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Inverse Upside Down Bathtub-shaped Hazard Function Distribution: Theory And Applications

By

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Abstract

In this study, we have proposed a three-parameter probability distribution called inverse upside down bathtub-shaped hazard function distribution. We have discussed some mathematical and statistical properties of the distribution such as the probability density function, cumulative distribution function and hazard rate function, survival function, quantile function, the skewness, and kurtosis measures. The model parameters of the proposed distribution are estimated using three well-known estimation methods namely maximum likelihood estimation (MLE), least-square estimation (LSE), and Cramer-Von-Mises estimation (CVME) methods. The goodness of fit of the proposed distribution is also evaluated by fitting it in comparison with some other existing life-time models using a real data set.

Keywords : Inverse distribution, Bathtub curve, MLE, LSE and CVME.
2010 AMS Subject Classification : 62F15, 65C05.

1. Introduction

Lifetime distributions are generally used to study the length of the life of components of a system, a device, and in general, reliability and survival analysis. Lifetime distributions are frequently used in fields like biological science, information technology, engineering, insurance, etc. Many continuous probability distributions such as Cauchy, exponential, gamma, Weibull have been frequently used in statistical literature to analyze lifetime data. The inverse distributions are often very helpful to investigate additional characteristics of the events that cannot be explored by non inverse distributions. Many inverse distributions have been created by using inverse transformation approach and found that they are flexible to analyze life-time datasets (Sheikh et al., 1987).

Some of the well-known models that are generated using inverse transformations are as follows:

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(i) Inverse Lomax distribution: The inverse Lomax(IL) distribution (Kleiber, 2004) has a lot of applications in stochastic modeling of decreasing failure rate life components, and life testing. Let X follow the IL random variable (rv), say $X \sim IL(\beta, \zeta)$. The CDF of X is given by

$$F_{IL}(t;\beta,\zeta) = \{1 + (\beta/t)\}^{-\zeta};\ t \geqslant 0,\ \ \beta > 0, \zeta > 0. \tag{1.1}$$

(ii) Voda (1972) has introduced the inverse Rayleigh distribution for the study of lifetimes of several types of experimental units. The cumulative density function of the inverse Rayleigh distribution with scale parameter ρ is

$$G(x;\rho) = \exp\left(-\rho x^{-2}\right); x, \rho > 0.$$

(iii) Inverse Weibull distribution The inverse Weibull distribution (Keller et al., 1982) is an important life-time distribution whose cumulative density function is

$$G(t;\alpha,\beta) = \exp\left(-\beta t^{-\alpha}\right) : t, \alpha, \beta > 0.$$

(iv) Inverse Exponential Distribution It is another important and flexible life-time model was introduced by (Keller et al., 1982) whose cumulative density function can be expressed as

$$G(t;\eta) = \exp\left(-\eta t^{-1}\right) : t, \eta > 0.$$

(v) Inverse Lindley Distribution The inverse Lindley distribution was introduced by (Sharma et al., 2015) whose cumulative density function can be expressed as

$$F(t,\lambda) = \left(1 + \frac{\lambda}{(1+\lambda)\,x}\right)\exp\left(-\lambda t^{-1}\right); t, \lambda > 0.$$

In the study of survival and reliability of a component or event or a system, we may encounter with three-step behavior of the failure rate will be observed at that situation a distribution with a bathtub and upside down bathtub-shaped failure rate would be appropriate (Rajarshi & Rajarshi, 1988). Dimitrakopoulou et al. (2007) has introduced a three-parameter life-time distribution with an increasing, decreasing, bathtub and upside down bathtub-shaped failure rate. The hazard function of this distribution is in the form of,

$$h(x) = \alpha\beta\lambda x^{\beta-1}\left(1 + \lambda x^{\beta}\right)^{\alpha-1}; x > 0, (\alpha\beta\lambda) > 0. \tag{1.2}$$

It is the special case of Weibull distribution, when $\alpha = 1$, it reduces to Weibull distribution. In this study we have presented the new model by applying the inverse transformation using (1.1) as parent distribution.

The objective of this study is to present a more flexible model that can have an increasing, decreasing, bathtub and upside down bathtub-shaped failure rate with a least number of parameters. The rest parts of the proposed study are organized as follows. In Section 2 we introduce a new distribution and talk about some distributional properties. We have considered some well-known estimation methods to estimate the parameters of the proposed distribution namely the maximum likelihood estimation (MLE), least-square estimation (LSE) and Cramer-Von-Mises estimation (CVME) methods. For the maximum likelihood (ML) estimate, we have constructed the asymptotic confidence intervals using the observed information matrix are presented in Section 3. In Section 4, a real data set has been analyzed to explore the applications and capability of the proposed distribution. The goodness of fit of the proposed distribution is evaluated by fitting it in contrast with some other existing distributions using a real data set. Finally, in Section 5 we present some concluding remarks.

2. New Distribution

A three-parameter new inverse upside down bathtub-shaped hazard function distribution is introduced. This new distribution is created by employing inverse transformation approach by taking CDF as parent distribution defined by (Dimitrakopoulou et al., 2007).

