PHYS2350 Final Exam Content Review

Section EV1, Fall 2017

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Abstract: This is a quick review of every equation in the formula sheet for the final exam for PHYS2350 EV1 during the Fall 2017 semester at Nova Southeastern University. The majority of the material hasn't been seen in a while, obviously, so it would be well worth your time to review what each formula means, when it can be applied, and the common problems it is applied to. I've done my best to review each formula, and the context in which the formulae apply, as quickly as possible, giving quick illustrative examples when necessary. Note that these examples are not meant to be solved in detail, or even solved at all, but are meant simply to provide an example of what a problem would look like. There is plenty of review material for the specific problems you are likely to see on the final exam on the course website:

http://www.blazartheory.com/teaching/phys2350fall17.html

Contents

1 Vectors

While the odds are very low that there will be a question that tests on vectors alone, the two equations presented on the formula sheet are for the dot and cross product of vectors. Remember that these are the two types of multiplication for vectors, with the **dot product** measuring how **parallel** two vectors are, and the **cross product** measuring how **perpendicular** two vectors are. The result of the dot product is a **scalar**, while the result of the cross product is a **vector**. This is why the dot product equation gives $\vec{A} \cdot \vec{B}$ directly, but the cross product equation only gives the *magnitude* of the cross product $|\vec{A} \times \vec{B}|$.

The figure included with the formulae gives the **cyclic permutations** of the cross product. Whenever computing a cross product, you distribute the components of a vector like any multiplication:

$$(3\hat{\imath} + 2\hat{\jmath}) \times (\hat{\imath} + 4\hat{k}) = (3\hat{\imath} \times \hat{\imath}) + (3\hat{\imath} \times 4\hat{k}) + (2\hat{\jmath} \times \hat{\imath}) + (2\hat{\jmath} \times 4\hat{k})$$

The only thing that's different about the cross product is how to compute the unit vector multiplications, e.g. $\hat{\imath} \times \hat{k}$. This is where the cyclic permutations come into play. You trace a circle in the direction of the first unit vector, e.g. $\hat{\imath}$, to the second unit vector, e.g. $\hat{k}$. This will tell you the **sign** of the cross product; the output will be the third unit vector, e.g. $\hat{\jmath}$, since the output of a cross product must be perpendicular to both vectors that are input (y is the only direction that's perpendicular to both x and z). The direction $\hat{\imath} \to \hat{k}$ is the negative direction, so $\hat{\imath} \times \hat{k} = -\hat{\jmath}$. Do this for each term and add, and you have the result of the cross product.

2 Kinematics

The most important thing to remember about kinematics is that **it only applies to cases of constant acceleration**. If the forces acting on an object are constant, you can use kinematics; if the object is sliding down a non-curved slope, you can use kinematics; if the problem doesn't imply anything about a varying acceleration, you can use kinematics. The most common problems that you **cannot** use kinematics for are ones where the motion is along a curved path, or problems where you can't assume the path is straight, because the acceleration must be changing in these problems. For instance, an object sliding down a curved slope, the swing of a pendulum, or the motion of a mass on a spring all represent problems where the acceleration isn't constant, so kinematics cannot be used.

Kinematics is one of the few methods of solving problems that **includes time**. When using energy conservation or momentum conservation, the solutions won't tell you a time interval for the motion. In kinematics, you can find or use the duration of the motion in the problem. So, broadly speaking, if the problem involves time, you should consider kinematics as a possible way to solve the problem. This obviously doesn't include problems like: "What is the average force applied when a bat delivers an impulse of J to a baseball over a time interval Δt?" This problem would be typically solved using $F = \frac{J}{\Delta t}$. (Recall that impulse is a change in momentum, so $J = \Delta p$.)

Kinematics involves 5 variables: distance Δx, initial speed v_0, final speed v, acceleration a, and time t. In order to solve a kinematics problem, you **need to know three** of these variables; then you can solve for either of the other two. Sometimes values of these variables might be implied: an object that starts from rest has $v_0 = 0$; and object that comes to a stop has $v = 0$; and object that is falling has $a = g$; etc. Once you know your three variables and the variables you're solving for, you'll have **one leftover variable**: the one you don't care about. You want to choose one of the kinematics equations that doesn't have the variable you don't care about. The three common kinematics equations are:

$$\begin{aligned}
\Delta x = v_0 t + \frac{1}{2}at^2 &\quad\leftarrow\quad \text{independent of final speed } v \\
v = v_0 + at &\quad\leftarrow\quad \text{independent of distance } \Delta x \\
v^2 = v_0^2 + 2a\Delta x &\quad\leftarrow\quad \text{independent of time } t
\end{aligned}$$

3 Forces

It's important to remember **Newton's three laws**:

1. An object will maintain its current state of motion (i.e. it's velocity $\vec{v}$) unless acted upon by a force.

2. The net force on an object is the rate of change of the object's momentum:

$$\sum \vec{F} = \frac{\Delta \vec{p}}{\Delta t}$$

3. For any force an object could put on a second object, the second object would put an equal and opposite force back on the first.

The first law just says an object will move in a straight line with a constant speed (which could be zero) unless acted upon by a force; this is often phrased (especially by me) as "an object *wants* to move in a straight line." The second law is the mathematical description of a force. Don't forget the more common representation, which is for the case when **mass is constant**:

$$\sum \vec{F} = m\vec{a}$$

And the third law defines a force as an interaction between two objects; there *have* to be two objects acting on one another for there to be a force. Pseudoforces like the Coriolis force or the centrifugal force (which aren't of interest in this course, but act as good examples) involve only a single object, and so violate Newton's third law and thus aren't real forces; sometimes they're referred to as "fictitious forces" to emphasize this fact. Newton's third law defines the concept of an **action-reaction pair** of forces, which can be a tricky concept. Remember that an action-reaction pair always involves **two objects** which experience the **same type of force** (e.g. gravity, a contact force, etc.) that are **equal in magnitude and opposite in direction**. For example, a person standing on a flat surface experiences a weight and a normal force, which are equal in magnitude and opposite in direction, but the weight and normal force are *not* an action-reaction pair; they both act on the person (i.e. only one object, not two) and they're different types of forces (one is gravity and the other is a contact force).

Typically, Newton's second law is used to find thee magnitude of the acceleration on an object, and then kinematics can be used to solve for different aspects of the motion produced by those forces. So **kinematics and forces are typically linked**. Free body diagrams are usually used to solve Newton's second law because it allows for easy vector addition. When drawing a free body diagram, you always consider 3 categories of forces:

1. Forces you're told are there (a pushing force, friction, etc.)
2. Forces you know are there (gravity, a spring force, etc.)
3. Forces that have to be there to satisfy Newton's second law; the so called **adaptive forces** (normal force, tension, static friction, etc.)

Think about these three categories of forces when drawing your free body diagram; if a force doesn't belong to any of these three categories, it doesn't belong on your free body diagram.

Whenever using Newton's second law, you always have to **be aware of the motion of the object**. Newton's second law links forces and motion. If you know an object is at rest, the net force *must* be zero. If you know an object is sliding along a flat surface, you know the vertical acceleration *must* be zero. Etc.

There are forces that can be described by an equation. The **weight** of an object is:

$$W = mg$$

and the **kinetic friction** (friction due to **sliding**) is:

$$f_k = \mu_k N$$

The **static friction** (friction to **prevent sliding**) is more complicated because it's an adaptive force. However, static friction can only get so large, so we define a **maximum** static friction:

$$f_{s,max} = \mu_s N$$

This equation is **not true in general**; in general, the static friction is less than or equal to this value. **Only** in the case of maximal static friction is this equation true; if the static friction is less than maximal, it's an adaptive force and the *only* way to solve for it is by solving Newton's second law.

4 Circular Motion

Typically, the circular motion considered is **uniform circular motion**, which is circular motion at a **constant speed**. Note that this is constant *speed*, not constant *velocity*; in order to move in a circle, the direction of motion must be continuously changing, so the direction of the velocity vector is continuously changing, meaning the velocity isn't constant. This means that there must be an acceleration on an object to undergo uniform circular motion. This is known as the **centripetal acceleration** or the **radial acceleration**:

$$a_c = \frac{v^2}{r}$$

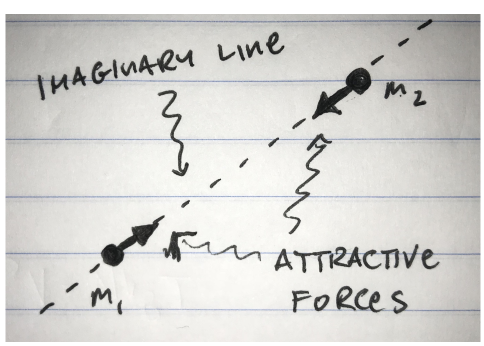
where r is the radius of the orbit. Note that in uniform circular motion, r is a constant (the definition of a circle) and v is a constant (the definition of uniform in uniform circular motion), so the **magnitude of the acceleration** is a constant. However, like the velocity, the acceleration isn't a constant because the direction is continuously changing. The acceleration always points **towards the center of the orbit**, and this direction is constantly changing as the object moves along the circle.

Because the motion is uniform, it's **periodic**, meaning that it repeats itself in uniform intervals. The **period** of an orbit is the amount of time it takes to complete one orbit. For example, and object moving at 20m/s will complete one circular orbit of .5m radius in 0.16s. This is just an application of speed = distance/time, with the distance being the circumference $2\pi r$ and the time being the period T. This is just the equation:

$$v = \frac{2\pi r}{T}$$

5 Gravity

The force of gravity is the only force in physics 1 that acts over a distance, i.e. is a non-contact force. Gravity is **always attractive**, meaning it pulls two masses towards each other. For any two masses, you can draw an imaginary line between them; the force of gravity will always act along this line, point towards each other. This can be seen in the following figure.



So that's the direction of the gravitational force; the magnitude is given by **Newton's law of gravitation**:

$$F = G\frac{m_1 m_2}{r^2}$$

where G is the gravitational constant and r is the **center-to-center distance**. I drew each mass as a point in the above figure, but often times you won't be considering two point-masses, but one very large mass and a point mass. Imagine, for instance, a satellite orbiting the Earth. The large, **attracting mass** will have a noticeable size that has to be taken into account. For example, if that satellite were at an **altitude** (the height above the surface) of 3 times the radius of the Earth, the center-to-center distance r is *four* times the radius of the Earth.

These types of satellite problems are the most common in gravitation. In this case, we'd consider one very large mass M, the attracting body, and one very small mass m, the satellite. The satellite is going to experience a noticeable acceleration, while the attracting body will be so massive that its acceleration will essentially be zero. In this case, we'd write Newton's law of gravitation in the form given in the formula sheet:

$$F = G\frac{mM}{r^2}$$

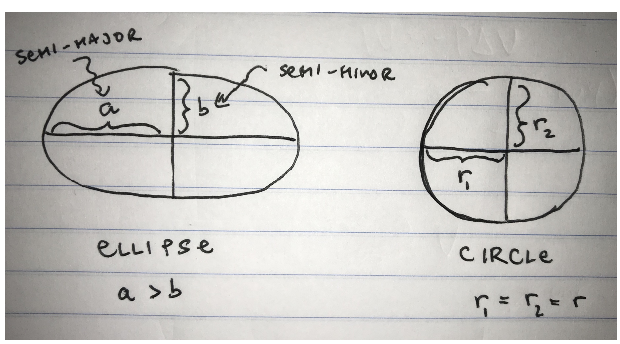
Since the satellite m is the mass accelerating, plugging the above force into Newton's second law will give us the acceleration of the satellite:

$$a_g = G\frac{M}{r^2}$$

These satellite motion problems are typically examples of uniform circular motion, so the gravitational acceleration can be set equal to the centripetal acceleration. Using the equation for the orbital speed in uniform circular motion, $v = \frac{2\pi r}{T}$, one can derive **Kepler's third law**:

$$T^2 = \frac{4\pi^2}{GM}a^3$$

where a is the **semi-major** axis of an elliptical orbit.



Satellite orbits are typically **ellipses, not circles**. An ellipse, as shown in the figure above, is essentially a circle with a non-constant radius, where **the largest radius is perpendicular to the smallest radius**. The semi-major axis of an ellipse is the largest radius while the semi-minor axis is the smallest. In a circle, both perpendicular radii are the same, so the semi-major axis is just equal to the radius of the circle $a = r$. It's important to remember that Kepler's third law will always be applied to circular orbits (at least in this course), so a is treated as the radius of the circle.

