Poisson Inverse Weibull Distribution with Theory and Applications

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Abstract

In this article, we have introduced a new three-parameter Poisson inverse Weibull distribution using the Poisson family of distribution. The mathematical and statistical properties of the proposed distribution such as probability density function, cumulative distribution function, survival function, hazard rate function, quantile, the measure of skewness, and kurtosis are illustrated. The parameters of the new distribution are estimated using maximum likelihood estimation (MLE). By using the maximum likelihood method, we have constructed the asymptotic confidence interval and the Fisher information matrix for the model parameters. All the computations are performed in R software. A real data set is analyzed for illustration and it has found that the proposed distribution is more flexible as compared to some selected lifetime models.

Keywords: Poisson distribution, Inverse Weibull, Maximum likelihood estimation, Hazard function.

1. INTRODUCTION

In the probability distribution and applied statistics literature, we can found so many continuous univariate distributions. Most of the classical distributions have been used widely over the past decades for modeling data in several areas such as actuarial, environmental and medical sciences, life sciences, demography, economics, finance, insurance, etc. However, in many applied areas like survival analysis, finance, and insurance, there is a clear need for a modified form of more flexible distributions to model real data that can address a high degree of skewness and kurtosis.

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The Weibull distribution has been used extensively in survival analysis and applications of several different fields. For a detailed study, the learners can go through Lai et al. (2003) and Nadarajah (2009). Even though its widespread use, it has some drawbacks that is the limited shape of its hazard rate function (HRF) that can only be monotonically increasing or decreasing or constant. Usually, practical problems require a wider range of possibilities in the medium risk, for example, when the lifetime data produces a bathtub shaped hazard function such as human mortality and machine's component life cycles.

The inverse Weibull distribution has been used to model, many real-life applications for example degradation of mechanical components such as pistons, crankshafts of diesel engines, as well as the breakdown of insulating fluid (Khan et al., 2008, Pararai et al., 2014). Akgül et al. (2016) has introduced the inverse Weibull distribution for modeling the wind speed data. Kumar & Kumar (2019) has presented the estimation of the parameters and reliability characteristics in inverse Weibull distribution based on the random censoring model.

Hence the researchers in the last few years developed various extensions and modified forms of the Weibull distribution to obtain more flexible distributions. Some generalizations of the Weibull (W) distribution are available in the statistical literature such as the exponentiated W (Mudholkar et al. 1996), additive W (Xie and Lai 1995), Marshall–Olkin extended W (Ghitany et al. 2005), modified W (Lai et al. 2003, Sarhan and Zaindin 2009), beta-W (Lee et al. 2007), beta modified W (Silva et al. 2010), transmuted W (Aryal and Tsokos 2011), Kumaraswamy inverse W (Shahbaz et al. 2012), exponentiated generalized W (Cordeiro et al. 2013), beta inverse W (Hanook et al. 2013), transmuted complementary W geometric (Afify et al. 2014), Marshall–Olkin additive W (Afify et al. 2018), Kumaraswamy transmuted exponentiated additive W (Nofal et al. 2016), Topp-Leone generated W (Aryal et al. 2017) and Kumaraswamy complementary W geometric (Afify et al. 2017) distributions. Recently, Okasha et al. (2017) has introduced the extended inverse Weibull distribution with reliability application, Cordeiro et al. (2018) has introduced the Lindley Weibull distribution which accommodates unimodal and bathtub, and a broad variety of monotone failure rates, Basheer (2019) also introduced the alpha power inverse Weibull distribution with reliability application and Abd EL-Baset and Ghazal (2020) has presented the exponentiated additive Weibull distribution. Lindley inverse Weibull has introduced by (Joshi & Kumar, 2020).

There are plenty of lifetime distributions which are obtained by compounding with Zero truncated Poisson distribution and found more flexible than the previous one, some of them are as follows,

Kus (2007) has defined the two-parameter exponential Poisson (EP) distribution by compounding exponential distribution with zero truncated Poisson distribution with a decreasing failure rate. The CDF of PE distribution is,

$$F(t;\beta,\lambda) = \frac{1}{\left(1-e^{-\lambda}\right)}\left[1-e^{\left\{-\lambda\left(1-e^{-\beta t}\right)\right\}}\right] \quad ; t>0, (\beta,\lambda)>0 \qquad (1.1)$$

