Geometric Proofs of Dot and Cross Product Distributivity

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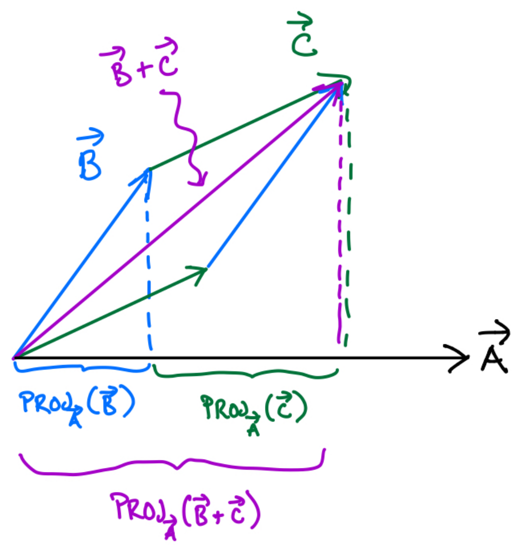
1 Dot Product Distributivity

By definition, the projection of a vector $\vec{v}$ onto a vector $\vec{u}$ is:

$$\text{proj}_{\vec{u}}(\vec{v}) = (\vec{v} \cdot \vec{u})\vec{u} \tag{1}$$

Referring to the figure below,



it is clear that

$$\text{proj}_{\vec{A}}(\vec{B} + \vec{C}) = \text{proj}_{\vec{A}}(\vec{B}) + \text{proj}_{\vec{A}}(\vec{C})$$

So, using equation (1), the above is equivalent to:

$$\left[(\vec{B} + \vec{C}) \cdot \vec{A}\right] \vec{A} = (\vec{B} \cdot \vec{A})\vec{A} + (\vec{C} \cdot \vec{A})\vec{A}$$

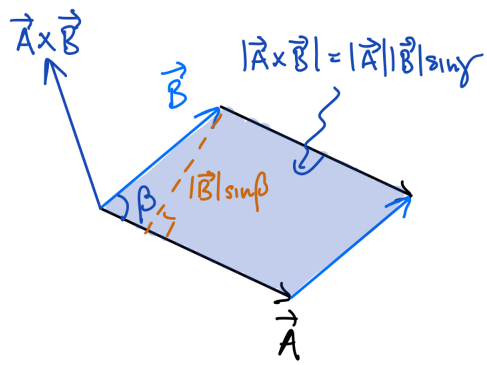
In order for the above to hold (for any non-zero $\vec{A}$), it is clear that the following must be true:

$$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C} \tag{2}$$

Thus, the dot product is distributive.

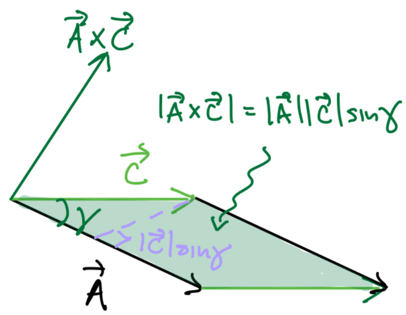
2 Cross Product Distributivity

Consider vectors $\vec{A}$ and $\vec{B}$ such that they form the plane shown in the following figure.



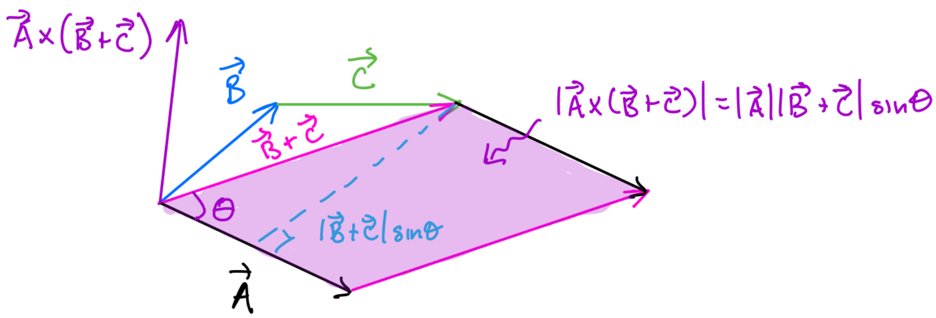
I'll say β is the angle between $\vec{A}$ and $\vec{B}$, so that a line drawn from the tip of $\vec{B}$ perpendicular to $\vec{A}$ has a length of $|\vec{B}|\sin\beta$. The area of this plane, as given by the cross product, is $|\vec{A}||\vec{B}|\sin\beta$.

In the next figure, consider a vector $\vec{C}$ such that it forms the following plane with the vector $\vec{A}$.



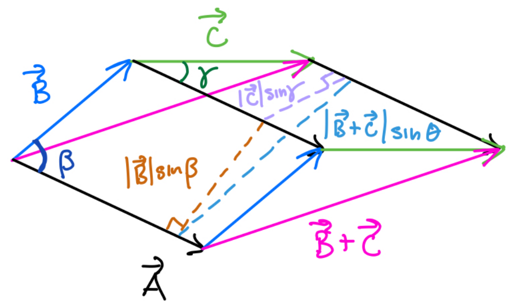
I'll say γ is the angle between $\vec{A}$ and $\vec{C}$, so that a line drawn from the tip of $\vec{C}$ perpendicular to $\vec{A}$ has a length of $|\vec{C}|\sin\gamma$. The area of this plane, as given by the cross product, is $|\vec{A}||\vec{C}|\sin\gamma$.