6 Work & Energy

There are three types of (mechanical) energy that we care about: **kinetic energy** K (the energy of motion), **potential energy** U (stored energy which can be converted into kinetic energy), and **total mechanical energy** E (the sum of kinetic and potential energy). The kinetic energy of **translational motion** is always given by:

$$K = \frac{1}{2}mv^2$$

The equation potential energy **depends on the force that produces the potential energy**. Only **conservative** forces produce a potential energy; non-conservative forces do not. By definition, conservative forces **conserve total mechanical energy**, while non-conservative forces don't (hence the names). What this means is that during some motion, if an object is only acted upon by conservative forces, its total energy E is the same everywhere along the path it takes and the same at any time during the motion, so if you know E anywhere you know E everywhere.

A consequence of a force being conservative is that the **work**, which is the **change in energy** due to a force, is path-independent; works due to non-conservative forces depend on the particular path taken. This means that if you considered closed loops, the work due to conservative forces will always be zero. You don't actually have to check this for each force you consider, though; there are only **two** conservative forces we consider in this course: **gravity** and the **spring force**. The easiest way to remember if a force is conservative or not is if it has a potential energy associated with it; if it doesn't, it's a non-conservative force. The two potential energies are:

$$U_g = mgy$$

$$U_{sp} = \frac{1}{2}kx^2$$

where y is the height above some chosen zero, k is the force constant (aka the spring constant), and x is the **displacement from equilibrium** (e.g. if a spring is normally 5cm and it's stretched to 6cm, $x = 1$cm). In the form given, U_g is only applicable **near the surface of the Earth**, where $g = 9.8\text{m/s}^2$. The actual form of U_g that works **anywhere** is:

$$U_g = -G\frac{mM}{r}$$

where this is given in the same context as satellite motion: one small satellite of mass m and one large attracting body of mass M. This form is, obviously, more complicated to use than the approximation near the surface of the Earth, which is why it's not typically used. One would use it to calculate the escape velocity of a planet ($K_i = \frac{1}{2}mv_{esc}^2$ and U_i is the potential energy at the surface of the planet, and $K_f = U_f = 0$; solve for v_{esc}) or the potential energy of a satellite in orbit around the Earth, but you'd never use it, say, to calculate the change in potential energy of an object sliding down an incline.

The **work due to any force** can be found using the equation:

$$W = F\Delta x \cos\theta$$

where θ is the angle between the displacement $\Delta\vec{x}$ and the force $\vec{F}$; note that this equation only works for a **constant force along a straight displacement**, meaning that θ will be easily defined. θ, for instance, would not be easily defined if the path was curved, because it would be constantly changing and you wouldn't know what to set it equal to. Of course, in this case, the above equation doesn't apply. Because of this restriction, the above equation can only be used in the case of a constant force along a straight displacement; if this isn't the case for your problem, you need to solve it another way.

Notice that if the force $\vec{F}$ points **along the displacement** – that is, if $-90^o \leq \theta \leq 90^o$ – the work is **positive**, and if the force points **against the displacement**, the work is **negative**. This is because a force along the displacement **increases the kinetic energy** of an object, and so is positive, while a force against the displacement **decreases the kinetic energy** of an object, and so is negative.

I said earlier that a work is a "change in the energy" of the object, but what does this mean? There are three different energies; which does a work change? In this regard, we can think of the types of works: a **total work** W_{tot}, a work due to **conservative forces** W_{cons}, and a work due to **non-conservative or "other" forces** W_{other}. The total work changes the kinetic energy, the conservative work changes the potential energy, and the other work changes the total energy:

$$W_{tot} = \Delta K$$

$$W_{cons} = -\Delta U$$

$$W_{other} = \Delta E$$

The negative sign in the equation for W_{cons} is very important. The conceptual way to think about these relations is this: *all* forces can change the kinetic energy of an object because all forces can produce an acceleration; only *conservative* forces can change the potential energy of an object because non-conservative forces don't produce a potential energy; and only *non-conservative* forces can change the total energy because that's the definition of a non-conservative force.

If the **work due to non-conservative forces is zero**, then we have energy conservation, i.e. that $E_i = E_f$ along the motion, so:

$$K_i + U_i = K_f + U_f$$

A more general equation is what I refer to as the **energy non-conservation equation**, which is true if $W_{other} \neq zero$. In this case, the above equation is modified slightly:

$$K_i + U_i + W_{nc} = K_f + U_f$$

Works due to non-conservative forces are typically difficult to calculate unless the motion is along a straight line and the force is constant, then you could use $W = F\Delta x \cos\theta$. Of course, in this case, **you could also use kinematics** to solve the problem, and you wouldn't need to use energy physics. The more interesting problems are when the forces are not constant, or when the motion isn't along a straight line, because then you *can't* use kinematics. Of course, in this case, you also can't compute W_{other} using $W = F\Delta x \cos\theta$; in these cases, W_{other} would have to be provided for you. For example, if a child went down a slide (don't assume it's non-curved) from some initial height starting at rest, the speed of the child at the bottom of the slide can be solved for using the above equation if $W_{friction}$ was given to you as a number (e.g. -70 J); since the slide isn't assumed to be straight, you wouldn't be able to calculate the work due to friction yourself, though, it would have to be given to you.

The last thing to know is the rate at which energy is changing, known as **power**:

$$P = \frac{\Delta E}{\Delta t}$$

At a constant velocity, an object uses a power equal to:

$$P = Fv$$

For example, to pull an object up an incline at a constant speed of v, a machine must operate at a power $P = mg\sin\theta v$ to combat gravity.

7 Linear Momentum

Momentum is another quantity that is often conserved (like energy), and is defined as:

$$\vec{p} = m\vec{v}$$

Unlike energy, though, momentum is a vector. While total energy is changed by non-conservative forces, momentum is changed by **external forces**:

$$\sum \vec{F}_{ext} = \frac{\Delta \vec{p}_{sys}}{\Delta t}$$

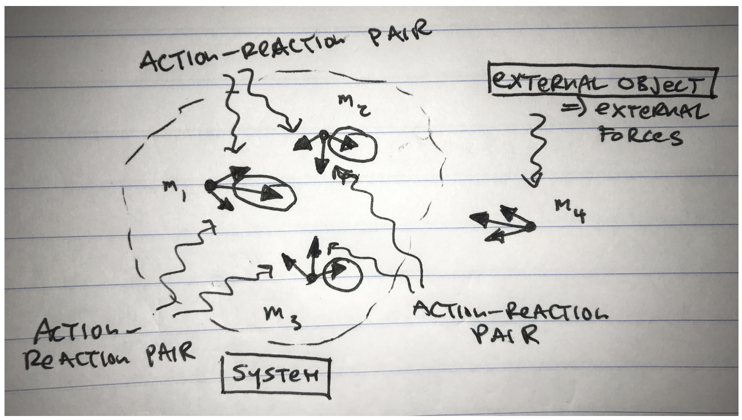
where $\vec{p}_{sys}$ is the momentum of a chosen **system** of objects:

$$\vec{p}_{sys} = \vec{p}_1 + \vec{p}_2 + ...$$

This is a direct consequence of Newton's third law; since all forces have an action-reaction partner, the **net internal force of a system is always zero**.

The figure below illustrates an example of this. If we consider m_1, m_2, and m_3 to be our system, while m_4 is external to the system, then the pairs of forces between m_1 and m_2, m_1 and m_3, and m_2 and m_3 are all action-reaction pairs, and cancel. The only forces that survive in the sum of forces are thus the forces between m_1 and m_4, m_2 and m_4, and m_3 and m_4, which are circled in the figure. Those are all external forces, because they are due to an object which is external to the system. This illustrates the fact that:

$$\sum \vec{F}_{sys} = \cancelto{0}{\sum \vec{F}_{int}} + \sum \vec{F}_{ext} = \sum \vec{F}_{ext}$$



If the net external force on a system is zero, then the **momentum of the system is conserved**. Momentum conservation is most commonly applied to **collisions**. Consider two masses, m_1 and m_2, colliding. Momentum conservation would say that the total momentum of the system, $\vec{p}_{sys} = \vec{p}_1 + \vec{p}_2$, would be a constant, i.e. that $\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}$. Using the definition of momentum, we write this equation as:

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$$

There are **three types of collisions**:

- **Elastic collisions**: Both momentum and kinetic energy are conserved in these collisions. These are special collisions. Don't assume a collision is elastic; a problem *must* specify if a collision is elastic. Instead of using the equation $K_{1i} + K_{2i} = K_{1f} + K_{2f}$ to represent kinetic energy conservation, which is a quadratic equation in velocity, use the **special elastic collision equation**:

  $$v_{1i} - v_{2i} = v_{2f} - v_{1f}$$

  that way you have to solve a system of two linear equations in velocity, instead of a linear and a quadratic equation. Trust me, this is easier. Be very careful with this equation, as well as the momentum conservation equation: v is a **velocity, not a speed**, so the sign matters.

- **Inelastic collisions**: These are your standard collisions, where only momentum is conserved. If a problem doesn't specify or imply any other type of collision, assume it's inelastic. Kinetic energy is always **lost** during an inelastic collision, never gained, and the kinetic energy is lost as **heat**.

- **Perfectly inelastic collisions**: These are inelastic collisions in which the **maximal** amount of kinetic energy is lost during the collision. In practical terms, this is always caused when the two colliding objects **stick together** after the collision. Imagine two cars that are wrecked such that their wreckages stick together, or two sticky balls that stick together after colliding; these are examples of perfectly inelastic collisions. A problem can be explicit in calling a collision perfectly inelastic, or can imply it by stating that the colliding objects stick together-post collision. The equation for a perfectly inelastic collision is just the momentum conservation equation, but you can write it as:

$$m_1 v_{1i} + m_2 v_{2i} = (m_1 + m_2) v_f$$

  since the objects stick together and move with one final velocity v_f post-collision.

A related definition is the **impulse** $\vec{J}$ delivered by a collision:

$$\vec{J} = \Delta \vec{p}$$

Impulse is a lot like work: work is the change in energy due to some force, and impulse is the change in momentum due to some collision. Using Newton's second law, we can say that the impulse delivered by some force $\vec{F}_{av}$ (technically the *average* force during a collision), if the collision took some time Δt, is:

$$\vec{J} = \vec{F}_{av} \Delta t$$

8 Rotational Motion

Because there are a lot of topics in rotational motion, I subdivided the equations into rotational kinematics, rolling without slipping, rotational dynamics, and moment of inertia equations. In rotational motion, almost every equation is an **analogy to a linear motion equation**. For example, the rotational analogue to mass is the moment of inertia I, and the rotational analogue to speed is angular speed ω, so the rotational kinetic energy is $K_{rot} = \frac{1}{2} I \omega^2$, which is just $\frac{1}{2}$ times (rotational) mass times (rotational) speed-squared. Almost every equation is defined exactly like this.

The **big difference** between rotational equations and linear equations is that rotational equations all depend on a **chosen axis of rotation**, so there are many (i.e. infinite) different ways of writing the same equation, each of which is for a different chosen rotational axis. We'll see that most rotational equations depend on the moment of inertia (just like most linear equations depend on the mass), and we only have a select few equations for the moment of inertia, so our choice of rotational axis is going to be **limited by the availability of**

a corresponding moment of inertia equation. For example, if you wanted to write an equation for a disc rotating about a point on its edge (like a disc rolling down a slope, where you consider the rotational axis to be the point of contact between the disc and the ground), you cannot write out your equation because you don't have a corresponding moment of inertia equation for a disc rotating about a point on its edge. All you have is the moment of inertia for a disc rotating about its center, so you're forced to write out your equation for a disc rotating about its center; while you can technically choose *any* rotational axis, you're practically restricted to choosing ones for which you have a corresponding moment of inertia equation.