The CDF of an inverse upside down bathtub-shaped (IUDB) distribution with parameters $\alpha > 0, \beta > 0$ and $\lambda > 0$, say $X \sim IUBD\,(\alpha, \beta, \lambda)$ is defined as

$$F\,(x; \alpha, \beta, \lambda) = \exp\left\{1 - \left(1 + \lambda x^{-\beta}\right)^{\alpha}\right\}\ ,\ x\ \in\ (0,\ \infty)\ . \tag{2.1}$$

The corresponding PDF of IUBD distribution is

$$f\,(x; \alpha, \beta, \lambda) = \alpha\beta\lambda x^{-(\beta+1)}\left(1 + \lambda x^{-\beta}\right)^{\alpha-1} \exp\left\{1 - \left(1 + \lambda x^{-\beta}\right)^{\alpha}\right\}\ ,\ x > 0. \tag{2.2}$$

Hazard Function:

$$h\,(x; \alpha, \beta, \lambda) = \frac{\alpha\beta\lambda x^{-(\beta+1)}\left(1 + \lambda x^{-\beta}\right)^{\alpha-1} \exp\left\{1 - \left(1 + \lambda x^{-\beta}\right)^{\alpha}\right\}}{1 - \exp\left\{1 - \left(1 + \lambda x^{-\beta}\right)^{\alpha}\right\}}. \tag{2.3}$$

Plots of probability density function and hazard rate function of the proposed distribution with different values of parameters are shown in Figure 1.

Reliability function/Survival function:

$$R\,(x; \alpha, \beta, \lambda) = 1 - \exp\left\{1 - \left(1 + \lambda x^{-\beta}\right)^{\alpha}\right\}. \tag{2.4}$$

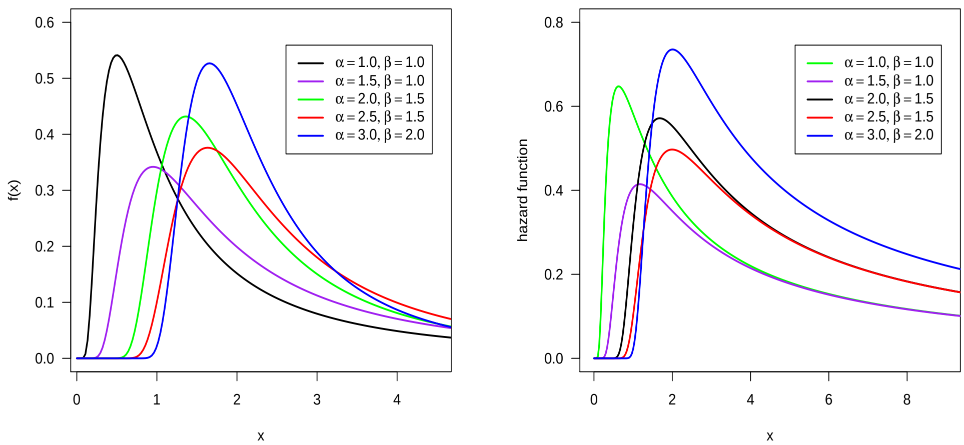

Figure 1. Plots of the probability density function(left panel) and hazard function (right panel), for λ=1 and different values of α and β.

The Quantile function:

$$x_p = \left[\frac{1}{\lambda}\left\{(1-\log p)^{1/\alpha} - 1\right\}\right]^{-1/\beta};0 < p < 1. \tag{2.5}$$

Random deviate generation:

$$x = \left[\frac{1}{\lambda}\left\{(1-\log u)^{1/\alpha} - 1\right\}\right]^{-1/\beta};0 < u < 1. \tag{2.6}$$

Skewness and Kurtosis:

The coefficient of skewness based on quantiles can be obtained by using the expression

$$S_k = \frac{Q_3 + Q_1 - 2Q_2}{Q_3 - Q_2}, \tag{2.7}$$

where Q_1, Q_2 *and* Q_3 are lower, median and lower quartiles respectively. The coefficient of kurtosis depends upon octiles was introduced by (Moors, 1988) which can be expressed as

$$M_{kurtosis} = \frac{Q\,(0.875) - Q\,(0.625) + Q\,(0.375) - Q\,(0.125)}{Q\,(3/4) - Q\,(1/4)}, \tag{2.8}$$

where the $Q(.)$ is the quantile function. The skewness and kurtosis measures based on quantiles like Bowley's skewness and Moors's kurtosis have several advantages over to classical measures of skewness and kurtosis because they are less sensitive to outliers and do not require the moments.

