

1. Introduction

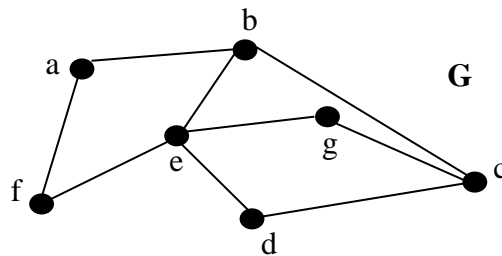
A **graph** consists of set of **nodes** (or **vertices**) and a set of **arcs** (or **edges**) that connect these vertices. A *graph* $G = (V, E)$ consists of V , a nonempty set of *vertices* and E , a set of *edges*. Each edge has either one or two vertices associated with it, called its *endpoints*. An edge is said to *connect* its endpoints.

2. Graph Types

2.1. Simple Graph

A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices is called a simple graph.

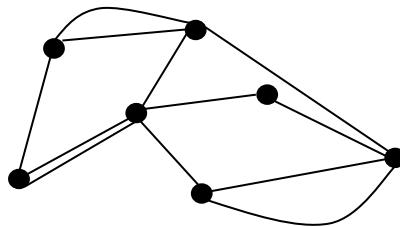
A simple graph $G = (V, E)$ consists of V , a nonempty set of vertices, and E , a set of unordered pairs of distinct elements of V called edges. This kind of graph has undirected edges, no loops and no multiple edges. The figure below is an example of simple graph.



In the above graph $G = (V, E)$, the set of vertices, $V(G)$ or $V = \{a, b, c, d, e, f, g\}$ and the set of edges $E(G)$ or $E = \{\{a, b\}, \{a, f\}, \{b, c\}, \{b, e\}, \{c, d\}, \{c, g\}, \{d, e\}, \{e, g\}, \{e, f\}\}$.

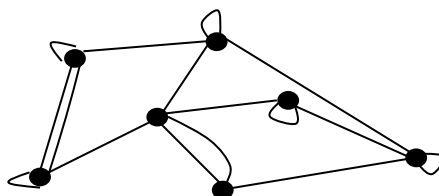
2.2. Multigraph

Graphs that may have **multiple edges** connecting the same vertices are called **multigraphs**. When there are m different edges associated to the same unordered pair of vertices $\{u, v\}$, we also say that $\{u, v\}$ is an edge of multiplicity m . That is, we can think of this set of edges as m different copies of an edge $\{u, v\}$. This kind of graph has undirected edges, and no loops. The figure below is an example of a multigraph.



2.3. Pseudograph

Some graphs have edges that connect a vertex to itself. Such edges are called **loops**, and sometimes we may even have more than one loop at a vertex. Graphs that may include loops, and possibly multiple edges connecting the same pair of vertices or a vertex to itself, are sometimes called **pseudographs**.



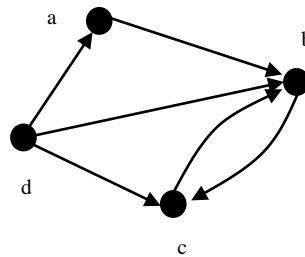
2.4. Directed Graph

The graphs have introduced so far are **undirected graphs**. Their edges are also said to be **undirected**. However, to construct a graph model, we may find it necessary to assign directions to the edges of a graph.

A *directed graph* (or *digraph*) (V, E) consists of a nonempty set of vertices V and a set of *directed edges* (or *arcs*) E . Each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u, v) is said to *start* at u and *end* at v .

A directed graph may contain loops and it may contain multiple directed edges that start and end at the same vertices. A directed graph may also contain directed edges that connect vertices u and v in both directions; that is, when a digraph contains an edge from u to v , it may also contain one or more edges from v to u .

When a directed graph has no loops and has no multiple directed edges, it is called a **simple directed graph**. The figure below is an example of simple directed graph.



In the above graph $G = (V, E)$, the set of vertices, $V(G)$ or $V = \{a, b, c, d\}$ and the set of edges $E(G)$ or $E = \{(a, b), (b, c), (c, b), (d, a), (d, b), (d, c)\}$.

Directed graphs that may have **multiple directed edges** and **loops** are called **directed multigraphs**. When there are m directed edges, each associated to an ordered pair of vertices (u, v) , we say that (u, v) is an edge of **multiplicity** m .

2.5. Mixed Graph

For some models we may need a graph where some edges are undirected, while others are directed. A graph with both directed and undirected edges is called a **mixed graph**.

Type	Edge	Multiple Edges	Loops
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes
Mixed graph	Directed and Undirected	Yes	Yes

3. Graph Terminologies

Two vertices u, v in an undirected graph G are called **adjacent** or **neighbors** in G if $\{u, v\}$ is an edge. The edge e is called **incident with** the vertices u and v if $e = \{u, v\}$. This edge is also said to **connect** u and v , where u and v are end points of the edge.

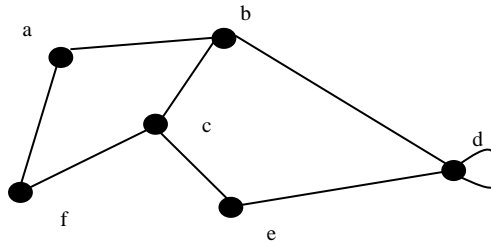
The set of all neighbors of a vertex v of $G = (V, E)$, denoted by $N(v)$, is called the *neighbourhood* of v . If A is a subset of V , we denote by $N(A)$ the set of all vertices in G that are adjacent to at least one vertex in A . So, $N(A) = \cup_{v \in A} N(v)$.

The **degree of a vertex** in an undirected graph is the number of edges incident with it, except a loop at a vertex. Loop in a vertex counts twice to the degree. Degree of a

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vertex v is denoted by $\deg(v)$. A vertex of degree zero is called **isolated** vertex and a vertex with degree one is called **pendant** vertex. A pendant vertex is adjacent to exactly one other vertex.

Example: What are the degree of the vertices in the following graph? What is $N(a)$ and $N(a, b)$?