Next, adding $\vec{B}$ and $\vec{C}$ allows us to form a plane between $\vec{A}$ and $\vec{B}+\vec{C}$, shown in the following figure.

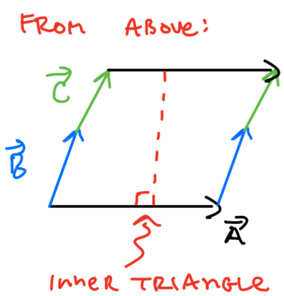


As with the previous two planes, I'll say θ is the angle between $\vec{A}$ and $\vec{B}+\vec{C}$, so that a line drawn from the tip of $\vec{B}+\vec{C}$ perpendicular to $\vec{A}$ has a length of $|\vec{B}+\vec{C}|\sin\theta$. The area of the plane is also just the cross product of the two vectors, $|\vec{A}||\vec{B}+\vec{C}|\sin\theta$.

Putting the four vectors, $\vec{A}$, $\vec{B}$, $\vec{C}$, and $\vec{B}+\vec{C}$, together forms a prism, as shown in the following figure. Within this prism, I can define a triangle by drawing a line perpendicular to $\vec{A}$ within the plane of $\vec{A}$ and $\vec{B}$, drawing a second line perpendicular to $\vec{A}$ within the plane of $\vec{A}$ and $\vec{C}$, and drawing a final line perpendicular to $\vec{A}$ within the plane of $\vec{A}$ and $\vec{B}+\vec{C}$. I'll call this the "inner" triangle, while the sides of the prism (formed by the vectors $\vec{B}$, $\vec{C}$, and $\vec{B}+\vec{C}$) I'll call the "outer" triangles.



Something that's easy to miss in this figure is that the inner triangle and the outer triangles are not the same. This is more easily seen taking a view "from above" the prism, as shown in the next figure.



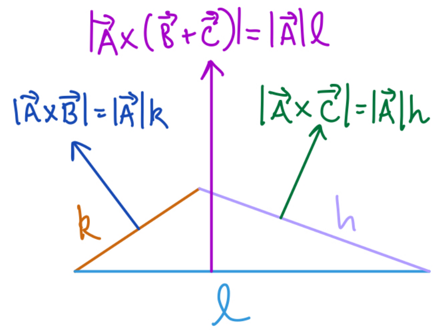
For simplicity while finishing the proof, I'll make the following substitutions:

$$|\vec{B}|\sin\beta = k \tag{3a}$$
$$|\vec{C}|\sin\gamma = h \tag{3b}$$
$$|\vec{B}+\vec{C}|\sin\theta = l \tag{3c}$$

I can plug these substitutions into the three cross products we are interested in:

$$|\vec{A}\times\vec{B}| = |\vec{A}|k \tag{4a}$$
$$|\vec{A}\times\vec{C}| = |\vec{A}|h \tag{4b}$$
$$|\vec{A}\times(\vec{B}+\vec{C})| = |\vec{A}|l \tag{4c}$$

Now I will draw the inner triangle from a "side" view. This is shown in the following figure. Notice that I've also included the three cross products we are interested in, which were included in the first three figures of this section. The important thing to consider here is that each of these cross products lies in the plane of this triangle. $\vec{A}\times\vec{B}$ and $\vec{A}\times\vec{C}$ both lie in the plane because they are (obviously) perpendicular to $\vec{A}$. This triangle was drawn specifically so that its plane is perpendicular to $\vec{A}$, so the two cross products lie in the same plane. The vector $\vec{A}\times(\vec{B}+\vec{C})$ lies in the plane of this triangle because it is perpendicular to the plane formed by $\vec{A}$ and $\vec{B}+\vec{C}$, as is the triangle.



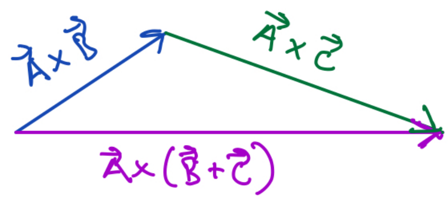
From this inner triangle, it is clear, according to the Pythagorean theorem, that

$$l^2 = k^2 + h^2$$

which also means that

$$|\vec{A}|^2 l^2 = |\vec{A}|^2 k^2 + |\vec{A}|^2 h^2 \tag{5}$$

For the last step, imagine rotating the vector $\vec{A}\times(\vec{B}+\vec{C})$ by 90^o clockwise. This would cause it to lie on the same line as l in the above figure. Likewise, rotating $\vec{A}\times\vec{B}$ and $\vec{A}\times\vec{C}$ by the same amount will cause them to lie on the same lines as k and h, respectively. This leads us to one final triangle, shown below.



I've drawn this as a closed triangle, when I've made no explicit guarantees that the end points of the vectors line up as such. However, look at equation (5): this says that these three vectors satisfy the Pythagorean theorem, and thus should form a proper triangle. Likewise, according to the above figure, the following must be true,

$$\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C} \tag{6}$$

proving that the cross product is distributive.