Unit 4: Multiple Integrals

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Contents

1 Double integrals

We will start out by assuming that the region in $\mathbb{R}^2$ is a rectangle which we will denote as follows,

$$R = [a, b] \times [c, d].$$

This means that the ranges for x and y are

$$a \leq x \leq b \text{ and } c \leq y \leq d.$$

Also, we will initially assume that $f(x, y) \geq 0$. Let's start out with the graph of the surface S given by graphing $f(x, y)$ over the rectangle R. Lets first ask what the volume of the region

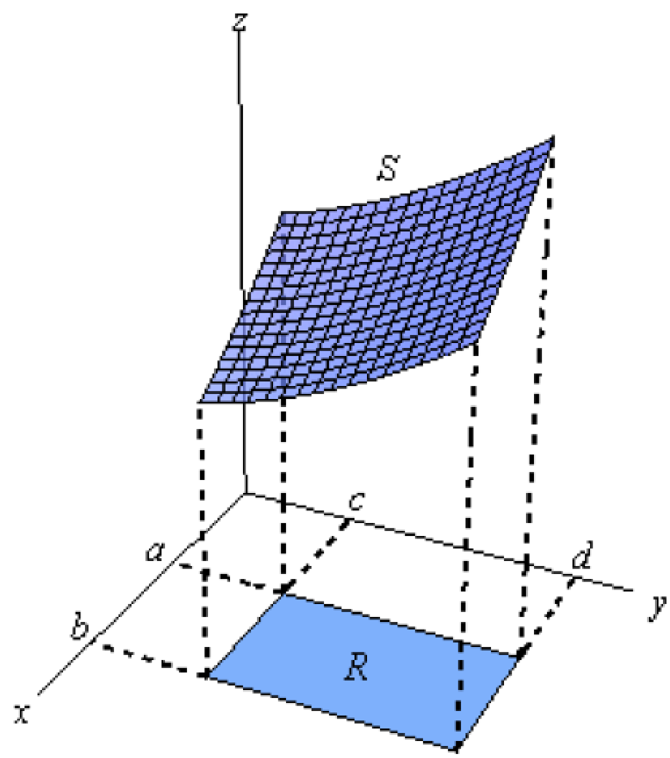


Figure 1

under S (and above the xy-plane of course) is.

We will approximate the volume much as we approximated the area above. We will first divide up $a \leq x \leq b$ into n subintervals and divide up $c \leq x \leq d$ into m subintervals. This will divide up R into a series of smaller rectangles and from each of these we will choose a point (x_i^*, y_j^*). Here is a sketch of this set up.

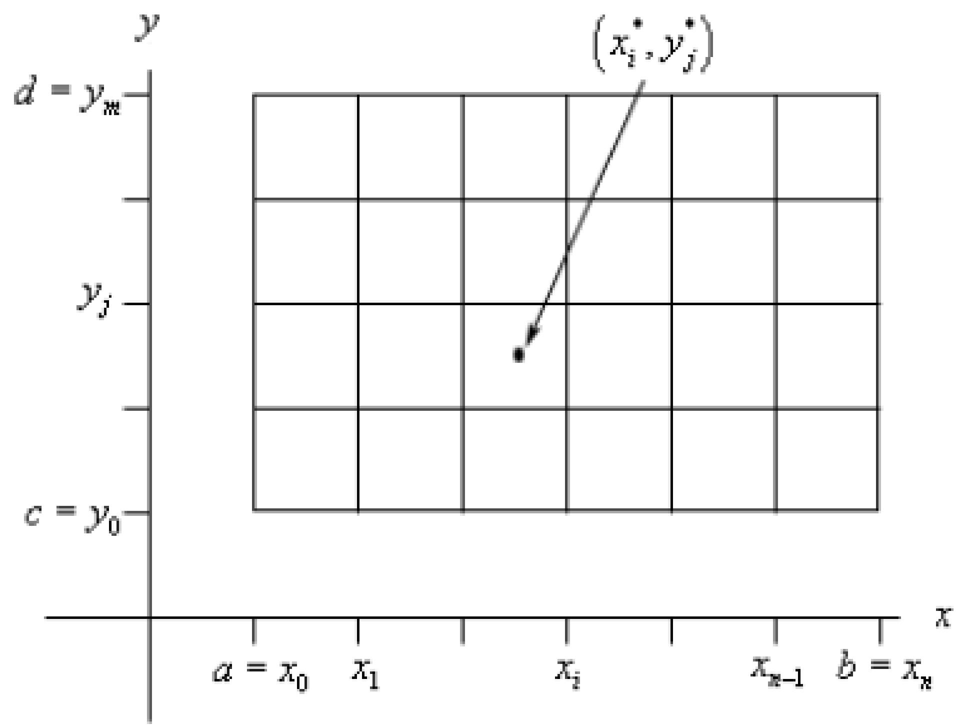


Figure 2

Now, over each of these smaller rectangles we will construct a box whose height is given by $f(x_i^*, y_j^*)$. Here is a sketch of that.

Each of the rectangles has a base area of ΔA and a height of $f(x_i^*, y_j^*)$ so the volume of each of these boxes is $f(x_i^*, y_j^*)\Delta A$. The volume under the surface S is then approximately,

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta A.$$

We will have a double sum since we will need to add up volumes in both the x and y directions. To get a better estimation of the

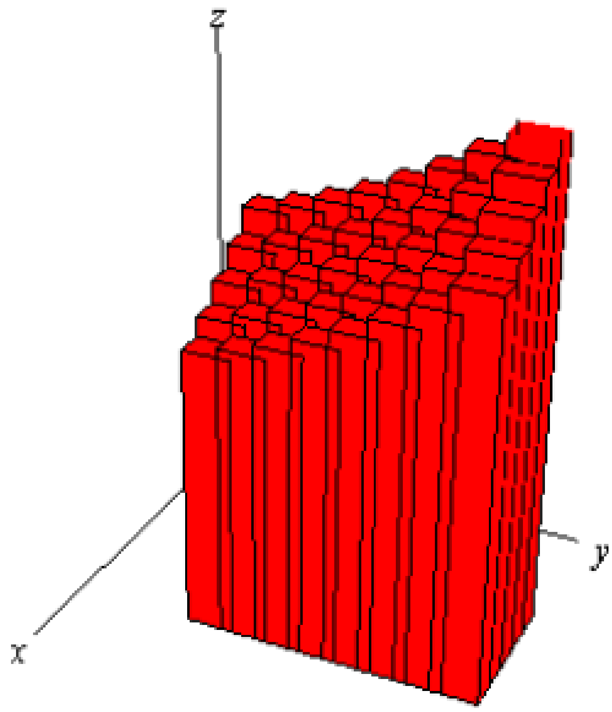


Figure 3

volume we will take n and m larger and larger and to get the exact volume we will need to take the limit as both n and m go to infinity. In other words,

$$V = \lim_{n,m\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta A.$$

This looks a lot like the definition of the integral of a function of single variable. In fact, this is also the definition of an integral of a function of two variables over a rectangle.

Here is the official definition of a double integral of a function of two variables over a rectangular region R as well as the notation that well use for it.

$$\iint_A f(x,y)\, dA = \lim_{n,m\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta A.$$

The sample point (x_i^*, y_j^*) in the definition can be chosen to be any point (x_i, y_j) in the subrectangle.

Note the similarities and differences in the notation to single integrals.

- We have two integrals to denote the fact that we are dealing with a two dimensional region and we have a differential here as well.
- the differential is dA instead of the dx and dy that we're used to seeing.
- We don't have limits on the integrals in this notation. Instead we have the R written below the two integrals to denote the region that we are integrating over.

The sum in the definition of the integral,

$$\lim_{n,m\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta A$$

is called a **double Riemann sum** and is used as an approximation to the value of the double integral.

If f happens to be a positive function, then the double Riemann sum represents the sum of volumes of columns, as in Figure 3, and is an approximation to the volume under the graph of f and above the rectangle R. Thus, we have the following difinition:

If $f(x, y) \geq 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x, y)$ is

$$\text{Volume } = \iint_A f(x, y)\, dA.$$

Example 1.

> Estimate the volume of the solid that lies above the square $R = [0, 2] \times [0, 2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares and choose the sample point to be the upper right corner of each square . Sketch the solid and the approximating rectangular boxes.

Solution. The paraboloid is the graph of $f(x, y) = 16 - x^2 - 2y^2$ and the area of each square is $\Delta A = 1$. The squares are shown in Figure 4. Approximating the volume by the Riemann

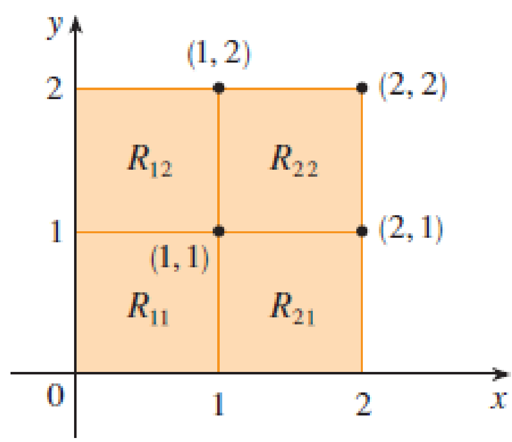


Figure 4

sum with $m = n = 2$, we have

$$
\begin{aligned}
V &\approx \lim_{n,m\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_i)\Delta A \\
&= f(1,1)\Delta A + f(1,1)\Delta A + f(1,2)\Delta A + f(2,2)\Delta A \\
&= 13(1) + 7(1) + 10(1) + 4(1) = 34.
\end{aligned}
$$

This is the volume of the approximating rectangular boxes shown in Figure 5. ◀

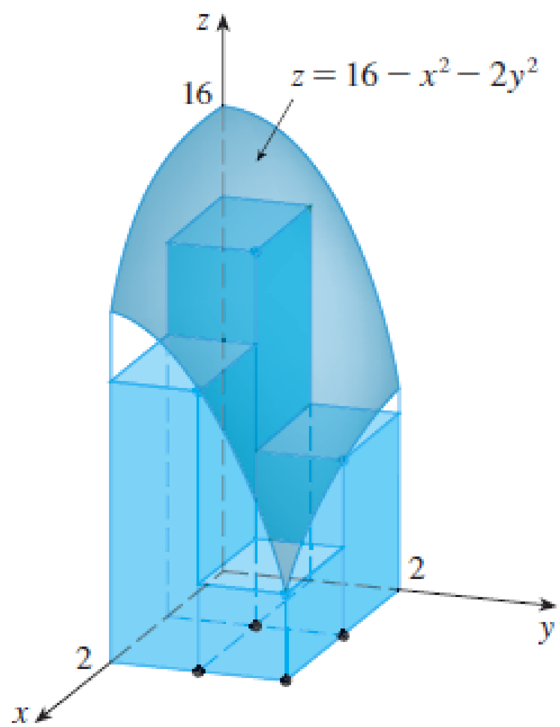


Figure 5

The Midpoint Rule

We use a double Riemann sum to approximate the double integral, where the sample point (x_i^*, y_j^*) in R_{ij} is chosen to be the center $(\overline{x}_i, \overline{y}_j)$ of R_{ij}. In other words, $\overline{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\overline{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.

Midpoint Rule for Double Integrals

$$\iint_A f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\overline{x}_i, \overline{y}_j)\Delta A,$$

where $\overline{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\overline{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.

Example 2.

Use the Midpoint Rule with $m = n = 2$ to estimate the value of the integral $\iint_R (x-3y)^2\,dA$, where $R = \{(x,y)|\ 0 \le x \le 2, 1 \le y \le 2\}$

Solution. In using the Midpoint Rule with $m = n = 2$, we evaluate $f(x,y) = (x-3y)^2$ at the centers of the four subrectangles shown in Figure 6.

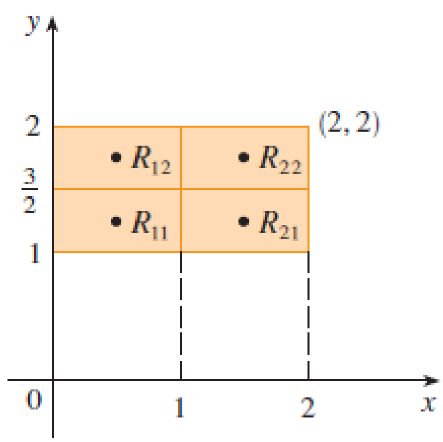


Figure 6

So $\overline{x}_1 = \frac{1}{2}, \overline{x}_2 = \frac{3}{2}, \overline{y}_1 = \frac{5}{4}$, and $\overline{y}_2 = \frac{7}{4}$. The area of each subrectangle is $\Delta A = \frac{1}{2}$. Thus

$$\begin{aligned}
\iint_R (x-3y)^2 \, dA &\approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\overline{x}_i, \overline{y}_j)\Delta A \\
&= f(\overline{x}_1, \overline{y}_1)\Delta A + f(\overline{x}_1, \overline{y}_2)\Delta A + f(\overline{x}_2, \overline{y}_1)\Delta A \\
&\quad + f(\overline{x}_2, \overline{y}_2)\Delta A \\
&= f(\tfrac{1}{2}, \tfrac{5}{4})\Delta A + f(\tfrac{1}{2}, \tfrac{7}{4})\Delta A + f(\tfrac{3}{2}, \tfrac{5}{4})\Delta A \\
&\quad + f(\tfrac{3}{2}, \tfrac{7}{4})\Delta A \\
&= (-\tfrac{67}{16})\tfrac{1}{2} + (-\tfrac{130}{16})\tfrac{1}{2} + (-\tfrac{51}{16})\tfrac{1}{2} + (-\tfrac{123}{16})\tfrac{1}{2} \\
&= -\tfrac{95}{8} = -11.875
\end{aligned}$$

◀

Average Value

We define the average value of a function f of two variables defined on a rectangle R to be

$$f_{ave} = \frac{1}{A(R)} \iint_R f(x,y) \, dA,$$

where $A(R)$ is the area of R.

If $f(x, y \geq 0$, the equation

$$A(R) \times f_{ave} = \iint_R f(x,y) \, dA$$

says that the box with base R and height f_{ave} has the same volume as the solid that lies under the graph of f.

Properties of Double Integrals

Here are some properties of the double integral. Note that all three of these properties are really just extensions of properties of single integrals that have been extended to double integrals.

1.
$$\iint_R [f(x,y)+g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$$

2. If c is a constant, then
$$\iint_R cf(x,y)\ dA = c\iint_R f(x,y)\ dA$$

3. If $f(x,y) \geq g(x,y)$ for all in $(x,y) \in R$, then
$$\iint_R f(x,y)\ dA \geq \iint_R g(x,y)\ dA$$

1.1 Iterated Integrals

Just like with the definition of a single integral, it is usually difficult to evaluate double integrals from first principles.

So we need to start looking into how we actually compute double integrals.