8.1 Rotational Kinematics

Just as with linear kinematics, you can only use rotational kinematics if the **angular acceleration is constant** for the motion. Also analogous to linear kinematics, there are five variables we care about: angular displacement $\Delta\theta$, initial angular speed ω_0, final angular speed ω, angular acceleration α, and time t. All three rotational kinematics equations are identical to the linear kinematics equations, just with the linear variables replaced by their rotational counterparts:

$$\begin{aligned} \Delta\theta = \omega_0 t + \frac{1}{2}\alpha t^2 \quad &\leftarrow \quad \text{independent of final angular speed } \omega \\ \omega = \omega_0 + \alpha t \quad &\leftarrow \quad \text{independent of angular displacement } \Delta\theta \\ \omega^2 = \omega_0^2 + 2\alpha\Delta\theta \quad &\leftarrow \quad \text{independent of time } t \end{aligned}$$

8.2 Rolling without Slipping

Objects don't have to simply rotate; objects can rotate *while* translating. This type of motion is calling **rolling**, exactly as you'd expect. A special case is that of **rolling without slipping**, which is when the point of contact between the object never slips along the ground. This occurs when static friction is large enough to prevent slipping. Often times problems will explicitly state that an object is rolling without slipping, but unless a problem **states that the object is slipping**, always assume it's rolling without slipping.

In the case of rolling without slipping, we can relate the rotational kinematic variables to the linear kinematic variables:

$$\begin{aligned} \Delta x &= R\Delta\theta \\ v &= R\omega \\ a &= R\alpha \end{aligned}$$

8.3 Rotational Dynamics

The rolling without slipping equations are very important when considering energy conservation for a rolling object. When an object, say, rolls down a hill of height h, all that initial potential energy $U_i = mgh$ becomes kinetic energy at the bottom of the hill, but it becomes a **mixture of translational and rotational** kinetic energy:

$$mgh = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2$$

where the rotational kinetic energy,

$$K_{rot} = \frac{1}{2}I\omega^2$$

is identical to the translational kinetic energy, with all linear variables replaced by their rotational counterparts.

The ability to relate ω and v means that you can re-write the above equation entirely in terms of one or the other. If a problem asks for you to solve for the angular speed at the bottom of the hill, you can use the equation:

$$mgh = \frac{1}{2}\left(I + mR^2\right)\omega^2$$

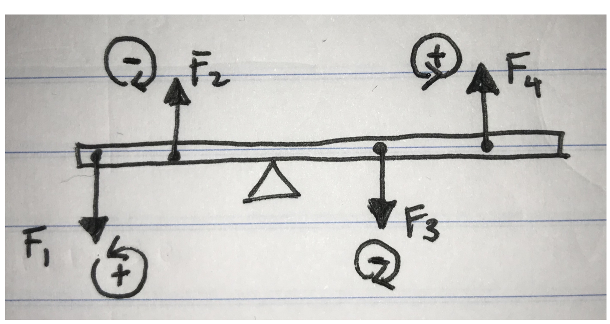
If the problem asks for the linear speed at the bottom of the hill, you can use the equation:

$$mgh = \frac{1}{2}\left(\frac{I}{R^2} + m\right)v^2$$

In linear motion, a force produces an acceleration, while in rotational motion a **torque** τ produces an angular acceleration. There is a Newton's second law-like equation, which is typically called the **rotational Newton's second law** (or some variant of this):

$$\sum \tau = I\alpha$$

Note that, like all our equations, this is just Newton's second law with its linear variables replaced by their rotational counterparts. Notice that I've written this as a non-vector equation. The torque τ and the angular acceleration α *are* vector quantities, but their directions are difficult to determine. A way to get around this is that if a torque produces a rotational motion in one direction (say counterclockwise) then any torque which produces a rotational motion in the opposite direction (say clockwise) will have an opposite sign.

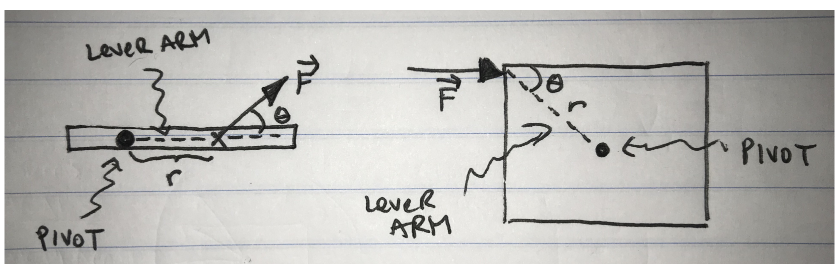


In the above figure, four forces act at various positions on a lever, with the fulcrum of the lever being the triangle near the center. F_1 is going to produce a counter-clockwise rotation on the lever, which I call a positive torque τ_1. As for any vector, which direction is positive is your choice; I've simply chosen counter-clockwise to be positive. As a result, the torque due to F_2, τ_2, is going to be negative, τ_3 is going to be negative as well, and τ_4 is going to be positive. The sign of a torque is very easy to determine by simply imagining the rotational motion of the object under the influence of that torque. As mentioned at the start of this section, most quantities in rotational motion are dependent upon your choice of rotational axis. If the fulcrum were to be moved (which represents a change of rotational axis), the signs of the torques might change as a result.

The magnitude of a torque τ due to some force $\vec{F}$ is given by:

$$\tau = rF\sin\theta$$

where θ is the **angle between the force and the lever arm**. Two examples of the geometry for the above equation are given in the following figure. The first example, on the left side of the figure, is for a 1-dimensional case: the lever arm is just the length along the 1-dimensional lever, and θ is easily defined as the angle between the force $\vec{F}$ and the lever. In the second example, on the right side of the figure, we consider a more complicated 2-dimensional case: the lever arm is always going to be the straight-line distance between the rotational axis (i.e. the pivot) and the point where the force is applied. With this in mind, the angle θ is easily defined.

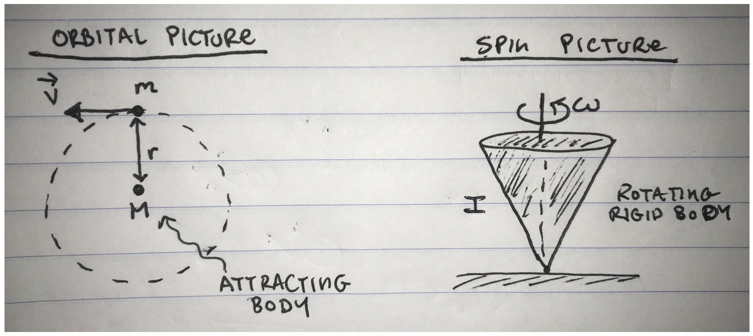


The last thing to define is the rotational analogue to momentum, the **angular momentum** L, just as we have a rotational analogue to energy. Our rotational form of Newton's second law will be re-written in terms of the angular momentum:

$$\sum \tau = \frac{\Delta L}{\Delta t}$$

just like Newton's second law, in general, is written in terms of Δp. So we have the same interpretation of torque in general: torque is the rate of change of the angular momentum of an object, just like force is the rate of change of an object's (linear) momentum.

To define angular momentum, it helps to consider two different "pictures" for the angular momentum: the **orbital** picture and the **spin** picture. The following figure illustrates the orbital and spin pictures. The orbital picture is meant to be applied to orbital motion (i.e.



satellite motion), when you have a small mass m orbiting a large attracting body M at some speed v with an orbital radius r. In this case, the "orbital" angular momentum is:

$$L_{orb} = rp = rmv$$

The spin picture is applied to a rigid body, of moment of inertia I, which is rotating at some angular speed ω, as shown in the figure above. The "spin" angular momentum is:

$$L_{spin} = I\omega$$

There are two things to note about the above equations. First, they are **equivalent** ways of defining the angular momentum, which is why in the formula sheet, I write $L = I\omega = rp$. The moment of inertia of a point mass m orbiting at a radius r is $I = mr^2$. The angular speed of a point mass moving at a linear speed of v, at a radius r, is $\omega = \frac{v}{r}$. (This is the same as the equation for rolling without slipping, though this is *not* an example of rolling motion.) Plugging this into the orbital angular momentum:

$$L_{orb} = rmv = (mr^2)\left(\frac{v}{r}\right) = I\omega = L_{spin}$$

As you can see, these equations are completely equivalent, it's just that one picture will always make more sense than the other picture for a particular problem.

The second thing to notice is that the spin angular momentum equation is identical to the linear angular momentum, mass times velocity, where you replace mass with moment of inertia and velocity with angular velocity. Once again, this is how almost every rotational equation is written: replacing the linear variables with their rotational counterparts.

8.4 Moment of Inertia

By definition, the moment of inertia of any collection of masses m_1, m_2, m_3, ..., each orbiting some common rotational axis at radii r_1, r_2, r_3, ..., respectively, is:

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + ...$$

Or, using the more compact summation notation:

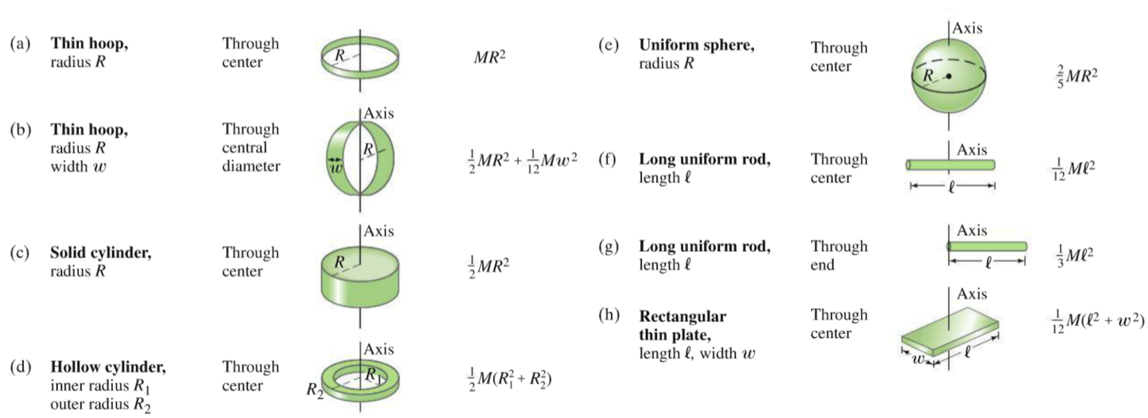
$$I = \sum_i m_i r_i^2$$

Essentially, the moment of inertia is a measurement of **how much mass you have far away from the rotational axis**. The more mass you have at a large r, the more moment of inertia you have. Consider a solid disc versus a hoop, which is just a disc with all its mass at the edge. If the disc and the hoop have the same radius R and mass M, we would expect the hoop to have a **larger** moment of inertia than the disc because the hoop has all of its mass located at R while the disc has its mass spread between 0 and R. This is held up by the equations for the moment of inertia, which is $\frac{1}{2}MR^2$ for a disc and MR^2 for a hoop, thus the hoop's moment of inertia is larger.

For a point mass m orbiting at a radius r, there is only a single term in the summation, and so the moment of inertia is just:

$$I = mr^2$$

For rigid bodies, we don't consider them as being made up of a finite number of atoms; for all intents and purposes, we consider them as being made up of an infinite number of atoms. The calculation of the moments of inertia involve calculus, so we're not interested in *how* we find the equations for each rigid object; we're interested only in what the equations are. From your textbook, the list of moment of inertia equations is given in the following figure.



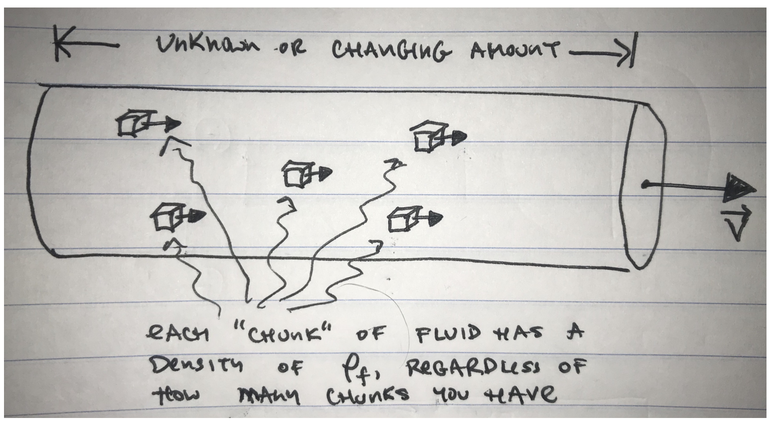
9 Fluids

A fluid is any phase of matter that can **conform to the shape of a container** it's in; these are either **liquids or gases**. Fluids can be further subdivided into fluids that can expand or contract to the exact volume of a container, known as **compressible** fluids, and fluids that have a fixed volume and can't be expanded or compressed, known as **incompressible** fluids; gasses are compressible and liquids are incompressible.