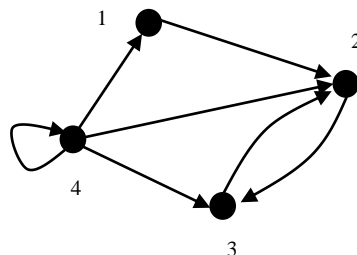


Solution: $\deg(a) = \deg(f) = \deg(e) = 2$; $\deg(b) = \deg(c) = 3$; $\deg(d) = 4$; $N(a) = \{b, f\}$; $N(a, b) = \{b, f, a, c, d\}$.

Let (u, v) be an edge representing edge of a directed graph G , u is called **adjacent to** v and v is called **adjacent from** u . The vertex u is called **initial vertex** and the vertex v is called **terminal or end vertex**. Loop has same initial and terminal vertex.

In directed graph the **in-degree** of a vertex v , denoted by $\deg^-(v)$, is the number of edges that have v as their terminal vertex. The **out-degree** of a vertex v , denoted by $\deg^+(v)$, is the number of edges that have v as their initial vertex. Loop at a vertex adds up both in-degree and out-degree to one more than calculated in-degree and out-degree.

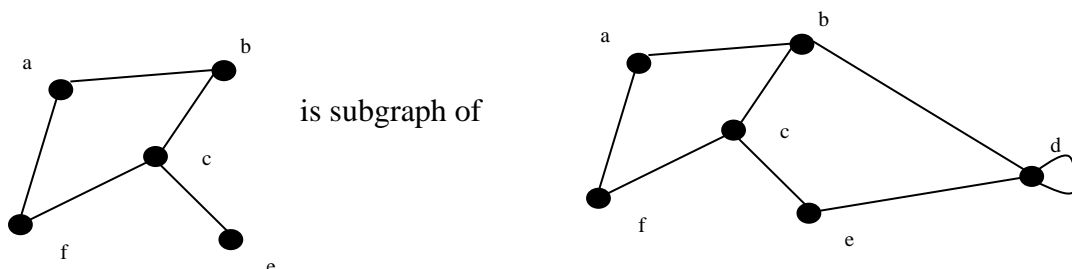
Example: Find the in-degree and out-degree of each vertex in the following graph.



Solution: In-degrees of a graph are $\deg^-(1) = \deg^-(4) = 1$; $\deg^-(2) = 3$; $\deg^-(3) = 2$ and the out-degrees of a graph are $\deg^+(1) = \deg^+(2) = \deg^+(3) = 1$; $\deg^+(4) = 4$.

A **subgraph** of a graph $G = (V, E)$ is a graph $H = (W, F)$ where $W \subseteq V$ and $F \subseteq E$.

Example:

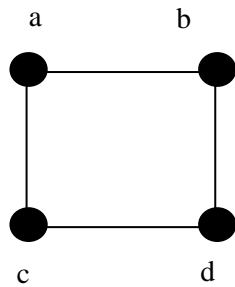


4. Graph Representation

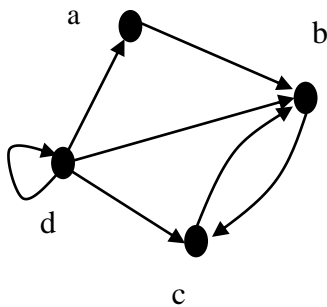
There are many useful ways to represent graphs. As explained earlier, one way to represent a graph without multiple edges is to list all the edges of this graph. Other techniques are **adjacency list**, **adjacency matrix**, and **incidence matrix**.

4.1. Adjacency List

This type of representation is suitable for the undirected graphs without multiple edges, and directed graphs. This representation looks as in the tables below.



Edge List for Simple Graph	
Vertex	Adjacent Vertices
a	b, c
b	a, d
c	a, d
d	b, c



Edge List for Directed Graph	
Initial Vertex	End Vertices
a	b
b	c
c	b
D	a, b, c, d

Carrying out graph algorithms using the representation of graphs by lists of edges, or by adjacency lists, can be cumbersome if there are many edges in the graph. To simplify computation, graphs can be represented using matrices. Two types of matrices commonly used to represent graphs will be presented here. One is based on the adjacency of vertices, and the other is based on incidence of vertices and edges.

4.2. Adjacency Matrix

Given a simple graph $G = (V, E)$ with $|V| = n$. Assume that the vertices of the graph are listed in some arbitrary order like v_1, v_2, \dots, v_n . The adjacency matrix A of G (A_G), with respect to the order of the vertices is n -by- n zero-one matrix ($A = [a_{ij}]$) with the condition,

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of } G, \\ 0 & \text{otherwise.} \end{cases}$$

Since there are n vertices and we may order vertices in any order there are $n!$ possible order of the vertices. The adjacency matrix depends on the order of the vertices, hence there are $n!$ possible adjacency matrices for a graph with n vertices.

Adjacency matrix for undirected graph is *symmetric*, in case of the pseudograph or multigraph the representation is similar but the matrix here is not zero-one matrix rather the $(i, j)^{\text{th}}$ entry of the matrix contains the number of edges appearing between that pair of vertices.

In case of the directed graph we can extend the same concept as in undirected graph as dictated by the relation

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \text{ is an edge of } G, \\ 0 & \text{otherwise.} \end{cases}$$

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The main difference is that the matrix may not be *symmetric*. Adjacency matrices can also be used to represent directed multigraphs. Again, such matrices are not zero–one matrices when there are multiple edges in the same direction connecting two vertices. In the adjacency matrix for a directed multigraph, a_{ij} equals the number of edges that are associated to (v_i, v_j) .

If the number of edges is few then the adjacency matrix becomes *sparse*. Sometimes it will be beneficial to represent graph with adjacency list in such a condition.