We will continue to assume that we are integrating $f(x,y)$ over the rectangle
$$R = [a,b] \times [c,d].$$
We use the notation
$$\int_c^d f(x,y)\ dy$$
to mean that x is held fixed and $f(x,y)$ is integrated with respect to y from c to d. This procedure is called **partial integration** with respect to y. Now $\int_c^d f(x,y)\ dy$ is a number that depends on the value of x, so it defines a function of :
$$A(x) = \int_c^d f(x,y)\ dy.$$

We now integrate the function A with respect to x from a to b. We then get

$$\int_a^b A(x)\ dx = \int_a^b \left[\int_c^d f(x,y)\ dy\right] dx.$$

The integral on the right side is called an **iterated integral**. Usually the brackets are omitted. Thus

$$\int_a^b \int_c^d f(x,y)\ dydx = \int_a^b \left[\int_c^d f(x,y)\ dy\right] dx.$$

means that we first integrate with respect to y from c to d and then with respect to$x$ from a to b.

Similarly, we define the iterated integral

$$\int_c^d \int_a^b f(x,y)\ dxdy = \int_c^d \left[\int_a^b f(x,y)\ dx\right] dy.$$

Example 3.

Evaluate the iterated integrals:

(a) $\int_0^1 \int_1^2 x^2 y\ dydx$ (b) $\int_1^2 \int_0^1 x^2 y\ dxdy$.

Solution. (a) Regarding x as a constant, we obtain

$$\int_1^2 x^2 y\ dy = \left[x^2 \frac{y^2}{2}\right]_{y=1}^{y=2}$$

◀

It doesn't matter which variable we integrate with respect to first, we will get the same answer regardless of the order of integration. To prove that let's work this one with each order to make sure that we do get the same answer.

The following theorem gives a practical method for evaluating a double integral by expressing it as an iterated integral (in either order).

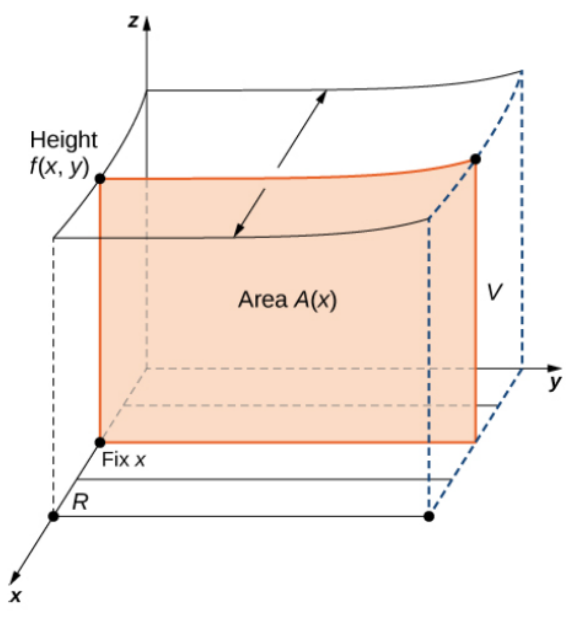
Theorem 1.1. *Suppose that $f(x, y)$ is a function of two variables that is continuous over a rectangular region*

$$R = \{(x, y) \in \mathbb{R}^2 | a \le x \le b,\ c \le y \le d\}.$$

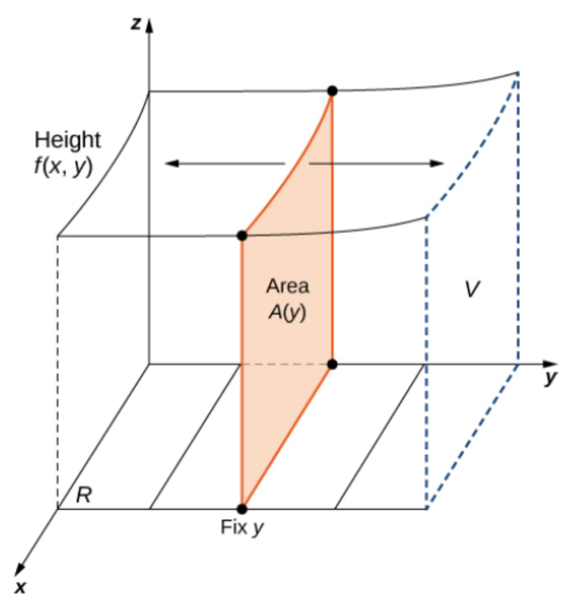
Then the double integral of f over the region equals an iterated integral, in symbols,

$$\iint_R f(x, y)\ dA = \iint_R f(x, y)\ dxdy$$
$$= \int_a^b \int_c^d f(x, y)\ dydx = \int_c^d \int_a^b f(x, y)\ dxdy.$$

More generally, Fubinis theorem is true if f is bounded on R and f is discontinuous only on a finite number of continuous curves. In other words, f has to be integrable over R.



Integrating first w.r.t. y and then w.r.t. x to find the area $A(x)$ and then the volume V.



Integrating first w.r.t. x and then w.r.t. y to find the area $A(y)$ and then the volume V.

Example 4.

> Compute the following double integral over the indicated rectangle.
>
> $$\iint_R x\,dA,\ \ R = [0,2] \times [0,1].$$

Solution. Fubini's theorem offers an easier way to evaluate the double integral by the use of an iterated integral. Note how the boundary values of the region R become the upper and lower limits of integration. ◀

Example 5.

> Compute the following double integral over the indicated rectangle.
>
> $$\iint_R (2x - 4y^3)\,dA,\ \ R = [-5,4] \times [0,3].$$

Solution. Fubini's theorem offers an easier way to evaluate the double integral by the use of an iterated integral. Note how the boundary values of the region R become the upper and lower limits of integration. ◀

Example 6.

> Compute the following double integral over the indicated rectangle.
>
> $$\iint_R y\sin(xy)\,dA,\ \ R = [1,2] \times [0,\pi].$$

Solution. It is easier to integrate first with respect to x and then with respect to y. ◀

Example 7.

Find the volume of the solid S enclosed by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and three coordinate planes.

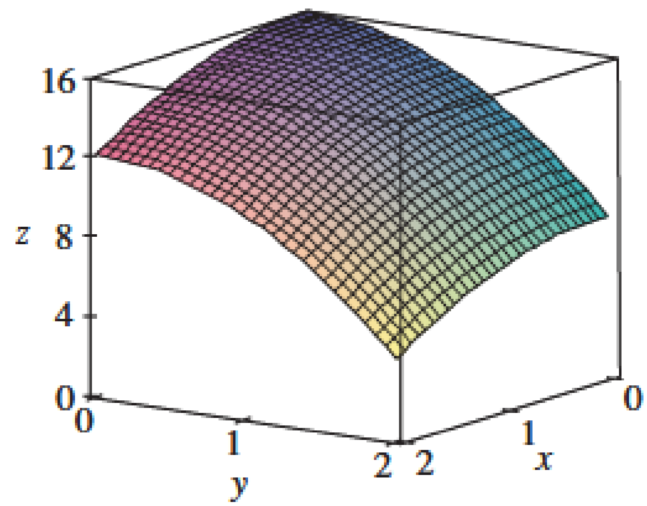


Figure 7

Solution. ◀

Example 8.

Find the volume of the solid enclosed by the planes $4x + 2y + z = 10, y = 3x, z = 0, x = 0$.

Solution. Notice that the planes $4x + 2y + z = 10$ is the top of the volume and the planes $z = 0$ and $x = 0$ indicate that the plane $4x + 2y + z = 10$ does not go past the xy-plane and the yz-plane. So we are really looking for the volume under the plane

$$z = 10 - 4x - 2y$$

and above the region R in the xy-plane. The second plane, $y = 3x$, gives one of the sides of the volume as shown below. The region R will be the region in the xy-plane (i.e. $z = 0$) that

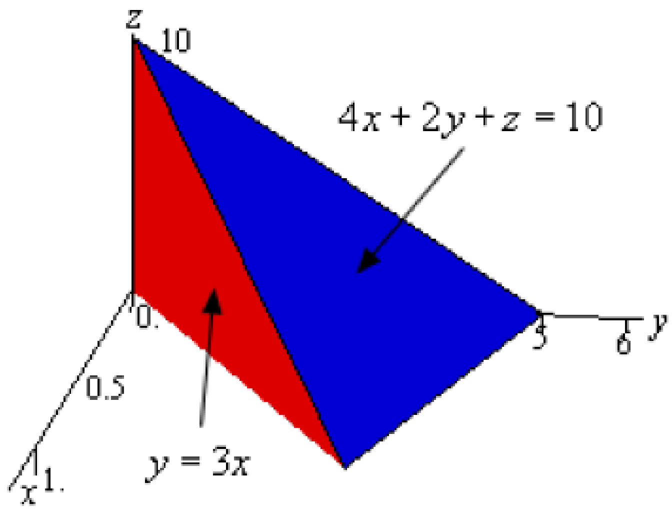


Figure 8

is bounded by $y = 3x, x = 0$, and the line where $4x+2y+z = 10$ intersects the xy-plane. We can determine where $4x+2y+z = 10$ intersects the xy-plane by plugging $z = 0$ into it.

$$4x + 2y + 0 = 10 \Rightarrow 2x + y = 5 \Rightarrow y = -2x + 5$$

So, here is a sketch the region R. The region R is really where

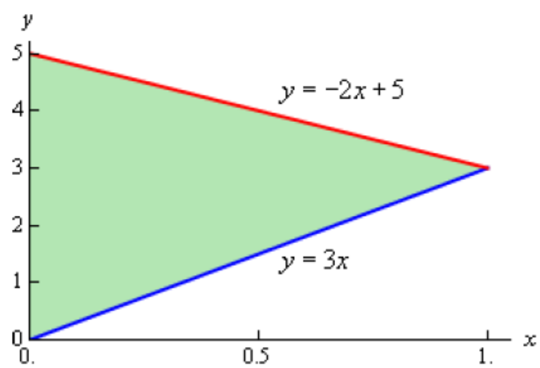


Figure 9

this solid will sit on the xy-plane and here are the inequalities that define the region.

$$\begin{aligned} &0 \le x \le 1 \\ &3x \le y \le -2x + 5. \end{aligned}$$

◀

A special case:

$$f(x, y) = f(x)h(y) \text{ on } R = [a, b] \times [c, d].$$

In this case,

$$\begin{aligned} \iint_R f(x, y)\, dA &= \int_a^b \int_c^d g(x)h(y)\, dydx \\ &= \int_a^b g(x)\, dx \int_c^d h(y)\, dy. \end{aligned}$$

Example 9.

Compute the double integral of

$$f(x, y) = \frac{1 + x^2}{1 + y^2},$$

in the rectangular region $R = [0, 2] \times [0, 1]$.

2 Double Integrals over General Regions

There are two types of regions that we need to look at. Here is a sketch of both of them. We will often use set builder notation to describe these regions. Here is the definition for the region in Case 1

$$D = \{(x, y)|\ a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

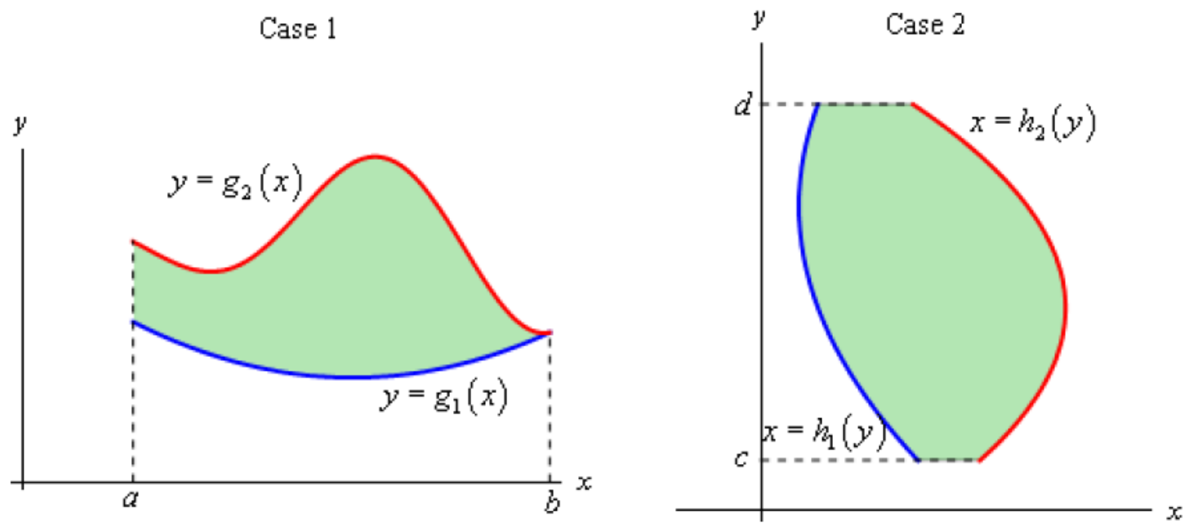


Figure 10

and here is the definition for the region in Case 2.

$$D = \{(x, y) |\ h_1(x) \leq x \leq h_2(x), c \leq y \leq d\}$$

This notation indicates that we are going to use all the points, (x, y), in which both of the coordinates satisfy the two given inequalities.

Calculating a double integral over a type I region

Example 10.

> Evaluate $\iint_D (x + 2y)\ dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

Solution. We have

$$2x^2 = 1 + x^2 \Rightarrow x = \pm 1.$$

Then $y = 2$. Thus, the parabolas intersect at $(-1, 2)$ and $(1, 2)$. We see that the region D is given by We also see that the lower boundary is $y = 2x^2$ and the upper boundary is $y = 1 + x^2$.

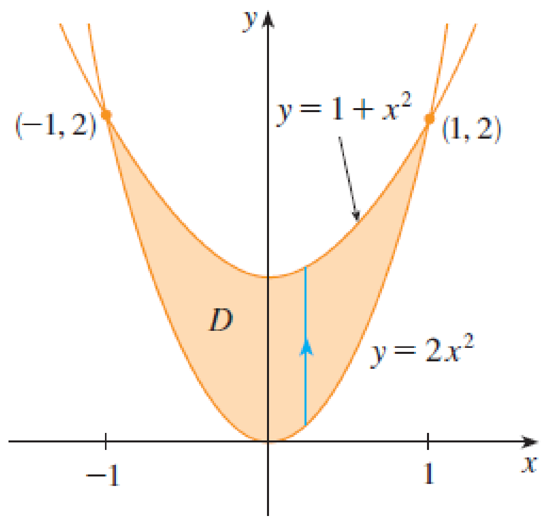


Figure 11: Type I region

Then we have

$$\iint_D (x + 2y)\, dA = \int_1^1 \int_{2x^2}^{1+x^2} (x + 2y)\, dydx$$
$$= \frac{12}{15}.$$

◀

Calculating a double integral over both type I and type II region

Example 11.

> Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

Solution. From the figure we see that D is a type I region and

$$D = \{(x, y) : \ 0 \leq x \leq 2, x^2 \leq y \leq 2x\}$$

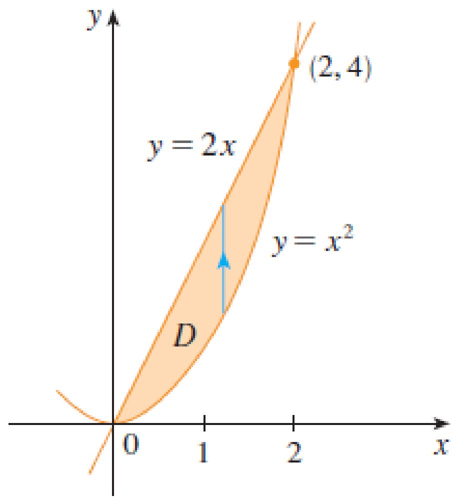


Figure 12: Type I region

Therefore the volume under $z = x^2 + y^2$ and above D is

$$\begin{aligned} V &= \iint_D (x^2 + y^2)\, dA \\ &= \int_0^2 \int_{x^2}^{2x} (x^2 + y^2)\, dydx \\ &= \frac{216}{35}. \end{aligned}$$

◀

Alternatively,

Solution. From the figure we see that D can also be written as a type II region:

$$D = \{(x, y) :\ 0 \le y \le 4, \tfrac{1}{2}y \le x \le \sqrt{y}\}$$

Therefore another expression for V is

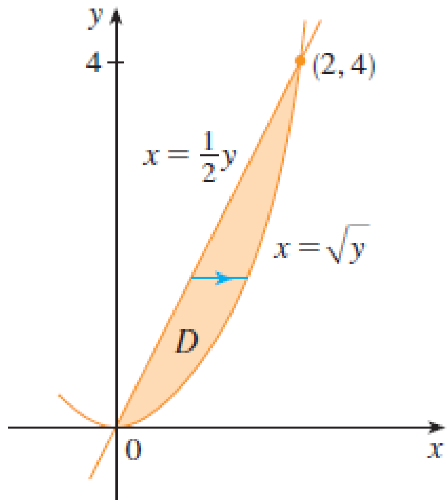


Figure 13: Type II region

$$\begin{aligned}
V &= \iint\limits_D (x^2 + y^2)\, dA \\
&= \int_0^4 \int_{\frac{1}{2}y}^{\sqrt{y}} (x^2 + y^2)\, dxdy \\
&= \frac{216}{35}.
\end{aligned}$$

◀

Choosing the better description of a region

Example 12.

Evaluate $\iint\limits_D xy\, dA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Solution.

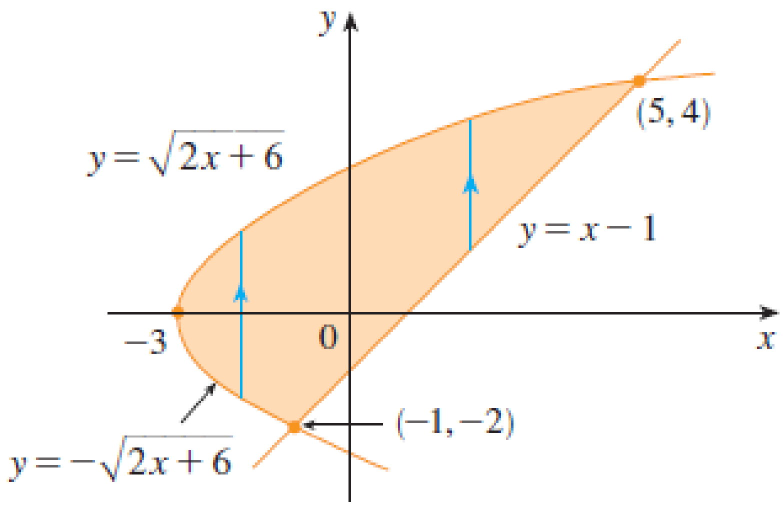


Figure 14: Type I region

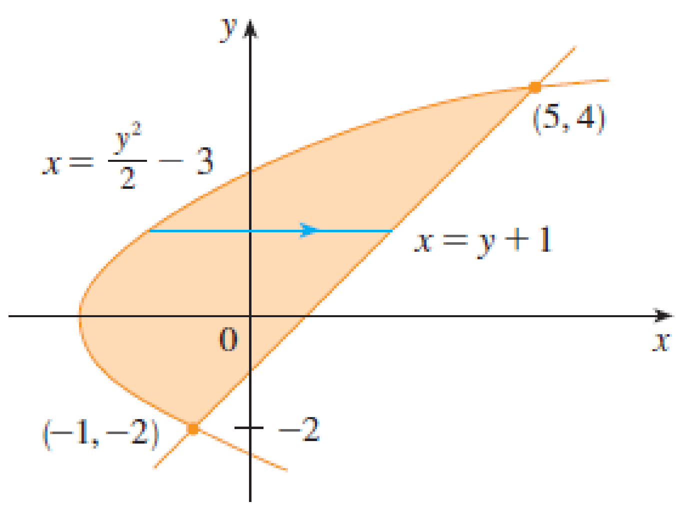


Figure 15: Type II region

The region D can be written as both type I and type II, but the description of D as a type I region is more complicated because the lower boundary consists of two parts. Therefore we prefer to express D as a type II region:

$$D = \{(x, y) : \ -2 \le y \le 4, \tfrac{1}{2}y^2 - 3 \le x \le y + 1\}$$

Then we have

$$\begin{aligned}\iint\limits_D xy \, dA &= \int_{-2}^{4} \int_{\frac{1}{2}y^2-3}^{y+1} xy \, dxdy \\ &= 36.\end{aligned}$$

If we had expressed D as a type I region using Figure 12(a), then we would have obtained

$$\iint\limits_D xy \, dA = \int_{-3}^{-1} \int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xy \, dydx + \int_{-1}^{5} \int_{x-1}^{\sqrt{2x+6}} xy \, dydx$$

◀

Reversing the order of integration

Example 13.

Find the iterated integral

$$\int_0^1 \int_1^x \sin(y^2) \, dy \, dx.$$

Solution. If we try to evaluate the integral as it stands, we are faced with the task of first evaluating $\int \sin(y^2) \, dy$. But it's impossible to do so in finite terms since $\int \sin(y^2) \, dy$ is not an elementary function. (See the end of Section 5.8.) So we must change the order of integration. This is accomplished by first expressing the given iterated integral as a double integral. We have

$$\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx = \iint_D \sin(y^2)\, dA,$$

where $D = \{(x, y) : \ 0 \le x \le 1, x \le y \le 1\}$. The sketch of this region D is as follows:

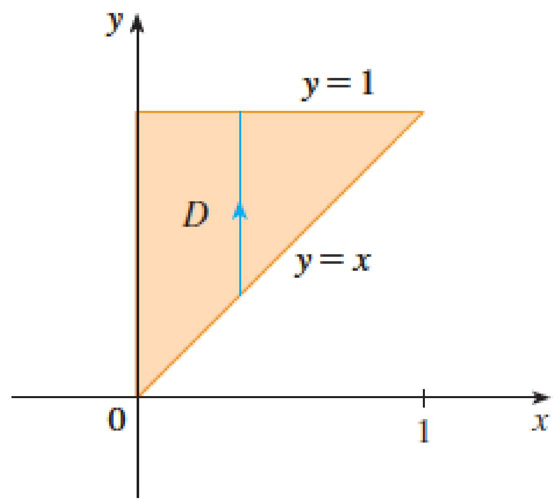


Figure 16: Type I region

An alternative description of D is as follows:

$$D = \{(x, y) : \ 0 \le y \le 1, 0 \le x \le y\}$$

This enables us to express the double integral as an iterated

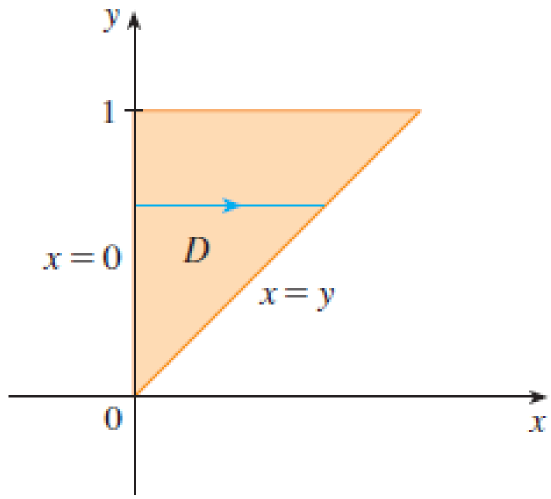


Figure 17: Type II region

integral in the reverse order:

$$\begin{aligned}\int_0^1 \int_x^1 \sin(y^2)\; dy\, dx &= \iint\limits_D \sin(y^2)\; dA \\ &= \int_0^1 \int_0^y \sin(y^2)\; dx\, dy \\ &= \tfrac{1}{2}(1-\cos 1).\end{aligned}$$

◀

Properties of Double Integrals

Here are some properties of the double integral. We assume that all of the following integrals exist. Note that all first three of these properties are really just generalizations of properties of double integrals over rectangles.

1.

$$\iint\limits_D [f(x,y)+g(x,y)]\, dA = \iint\limits_D f(x,y)\, dA + \iint\limits_D g(x,y)\, dA$$

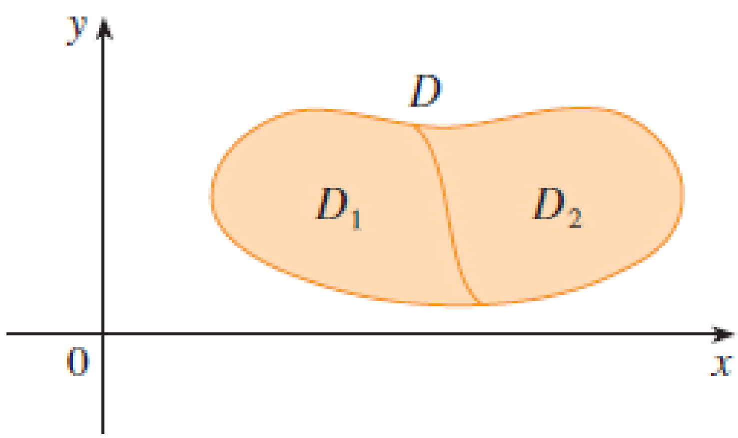
2. If c is a constant, then

$$\iint_D cf(x,y)\,dA = c\iint_D f(x,y)\,dA$$

3. If $f(x,y) \geq g(x,y)$ for all in $(x,y) \in D$, then

$$\iint_D f(x,y)\,dA \geq \iint_D g(x,y)\,dA$$

Assume that $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries. See the figure.



Then

4.

$$\iint_D [f(x,y)+g(x,y)]\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$

5.

$$\iint_D 1\,dA = A(D),$$

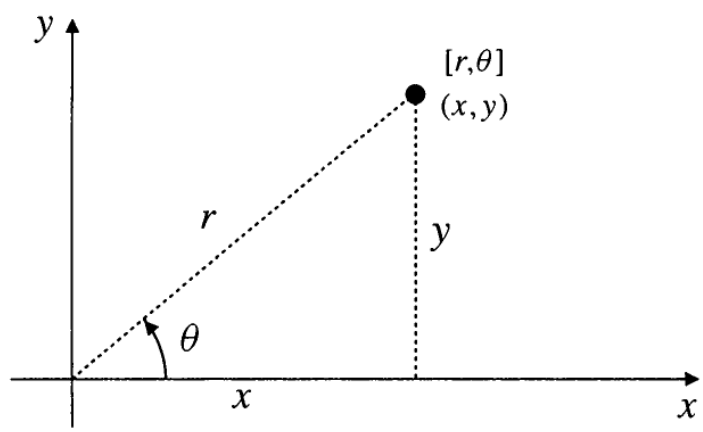
where $A(D)$ is the area of D.

6. If $m \leq f(x,y) \leq M$ for all in $(x,y) \in D$, then

$$mA(D) \leq \iint_D f(x,y)\, dA \leq MA(D)$$

3 Polar coordinates

Recall that the polar representation of a point P is the ordered pair (r,θ), where r is the distance from the origin to P and θ is the angle the ray through the origin and P makes with the positive x-axis.



The polar coordinates r and θ of a point (x,y) in rectangular coordinates satisfy

$$r = \sqrt{x^2+y^2} \qquad \text{and} \qquad \tan\theta = \frac{y}{x}.$$

the rectangular coordinates x and y of a point (r,θ) in polar coordinates satisfy

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta.$$

For many applications of double integrals, either the domain of integration, the integrand function, or both may be more easily expressed in terms of polar coordinates than in terms of Cartesian coordinates.

Consider the double integral

$$\iint_D e^{x^2+y^2}\ dA,$$

where D is the unit disk. While we cannot directly evaluate this integral in rectangular coordinates, a change to polar coordinates will convert it to one we can easily evaluate.

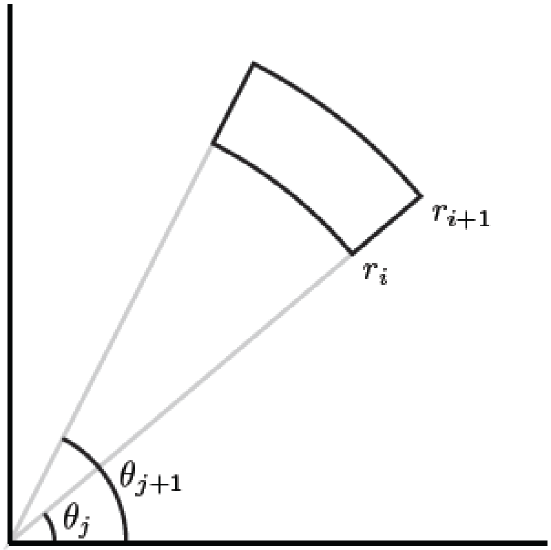
We have seen how to evaluate a double integral $\iint_D f(x, y)\, dA$ as an iterated integral of the form

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\ dA$$

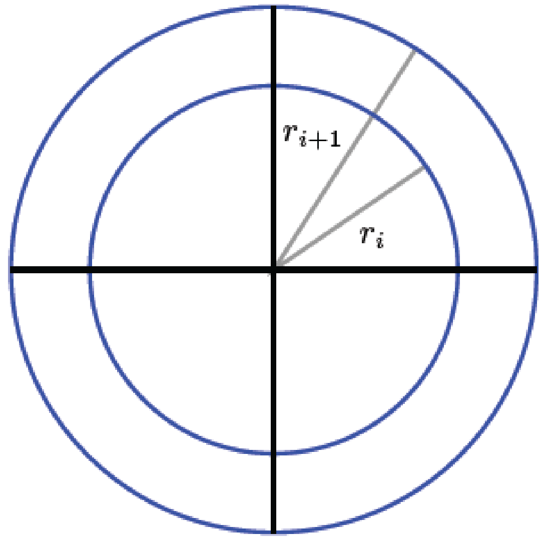
in rectangular coordinates, because we know that $dA = dydx$ in rectangular coordinates. To make the change to polar coordinates, we need to

- represent the variables x and y in polar coordinates,
- write the area element, dA, in polar coordinates. That is, we must determine how the area element dA can be written in terms of dr and $d\theta$ in the context of polar coordinates.

