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On Stock Trading Using a PI Controller in an Idealized Market: The Robust Positive Expectation Property

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***Abstract*— In a number of recent papers, a new line of research has been unfolding which is aimed at using classical linear feedback control in a model-free stock trading context. The salient feature of this approach is that no model for stock price dynamics is used to determine the dollar amount invested $I(t)$. Instead, the investment level is performance driven and generated in a model-free manner via an adaptive feedback on the cumulative gains and losses $g(t)$. One of the main results obtained to date is paraphrased as follows: Under idealized market conditions with stock prices governed by a non-trivial Geometric Brownian Motion (GBM), a combination of two static linear feedbacks, one long and one short, leads to a positive expected value for the trading gain $g(t)$ for all $t > 0$. Since this holds independently of the parameters underlying the GBM process, it is called the "robust positive expectation" property. Working in this same GBM setting, the main objective in this paper is to generalize this result from static to dynamic feedback. To this end, we consider a Proportional-Integral (PI) controller for the investment function $I(t)$. Subsequently, we reduce the stochastic trading equations for the expectation of $g(t)$ to a classical second order system and use the closed-form solution to prove that the robust positive expectation property still holds. We also consider a number of other issues such as the analysis of the variance of $g(t)$ and the monotonic dependence of $g(t)$ on the feedback gains. Finally, we provide simulations showing how the PI controller performs in a real market with prices obtained from historical data.**

I. INTRODUCTION

Over the last several years, a new line of research has been developing which is aimed at using classical linear feedback control concepts in a stock trading context; e.g., see [1]-[20]. Perhaps the most salient feature of the most recent papers in this direction is the fact that the feedback rules defining the strategies are "model-free." That is, instead of determining the investment level $I(t)$ based on some parameterized model of stock prices which might be estimated over time, the controller is performance driven; i.e., $I(t)$ is adaptively updated based on the cumulative profits and losses $g(t)$ up to time t without concern for prediction of future prices; e.g., see [1]-[6]. To illustrate, in the case of static "long" linear feedback, the amount invested $I(t)$ at time t is

$$I(t) = I_0 + Kg(t)$$

where $I_0 > 0$ is the initial investment and $K \geq 0$ is the feedback gain. It is also possible to include short selling in this formulation by allowing $I(t) < 0$; e.g., consider $I_0 < 0$ and $K \leq 0$ above.

Working in this type of setting, the main objective in this paper is to generalize the "robust positive expectation" result given in [1] and [2]. More specifically, in these papers, using a combination of two linear feedbacks, one long and one short, the so-called *Simultaneous Long-Short* (SLS) controller is seen to have a remarkable property: In an idealized market with benchmark prices that follow Geometric Brownian Motion (GBM) with non-zero drift μ, non-zero feedback gain K and volatility σ, irrespective of the sign and magnitude of μ and the magnitude of σ, the SLS static linear feedback controller leads to a trading gain with positive expected value $\mathbb{E}[g(t)]$ for all $t > 0$. It is established that

$$\mathbb{E}[g(t)] = \frac{I_0}{K}\left[e^{K\mu t} + e^{-K\mu t} - 2\right]$$

which is readily seen to be positive for all $t > 0$. We call this the *robust positive expectation property*.

In the sequel, we generalize the static feedback analysis to handle the case when the controller includes dynamics. We consider a classical Proportional-Integral (PI) controller

$$I(t) = I_0 + K_P g(t) + K_I \int_0^t g(\tau)d\tau,$$

noting that we use differentiator-free dynamics to avoid problems associated with price signals which typically include high-frequency components. Our goal is to derive expressions for the mean and variance of $g(t)$. Working with a long-short version of the PI controller above, our main result is that the robust positive expectation result $\mathbb{E}[g(t)] > 0$ still holds except for the trivial break-even case with either both feedback gains $(K_P, K_I) = (0, 0)$ or drift $\mu = 0$.

Our analysis of the PI trading strategy involves a number of technical issues: First, we show that when dealing with $\mathbb{E}[g(t)]$, the stochastic equations for trading can be reduced to a classical second order differential equation with parameters being simple functions of I_0, μ, K_P and K_I. Hence, in addition to positivity of the expectation, many aspects of the behavior of $\mathbb{E}[g(t)] > 0$ such as damping, overshoot and oscillations become straightforward to study. The paper also considers a number of additional items such as the analysis of variance, monotonicity properties of the expected value of $g(t)$ and a suggestion for further research

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involving the modification of the analysis to accommodate exponential discounting. Finally, to demonstrate the application of the main ideas in this paper, using historical data, we provide simulations to show how the controller performs in a real market when trades are conducted every two minutes.