While Barreto-Souza and Cribari-Neto (2009) have introduced generalized EP distribution having the decreasing or increasing or upside-down bathtub shaped failure rate. This is the generalization of the distribution proposed by Kus (2007) adding a power parameter to this distribution. Following a similar approach, Percontini et al. (2013) have proposed the five-parameter beta Weibull Poisson distribution, which is obtained by compounding the Weibull Poisson and beta distributions. Following the same trend, Cancho (2011) has developed a new distribution family also based on the exponential distribution with an increasing failure rate function known as Poisson exponential (PE) distribution. The cumulative distribution function of PE distribution can be expressed as

$$F(y;\lambda,\theta) = 1 - \frac{1 - e^{\left\{-\theta\left(1-e^{-\lambda y}\right)\right\}}}{\left(1-e^{-\lambda}\right)} \quad ; y > 0, (\lambda,\theta) > 0 \tag{1.2}$$

A two-parameter Poisson-exponential with increasing failure rate has been defined by (Louzada-Neto et al., 2011) by using the same approach as used by (Cancho, 2011) under the Bayesian approach. Alkarni and Oraby (2012) have introduced a new lifetime family of distribution with a decreasing failure rate which is obtained by compounding truncated Poisson distribution and a lifetime model. The cumulative distribution function of the Poisson generating family is given by,

$$F_P(x;\lambda,\underline{\delta}) = 1 - \frac{1 - e^{\left\{-\lambda G(y,\underline{\delta})\right\}}}{\left(1-e^{-\lambda}\right)} \quad ; \lambda > 0 \tag{1.3}$$

Where $\underline{\delta}$ the parameter is space and $G(y,\underline{\delta})$ is the cumulative distribution function of any distribution. Using a parallel approach the Weibull power series class of distributions with Poisson has presented by (Morais & Barreto-Souza, 2011). Mahmoudi and Sepahdar (2013) have defined a new four-parameter distribution with increasing, decreasing, bathtub-shaped, and unimodal failure rate called as the exponentiated Weibull–Poisson (EWP) distribution which has obtained by compounding exponentiated Weibull (EW) and Poisson distributions. Similarly, Lu and Shi (2012) have created the new compounding distribution named the Weibull–Poisson distribution having the shape of decreasing, increasing, upside-down bathtub-shaped, or unimodal failure rate function. Furthur Kaviayarasu and Fawaz (2017) have made an extensive study on Weibull–Poisson distribution through a reliability sampling plan. Kyurkchiev et al. (2018) has used the exponentiated exponential-Poisson as the software reliability model. Louzada et al. (2020) has used different estimation methods to estimate the parameter of exponential-Poisson distribution using rainfall and aircraft data.

 In this study, we propose a new distribution based on the inverse Weibull distribution have developed by (Keller et al. 1982)) to study the shapes of the density and failure rate function. The different sections of this study are arranged as follows; in Section 2

we present the new distribution Poisson inverse Weibull (PIW) with its mathematical and statistical properties. We comprehensively discuss the maximum likelihood estimation method in Section 3. In Section 4 using a real dataset, we present the estimated values of the model parameters and their corresponding asymptotic confidence intervals and fisher information matrix. Also, we have illustrated the different test criteria to assess the goodness of fit of the proposed model. Some concluding remarks are presented in Section 5.

2. POISSON INVERSE WEIBULL (PIW) DISTRIBUTION

Alkarni and Oraby (2012) have introduced a new lifetime class with a decreasing failure rate which is obtained by compounding truncated Poisson distribution and a lifetime distribution, where the compounding procedure follows the same way that was previously carried out by (Adamidis & Loukas, 1998). Let $G(x)$ and $g(x)$ be the baseline cumulative distribution function and probability density function respectively then the CDF of Poisson family may be written as,

$$F\left(x;\lambda,\underline{\omega}\right)=\frac{1}{\left(1-e^{-\lambda}\right)}\left[1-\exp\left\{-\lambda G(x;\underline{\omega})\right\}\right] \quad ; x>0, \lambda>0 \tag{2.1}$$

And its corresponding PDF is

$$f\left(x;\lambda,\underline{\omega}\right)=\frac{1}{\left(1-e^{-\lambda}\right)}\lambda g(x;\underline{\omega})\exp\left\{-\lambda G(x;\underline{\omega})\right\} \quad ; x>0, \lambda>0 \tag{2.2}$$

here $\underline{\omega}$ is the parameter space of baseline distribution.