We address this question in the following activity.



A polar rectangle



An annulus

A polar rectangle is almost, but not quite a rectangle. The area of this piece is ΔA. The two sides of this piece both have length $\Delta r = r_{i+1} - r_i$ where r_{i+1} is the radius of the outer arc and r_i is the radius of the inner arc. Basic geometry then tells us that the length of the inner edge is $r_{i+1}\Delta\theta$ while the length of the out edge is $r_i\Delta\theta$ where $\Delta\theta$ is the angle between the two radial lines that form the sides of this piece.

Now, we take the mesh so small that we can assume that $r_{i+1} \approx r_i = r$ and with this assumption we can also assume that our piece is close enough to a rectangle that we can also then assume that

$$dA \approx \Delta A \qquad d\theta \approx \Delta\theta \qquad dr \approx \Delta r.$$

With these assumptions we then get

$$dA \approx r\,\Delta r\,\Delta\theta.$$

In the limiting case we have

$$\boxed{dA = r\,dr\,d\theta.}$$

The reason the additional factor of r in the polar area element is due to the fact that in polar coordinates, the cross sectional area element increases as r increases, while the cross sectional area element in rectangular coordinates is constant.

So, given a double integral $\iint\limits_D f(x,y)\,dA$ in rectangular coordinates, to write a corresponding iterated integral in polar coordinates, we replace

- x with $r\cos\theta$,
- y with $r\sin\theta$ and
- dA with $r\,dr\,d\theta$.

Of course, we need to describe the region D in polar coordinates as well. To summarize:

If f is continuous on a polar rectangle R given by $0 \le a \le r \le b, \alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then

$$\iint\limits_D f(x,y)\,dA = \iint\limits_D f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

This formula says that we change from rectangular coordinates to polar coordinates in a double integral by substituting $r\cos\theta$ for x and $r\sin\theta$ for y, and , using the appropriate limits of integration for r and θ, and replacing dA by $r\,dr\,d\theta$.

Example 14.

> Let $f(x,y) = e^{x^2+y^2}$ on the disk $D = \{(x,y) : x^2 + y^2 \leq 1\}$. Evaluate $\iint\limits_D f(x,y)\,dA$.

Solution. ◀

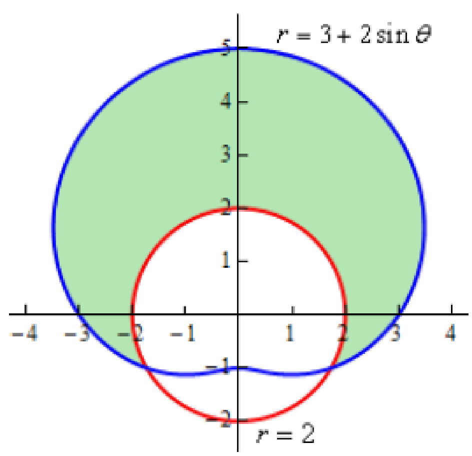
While there is no firm rule for when polar coordinates can or should be used, they are a natural alternative anytime the domain of integration may be expressed simply in polar form, and/or when the integrand involves expressions such as

$$\sqrt{x^2+y^2}.$$

Example 15.

> Determine the area of the region that lies inside $r = 3 + 2\sin\theta$ and outside $r = 2$.

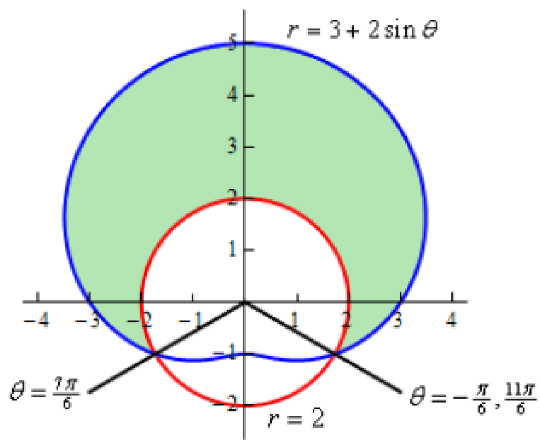
Solution. Here is a sketch of the region, D, that we want to determine the area of.



To determine this area we'll need to know that the value of θ for which the two curves intersect. We can determine these points by solving the two equations. We have

$$\begin{aligned} 3+2\sin\theta &= 2 \\ \Rightarrow \quad \sin\theta &= -\frac{1}{2} \\ \Rightarrow \quad \theta &= \frac{7\pi}{6}, \frac{11\pi}{6}. \end{aligned}$$

Here is a sketch of the figure with these angles added.



Note as well that we've acknowledged that $-\frac{\pi}{6}$ is another representation for the angle $-\frac{11\pi}{6}$. This is important since we need the range of θ to actually enclose the regions as we increase from the lower limit to the upper limit. If we'd chosen to use $-\frac{11\pi}{6}$, then as we increase from $\frac{7\pi}{6}$ to $\frac{11\pi}{6}$ we would be tracing out the lower portion of the circle and that is not the region that we are after.

So, here are the ranges that will define the region.

$$\begin{gathered} -\frac{\pi}{6} \leq \theta \leq \frac{7\pi}{6} \\ 2 \leq r \leq 3+2\sin\theta. \end{gathered}$$

To get the ranges for r the function that is closest to the origin is the lower bound and the function that is farthest from the origin is the upper bound.

The area of the region D is then

$$\begin{aligned} A &= \iint\limits_D dA \\ &= \int_{-\pi/6}^{\pi/6} \int_2^{3+2\sin\theta} dr\, d\theta \\ &= \frac{11\sqrt{3}}{2} + \frac{14\pi}{3}. \end{aligned}$$

◀

Example 16.

Determine the volume of the region that lies under the sphere $x^2 + y^2 + z^2 = 9$, above the plane $z = 0$ and inside the cylinder $x^2 + y^2 = 5$.

Solution. We know that the formula for finding the volume of a region is

$$V = \iint\limits_D f(x, y)\, dA.$$

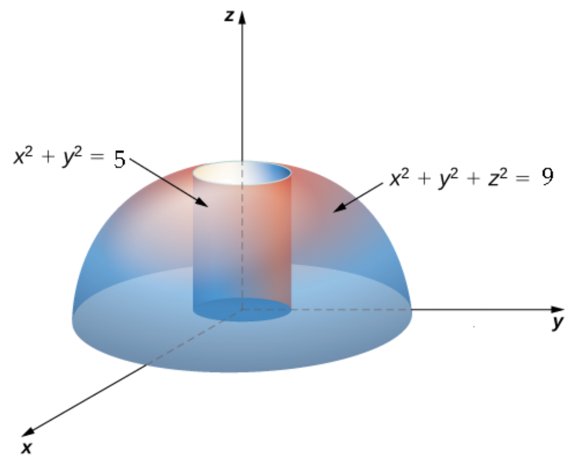
We have

$$f(x, y) = z = \sqrt{9 - x^2 - y^2} = \sqrt{9 - r^2}.$$

The region D is the bottom of the cylinder given by $x^2 + y^2 = 5$, that is, the disk

$$D = \{(x, y) |\ x^2 + y^2 \le 5\}$$

in the xy-plane. So, the region that we want the volume for is really a cylinder with a cap that comes from the sphere.



Now, the limits (in polar coordinates) for the region are

$$0 \le \theta \le 2\pi$$
$$0 \le r \le \sqrt{5}.$$

The volume is then

$$V = \iint\limits_D \sqrt{9 - x^2 - y^2}\, dA = \int_0^{2\pi} \int_0^{\sqrt{5}} r\sqrt{9 - r^2} dr\, d\theta = 38\pi/3.$$

◀

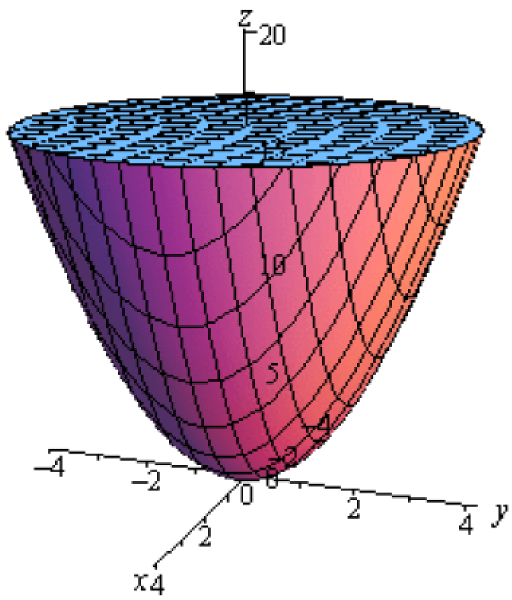
Example 17.

Find the volume of the region that lies inside $z = x^2 + y^2$ and below the plane $z = 16$.

Solution. Let's start this example off with a sketch of the region.

Now, in this case the standard formula is not going to work. The formula

$$\iint\limits_D f(x, y)\, dA.$$



finds the volume under the function $f(x, y)$ and we're actually after the volume that is above a function. However, notice that the formula

$$\iint_D 16 \, dA.$$

will be the volume under $z = 16$ while the formula

$$\iint_D (x^2 + y^2) \, dA.$$

is the volume under $z = x^2 + y^2$, using the same D. Hence the required volume is

$$V = \iint_D 16 \, dA - \iint_D (x^2 + y^2) \, dA = \iint_D (16 - x^2 - y^2) \, dA.$$

Now all that we need to do is to determine the region D and then convert everything over to polar coordinates.

We see that the top of the region, where the elliptic paraboloid intersects the plane $z = 16$, is the widest part of the region. So, setting $z = 16$ in the equation of the paraboloid gives,

$$16 = x^2 + y^2,$$

which is the equation of a circle of radius 4 centered at the origin.

Here are the inequalities for the region and the function we'll be integrating in terms of polar coordinates.

$$0 \le \theta \le 2\pi, \qquad 0 \le r \le 4, \qquad z = 16 - r^2.$$

The volume is then,

$$\begin{aligned} V &= \iint_D (16 - x^2 - y^2)\, dA \\ &= \int_0^{2\pi} \int_0^4 r(16 - r^2)\, dr\, d\theta \\ &= 128\pi. \end{aligned}$$

◀

Example 18.

> Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

4 Applications of double integrals

Mass

We saw before that the double integral over a region of the constant function 1 measures the area of the region. If the region has uniform density 1, then the mass is the density times the area which equals the area. What if the density is not constant. Suppose that the density is given by the continuous function

$$\text{Density } = \rho(x, y)$$

In this case we can cut the region into tiny rectangles where the density is approximately constant. The area of mass rectangle is given by

$$\text{Mass } = (\text{ Density })(\text{ Area }) = \rho(x, y)\Delta x \, \Delta y$$

You probably know where this is going. If we add all to masses together and take the limit as the rectangle size goes to zero, we get a double integral.

Mass
Let $\rho(x, y)$ be the density of a lamina (flat sheet) D at the point (x, y). Then the total mass of the lamina is the double integral

$$\text{Mass } = \iint_D \rho(x, y) \, dy \, dx.$$

Moments and Center of Mass

We know that the moments about an axis are defined by the product of the mass times the distance from the axis.

$$M_x = (\text{ Mass })(y), \qquad M_y = (\text{ Mass })(x).$$

If we have a region D with density function $\rho(x, y)$, then we do the usual thing. We cut the region into small rectangles for which the density is constant and add up the moments of each of these rectangles. Then take the limit as the rectangle size approaches zero. This will give us the total moment.

Definition of Moments of Mass and Center of Mass

Suppose that $\rho(x, y)$ is a continuous density function on a lamina D. Then the moments of mass are

$$M_x = \iint\limits_D \rho(x, y)y \; dy \; dx, \qquad M_y = \iint\limits_D \rho(x, y)x \; dy \; dx.$$

and if m is the mass of the lamina, then the center of mass is

$$(\overline{x}, \overline{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m} \right)$$

Example 19.

> Set up the integrals that give the center of mass of the rectangle with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$ and density function proportional to the square of the distance from the origin.

Solution. Since the density function $\rho(x, y)$ is proportional to the square of the distance from the origin, $x^2 + y^2$, the mass is given by

$$m = \int_0^1 \int_0^1 k(x^2 + y^2) \; dy \; dx = \frac{2k}{3}.$$

The moments are given by

$$M_x = \int_0^1 \int_0^1 k(x^2 + y^2)y \; dy \; dx = 5k/12$$

$$M_y = \int_0^1 \int_0^1 k(x^2 + y^2)x \; dy \; dx = 5k/12$$

It should not be a surprise that the moments are equal since there is complete symmetry with respect to x and y. Finally, we divide to get

$$(\overline{x}, \overline{y}) = (5/8, 5/8)$$

This tells us that the metal plate will balance perfectly if we place a pin at (5/8 , 5/8). ◀

Example 20.

> Find the mass and center of mass of a triangular lamina with vertices $(0,0), (1,0)$, and $(0,2)$ if the density function is
>
> $$\rho(x, y) = 1 + 3x + y.$$

Solution. content... ◀

Moments of Inertia

We often call M_x and M_y the first moments. They have first powers of y and x in their definitions and help find the center of mass. We define the moments of inertia (or second moments) by introducing squares of y and x in their definitions. The moments of inertia help us find the kinetic energy in rotational motion. Below is the definition.

Moments of Inertia

Suppose that $\rho(x, y)$ is a continuous density function on a lamina D. Then the moments of inertia about the x-axis and the y-axis are

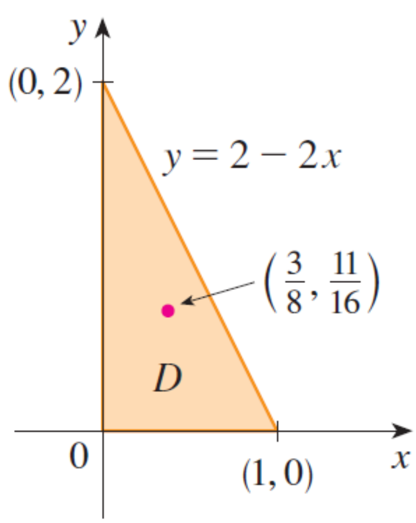
$$I_x = \iint_D \rho(x, y) y^2 \, dy \, dx, \qquad I_y = \iint_D \rho(x, y) x^2 \, dy \, dx.$$

It is also of interest to consider the moment of inertia about the origin, also called the **polar moment of inertia**:

$$I_x = \iint_D \rho(x, y)(x^2 + y^2) \, dy \, dx.$$

Example 21.

