A Brief Review on Floer Homology

Dongho Lee

February 16, 2021

Abstract

Floer homology could be understood as an infinite-dimensional analogy of Morse homology, defined on certain type of symplectic manifolds. The purpose of this note is to provide guideline for the construction of Floer homology. In section 1, we provide some basic notions of symplectic geometry which is crucial for the construction of Floer homology. In section 2, we introduce how to construct Floer homology on a symplectic manifold with some assumptions, which is same setting as [AD]. We also introduce the Arnold's conjecture, and provide a proof using Floer homology in a specific case. The main reference of this note is [AD].

Contents

1 Basic Notions of Symplectic Geometry

The symplectic geometry has its origin at classical mechanics, in particular Hamiltonian mechanics. In short, a symplectic manifold is generalization of a *phase space* which appears in the mechanics, which can be converted to a cotangent bundle with a nondegenerate 2-form in mathematics. [A] provides a nice introduction to symplectic geometry starting from the classical mechanics. [dS] is also an excellent guide into the symplectic world for the novices.

This section provides the basic notions of symplectic geometry briefly, in particular the notions which are crucial to define Floer homology. Those are symplectic manifolds, almost complex structure, symplectic action functional and Hamiltonian orbits. In the last subsection, we introduce the conjecture of Arnold, which is one of the central topic in symplectic geometry and ignited the Floer theory. There are many interesting topics missing, for example Moser tricks, Noether's theorem, etc. The reader might look for the traditional textbooks to study symplectic geometry deeply.

1.1 Symplectic Manifolds

A **symplectic manifold** is a manifold M equipped with 2-form $\omega \in \Omega^2(M)$, which is called **symplectic form**, satisfies followings:

- ω is **non-degenerate**, i.e. for each $p \in M$, $\omega_p(X, Y) = 0$ for any $Y \in T_pM$ implies that $X = 0$. It's equivalent to say that the following map is an isomorphism for any $p \in M$.

$$\begin{aligned} \tilde{\omega} : T_pM &\to T_p^*M \\ X &\mapsto \omega_p(X, -) \end{aligned}$$

- ω is closed, i.e. $d\omega = 0$.

Example 1.1. The prototype of a symplectic manifold is a cotangent bundle T^*M for any manifold M. Using local coordinate $(q, p) \in T^*U \simeq U \times \mathbb{R}^n$, define **canonical 1-form** by

$$\lambda = pdq = \sum p_j dq_j$$

Then $d\lambda = \omega = dp \wedge dq$ becomes a symplectic form on T^*M.

There are many consequences directly follow from the definition. Let (M, ω) be a symplectic manifold.

1. M should be even-dimensional, since a nondegenerate alternating bilinear form can only exist in even dimensions. So we usually say that M is $2n$-manifold.

2. ω^n is nowhere vanishing n-form, which is usually called a volume form. Hence M should be oriented.

3. In particular, any oriented surfaces admit a symplectic structure, where ω is given by its volume form.

4. If M is closed, for any exact 2-form $\omega = d\lambda$, we have

$$\int_M (d\lambda)^n = 0$$

by Stokes' theorem. It's impossible by the first observation. It means that if M is closed, then the symplectic form should represent nontrivial class in de Rham cohomology $H^2(M)$. It follows that closed manifold with $H^2(M) = 0$ does not admit a symplectic structure, for example S^{2n} with $n \geq 2$.

5. We can identify TM with T^*M canonically. That is, if we have 1-form $\lambda \in \Omega(M)$, then we have a vector field X_λ defined by an equation

$$i_{X_\lambda}\omega = \lambda.$$

We call such vector field X_λ **symplectic vector field**. If $\lambda = dH$, we call $X_{dH} = X_H$ **Hamiltonian vector field**.

There is a notion of equivalence, as you might expect. If a diffeomorphism $f : (M_0, \omega_0) \to (M_1, \omega_1)$ satisfies $f^*\omega_1 = \omega_0$, we call f **symplectomorphism**. There are symplectic manifolds which are diffeomorphic but not symplectomorphic, so the symplectic structure gives more information then the smooth structure itself.

One can also think about the bilinear forms. Given a bilinear form A, we can decompose A by symmetric part and alternating part. The symmetric part gives an inner product, which allows negative-definite subspace and null subspace. If we ignore the null subspace, this gives rise to pseudo-Riemannian geometry. On the other hand, if the dimension is even, alternating part gives rise to symplectic geometry. If the dimension is odd, the alternating part must have null subspace of dimension$\geq$1. If the dimension of null subspace is exactly 1, this gives rise to contact geometry which is a relative of symplectic geometry in odd dimensions.

