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Introduction to Energy

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Abstract: These notes present a brief introduction to energy physics, covering the work-energy theorem, the existence of potential energies, and total mechanical energy conservation. These derivations and proofs would be introduced in any introductory physics course, and be covered at this level in any upper-division classical mechanics course.

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1 Work-Energy Theorem

A particle at rest is accelerated by some force $\vec{F}$ along some curve C. For an infinitesimal displacement $d\vec{s}$ along C, the work done, by definition, is

$$dW = \vec{F} \cdot d\vec{s} \tag{1.1}$$

Integrating along C will give us the total work done:

$$W = \int_C \vec{F} \cdot d\vec{s}$$

We can expand the integrand using chain rule:

$$d\vec{s} = \vec{v}dt$$

where the direction of v changes along the curve, so we need to maintain the dot product.

The work is now:

$$W = \int_C \vec{F} \cdot \vec{v}dt$$

Substituting in Newton's second law (for the constant mass case) into the above equation:

$$W = \int_C m\dot{\vec{v}} \cdot \vec{v}dt$$

There is a great derivative trick[1] that we should use:

$$\frac{d}{dt}\left(\vec{v} \cdot \vec{v}\right) = \frac{d\vec{v}}{dt} \cdot \vec{v} + \vec{v} \cdot \frac{d\vec{v}}{dt} = 2\dot{\vec{v}} \cdot \vec{v} \tag{1.2}$$

We can just call $\vec{v} \cdot \vec{v} = v^2$; there's no point in being that formal with our notation. Substituting this into our last equation for the work along C:

[1] This trick is used a lot, in particular when solving problems in classical mechanics. I'd memorize it.

$$W = \frac{1}{2}\int_C m\frac{d}{dt}\left(v^2\right)dt$$

Now, we can reverse the process of chain rule, arriving at

$$W = \frac{1}{2}\int_C md\left(v^2\right)$$

We're going to assume that the particle gains some speed, v, along the curve C. Now, just to be completely correct about the answer, we're going to assume that $v \ll c$, this way the mass m remains constant[2] and we can pull it out of the integral. Then, the integral along C of $d(v^2)$ is just v^2, the final speed at the end of the curve. So, the work done on the object along the curve C was

$$W = \frac{1}{2}mv^2$$

which is just the kinetic energy gained, T. It's also easy enough to say that, instead of starting from rest, the particle started at some speed v_0 and ended at a speed of v. In this case, the integral would result in a v^2 being evaluated from v_0 to v, which is just $v^2 - v_0^2$, so

$$W = \frac{1}{2}m\left(v^2 - v_0^2\right) = \Delta T \tag{1.3}$$

This is known as the **work-energy theorem**.

2 On the Existence of Potential Energies

Definition 2.1. A conservative force is a force whose work is **path-independent**.

Theorem 2.1. *Any conservative force $\vec{F}$ has an associated scalar function, V, which has units of J.*

Along some curve C, an infinitesimal displacement $d\vec{s}$ along C will produce a work given by equation (1.1):

$$dW = \vec{F}\cdot d\vec{s}$$

For $\vec{F}$ to be a conservative force, then, the line integral around a closed loop C must be zero:

$$\oint_C \vec{F}\cdot d\vec{s} = 0 \tag{2.1}$$

We can use one of the most important relationships in vector calculus to help us solve this problem – Stokes' theorem – which says

[2] If you haven't studied special relativity, the mass is actually a function of speed $m = m_0(1-\beta^2)^{-1/2}$, where β is the particle's speed, in units of c, and m_0 is the particle's mass measured at rest.

$$\int_{C_S} \vec{A} \cdot d\vec{s} = \iint_S \left(\vec{\nabla} \times \vec{A}\right) \cdot d\vec{S} \tag{2.2}$$

For a loop C_S enclosing an area S. As we always define it, the direction of the area vector $d\vec{S}$ is normal to the surface S. Plugging this into (2.1), we get