For a moving fluid, it doesn't make much sense to think about the mass of the fluid, because you don't really have a fixed amount of the fluid. What makes more sense is to think about the **density** of the fluid ρ, defined as:

$$\rho = \frac{m}{V}$$

Instead of thinking about a fixed amount of fluid with some mass m, you can think about a fluid being broken up into a bunch of small "chunks" of fluid, each of which has the same density. Then it doesn't matter how many chunks you happen to have at any given moment of time, or whether those chunks or moving or are at rest; each chunk has that same density ρ_f. (The subscript f is for "fluid; we can, and will, define densities for solid objects, which we would typically refer to as ρ_{obj} for "object.") This way of viewing fluids is illustrated in the figure below.



A fluid will conform to the shape of a container that it's in because the molecules aren't bound together like in a solid, but are loosely interacting with one another. In water, for instance, the molecules interact via hydrogen "bonds," while in an ideal gas, there is basically no interaction what-so-ever. This means that when you pour a fluid into a container, the weight of the fluid pushes all the molecules apart, trying to make space for more molecules to lower themselves, since gravity is trying to pull them down. This causes a force to be applied to all surfaces of the container in contact with the fluid. (For a gas, this will be all the surfaces of the container, since a gas fills the container, but for a liquid, this will only be some of the interior surfaces, depending on the volume of the liquid.)

When considering the contact force of a solid object, it makes sense to talk about the force itself, since it's a single force acting at a single point. But when you have all this fluid putting a force across an entire surface (think about this in terms of chunks of the fluid, with each chunk putting a small force on a separate part of the surface), it doesn't make a lot of sense to think about a single force because it's not acting on a single point on the surface, but is smeared across the entirety of the surface. In this case, we define a more sensible measruable known as the **pressure**, which is the force **per unit area** on a surface:

$$P = \frac{F}{A}$$

For a fluid, the pressure on a surface is due to the weight of the fluid above the surface. If you're considering some surface at a depth D in a fluid, you can imagine having a column of fluid rising above that surface to a height D. In this case, the **fluid pressure**, which is due to the weight of that column of fluid, is given by:

$$P_f = \rho_f g D$$

Note that this does **not** depend upon the height of the surface, but the depth beneath the fluid.

This isn't the only thing putting a weight on this submerged surface, though. If the container was open to the atmosphere, you could think about all that air above the fluid as also putting a weight on the surface; it's a small weight, but it exists non-the-less. We could define the **total pressure** to be the sum of the fluid pressure and the atmospheric pressure P_{atm}:

$$P_{tot} = P_f + P_{atm}$$

The above isn't a fixed definition, but more of an illustration of the concept of total pressure. If you had a surface submerged in water to a depth D, but that water had a layer of oil above it of thickness H, and then the oil was exposed to the atmosphere, you'd have a total pressure of:

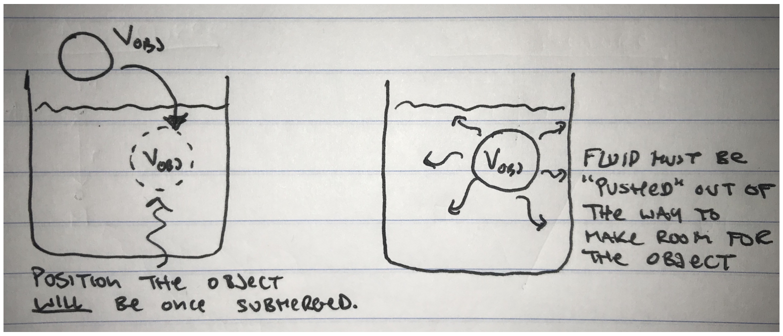
$$P_{tot} = P_{water} + P_{oil} + P_{atm}$$

where the water pressure would be given by the fluid pressure equation, $\rho_f g D$. This idea holds for any number of layered fluids.

Because fluid pressure depends on the depth, a solid object submerged in a fluid is going to experience a different pressure on its top surface than its bottom surface. The bottom surface will experience a greater pressure (due to the fact that it's deeper), and thus a greater force. The force on the bottom surface will point upwards while the force on the top surface will point downwards; since the force on the bottom surface (i.e. the upward force) is larger, there will be a net force upwards. This net upward force is known as the **buoyant force**. The magnitude of the buoyant force B is given by **Archimedes' Principle**:

> **The buoyant force on a submerged object is equal to the weight of the fluid displaced by that object.**

It's important to understand the concept of **fluid displacement**. Consider the figure below. On the left, there is an object with some volume V_{obj} that is going to be submerged; I've drawn a dashed outline of where the object *will* be once submerged. Currently, that volume is occupied by fluid, and in order for that volume to be occupied by the object, that fluid must be moved. On the right of the figure, I've shown the object, once submerged, "pushing" that fluid out of the way. This is the concept of fluid displacement.



Using the definition of density, we can express any mass in terms of the density and volume of a substance:

$$m = \rho V$$

When an object is submerged, the volume submerged V_{sub} (and thus the volume displaced) **doesn't have to be the entire volume** of the object. For an object that floats, $V_{sub} < V_{obj}$. Note, though, that V_{sub} will never be greater than V_{obj}. So, we can say that the mass of the fluid displaced is:

$$m_{disp} = \rho_f V_{sub}$$

and thus the weight of the fluid displaced, $m_{disp}g$, which is the **buoyant force** B by Archimedes' Principle, is:

$$B = \rho_f g V_{sub}$$

Any object whose density is **less** than the fluid's density will **float** in that fluid, due to the fact that there will be a non-zero upward net force $\sum F = B - W$ (the buoyant force will be greater than the weight), and any object whose density is **greater** than the fluid's density will **sink** in that fluid, due to a non-zero downward net force $\sum F = W - B$ (the weight will be greater than the buoyant force). For an object that floats, as the volume submerged decreases, so does the buoyant force, until the buoyant force is balanced with the weight. By Newton's second law, when these are balanced, we'd arrive at the equation:

$$\frac{V_{sub}}{V_{obj}} = \frac{\rho_{obj}}{\rho_f}$$

Note that this equation makes no sense if $\rho_{obj} > \rho_f$; this is because the object would sink in this case, and so the equation wouldn't apply.

Everything up to this point has been for fluids at rest; next we need to consider fluids in motion. For fluids in motion, we will only consider **ideal fluids**. Ideal fluids have 4 properties they must satisfy, but the most important one is that an ideal fluid is incompressible. (Note that you don't need to make sure that a fluid in a problem is ideal; always consider fluids to be ideal.) Because an ideal fluid is incompressible, the amount of volume passing any point ΔV during any given time interval Δt **must be a constant**:

$$\frac{\Delta V}{\Delta t} = \text{constant}$$

The above quantity, the rate at which volume moves in a fluid, is known as the **flow rate**, and it must be **conserved** throughout a fluid. This means that if a pipe that a fluid is moving through changes area, then the speed that the fluid moves at must change in order for the same volume to pump through that pipe per second. The equation relating area to speed is:

$$A_1 v_1 = A_2 v_2$$

Finally, we have an equation for energy conservation for a fluid, known as **Bernoulli's equation**. Just like energy conservation for a solid, we'll define two points: some initial point 1 and some final point 2. We want to know the initial speed (to know the initial kinetic energy) and the initial height (to know the initial potential energy) to find the final speed at whatever the final height happens to be. For solids, we would then say $K_1 + U_1 = K_2 + U_2$.

One main difference between energy physics for solids and fluids is that for a fluid, we don't use mass m, but we use density ρ. This means that we won't be defining a kinetic energy $\frac{1}{2}mv^2$ or a potential energy mgy, but a **kinetic energy density** (kinetic energy per unit volume) $\frac{1}{2}\rho v^2$ and a **potential energy density** ρgy.

The second main difference between energy physics for solids and fluids is that the equation isn't just $K_1 + U_1 = K_2 + U_2$, but we have an **additional term**: the pressure. The equation is going to be $P + K + U$ is a constant, or:

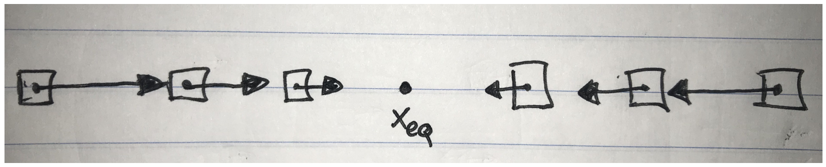
$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2$$

As with energy conservation for solids, you are free to choose any $y = 0$ that you'd like, and define all y's from this zero. Further, you can relate v_1 and v_2 using the flow rate conservation equation $A_1 v_1 = A_2 v_2$. Then, it's just a matter of labeling the initial and final points and finding all your variables (P_1, v_1, y_1, P_2, v_2, and y_2) and solving the equation, just like energy conservation for solids.

10 Oscillations

The study of oscillations, which is the most common example of periodic motion (recall that periodic motion is a motion that **repeats itself**), is restriction to **simple harmonic motion** in this class. A harmonic motion is going to be produced by any force that points towards some **equilibrium position** (a point where the force is zero), and gets larger the further away you get from that equilibrium position. The figure below illustrates this concept; in the figure, each arrow represents the force, and the length of the arrow represents the magnitude of the force. Forces that produce harmonic motion are known as **restoring forces**, because they try to bring the object back towards equilibrium.

Specifically, **simple harmonic motion** is produced by a restoring force that is proportional to the displacement:



$$F \propto x$$

The most common example of a simple harmonic motion is that of oscillations of a mass attached to a spring. In this case, the restoring force is given by **Hooke's law**:

$$F_{sp} = -kx$$

where x is the **displacement from equilibrium**, *not* the length of the spring, and k is the **force constant** of the spring (aka the spring constant). The negative sign is to indicate that the force is restoring; if you stretch the spring, i.e. $x > 0$, then $F < 0$ pulling the spring in the opposite direction, and if you compress the spring, i.e. $x < 0$, then $F > 0$, pushing the spring in the direction opposite the compression. In either instance, the force tries to restore the spring to its **natural length**; the natural length of the spring defines the equilibrium position. Since the sign is simply to indicate direction (as with any vector), we can ignore it and compute the magnitude and direction separately, which is how we treat any force.

The spring force is **conservative**, as mentioned back in Section 6, on work and energy, with a potential energy of:

$$U_{sp} = \frac{1}{2}kx^2$$

At any point in the motion of a mass m oscillating on a spring of force constant k, we can define the **total energy** as:

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

So, if you know the speed at any displacement, you can find the total energy of the oscillations. Since energy is conserved in the case of a spring, you'd therefore know the energy **everywhere** in the motion.

There are two points of particular interest: the equilibrium position and the furthest displacement from the equilibrium position, known as the **amplitude** of the oscillation. At the amplitude, the mass won't be moving anymore; this is analogous to the fact that an object's vertical velocity is zero at the peak of a trajectory. In this case, the total energy is just potential energy, and that potential energy is the **maximum** value. Calling the amplitude A, the total energy is then:

$$E = U_{max} = \frac{1}{2}kA^2$$

The restoring force F_{sp} is going to pull the mass from the amplitude towards the equilibrium position. As the mass moves towards the equilibrium position, it's going to be accelerating the whole time, because the force always points towards the equilibrium position (see the previous picture). Once it crosses the equilibrium position, the force will flip directions and the mass will begin *decelerating*. So the mass accelerates the entire way from the amplitude towards the equilibrium position, and so the **speed is maximal at the equilibrium position**. At x_{eq}, all of the energy is kinetic ($x_{eq} = 0$ by definition, so $\frac{1}{2}kx_{eq}^2 = 0$), and this kinetic energy will be a **maximum** since the speed is a maximum:

$$E = K_{max} = \frac{1}{2}mv_{max}^2$$

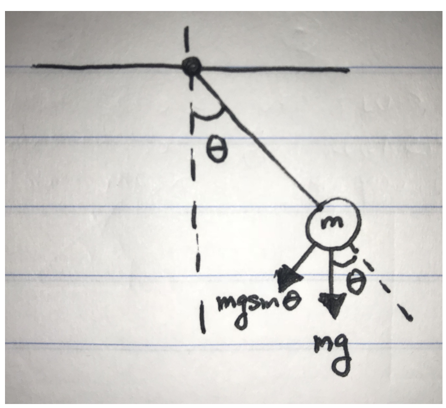
The key thing here is that both this equation and the previous equation for E (in terms of U_{max}) are true. Along with the general equation for the total energy, we have three ways of finding E:

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \quad \leftarrow \quad \text{at any displacement}$$

$$E = \frac{1}{2}kA^2 \quad \leftarrow \quad \text{at the amplitude}$$

$$E = \frac{1}{2}mv_{max}^2 \quad \leftarrow \quad \text{at the equilibrium position}$$

In addition to the simple harmonic motion of a spring, we can consider the simple harmonic motion of a **pendulum**, which is just a massless rod of length l with a "bob" of mass m attached at its end, oscillating back-and-forth to some θ_{max} (measured from the vertical). The restoring force in the case of the pendulum is gravity, and the equilibrium position is $\theta = 0$. (The gravitational force always pulls the mass towards the lowest position, which is $\theta = 0$ measured from the vertical.)



