TRIBHUVAN UNIVERSITY

INSTITUTE OF SCIENCE AND TECHNOLOGY

SCHOOL OF MATHEMATICAL SCIENCES

First Assessment 2078

Subject: **Mathematics for Data Science**
Course No.: MDS 504
Level: **Master in Data Science**/I Semester

Full Marks: 45
Pass Marks: 22.5
Time: 2:00 hr

*Attempt **ALL** questions. Write your answer in detail as far as possible.*

Group A $[3 \times 5 = 15]$

1. Show that
   (a) The line $x_2 = ax_1$ (in usual notations, $y = ax$) is a subspace $\mathbb{R}^2$.
   (b) The line $x_2 = ax_1 + b$ (perhaps more familiar as $y = ax + b$) is not a subspace $\mathbb{R}^2$ for $b \neq 0$.
2. Show that any vector in $\mathbb{R}^3$ can be expressed as a linear combination of the three unit basis vectors in $\mathbb{R}^3$. Also, show that a linear combination of the three unit basis vectors in $\mathbb{R}^3$ equals to 0 if and only if all coefficients in the linear combination are zeros.
3. What is the parallel coordinates method? Explain with explain with an example. What is the use of this method in data science?
4. Find a basis for the solution space of the equation $x + y - z = 0$.
5. Let $u_1 = (1, 2, 2, -1)$, $u_2 = (1, 1, -1, 1)$, $u_3 = (-1, 1, -1, -1)$ and $B = \{u_1, u_2, u_3\}$ an orthogonal basis for $V = \text{span}(u_1, u_2, u_3)$. Find the projection of $w = (0, 1, 2, 3)$ onto V.

Group B $[6 \times 5 = 30]$

6. (a) By showing that the L_∞-norm satisfies each of the conditions in the definition of a norm prove this is a vector norm for $\mathbb{R}^n$.
   (b) Let $x = (x_1, ..., x_n) \in \mathbb{R}^n$ be a vector with $x_i = i^{-1}$. Compute the 1-norm, the 2-norm, and the ∞-norm of x.

OR

Prove that if x and y are vectors in $\mathbb{R}^n$, then

(a) $|x \cdot y| \leq \|x\|_2 \|y\|_2$.

(b) Equality holds iff $x = \alpha y$ for $\alpha \in \mathbb{R}$.

7. Define Hadamard product of matrices and Matrix multiplication with examples. Prove that if A is an $m \times n$ matrix, B an $n \times p$ matrix, and C a $p \times q$ matrix, so that $(AB)C$ and $A(BC)$ are defined, then $(AB)C = A(BC)$.

8. Let $v_1, \ldots, v_k$ be vectors in a vector space V and v_{k+1} a linear combination of the vectors $v_1, \ldots, v_k$. Prove that

$$\operatorname{span}(v_1, \ldots, v_k) = \operatorname{span}(v_1, \ldots, v_k, v_{k+1}).$$

If $v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $v_4 = \begin{pmatrix} 15 \\ 5 \\ -2 \end{pmatrix}$, show that $\operatorname{span}(v_1, v_2, v_3) = \operatorname{span}(v_1, v_2, v_3, v_4)$.

9. Prove that if $B = \{v_1, v_2, \ldots, v_k\}$ be an orthogonal basis for a vector space V, then for any vector $w \in V$,

$$w = \frac{w \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{w \cdot v_2}{v_2 \cdot v_2} v_2 + \ldots + \frac{w \cdot v_k}{v_k \cdot v_k} v_k,$$

and moreover, if B is an orthonormal basis for a vector space V, then for any vector $w \in V$,

$$w = (w \cdot v_1)w_1 + (w \cdot v_2)w_2 + \ldots + (w \cdot v_k)w_k.$$

10. Let $u_1 = (1, 2, 2, -1)$, $u_2 = (1, 1, -1, 1)$, $u_3 = (-1, 1, -1, -1)$, $u_4 = (-2, 1, 1, 2)$

(a) Obtain an orthonormal set S' relative to orthogonal set S.

(b) Is S' an orthonormal basis for $\mathbb{R}^4$? Justify your answer.

OR

Use Gram-Schmidt Process to transform the basis $\{(2, 1, 0, 0), (-1, 0, 0, 1), (2, 0, -1, 1), (0, 0, 1, 1)\}$ for $\mathbb{R}^4$ to an orthonormal basis.

MDS504-2078✡

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Master Level / I Year /First Semester/ Science

Data Science (MDS 504)

(Mathematics for Data Science)

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Attempt All Questions. Write your answer in detail as far as possible.

Group A

(3×5=15)

1. Show that
   a. The line $x_2 = ax_1$ is a subspace $\mathbb{R}^2$.
   b. The set of points that is the union of two lines through the origin is not a subspace.

2. Let

$$v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \quad v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}.$$

   Show that $B = \{v_1, v_2\}$ is an orthonormal basis for $\mathbb{R}^2$. Find a vector $x \in \mathbb{R}^2$ with respect to the basis B.

3. Without calculation, find one eigenvalue and two linearly independent eigenvectors of

$$A = \begin{pmatrix}4 & 4 & -4\\4 & 4 & -4\\4 & 4 & -4\end{pmatrix}.$$

   Justify your answer.

4. Let $Q(x) = 3x_1^2 + 9x_2^2 + 8x_1x_2$. Find (a) the maximum value of $Q(x)$ subject to the constraint $x^Tx = 1$, (b) a unit vector u where this maximum is attained, and (c) the maximum of $Q(x)$ subject to the constraints $x^Tx = 1$ and $x^Tu = 0$.

5. Describe and compare the solution sets of $x_1 + 9x_2 - 4x_3 = 0$ and $x_1 + 9x_2 - 4x_3 = 2$

Group B

(6×5=30)

6. Give a geometric description of span (v) and span (u, v). Consider the vectors $u = \begin{pmatrix}1\\2\end{pmatrix}$ and $v = \begin{pmatrix}-1\\1\end{pmatrix}$.
   a. Write the vector $\begin{pmatrix}3\\2\end{pmatrix}$ in terms of the vectors u and v.
   b. Show that the vectors u and v span $\mathbb{R}^2$.

7. Let $u_1 = (2\ 0\ 0)^T, u_2 = (0\ 1\ 1)^T$ and $u_3 = (0\ 1\ -1)^T$. Find the orthonormal set associated with the set $S = \{u_1, u_2, u_3\}$.

Prove that an orthogonal set of nonzero vectors in a vector space is linearly independent.

8. consider the matrix : $A = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$. What can you say about the action of A on an arbitary vector? What are examples of eigenvalues/eigenvectors of this matrix? What does this discussion for this example illustrate?

OR

Let v_1, v_2 be the eigenvectors associated with the eigenvalues λ_1, λ_2 of a 2×2 symmetric matrix A respectively. Prove that $A = \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T$.

(1)

9. Prove that if B is a symmetric bilinear function on $\mathbb{R}^n$, then it is of the form $B = B_A(v, w) = v^T A w$, for some unique symmetric matrix A.

OR

Express the quadratic form $Q(x) = x_1x_2 - x_1x_3 + x_2x_3$ as a sum of squares.

10. Find the SVD of $A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$. If A an invertible $n \times n$ matrix, what is the relationship between the singular values of A and A^{-1}?