3. Methods of Parameter estimation

In this section, we have employing three well-known estimation methods, namely

- Maximum likelihood
- Least square
- Cramer-Von-Mises

(a) Maximum likelihood estimation. In this section, we determine the maximum likelihood estimates of the model parameters and asymptotic confidence intervals. Let $\underline{x} = (x_1, \ldots, x_n)$ be a sample from a distribution with probability density function (2.1). The likelihood function of the parameter $\ell(\alpha, \beta, \lambda|\underline{x})$ is given by

$$\begin{aligned}\ell(\alpha, \beta, \lambda|\underline{x}) = n\ln\alpha + n\ln\beta + n\ln\lambda + (\alpha-1)\sum_{i=1}^{n}\ln\left(1+\lambda x_i^{-\beta}\right)\\ -(\beta+1)\sum_{i=1}^{n}\ln x_i + n - \sum_{i=1}^{n}\left(1+\lambda x_i^{-\beta}\right)^{\alpha}.\end{aligned} \tag{3.1}$$

By differentiating (3.1) with respect to unknown parameters α, β and λ , we get

$$\begin{aligned}\frac{\partial\ell}{\partial\alpha} &= \frac{n}{\alpha} + \sum_{i=1}^{n}\ln\left(1+\lambda x_i^{-\beta}\right) - \sum_{i=1}^{n}\left(1+\lambda x_i^{-\beta}\right)^{\alpha}\ln\left(1+\lambda x_i^{-\beta}\right),\\ \frac{\partial\ell}{\partial\beta} &= \frac{n}{\beta} - \sum_{i=1}^{n}\ln x_i - (\alpha-1)\lambda\sum_{i=1}^{n}\frac{x_i^{-\beta}\ln x_i}{\left(1+\lambda x_i^{-\beta}\right)}\\ &\qquad + \alpha\lambda\sum_{i=1}^{n}x_i^{-\beta}\left(1+\lambda x_i^{-\beta}\right)^{\alpha-1}\ln x_i,\\ \frac{\partial\ell}{\partial\lambda} &= \frac{n}{\lambda} + (\alpha-1)\sum_{i=1}^{n}\frac{x_i^{-\beta}}{\left(1+\lambda x_i^{-\beta}\right)} - \alpha\sum_{i=1}^{n}x_i^{-\beta}\left(1+\lambda x_i^{-\beta}\right)^{\alpha-1}.\end{aligned}$$

After equating these non-linear equations to zero and solving for the unknown parameters (α, β, λ) we will obtain the ML estimators of the proposed distribution. Manually, it is difficult to solve hence by add of appropriate computer software one can solve these equations. Let us denote the parameter vector by $\underline{\Delta} = (\alpha, \beta, \lambda)$ and the corresponding MLE of $\underline{\Delta}$ as $\hat{\underline{\Delta}} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda})$, then the asymptotic normality results in, $\left(\hat{\underline{\Delta}} - \underline{\Delta}\right) \to N_3 \left[0, (K(\underline{\Delta}))^{-1}\right]$ where $K(\underline{\Delta})$ is the Fisher's information matrix given by, (Casella & Berger, 2002)

$$K(\underline{\Delta}) = -\begin{pmatrix} E\left(\frac{\partial^2 l}{\partial \alpha^2}\right) & E\left(\frac{\partial^2 l}{\partial \alpha \partial \beta}\right) & E\left(\frac{\partial^2 l}{\partial \alpha \partial \lambda}\right) \\ E\left(\frac{\partial^2 l}{\partial \beta \partial \alpha}\right) & E\left(\frac{\partial^2 l}{\partial \beta^2}\right) & E\left(\frac{\partial^2 l}{\partial \beta \partial \lambda}\right) \\ E\left(\frac{\partial^2 l}{\partial \alpha \partial \lambda}\right) & E\left(\frac{\partial^2 l}{\partial \beta \partial \lambda}\right) & E\left(\frac{\partial^2 l}{\partial \lambda^2}\right) \end{pmatrix}.$$

Again taking second order derivatives with respect to $(\alpha, \beta, \lambda\ )$, we get

$$\begin{aligned}
\frac{\partial^2 \ell}{\partial \alpha^2} &= -\frac{n}{\alpha^2} - \sum_{i=1}^{n} (A(x_i))^{\alpha} \{\ln A(x_i)\}^2 \\
\frac{\partial^2 \ell}{\partial \beta^2} &= -\frac{n}{\beta^2} + (\alpha - 1)\lambda \sum_{i=1}^{n} \frac{x_i^{-\beta} (\ln x_i)^2}{\left(\lambda + x_i^{\beta}\right)^2} - \sum_{i=1}^{n} \alpha\lambda (\ln x_i)^2 x_i^{-\beta} \\
&\qquad \left[(\alpha - 1)\lambda (A(x_i))^{\alpha-2} + (A(x_i))^{\alpha-1}\right] \\
\frac{\partial^2 \ell}{\partial \lambda^2} &= -\frac{n}{\lambda^2} - (\alpha - 1) \sum_{i=1}^{n} \frac{x_i^{-2\beta}}{(A(x_i))^2} - \alpha(\alpha - 1) \sum_{i=1}^{n} x_i^{-2\beta} A(x_i) \\
\frac{\partial \ell}{\partial \alpha \partial \beta} &= \lambda \sum_{i=1}^{n} x_i^{-\beta} \ln x_i \left[A(x_i)^{\alpha-1} + \alpha (A(x_i))^{\alpha-1} \ln (A(x_i))\right] \\
&\quad -\lambda \sum_{i=1}^{n} \frac{x_i^{-\beta} \ln x_i}{A(x_i)} \\
\frac{\partial \ell}{\partial \alpha \partial \lambda} &= \sum_{i=1}^{n} \frac{x_i^{-\beta}}{A(x_i)} - \sum_{i=1}^{n} x_i^{-\beta} \left[(A(x_i))^{\alpha-1} + \alpha (A(x_i))^{\alpha-1} \ln (A(x_i))\right] \\
\frac{\partial \ell}{\partial \beta \partial \lambda} &= -(\alpha - 1) \sum_{i=1}^{n} \frac{x_i^{-\beta} \ln x_i}{\left(\lambda + x_i^{\beta}\right)^2} + \alpha \sum_{i=1}^{n} x_i^{-\beta} \ln x_i \\
&\qquad \left[(A(x_i))^{\alpha-1} + (\alpha - 1)\lambda x_i^{-\beta} (A(x_i))^{\alpha-2}\right]
\end{aligned}$$