Consider the above figure of a pendulum; we can see that for an angle θ, the restoring force is not all of gravity mg, but the sine component $mg\sin\theta$. This means that the resorting force is:

$$F_{pend} = mg\sin\theta$$

Notice, though, that this **does not** meet the criteria for simple harmonic motion. In simple harmonic motion, the force needs to be proportional to the displacement, so we'd need $F \propto \theta$; instead, we have $F \propto \sin\theta$. However, there is something known as the **small angle approximation**, which says for angles of about 10^o or less,

$$\sin\theta \approx \theta$$

Caution! The above equation is only true if θ is in radians, not degrees! Using the small angle approximation, our restoring force becomes:

$$F_{pend} = mg\theta$$

And so we have exactly the condition for simple harmonic motion. So, simple harmonic motion will exist for a spring under any conditions, but it will only exist for a pendulum if the **amplitude of oscillations θ_{max} is small**.

Because simple harmonic motion is periodic, we can define a frequency, period, and **angular frequency** of oscillations. The angular frequency is defined as:

$$\omega = 2\pi f$$

and the period and frequency are related by:

$$f = \frac{1}{T}$$

For a spring and pendulum, the angular frequencies are defined as:

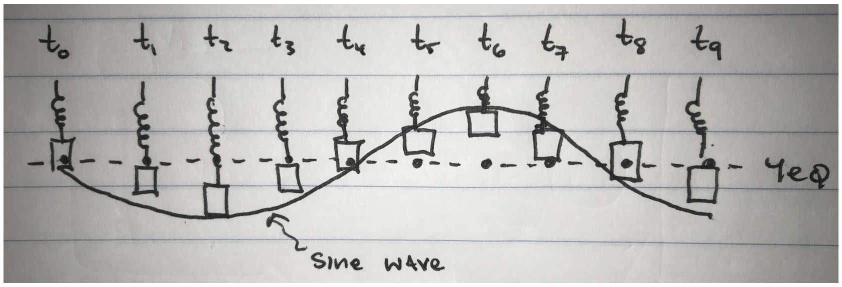
$$\omega_{sp} = \sqrt{\frac{k}{m}} \quad \text{and} \quad \omega_{pend} = \sqrt{\frac{g}{l}}$$

Though the equations aren't included in the formula sheet, it's easy to find the periods of the oscillations: all you do is flip the fraction for angular frequency and then multiply by 2π:

$$T_{sp} = 2\pi\sqrt{\frac{m}{k}} \quad \text{and} \quad T_{pend} = 2\pi\sqrt{\frac{l}{g}}$$

11 Waves & Sound

A wave is an example of an oscillation that is also **moving through space**, or **propagating**. The best way to think about a wave is illustrated in the following figure. Imagine a vertical harmonic oscillator made of a mass/spring system, which is oscillating with some period T (or some frequency f, however you want to describe the oscillations). Normally, we'd consider the oscillator has being fixed in place, so all we can describe are oscillations over time. However, if we consider the **equilibrium point to be moving through space**, then we also have **oscillations over space**. This is the idea of a propagating wave: an oscillation over time and an oscillation over space.



Each oscillation is going to be described as a sine wave. The exact mathematics describing the wave aren't relevant. However, like we define a period as the amount of time per cycle we need to describe an analogous quantity for the oscillations over space. This is the **wavelength** λ of the oscillations, which is the **distance traveled per cycle**. If, in one cycle, the wave travels a distance of λ (by definition) in a time T (by definition), then the speed is just λ/T, or:

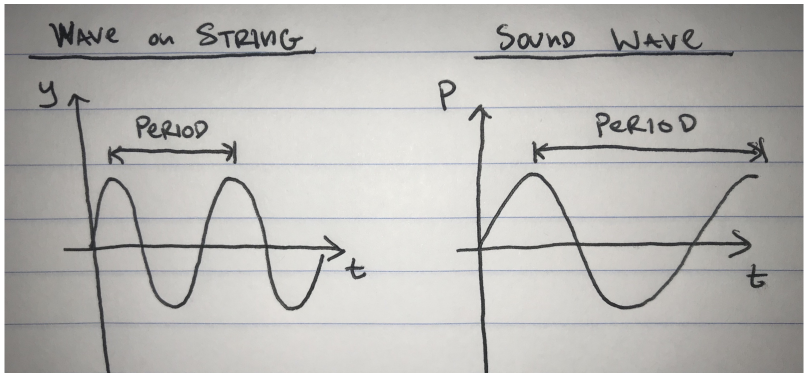
$$v = \lambda f$$

This equation is sometimes referred to simple as the **wave equation**. (Technically, the "wave equation" is a much more complicated equation describing a wave, but we won't go anywhere near the true wave equation in this class, so we can use the name for the above equation without confusion.)

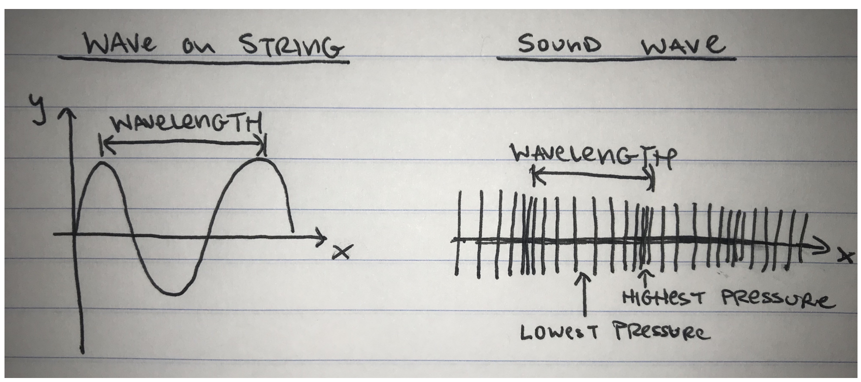
Recall that there are two types of waves: **transverse** waves, in which the oscillations are perpendicular to the propagation, and **longitudinal** waves, in which the oscillations are parallel to the propagation. All relevant waves **except sound** are transverse waves, including waves on a string, light, waves on a string, etc., while sound is a longitudinal wave. In fact, all compression waves (sound being one example) are longitudinal, so other types of compression waves like a shockwave from an explosion would also be longitudinal. Despite this fundamental difference between transverse and longitudinal waves, **the physics of transverse and longitudinal waves is the same**, so all the relevant equations and results presented in this course are identical regardless of whether a wave is transverse or longitudinal; the second-half of chapter 11 and chapter 12 are, essentially, the same.

Waves can be difficult to visualize without any multimedia aids, even when images are presented; the best aid to visualizing waves are graphics. You can find a GIF of a transverse wave at this link, and a GIF of a longitudinal wave this link.

Plotting wave characteristics on graphs can be a useful tool for describing waves. For **either** transverse of longitudinal waves, the plot of oscillations vs. time look identical; this is illustrated in the figure below for a wave on a string, for which the oscillations are in the vertical position y, and sound, for which the oscillations are in the pressure of the medium P. Each wave is sinusoidal, and the period is just the horizontal separation of two peaks (or two troughs). (Since the horizontal axis is time, this separation is a measurement of time, and since two peaks are separated by one cycle, this measurement is the period.)



Plotting oscillations versus position (along the propagation direction) is more perilous when it comes to longitudinal waves, since the oscillations occur along the propagation direction. The way around this is to draw vertical lines representing, for sound, lines of actual atoms in the medium. Pressure is then represented by how near two consecutive lines are; at high pressure, the lines are very near one another, and at low pressure, the lines are far apart from one another. The point of highest pressure is known as a **compression**, and the point of lowest pressure is known as a **rarefaction**. The wavelength of sound would then be the distance between two points of compression (or two points of rarefaction). These features of oscillations versus position graphs are illustrated in the following figure.



There are two very important facts about waves to remember:

1. A wave's speed depends on the type of wave and the properties of the medium the wave propagates in, **not** on the wavelength or the frequency of the wave. The wave equation should be thought of, conceptually, as a way to relate λ and f assuming you know v; it should **never** be thought of, conceptually, as an equation that shows that v depends on λ and f.

2. When a wave crosses a boundary between two media, the frequency of that wave **remains the same**; if the speed of the wave is different in the new medium (which is

almost always the case), then the wavelength must be different in the new medium, but the frequency will always remain the same.

The most common example of fact 1 is the equation for the speed of **waves on a string**:

$$v = \sqrt{\frac{T}{m/L}}$$

where T is the tension in the string, and m/L is the mass per unit length (or the **linear mass density**) of the string. This fits fact 1 because the equation itself is determined by the type of wave (i.e. a wave on a string), and T and m/L are properties of the medium that the wave propagates in (i.e. the particulars of the string itself). Another example would be the speed of sound, for which the speed depends on the compressibility of the medium (known as the bulk modulus) and the density of the medium; the "stiffer" the medium is, i.e. the lower the bulk modulus, the faster sound travels, and the higher the density, the slower sound travels. A final example is the speed of light, which is determined by a single factor known as the index of refraction; the index of refraction of the vacuum is 1, and all other media have an index greater than 1, such that light travels fastest in a vacuum and slower in all other media.

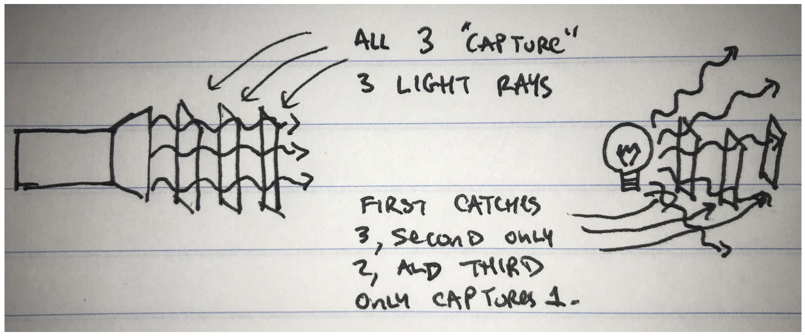
Next, it's important to discuss the energy carried by waves. A wave is thought of as a potentially infinitely long oscillation – in theory, they could extend forever. When a wave is produced, however long the wave is in production for determines how many wavelengths of that wave are produced. For instance, if a wave moves at 20m/s at a wavelength of 1m, if it's produced for 1s, 20 wavelengths of that wave are produced. Similarly to our approach with fluids, when we saw that mass itself wasn't a good variable to describe a fluid, but density was a great variable, the total energy of a wave isn't particularly good for describing waves, but the **energy per unit wavelength** E_λ is a good way to describe waves.

Equivalently, instead of describing the energy per unit wavelength, we could give the **energy per unit period** E_T. Think about what this quantity is, though: it's an energy divided by time. This is just **power**. So, while energy itself isn't a great quantity for describing waves, power absolutely is. Instead of knowing what the equation for power is, which isn't particularly useful in this course, we just need to know that **power is proportional to the amplitude-squared** of a wave:

$$P \propto A^2$$

where the amplitude is the maximum magnitude of the oscillation, exactly as it was for simple harmonic motion.

Just like there was a catch when trying to describe the energy carried by a wave, it turns out there's a catch when trying to describe the power carried by a wave, too. Look at the following figure. In it, we consider two cases: a flashlight, which emits light in rays that are all parallel, and a lightbulb, which emits rays of light equally in all directions (known as **isotropic** emission). If we were to imagine "capturing" some light on a piece of film, which



has some surface area S, then no matter how far we placed that film from the flashlight, it would capture the same number of light rays. The number of light rays is important because it determines how bright the light is; more properly, it determines the **intensity** of the light:

$$I = \frac{P}{S}$$

which is the power per unit surface area of the film (or any instrument that can measure the power carried by a wave). However, look at the case of the lightbulb: in isotropic emission, as you place the film further and further away from the lightbulb, it captures less and less light rays, meaning the intensity drops with distance.

