Differentiation

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Slope of a line

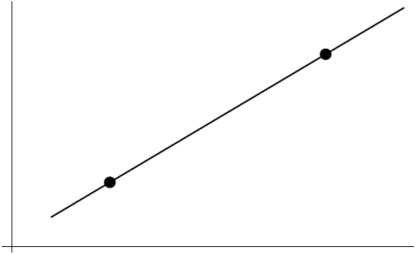


Figure 1: Slope of a line $\frac{y_2 - y_1}{x_2 - x_1}$

measures how steep a line is.

Slope cont'd

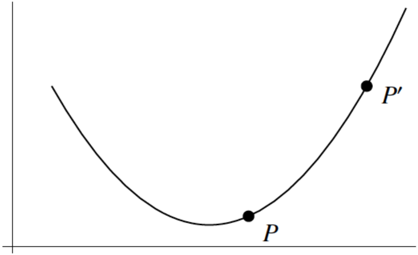


Figure 2: in a curve, slope changes along curve; slope at P' is steep, while at P pretty much gentle. A slope is a piece of ground going up and down.

Slope cont'd

To define the slope of curve $\mathcal{C}$ at point P,

1. Take a point Q on $\mathcal{C}$ different from P. The line PQ is called a *secant* line at P.
2. The slope of PQ, m_{PQ}, can be found using the coordinates of P and Q.
3. If we move Q along the curve, the slope m_{PQ} changes.

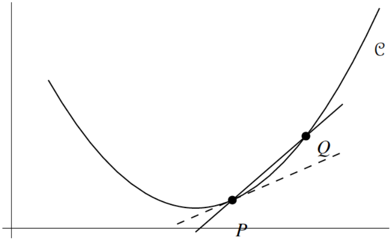


Figure 3: The secant line PQ.

When Q approaches P, m_{PQ} approaches a fixed value. This value is called m_P, which is the slope of $\mathcal{C}$ at P. The line with slope m_P and going through P is called *tangent* line.

Slope cont'd

Based on the curve $\mathcal{C}$ in Figure 3, in the concept of limit we may write,

$$\lim_{Q \to P} m_{PQ} = m_{\mathcal{C}P}. \tag{1}$$

Note that $Q \to P$ refers to as Q approaches P along the curve $\mathcal{C}$.

Suppose $\mathcal{C}$ defined as $y = f(x)$, and $P(x_0, f(x_0))$ is a point at $\mathcal{C}$. The x-coordinate of any point Q in $\mathcal{C}$, where $Q \neq P$, can be written as $Q(x_0 + h, f(x_0 + h))$, with $h \neq 0$. Thus the slope m_{PQ} of the secant line PQ is

$$m_{PQ} = \frac{f(x_0 + h) - f(x_0)}{(x_0 + h) - x_0} \tag{2}$$

$$= \frac{f(x_0 + h) - f(x_0)}{h} \tag{3}$$

Slope cont'd

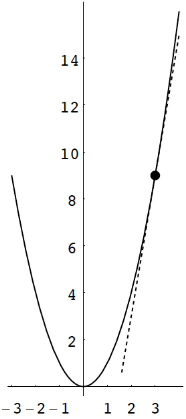
As Q approaches P, h approaches 0. We can thus write the slope of $\mathcal{C}$ at P as

$$\lim_{h \to 0} m_P = \frac{f(x_0 + h) - f(x_0)}{h} \tag{4}$$

Example

To find the slope of a curve defined by $y = x^2$ at the point $(3, 9)$,

$$
\begin{aligned}
m_P &= \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \\
&= \lim_{h \to 0} \frac{f(3 + h) - f(3)}{h} \\
&= \lim_{h \to 0} \frac{(3 + h)^2 - 3^2}{h} \\
&= \lim_{h \to 0} \frac{(9 + 6h + h^2) + 9}{h} \\
&= \lim_{h \to 0} \frac{6h + h^2}{h} \\
&= \lim_{h \to 0} 6 + h \\
&= 6.
\end{aligned}
$$



Definition

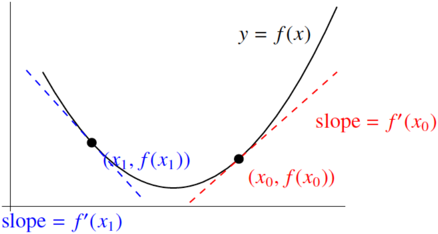
Let x_0 be a real number and let f be a function defined on an open interval containing x_0. Suppose the limit in (4) exists, then we say that f is differentiable at x_0.