One of the essential difference of symplectic geometry and Riemannian geometry is the existence of *local* geometry. In the Riemannian geometry, there is curvature which describes the local geometry of our manifold. Conversely, we have following theorem which asserts that the symplectic manifold has no local geometry i.e. we need to look for global invariants.

Theorem 1.2 (Darboux)**.** For any point p in a symplectic manifold (M, ω), there exists a local coordinate (q, p) near p such that

$$\omega = \sum dp_j \wedge dq_j.$$

Proof. See Theorem 8.1. of [dS]. □

1.2 Almost Complex Structures

In this subsection, we introduce *compatible triple* on a symplectic manifold (M, ω). In short, those are a symplectic form ω, a Riemannian metric $\langle -, - \rangle$ and almost complex structure J. They must satisfy some reasonable relation. We first motivate the notion of almost complex structure, and then introduce the notion.

We can equip some additional structures on symplectic manifolds. For example, one might define Riemannian metric g on symplectic manifold (M, ω). A priori, there is no relationship that should be satisfied between g and ω. We need some observation from the linear Lie groups.

Let's forget the symplectic structure on M for now, and just think M as a smooth orientable $2n$-manifold. Consider the frame bundle $Fr(M) \to M$, whose fiber at p is the local frame, i.e. the choice of ordered basis of T_pM. This is a principal $GL_{2n}(\mathbb{R})$-bundle with canonical action. If we equip Riemannian metric, we only need to consider the *orthonormal* frames on T_pM, using Gram-Schmidt process. Heuristically, we can reduce the fiber which are isomorphic to $GL_{2n}(\mathbb{R})$ to its subgroup $O(2n)$, which is

$$\begin{aligned} O(2n) &= \{A \in GL_{2n}(\mathbb{R}) : \langle Av, Aw \rangle = \langle v, w \rangle\} \\ &= \left\{A \in GL_{2n}(\mathbb{R}) : A^t A = \mathrm{Id}_{2n}\right\} \end{aligned}$$

Here, we used canonical inner product represented by Id_{2n}. The first equation $\langle Av, Aw \rangle = \langle v, w \rangle$ means that A *preserves* g_p as a linear map on T_pM, and in the matrix form we have the latter one. (Note that, there is no group structure on the fibers; they are sets with a free group action.) Such process is called *reduction of stucture group*. With our additional assumption on M that M is orientable, we can reduce the fibers so that every fiber is positively oriented orthonormal frames. That is the reduction of principal $O(n)$-bundle to principal $SO(n)$-bundle, where

$$SO(2n) = \{A \in O(2n) : \det A = 1\}$$

Likewise, let a symplectic structure ω on M is given. Using Darboux's theorem, we might regard ω_p in a matrix form in a local basis

$$\omega_p = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} := J$$

The **symplectic matrix** then represents the symplectic basis with respect to J, and such matrices form a group called **symplectic group**

$$\begin{aligned} Sp(2n) &= \{A \in GL_{2n}(\mathbb{R}) : \omega(Av, Aw) = \omega(v, w)\} \\ &= \left\{A \in GL_{2n}(\mathbb{R}) : A^t J A = J\right\} \end{aligned}$$

Our basic observation is following.

Lemma 1.3. $SO(2n) \cap Sp(2n) = U(n)$, where $U(n)$ is a **unitary group**

$$U(n) = \{A \in GL_n(\mathbb{C}) : A^* A = \mathrm{Id}\}$$

Here, we identified $GL_n(\mathbb{C})$ as a subgroup of $GL_{2n}(\mathbb{R})$. (This step will be illustrated in next paragraph.) The proof of lemma is quite elementary algebra. Since the complex structure is involved, we need a structure corresponding to this.

A **complex structure** on $2n$-dimensional $\mathbb{R}$-vector space V is an endomorphism $J_0 : V \to V$ such that $J_0^2 = -\mathrm{Id}$. One can check that there exists a basis of V in which

$$J_0 = \begin{pmatrix} 0 & -\mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix}$$

Then we can define the *complex basis* to satisfy that $A(Jv) = JA(v)$, i.e. A is $\mathbb{C}$-linear. In matrix form, we have

$$GL_n(\mathbb{C}) = \{A \in GL_{2n}(\mathbb{R}) : AJ_0 = J_0 A\}$$

The reduction of $Fr(M)$ to $GL_n(\mathbb{C})$-bundle corresponds to equipping an endomorphism $J : TM \to TM$ such that $J^2 = -\mathrm{Id}$. Such endomorphism is called **almost complex structure**. Note that, if M itself is a *complex manifold*, then we can take J simply by 'multiplying i'. The situation is quite similar; the tangent spaces are complex vector spaces. But there is a great difference; J is intrinsic (came from the atlas of M) or not. In our case, we regard J as a structure we equipped on.