It turns out that the **intensity of isotropic emission** has the following relationship:

$$I \propto \frac{1}{r^2}$$

where r is the distance from the source. This equation isn't only true for light, but is true for **any wave emitted isotropically**. Another common example of isotropic emission is sound from a speaker. Often times, the above equation isn't useful on its own for solving problems, because most problems ask for the intensity of a wave at one point to be compared to the intensity of a wave at a second point. You can divide I_1 by I_2 an get $\frac{1/r_1^2}{1/r_2^2}$. After some manipulation, you arrive at the **more useful form** of the equation for the intensity of isotropic emission:

$$\frac{I_1}{I_2} = \left(\frac{r_2}{r_1}\right)^2$$

Up to this point, as I said, all of the physics applies equally to transverse and longitudinal waves of any kind. Now I want to talk specifically about sound for a second. It turns out that for sound, the intensity on its own it's a useful measurement. The **volume** of sound doesn't scale linearly (meaning if the intensity doubles, the loudness that we perceive doesn't double), but scales **logarithmically**. Because of this, we need to use the **intensity level** β to describe the volume of sound:

$$\beta = (10\text{dB}) \log\left(\frac{I}{I_0}\right)$$

where I_0 is the **threshold of human hearing** – the lowest-intensity (i.e. quietest) sound a human ear can perceive – which is:

$$I_0 = 10^{-12}\ \frac{\mathrm{W}}{\mathrm{m}^2}$$

Manipulating the β equation can be done by solving the logarithmic equation outright for intensity at a first location for some volume, then using $I_1/I_2 = (r_2/r_1)^2$ to compare two intensities, and finally plugging I_2 back into the β equation to find the volume at a second location. However, using some tricks with logarithms can make most problems more easily solvable. Recall the following identities for logarithms:

$$\log(A) + \log(B) = \log(AB)$$
$$\log(A) - \log(B) = \log\left(\frac{A}{B}\right)$$
$$\log(A^x) = x\log(A)$$

Imagine this set up to a problem: you know the volume β_1 at some r_1 away from a source of sound, and you want to find the volume β_2 at some other r_2 from the source. The fastest way to solve this problem is to exploit the above identities for logarithms:

$$\beta_1 - \beta_2 = \Delta\beta = 10\log\left(\frac{I_1}{I_0}\right) - 10\log\left(\frac{I_2}{I_0}\right) = 10\log\left(\frac{I_1/I_0}{I_2/I_0}\right) = 10\log\left(\frac{I_1}{I_2}\right)$$
$$= 10\log\left[\left(\frac{r_2}{r_1}\right)^2\right] = 20\log\left(\frac{r_2}{r_1}\right)$$

If the argument of a logarithm is greater than 1, the output is positive, but if the argument is less than 1, the output is negative. This means that if $r_2 > r_1$, $\Delta\beta > 0$, and if $r_2 < r_1$, $\Delta\beta < 0$. This gives us exactly what we'd expect: if you move further away from the source of the sound, the volume decreases, and if you move closer to the source of the sound, the volume increases. You should always reality-check your answers and make sure they are at least sensible.

Now we can go back to discussing waves in general. All waves exhibit a phenomenon known as the **Doppler effect**, which is the changing of observed frequencies (or wavelengths) of waves depending on the relative motion of the source and the observer. The Doppler effect can be summarized by:

> **When a source and observer are receding, i.e. getting further apart, the observed frequency is less than the source frequency. When a source and observer are approaching, i.e. getting closer together, the observed frequency is greater than the source frequency.**

Always use this to reality-check your answers.

While all waves exhibit the Doppler effect, the only application of it that we're going to see in this course is the Doppler effect of sound. The equation for the Doppler effect of sound is:

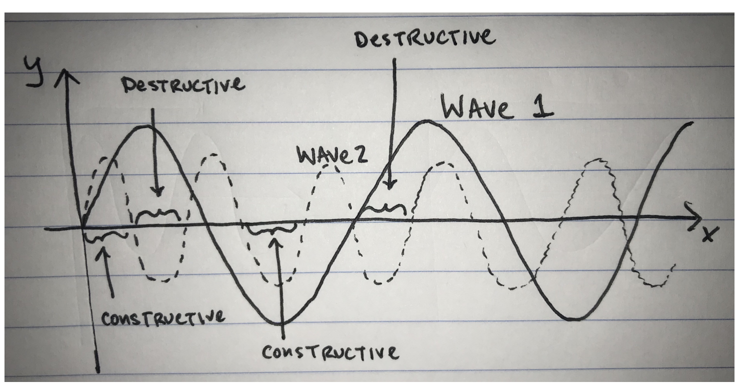
$$f_{obs} = \frac{v \pm v_{obs}}{v \mp v_s} f_s$$

where the subscript *obs* is for **observer**, the subscript s is for **source**, and v is the speed of sound. We'll always use $v = 350$ m/s, as given in the formula sheet, though the speed varies between 340 m/s and 350 m/s depending on the temperature. The speed of sound depends on altitude, as well, but the variance is a lot greater than these 10 m/s; the range of 340 m/s to 350 m/s is for the speed of sound **at sea level**.

The signs of the equation for the Doppler effect must be chosen carefully. The equation is set up such that you choose the signs with the mnemonic: **top is towards**. What this means is that, for both v_{obs} and v_s, you choose the **top** sign if the object (either the observer or the source) is moving **towards** the other object. For example, if an ambulance is driving at some speed v_1, and you're an ambulance chaser driving behind the ambulance at some speed v_2, then you (the observer) are moving towards the ambulance, and the ambulance (the source) is moving away from you. So you'd choose the top sign for v_{obs} and the bottom sign for v_s. If the frequency of the ambulance's siren is some f_0, the equation would be:

$$f_{obs} = \frac{v + v_2}{v + v_1} f_0$$

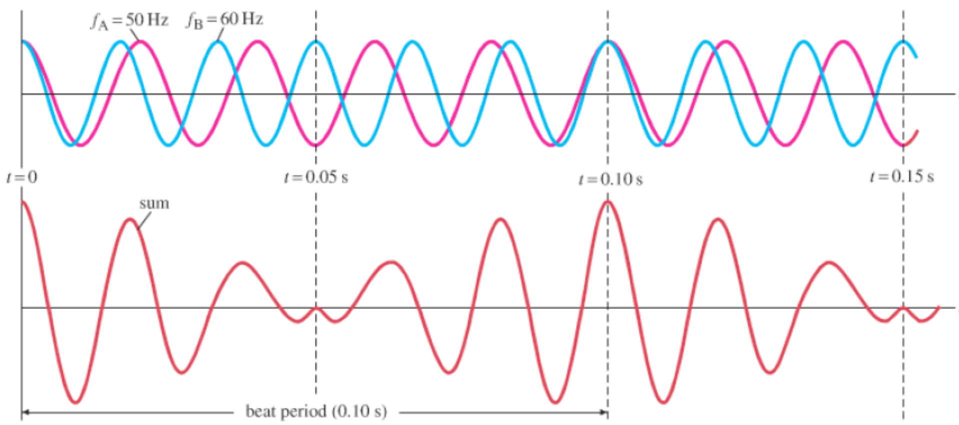
Finally, the last phenomenon to discuss about waves is **interference**. All waves exhibit interference; it's one of the characteristics used to prove that something is a wave. (In the 1800s, light was shown to exhibit interference, which convinced physicists at the time that light was a wave instead of a particle. However, it turns out that the question of whether light is a wave of a particle is a very complicated one.) Interference is simply the summing of oscillations of two waves that occupy the same place at the same time.



If you consider a transverse wave, for instance a wave on a string (like the ones shown in the above figure), then you would define the oscillations to be the vertical displacement

$y(x,t)$, where x is the position of the wave and t is the time. The interference of two waves simply produces a new vertical displacement $y_I(x,t) = y_1(x,t) + y_2(x,t)$. If the two waves are oscillating in the **same direction**, then y_I is larger than either y_1 or y_2 and the interference is **constructive**; if the two waves are oscillating in **opposite directions**, then y_I will be smaller than the larger of y_1 and y_2, and the interference is **destructive**.

Notice in the figure above the alternating areas of constructive and destructive interference. This means that the amplitude of the oscillations is going to oscillate. This is illustrated in the figure below. With the oscillations in amplitude come the oscillations in power (recall that $P \propto A^2$), which lead to oscillations in intensity, which, **for sound**, lead to oscillations in **volume**.



These oscillations in volume are known as **beats**, and they have a characteristic frequency, known as the **beat frequency**:

$$f_{beat} = |f_1 - f_2|$$

Note that knowing the beat frequency and f_1 isn't enough to determine f_2. For instance, consider a speaker emitting a sound at 500 Hz which interferes with a sound of unknown frequency from a second speaker, producing a beat frequency of 30 Hz. The frequency of the sound from the unknown speaker could be **either** 470 Hz or 530 Hz, because the magnitude of the separation in frequency in both cases is 30 Hz, so the beat frequency would be 30 Hz in either case. One way to determine for sure which of the two frequencies is the actual frequency of the second speaker is to vary the known frequency, say from 500 Hz to 510 Hz, and see how the beat frequency changes. If the beat frequency increases to 40 Hz, then the second speaker *must* emit sound at 470 Hz; if the beat frequency decreases to 20 Hz, then the second speaker *must* emit sound at 530 Hz.

Finally, when waves of equal frequency pass over one another in opposite directions, they can produce **standing waves**. "Can" is the operative word in the previous sentence, because standing waves **aren't always produced**; rather they are only produced at specific wavelengths (or frequencies) known as **harmonic wavelengths** (or frequencies). There are two types of standing waves that you need to be aware of: **node-node** standing waves and **node-antinode** standing waves. A node is a point of **no-d**isplacement (a mnemonic to remember what a node is), and an anti-node is a point of maximal displacement. The easiest way to tell if your standing waves are node-node or node-antinode is this:

> **If the set up is the same at both ends, the standing waves are node-node; if the set up is different at each end, the waves are node-antinode**.

So a string fixed at both ends would produce node-node standing waves, a tube that's open at both ends would produce node-node standing waves, but a tube that's open at one end and closed at the other would produce node-antinode standing waves. The equations for standing waves are:

$$\lambda_n = \frac{2L}{n}, \;\; f_n = \frac{nv}{2L}, \;\; n = 1, 2, 3, ... \;\; \text{(node-node)}$$

$$\lambda_n = \frac{4L}{n}, \;\; f_n = \frac{nv}{4L}, \;\; n = 1, 3, 5, ... \;\; \text{(node-antinode)}$$

Only at the specific values of n, the **harmonic number**, indicated will standing waves be produced.

12 Temperature & Kinetic Theory

Thermodynamics is the topic that encompasses the final chapters, chapters $13 - 15$, of the course. Thermodynamics can be viewed in two equivalent ways: **macroscopically**, in which case we care about the macroscopic measurements of temperature T, pressure P, volume V, etc., or **microscopically**, in which case we care about the microscopic measurements of v_{rms}, K_{av}, m, etc. The study of the relationship between these microscopic measurables and the macroscopic measurables is known as kinetic theory: you envision a gas not as a whole quantity, but as being made up of individual atoms or molecules, each of which is moving with some velocity that is randomly oriented with an rms magnitude of v_{rms}. Then you can define the macroscopic quantity T in terms of the average kinetic energy per particle, using the **equipartition theorem**:

$$K_{av} = \frac{3}{2} k_B T$$

and you can define the macroscopic quantity P in terms of the frequency with which particles in the gas collide with the surfaces of the container. The above equation for the equipartition theorem leads to the equation for the **rms speed**:

$$v_{rms} = \sqrt{\frac{3k_B T}{m}}$$

where m is the mass of the particles making up the gas.

In the context of kinetic theory, it is easy to explain the phenomenon of **thermal expansion**. (Remember that we're only concerned with linear thermal expansion in this course.)