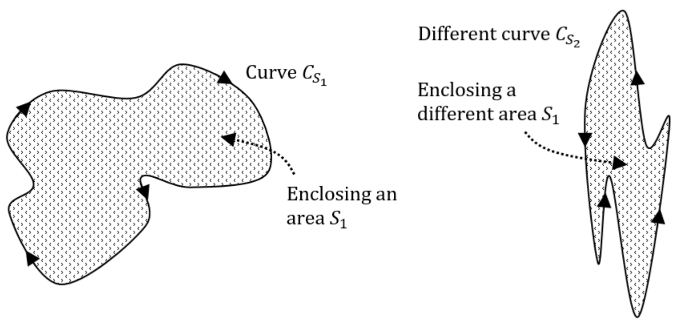
$$\iint_S \left(\vec{\nabla} \times \vec{F}\right) \cdot d\vec{S} = 0 \tag{2.3}$$

There are two way that equation (2.3) can be zero:

1. The dot product between the curl of $\vec{F}$ and $d\vec{S}$ is zero

2. The force is curl-less

In order for the force to be conservative, the work around a loop must **always** be zero. This is illustrated in the following figure.



Note that the surface on the left is supposed to be in a perpendicular plane to the surface on the left. Around both curves, the work must be zero. This means for **any** loops enclosing **any** area, one of those two conditions must be satisfied. Obviously, it can't be condition 1. Condition 1 is going to depend on our choice of S, but as I said, there must be something fundamental (i.e. independent of our choice of S) about $\vec{F}$ that makes it a conservative force. So, it must be that condition 2 is always true for a conservative force, because then the work around a loop will be zero regardless of what loop or enclosed area we choose.

Lemma 2.1. *The curl of the gradient of a scalar field ϕ is always zero,*

$$\vec{\nabla} \times \left(\vec{\nabla}\phi\right) \equiv 0 \tag{2.4}$$

Proof. By the definition of a cross product,

$$\left[\vec{\nabla} \times \left(\vec{\nabla}\phi\right)\right]_i = \sum_{jk} \epsilon_{ijk}\partial_j\left(\partial_k\phi\right)$$

Partial derivative operates commute, so we can change the order:

$$\sum_{jk} \epsilon_{ijk}\partial_j\left(\partial_k\phi\right) = \sum_{jk} \epsilon_{ijk}\partial_k\left(\partial_j\phi\right) \tag{2.5}$$

Now, what I'm about to do is an extremely useful trick in these vector calculus proofs, so it's important that you understand it completely. **These indices, i, j, and k, are "dummy" indices; it doesn't matter what you call them, because you're going to sum over them anyways. So I can, at will, relabel any index I want.** I will exercise that right I have, and relabel k as j and j as k:

$$\underbrace{\sum_{jk}}_{j\leftrightarrow k} \underbrace{\epsilon_{ijk}\partial_j(\partial_k}_{j\leftrightarrow k}\phi) = \sum_{kj} \epsilon_{ikj}\partial_j\left(\partial_k\phi\right) \tag{2.6}$$

Now, the last thing to do is to explout the antisymmetry of the Levi-Civita symbol,

$$\epsilon_{ikj} = -\epsilon_{ijk}$$

I cannot emphasize this point enough: **this last step, exploiting the antisymmetry, is not the same as swapping dummy indices. In this step, I am actually exchanging rows and columns, whereas before I was just re-labing the indices. Notice that when I relabeled the indices, all j's and k's swapped, whereas in this step, only the j and k in the Levi-Civita symbol will swap places**. So, putting it all together

$$\left[\vec{\nabla} \times \left(\vec{\nabla}\phi\right)\right]_i = \sum_{jk} \epsilon_{ijk}\partial_j\left(\partial_k\phi\right) = -\sum_{jk} \epsilon_{ijk}\partial_j\left(\partial_k\phi\right)$$

Since the number equals the negative of itself, it must be equal to zero. So, we've proven equation (2.4).

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Now, equation (2.4) with our condition to satisfy equation (2.3), we can say that every conservative force must be associated with a scalar function, V, as given by the following equation

$$\vec{F} = -\vec{\nabla}V \tag{2.7}$$

The negative sign is a convention, which will make sense once we assign meaning to the function V, which as of right now it doesn't have. It has the units of Nm, but is it an energy or a torque? Both have these units.