II. IDEALIZED MARKETS

In this paper, we work in an idealized market, see [1]-[3], where it is assumed that a trader has the ability to transact continuously with perfect liquidity. A second assumption is that the trader is a price taker. We now elaborate on these terms: By *continuous trading*, we mean that the amount invested $I(t)$ can be continuously updated. Furthermore, in making such trades, the *perfect liquidity condition* guarantees no gap between the bid and ask price as shares are acquired or sold. As far as being a *price taker* is concerned, it is assumed that the trader is "small enough" so that the stock price is not changed during the course of a transaction. In practice, this assumption and the perfect liquidity condition could be violated for the case of a large fund buying many shares in a single transaction. That is, if the trade being consummated involves a significant percentage of total daily volume, a lack of liquidity could mean that the "last" shares are more costly to acquire than the "first." Finally, it should be noted that markets satisfying continuity and liquidity assumptions of the sort described above are referred to as "frictionless" in the finance literature; e.g., see [21]-[24].

The remaining two assumptions, *zero transaction costs* and *adequacy of resources* are easy to understand: First, no commissions, fees or taxes are involved on any trade. For active traders who are frequently placing orders of reasonable size, say several thousand dollars per trade, nowadays, commissions are sufficiently low so as to be, more or less, a non-issue. Our final assumption is that the trader has adequate resources on the "sidelines" to make whatever trade is dictated by the feedback law leading to $I(t)$ without triggering a margin call or forced liquidation due to lack of collateral. For simplicity of exposition, with regard to cash balances in the account, it is also assumed that no interest is accrued and no margin charges are incurred. Hence, in this idealized setting, changes in the trader's account value $V(t)$ correspond to changes in the profit or loss level $g(t)$. That is, since $g(0) = 0$, we have $V(t) = V_0 + g(t)$.

III. DERIVATION OF DYNAMICS FOR EXPECTATION

We now consider the analysis of the expected gain or loss $\mathbb{E}[g(t)]$ associated with a PI controller operating in an idealized market with prices $p(t)$ generated by a Geometric Brownian Motion (GBM) having stochastic differential equation

$$\frac{dp}{p} = \mu dt + \sigma dZ$$

where $Z(t)$ is a standard Wiener process, μ is the drift and σ is the volatility.

Interaction of Prices and PI Controller: It is important to note that neither μ nor σ is known to the trader. Furthermore, as previously stated, the feedback control law underlying the investment $I(t)$ is model-free and no attempt is made to estimate μ and σ on the fly. In order to differentiate between long and short positions, we use subscripts "L" and "S" for the investment function $I(t)$ and trading gain $g(t)$. Accordingly, with initial condition given by $I_L(0) = I_0 > 0$, PI trader who is initially long works with investment

$$I_L(t) = I_0 + K_P g_L(t) + K_I \int_0^t g_L(\tau) d\tau$$

where $K_P \geq 0$ and $K_I \geq 0$. Similarly, on the short side, the trader begins with $I_S(0) = -I_0$ and investment

$$I_S(t) = -I_0 - K_P g_S(t) - K_I \int_0^t g_S(\tau) d\tau.$$

The theory to follow allows us to consider three scenarios: Initially Long, Initially Short and *Initially Long-Short* (ILS). Our use of the word "initially" in describing these trades is based on a fundamental difference between static versus dynamic trading. That is, in the static case with $K_I = 0$, as seen in our earlier work, the signs of $I_L(t)$ and $I_S(t)$ remain invariant over the course of the trade. However, when integrator action is included with $K_I \neq 0$, one can end up with either $I_L(t)$ or $I_S(t)$ changing sign; e.g., the initially long position can be "morphed" into a short. To conclude, in the ILS case, the control, being the sum of long and short positions, reduces to

$$I(t) = K_P(g_L(t) - g_S(t)) + K_I \int_0^t (g_L(\tau) - g_S(\tau)) d\tau.$$

Next, we consider the stochastic differential equation for the trading gain g_L noting that a nearly identical analysis applies to g_S. Indeed, noting that the incremental trading gain or loss dg_L is the percentage change in price dp/p times the amount invested $I(t)$, we have

$$dg_L = \frac{dp}{p} I = (\mu dt + \sigma dZ)\left(I_0 + K_P g_L(t) + K_I \int_0^t g_L(\tau) d\tau\right).$$