Now our interest is the relation between these three groups. We have following simple result.

Proposition 1.4.

$$Sp(2n) \cap SO(2n) = SO(2n) \cap GL_n(\mathbb{C}) = GL_n(\mathbb{C}) \cap Sp(2n) = U(n).$$

Proof. See Proposition 5.6.2 of [AD], or Section 12 of [dS]. □

Hence if two of symplectic structure ω, Riemannian metric $\langle -, - \rangle$ and almost complex structure J are given, the another can be determined. Our desired relation is following:

$$\langle -, - \rangle = \omega(-, J-)$$

The most usual viewpoint is that $\omega(-, J-)$ defines a Riemannian metric on M. In this case, we call $(\omega, J, \langle -, - \rangle)$ **compatible triple**, and call J is **compatible with** ω. If M also admit a complex structure and J came from the intrinsic complex structure of M, we call M equipped with compatible triple a **Kähler manifold**.

Note that the space of metrics on M is nonempty and contractible (convex), and therefore the space of ω-compatible J's also is.

Remark 1.5. Here, we have $J = -J_0$ (as matrices) and $\omega(-, J-)$ to be a metric. This is the one which [AD] uses. However, this is not the only sign convention. For example, [dS] have $J = J_0$ in our notation. This depends on the definition of compatibility; $\omega(-, J-)$ or $\omega(J-, -)$ is a metric. The choice of sign is not an essential point, but one should take care.

1.3 Hamiltonian Orbits

1.4 Arnold's Conjecture

2 Floer Homology

In this section, we define Floer complex and its homology on a symplectic manifold (M, ω). To avoid difficulties, [AD] assumed two conditions; that are

- M is closed. That is, M is compact manifold without boundary.
- For every smooth map $f : S^2 \to M$, we have
$$\int_{S^2} f^*\omega = 0.$$
This might be expressed as $\langle \omega, \pi_2(M) \rangle = 0$.
- For every smooth map $f : S^2 \to M$, the pullback bundle f^*TM on S^2 has a symplectic trivialization. It can be expressed as $\langle c_1(M), \pi_2(M) \rangle = 0$, where $c_1(M) = c_1(TM)$ is the first Chern class of M.

One might easily assume that $\pi_2(M) = 0$. I guess that this is the simplest case for the Floer theory.

Of course, there are ways to define the Floer homology on the symplectic manifolds with weak conditions. For example, [S] deals with the *monotone* case, in which there exists some constant τ such that

$$\int_{S^2} f^*c_1(M) = \tau \int_{S^2} f^*\omega.$$

One might ask: why don't we just construct one in the full generality? There are many problems to do so: for example, the Gromov's compactness theorem, which is crucial to show that compactness of our moduli space of Floer trajectories, have different form which allows more and more exceptional cases as we make the restriction weaker, and the theory gets more complicated. The grading of chains, which is given by Maslov index, is also should be modified under weaker assumptions. Our goal is to introduce the idea of Floer theory and outline the construction in the easiest case. Instead, we'll emphasize the points when our assumptions are used critically by writing **Caution**. If one meets the corresponding statements in other cases, then one should read that part more carefully.

2.1 Overview

The construction of Floer homology is parallel to the construction of Morse homology. It's very helpful to have a picture of Morse homology in mind as a prototype. [Mat] describes a construction, and there's a nice explanation in the first few chapters of [AD]. There is also a brief note about Morse theory uploaded in my homepage, which is mainly based on [AD] and [Mil].

Let (M, ω) be a symplectic manifold which satisfies our assumptions. To define Floer homology, we need extra structures. Those are symplectic action functional $\mathcal{A}_H$ and gradient-like vector field of H, which correponds to the

Morse function and pseudo-gradient in Morse homology. A (time-dependent) Hamiltonian function $H : M \times \mathbb{R} \to \mathbb{R}$ determines $\mathcal{A}_H$, and gradient-like vector field would be defined by $J_t \text{grad} H$ where $J = J_t$ is (time-dependent) almost complex structure compatible with ω. Thus we can say that, we will define Floer homology with (M, ω, H, J). However, in the last subsections, we'll see that the homology is independent of a choice of (H, J).

Here's an outline. Compare this with the construction of Morse homology.

1. On the loopspace $\mathcal{L}M$, we define symplectic action functional $\mathcal{A}_H$. $\text{Crit}(\mathcal{A}_H)$ are Hamiltonian 1-orbits. This would be the generators of our chain group $CF_*(H, J)$. As in Morse function, we need nondegeneracy assumption.

