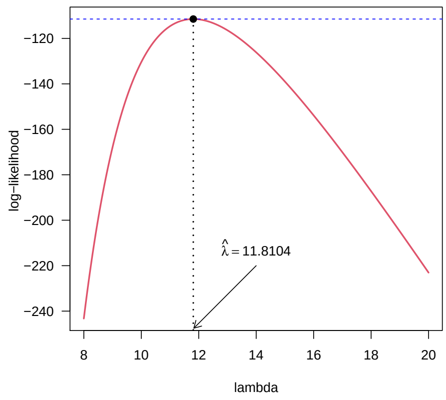


Figure 2. Profile log-likelihood functions of α, β and λ.

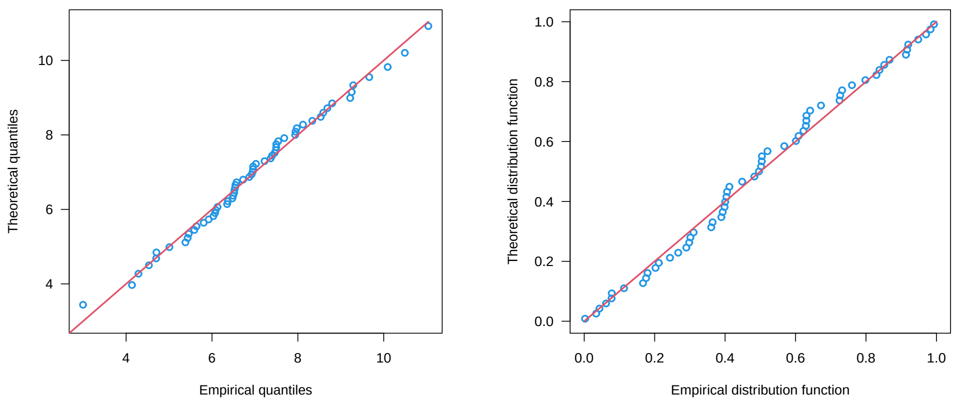


Figure 3. PP plot(left panel) and QQ plot(right panel).

We have considered some alternatives distributions for the comparison of goodness of fit and flexibility of the observed distribution are as follows.

(i) Weibull distribution: The probability density function of Weibull (W) distribution is

$$f_W\left(x\right) = \frac{\theta}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta-1} e^{-(x/\lambda)^{\theta}};\, (\lambda, \theta) > 0, x \geqslant 0.$$

(ii) Weibull Extension (WE) Model: The probability density function of Weibull extension (WE) distribution (Tang et al., 2003) with three parameters (α, β, λ) is

$$f_{WE}(x;\alpha,\beta,\lambda) = \lambda\beta\left(\frac{x}{\alpha}\right)^{\beta-1}\exp\left(\frac{x}{\alpha}\right)^{\beta}$$
$$\exp\left\{-\lambda\alpha\left(\exp\left(\frac{x}{\alpha}\right)^{\beta}-1\right)\right\} \quad ;\ x>0,(\alpha,\beta,\lambda)>0.$$

(iii) Generalized Exponential (GE) distribution: The probability density function of generalized exponential distribution (Gupta & Kundu, 1999) is.

$$f_{GE}\left(x;\,\alpha,\,\lambda\right) = \alpha\,\lambda\,e^{-\lambda\,x}\left\{1-e^{-\lambda\,x}\right\}^{\alpha-1};\ (\alpha,\ \lambda)>0,\ x>0.$$

(iv) Inverse Weibull (IW) distribution The probability density function of IW distribution is

$$f_{IW}\left(x\right) = \alpha\beta x^{-(\beta+1)}\exp\left(-\alpha x^{-\beta}\right);\ x\geqslant 0,\ \alpha>0,\ \beta>0.$$

By using MLE method we estimate the parameter of each of these distributions. For the goodness of fit purpose we use log-likelihood $(l(\hat{\theta}))$ where $\hat{\theta} = (\hat{\alpha},\hat{\beta},\hat{\lambda})$ to compute Akaike information criterion (AIC), Bayesian information criterion (BIC), Corrected Akaike Information criterion (CAIC) and Hannan-Quinn information criterion (HQIC), statistic to select the best model among selected models. The expressions to calculate AIC, BIC, CAIC and HQIC are listed below:

$$AIC = -2l(\hat{\theta}) + 2k\,,$$
$$BIC = -2l(\hat{\theta}) + k\log\left(n\right)\ ,$$
$$CAIC = AIC + \frac{2k\left(k+1\right)}{n-k-1}\,,$$
$$HQIC = -2l(\hat{\theta}) + 2k\log\left[\log\left(n\right)\right]\ ,$$