State Space Representation: We now create a state-space representation using the two-dimensional state vector

$$x(t) \doteq \begin{bmatrix} \int_0^t g_L(\tau) d\tau \\ g_L(t) \end{bmatrix},$$

with initial condition $x(0) = 0$. Next, we reduce the gain dynamics to the first order stochastic equation

$$\begin{aligned} dx_1 &= x_2 dt; \\ dx_2 &= (\mu dt + \sigma dZ)(I_0 + K_P x_2 + K_I x_1). \end{aligned}$$

To more clearly see the structure of these equations, we view the initial investment as a unit step input $u(t) \equiv I_0$ for $t \geq 0$ and express the increment above in the classical form

$$dx = (Ax + bu)dt + (Cx + du)dZ$$

with matrices having the structure

$$A \doteq \left[\begin{array}{cc} 0 & 1 \\ \mu K_I & \mu K_P \end{array} \right]; \quad b \doteq \left[\begin{array}{c} 0 \\ \mu \end{array} \right];$$

$$C \doteq \left[\begin{array}{cc} 0 & 0 \\ \sigma K_I & \sigma K_P \end{array} \right]; \quad d \doteq \left[\begin{array}{c} 0 \\ \sigma \end{array} \right].$$

This is a linear system with multiplicative noise; for example, see see [25] and [26].

Gain Expectation Dynamics: Denoting the unconditional expectation of the state at time t as

$$\bar{x}(t) \doteq \mathbb{E}[x(t)],$$

the differential equation for the expected-state dynamics is obtained by taking the expectation of both sides of the stochastic differential equation for $x(t)$. To accomplish this, a formal argument requires commutation of derivative and expectation operations above, for example, see [27], and exploiting the zero-mean property of the Wiener process to eliminate the term multiplied by dZ. Hence, we arrive at

$$\begin{aligned} \frac{d\bar{x}}{dt} &= A\bar{x} + bu \\ &= \left[\begin{array}{cc} 0 & 1 \\ \mu K_I & \mu K_P \end{array} \right] \bar{x}(t) + \left[\begin{array}{c} 0 \\ \mu \end{array} \right] I_0 \end{aligned}$$

with output of interest $y = g_L$ given by

$$y = c^T \bar{x} = [0 \ \ 1]\bar{x}.$$

The system above is now straightforward to analyze using the classical analysis for second-order systems. To this end, the transfer function from the investment to the trading gain is immediately calculated to be

$$H(s) = c^T(sI - A)^{-1}b = \frac{\mu s}{s^2 - \mu K_P s - \mu K_I},$$

with associated eigenvalues

$$\lambda_{\pm} = \frac{\mu K_P \pm \sqrt{\mu^2 K_P^2 + 4\mu K_I}}{2}.$$

The simplicity of the transfer function above makes it possible to analyze various scenarios using the Final Value Theorem; e.g., if a trader is initially long and $\mu < 0$, $K_I > 0$ and $K_P > 0$, then despite being on the "wrong side of the market," it is easy to verify that $lim_{t\to\infty}\mathbb{E}[g_L(t)] = 0$. That is, with the integrator, an initially "bad trade" eventually turns into in a break-even situation.

IV. Closed-Form Solution Possibilities

For the second order system above, the classical solution possibilities are readily enumerated and closed-form solutions for $\mathbb{E}[g_L(t)]$ can easily be obtained by considering two possibilities, $\mu > 0$ or $\mu < 0$ for the sign of the drift and for the discriminant, $\Delta \doteq \mu^2 K_P^2 + 4\mu K_I$, there are three cases: $\Delta < 0, \Delta = 0$ and $\Delta > 0$. We now demonstrate by showing the solution for two of the most important cases and then provide a convenient compact formula which covers all cases in one fell swoop.

The Oscillatory Case: Suppose $\mu < 0$, $\Delta < 0$, $K_I > 0$ and $K_P \geq 0$. Then using the formulae above, the expected value of the trading gain is a damped harmonic; i.e., the damping ratio $\zeta < 1$ is given by

$$\zeta = \frac{1}{2} K_P \sqrt{\frac{-\mu}{K_I}},$$

the undamped natural frequency is $\omega_n = \sqrt{-\mu K_I}$, and, inverting the Laplace transform, we easily obtain

$$\mathbb{E}[g_L(t)] = -\left(\frac{I_0}{K_I}\right) \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2} t).$$

