PHY2048 Spring 2018 Homework Assignment #2

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Abstract

In this homework assignment, you're going to be solving problems on the various forms of mathematics we reviewed in the first couple of lectures. This homework set is due **January 17**.

1. Take the first derivative, with respect to x, of the function $f(x) = x^4 - 7x^3 + 4x^2 - 7$.

2. Take the first and second derivatives, with respect to y, of the function $f(y) = 3y^2 \ln y$.

3. Integrate the function $f(x) = 3x^2 - 4x$ from 0 to 4.

4. This problem is going to be all about **interpretation of a physical results**, so I need to define a few things first. Consider a sky-diver dropping out of an airplane at some initial height H. We'll assume, for practical reasons, that his chute is opened immediately after jumping out of the airplane, so his chute is open at a height H. We'll define the **distance he falls** as $y(t)$, where y is a positive quantity when he moves downwards. The **speed at which he falls**, $v(t)$, is the rate of change of his position, $y(t)$, so:

$$v(t) = \frac{dy}{dt}$$

We'll discuss how to get this result later on in the course, but for now, let's say "due to physics," his velocity as a function of time looks like:

$$v(t) = Ae^{-\lambda t} + B, \quad \text{where } \lambda > 0$$

The last thing we need is the idea of an **initial value**. Recall that every time you take an indefinite integral, you pick up an integration constant, e.g.:

$$y(t) = \int f(t)dt = F(t) + C$$

There are multiple ways of dealing with the constant C (in your calculus class, you more-than-likely used a definite integral). What physicists like to use are **initial values**: if we know that, initially, $y(t = 0) = 0$ (a popular initial value), and we define $F(t = 0) = F_0$, then:

$$y(t = 0) = F(t = 0) + C = 0 \;\Rightarrow\; C = -F_0 \;\Rightarrow\; y(t) = F(t) - F_0$$

and your integration constant is gone! The two initial values in this problem are easy to come up with because they're the only ones that **make physical sense**:

$$y(t = 0) = H$$

$$v(t = 0) = 0$$

That is, the sky diver starts at rest at a height of H, exactly as we would expect. Now that we have this background information (you might need to read over it a few times), answer the following questions (make sure you **justify each answer**):

(a) What is the physical interpretation of B? *Hint: you can figure out what **type of quantity** B is by figuring out its units; for instance, if you figure out its units are seconds, then it's a time value. You can determine the **value** of B by taking the limit at $t \to \infty$. The combination of what type of quantity and its asymptotic value is enough to give a physical meaning to B.*

(b) What is the physical interpretation of λ? *You should phrase your answer not as "λ measures blah blah because it has units of blah blah." Rather, you should phrase your answer as "because of the way λ **affects the behavior of $y(t)$**, it's clear that λ **depends** on blah blah." Also, the best way to tell the behavior of $y(t)$ is to remember what the function e^{-x} looks like.*

(c) Figure out the value of A by using the initial values.

(d) Perform an indefinite integral on $v(t)$ to find $x(t)$. What is the interpretation of the integration constant? What is the value of the integration constant?

5. For some functions, the usual differentiation rules don't work. One trick for solving some of these derivatives is known as **logarithmic differentiation**: first you take the logarithm of the function, so that if $f(x) = y$ (where y just represents the function of x, e.g. $y = 2x^2$) then $\ln f = \ln y$. You then take the *implicit* derivative of both sides, such that:

$$d(\ln f) = d(\ln y) \;\;\Rightarrow\;\; \frac{1}{f}df = \mathsf{implicit}(\ln y)dx$$

where $\mathsf{implicit}(\ln y)$ is just the implicit derivative of $\ln y$, which is a function of x, thus the factor of dx (e.g. for $y = 2x^2$, $\mathsf{implicit}(\ln y) = \mathsf{implicit}(\ln(2x^2) = (1/2x^2) * 4xdx = 2x^{-1}dx)$. Then, you can simply solve for df/dx:

$$\frac{df}{dx} = f(x)\mathsf{implicit}(y) = y * \mathsf{implicit}(y)$$

where you'd substitute in whatever y is (e.g. if $y = 2x^2$, $df/dx = (2x^2)(2x^{-1}) = 4x$, which is exactly what you'd get by just applying regular polynomial differentiation). Using the process of logarithmic differentiation, prove that:

$$\text{for } f(x) = \ln\left(3^{x^2}\right), \qquad \frac{df}{dx} = 2\ln(3)x3^{x^2}$$

Note that you couldn't solve this differentiation with the typical approach: chain rule. Chain rule would require you to take the derivative of the "function" 3^x, but the derivative of 3 is zero, because it's a constant, so it would seem like a dead end. Logarithmic differentiation is the only way to approach this problem.