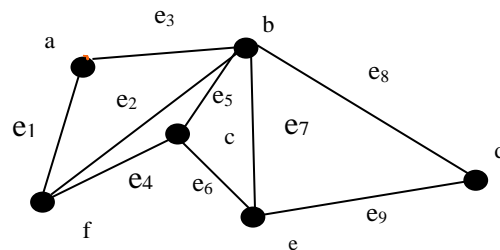
4.3. Incidence Matrix

This is another way of representing graph. Given an undirected graph $G = (V, E)$. Assume that the vertices of the graph are v_1, v_2, \dots, v_n and the edges of the graph are e_1, e_2, \dots, e_m . The incidence matrix of a graph with respect to the above ordering of V and E is n -by- m matrix $M = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incident with } v_i \\ 0 & \text{otherwise.} \end{cases}$$

When the graph is not simple then also the graph can be represented by using incidence matrix where multiple edges corresponds to two different columns with exactly same entries. Loops are represented with column with only one entry.

Example 1: Represent the following graph using adjacency matrix and incidence matrix.



Solution: Let the order of the vertices be a, b, c, d, e, f and edges order be $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9$.

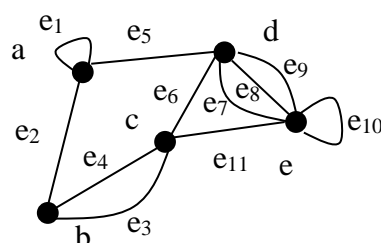
$$\begin{matrix} & a & b & c & d & e & f \\ \begin{matrix} a \\ b \\ c \\ d \\ e \\ f \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

Adjacency Matrix

$$\begin{matrix} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 \\ \begin{matrix} a \\ b \\ c \\ d \\ e \\ f \end{matrix} & \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

Incidence Matrix

Example 2: Represent the following graph using adjacency matrix and incidence matrix.



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Solution: Let the order of the vertices be a, b, c, d, e and edges order be $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}$.

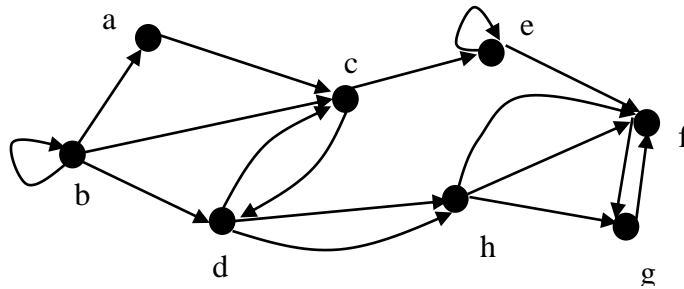
$$\begin{array}{c}
 a \\
 b \\
 c \\
 d \\
 e
 \end{array}
 \begin{array}{ccccc}
 a & b & c & d & e \\
 \begin{bmatrix}
 1 & 1 & 0 & 1 & 0 \\
 1 & 0 & 2 & 0 & 0 \\
 0 & 2 & 0 & 1 & 1 \\
 1 & 0 & 1 & 0 & 3 \\
 0 & 0 & 1 & 3 & 1
 \end{bmatrix}
 \end{array}$$

Adjacency Matrix

$$\begin{array}{c}
 a \\
 b \\
 c \\
 d \\
 e
 \end{array}
 \begin{array}{cccccccccccc}
 e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} & e_{11} \\
 \begin{bmatrix}
 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1
 \end{bmatrix}
 \end{array}$$

Incidence Matrix

Example 3: Represent the following directed graph using adjacency matrix.



Solution: Let the order of the vertices be a, b, c, d, e, f, g and h. The adjacency matrix is

$$\begin{array}{c}
 a \\
 b \\
 c \\
 d \\
 e \\
 f \\
 g \\
 h
 \end{array}
 \begin{array}{cccccccc}
 a & b & c & d & e & f & g & h \\
 \begin{bmatrix}
 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 \\
 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0
 \end{bmatrix}
 \end{array}$$

5. Shortest Path Algorithm

Many problems can be modeled using graphs with weights assigned to their edges. Graphs that have a number assigned to each edge are called **weighted graphs**. Several types of problems involving weighted graphs arise frequently. Determining a path of least length (**i.e. shortest path**) between two vertices in a graph is one such problem.

There are several different shortest path algorithms to find shortest path between two vertices in a graph. Here, we will present a **greedy algorithm** discovered by the Dutch mathematician Edsger Dijkstra in 1959. The version we will describe solves this problem in undirected weighted graphs where all the weights are positive. It is easy to adapt it to solve shortest-path problems in directed graphs.

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Dijkstra's Algorithm

procedure *Dijkstra*(G : weighted connected simple graph, with all weights positive)
 $\{G$ has vertices $a = v_0, v_1, \dots, v_n = z$ and lengths $w(v_i, v_j)$ where $w(v_i, v_j) = \infty$ if $\{v_i, v_j\}$ is not an edge in $G\}$

for $i := 1$ **to** n

$L(v_i) := \infty$

$L(a) := 0$

$S := \emptyset$

$\{$ the labels are now initialized so that the label of a is 0 and all other labels are ∞ , and S is the empty set $\}$

while $z \notin S$

$u :=$ a vertex not in S with $L(u)$ minimal

$S := S \cup \{u\}$

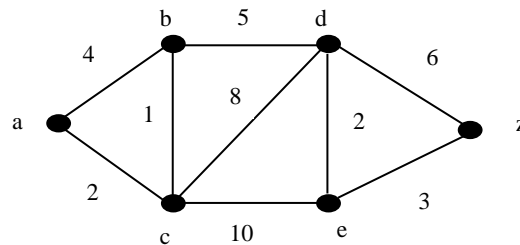
for all vertices v not in S

if $L(u) + w(u, v) < L(v)$ **then** $L(v) := L(u) + w(u, v)$

$\{$ this adds a vertex to S with minimal label and updates the labels of vertices not in $S\}$

return $L(z)$ $\{L(z) =$ length of a shortest path from a to $z\}$

Example: Use Dijkstra's algorithm to find the length of shortest path between the vertices a and z in the weighted graph given below.