Perhaps the most important feature of the solution above is that it adapts to the trader's "error" in the assessment of the market's direction. By this, we mean the following: Suppose $K_P > 0$ and the trader begins with a long position $I_L(0) = I_0 > 0$ in a market which is drifting downward with $\mu < 0$. As losses build up, the integration action eventually forces $I_L(t) < 0$. That is, the trader finally "gets it right" in a falling market by switching from a long to a short position. To see this effect more clearly, we calculate the expected value of the investment, and, via a straightforward calculation, we obtain

$$\mathbb{E}[I_L(t)] = \frac{I_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos\left(\omega_n \sqrt{1-\zeta^2} t + \theta\right)$$

where

$$\theta \doteq \arctan\left(\frac{-\zeta}{\sqrt{1-\zeta^2}}\right).$$

The adaptation phenomenon described above is now seen in Figure 1 where $\mathbb{E}[g_L(t)]$ and $\mathbb{E}[I_L(t)]$ are plotted using sample parameter values $I_0 = 1, \mu = -3, K_P = .5$ and $K_I = 4$. A key observation is that three times over the duration of the trade, $I_L(t)$ switches from long to short and eventually, per discussion of the Final Value Theorem above, turns a losing trade into a break-even situation.

The Purely Exponential Case: Continuing with the initially-long trader in a falling market with $\mu < 0$, when the discriminant Δ is positive, that is, $\mu^2 K_P^2 + 4\mu K_I > 0$, we obtain the solution

$$\mathbb{E}[g_L(t)] = \frac{\mu I_0}{2\alpha} e^{\mu K_P t/2} \left[e^{\alpha t} - e^{-\alpha t}\right],$$

where

$$\alpha \doteq \frac{\sqrt{\mu^2 K_P^2 + 4\mu K_I}}{2}.$$

We note that the initially-long trader is on the wrong side of the market but the controller adapts by switching from long to short as losses increase and ultimately breaks even.

Convenient Compact Solution Representation: The two cases considered above do not cover all the solution possibilities. Rather than enumerating them all, we simply provide a compact formula which covers all cases. Below, the understanding is that arguments under the square root sign can be negative. In such cases, using de Moivre's formula, we

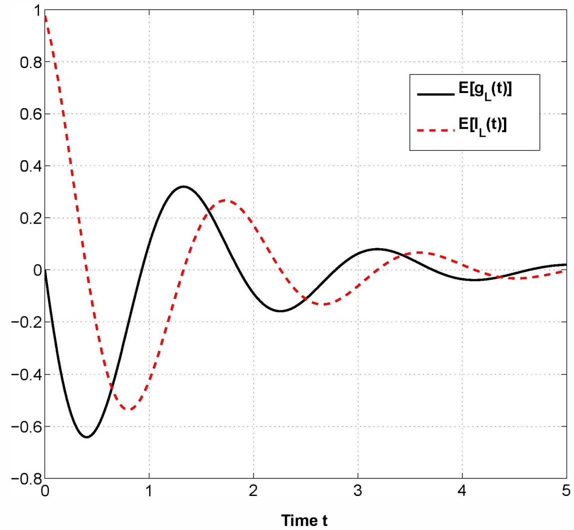

Fig. 1. Trading Gain and Investment for the Oscillatory Case

obtain the appropriate interpretation in terms of harmonics. Furthermore, since it is easily verified that the expected gain or loss is an even function of the drift μ, the formulae below are given for $\mu \geq 0$. Accordingly, for the case of PI control, we obtain

$$\mathbb{E}[g_L(t)] = \frac{\mu I_0}{\alpha} e^{\mu K_P t/2} \sinh(\alpha t);$$

$$\mathbb{E}[g_S(t)] = \frac{-\mu I_0}{\beta} e^{-\mu K_P t/2} \sinh(\beta t),$$

where

$$\beta \doteq \frac{\sqrt{\mu^2 K_P^2 - 4\mu K_I}}{2}.$$

Initially Long-Short (ILS) Controller: Motivated by results in the purely proportional feedback case, see [1] and [3], we consider a trading strategy that implements both a long and short version of the PI strategy, simultaneously. The expected gain from this strategy is simply the sum of the gain from the long and short strategies which is found to be

$$\mathbb{E}[g(t)] = \mu I_0 e^{-\mu K_P t/2} \left[e^{\mu K_P t} \frac{\sinh(\alpha t)}{\alpha} - \frac{\sinh(\beta t)}{\beta} \right].$$

Recalling the examples in the previous section, we know that a long controller which begins with $I_L(0) = I_0 > 0$, can eventually become a short over the course of the trade. Similarly, the short side can switch to long. Hence, we refer to this as an *Initially Long-Short (ILS)* PI controller. In our earlier work with a static controller, corresponding to the special case $K_I = 0$, this switch between long and short could not occur. Hence, we used the terminology *Simultaneous Long-Short* (SLS) to describe the controller.