Skipping ahead, because I'm sure you know what it is already, the function V is the potential energy. The force on a particle at any point is going to be equal to the slope of V vs. x at that point. If V is increasing at that point, the slope is positive, and the force will be negative. This works to always pull a particle towards lower values of potential energy; since V is increasing at this point, the force is in the direction opposite to the increase.

We could also consider the particle at a point where V is decreasing. Then the slope would be negative, and the force would be positive, pushing the particle along the path that decreases potential energy. It's a universal fact that a particle under the influence of conservative forces will **always** move towards lower potential energy; this is just a manifestation of our choice of sign for equation (2.7).

3 Total Mechanical Energy and Energy Conservation

Consider the function V given by equation (2.7). If this is simply a function of position (it could also be a function of time), then we can use chain rule to claim

$$\dot{V} = \sum_i \frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial t}$$

This is clearly the dot product between the gradient $\vec{\nabla} V$ and the velocity $\dot{\vec{x}}$ [3]:

$$\dot{V} = \vec{\nabla} V \cdot \dot{\vec{x}} \tag{3.1}$$

Now, let's consider the time derivative of the kinetic energy. Recalling out derivative trick, equation (1.2), written with $\vec{v} = \dot{\vec{x}}$,

$$\dot{T} = \frac{1}{2}m\frac{d}{dt}(\dot{x}^2) = \frac{1}{2}m(2\ddot{\vec{x}}\cdot\dot{\vec{x}}) = m\ddot{\vec{x}}\cdot\dot{\vec{x}}$$

which means that

$$\dot{T} = \vec{F}\cdot\dot{\vec{x}} \tag{3.2}$$

In the above equation, we didn't specify whether the force $\vec{F}$ is conservative or not. In fact, when we derived the work-energy theorem, we didn't specify whether $\vec{F}$ was conservative or not, either. That's because we're not considering an *individual* force acting on a particle; we want to consider the force on the particle that's changing its velocity (i.e. producing an acceleration), and that's the total force on the particle. We can simply break this total force up into the conservative and non-conservative forces acting on the particle:

$$\vec{F} = \vec{F}_{cons} + \vec{F}_{nc} \tag{3.3}$$

[3] If you remember your vector calculus class, this will be obvious, since it's the definition of a **directional derivative**.

We know equation (2.7) allows us to express $\vec{F}_{cons}$ in terms of the function V, so we can say

$$\vec{F} = -\vec{\nabla} V + \vec{F}_{nc} \tag{3.4}$$

and if we take the dot product of the above equation with velocity,

$$\vec{F} \cdot \dot{\vec{x}} = -\vec{\nabla} V \cdot \dot{\vec{x}} + \vec{F}_{nc} \cdot \dot{\vec{x}}$$

We know from equations (3.1) and (3.2) that we can re-write the above equation (moving the V term to the other side) as

$$\vec{F}_{nc} \cdot \dot{\vec{x}} = \dot{T} + \dot{V} = \frac{d}{dt}(T + V)$$

Now, consider two possibilities:

1. $\vec{F}_{nc} = 0$, or there are no conservative forces acting on the particle
2. $\vec{F}_{nc} \cdot \dot{\vec{x}} = 0$, or there are no conservative forces acting along the velocity of the particle

If either of these are true, then the quantity $T + V$ is conserved. Clearly this is some kind of energy, because it's conserved in this motion and has units of J. We'll define it as E, the **total mechanical energy**:

$$E = T + V \tag{3.5}$$

This allows us to also define V as a type of energy, which we refer to as the **potential energy**.

Notice that if the second condition given above is satisfied, then the non-conservative forces are doing no work. If a force never points along the velocity, i.e. never points along the direction of motion, then it's never doing any work. So, we can actually re-state the two conditions as one condition: if non-conservative forces aren't doing any work, then total mechanical energy is conserved[4].

[4]Obviously, if $F_{nc} = 0$, then non-conservative forces aren't doing any work.