Keller et al. (1982) study the shapes of the density and failure rate functions for the basic inverse model. The inverse Weibull distribution with parameters α (scale parameter) and β (shape parameter) with cumulative distribution function (CDF) and the probability density function (PDF) of a random variable X are respectively given by

$$G\left(x;\alpha,\beta\right)=\exp\left(-\alpha x^{-\beta}\right); \ x\geq 0, \ \alpha>0, \ \beta>0 \tag{2.3}$$

and

$$g\left(x\right)=\alpha\beta x^{-(\beta+1)}\exp\left(-\alpha x^{-\beta}\right); \ x\geq 0, \ \alpha>0, \ \beta>0 \tag{2.4}$$

Substituting the equations (2.3) and (3.4) in equations (2.1) and (2.2) then we can define Poisson inverse Weibull distribution as,

Let X be a nonnegative random variable representing the survival time of an item or component or a system in some population. The random variable X is said to follow the PIW distribution with parameters $\left(\alpha,\beta,\lambda\right)>0$ if its cumulative distribution function is given by

$$F(x;\alpha,\beta,\lambda) = \frac{1}{\left(1-e^{-\lambda}\right)}\left[1-\exp\left\{-\lambda\exp\left(-(\alpha/x)^{\beta}\right)\right\}\right]; x > 0 \qquad (2.5)$$

And its corresponding probability density function is

$$f(x;\alpha,\beta,\lambda) = \frac{\alpha^{\beta}\beta\lambda}{\left(1-e^{-\lambda}\right)}x^{-(1+\beta)}\exp\left(-\lambda\exp\left(-(\alpha/x)^{\beta}\right)-(\alpha/x)^{\beta}\right); x > 0 \qquad (2.6)$$

The survival function of PIW distribution is

$$S(x) = 1 - F(x) = 1 - \frac{1}{\left(1-e^{-\lambda}\right)}\left[1-\exp\left\{-\lambda\exp\left(-(\alpha/x)^{\beta}\right)\right\}\right]; x > 0 \qquad (2.7)$$

And its hazard rate function can be expressed as

$$h(x) = \frac{f(x)}{S(x)} = \frac{\alpha^{\beta}\beta\lambda x^{-(1+\beta)}\exp\left(-\lambda\exp\left(-(\alpha/x)^{\beta}\right)-(\alpha/x)^{\beta}\right)}{\exp\left\{-\lambda\exp\left(-(\alpha/x)^{\beta}\right)\right\}-e^{-\lambda}}; x > 0 \qquad (2.8)$$

In Figure 1 we have displayed plots for the PDF and hazard function of PIW for several parameter values. Figure 1 shows that the PDF has various shapes while the hazard function has very flexible shapes, such as decreasing, increasing, constant and inverted bathtub.
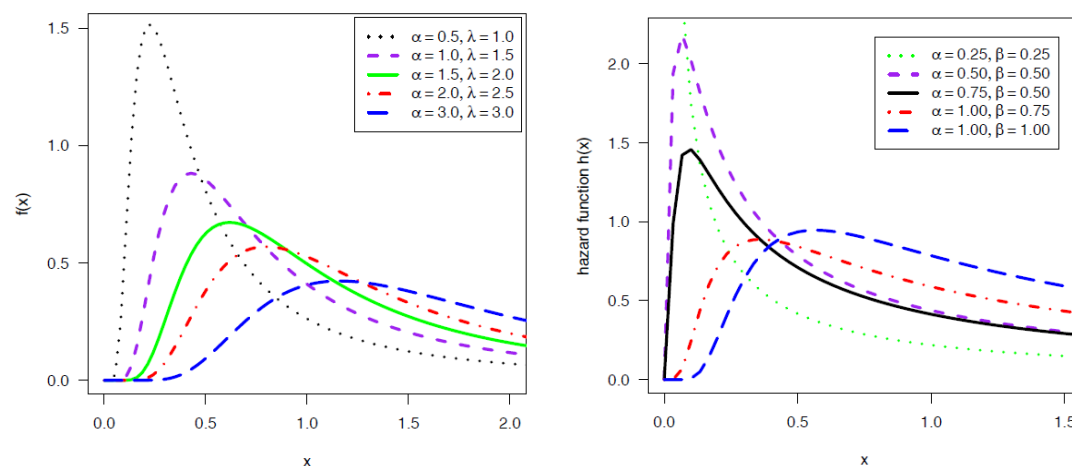

Figure 1. Graph of PDF (left panel) and hazard function (right panel) for different values of α, β, and λ.