Solution:

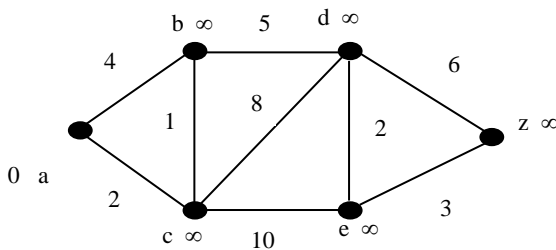


Fig (a)

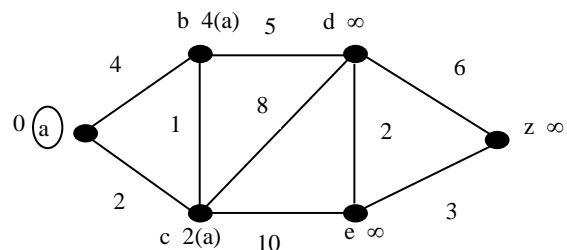


Fig (b)

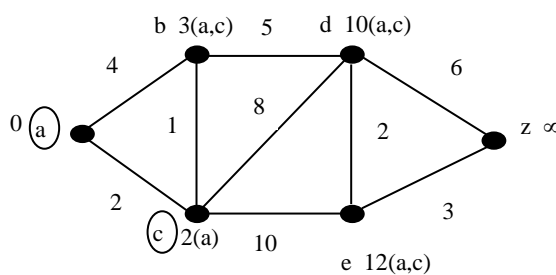


Fig (c)

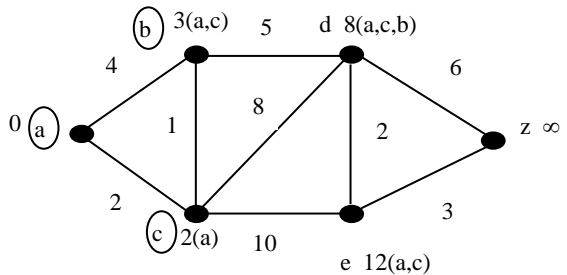


Fig (d)

Unit 8.2: Graphs

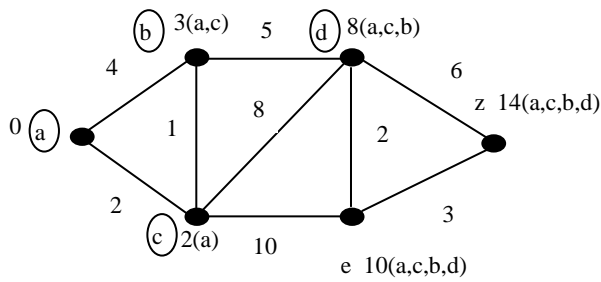
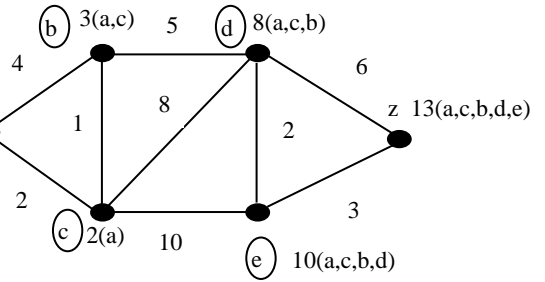


Fig (e)



Fig(f)

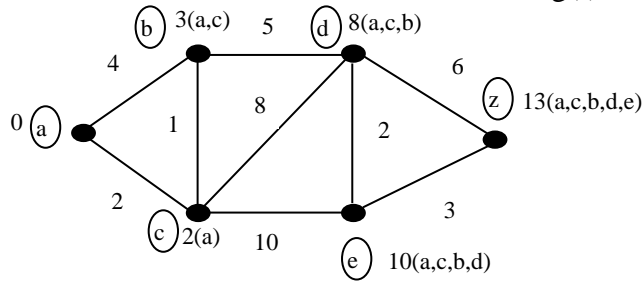
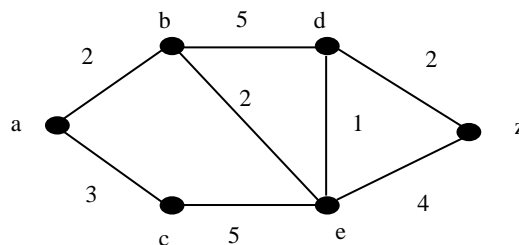


Fig (g)

Hence, the length of the shortest path is 13 and the shortest path is a, c, b, d, e, z.

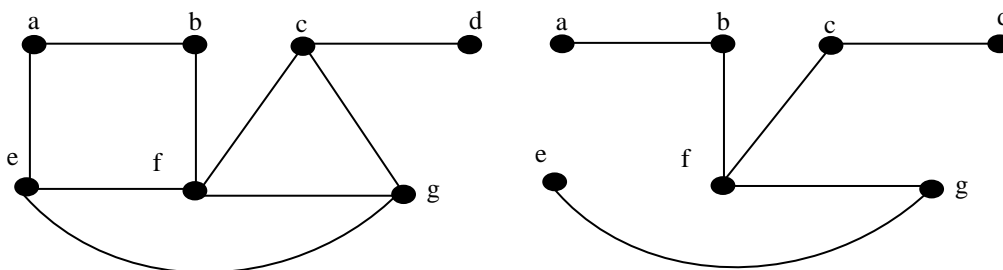
Exercise: Find the length of the shortest path between a and z in the given graph.



6. Spanning Trees and Minimum Spanning Trees

Let G be a simple connected graph. A spanning tree of G is a subgraph of G that is a tree containing every vertex of G .

Example: Find a spanning tree of the simple graph G on the left side below.



Solution: The left graph is connected but is not tree because it contains simple circuits. By removing the edges $\{a, e\}$, $\{e, f\}$ and $\{c, g\}$ from the left graph, we obtain a simple graph with no simple circuits on the right side. This subgraph is a spanning tree.

This tree is not the only spanning tree of the graph. The graph can also have other spanning trees.

A **minimum spanning tree** in a connected weighted graph is a spanning tree that has the smallest possible sum of weights of its edges.

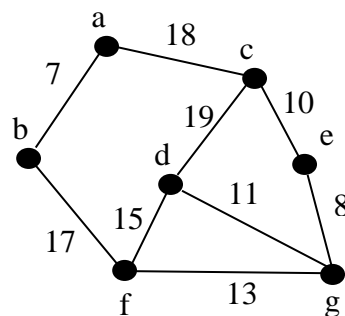
6.1. Prim's Algorithm

Prim's Algorithm is a greedy algorithm that is used to find the minimum spanning tree from a graph. Prim's algorithm finds the subset of edges that includes every vertex of the graph such that the sum of the weights of the edges can be minimized. Prim's algorithm starts with a randomly chosen the single vertex and explores all the adjacent nodes with all the connecting edges at every step. The edges with the minimal weights causing no cycles in the graph got selected.