As previously stated, in the purely proportional SLS case in [1], the strategy is shown to achieve a positive expected trading gain under any GBM price process with $K\mu \neq 0$. It is reasonable to ask whether this same robustness property holds in the ILS case for the PI controller. We can see immediately the potential for such a result in the ILS case by considering the pure integrator case obtained with $K_P = 0$. Indeed, using the compact solution representation above,

$$\mathbb{E}(g(t)) = \sqrt{\frac{|\mu|}{K_I}} I_0 \left[\sinh(\sqrt{|\mu| K_I} t) - \sin(\sqrt{|\mu| K_I} t) \right]$$

which is readily verified to be positive except for the trivial break-even case when $\mu = 0$ or $K_I = 0$. In the section to follow, we establish that this robust positive expectation property holds in the more general case when both K_P and K_I are in play. In addition we establish the monotonic dependence of the expected value of $g(t)$ on K_P and K_I.

V. Main Results

For the PI controller above, we now provide two theorems.

Robust Positive Expectation Theorem: *Consider the ILS PI controller with $K_I \geq 0$ and $K_P \geq 0$ in an idealized market with GBM prices. Then, except for the trivial break-even case obtained when either $\mu = 0$ or $(K_P, K_I) = (0, 0)$, the expected gain $\mathbb{E}[g(t)]$ is strictly increasing in t. Moreover, since $\mathbb{E}(g(0)) = 0$, it follows that*

$$\mathbb{E}(g(t)) > 0$$

for all $t > 0$.

Proof: To establish that the expected return is increasing with respect to t, we use the compact solution representation given in Section 4. Additionally, we use the notation,

$$f(x) \doteq \frac{\sinh(x)}{x}.$$

Recalling that the expected gain is an even function of μ, without loss of generality we assume that $\mu > 0$. To prove that the expected gain is increasing in time, we will show that its derivative is strictly positive. Indeed, differentiating with respect to time t and rearranging terms gives

$$\begin{aligned} \frac{d\mathbb{E}(g(t))}{dt} &= \frac{\mu^2 K_P I_0 t e^{-\mu K_P t/2}}{2} \left[e^{\mu K_P t} f(\alpha t) + f(\beta t) \right] \\ &\quad + \mu I_0 e^{-\mu K_P t/2} \left[e^{\mu K_P t} \cosh(\alpha t) - \cosh(\beta t) \right] \end{aligned}$$

where α and β are defined in the previous section. To show that this quantity is strictly positive, we consider the following two cases.

Case 1: β is real, that is $\mu^2 K_P^2 - 4\mu K_I \geq 0$. Then using the following facts proves that the derivative is positive:

1) $f(x) \geq 0$ for $x \geq 0$.
2) $e^{\mu K_P t} \geq 1$.
3) $\alpha > \beta$ and cosh is an increasing function.

Case 2: β is pure imaginary, that is $\mu^2 K_P^2 - 4\mu K_I < 0$. Then $\beta = \gamma j$ where $\gamma = \frac{\sqrt{4\mu K_I - \mu^2 K_P^2}}{2}$. Rewriting the derivative,

$$\begin{aligned} \frac{d\mathbb{E}(g(t))}{dt} &= \frac{\mu^2 K_P I_0 t e^{-\mu K_P t/2}}{2} \left[e^{\mu K_P t} f(\alpha t) + \mathrm{sinc}(\gamma t) \right] \\ &\quad + \mu I_0 e^{-\mu K_P t/2} \left[e^{\mu K_P t} \cosh(\alpha t) - \cos(\gamma t) \right]. \end{aligned}$$

Again the following facts prove the time derivative of the expected return to be strictly positive in this case:

1) $e^{\mu K_P t} > 1$.
2) $f(x) \geq 1$ for $x > 0$.
3) $|\text{sinc}(x)| \leq 1$ for all x.
4) $\cosh(x) \geq 1$ for all x.
5) $|\cos(x)| < 1$ for all x.