where k is the number of parameters and n is the size of the sample in the model under consideration. Further, in order to evaluate the fits of the WIE distribution with some selected distributions we have taken the Kolmogorov-Simnorov (KS), the Anderson-Darling (W) and the Cramer-Von Mises (A^2) statistic. These statistics are widely used to compare non-nested models and to illustrate how closely a specific CDF fits the empirical distribution to the given data set. These statistics are calculated as

$$KS = \max_{1\leqslant i\leqslant n}\left(d_i - \frac{i-1}{n}, \frac{i}{n} - d_i\right),$$

$$W = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\ln d_i + \ln\left(1-d_{n+1-i}\right)\right],$$

$$A^2 = \frac{1}{12n} + \sum_{i=1}^{n}\left[\frac{(2i-1)}{2n} - d_i\right]^2,$$

where $d_i = CDF(x_i)$; the x_i's being the ordered observations, (D'Agostino and Stephens,1986).

For the assessment of potentiality of the proposed model we have calculated the Akaike information criterion (AIC), Bayesian information criterion (BIC), Corrected Akaike information criterion (CAIC) and Hannan-Quinn information criterion (HQIC) and these are presented in Table 2.

Table 2

Log-likelihood (LL), AIC, BIC, CAIC and HQIC

Model	-LL	AIC	BIC	CAIC	HQIC
WIE	111.4818	228.9637	235.1963	229.4	231.3966
Weibull	112.4973	228.9946	233.1496	229.2088	230.6165
WE	113.6745	233.3491	239.5817	233.7855	235.7821
GE	114.9473	233.8946	238.0497	234.1089	235.5166
IW	126.6335	257.2671	261.4221	257.4813	258.889

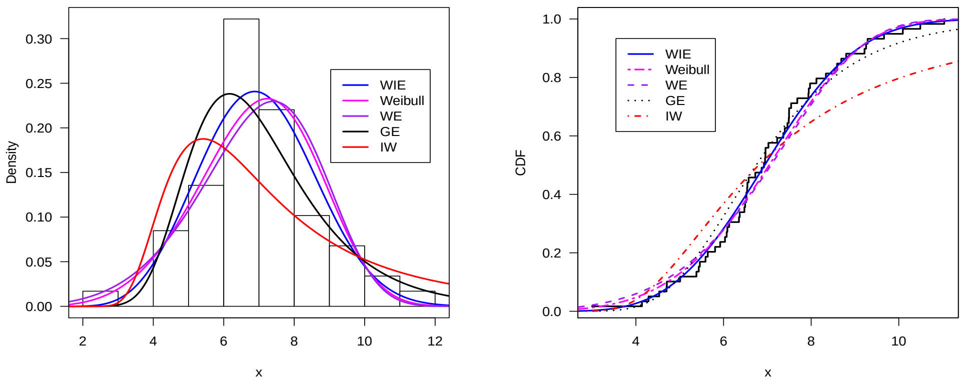


Figure 4. The Histogram and the PDF of fitted distributions (left panel); Empirical CDF with estimated CDF(right panel).

The Histogram and the density function of fitted distributions and Empirical distribution function with estimated distribution function of WIE, generalized Gompertz (GG), generalized exponential extension (GEE), exponential extension

(EE), Weibull and EEP distributions are presented in Figure 4. To compare the goodness-of-fit of the WIE distribution with other competing distributions we have presented the value of Kolmogorov-Simnorov (KS), the Anderson-Darling (AD) and the Cramer-Von Mises (CVM) statistics. These three statistics are widely used to compare non-nested models and to illustrate how closely a specific CDF fits the empirical distribution of a given data set. From Table 3 the result shows that the WIE distribution has the minimum value of the test statistic and higher p-value hence we conclude that the WIE distribution gets quite better fit and more consistent and reliable results from others taken for comparison.

Table 3

The KS, A^2 and W statistics with p-value

Model	*KS(p-value)*	*AD(p-value)*	*CVM(p-value)*
WIE	0.0704(0.9117)	0.0424(0.9220)	0.2411(0.9748)
Weibull	0.0956(0.6194)	0.0840(0.6707)	0.4773(0.7693)
WE	0.1067(0.4797)	0.1154(0.5160)	0.6800(0.5751)
GE	0.1042(0.5103)	0.1173(0.5079)	0.7368(0.5282)
IW	0.1774(0.0427)	0.5830(0.0243)	3.5372(0.0148)

Dataset II: For the illustration, we are using a real data set that was used by Hinkley (1977). The data represents thirty successive values of March precipitation (inches) for Minneapolis/St Paul.