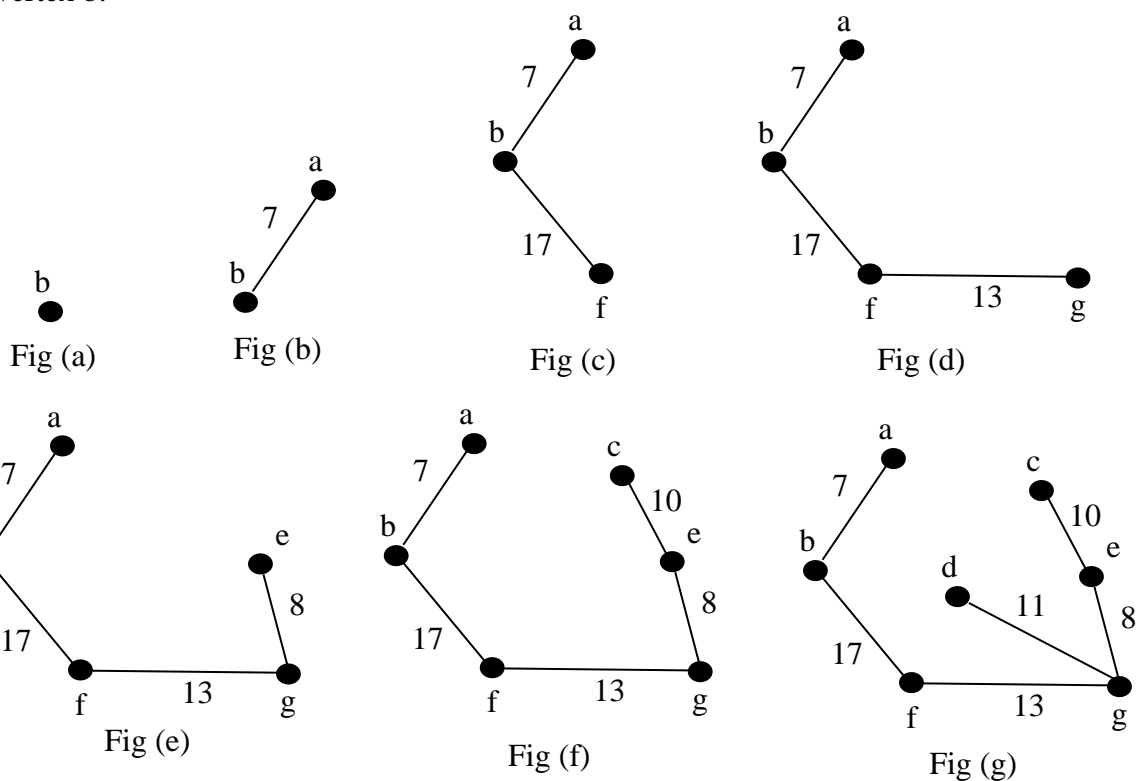
Algorithm:

1. Initialize minimum spanning tree with the randomly chosen vertex.
2. Find all the edges that connect the tree in the above step with the new vertices. From the edges found, select the minimum edge and add it to the tree.
3. Repeat step 2 until minimum spanning tree is formed.

Example: Find the minimum spanning tree of the following graph using Prim's algorithm.



Solution: First, we have to choose a vertex randomly from the graph. Let's choose vertex b.



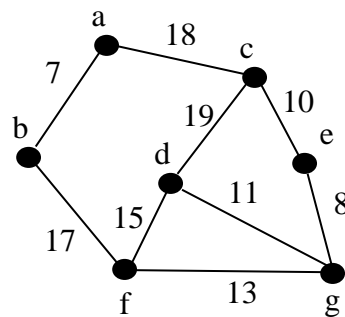
6.2. Kruskal's Algorithm

Kruskal's algorithm is also used to construct minimum spanning trees. To carry out Kruskal's algorithm, choose an edge in the graph with minimum weight. Successively add edges with minimum weight that do not form a simple circuit with those edges already chosen. Stop after $n-1$ edges have been selected.

Algorithm:

1. Take the edge with the lowest weight and add it to the spanning tree. If the edge to be added creates a cycle, then reject the edge.
2. Continue to add the edges until we reach all vertices, and a minimum spanning tree is created.

Example: Find the minimum spanning tree of the following graph using Kruskal's algorithm.



Solution:

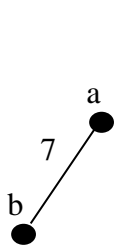


Fig (a)

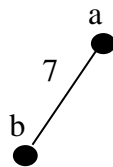


Fig (b)



Fig (c)

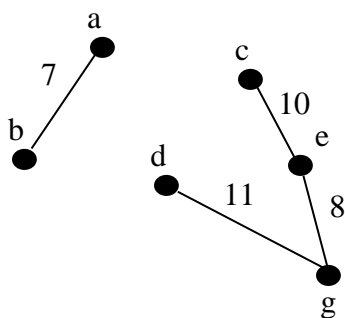
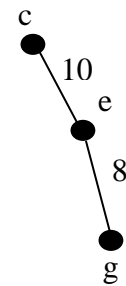


Fig (d)

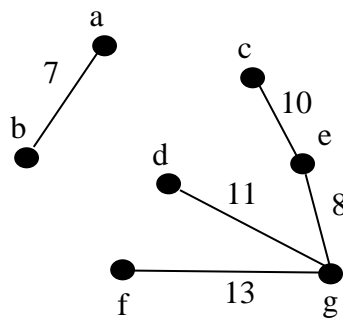


Fig (e)