Since these two cases are mutually exclusive and exhaustive, this completes the proof. □

Theorem (Monotonicity in the Control Parameters): *Consider the ILS PI controller with $K_I \geq 0$ and $K_P \geq 0$ in an idealized market with GBM prices. Then, except for the trivial break-even case of $\mu = 0$, for $t > 0$, the expected return $\mathbb{E}[g(t)]$ is increasing in K_P and in K_I.*

Proof: We begin from the convenient compact solution provided in the previous section. Rearranging, we obtain

$$\mathbb{E}\left(g(t)\right) = \mu I_0 t \left[e^{\mu K_P t/2} f(\alpha t) - e^{-\mu K_P t/2} f(\beta t)\right].$$

Recalling that the expected gain is an even function of μ, without loss of generality we assume $\mu > 0$. Using the Taylor expansion for the function $f(x)$ introduced in the proof of the last theorem, we obtain

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)!}; \quad f'(x) = \sum_{n=1}^{\infty} \frac{2nx^{2n-1}}{(2n+1)!}.$$

Taking the derivative of the expected gain and replacing $f(x)$ and $f'(x)$ by the expressions above, after further simplification and reordering of terms, we obtain

$$\begin{aligned} \frac{\partial \mathbb{E}(g)}{\partial K_I} &= \mu^2 I_0 t^3 e^{-\mu K_P t/2} \\ &\times \sum_{n=1}^{\infty} \frac{n}{(2n+1)!} \left[e^{\mu K_P t} (\alpha t)^{2n-2} + (\beta t)^{2n-2})\right]. \end{aligned}$$

Noting that $\alpha^2 \geq |\beta^2|$ and the fact that $e^{\mu K_P t} > 1$, one may verify that each term in the sum is positive, and therefore, the derivative is positive. Similarly, for K_P, we obtain

$$\begin{aligned} \frac{\partial \mathbb{E}(g)}{\partial K_P} &= \frac{\mu^2 I_0 t^2}{2} e^{-\mu K_P t/2} \\ &\times \left\{ \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \left[e^{\mu K_P t} (\alpha t)^{2n} + (\beta t)^{2n}\right] \right\} \\ &+ \frac{\mu^3 I_0 K_P t^3}{2} e^{-\mu K_P t/2} \\ &\times \left\{ \sum_{n=1}^{\infty} \frac{n}{(2n+1)!} \left[e^{\mu K_P t} (\alpha t)^{2n-2} - (\beta t)^{2n-2})\right] \right\}. \end{aligned}$$

Again, using the same reasoning, each term in each of the sums is positive and therefore the derivative is positive. □

VI. Analysis of the Covariance Matrix

In this section, we analyze the variance and covariance of the ILS PI strategy. To begin, we combine the states for the initially-long and initially-short cases into

$$x \doteq \begin{bmatrix} x_L \\ x_S \end{bmatrix}$$

and write the combined dynamics for x as

$$dx = (Ax + bu)\,dt + (Cx + du)\,dZ$$

with

$$A \doteq \begin{bmatrix} 0 & 1 & 0 & 0 \\ \mu K_I & \mu K_P & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\mu K_I & -\mu K_P \end{bmatrix}; \quad b \doteq \begin{bmatrix} 0 \\ \mu \\ 0 \\ -\mu \end{bmatrix};$$

$$C \doteq \begin{bmatrix} 0 & 0 & 0 & 0 \\ \sigma K_I & \sigma K_P & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\sigma K_I & -\sigma K_P \end{bmatrix}; \quad d \doteq \begin{bmatrix} 0 \\ \sigma \\ 0 \\ -\sigma \end{bmatrix}.$$

Differential Equation for the Covariance Matrix: To analyze the covariance matrix of the state x, we take

$$\bar{x}(t) \doteq \mathbb{E}[x(t)]; \quad e \doteq x - \bar{x}; \quad P(t) \doteq \mathbb{E}[e(t)e^T(t)].$$

Now, similar to the previous initially-long analysis, a straightforward calculation leads to the stochastic increment for the error

$$de = Aedt + (Ce + C\bar{x} + du)dZ.$$

By Ito's lemma, we can calculate dee^T, which gives

$$\begin{aligned} dee^T &= Aee^T dt + ee^T A^T dt \\ &\quad + (Ce + C\bar{x} + du)(Ce + C\bar{x} + du)^T dt \\ &\quad + ((Ce + C\bar{x} + du)e^T + e(Ce + C\bar{x} + du)^T)dZ. \end{aligned}$$