Find the moments of inertia for the square metal plate with vertices $(0, 0), (1, 0), (1, 1)$, and $(0, 1)$.



Solution. ◀

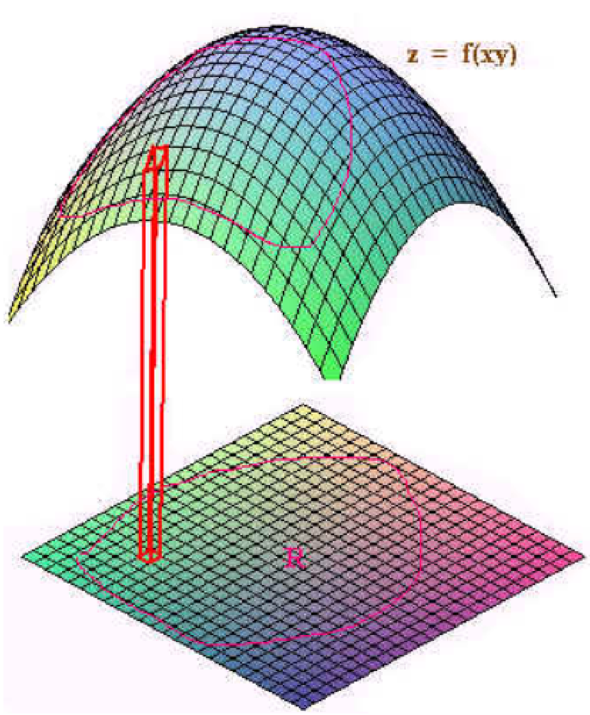
Example 22.

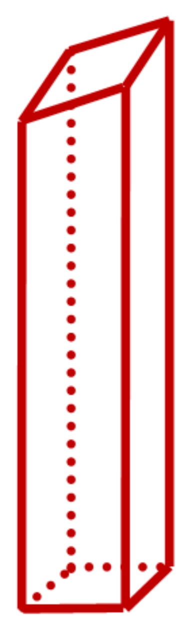
> Find the moments of inertia I_x, I_y, and I_o of a homogeneous disk with density $\rho(x, y) = \rho$, center the origin, and radius a.

Solution. The boundary of D is the circle $x^2 + y^2 = a$ and in polar coordinates D is described by $0 \leq \theta \leq 2\pi, 0 \leq r \leq a$. First compute I_o. Then use $I_x + I_y = I_o$ and $I_x = I_y$ (due to the symmetry of the problem) to find I_x and I_y. ◀

5 Surface area

Let $z = f(x, y)$ be a surface in $\mathbb{R}^3$ defined over a region D in the xy-plane. cut the xy-plane into rectangles. Each rectangle will project vertically to a piece of the surface as shown in the figure below.





Although the area of the rectangle in D is

$$\text{Area } = \Delta y \Delta x.$$

The area of the corresponding piece of the surface will not be $\Delta y \Delta x$ since it is not a rectangle. Even if we cut finely, we will still not produce a rectangle, but rather will approximately produce a parallelogram. With a little geometry we can see that the two adjacent sides of the parallelogram are (in vector form)

$$u = \Delta x \vec{i} + f_x(x, y)\Delta x \vec{k}$$

and

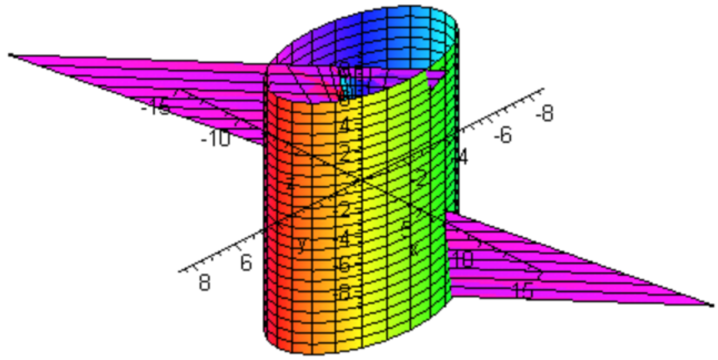
$$v = f_y(x, y)\Delta y \vec{i} + \Delta y \vec{k}$$

We can see this by realizing that the partial derivatives are the slopes in each direction. If we run Δx in the $\vec{i}$ direction, then we will rise $f_x(x, y)\Delta x$ in the $\vec{k}$ direction so that

$$\frac{\text{rise}}{\text{run}} = f_x(x, y),$$

which agrees with the slope idea of the partial derivative. A similar argument will confirm the equation for the vector v. Now that we know the adjacent vectors we recall that the area of a parallelogram is the magnitude of the cross product of the two adjacent vectors. We have

$$\begin{aligned}
|v \times w| &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \Delta x & 0 & f_x(x, y)\Delta x \\ 0 & \Delta y & f_y(x, y)\Delta y \end{vmatrix} \\
&= | - (f_y(x, y)\Delta y \Delta x)\vec{i} - (f_x(x, y)\Delta y \Delta x)\vec{j} + (\Delta y \Delta x)\vec{k}| \\
&= \sqrt{f_y^2(x, y)(\Delta y \Delta x)^2 + f_x^2(x, y)(\Delta y \Delta x)^2 + (\Delta y \Delta x)^2} \\
&= \sqrt{f_y^2(x, y) + f_x^2(x, y) + 1}\, \Delta y \Delta x.
\end{aligned}$$



This is the area of one of the patches of the quilt. To find the total area of the surface, we add up all the areas and take the limit as the rectangle size approaches zero. This results in a double Riemann sum, that is a double integral. We state the definition below.

Definition of Surface Area

Let $z = f(x, y)$ be a differentiable surface defined over a region D. Then its surface area is given by

$$\text{Surface Area } = \iint_D \sqrt{1 + f_x^2(x, y) + f_y^2(x, y)}\, dy\, dx.$$

Example 23.

Find the surface area of the part of the plane

$$z = 8x + 4y$$

that lies inside the cylinder

$$x^2 + y^2 = 16.$$

Solution. We calculate partial derivatives

$$f_x(x, y) = 8, \qquad f_y(x, y) = 4$$

so that

$$1 + f_x^2(x, y) + f_y^2(x, y) = 1 + 64 + 16 = 81$$

Taking a square root and integrating, we get

$$\iint\limits_D 9 \, dy \, dx.$$

We could work this integral out, but there is a much easier way. The integral of a constant is just the constant times the area of the region. Since the region is a circle, we get

$$\text{Surface Area } = 9(16\pi) = 144\pi.$$

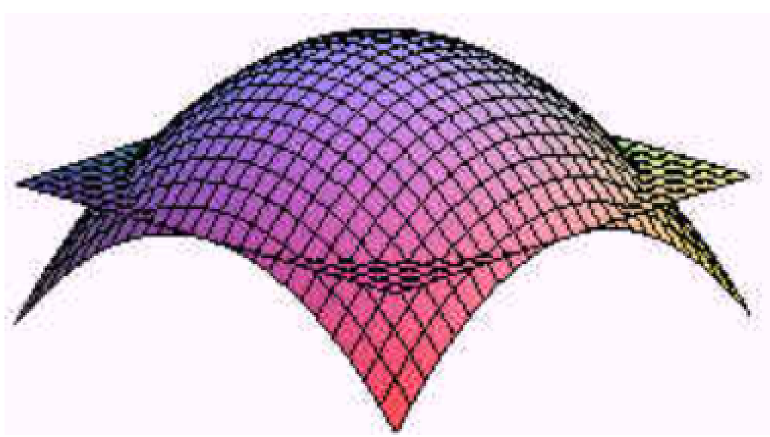
◀

Example 24.

Find the surface area of the part of the paraboloid

$$z = 25 - x^2 - y^2$$

that lies above the xy-plane.



Solution. We calculate partial derivatives

$$f_x(x,y) = -2x \qquad f_y(x,y) = -2y$$

so that

$$1 + f_x^2(x,y) + f_y^2(x,y) = 1 + 4x^2 + 4y^2.$$

Using "Polar Coordinates", we realize that the region is just the circle

$$r = 5$$

Now convert the integrand to polar coordinates to get

$$\int_0^{2\pi} \int_0^5 \sqrt{(}1 + 4r^2)\ rdr\ d\theta$$

Now let

$$u = 1 + 4r^2, \qquad du = 8rdr$$

and substitute

$$\frac{1}{8}\int_0^{2\pi} \int_1^{101} u^{1/2}\ du\ d\theta = \frac{1}{12}\int_0^{2\pi} \left[u^{3/2}\right]_1^{101}\ d\theta \approx 169.3\pi.$$

◀

6 Triple integrals

We have seen that the geometry of a double integral involves cutting the two dimensional region into tiny rectangles, multiplying the areas of the rectangles by the value of the function there, adding the areas up, and taking a limit as the size of the rectangles approaches zero. We have also seen that this is equivalent to finding the double iterated iterated integral.

We will now take this idea to the next dimension. Instead of a region in the xy-plane, we will consider a solid in xyz-space. Instead of cutting up the region into rectangles, we will cut up the solid into rectangular solids. And instead of multiplying the function value by the area of the rectangle, we will multiply the function value by the volume of the rectangular solid.

We can define the triple integral as the limit of the sum of the product of the function times the volume of the rectangular solids.

Instead of the double integral being equivalent to the double iterated integral, the triple integral is equivalent to the triple iterated integral.

Definition of the Triple Integral
Let $f(x, y, z)$ be a continuous function of three variables defined over a solid B. Then the triple integral over B is defined as

$$\iiint_B f(x, y, z)\, dxdydz = \lim \sum f(x, y, z)\Delta x\Delta y\Delta z,$$

where the sum is taken over the rectangular solids included in the solid B and lim is taken to mean the limit as the side lengths of the rectangular solid.

This definition is only practical for estimating the triple integral when a data set is given. Just as for double integrals, the practical method for evaluating triple integrals is to express them as iterated integrals as follows.

Fubinis Theorem for Triple Integrals
If f is continuous on a rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z)\, dxdydz = \int_a^b \int_c^d \int_r^s f(x, y, z)\, dxdydz.$$

Remark 1. As with double integrals the order of integration can be changed with care.

Triple integral over a general bounded region E

A solid region is said to be of **type 1** if it lies between the graphs of two continuous functions of x and y, that is,

$$E = \{(x, y, z) : \ (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$

where D is the projection of E onto the xy-plane as shown below.

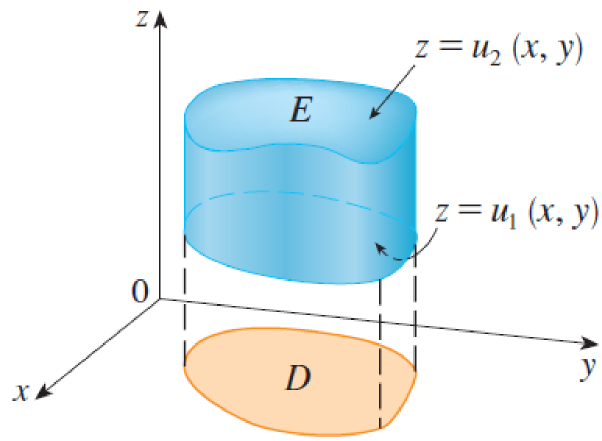


Figure 18: A type 1 solid region

Notice that the upper boundary of the solid E is the surface with equation $z = u_2(x, y)$, while the lower boundary is the surface $z = u_1(x, y)$. Then the triple integral takes the form as given below.

$$\iiint\limits_B f(x, y, z)\ dxdydz$$

$$= \iint\limits_D \left[\int_{u_1(x)}^{u_2(x)} f(x, y, z)\ dz \right] dA.$$

In particular, if the projection D of E onto the xy-plane is a type I plane region (as in Figure 3),

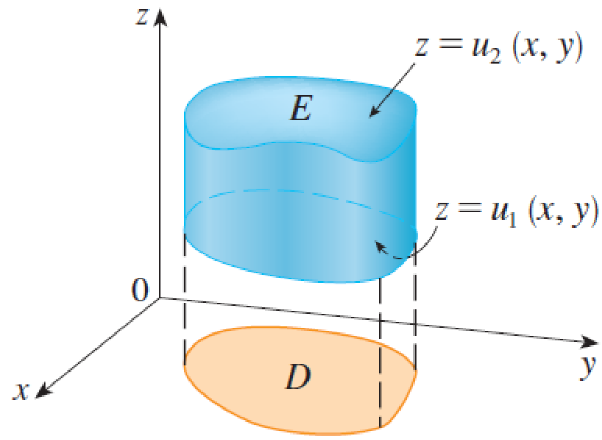


Figure 19: A type 1 solid region where the projection D is a type I plane region

then the triple integral becomes as in the following theorem:

Theorem for Evaluating Triple Integrals

Let $f(x, y, z)$ be a continuous function over a solid E defined by

$$E = \{(x, y, z) : \ a \leq x \leq b, g_1(x) \leq y \leq g_2(x), \\ u_1(x, y) \leq z \leq u_2(x, y)\}.$$

Then the triple integral is equal to the triple iterated integral.

$$\iiint_E f(x, y, z) \; dxdydz \\ = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z) \; dzdydx.$$

Example 25.

Evaluate

$$\iiint_E f(x, y, z)\ dzdydx,$$

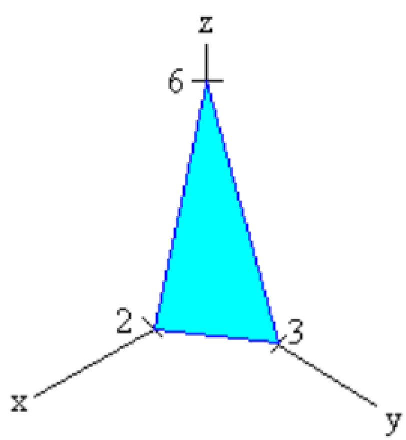
where

$$f(x, y, z) = 1 - x$$

and E is the solid that lies in the first octant and below the plain

$$3x + 2y + z = 6.$$

Solution. The picture of the region is The challenge here is



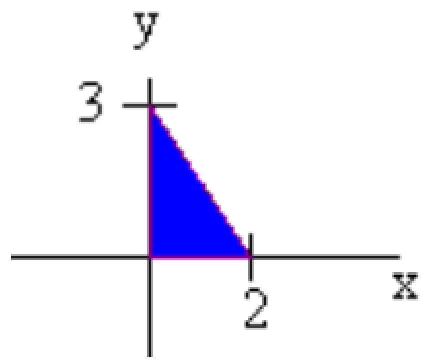
to find the limits. We work on the innermost limit first which corresponds with the variable "z". Think of standing vertically. Your feet will rest on the lower limit and your head will touch the higher limit. The lower limit is the xy-plane or

$$z = 0.$$

The upper limit is the given plane. Solving for z, we get

$$z = 6 - 3x - 2y.$$

Now we work on the middle limits that correspond to the variable "y". We look at the projection of the surface in the xy-plane.