Open interval, by convention, is denoted by (a, b), which can be one of the following:

- (a, b) where $a, b \in \mathbb{R}$, with $a < b$;
- $(-\infty, b)$, where $a = -\infty$ and $b \in \mathbb{R}$;
- $(a, +\infty)$m, where $a \in \mathbb{R}$ and $b = \infty$;
- $(-\infty, \infty)$, where $a = -\infty$ and $b = \infty$.

A function f is *differentiable*, meaning that for every $x \in dom(f)$, the limit of difference quotient $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ exists. We thus get a *derivative* function of f, denoted by f', from $dom(f)$ to $\mathbb{R}$.

Having f differentiable implies that every point on the curve has existing slope. The slope of point whose x-coordinate is x_0 is $f'(x_0)$, and thus f' can be considered as a *slope function*.



Derivative

Let f be a function that is differentiable at some points belonging to its domain. Then the derivative of f, denoted by f', is the function (from a subset of $dom(f)$ into R) given by,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \tag{5}$$

where the domain of f' is
$\{x \in dom(f) : \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \text{ exists}\}$.

Derivative

To find derivative of f means to find the domain of f' and find a formula for $f'(x)$. For example, given $f(x) = x^2$, find the derivative of f.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{(x^2+2xh+h^2)-x^2}{h} \\
&= \lim_{h\to 0} \frac{2xh+h^2}{h} \\
&= \lim_{h\to 0} (2x+h) \\
&= 2x.
\end{aligned}$$

We find that the domain of f' is $\mathbb{R}$ and $f'(x) = 2x$.

Exercise

With same step as above, find the derivative of $f(x) = x^3$.

Exercise

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h\to 0} \frac{(x^3+3x^2h+3xh^2+h^3)-x^3}{h} \\
&= \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3}{h} \\
&= \lim_{h\to 0} (3x^2+3xh+h^2) \\
&= 3x^2.
\end{aligned}
$$

The domain of f' is $\mathbb{R}$ and $f'(x) = 3x^2$.

Differentiation

The process of finding derivatives is called *differentiation*.

To denote the derivative of function f, we can use the notations f', y', $\frac{\mathrm{d}y}{\mathrm{d}x}$, Df, Dy, $f'(x)$, and $\frac{\mathrm{d}}{\mathrm{d}x}f(x)$.

To denote the derivative of function f at x_0, we can use notations $f'(x_0)$, $y'(x_0)$, and so on.

Rule example

$$\frac{\mathrm{d}}{\mathrm{d}x} c = 0, \tag{6}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} x = 1, \tag{7}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} x^n = n x^{n-1}, \tag{8}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} (kf)(x) = k \cdot \frac{\mathrm{d}}{\mathrm{d}x} f(x), \tag{9}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} (f+g)(x) = \frac{\mathrm{d}}{\mathrm{d}x} f(x) + \frac{\mathrm{d}}{\mathrm{d}x} g(x), \tag{10}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \sqrt{x} = \frac{1}{2\sqrt{x}}. \tag{11}$$

Exercise

Given the functions below, find the corresponding derivatives.

1. $y = 2x^5$
2. $y = x^2 + 4$
3. $y = x^5 - 6x^7$

Example in R

An example of implementing Rule (8),

```
1 rule8 <- function(x, n) {
2     return(n * x^(n-1))
3 }
4
5 #using Ryacas package
6 #returns the derivative formula
7 library(Ryacas)
8 x <- Sym("x")
9 Simplify(deriv(x^2, x))
```

Exercise

1. Implement Rules (11) in R as Lines 1-3 above, i.e., to return the value when x and n are given.
2. Implement exercises 1-3 in R (with Ryacas package) to find the derivatives formula.

References

S.K. Chung (2014), "Understanding Basic Calculus".
Available at http://www.math.nagoya-u.ac.jp/~richard/teaching/f2015/BasicCalculus.pdf