Taking the expectation of both sides leads to

$$\frac{dP}{dt} = AP + PA^T + CPC^T + (C\bar{x} + du)(C\bar{x} + du)^T.$$

This is a Lypunov-type linear matrix differential equation in $P(t)$ and thus can be easily solved. In particular, we are interested in the variance of the individual components $g_L(t)$ and $g_S(t)$, P_{22} and P_{44} respectively, of the ILS strategy. In addition, using $P(t)$ we obtain the variance of the overall ILS trading gain $g(t) = g_L(t) + g_S(t)$ as

$$Var[g(t)] = h^T P(t) h \text{ with } h \doteq [0, 1, 0, 1]^T.$$

The calculation of the variance, as prescribed above, is straightforward to implement numerically. Given the emphasis here on the robust positive expectation property, we do not provide an illustrative plot but make note of the fact that variance of $g(t)$ for the ILS strategy is lower than the sum of the variances of g_L and g_S. This is due to the fact that g_L and g_S are negatively correlated.

VII. A Back-Test Using Historical Data

In this section, we illustrate the application of the ILS PI feedback control strategy using a data set consisting of about $35,000$ price quotes for Apple (Ticker: AAPL) stock. The data covers 175 days from June 18, 2012 to February 28, 2013 with each new price quote following its

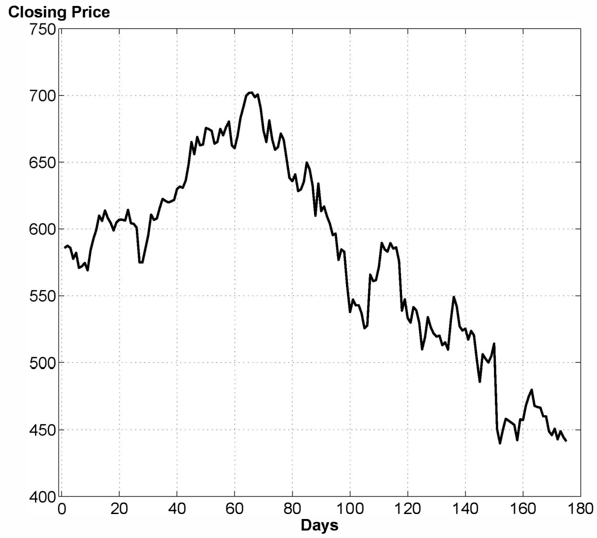

Fig. 2. Closing Prices for AAPL

predecessor by about two minutes[1]. In Figure 2, the daily closing prices are plotted.

To more closely approximate the idealized market conditions underlying the theory in this paper, we consider the case when the ILS position is initiated at the market open and vacated at the close; i.e., the trader is entirely in cash overnight. This increases the likelihood that price paths are more nearly continuous than would be the case if a position is carried overnight. Our interpretation is that each day provides a 200-point sample path from some random process governing the price of Apple. Figure 3 provides typical daily sample paths with the opening price normalized to \$1.

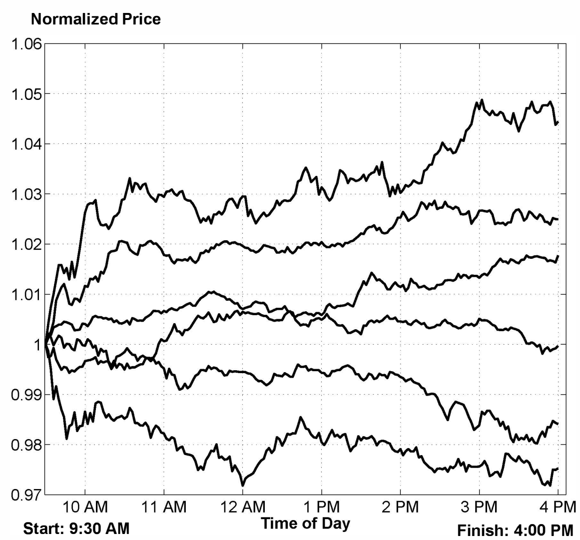

Fig. 3. Sample Paths for AAPL

[1]The authors express their thanks to Mr. Amin Farmahini-Farahani for help with the acquisition and processing of intraday stock prices.

The first scenario we consider is as follows: The investor begins with initial account value of \$10,000. Then, each day, the ILS position is initiated with $I_0 = 10,000$. We further assume no transaction costs and the ability to trade about every two minutes as a new data point arrives. In the first set of simulations, we held the proportional gain $K_P = 2$ fixed and varied the integrator gain in the interval $0 \leq K_I \leq 2$. The results are given in the table below. In Figure 4, the evolution of $g(t)$ is shown for $K_I = 2$.