where $A(x_i) = 1 + \lambda x_i^{-\beta}$. In practice, we do not know $\underline{\Delta}$ hence it is useless that the MLE has an asymptotic variance $(K(\underline{\Delta}))^{-1}$. Hence, we approximate the asymptotic variance by plugging in the estimated value of the parameters. The observed Fisher information matrix $O(\hat{\underline{\Delta}})$ is used as an estimate of the information matrix $K(\underline{\Delta})$ given by

$$O(\underline{\hat{\Delta}}) = -\begin{pmatrix} \frac{\partial^2 l}{\partial \alpha^2} & \frac{\partial^2 l}{\partial \alpha \partial \beta} & \frac{\partial^2 l}{\partial \alpha \partial \lambda} \\ \frac{\partial^2 l}{\partial \beta \partial \alpha} & \frac{\partial^2 l}{\partial \beta^2} & \frac{\partial^2 l}{\partial \beta \partial \lambda} \\ \frac{\partial^2 l}{\partial \alpha \partial \lambda} & \frac{\partial^2 l}{\partial \beta \partial \lambda} & \frac{\partial^2 l}{\partial \lambda^2} \end{pmatrix}_{|(\hat{\alpha},\hat{\beta},\hat{\lambda})} = -H(\underline{\Delta})_{|(\underline{\Delta}=\underline{\hat{\Delta}})},$$

where H is the Hessian matrix. The Newton-Raphson algorithm to maximize the likelihood produces the observed information matrix. Therefore, the variance-covariance matrix is given by

$$\left[-H(\underline{\Delta})_{|(\underline{\Delta}=\underline{\hat{\Delta}})}\right]^{-1} = \begin{pmatrix} \mathrm{var}(\hat{\alpha}) & \mathrm{cov}(\hat{\alpha},\hat{\beta}) & \mathrm{cov}(\hat{\alpha},\hat{\lambda}) \\ \mathrm{cov}(\hat{\alpha},\hat{\beta}) & \mathrm{var}(\hat{\beta}) & \mathrm{cov}(\hat{\lambda},\hat{\beta}) \\ \mathrm{cov}(\hat{\alpha},\hat{\lambda}) & \mathrm{cov}(\hat{\lambda},\hat{\beta}) & \mathrm{var}(\hat{\lambda}) \end{pmatrix}. \tag{3.2}$$

Hence from the asymptotic normality of MLEs, approximate $100(1-\alpha)\%$ confidence intervals for α, β and λ can be constructed as,

$$\hat{\alpha} \pm z_{\alpha/2}\sqrt{\mathrm{var}(\hat{\alpha})},\ \hat{\beta} \pm z_{\alpha/2}\sqrt{\mathrm{var}(\hat{\beta})} \text{ and } \hat{\lambda} \pm z_{\alpha/2}\sqrt{\mathrm{var}(\hat{\lambda})}, \tag{3.3}$$

where $z_{\alpha/2}$ is the upper percentile of standard normal variate.

(b) Least-square estimation method

The estimation of the unknown parameters α, β and λ of IUDB distribution employing least-square estimation(LSE) method can be obtained by minimizing

$$B\left(x;\alpha,\beta,\lambda\right) = \sum_{i=1}^{n}\left[F(t_i) - \frac{i}{n+1}\right]^2 \tag{3.4}$$

with respect to unknown parameters α, β and λ.

Consider the CDF of the ordered random variables $X_{(1)} <, \ldots, < X_{(n)}$ where $\{X_1, \ldots, X_n\}$ is a random sample of size n from a distribution function $F(.)$. The least square estimators of α, β and λ can be calculated by minimizing

$$B\left(X;\alpha,\beta,\lambda\right) = \sum_{i=1}^{n}\left[\exp\left\{1 - \left(1+\lambda x^{-\beta}\right)^{\alpha}\right\} - \frac{i}{n+1}\right]^2, \tag{3.5}$$

with regard to α, β and λ.