0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20,
3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81,
2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05

We have obtained the MLEs with their standard errors (SE) in parenthesis of WIE distribution as $\hat{\alpha} = 0.5307(0.9969), \hat{\beta} = 1.3575(0.4111), \hat{\lambda} = 0.8822(0.8385)$ and corresponding value of log-likelihood is -37.8602.
In Figure 5 we have displayed the graph of profile log-likelihood functions of ML estimates of α, β and λ. We have noticed that ML estimates of α, β and λ exist and can be obtained uniquely.

To evaluate the goodness of fit of a given distribution we generally use the PDF and CDF plot. To get the additional information we have to plot Q-Q and P-P plots. In particular, the Q-Q plot may provide information about the lack-of-fit at the tails of the distribution, whereas the P-P plot emphasizes the lack-of-fit. From Figure 6 we have shown that the WIE model fits the data very well.

For the assessment of potentiality of the proposed model we have calculated the Akaike information criterion (AIC), Bayesian information criterion (BIC), Corrected Akaike information criterion (CAIC) and Hannan-Quinn information criterion (HQIC) and these are presented in Table 4.

Table 4

Log-likelihood (LL), AIC, BIC, CAIC and HQIC

Model	-LL	AIC	BIC	CAIC	HQIC
WIE	37.8602	81.7204	85.924	82.6434	83.0651
Weibull	38.6433	81.2866	84.089	81.731	82.1831
WE	38.6654	83.3308	87.5344	84.2539	84.6756
GE	38.0943	80.1885	82.9909	80.633	81.085
IW	41.917	87.834	90.6364	88.2785	88.7305

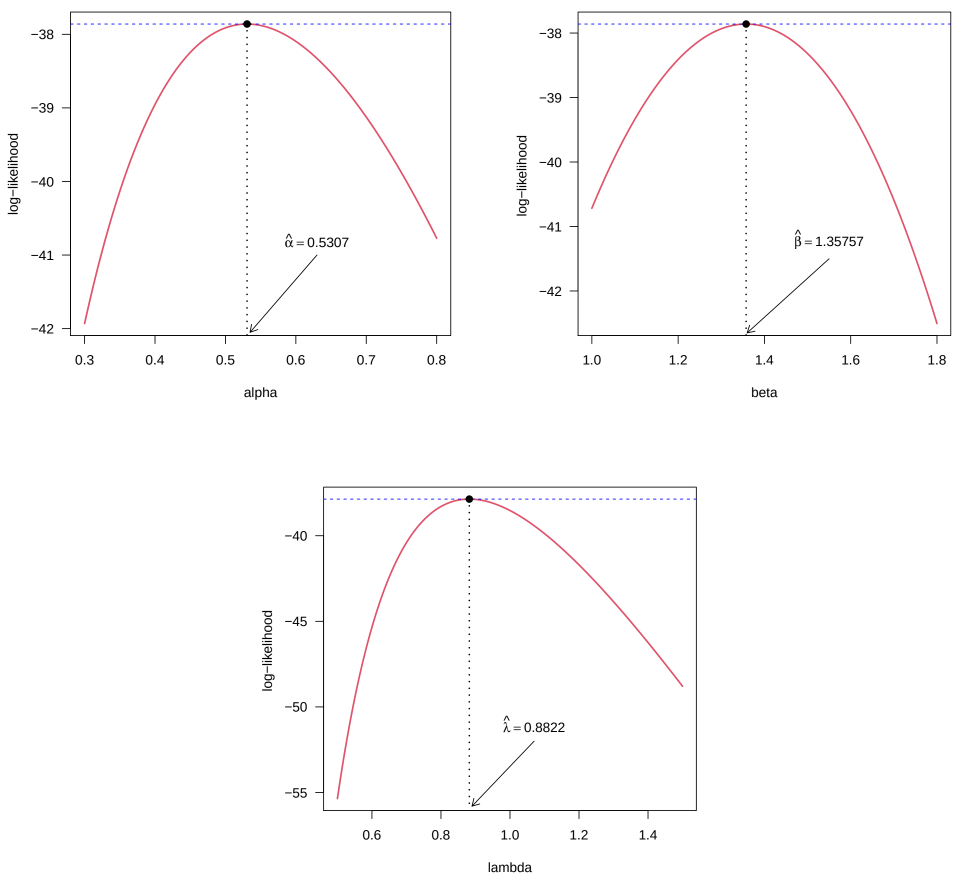


Figure 5. Profile log-likelihood functions of α, β and λ.