K_I	Ending Account Value	Average Investment
0	\$10,098	\$335
0.2	\$11,950	\$3,005
0.5	\$14,726	\$7,025
1	\$19,357	\$13,630
2	\$28,686	\$26,406

Looking at the investment levels in the table above, we raise the possibility of a difference which might arise between theory and practice. The trader may not have adequate cash reserves to fund the required investment level $I(t)$. For example, the gain $K_I = 2$ leads to an average daily investment of over \$26,000. If the account value is much less than this investment level, there would be a requirement to pay margin interest or worse yet, a margin call by the broker could occur if the account is not adequately collateralized. In

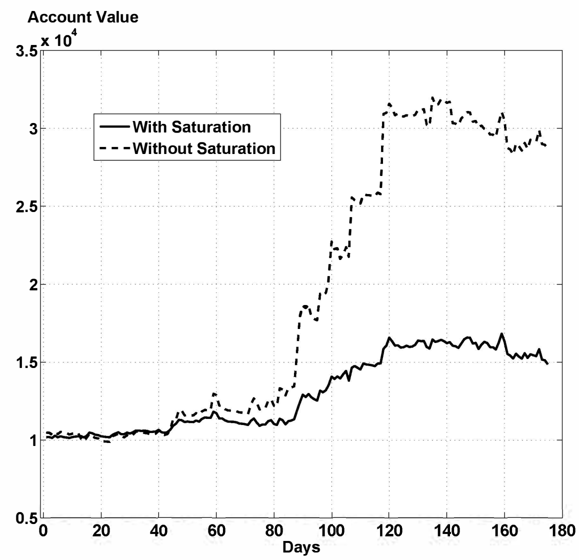

Fig. 4. Cumulative Return for PI Controller

view of these practical considerations, we also considered the case $K_I = K_P = 2$ taking the account value (wealth) of the trader into account in restricting the admissible investment level. If we denote the account value of the trader at time t by $V(t)$, a simple but practical constraint would be the following: Recalling that $I(t) = I_L(t) + I_S(t)$ is the net investment, we impose the constraint $|I(t)| \leq 2V(t)$. In other words, if the ILS control algorithm dictates an investment level no larger than $2V(t)$, we use the saturated control $I_{sat}(t) \doteq 2V(t)\text{sign } I(t)$. In Figure 4, with saturation included, the trading gain is seen to evolve more modestly versus the unconstrained case.

VIII. Conclusion and Further Research

The reader should not erroneously misconstrue the results in this paper to mean that the ILS PI controller is necessarily a good strategy in practice. The main point of this paper is that the analysis is tractable using classical control-theoretic tools. A positive expected value does not necessarily mean that the probability of winning is significant. In fact, based on previous work for the static case, it is known that the probability density function for gains and losses can be highly skewed and that individual sample paths might exhibit significant drawdown; see [5] and [6]. The positive expectation result is useful but in practice, it should also be counterbalanced by including a measure of risk.

By way of future research, it would be of interest to formulate and solve an *optimal gain selection problem* for K_P and K_I which takes both risk and return considerations into account. In other words, we view the results in this paper as analysis tools which will be helpful going forward – as opposed to a recipe for "winning" in the stock market.

For practical purposes, one immediate direction for further work involves generalizing the results in this paper to allow for more emphasis on recent data. That is, to more heavily weight more recent data, it is of interest to consider the so-called *exponentially weighted moving average* with control

$$I(t) = I_0 + K_P g(t) + K_I \int_0^t e^{-\gamma(t-\tau)} g(\tau) d\tau.$$

where $\gamma \geq 0$ is chosen by the trader. Note that by choosing $\gamma = 0$, we recover integral feedback.

The analysis for this more general case begins in much the same way as for $\gamma = 0$. That is, we first define the state

$$x(t) \doteq \begin{bmatrix} \int_0^t e^{-\gamma(t-\tau)} g(\tau) d\tau \\ g(t) \end{bmatrix}$$

and then impose the GBM price dynamics which can be used to develop a state-space model for $\bar{x} = \mathbb{E}[g(t)]$. Via a straightforward calculation, this turns out to be $\dot{\bar{x}} = A_\gamma \bar{x} + bu$ where and b and u are the same as in the earlier analysis and

$$A_\gamma \doteq \begin{bmatrix} -\gamma & 1 \\ \mu K_I & \mu K_P \end{bmatrix}$$

has eigenvalues that are readily found. Then, the analysis of this second order system can be continued in much the same manner as was done for the uniformly weighted case.

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