You can picture a solid as being a container of some length l_0 holding some gas. However, the container isn't perfectly rigid – it can be stretched, but the walls of the container are elastic like a spring, meaning to continue stretching it you have to have successively larger forces. As the temperature of the gas increases, the pressure of the gas is going to increase, putting progressively larger and larger forces on the walls of the container, causing the container to stretch (i.e. expand). As the temperature decreases, the pressure of the gas drops, and the container will relax and reduce its length. The "spring constant" of the solid in this analogy would be given by the **thermal expansion coefficient** α. For a solid of initial length l_0 and thermal expansion coefficient of α, a change in temperature of ΔT yields an expansion of:

$$\Delta l = \alpha l_0 \Delta T$$

Speaking of temperature, there are three important temperature scales: **Fahrenheit**, ^{o}F, **Celsius**, ^{o}C, and **Kelvin**, K. Recall that Kelvin defines the coldest possible temperature: **absolute zero**, 0K. Recall as well that absolute zero is impossible to reach, even theoretically; you could use a refrigerator to try and cool it down, but it would take an infinite amount of work to cool a gas down to 0K, making it impossible. The three temperature scales are related by the following equations:

$$T_K = T_{^oC} + 273$$

$$T_{^oF} = \frac{9}{5} T_{^oC} + 32$$

It's important to note that since Celsius and Kelvin differ by a constant, the **difference** in temperature is the same in both units:

$$\Delta T_{^o\mathrm{C}} = \Delta T_K$$

So for equations like the above equation for thermal expansion, or the equation for calorimetry $Q = mc\Delta T$, the value of ΔT can be given in either Celsius or Kelvin, since they are the same number.

The final topic in this section is the **ideal gas law**, which relates the macroscopic measurables temperature T, pressure P, and volume V:

$$PV = Nk_BT$$

where N is the number of particles in the gas. Equivalently, we could give the ideal gas law in terms of the number of **moles** of a gas:

$$PV = nRT$$

where R is the ideal gas constant. Recall that the number of moles of a substance times **Avagadro's number**, $N_A = 6.02 \times 10^{23}$, is equal to the number of particles of a substance:

$$N = nN_A$$

The ideal gas law allows us to compute changes in the macroscopic thermodynamic variables P, V, and T during **thermal processes** that the gas can undergo. In any thermal process, *something* is going to remain constant. We're (almost) always going to consider processes that keep the number of particles in the gas N constant, so we'll assume that Nk_B is a constant. Then, depending on the problem, one of the three thermodynamic quantities might be a constant. For instance, if I say we have a gas at some initial volume V_1 and initial temperature T_1 expanding **isobarically** (at a constant pressure) to some new volume V_2, we could find the new temperature T_2 by manipulating the ideal gas law such that everything constant is on one side and everything variable is on the other:

$$PV = Nk_BT \;\; \Rightarrow \;\; \frac{V}{T} = \frac{Nk_B}{P} = \text{constant}$$

This means that V/T initial must equal V/T final, since that quantity is a constant:

$$\frac{V_1}{T_1} = \frac{V_2}{T_2}$$

This approach can applied identically to any thermal process; just manipulate $PV = Nk_BT$ such that the left-hand-side is entirely made of variables and the right-hand-side is entirely made of constants. Your equation will be the left-hand-side at 1 equals the left-hand-side at 2.

13 Thermodynamics

This is the final section of content in the course, and it's broken up into 5 subsections: heat transfer, calorimetry, the first law of thermodynamics, the second law of thermodynamics, and heat engines & refrigerators.

13.1 Heat Transfer

There are three methods of heat transfer: one direct method, known as conduction, and two indirect methods, known as convection and radiation. **Conduction** is the transfer of heat via direct contact between two objects of different temperature; heat will flow from the hot object to the cold object until their temperatures equalize. The rate at which heat will cross a boundary is:

$$\frac{Q}{\Delta t} = kA\frac{\Delta T}{L}$$

where k is the **thermal conductivity** of the substance the boundary is composed of, A is the area of the boundary, L is the width of the boundary, and ΔT is the temperature difference across the boundary.

Convection is the transfer of heat across a distance via the rising of a hot gas. Imagine a candle; the flame of the candle heats up the air in the immediate vicinity via conduction.

That hot air is now less dense than the cooler air around it, so it's buoyant and rises, carrying heat away from the flame with it. This is why you can get burned by placing your hand over a flame without directly touching the flame. Convection is a very complicated process, and so doesn't have an equation associated with it.

Radiation is the transfer of heat across a distance by the production of electromagnetic radiation. It is well known that hot objects glow (i.e. emit light); for instance, the coils on an electric stove will glow red when heated. Objects that emit electromagnetic radiation as a way of expelling heat are known as **blackbodies** and **graybodies**; a blackbody is the ideal, expelling the largest amount of heat as electromagnetic radiation as possible for a given temperature, while a graybody is the realistic case, emitting a value less than the ideal blackbody. The equation for the intensity of this **blackbody radiation** is:

$$I = \epsilon\sigma T^4$$

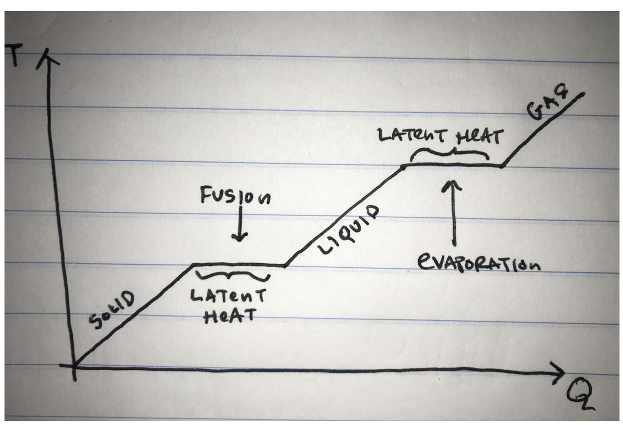
where ϵ is the **emissivity** of the object (equal to 1 for a blackbody and less than one for a graybody) and σ is the Stefan-Boltzmann constant. Blackbody emission is isotropic, so it obeys the usual equation for the intensity of isotropic emission $I_1/I_2 = (r_2/r_1)^2$.

13.2 Calorimetry

Calorimetry is the study of how heat changes the temperature of a substance. For a substance in a **single phase** (e.g. a solid), the relationship between heat absorbed/emitted and the change in temperature of the substance is:

$$Q = mc\Delta T$$

where c is the **specific heat** of the substance, which is the amount of heat per unit mass required to raise or lower the temperature by 1 K. The specific heat **depends on the phase** of the substance; for instance, liquid water has a specific heat of 4184 J/kgK, but frozen water (ice) has a specific heat of 2100 J/kgK.



It's important to note that the above calorimetry equation applies only if the substance is in a single phase. This is because at the boundary between two phases, there is a bunch of heat required to change the phase of the substance without even changing the temperature. This is known as the **latent heat** of the phase transition. The above figure shows how the temperature changes with heat added; it's clear that the latent heats of each phase transition are the flat parts of the line, since those represent periods where heat is being added by the temperature remains a constant. Latent heat L is typically given as the amount of heat per unit mass to transition from one phase to another, meaning the total heat required for the phase transition is:

$$Q = mL$$

As expected, the latent heat L depends on the phase transition. The latent heat of fusion (solid to liquid or liquid to solid) for water is 333 kJ/kg, while the latent heat of vaporization (liquid to gas or gas to liquid) for water is 2260 kJ/kg. Note that the equation for a constant phase $Q = mc\Delta T$ will give you the sign of Q explicitly: if the temperature decreases, $\Delta T < 0$ and so $Q < 0$, and if the temperature increases, $\Delta T > 0$ and $Q > 0$. However, there is nothing built into the latent heat equation $Q = mL$ to determine the sign of Q, since L is the same for either direction of a phase transition; you just have to know that if you're transitioning from a cold phase to a hot phase (solid to liquid or liquid to gas) you have to **add heat**, meaning $Q > 0$, and if you're transitioning from a hot phase to a cold phase (gas to liquid or liquid to solid), you have to **remove heat**, meaning $Q < 0$.

13.3 The First Law of Thermodynamics

As mentioned in Section 6 on work & energy, the first law of thermodynamics is basically an updated version of the work-energy theorem. What it says, conceptually, is that **energy cannot be created nor destroyed; only converted from one form to another**. In the study of mechanical energy, we considered energy as being conserved if non-conservative forces did no work, and energy as being "lost" if non-conservative forces do work. In reality, energy is never lost; what we considered was a loss in *mechanical* energy, which is $K + U$, but if mechanical energy is lost, it's simply transformed into heat. The first law of thermodynamics accounts for this.

In order to give the first law of thermodynamics, we need to define a new energy known as the **internal energy**. If we view a gas microscopically, the internal energy of the gas is just the total kinetic energy of all the gas particles. Recalling that the equipartition theorem says the average kinetic energy per particle is $\frac{3}{2}k_BT$, if the gas has N particles, then the internal energy U is:

$$U = \frac{3}{2}Nk_BT$$

Though the variable is U for both, don't get internal energy confused with potential energy. We're only ever going to consider ideal gases, which are specifically non-interacting gases, meaning they don't have any potential energy. So when studying thermodynamics, we never consider potential energy (at least in our course), and U is always the internal energy of the gas.

With our definition of internal energy, the **first law** of thermodynamics is:

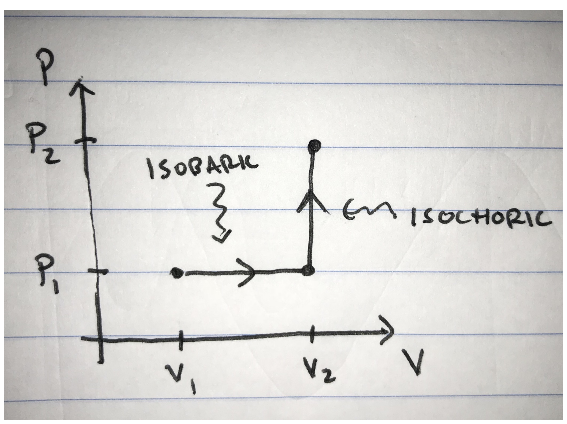
$$\Delta U = Q + W$$

The plus sign between the Q and W is really important, because it determines the interpretation of W. Some references use a negative sign, which would reverse the interpretation of W. For our interpretation, a **positive work** means that the gas **gains energy** and a negative work means that the gas **loses energy**. This is

Note that the ideal gas law says that $PV = Nk_BT$, so we can give the internal energy as:

$$U = \frac{3}{2}PV$$

This form of the equation is often times more useful for solving problems than the other form. The state of an ideal gas is completely defined by P and V, so thermal processes (which we discussed in the previous section) can be described completely as paths on a PV diagram. For instance, a gas undergoing an isobaric expansion (an increase in volume at a constant pressure) followed by an **isochoric** (constant volume) increase in pressure can be given by PV diagram in the following figure.



For an isobaric process, the work is equal to:

$$W = -P\Delta V$$

The work is always going to depend on a change in the volume of a gas, so for an isochoric process, the work will be **zero**.

There is a lingo that goes along with work for a gas: if the work is positive, i.e. the energy of the gas **increased**, then we say that **work was done on the gas**; if the work is negative, i.e. the energy of the gas **decreased**, then we say that **work was done by the gas**. That is, if work is done on a gas, the gas' energy increases, and if a gas does work, it has to expend energy.

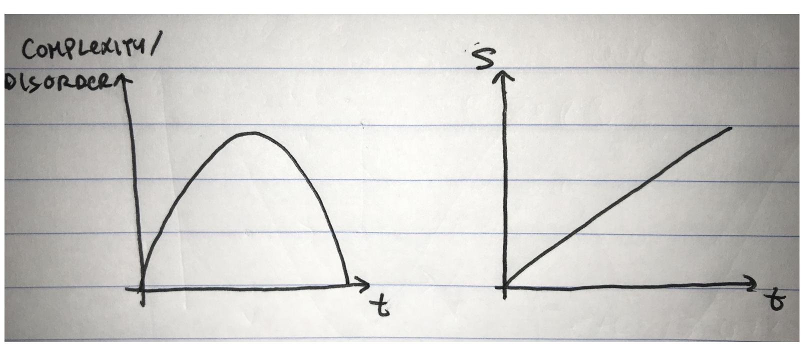
13.4 The Second Law of Thermodynamics

In order to discuss the second law of thermodynamics, we have to introduce a new, energy-like quantity known as **entropy** S. For an **isothermal** process, i.e. a process performed at a constant temperature, the change in entropy is:

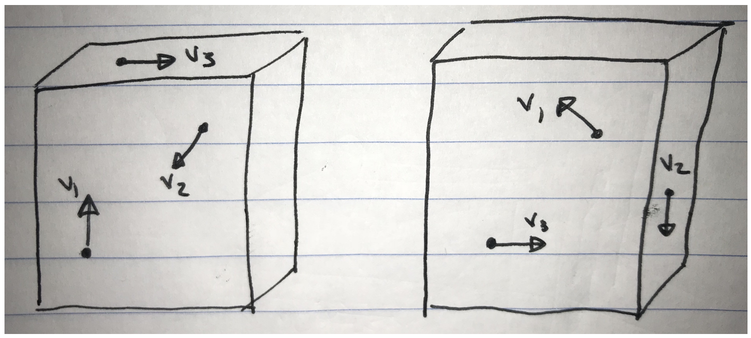
$$\Delta S \geq \frac{Q}{T}$$

The second law of thermodynamics, essentially, requires that the above statement be a $\geq$ inequality; that is, ΔS can **never** be less than Q/T. Another way of phrasing the second law is that the entropy of a **closed system** must always increase. A closed system is one in which heat cannot enter or leave. This could be a perfectly insulated gas, or (more realistically, since perfect insulators don't exist) the entire universe.