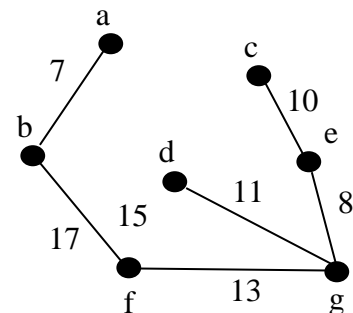
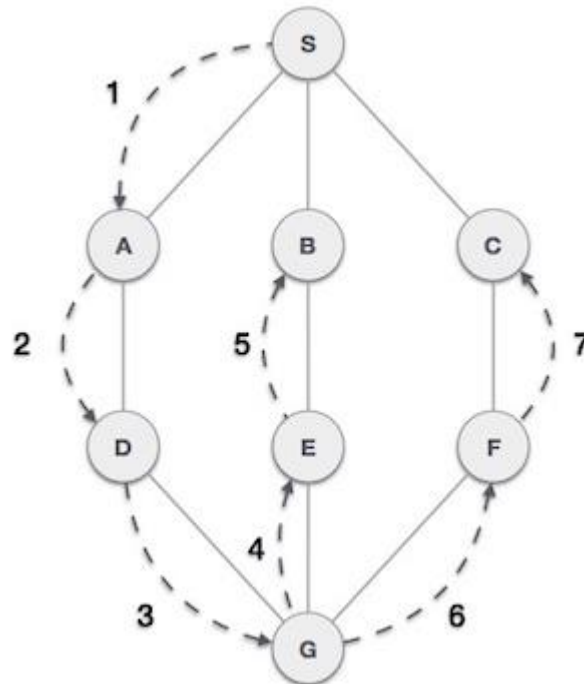


Fig (f)

7. Graph Traversal

7.1. Depth First Search (DFS)

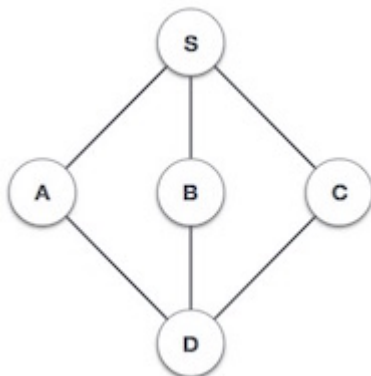
Depth First Search (DFS) algorithm traverses a graph in a depthward motion and uses a stack to remember to get the next vertex to start a search, when a dead end occurs in any iteration.



As in the example given above, DFS algorithm traverses from S to A to D to G to E to B first, then to F and lastly to C. It employs the following rules.

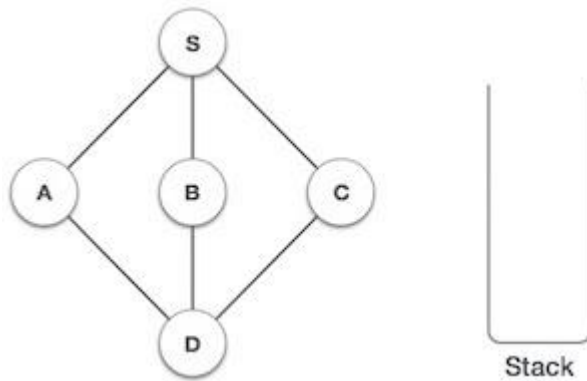
- **Rule 1** – Visit the adjacent unvisited vertex. Mark it as visited. Display it. Push it in a stack.
- **Rule 2** – If no adjacent vertex is found, pop up a vertex from the stack. (It will pop up all the vertices from the stack, which do not have adjacent vertices.)
- **Rule 3** – Repeat Rule 1 and Rule 2 until the stack is empty.

Example: Traverse the graph below using depth first search.

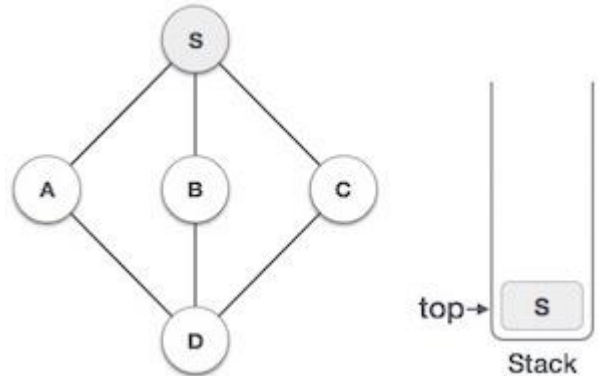


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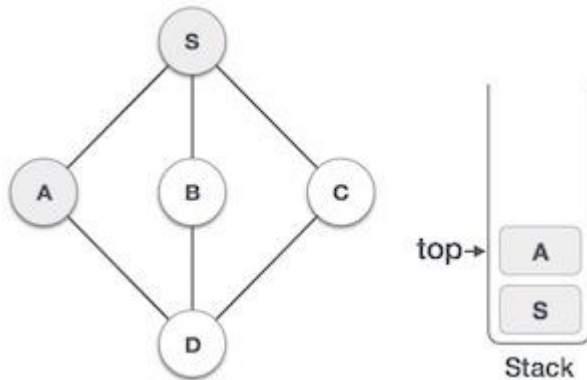
Step 1: Initialize stack



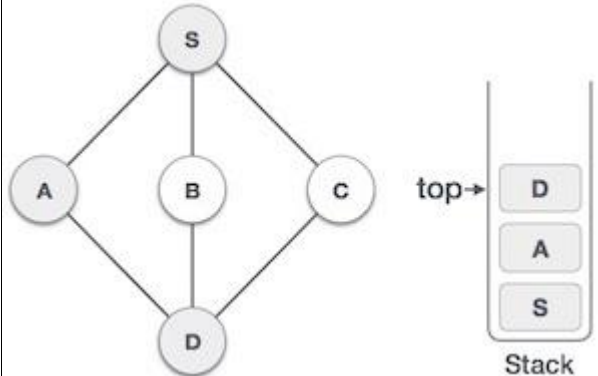
Step 2: Mark **S** as visited and put it onto the stack. Explore any unvisited adjacent node from **S**. We have three nodes and we can pick any of them. For this example, we shall take the node in an alphabetical order.