Quantile function of PIW distribution

The quantile function of PIW (α, β, λ) can be obtained as

$$Q(u) = F^{-1}(u)$$

Hence the quantile function can be written as,

$$Q(p) = \alpha \left[-\ln \left[-\frac{1}{\lambda} \ln \left\{ 1 - \left(1 - e^{-\lambda} \right) p \right\} \right] \right]^{-1/\beta} ; 0 < p < 1 \tag{2.9}$$

Random numbers generation:

Random numbers can be generated for the PIW (α, β, λ) distribution, for this let, simulating values of random variable X with the CDF (2.5) and W denote a uniform random variable in (0, 1), then the simulated values of X are obtained by as,

$$X = \alpha \left[-\ln \left[-\frac{1}{\lambda} \ln \left\{ 1 - \left(1 - e^{-\lambda} \right) w \right\} \right] \right]^{-1/\beta} ; 0 < w < 1 \tag{2.10}$$

Skewness and Kurtosis:

Skewness and Kurtosis are mostly used in data analysis to study the nature of the distribution or data set. Skewness and Kurtosis based on quantile function are

$$Skewness(B) = \frac{Q_3 + Q_1 - 2Q_2}{Q_3 - Q_1},$$

where Q_1, Q_2 *and* Q_3 is the first quartile, median and third quartile respectively. The Coefficient of kurtosis based on octiles defined by (Moors, 1988) is

$$K_u(M) = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)}$$

3. MAXIMUM LIKELIHOOD ESTIMATION (MLE)

Let $X_1, X_2, ..., X_n$ be a sample of size 'n' independently and identically distributed random variables from the PIW distribution with unknown parameters, α β and λ defined previously.

The likelihood function of the PIW using the PDF in equation (2.4) is given by:

$$L(\alpha, \beta, \lambda \mid \underline{x}) = \frac{\alpha^\beta \beta \lambda}{\left(1 - e^{-\lambda} \right)} \prod_{i=1}^{n} x_i^{-(1+\beta)} \exp \left(-\lambda \exp \left(-(\alpha / x_i)^\beta \right) - (\alpha / x_i)^\beta \right)$$

It is easy to deals with natural logarithm, hence let $l(\alpha, \beta, \lambda)$ be log-likelihood function,

$$l(\alpha, \beta, \lambda) = n\beta \ln \alpha + n \ln(\beta) + n \ln \left(\frac{\lambda}{1 - e^{-\lambda}} \right)$$

$$-(\beta + 1) \sum_{i=1}^{n} \ln x_i - \lambda \sum_{i=1}^{n} e^{-(\alpha / x_i)^\beta} - \sum_{i=1}^{n} (\alpha / x_i)^\beta \tag{3.1}$$

To estimate the unknown parameters of the $PIW(\alpha, \beta, \lambda)$, we have to solve the following nonlinear equations equating to zero.

$$\frac{\partial l}{\partial \alpha} = \frac{n\beta}{\alpha} - \beta \sum_{i=1}^{n} \frac{1}{x_i} (\alpha/x_i)^{\beta-1} - \beta\lambda \sum_{i=1}^{n} \frac{1}{x_i} e^{-(\alpha/x_i)^{\beta}} (\alpha/x_i)^{\beta-1}$$

$$\frac{\partial l}{\partial \beta} = n\ln\alpha + \frac{n}{\beta} - \sum_{i=1}^{n} \ln x_i + \lambda \sum_{i=1}^{n} (\alpha/x_i)^{\beta} e^{-(\alpha/x_i)^{\beta}} \ln(\alpha/x_i) - \sum_{i=1}^{n} (\alpha/x_i)^{\beta} \ln(\alpha/x_i)$$

$$\frac{\partial l}{\partial \lambda} = \frac{n(e^{\lambda} - \lambda - 1)}{\lambda(e^{\lambda} - 1)} - \sum_{i=1}^{n} e^{-(\alpha/x_i)^{\beta}}$$