Differentiating (3.5) with regard to α, β and λ, we get

$$\frac{\partial B}{\partial \alpha} = -2\sum_{i=1}^{n}\left[\exp\left\{1-\left(A\left(x_i\right)\right)^{\alpha}\right\} - \frac{i}{n+1}\right] \left(A\left(x_i\right)\right)^{\alpha}\exp\left\{1-\left(A\left(x_i\right)\right)^{\alpha}\right\}\ln\left(A\left(x_i\right)\right)$$

$$\frac{\partial B}{\partial \beta}=2\alpha\lambda\sum_{i=1}^{n}\left[\exp\left\{1-\left(A\left(x_i\right)\right)^{\alpha}\right\}-\frac{i}{n+1}\right]x_i^{-\beta}\ln(x_i)$$
$$\exp\left\{1-\left(A\left(x_i\right)\right)^{\alpha}\right\}\left(A\left(x_i\right)\right)^{\alpha-1}$$

$$\frac{\partial B}{\partial \lambda}=2\alpha\sum_{i=1}^{n}\left[\exp\left\{1-\left(A\left(x_i\right)\right)^{\alpha}\right\}-\frac{i}{n+1}\right]x_i^{-\beta}\exp\left\{1-\left(A\left(x_i\right)\right)^{\alpha}\right\}\left(A\left(x_i\right)\right)^{\alpha-1}.$$

Applying the similar procedure as above, we can calculate the weighted least square estimators by minimizing

$$C\left(X;\alpha,\beta,\lambda\right)=\sum_{i=1}^{n}w_i\left[F(X_{(i)})-\frac{i}{n+1}\right],$$

with respect to α, β and λ. The weights w_i are

$$w_i=\frac{1}{Var(X_{(i)})}=\frac{\left(n+1\right)^2\left(n+2\right)}{i\left(n-i+1\right)}.$$

Hence, the weighted LS estimators for α, β and λ respectively can be obtained by minimizing,

$$C\left(X;\alpha,\beta,\lambda\right)=\sum_{i=1}^{n}\frac{\left(n+1\right)^2\left(n+2\right)}{i\left(n-i+1\right)}\left[\exp\left\{1-\left(1+\lambda x_i^{-\beta}\right)^{\alpha}\right\}-\frac{i}{n+1}\right]^2, \quad (3.6)$$

with respect to α, β and λ.

(c) Cramer-Von-Mises estimation Method

The Cramer-Von-Mises(CVM) estimators of α, β and λ can be obtained by minimizing the function

$$\begin{aligned} M\left(x;\alpha,\beta,\lambda\right)&=\frac{1}{12n}+\sum_{i=1}^{n}\left[F\left(x_{i:n}|\alpha,\beta,\lambda\right)-\frac{2i-1}{2n}\right]^2\\ &=\frac{1}{12n}+\sum_{i=1}^{n}\left[\exp\left\{1-\left(1+\lambda x_i^{-\beta}\right)^{\alpha}\right\}-\frac{2i-1}{2n}\right]^2 \end{aligned} \quad (3.7)$$

Differentiating (3.7) with respect to α, β and λ, we get

$$\frac{\partial M}{\partial \alpha}=-2\sum_{i=1}^{n}\left[\exp\left\{1-\left(A\left(x_i\right)\right)^{\alpha}\right\}-\frac{2i-1}{2n}\right]$$
$$\left(A\left(x_i\right)\right)^{\alpha}\exp\left\{1-\left(A\left(x_i\right)\right)^{\alpha}\right\}\ln\left(A\left(x_i\right)\right)$$

$$\frac{\partial M}{\partial \beta} = 2\alpha\lambda \sum_{i=1}^{n} \left[\exp\left\{1 - (A(x_i))^{\alpha}\right\} - \frac{2i-1}{2n} \right] x_i^{-\beta} \ln(x_i)$$
$$\exp\left\{1 - (A(x_i))^{\alpha}\right\} (A(x_i))^{\alpha-1}$$

$$\frac{\partial M}{\partial \lambda} = 2\alpha \sum_{i=1}^{n} \left[\exp\left\{1 - (A(x_i))^{\alpha}\right\} - \frac{2i-1}{2n} \right] x_i^{-\beta} \exp\left\{1 - (A(x_i))^{\alpha}\right\} (A(x_i))^{\alpha-1}$$

where $A(x_i) = 1 + \lambda x_i^{-\beta}$. We will get the CVM estimators after solving non-linear equations simultaneously

$$\frac{\partial M}{\partial \alpha} = 0, \frac{\partial M}{\partial \beta} = 0 \ and \ \frac{\partial M}{\partial \lambda} = 0. \tag{3.8}$$

4. Data Analysis: Application

In this section, we demonstrate the applicability of the IUDB by using a real dataset used by earlier researchers. For the illustration, we are using a real data set that was used by, (Hinkley, 1977). The data represents thirty successive values of March precipitation (inches) for Minneapolis/St Paul.

0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05

The MLEs are calculated by utilizing the optim() function in R software (R Core Team, 2015) and (Rizzo, 2008) by maximizing the likelihood function (3.1). We have obtained log-likelihood value is $\ell(\hat{\theta}) = -38.0370$ and the MLE?s with their standard errors (SE) and 95% asymptotic confidence interval for α, β and λ are displayed in Table 1.