It is shown below. Now we find the limits just as we found the limits of double integrals. The lower limit is just

$$y = 0.$$

If we set $z = 0$ and solve for y, we get for the upper limit

$$y = 3 - (3/2)x.$$

Next we find the outer limits, corresponding to the variable "x". The lowest x gets is 0 and highest x gets is 2. Hence

$$0 < x < 2.$$

The integral is thus

$$\begin{aligned}
&\int_0^2\int_0^{3-3x/2}\int_0^{6-3x-2y}(1-x)\;dzdydx\\
=&\int_0^2\int_0^{3-3x/2}[z-xz]_{z=0}^{6-3x-2y}\;dydx\\
=&\int_0^2\int_0^{3-3x/2}[(6-3x-2y)-(6x-3x^2-2xy)]\;dydx\\
=&\int_0^2\int_0^{3-3x/2}(6-9x-2y+3x^2+2xy)\;dydx\\
=&\int_0^2\left[6y-9xy-y^2+3x^2y+xy^2\right]_{y=0}^{3-3x/2}\;dx\\
=&\int_0^2(9-18x+(45/4)x^2-(9/4)x^3)\;dx\\
=&\left[9x-9x^2+(15/4)x^3-(9/16)x^4\right]_0^2\\
=&3.
\end{aligned}$$

◀

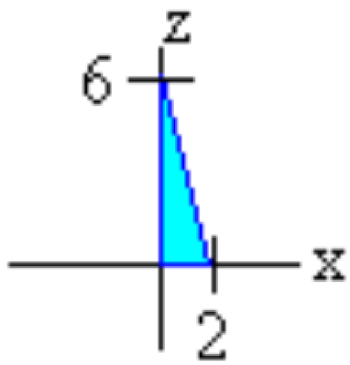
Example 26.

> Switch the order of integration from the previous example so that $dydxdz$ appears.

Solution. This time we work on the "y" variable first. The lower limit for the y-variable is 0. For the upper limit, we solve for y in the plane to get

$$y=3-3/2x-1/2z$$

To find the "x" limits, we project onto the xz-plane as shown below The lower limit for x is 0. To find the upper limit we set



$y = 0$ and solve for x to get

$$x = 2 - (1/3)z$$

Finally, to get the limits for z, we see that the smallest z will get is 0 and the largest z will get is 6. We get

$$0 < z < 6$$

We can write

$$\int_0^6 \int_0^{2-z/3} \int_0^{6-3x/2-z=/3} (1-x)\; dydxdz.$$

◀

Applications of Triple Integrals

Let's begin with the special case where $f(x, y, z) = 1$ for all points in E. Then the triple integral does represent the volume of E :

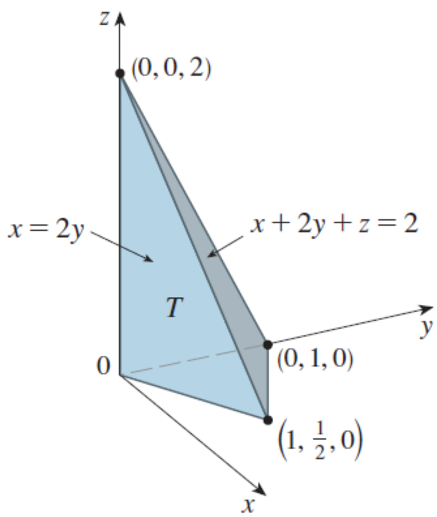
$$\boxed{V(E) = \iiint\limits_E dV.}$$

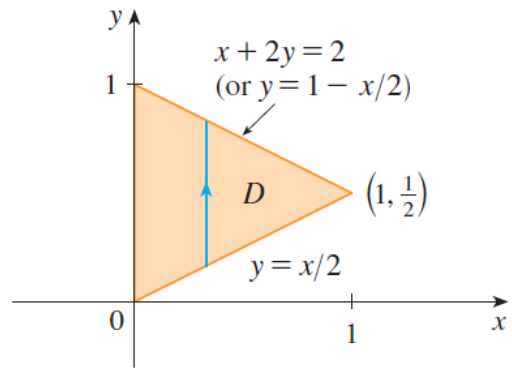
Example 27.

Use a triple integral to find the volume of the tetrahedron T bounded by the planes

$$x + 2y + z = 2, \;\; x = 2y, \;\; z = 0.$$

Solution. The tetrahedron T and its projection D onto the XY-plane are shown in the figure. The lower boundary of T is





the plane $z = 0$ and the upper boundary is the plane $x+2y+z = 2$, that is, $z = 2 - x - 2y$. Therefore we have

$$V(E) = \iiint\limits_{E} \, dV = \int_0^1 \int_{x/2}^{1-x/2} \int_0^{2-x-2y} \, dzdydx$$

$$= \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y) \; dydx = \frac{1}{3}.$$

◀

Mass, Center of Mass, and Moments of Inertia

For a three dimensional solid E with constant density, the mass is the density times the volume. If the density is not constant but rather a continuous function of x, y, and z, then we can cut the solid into very small rectangular solids so that on each rectangular solid the density is approximately constant. The volume of the small rectangular solid is

$$\Delta\text{Mass} \;=\; (\text{Density})(\Delta\text{Volume}) \;= f(x,y,z)\Delta x\Delta y\Delta z$$

Now do the usual thing. We add up all the small masses and take the limit as the rectangular solids get small. This will give us the triple integral

$$\text{Mass} \;= \iiint\limits_E f(x,y,z)\; dzdydx.$$

We find the center of mass of a solid just as we found the center of mass of a lamina. Since we are in three dimensions, instead of the moments about the axes, we find the moments about the coordinate planes. We state the definitions from physics below.

Definition: **Moments and Center of Mass**

Let $\rho(x, y, z)$ be the density of a solid E. Then the first moments about the coordinate planes are

$$M_{yz} = \iiint\limits_E x\rho(x, y, z)\ dzdydx$$

$$M_{xz} = \iiint\limits_E y\rho(x, y, z)\ dzdydx$$

$$M_{xy} = \iiint\limits_E z\rho(x, y, z)\ dzdydx$$

and the center of mass is given by

$$(x, y, z) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m}\right)$$

Notice that letting the density function being identically equal to 1 gives the volume

$$\text{Volume } = \iiint\limits_E\ dzdydx.$$

Just as with lamina, there are formulas for moments of inertial about the three axes. They involve multiplying the density function by the square of the distance from the axes. We have

Definition: Moments and Center of Mass

Let $\rho(x, y, z)$ be the density of a solid E. Then the first moments of inertia about the coordinate axes are

$$I_x = \iiint\limits_E (y^2 + z^2)\rho(x, y, z)\ dzdydx$$

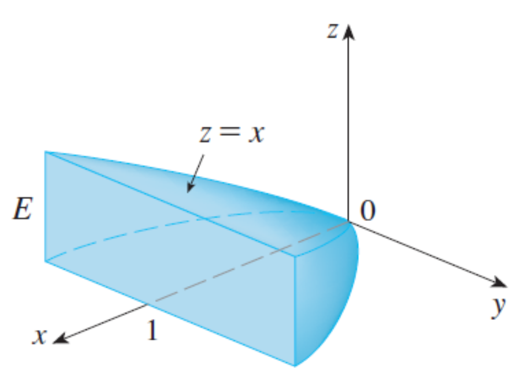
$$I_y = \iiint\limits_E (x^2 + z^2)\rho(x, y, z)\ dzdydx$$

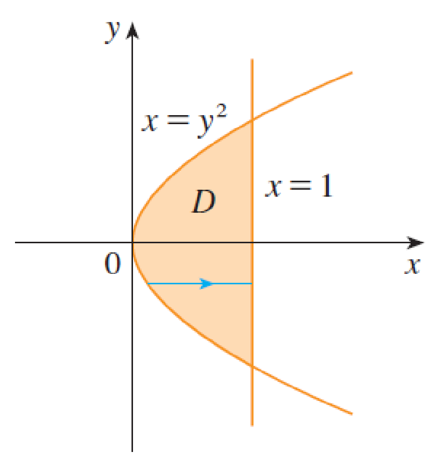
$$I_z = \iiint\limits_E (x^2 + y^2)\rho(x, y, z)\ dzdydx$$

Example 28.

Find the center of mass of a solid E of constant density that is bounded by the parabolic cylinder $x = y^2$ and the planes $x = z$, $z = 0$, and $x = 1$.

Solution. The solid E and its projection D onto the xy-plane are shown in the figure given below.





The lower and upper surfaces of E are the planes $z = 0$ and $z = x$, so we describe as a type 1 region:

$$E = \{(x, y, z) : \; -1 \le y \le 1, y^2 \le x \le 1, 0 \le z \le x\}.$$

Then, if the density is $\rho(x, y, z) = \rho$, the mass is

$$m = \iiint\limits_E \rho \, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_{0}^{x} \rho \, dzdydx = \frac{4\rho}{5}.$$

Because of the symmetry of E and ρ about the xz-plane, we can immediately say that M_{xz} and therefore $\overline{y} = 0$. The other moments are

$$M_{yz} = \iiint\limits_E x\rho \, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_{0}^{x} x\rho \, dzdydx = \frac{4\rho}{7}$$

$$M_{xy} = \iiint\limits_E z\rho \, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_{0}^{x} z\rho \, dzdydx = \frac{2\rho}{7}$$

◀

7 Change of variables in multiple integrals

Review of the Idea of Substitution

Consider the integral

$$\int_0^2 x \cos x^2 \, dx.$$

To evaluate this integral we use the substitution

$$u = x^2$$

This substitution sends the interval $[0, 2]$ onto the interval $[0, 4]$. We can see that there is stretching of the interval. The stretching is not uniform. In fact, the first part $[0, 0.5]$ is actually contracted. This is the reason why we need to find du.

$$\frac{du}{dx} = 2x \quad \text{or} \quad \frac{dx}{du} = \frac{1}{2x}$$

This is the factor that needs to be multiplied in when we perform the substitution. Notice for small positive values of x, this factor is greater than 1 and for large values of x, the factor is smaller than 1. This is how the stretching and contracting is accounted for.

Jacobians

We have seen that when we convert to polar coordinates, we use

$$dydx = rdrd\theta$$

With a geometrical argument, we showed why the "extra r" is included. Taking the analogy from the one variable case, the transformation to polar coordinates produces stretching and contracting. The "extra r" takes care of this stretching and contracting.

The goal for this section is to be able to find the "extra factor" for a more general transformation. We call this "extra factor" the *Jacobian of the transformation*. We can find it by taking the determinant of the 2×2 matrix of partial derivatives.

Definition of the Jacobian

Let

$$x = g(u, v) \quad \text{and} \quad y = h(u, v)$$

be a transformation of the plane. Then the Jacobian of this transformation is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Remark 2. A useful fact is that the Jacobian of the inverse transformation is the reciprocal of the Jacobian of the original transformation.

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\dfrac{\partial(u, v)}{\partial(x, y)}}$$

This is a consequence of the fact that the determinant of the inverse of a matrix A is the reciprocal of the determinant of A.

Example 29.

Find the Jacobian of the polar coordinates transformation

$$x(r, \theta) = r\cos\theta, \qquad y(r, \theta) = r\sin\theta.$$

Solution. We have

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

◀

Remark 3. This example confirms that

$$dydx = rdrd\theta$$

Double Integration and the Jacobian

Theorem: **Integration and Coordinate Transformations**

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$x = g(u, v), \qquad y = h(u, v)$$

be a transformation on the plane that is one to one from a region S to a region R. If g and h have continuous partial derivatives such that the Jacobian is never zero, then

$$\iint\limits_R f(x, y)\, dy\, dx = \iint\limits_S f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dv\, du$$

Remark 4. Note that a small region $\Delta A = \Delta x \Delta y$ in the xy-plane is related to a small region in the uv-plane whose area is the product $\Delta u \Delta v$, that is,

$$\Delta A \approx \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u\, \Delta v.$$

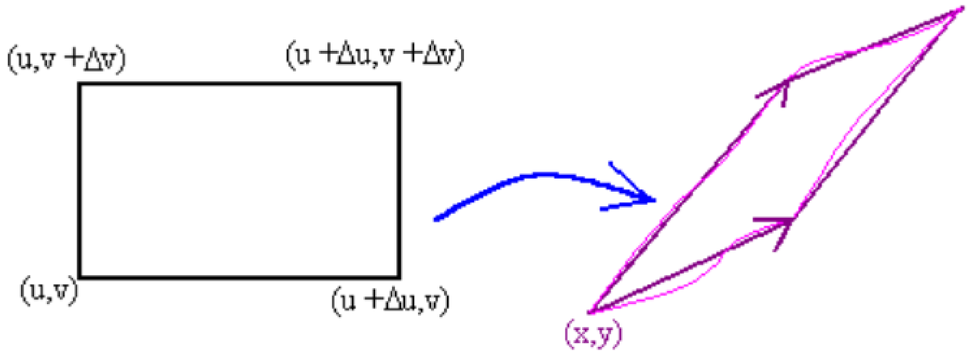
In the limiting case we have

$$\boxed{dx\, dy = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv.}$$

The additional factor of $\left| \frac{\partial(x,y)}{\partial(u,v)} \right|$ tells us how area changes under the map T.

Idea of the Proof

As usual, we cut S up into tiny rectangles so that the image under T of each rectangle is a parallelogram.



We need to find the area of the parallelogram. Considering differentials, we have

$$T(u+\Delta u, v) \approx T(u,v) + (x_u \Delta u, y_u \Delta u)$$
$$T(u, v+\Delta v) \approx T(u,v) + (x_v \Delta v, y_v \Delta v)$$

Thus the two vectors that make the parallelogram are

$$P = g_u \Delta u \vec{i} + h_u \Delta u \vec{j}$$
$$Q = g_v \Delta v \vec{i} + h_v \Delta v \vec{j}$$

To find the area of this parallelogram we just cross the two vectors.

$$\begin{aligned} |P \times Q| = &\ \text{abs} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_u \Delta u & y_u \Delta u & 0 \\ x_v \Delta v & y_v \Delta v & 0 \end{vmatrix} \\ &= |(x_u y_v - x_v y_u)\Delta u \Delta v| \\ &= \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta. \end{aligned}$$

and the extra factor is revealed.

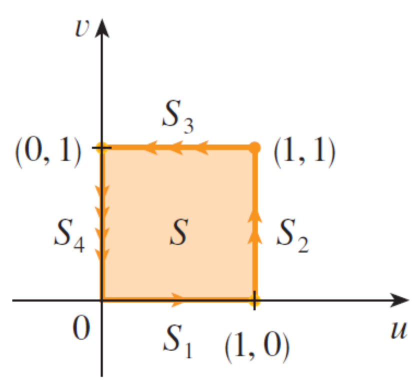
Example 30.