It is difficult to solve three equations simultaneously for α, β *and* λ, so by using the computer software R, Mathematica, Matlab or any other suitable software one can solve these equations. Let us denote the parameter vector by $\underline{\phi} = (\alpha, \beta, \lambda)$ and the corresponding MLE of $\underline{\phi}$ as $\underline{\hat{\phi}} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda})$, then the asymptotic normality results in,

$$(\underline{\hat{\phi}} - \underline{\phi}) \rightarrow N_3\left[0, \left(I(\underline{\phi})\right)^{-1}\right]$$ where $I(\underline{\phi})$ is the Fisher's information matrix given by,

$$I(\underline{\phi}) = -\begin{pmatrix} E\left(\frac{\partial^2 l}{\partial \alpha^2}\right) & E\left(\frac{\partial^2 l}{\partial \alpha \partial \beta}\right) & E\left(\frac{\partial^2 l}{\partial \alpha \partial \lambda}\right) \\ E\left(\frac{\partial^2 l}{\partial \alpha \partial \beta}\right) & E\left(\frac{\partial^2 l}{\partial \lambda^2}\right) & E\left(\frac{\partial^2 l}{\partial \lambda \partial \beta}\right) \\ E\left(\frac{\partial^2 l}{\partial \alpha \partial \lambda}\right) & E\left(\frac{\partial^2 l}{\partial \lambda \partial \beta}\right) & E\left(\frac{\partial^2 l}{\partial \lambda^2}\right) \end{pmatrix}$$

Further differentiating we get,

$$\frac{\partial^2 l}{\partial \alpha^2} = -\frac{n\beta}{\alpha^2} - \beta(\beta-1) \sum_{i=1}^{n} \frac{1}{x_i^2} (\alpha/x_i)^{\beta-2} - \beta(\beta-1) \frac{\lambda}{\alpha^2} \sum_{i=1}^{n} e^{-(\alpha/x_i)^{\beta}} (\alpha/x_i)^{\beta} \left\{\beta(\alpha/x_i)^{\beta} - \beta + 1\right\}$$

$$\frac{\partial^2 l}{\partial \beta^2} = -\frac{n}{\beta^2} - \lambda \sum_{i=1}^{n} (\alpha/x_i)^{\beta} e^{-(\alpha/x_i)^{\beta}} (\ln(\alpha/x_i))^2 \left\{(\alpha/x_i)^{\beta} - 1\right\} - \sum_{i=1}^{n} \frac{1}{x_i^2} (\alpha/x_i)^{\beta} (\ln(\alpha/x_i))^2$$

$$\frac{\partial^2 l}{\partial \lambda^2} = -\frac{ne^{-\lambda}\{e^{2\lambda}(\lambda^2 + \lambda + 1) - 2(1+\lambda)e^{\lambda} + \lambda + 1\}}{\lambda^2(e^{\lambda} - 1)^2}$$

$$\frac{\partial^2 l}{\partial\alpha\partial\beta} = \frac{n}{\alpha} + \frac{\lambda}{\alpha} \sum_{i=1}^{n} (\alpha/x_i)^\beta \, e^{-(\alpha/x_i)^\beta} \, [\beta\{(\alpha/x_i)^\beta - 1\}\ln(\alpha/x_i) - 1]$$

$$- \frac{1}{\alpha} \sum_{i=1}^{n} (\alpha/x_i)^\beta [\beta\ln(\alpha/x_i) + 1]$$

$$\frac{\partial^2 l}{\partial\alpha\partial\lambda} = \beta \sum_{i=1}^{n} x_i^{-\beta} (\alpha/x_i)^{\beta-1} e^{-(\alpha/x_i)^\beta}$$

$$\frac{\partial^2 l}{\partial\beta\partial\lambda} = \sum_{i=1}^{n} (\alpha/x_i)^\beta \, e^{-(\alpha/x_i)^\beta} \ln(\alpha/x_i)$$

In practice, it is useless that the MLE has asymptotic variance $\left(I(\underline{\phi})\right)^{-1}$ because we don't know $\underline{\phi}$. Hence we approximate the asymptotic variance by plugging in the estimated value of the parameters. The common procedure is to use the observed Fisher information matrix $\Omega(\hat{\underline{\phi}})$ as an estimate of the information matrix $I(\underline{\phi})$ given by

$$\Omega(\hat{\underline{\phi}}) = - \begin{pmatrix} \dfrac{\partial^2 l}{\partial\alpha^2} & \dfrac{\partial^2 l}{\partial\alpha\partial\beta} & \dfrac{\partial^2 l}{\partial\alpha\partial\lambda} \\[2ex] \dfrac{\partial^2 l}{\partial\alpha\partial\beta} & \dfrac{\partial^2 l}{\partial\beta^2} & \dfrac{\partial^2 l}{\partial\lambda\partial\beta} \\[2ex] \dfrac{\partial^2 l}{\partial\alpha\partial\lambda} & \dfrac{\partial^2 l}{\partial\lambda\partial\beta} & \dfrac{\partial^2 l}{\partial\lambda^2} \end{pmatrix}_{(\hat{\alpha},\hat{\beta},\hat{\lambda})} = -H(\underline{\phi})\Big|_{\left(\underline{\phi}=\hat{\underline{\phi}}\right)}$$

where H is the Hessian matrix.