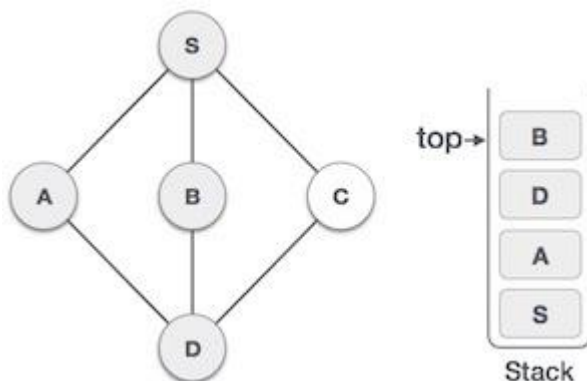
Step 3: Mark **A** as visited and put it onto the stack. Explore any unvisited adjacent node from **A**. Both **S** and **D** are adjacent to **A** but we are concerned for unvisited nodes only.



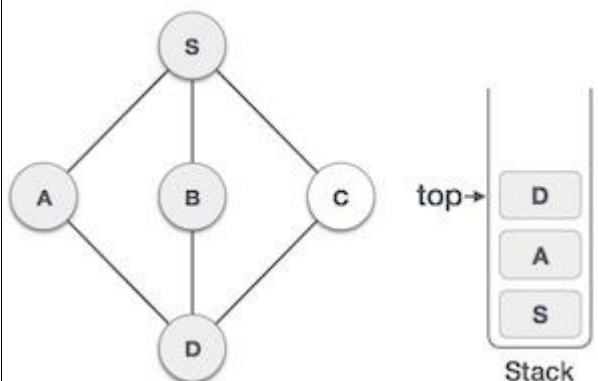
Step 4: Visit **D** and mark it as visited and put onto the stack. Here, we have **B** and **C** nodes, which are adjacent to **D** and both are unvisited. However, we shall again choose in an alphabetical order.



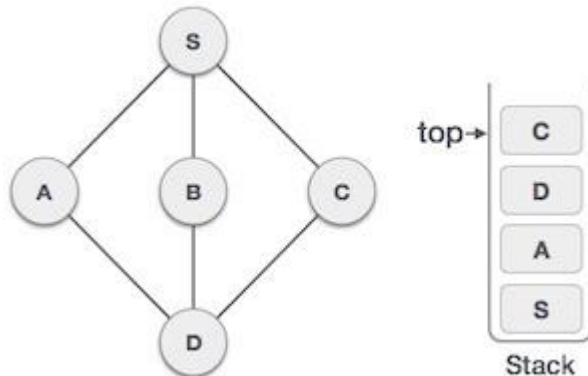
Step 5: We choose **B**, mark it as visited and put onto the stack. Here **B** does not have any unvisited adjacent node. So, we pop **B** from the stack.



Step 6: We check the stack top for return to the previous node and check if it has any unvisited nodes. Here, we find **D** to be on the top of the stack.



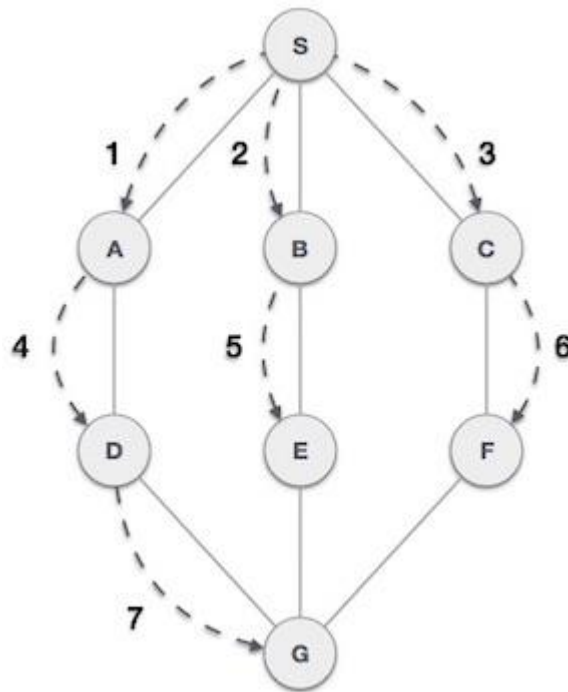
Step 7: Only unvisited adjacent node is from **D** is **C** now. So we visit **C**, mark it as visited and put it onto the stack.



As **C** does not have any unvisited adjacent node so we keep popping the stack until we find a node that has an unvisited adjacent node. In this case, there's none and we keep popping until the stack is empty. Hence, depth first search is **S, A, D B, C**.

7.2. Breadth Frist Search (BFS)

Breadth First Search (BFS) algorithm traverses a graph in a breadthward motion and uses a queue to remember to get the next vertex to start a search, when a dead end occurs in any iteration.

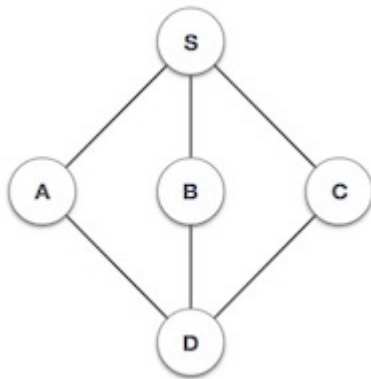


As in the example given above, BFS algorithm traverses from A to B to E to F first then to C and G lastly to D. It employs the following rules.

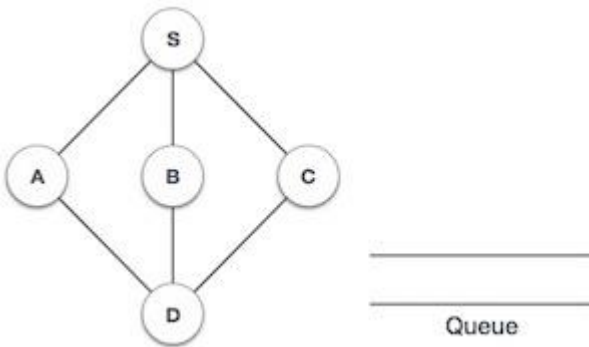
- **Rule 1** – Visit the adjacent unvisited vertex. Mark it as visited. Display it. Insert it in a queue.
- **Rule 2** – If no adjacent vertex is found, remove the first vertex from the queue.
- **Rule 3** – Repeat Rule 1 and Rule 2 until the queue is empty.