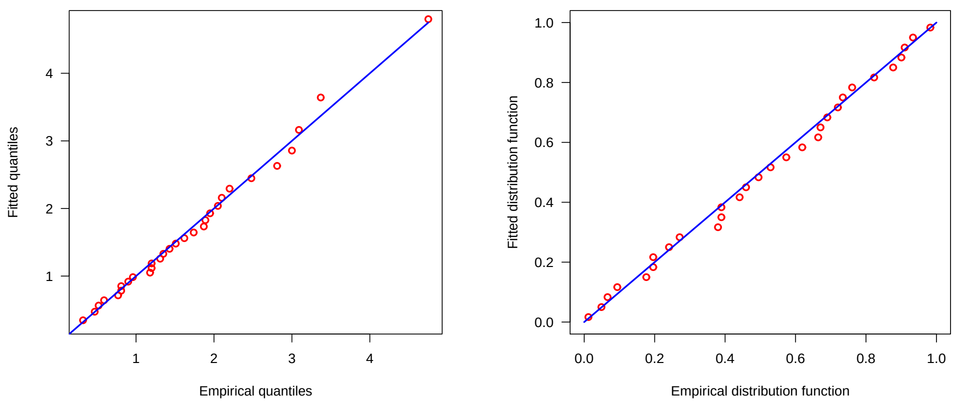

Figure 2. QQ plot(left panel) and PP plot (right panel).

Table 1
MLE, SE and 95% confidence intervals

Parameter	MLE	SE	95% ACI
alpha	0.1804	0.0209	(0.1394, 0.2214)
beta	4.3216	0.5062	(3.3294, 5.3138)
lambda	85.13	7.0427	(71.3263, 98.9337)

To assess the goodness of fit of a given distribution we normally use the PDF and CDF plot. To get the additional information we have to plot quantile-quantile(QQ) plot and probability-probability(PP) plot. In particular, the QQ plot may provide information about the lack-of-fit at the tails of the distribution, whereas the PP plot emphasizes the lack-of-fit. From Figure 2 we have shown that the IUBD model fits the data very well, (Kumar & Ligges, 2011).

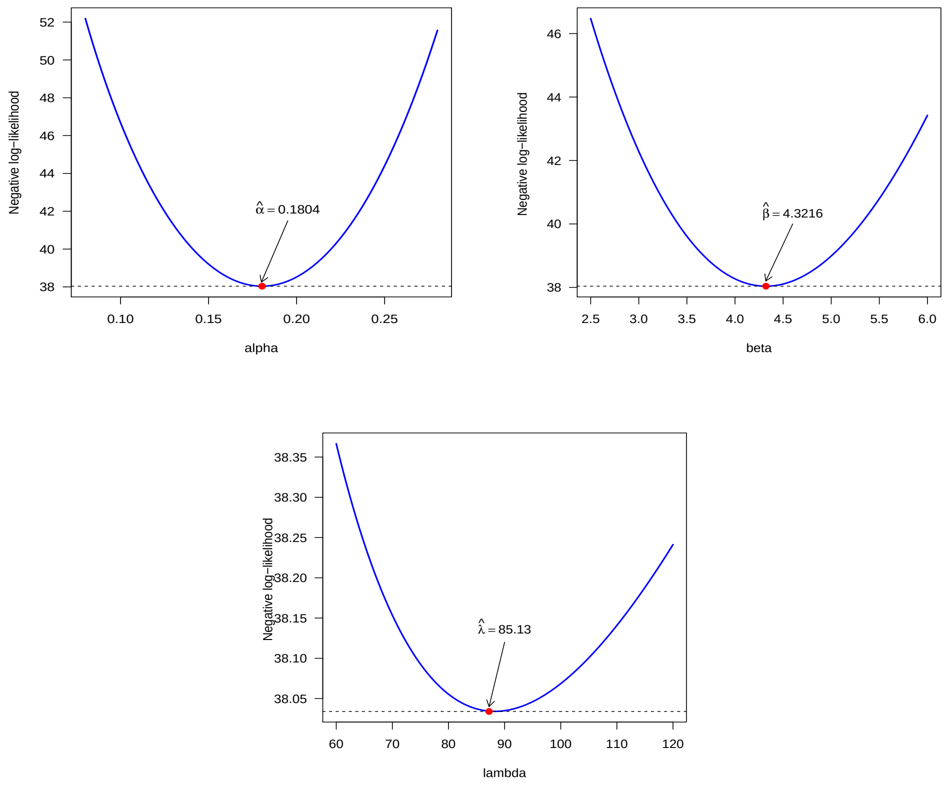

Figure 3. Profile log-likelihood functions of α and λ.

In Figure 3 we have displayed the graph of profile log-likelihood functions of ML estimates of α, β and λ. We have noticed that ML estimates of α, β and λ exist and can be obtained uniquely.