The Newton-Raphson algorithm to maximize the likelihood produces the observed information matrix. Therefore, the variance-covariance matrix is given by,

$$\left[-H(\phi)\big|_{(\phi=\hat{\phi})}\right]^{-1} = \begin{pmatrix} \text{var}(\hat{\alpha}) & \text{cov}(\hat{\alpha},\hat{\beta}) & \text{cov}(\hat{\alpha},\hat{\lambda}) \\ \text{cov}(\hat{\alpha},\hat{\beta}) & \text{var}(\hat{\beta}) & \text{cov}(\hat{\beta},\hat{\lambda}) \\ \text{cov}(\hat{\alpha},\hat{\lambda}) & \text{cov}(\hat{\beta},\hat{\lambda}) & \text{var}(\hat{\lambda}) \end{pmatrix}$$

Hence from the asymptotic normality of MLEs, approximate 100(1-α) % confidence intervals for α, β and λ can be constructed as,

$\hat{\alpha} \pm Z_{\alpha/2}\sqrt{\text{var}(\hat{\alpha})}$, $\hat{\beta} \pm Z_{\alpha/2}\sqrt{\text{var}(\hat{\beta})}$ and $\hat{\lambda} \pm Z_{\alpha/2}\sqrt{\text{var}(\hat{\lambda})}$ where $Z_{\alpha/2}$ is the upper percentile of standard normal variate.

4. ILLUSTRATION WITH A REAL DATA ANALYSIS

For the data analysis, we are using a real data set that was used by Bader and Priest (1982). The data given represent the strength measured in GPA for single carbon fibers of 10mm in gauge lengths with sample size 63 and they are as follows:

1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474,

2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738,

2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145,

3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435,

3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027,

4.225, 4.395, 5.020

We have calculated the maximum likelihood estimates of PIW distribution by using the log-likelihood function (3.1), directly by using R software (R Core Team, 2020) and (Dalgaard, 2008). From the above data set, we have calculated $\hat{\alpha} = 5.5146$, $\hat{\beta} = 1.8811$ and $\hat{\lambda} = 16.2341$ corresponding Log-Likelihood value is -56.54611. In Table 1 we have demonstrated the MLE's with their standard errors (SE) and 95% confidence interval for parameters α, β and λ.

Table 1

MLE, SE and 95% confidence interval

Parameter	MLE	SE	95% ACI
alpha	5.5146	0.6044	(4.3300, 6.6992)
beta	1.8811	0.2276	(1.4350, 2.3272)
lambda	16.2341	3.8506	(8.6869, 23.7813)

An estimate of the variance-covariance matrix by using MLEs, using equation (3.2) is

$$\left[-H(\phi)_{(\phi=\hat{\phi})} \right]^{-1} = \begin{pmatrix} 0.3653002 & -0.12803594 & 1.9077586 \\ -0.1280359 & 0.05179758 & -0.5663192 \\ 1.9077586 & -0.56631925 & 14.8272317 \end{pmatrix}$$

The plots of profile log-likelihood function for the parameters α, β and λ have been displayed in Figure 2 and noticed that the ML estimates can be uniquely estimated.
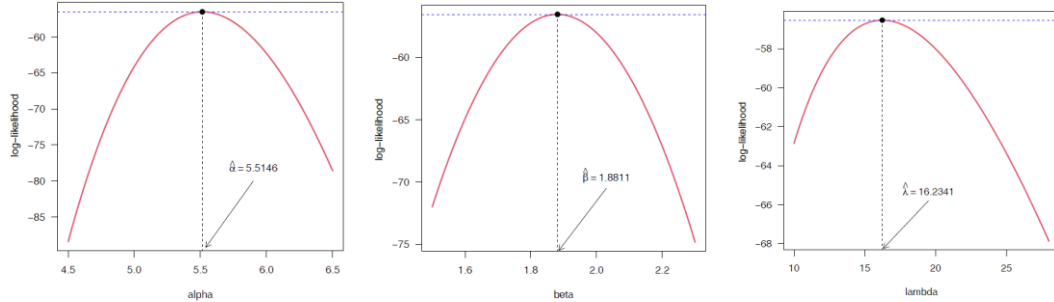

Figure 2. Graph of Profile log-likelihood function for the parameters α, β and λ.

