

**Johnson SU distribution**  
**MLE (Maximum Likelihood Estimate) fit to determine parameters**  
**LR (Likelihood Ratio) approach to find tolerance limit**

## Pseudo-Code Used to Describe Process

### Input

```
x      <- iris$Sepal.Width
sided  <- 1
alpha  <- 0.01 # confidence = 1 - alpha / sided
P      <- 0.99 # proportion or coverage
```

```
> x
 [1] 3.5 3.0 3.2 3.1 3.6 3.9 3.4 3.4 2.9 3.1 3.7 3.4 3.0 3.0 4.0 4.4 3.9 3.5 3.8 3.8
[21] 3.4 3.7 3.6 3.3 3.4 3.0 3.4 3.5 3.4 3.2 3.1 3.4 4.1 4.2 3.1 3.2 3.5 3.6 3.0 3.4
[41] 3.5 2.3 3.2 3.5 3.8 3.0 3.8 3.2 3.7 3.3 3.2 3.2 3.1 2.3 2.8 2.8 3.3 2.4 2.9 2.7
[61] 2.0 3.0 2.2 2.9 2.9 3.1 3.0 2.7 2.2 2.5 3.2 2.8 2.5 2.8 2.9 3.0 2.8 3.0 2.9 2.6
[81] 2.4 2.4 2.7 2.7 3.0 3.4 3.1 2.3 3.0 2.5 2.6 3.0 2.6 2.3 2.7 3.0 2.9 2.9 2.5 2.8
[101] 3.3 2.7 3.0 2.9 3.0 3.0 2.5 2.9 2.5 3.6 3.2 2.7 3.0 2.5 2.8 3.2 3.0 3.8 2.6 2.2
[121] 3.2 2.8 2.8 2.7 3.3 3.2 2.8 3.0 2.8 3.0 2.8 3.8 2.8 2.8 2.6 3.0 3.4 3.1 3.0 3.1
[141] 3.1 3.1 2.7 3.2 3.3 3.0 2.5 3.0 3.4 3.0
```

### Determine Johnson Su Parameters (this step not needed other than for comparison)

```
## define nll (negative log likelihood) function to fit
## parameters: gamma, delta, xi, lambda
nll <- function() {
  pdf <- delta / ( lambda * sqrt(2 * pi) ) *
    1 / sqrt(1 + ( (x-xi)/lambda )^2 ) *
    exp( -0.5*(gamma + delta * asinh( (x-xi)/lambda ))^2 )
  nll <- -sum(log(pdf))
}
```

```
## MLE fit to minimize nll() returns gamma, delta, xi, and lambda
```

	gamma	delta	xi	lambda	quant
standard fit	-3.306484	5.319412	1.784619	1.887725	NA

### Determine Equivalent Johnson SU Fit Using Quantile as a Parameter Instead of Gamma

```
## define nll.q function to fit alternate parameters: quant, delta, xi, lambda
## where P = coverage
## quant = quantile associated with coverage
nll.q <- function() {
  gamma <- qnorm(P) - delta * asinh( (quant-xi)/lambda )
  pdf <- delta / ( lambda * sqrt(2 * pi) ) *
    1 / sqrt(1 + ( (x-xi)/lambda )^2 ) *
    exp( -0.5*(gamma + delta * asinh( (x-xi)/lambda ))^2 )
  nll.q <- -sum(log(pdf))
}
```

```
## MLE fit to minimize nll.q() returns quant, delta, xi, and lambda for given P
```

	gamma	delta	xi	lambda	quant
standard fit	-3.306484	5.319412	1.784619	1.887725	NA
fit on quantile at 1-P	-3.306500	5.319415	1.784613	1.887714	2.134413
fit on quantile at P	-3.307633	5.319502	1.784351	1.887396	4.178574

## Calculate Confidence Limits Using LR (Likelihood Ratio)

```
## Find peak (ll.max) of log likelihood function (-nll.q())
```

```
ll.max = -86.55915
```

```
## Reduce peak by chi-squared
```

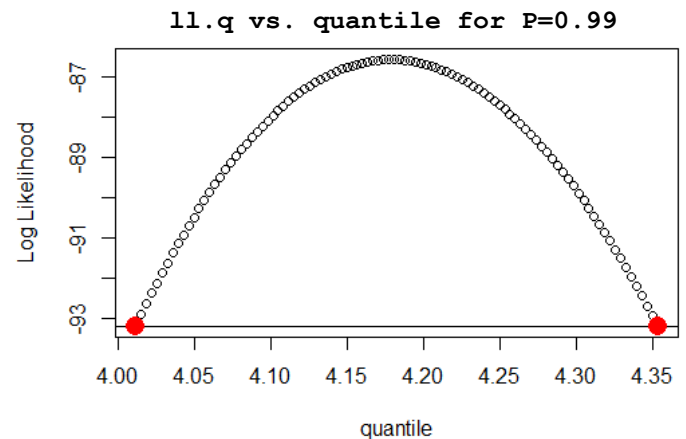
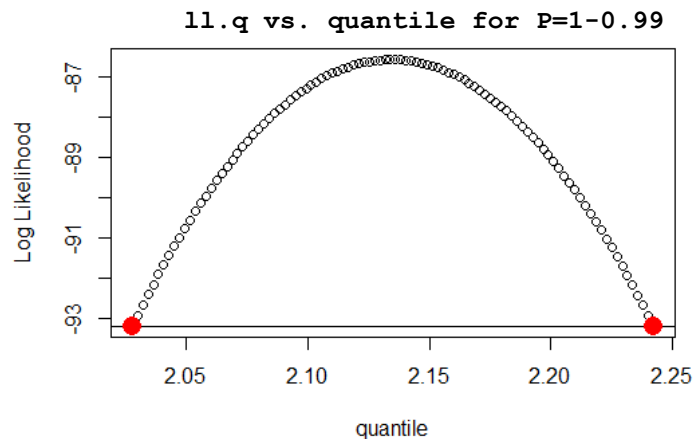
```
ll.tol <- ll.max - qchisq(1 - alpha/sided, 1) # qchisq(1-0.01/1, 1) = 6.634897
```

```
ll.tol = -93.19405
```

```
## confidence limits are the intersection of ll.tol and the log likelihood function
```

```
## for given level of coverage, P
```

```
ll.q <- -nll.q(x, P, quantile, delta, xi, lambda)
```