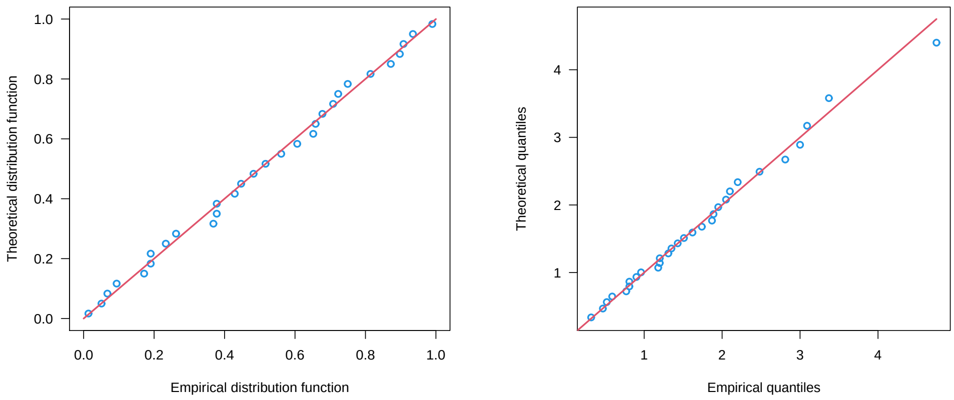


Figure 6. The P-P plot (left panel) and Q-Q plot (right panel).

The Histogram and the density function of fitted distributions and Empirical distribution function with estimated distribution function of WIE, generalized Gompertz (GG), generalized exponential extension (GEE), exponential extension (EE), Weibull and EEP distributions are presented in Figure 7.

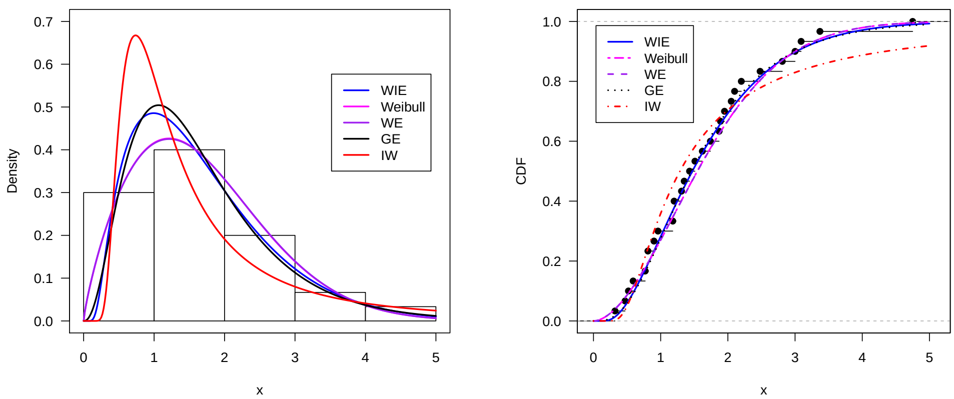


Figure 7. The Histogram and the density function of fitted distributions (left panel) and Empirical distribution function with estimated distribution function (right panel).

To compare the goodness-of-fit of the WIE distribution with other competing distributions we have presented the value of Kolmogorov-Simnorov (KS), the Anderson-Darling (AD) and the Cramer-Von Mises (CVM) statistics. These three statistics are widely used to compare non-nested models and to illustrate how closely a specific CDF fits the empirical distribution of a given data set. From Table 5 shows that the WIE distribution has the minimum value of the test statistic and higher p-value hence we conclude that the WIE distribution gets quite better fit and more consistent and reliable results from others taken for comparison.

Table 5

The goodness-of-fit statistics and their corresponding $p-$value

Model	*KS(p-value)*	*AD(p-value)*	*CVM(p-value)*
WIE	0.0684(0.9990)	0.0141(0.9998)	0.1033(0.9999)
Weibull	0.0690(0.9988)	0.0207(0.9969)	0.1631(0.9975)
WE	0.0698(0.9986)	0.0212(0.9963)	0.1670(0.9970)
GE	0.0655(0.9995)	0.0157(0.9996)	0.1109(0.9999)
IW	0.1523(0.4894)	0.1202(0.4973)	0.7596(0.5099)

8. Conclusion

In this study, we have studied the three-parameter Weibull inverse exponential (WIE) distribution. For our study, we have provided nature of the PDF, CDF, and the shapes of the hazard function. The shape of the PDF of the WIE model is unimodal and positively skewed, while the hazard function of the proposed model is increasing. The P-P and Q-Q plots showed that the purposed distribution is quite better for fitting the real dataset. Finally, using two real data sets, we have employed the well-known estimation method that is maximum likelihood estimation (MLE) method, and we also construct the asymptotic confidence interval for MLEs. The application illustrate that the proposed model provides consistently better fit then other underlying models. We expect that this model will contribute in the field of survival analysis and probability theory.

Trichandra Multiple Campus
Tribhuvan University, Kathmandu, Nepal
e-mail: rmshjoshi@gmail.com

Public Youth Campus
Tribhuvan University, Kathmandu, Nepal
e-mail: skstha09@gmail.com

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