One way to assess how well a particular theoretical model describes a data distribution is to plot the data quantiles against theoretical quantiles. In Fig. 3 we have plotted the P-P and Q-Q plot and verified that the new proposed model fits the data very well.
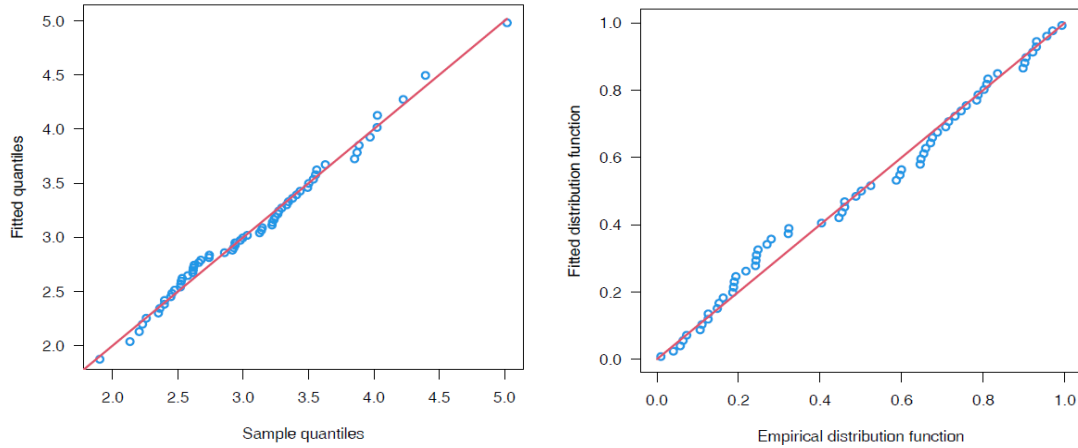

Figure 3. The graph of the P-P plot (left panel) and Q-Q plot (right panel) of the PIW distribution

For the illustration purpose, we have fitted the following probability distributions

I. Exponentiated Exponential Poisson (EEP):

The probability density function of EEP (Ristić & Nadarajah, 2014) can be expressed as

$$f(x) = \frac{\alpha\beta\lambda}{\left(1 - e^{-\lambda}\right)} e^{-\beta x} \left(1 - e^{-\beta x}\right)^{\alpha-1} \exp\left\{-\lambda\left(1 - e^{-\beta x}\right)^{\alpha}\right\} \quad ; x > 0, \alpha > 0, \lambda > 0$$

II. Generalized Exponential (GE) distribution:

The probability density function

$$f_{GE}(x;\alpha,\lambda) = \alpha\,\lambda\,e^{-\lambda\,x}\left\{1-e^{-\lambda\,x}\right\}^{\alpha-1} ; (\alpha,\lambda) > 0, x > 0.$$

III. Weibull Extension (WE) Distribution:

The probability density function of Weibull extension (WE) distribution (Tang et al., 2003) with three parameters (α, β, λ) is

$$f_{WE}(x;\alpha,\beta,\lambda) = \lambda\beta\left(\frac{x}{\alpha}\right)^{\beta-1}\exp\left(\frac{x}{\alpha}\right)^{\beta}\exp\left\{-\lambda\alpha\left(\exp\left(\frac{x}{\alpha}\right)^{\beta}-1\right)\right\} ; x > 0$$

$\alpha > 0, \beta > 0$ and $\lambda > 0$

IV. Exponential power (EP) distribution:

The probability density function Exponential power (EP) distribution (Smith & Bain, 1975) is

$$f_{EP}(x) = \alpha\,\lambda^{\alpha}\,x^{\alpha-1}\,e^{(\lambda\,x)^{\alpha}}\exp\left\{1-e^{(\lambda\,x)^{\alpha}}\right\} ; (\alpha,\lambda) > 0, \quad x \geq 0.$$

where α and λ are the shape and scale parameters, respectively.