6. For the following function:

$$f(x) = 2x^4 - 3x^3 + 7x^2 + 1$$

answer the following questions:

(a) How many local extrema do you expect $f(x)$ to have? Justify your answer.

(b) What are the positions of the local extrema?

(c) Which of these positions are local minima? Which are local maxima? Are any points false extrema? Once again, justify your answers.

7. A homeowner is painting her house. At some point, she's standing on a ladder, which leans against a wall, when the ladder starts slipping. At this point, the top of the ladder is 5ft off the ground and the bottom of the ladder is 1ft away from the wall. How fast would the bottom of the ladder be sliding away from the wall if the top of the ladder fell at 1ft per 20s?

8. Consider three vectors:

$$\vec{a} = \hat{\imath} + 3\hat{\jmath}$$

$$\vec{b} = -4\hat{\imath} + 2\hat{\jmath}$$

$$\vec{c} = 5\hat{\jmath}$$

Given these vectors, compute the following quantities:

(a) $\vec{d} = \vec{a} + \vec{b} + \vec{c}$

(b) θ, the angle between $\vec{a}$ and $\vec{b}$

(c) ϕ, the angle of $\vec{d}$ measured **counter-clockwise** from the $+x$-axis.

9. Consider the two vectors:

$$\vec{A} = 3\hat{\imath} - 2\hat{\jmath}$$

$$\vec{B} = 6\hat{\jmath} + \hat{k}$$

Compute the following quantities:

(a) $|\vec{A} + \vec{B}|$

(b) $\vec{A} \cdot \vec{B}$

(c) $\vec{A} \times \vec{B}$

(d) θ, the angle between $\vec{A}$ and $\vec{B}$

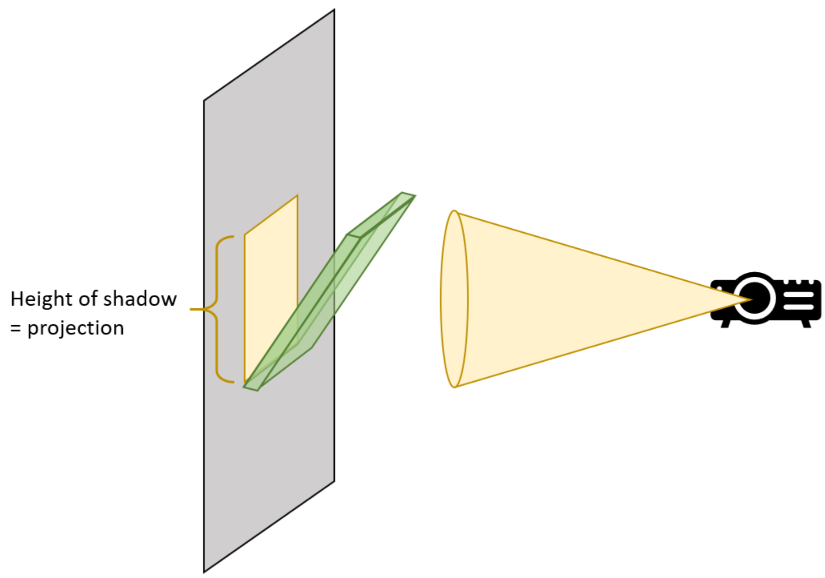
10. Consider the two vectors:

$$\vec{A} = 4\hat{\imath}$$

$$\vec{B} = 2\hat{\imath} + 3\hat{\jmath}$$

(a) Draw the two vectors in an xy-coordinate system.

(b) Define the **projection of $\vec{B}$ onto $\vec{A}$** as the length of $\vec{B}$ that runs parallel to $\vec{A}$. What is the projection of $\vec{B}$ onto $\vec{A}$ in this case? To help visualize what the projection of a vector represents, imagine an *actual* projector shining a light on a plank of wood, as shown in the following figure. The shadow of that plank of wood is what the projection projection represents. However, to imagine a vector in this analogy, we'd need the width of the board to be zero; so the projection of a vector (the plank of wood) onto another vector (the wall) is analogous to the *height/length* of the shadow.


Height of shadow
= projection