Unit 8.2: Graphs

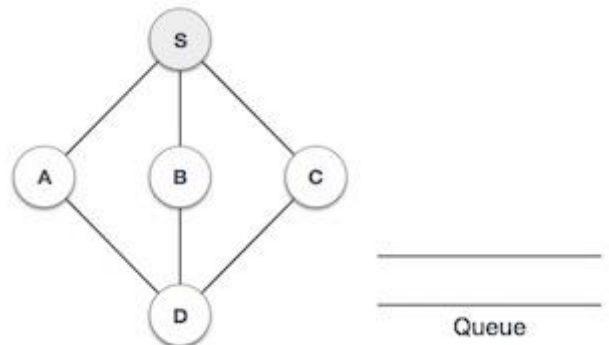
Example: Traverse the graph below using breadth first search.



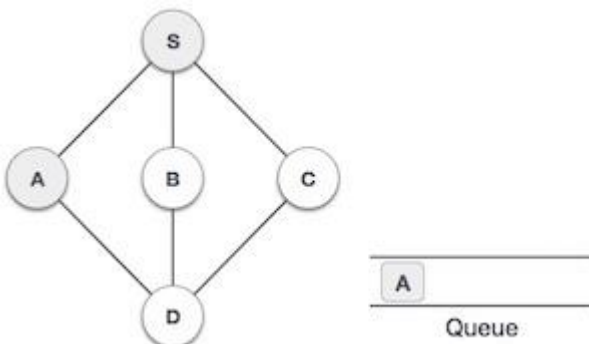
Step 1: Initialize queue.



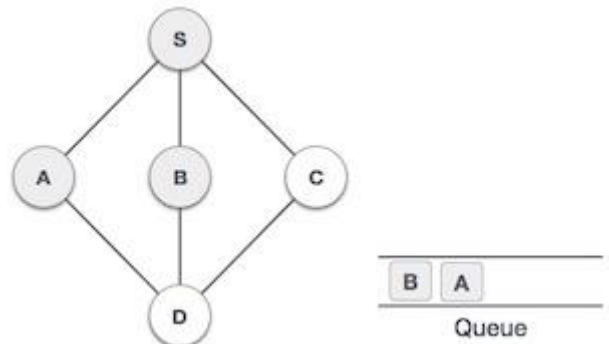
Step 2: We start from visiting **S** (starting node), and mark it as visited.



Step 3: We then see an unvisited adjacent node from **S**. In this example, we have three nodes but alphabetically we choose **A**, mark it as visited and enqueue it.



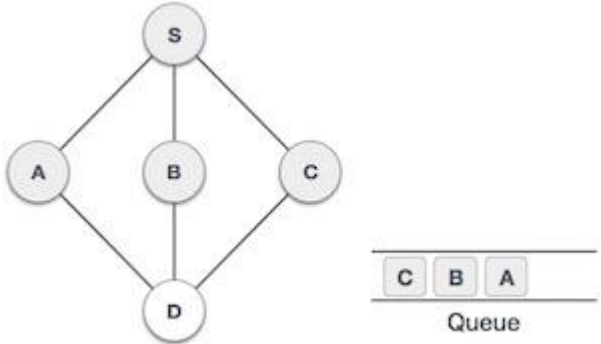
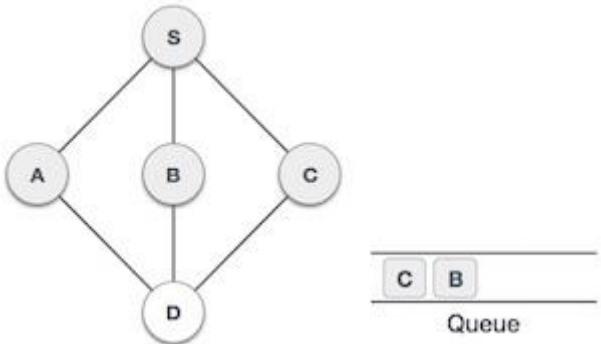
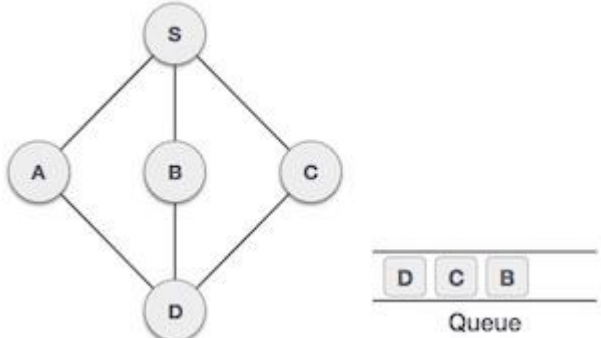
Step 4: Next, the unvisited adjacent node from **S** is **B**. We mark it as visited and enqueue it.



Step 5: Next, the unvisited adjacent node from **S** is **C**. We mark it as visited and enqueue it.

Step 6: Now, **S** is left with no unvisited adjacent nodes. So, we dequeue and find **A**.

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 <p>Graph diagram showing nodes S, A, B, C, and D. S is the source node at the top. A, B, and C are intermediate nodes, and D is the destination node at the bottom. The queue contains nodes C, B, and A.</p>	 <p>Graph diagram showing nodes S, A, B, C, and D. S is the source node at the top. A, B, and C are intermediate nodes, and D is the destination node at the bottom. The queue contains nodes C and B.</p>
<p>Step 7: From A we have D as unvisited adjacent node. We mark it as visited and enqueue it.</p>  <p>Graph diagram showing nodes S, A, B, C, and D. S is the source node at the top. A, B, and C are intermediate nodes, and D is the destination node at the bottom. The queue contains nodes D, C, and B.</p>	<p>At this stage, we are left with no unmarked (unvisited) nodes. But as per the algorithm we keep on dequeuing in order to get all unvisited nodes. When the queue gets emptied, the program is over.</p> <p>Hence, depth first search is S, A, B, C, D.</p>