For the test of goodness of fit and adequacy of the proposed model, Akaike information criterion (AIC), Bayesian information criterion (BIC), Corrected Akaike information criterion (CAIC) and Hannan-Quinn information criterion (HQIC) are calculated and presented in Table 3.

Table 3

Log-likelihood (LL), AIC, BIC, CAIC and HQIC

Model	-LL	AIC	BIC	CAIC	HQIC
PIW	56.5461	119.0922	125.5216	119.4990	121.6209
EEP	57.0630	120.1261	126.5555	120.5328	122.6548
GE	57.1387	118.2775	122.5638	118.4775	119.9633
WE	61.9865	129.9731	136.4025	130.3798	132.5018
EP	69.3299	142.6598	146.9461	142.8533	144.3456

We have displayed the histogram and the fitted probability density functions and the empirical cumulative distribution function with the estimated distribution function in Figure 4. For the given data set we have found that the proposed distribution provides a better fit and more reliable results than the selected ones.
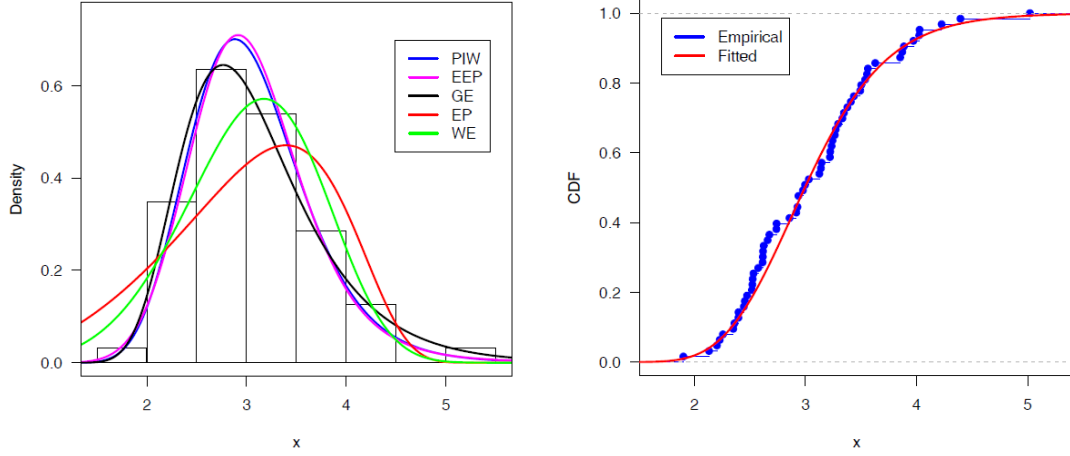

Figure 4. The Histogram and the PDF of fitted distributions (left panel) and Empirical CDF with estimated CDF (right panel).

We have reported the test statistics and their corresponding p-value of the PIW distribution and competing models in Table 4. The result shows that the PIW distribution has the minimum value of the test statistic and higher *p*-value hence we conclude that the PIW distribution gets quite better fit and more consistent and reliable results from others taken for comparison.

Table 4

The Goodness-of-Fit Statistics and Their Corresponding p-Value

Model	KS(p-value)	A^2(p-value)	W(p-value)
PIW	0.0856(0.7456)	0.0647(0.7862)	0.3534(0.8929)
EEP	0.0907(0.6784)	0.0714(0.7451)	0.4002(0.8480)
GE	0.0645(0.9559)	0.0538(0.8548)	0.3944(0.8538)
WE	0.0879(0.7148)	0.1250(0.4771)	0.9381(0.3911)
EP	0.1443(0.1452)	0.3504(0.0978)	2.3516(0.0595)

5   CONCLUSION

We have developed the Poisson inverse Weibull (PIW) distribution generated by a new class of Poisson generated distributions. We have discussed the important properties of the PIW distribution like survival function, hazard rate function, quantile

function, and skewness and kurtosis. The shape of the probability density function of the PIW distribution is unimodel and positively skewed, while the hazard function of the PIW distribution is increasing, decreasing, decreasing–increasing, increasing–decreasing and bathtub shaped. The unknown parameters of the proposed distribution are estimated by using the maximum likelihood method and we have constructed the asymptotic confidence interval and fisher information matrix. The importance of the proposed distribution is illustrated by using a real dataset and found that it provides a better fitting in comparison with other lifetime distributions.

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