> **Determining the image of a region under a transformation**
> A transformation is defined by the equations
> $$x = u^2 - v^2, \qquad y = 2uv.$$
> Find the image of the square $S = \{(u, v) : \ 0 \le u \le 1, \ 0 \le v \le 1\}$.

Solution. The transformation maps the boundary of S into the boundary of the image. So we begin by finding the images of the sides of S. The first side is given by

$$S_1 = \{(u, 0) : \ 0 \le u \le 1\}.$$

(See Figure 2.) From the given equations we have



$$x = u^2, \qquad y = 0$$

and so $0 \le x \le 1$. Thus, S_1 is mapped into the line segment from $(0, 0)$ to $(1,0)$ in the xy-plane.

The second side is

$$S_2 = \{(1, v) : \ 0 \le v \le 1\}$$

and, putting $u = 1$ in the given equations, we get

$$x = 1 - v^2, \qquad y = 2v$$

Eliminating v, we obtain

$$x = 1 - \frac{y^2}{4} \qquad 0 \leq x \leq 1, \tag{1}$$

which is part of a parabola. Similarly, S_3 is given by

$$S_3 = \{(u, 1): \ 0 \leq u \leq 1\},$$

whose image is the parabolic arc

$$x = \frac{y^2}{4} - 1 \qquad -1 \leq x \leq 0, \tag{2}$$

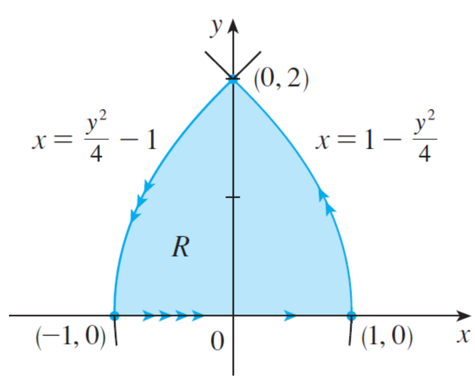
Finally, S_4 is given by

$$S_3 = \{(0, v): \ 0 \leq v \leq 1\},$$

whose image is

$$x = -v^2, y = 0,$$

that is, $-1 \leq x \leq 0$. (Notice that as we move around the



square in the counterclockwise direction, we also move around the parabolic region in the counterclockwise direction.) The image of S is the region (shown in Figure 2) bounded by the x-axis and the parabolas .

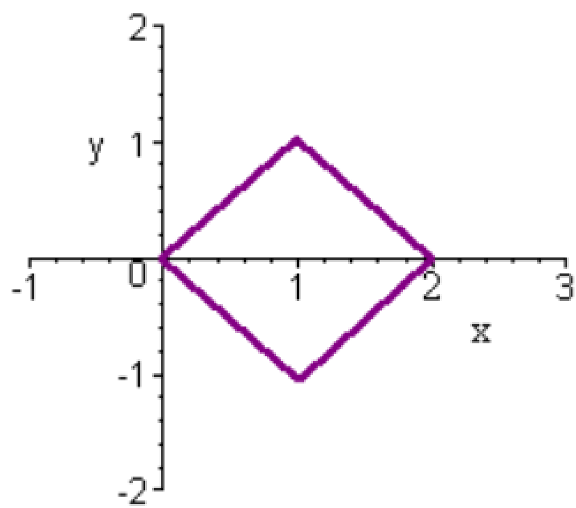
◀

Example 31.

Use an appropriate change of variables to find the volume of the region below

$$z = (x - y)^2$$

above the x-axis, over the parallelogram with vertices $(0,0), (1,1), (2,0)$, and $(1,-1)$.



Solution. We find the equations of the four lines that make the parallelogram to be

$$y = x, \quad y = x - 2, \quad y = -x, \quad y = -x + 2,$$

that is,

$$x - y = 0, \quad x - y = 2, \quad x + y = 0, \quad x + y = 2$$

The region is given by

$$0 < x - y < 2 \text{ and } 0 < x + y < 2$$

This leads us to the inverse transformation

$$u(x, y) = x - y, \quad v(x, y) = x + y$$

The Jacobian of the inverse transformation is

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2.$$

Since the Jacobian is the reciprocal of the inverse Jacobian we get

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2}.$$

The region is given by

$$0 < u < 2 \text{ and } 0 < v < 2$$

and the function is given by

$$z = u^2$$

Putting this all together, we get the double integral

$$\begin{aligned}\int_0^2 \int_0^2 u^2 \frac{1}{2}\, du\, dv &= \int_0^2 \left[\frac{u^3}{6}\right]_0^2 dv \\ &= \int_0^2 \frac{4}{3}\, dv = \frac{8}{3}.\end{aligned}$$

◀

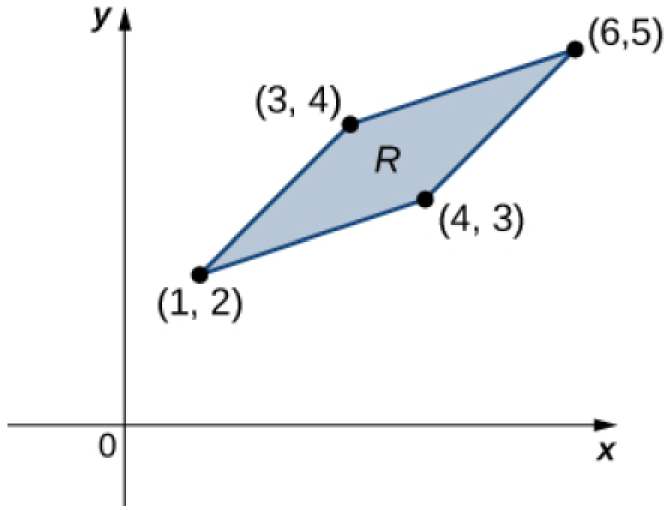
Example 32.

Changing Variables

Consider the integral

$$\iint_R (x-y)dy\, dx,$$

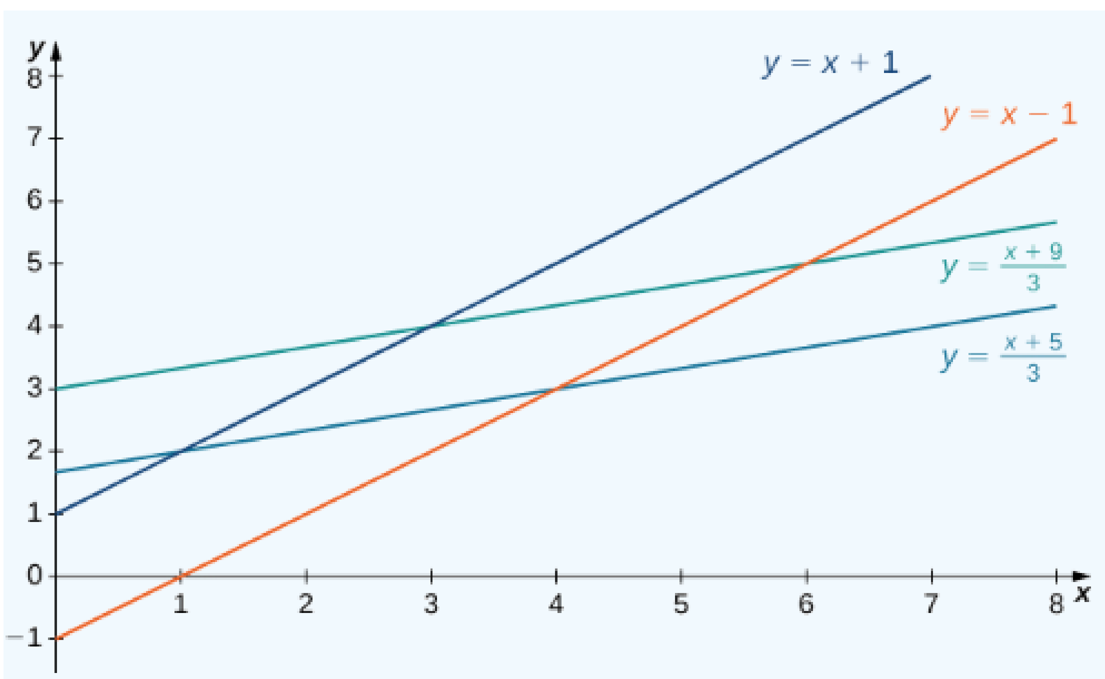
where R is the parallelogram joining the points $(1,2)$, $(3,4)$, $(4,3)$, and $(6,5)$ (See the figure given below). Make appropriate changes of variables, and write the resulting integral.



Solution. First, we need to understand the region over which we are to integrate. The sides of the parallelogram are

$x - y + 1,\ x - y - 1 = 0,\ x - 3y + 5 = 0$ and $x - 3y + 9 = 0$

See the figure.



Another way to look at them is

$x - y = -1,\ x - y = 1,\ x - 3y = -5,$ and $x - 3y = -9.$

Clearly the parallelogram is bounded by the lines

$$y = x + 1,\ y = x - 1,\ y = \frac{1}{3}(x + 5),\ y = \frac{1}{3}(x + 9).$$

Notice that if we were to make $u = x - y$ and $v = x - 3y$, then the limits on the integral would be

$$-1 \le u \le 1 \text{ and } -9 \le v \le -5.$$

To solve for x and y, we multiply the first equation by 3 and subtract the second equation,

$$3u - v = (3x - 3y) - (x - 3y) = 2x.$$

Then we have

$$x = \frac{3u - v}{2}.$$

Moreover, if we simply subtract the second equation from the first, we get

$$u - v = (x - y) - (x - 3y) = 2y \text{ and } y = \frac{u - v}{2}.$$

Thus, we can choose the transformation

$$T(u, v) = \left(\frac{3u - v}{2}, \frac{u - v}{2}\right)$$

and compute the Jacobian J(u,v). We have

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 3/2 & -1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{3}{4} + \frac{1}{4} = -\frac{1}{2}$$

Therefore, $|J(u,v)| = \frac{1}{2}$. Also, the original integrand becomes

$$x - y = \frac{1}{2}[3u - v - u + v] = \frac{1}{2}[3u - u] = \frac{1}{2}[2u] = u.$$

Therefore, by the use of the transformation T, the integral changes to

$$\iint_R (x-y)dy\,dx = \int_{-9}^{-5}\int_{-1}^{1} J(u,v)u\,du\,dv = \int_{-9}^{-5}\int_{-1}^{1}\left(\frac{1}{2}\right)u\,du\,dv,$$

which is much simpler to compute. ◀

Jacobians and Triple Integrals

For transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$, we define the Jacobian in a similar way

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}.$$

Example 33.

Evaluating a Triple Integral with a Change of Variables

Evaluate the triple integral

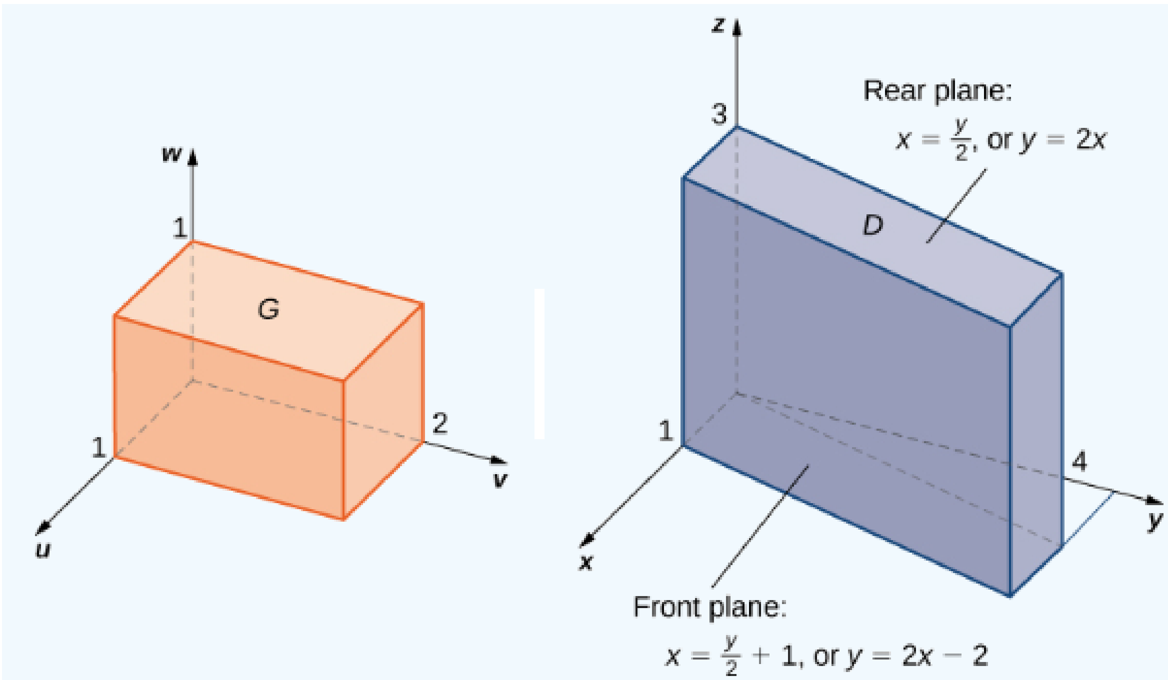
$$\int_0^3\int_0^4\int_{y/2}^{(y/2)+1}\left(x + \frac{z}{3}\right)dx\,dy\,dz$$

In xyz-space by using the transformation

$$u = (2x - y)/2,\ v = y/2, and w = z/3.$$

Then integrate over an appropriate region in uvw-space.

Solution. As before, some kind of sketch of the region G in xyz-space over which we have to perform the integration can help identify the region D in uvw-space (see the figure PageIndex13). Clearly G in xyz-space is bounded by the planes



$$x = y/2,\ x = (y/2) + 1,\ y = 0,\ y = 4,\ z = 0,\ \text{and } z = 4.$$

We also know that we have to use

$$u = (2x - y)/2,\ v = y/2,\ \text{and } w = z/3$$

for the transformations. We need to solve for x, y and z. Here we find that

$$x = u + v,\ y = 2v,\ \text{and } z = 3w.$$

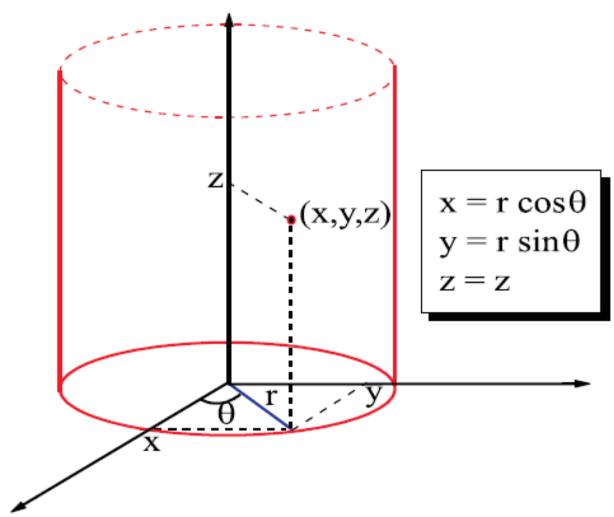
◀

Cylindrical Coordinates

When we were working with double integrals, we saw that it was often easier to convert to polar coordinates. For triple integrals we have been introduced to three coordinate systems. The rectangular coordinate system (x, y, z) is the system that we are used to. The other two systems are cylindrical coordinates (r, θ, z) and spherical coordinates (r, θ, ϕ).