What entropy actually *is* is irrelevant to the study of thermodynamics, but I'll explain it anyways. I only point this out because entropy has a fairly complicated definition. Typically, entropy is defined as "the amount of disorder" or "the amount of complexity" of a system. Neither of these are very good definitions of entropy. Imagine a cup of coffee that you're going to mix cream into. Initially, when the cream and coffee are completely separate, the system isn't very complex at all – you have the coffee on one side and the cream on the other. After mixing completely, you have a homogeneous mixture of coffee and cream, and the system is also not very complex at all – it all looks the same. It's in between these two states, when the cream is in the process of mixing, that the system is very complex and disordered: there will be vortices formed during the mixing process, pockets where there's a lot of cream and little coffee, pockets where there's a lot of coffee and little cream, variations in temperature all over the place (assuming that cold cream was added to hot coffee), etc. However, the entropy isn't the highest when the coffee is mixing; the entropy is highest at the end, when they are completely mixed. So the apparent complexity/disorder increases during the mixing, then peaks and decreases until the coffee and cream are fully mixed, but the entropy increases the entire time. The figure below illustrates this fact as plots of complexity/disorder versus time and entropy S versus time, with the initial time being the start of the mixing and the final time being the end of the mixing.



In order to **properly** define entropy, we need to go back to our microscopic versus macroscopic viewpoint of a gas; i.e. take a look back at kinetic theory. In kinetic theory there is the idea of a **macrostate**, which is a state defined by macroscopic measurables like P, V, T, etc. For an ideal gas, the macrostate is defined completely by P and V. In contrast to a macrostate, a gas also has a **microstate**, which is the microscopic arrangement of the individual atoms that corresponds to a **single** macrostate.

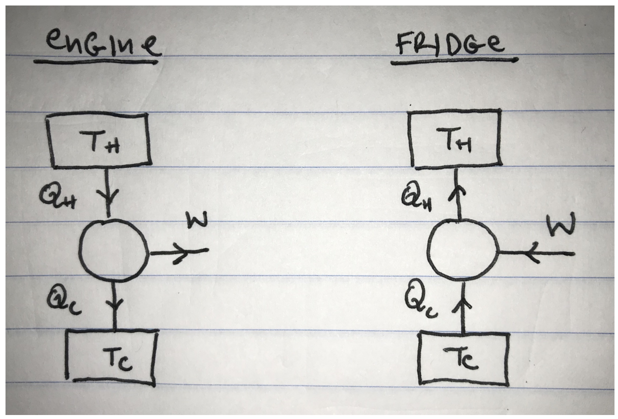


Consider a gas composed of three particles, moving at speeds v_1, v_2, and v_3, stored in a container of volume V. In the figure above, there are two possible arrangements of this gas that each correspond to the same pressure P. Since the volume of each arrangement is the same, the macrostate (V, P) is the same for each gas. These are two unique microstates for the same macrostate. Now, think about how many different microstates there are (how many different ways to arrange these three particles) – it's a *lot* of different ways. What entropy is is a measurement of the **number of different microstates corresponding to a single macrostate**.

One of the fundamental concepts of thermodynamics is the idea that all microstates are equally probable; in the above figure, this means that all locations and directions of the three gas atoms are equally probable, which is perfectly reasonable. Given that all microstates are equally probably, a system will **always** tend towards a state with the greatest possible number of microstates; that is, a system will always increase its entropy. If you have a system with two macrostates, macrostate 1 having two microstates and macrostate 2 having five microstates, which macrostate is more probable? There are 7 total microstates, so the probability of being in macrostate 1 is $2/7 = 29\%$ while the probability of being in macrostate 2 is $5/7 = 71\%$; clearly, macrostate 2 is more probably than macrostate 1, so given a long enough time, the system will **always** end up in macrostate 2 – the state of higher entropy. This is what is known as the **statistical interpretation of entropy**, and it's the foundation of a formulation of thermodynamics known as **statistical mechanics**.

13.5 Heat Engines & Refrigerators

Finally, the last bit of physics of the course. This subsection covers the application of the first and second laws of thermodynamics to heat engines and refrigerators. A **heat engine** is a machine that uses the natural flow of heat from a hot substance to a cold substance to produce work. A common example is the engine in a car, which burns fuel to raise the temperature of a gas to very high degree. This extremely hot gas then wants to release heat into the significantly colder atmosphere, and the engine in your car uses this flow of heat to power the wheels. A **refrigerator** does the opposite, utilizing work to reverse the natural flow of heat, removing heat from a cold substance and dumping it into a hot substance. Inside an actual fridge, you want to keep the air cold, which means continuously removing heat from it and dumping it into the kitchen, the air in which is much hotter than the air inside the fridge. To reverse the natural flow of heat, you need to use work, which the fridge gets by drawing power from an electrical outlet.



Heat flow diagrams, which are exactly what they sound like – diagrams that show the direction of heat flow – are extremely useful for solving engine and refrigerator problems. The figure above shows a heat flow diagram for both an engine and a fridge. The boxes labeled T_H and T_C are **thermal reservoirs**, which are basically giant sources or sinks of heat, whose temperatures never change. Imagine the ocean; adding a little bit of heat won't change the temperature of the ocean because the ocean is so massive. That's the idea of a reservoir. The circles in the diagrams represent the actual machine, which doesn't need to be specified to solve problems; it's just *something* that produces the heat flow diagram we want. We want apply conservation of energy (not mechanical energy, but total energy, including heat) to the above diagrams: whatever energy goes into the machine is equal to the energy that comes out of the machine. We see that we get the same energy conservation equation for both an engine and a fridge:

$$Q_H = W + Q_C$$

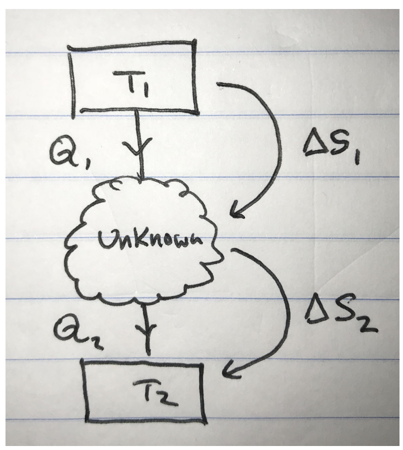
This equation ignores the signs of all energies; the signs are indicated by the direction of the energy flow in the diagram. For instance, $W > 0$ for a fridge – that is, the fridge does work on the gas in the machine – but this is indicated by the arrow for the work pointing towards the machine, indicating that energy is flowing into the gas within the machine.

The first law of thermodynamics will tell us how to figure out how much work we'll get out for a given process performed on a gas. We can draw a PV diagram and say that the gas will undergo such-and-such a process, and we'll be able to calculate ΔU, Q and W for this process, allowing us to put numbers onto these heat flow diagrams. Note that for both an engine and a fridge, the process that operates the machine must be **cyclic**; if the gas doesn't start and end in the same state, then you can't reuse the machine. In order to continue to use the machine over and over, the gas must start and end in the same state.

So the first law of thermodynamics tells us how to put numbers onto the heat flow diagrams, while the second law of thermodynamics will tell us how **efficient** our machines can be. Essentially, the second law of thermodynamics says it's impossible for an engine to convert 100% of Q_H into W (i.e. to operate with $Q_C = 0$), and that it's impossible for a fridge to operate without any input of work (i.e. that heat flow will never spontaneously reverse direction). Both of these results can be seen by applying our equation for ΔS at a constant temperature, $\Delta S \geq Q/T$. Imagine heat flowing from some reservoir T_1 to some reservoir T_2. The heat leaving T_1, let's call it Q_1, produces a change in entropy of ΔS_1, and the heat entering T_2, let's call it Q_2, produces a change in entropy of ΔS_2, where:

$$\Delta S_1 = \frac{Q_1}{T_1} \quad \text{and} \quad \Delta S_2 = \frac{Q_2}{T_2}$$

This process is diagrammed in the following figure, where the bubble labeled "unknown" is some unknown process – it could be an engine, it could be a refrigerator, or it could be nothing.



The total change in entropy for this process is the amount of entropy leaving the unknown process minus the amount of entropy entering:

$$\Delta S_{tot} = \Delta S_2 - \Delta S_1 = \frac{Q_2}{T_2} - \frac{Q_1}{T_1}$$

The second law of thermodynamics says that this must be greater than or equal to zero, so let's see what implications this has on engines and refrigerators. First of all, if the unknown process is nothing (literally nothing happens except for the natural flow of heat), then $Q_1 = Q_2 = Q$, so:

$$\Delta S_{tot} = \frac{Q}{T_2} - \frac{Q}{T_1} \geq 0 \;\Rightarrow\; \frac{1}{T_2} - \frac{1}{T_1} \geq 0$$

Since $1/T_2$ must be *greater* than $1/T_1$, T_2 must be **less** than T_1 – that is, **heat must flow from hot to cold**. This is exactly what we'd expect, and we can clearly see that it's a direct consequence of the second law of thermodynamics. What implications does this have? Well, it means that a refrigerator **cannot** operate with $W = 0$, because then heat will always flow from hot to cold, and never from cold to hot, as required for a fridge.

Does the above analysis have any implications on engine? Well, let's consider the case of the "perfect" engine, one which can convert 100% of Q_H into W. This would mean, in our heat flow diagram, that $Q_2 = 0$, so the total change in entropy is:

$$\Delta S_{tot} = \frac{Q_2}{T_2} - \frac{Q_1}{T_1} = -\frac{Q_1}{T_1} \leq 0$$

This is in direct violation of the second law of thermodynamics, so an engine can **never convert 100% of Q_H into work**.

The second law of thermodynamics puts limits on how good an engine and a fridge can be. These limits are known as the **efficiency of an engine**, e, defined as:

$$e = \frac{W}{Q_H}$$

and the **coefficient of performance of a refrigerator**, COP, defined as:

$$COP = \frac{Q_C}{W}$$

Clearly, the larger the work output W for some given heat input Q_H, the more efficient an engine is, i.e. the closer to 1 it becomes. For a fridge, the larger the heat removed from the cold reservoir Q_C for a given work input W, the more efficient the fridge, and the larger the COP is. So for an engine, you want an **efficiency near 1**, and for a refrigerator, you want a **coefficient of performance as large as possible**.

The **best case scenario** for both a fridge and an engine is known as the **Carnot** cycle (specifically a Carnot engine and a Carnot refrigerator). A Carnot cycle is one in which the change in entropy is **zero**, which is the most efficient either an engine or a refrigerator can possible get. The **Carnot efficient**, or the maximum theoretical efficiency of an engine, is given by:

$$e_{\text{Carnot}} = 1 - \frac{T_C}{T_H}$$

and the **Carnot coefficient of performance**, which is the maximum theoretical coefficient of performance of a refrigerator, is given by:

$$COP_{\text{Carnot}} = \frac{T_C}{T_H - T_C}$$

The Carnot engine gets more efficient the larger the temperature difference $T_H - T_C$ gets. If $T_H \gg T_C$, then T_C/T_H is essentially zero (imagine $T_H = 1{,}000{,}000$ and $T_C = 1$; $T_C/T_H = 0.000001$, which is basically zero) and the efficiency will be near 100%. Only if T_H becomes infinite would the efficiency become 100%. Conversely, a Carnot refrigerator gets more efficient as T_C and T_H become equal. As they get closer and closer to one another, the denominator becomes closer and closer to zero. The smaller the denominator gets, the larger the number becomes. COP_{Carnot} will become infinite only when $T_C = T_H$; in this case, though, you don't have a fridge, but two systems in thermal equilibrium, so nothing needs to happen.