SEPARABLE TOPOLOGICAL SPACE OF HEREDITARY

R. N. Yadav[1*], B. P. Regmi[2] and S. R. Pathak[3]

[1]Department of Mathematics, MMAM Campus, Biratnagar, Tribhuvan University, Nepal

[1]Department of Mathematics, MD Campus, Sunwal, Tribhuvan University, Nepal

[1]Department of Mathematics, AS Campus, Tribhuvan University, Nepal

**Email : rajnarayanyadav1961@gmail.com*

Abstract : A property of a topological space is termed hereditary if and only if every subspace of a space with the property also has the property. The purpose of this article is to prove that the topological property of separable space is hereditary.

Key words : topological space, separable space, hereditary

Definition 1. Let (X, Ω) be a topological space. Then (X, Ω) is a second axiom space if and only if there exists a countable base for Ω. This is known as the second axiom of countability.

Axiom 1. Let (X, Ω) be a second axiom space and $Z \subseteq X$. Then $(Z, Z \cap \Omega)$ is a second axiom space.

Definition 1. Let (X, Ω) be a topological space and $E \subseteq X$. Then E is dense in X if and only if $\kappa E = X$.

Definition 2. Let (X, Ω) be a topological space. Then (X, Ω) is separable if and only if there exists a countable dense subset of X.

We will show that the property of being separable is not hereditary by showing that every topological space is a subspace of a separable topological space. [2, p. 84]

Theorem 1. Let (X, Ω) be a topological space (in particular a non-separable space). Let ∞ be a point such that $\infty \notin X$. Then $X^* = X \cup (\infty)$ with the topology

$$\Omega^* = \{o^* : o^* = o \cup (\infty) \text{ for } o \in \Omega\} \cup \{\phi\}$$

is a separable topological space and (X, Ω) is a subspace.

Proof: First we will show that (X^*, Ω^*) is a topological space.

(i) Let $\lambda^* \subseteq \Omega^*$. If $\lambda^* = \phi$ then

$$\bigcup_{H \in \lambda^*} H = \phi \in \Omega^*.$$

Assume $\lambda^* \neq \phi$ and $\lambda = \{o : o \in \Omega$ such that $o \cup (\infty) = o^*$ for $o^* \in \lambda^*$. Then $\bigcup_{o \in \lambda^*} o^* = \bigcup_{o \in \lambda} o \cup (\infty)) = \bigcup_{o \in \lambda} o \cup (\infty) \in \Omega^*,$ since $\bigcup_{o \in \lambda} o \in \Omega..$

(ii) Now suppose $\lambda^* \subseteq \Omega^*$ with λ^* finite and $\phi \notin \lambda^*$. If $\lambda^* \neq \phi$ then

$$\bigcap_{H \in \lambda^*} o = x^* \in \Omega^*.$$

Otherwise

$$\bigcap_{o \in \lambda^*} o^* = \bigcap_{o \in \lambda} (o \cup (\infty)) = \left(\bigcap_{o \in \lambda} o \right) \cup (\infty) \in \Omega^*,$$

since

$$\bigcap_{o \in \lambda} o \in \Omega$$

as λ^* finite implies λ is finite also. Therefore (X^*, Ω^*) is a topological space.

Now we will show that (X*, Ω*) is separable. Let o* ∈ Ω*. Then (∞) ∈ o*, since $o^* = o \cup (\infty)$ where $o \in \Omega$..

Let $x \in X^*$. Then for every o* such that $x \in o^*$ we have

$\phi \neq o^* \cup (\infty)$

and $x \in \kappa$ whence $X^* \subseteq \kappa(\infty)$. Thus $\kappa(\infty) = X^*$, and {∞} is a countable dense subset of X. Therefore (X, Ω) is separable.

Clearly (X, Ω) is a subspace of (X*, Ω*), since

$X \subseteq x^* = X \cup (\infty)$

and

$$\begin{aligned} X \cap \Omega^* &= \{X \cap o^* : o^* \in \Omega^*\} \\ &= \{X \cap (o(o \cup (\infty)) : o \in \Omega\} \\ &= \{X \cap o : o \in \Omega\} \\ &= (o : o \in \Omega\} = \Omega. \end{aligned}$$

Theorem 2. [3, p. 60) In every second axiom topological space separability is hereditary.

Proof. Since each sub space of a second axiom space is also a second axiom space, we need to show that every second axiom space is separable. Now (X, Ω) second axiom implies that there is a countable base β for Ω. Thus for every $o \in \Omega$ there exists a countable family -$\beta^* \subseteq \beta$ such that $o = \bigcup_{B \in \beta^*} B$.

Let β = {B_n : n is a positive integer} . Choose $x_n \in B_n$.

We may assume $n \geq 1$ implies that $B_n \neq \phi$. Put

$$E = \bigcup_{n=1}^{\infty} (x_n).$$

Clearly E is a countable subset of X.

Now let $x \in X$ and $\beta_x \in \{B : B \in \beta\}$ and $x \in \beta\}$. Here β_x is a base at x and $B \in \beta_x$ implies that there exists an n such that $B = B_n$, whence

$x_n \in B_n \cap E = B \cap E$

so that $B \cap E \neq \phi$. Thus $x \in \kappa E$, and $x \subseteq \kappa E$. Therefore $\kappa E = X$, and E is a countable dense subset of X, whence (X, Ω) is separable.

Theorem 3. Every separable metric space is hereditarily separable.

Proof. This follows immediately from theorem 3 and the fact that a metric space is separable if and only if it is second axiom. [1, p. 121]

References

[1] Gaal, S. A. (1964). Point Set Topology. New York: Academic Press Inc.

[2] Pervin, W. J. (1964). Foundations of General Topology. New York: Academic Press Inc.

[3] Sierpinski, W. (1952). General Topology, trans. C. Cecilia Krieger. Toronto: University of Toronto Press.