Cylindrical coordinates are denoted by r, θ and z, and are defined by

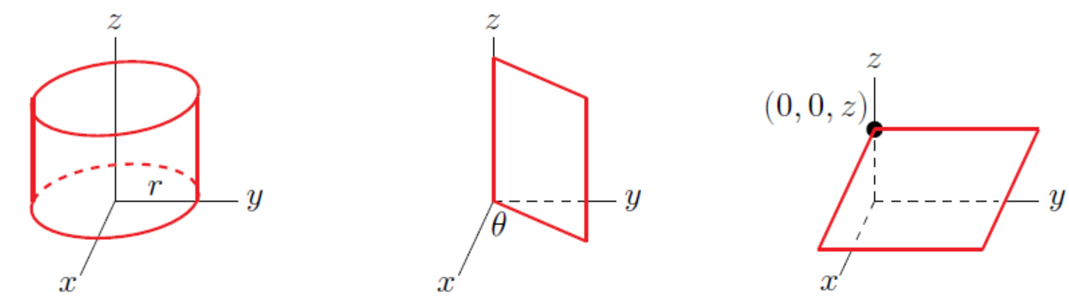
$$\begin{aligned} r = &\ \text{the distance from } (x, y, 0) \text{ to } (0, 0, 0) \\ = &\ \text{the distance from } (x, y, z) \text{ to the } z\text{-axis} \\ \theta = &\ \text{the angle between the positive } x\text{-axis and} \\ &\quad \text{the line joining } (x, y, 0) \text{ to } (0, 0, 0) \\ z = &\ \text{the signed distance from } (x, y, z) \text{ to the } xy\text{-plane} \end{aligned}$$



Here are sketches of surfaces of constant r, constant θ, and constant z.

The Cartesian and cylindrical coordinates are related by

Recall that cylindrical coordinates are most appropriate when



surface of constant r (a cylindrical shell) surface of constant θ (a plane) surface of constant z (a plane)

the expression

$$x^2 + y^2$$

occurs. The construction is just an extension of polar coordinates.

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

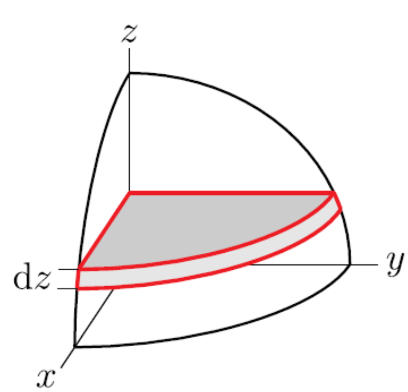
The Volume Element in Cylindrical Coordinates

We now establish that

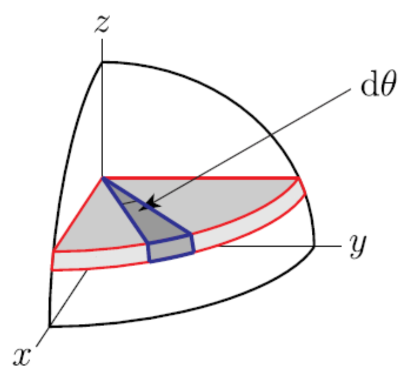
$$dV = r dr\, d\theta\, dz.$$

If we cut up a solid by

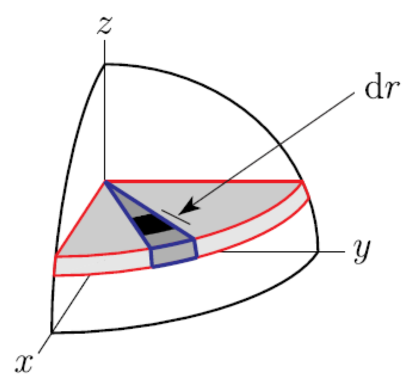
- first slicing it into horizontal plates of thickness dz by using planes of constant z,



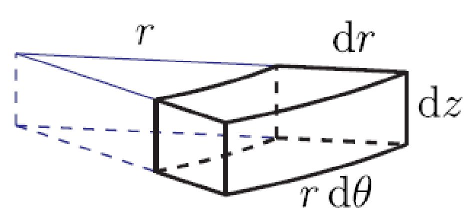
- and then subdividing the plates into wedges using surfaces of constant θ, say with the difference between successive θ's being $d\theta$,



- and then subdividing the wedges into approximate cubes using surfaces of constant r, say with the difference between successive r's being dr,



we end up with approximate cubes that look like



When we introduced slices using surfaces of

- constant r, the difference between the successive r's was dr, so the indicated edge of the cube has length dr.
- constant z, the difference between the successive z's was dz, so the vertical edges of the cube have length dz.
- constant θ the difference between the successive θ's was $d\theta$ so the remaining edges of the cube are circular arcs of radius essentially r that subtend an angle θ and so have length $rd\theta$.

Example 34.

Find the Jacobian for the cylindrical coordinate transformation

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z$$

Solution. We compute the Jacobian

$$\begin{aligned} \frac{\partial(x,y,z)}{\partial(r,\theta,z)} &= \begin{vmatrix} x_r & x_\theta & 0 \\ y_r & y_\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \\ &= r. \end{aligned}$$

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This example indicates that in the case of the spherical coordinate transformation we have

$$dxdydz = rdrd\theta dz.$$

This leads us to the following theorem:

Theorem (Integration With Cylindrical Coordinates): Let $f(x, y, z)$ be a continuous function on a solid B. Then

$$\iiint_B f(x, y, z)\ dzdydx$$

$$= \iiint_B f(r\cos\theta, r\sin\theta, z)\ rdzdrd\theta.$$

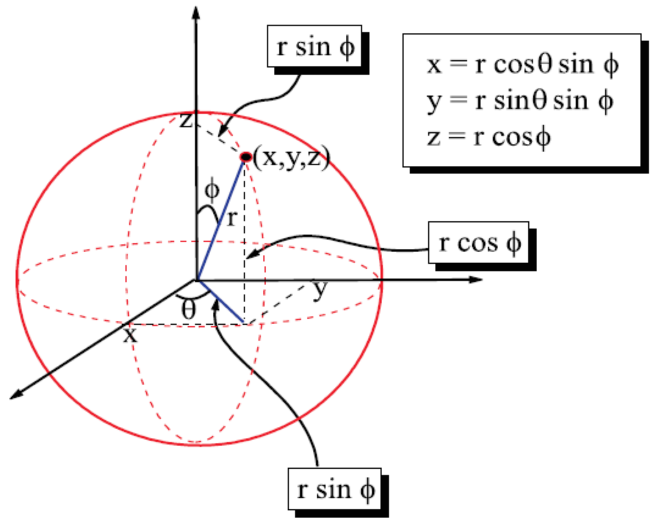
Triple Integrals in Spherical Coordinates

Another coordinate system that often comes into use is the spherical coordinate system.

Spherical coordinates are denoted by r, θ and ϕ, and are defined by

$$\begin{aligned} r = &\ \text{the distance from } (0,0,0) \text{ to } (x,y,z) \\ \theta = &\ \text{the angle between the positive } x\text{-axis and} \\ &\quad \text{the line joining } (x,y,0) \text{ to } (0,0,0) \\ \phi = &\ \text{the angle between the positive } z\text{-axis and} \\ &\quad \text{the line joining } (x,y,z) \text{ to } (0,0,0) \end{aligned}$$

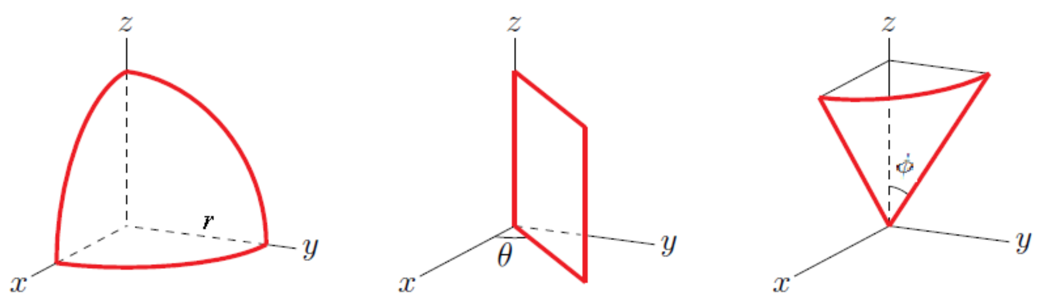
where $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$.



The spherical coordinate θ is the same as the cylindrical coordinate θ. The spherical coordinate ϕ is new. It runs from 0 (on the positive z-axis) to π (on the negative z-axis). The Cartesian and spherical coordinates are related by

$$x = r\cos\theta\sin\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\phi,$$
$$r^2 = x^2 + y^2 + z^2, \quad \theta = \arctan\tfrac{y}{x}, \quad \phi = \arctan\tfrac{\sqrt{x^2+y^2}}{z}.$$

Here are sketches of surfaces of constant r, constant θ, and constant ϕ.

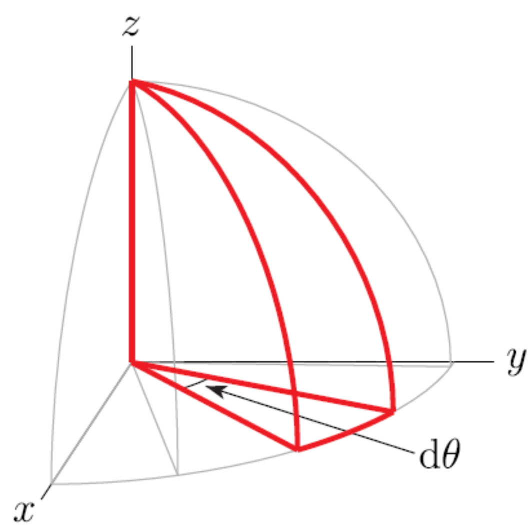


surface of constant r (a sphere) | surface of constant θ (a plane) | surface of constant ϕ (a cone)

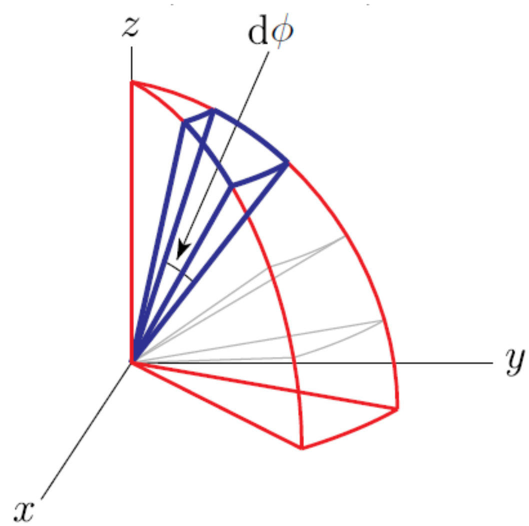
The Volume Element in Spherical Coordinates

If we cut up a solid by

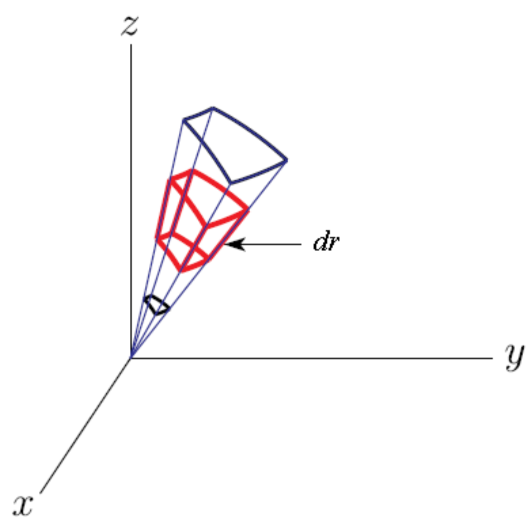
- first slicing it into segments (like segments of an orange) by using planes of constant θ, say with the difference between successive θ's being $d\theta$,



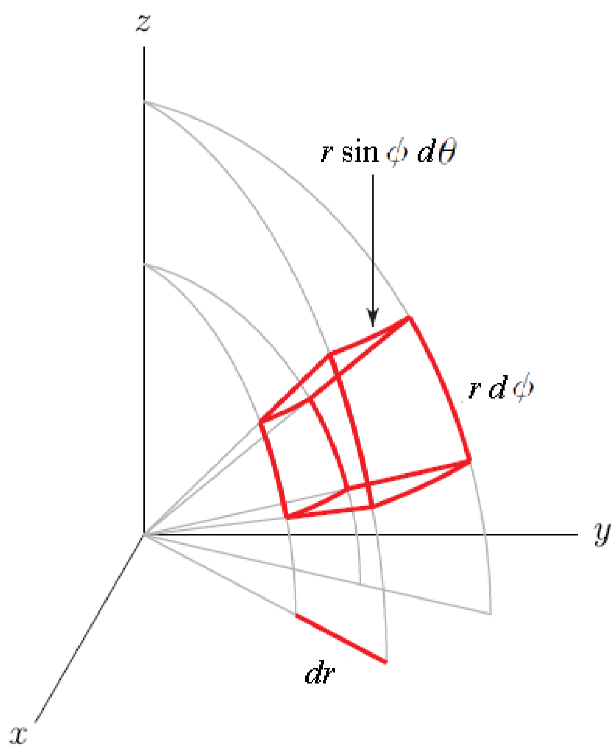
- and then subdividing the segments into "searchlights" (like the searchlight outlined in blue in the figure below) using surfaces of constant ϕ, say with the difference between successive ϕ's being$d\phi$,



- and then subdividing the searchlights into approximate cubes using surfaces of constant r, say with the difference between successive r's beingdr,



we end up with approximate cubes that look like the red one in the figure given above.



Example 35.

Find the Jacobian for the spherical coordinate transformation

$$x = r\cos\theta\sin\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\phi,$$

where $0 \le \theta \le 2$ and $0 \le \phi \le \pi$ and $r^2 = x^2 + y^2 + z^2$.

Solution. We compute the Jacobian

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \cos\theta\sin\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\phi & 0 & -r\sin\phi \end{vmatrix}$$
$$= r^2\sin\phi.$$

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