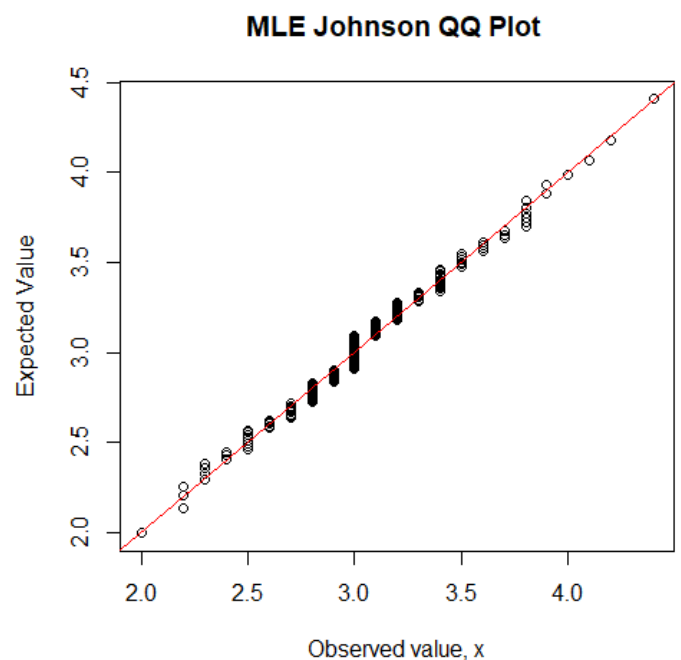
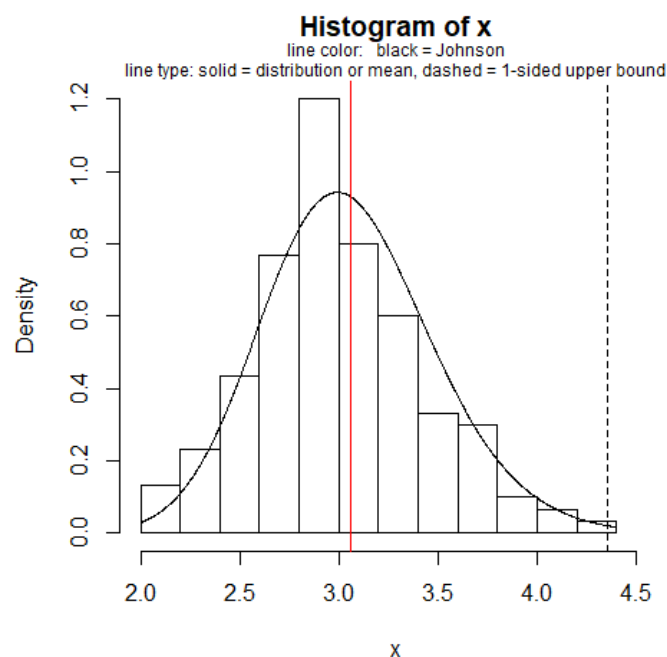
```
Final confidence interval for P= 0.01
2.027558 2.242365
```

```
Final confidence interval for P= 0.99
4.011839 4.352955
```

```
## Based on the above:
```

```
alpha  P sided tol.lower tol.upper
1 0.01 0.99 1 2.027558 4.352955
```

## Histogram with Upper, 1-Sided, 99/99 Tolerance Limit and QQ Plot



## Actual Coding Used to Recreate Above Output

```
## put common packages and functions defined in folder "modules" into namespace
## "modules" folder contains the two functions used below:
##     mle.johnsonsu() and hist_nwj()
source("setup.r")

## define input
x     <- iris$Sepal.Width

## use MLE to determine fit and LR to determine confidence limits
## using defaults: alpha=0.01, P=0.99, and sided=1
out.fit <- mle.johnsonsu(x, plots=TRUE)

## create histogram with Johnson SU fit (type='j') and show upper tolerance limit
out.hist <- hist_nwj(x, type='j')

## create QQ plot for Johnson SU fit
out.qq   <- qqplot_nwj(x, type='j')
```

## Acknowledgement

The approach described in the pseudo-code section is based on the approach found here:  
[https://personal.psu.edu/abs12/stat504/Lecture/lec3\\_4up.pdf](https://personal.psu.edu/abs12/stat504/Lecture/lec3_4up.pdf)