The models are compared via the Akaike Information Criterion (AIC), the Corrected Akaike Information criterion (CAIC), Bayesian information criterion (BIC) and Hanann-Quinn information criterion(HQIC) which are used to select the best model among several models, (D'Agostino & Stephens, 1986). The definitions of AIC, BIC, CAIC and HQIC are given below:

$$AIC = -2\,\ell(\hat{\theta}) + 2\,k$$
$$BIC = -2\,\ell(\hat{\theta}) + \,k\,\log{(n)}$$
$$CAIC = AIC\, + \,\frac{2k\,(k+1)}{n-k-1}$$
$$HQIC = -2\,\ell(\hat{\theta})\, + \,2\,k\,log(log(n))$$

where k is the number of parameters in the model under consideration.

Moreover, perfection of competing models is also tested via the Kolmogrov-Simnorov(K-S), the Anderson-Darling (A^2) and the Cramer-Von Mises (W) statistics. The mathematical expressions for the statistics above are given below

$$KS = \max_{1\leqslant i\leqslant n}\left(z_i - \frac{i-1}{n}, \frac{i}{n} - z_i\right)$$

$$A^2 = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\ln z_i + \ln{(1 - z_{n+1-i})}\right\}$$ where $z_i \;=\; CDF\,(x_i)$; the

$$W = \frac{1}{12n} + \sum_{i=1}^{n}\left\{\frac{(2i-1)}{2n} - z_i\right\}^2$$

x_i's being the ordered observations, D'Agostino and Stephens(1986).

We have displayed the estimated value of the parameters of IUBD distribution using MLE, LSE and CVE method and their corresponding negative log-likelihood, and AIC criterion in Table 2.

Table 2

Estimated parameters, log-likelihood, and AIC

Method	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	-LL	AIC
MLE	0.1804	4.3216	85.13	-38.037	82.0739
LSE	0.1691	4.3317	114.443	-38.084	82.1672
CVME	0.2141	3.7524	44.3436	-38.228	82.4552

In Table 3 we have displayed the KS, W and A^2 statistics with their corresponding $p-$value of MLE, LSE and CVE estimates.

Table 3

The KS, W and A^2 statistics with a $p-$value

Method	*KS(p-value)*	*W(p-value)*	A^2*(p-value)*
MLE	0.0799(0.9910)	0.0184(0.9986)	0.1205(0.9998)
LSE	0.0752(0.9958)	0.0163(0.9994)	0.1145(0.9999)
CVE	0.0719(0.9978)	0.0169(0.9992)	0.1341(0.9995)

To illustrate the goodness of fit of the IUBD distribution, we have select some well known distribution for comparison purpose which are listed below:

(i) Generalized Rayleigh distribution:
The probability density function of Generalized Rayleigh (GR) distribution (Kundu & Raqab, 2005) is

$$f_{GR}\,(\mathrm{x};\,\alpha,\,\lambda) = 2\,\alpha\,\lambda^2\,\mathrm{x}\,\mathrm{e}^{-(\lambda x)^2}\left\{1\;-\;\mathrm{e}^{-(\lambda x)^2}\right\}^{\alpha-1};\;x>0.$$

Here $\alpha > 0$ and $\lambda > 0$ are the shape and scale parameters, respectively.

(ii) Exponential power distribution:
The probability density function Exponential power (EP) distribution (Smith & Bain, 1975) with parameters $\alpha > 0$ and $\lambda > 0$ is

$$f_{EP}(x) = \alpha\,\lambda^{\alpha}\,x^{\alpha-1}\,e^{(\lambda\,x)^{\alpha}}\,\exp\left\{1-e^{(\lambda\,x)^{\alpha}}\right\}\,,\quad x>0.$$

where α and λ are the shape and scale parameters, respectively.

(iii) Gompertz distribution:
The probability density function of Gompertz distribution(GZ) (Murthy et al., 2003) with parameters $-\infty < \alpha < \infty$ and $\theta > 0$ is

$$f_{GZ}\,(x) = \theta\;\;e^{\alpha\,x}\;\exp\left\{\frac{\theta}{\alpha}\,(1-e^{\alpha\,x})\right\}\quad;\,x>0.$$

(iv) NHE distribution:
The density of exponential extension distribution with parameters $\alpha > 0$ and $\lambda > 0$, (Nadarajah & Haghighi, 2011) is given by

$$f_{NHE}(x) = \alpha\,\lambda\,(1+\lambda x)^{\alpha-1}\;\exp\left\{1-(1+\lambda x)^{\alpha}\right\}\quad;\,x>0.$$

The values of Akaike information criterion (AIC), Bayesian information criterion (BIC), Corrected Akaike information criterion (CAIC) and Hannan-Quinn information criterion (HQIC) are calculated for the assessment of potentiality of the proposed model which are presented in Table 4.

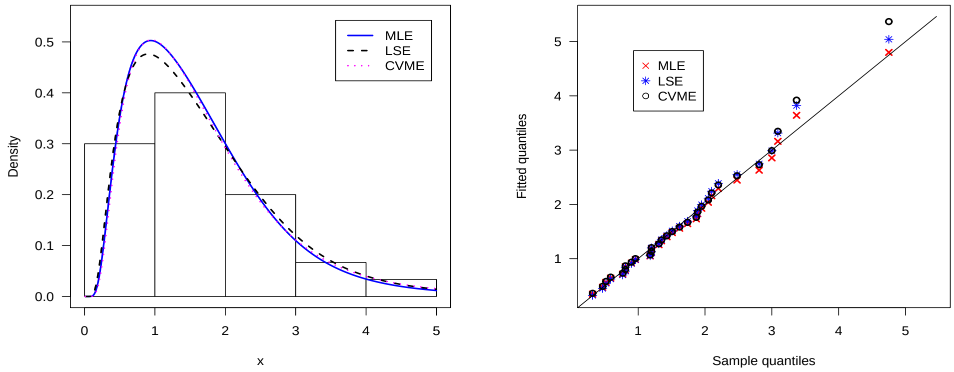

Figure 4. The Histogram and the density function of fitted distributions (left panel) and QQ plot (right panel) for MLE, LSE and CVM.

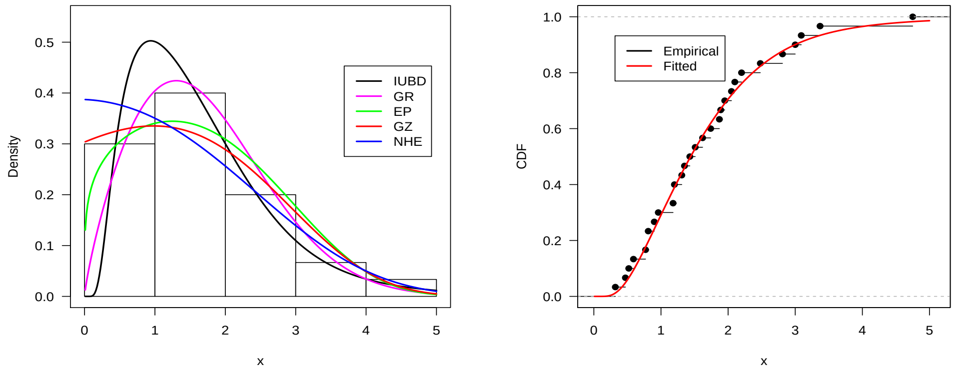

Figure 5. The Histogram and the density function of fitted distributions (left panel) and Empirical distribution function with estimated distribution function (right panel).

The Histogram and the density function of fitted distributions and Empirical distribution function with estimated distribution function of IUBD, generalized Raileigh (GR), exponential power (EP), exponential extension (NHE), and Gompertz (GZ) distributions are presented in Figure 5.

The values of Kolmogorov-Smirnov(KS), Anderson-Darling(A^2) and Cramer-Von Mises (W) statistic with their respective p-value of different models are reported in Table 3. As we can see in Table 3, the proposed model has the minimum values of the test statistics and higher p-value. Figure 4 (left panel) displays the histogram and the fitted density functions, which support the results in Tables 2 and 3. Also, Figure 4(right panel) which compares the distribution functions for the different models with the empirical distribution function reveals the same. Therefore, for the given data set shows the proposed distribution gets better fit and more reliable solutions from other alternatives.

Table 4
Log-likelihood (LL), AIC, BIC, CAIC and HQIC of IUDB distribution

Distribution	LL	AIC	BIC	CAIC	HQIC
IUBD	-38.037	82.0739	86.2775	82.997	83.4187
GR	-38.8284	81.6568	84.4592	82.1012	82.5533
EP	-40.4769	84.9537	87.7561	85.3675	85.8502
GZ	-41.0762	86.1523	88.9547	86.5968	87.0488
NHE	-41.4221	86.8442	89.6466	87.258	87.7407

Table 5
The KS, A^2 and W statistics with p-value

Distribution	*KS(p-value)*	*W(p-value)*	*A2(p-value)*
IUBD	0.0799(0.9910)	0.0184(0.9986)	0.1205(0.9998)
GR	0.0770(0.9942)	0.0341(0.9631)	0.2267(0.9813)
EP	0.1164(0.8108)	0.0738(0.7321)	0.5165(0.7286)
GZ	0.1149(0.8230)	0.0836(0.6749)	0.6440(0.6060)
NHE	0.1584(0.4390)	0.1571(0.3702)	1.0251(0.3437)

5. Conclusion

A new three-parameter inverse upside down bathtub-shaped hazard function distribution is presented. Some important statistical properties of the proposed distribution are presented such as the shapes of the probability density, cumulative density and hazard rate functions, survival function, reverse hazard rate function. Further quantile function, the skewness, and kurtosis measures are derived and established. We have employed three well-known estimation methods namely maximum likelihood estimation (MLE), least-square estimation (LSE), and Cramer-Von-Mises estimation (CVME) methods to estimate the model parameters and we concluded that the MLEs are quite better than LSE, and CVM estimators. A real dataset is considered to explore the applicability and suitability of the proposed distribution and found that the proposed distribution is quite better than other lifetime model taken into consideration. We hope this distribution may be an alternative in the field of survival analysis, probability theory and applied statistics.

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