2. To make $CF_*(H, J)$ into graded module, we need to assign each critical point a grade. (As in my previous note on Morse homology, we omit the definition of orientation and just think $CF_*(H, J)$ as $\mathbb{Z}/2\mathbb{Z}$-module.) The grading would be given by *Maslov index*, denoted by $\mu(x)$.

3. Consider the trajectory of gradient-like vector fields,

$$u : Z = \mathbb{R} \times S^1 \to M.$$

   This map should satisfy an equation, which is called *Floer equation*. We can show that in the limit such cylinder u *converges* to the Hamiltonian orbit under some assumptions, i.e. there exists $x_\pm \in \text{Crit}(\mathcal{A}_H)$ such that

$$\lim_{s \to \pm\infty} u(s, -) = x_\pm(-).$$

   Quotient by natural $\mathbb{R}$-action on cylinder, we get a *moduli space* of Floer trajetories, $\mathcal{M}$, and in particular $\mathcal{M}(x_+, x_-)$. This will define the differential ∂_X in our chain complex.

4. We first need to show that $\mathcal{M}$, which is the space of solutions of Floer equation, is a smooth manifold. Of course, this does not hold in every case. We need to perturb our H and J to get desired property. Some Fredholm theory will play a role, and we can see that

$$\dim \mathcal{M}(x_+, x_-) = \mu(x_+) - \mu(x_-) - 1.$$

5. For the compactness of $\mathcal{M}$, we need a celebrated *Gromov compactness theorem*. The theorem concerns with the compactness of moduli space of J-holomophic curves, and will give desired answer for us in our case. In particular, it follows that if $\mu(x_+) - \mu(x_-) = 1$, then the space $\mathcal{M}(x_+, x_-)$ is a finite set. Now we can define our differential by

$$\partial_X(x_+) = \sum_{\mu(x_-) = \mu(x_+) - 1} \#\mathcal{M}(x_+, x_-) x_-.$$

6. To show that $\partial_X^2 = 0$, we need to show that for any periodic orbits x_+ and x_- such that $\mu(x_+) = \mu(x_-) - 2$,

$$\sum_{\mu(x)=\mu(x_+)-1} \#\mathcal{M}(x_+, x) \cdot \#\mathcal{M}(x, x_-) = 0.$$

   This could be shown by proving that $\mathcal{M}(x_+, x) \times \mathcal{M}(x, x_-)$ is equal to the boundary of $\mathcal{M}(x_+, x_-)$, which is compact 1-manifold. Such kind of process is called *gluing*. After this, we can define Floer homology $HF_*(H, J)$.

7. Then we show that $HF_*(H, J)$ is independent of the choice of (H, J). The idea and the proof is parallel to the Morse case which is described in the Section 3.4 of [AD]. That is, making a Floer complex with homotopies connecting (H^a, J^a) and (H^b, J^b), and consider the differential of that complex. However, the exactly same method might now work since we cannot work on $M \times [0, 1]$ or $M \times \mathbb{R}$ instead of M under our assumption. For this, we need to check the compactness of new trajectories and gluing areguments again.

8. Finally, we show that for C^2-small H, the elements of $\mathrm{Crit}(\mathcal{A}_H)$ are exactly the critical points of Hamiltonian (i.e. the constant orbits) and the Floer trajectories between (x_+, x_-) are exactly the Morse trajectory between them. Combined with the previous statement, we can now say that $HF_*(M, \omega)$ is isomorphic to $HM_*(M)$, and moreover isomorphic to its singular homology.

2.2 Symplectic Action Functional $\mathcal{A}_H$

Our convention is $S^1 = \mathbb{R}/\mathbb{Z}$. Let $H : M \times S^1 \to \mathbb{R}$ be a periodic Hamiltonian equipped on our (M, ω). We define **symplectic action** $\mathcal{A}_H$ on the loop space $\mathcal{L}M$, which consists of *contractible* smooth loops $x : S^1 \to M$ and equipped with compact-open topology, by following formula.

$$\begin{aligned} \mathcal{A}_H : \mathcal{L}M &\to \mathbb{R} \\ x &\mapsto -\int_{D^2} \tilde{x}^*\omega + \int_0^1 H_t(x(t))dt. \end{aligned}$$

Here, $\tilde{x} : D^2 \to M$ is an extension of x to a unit disk $D^2 \subset \mathbb{C}$, i.e. $\tilde{x}(e^{2\pi it}) = x(t)$ for any $t \in S^1$. Let's see that this is well-defined. Our x is contractible loop, so such extension exists. If we have another extension of x, say $\tilde{x}_1$, then we have

$$\int_{D^2} \tilde{x}^*\omega - \int_{D^2} \tilde{x}_1^*\omega = \int_{S^2} f^*\omega$$

where $f : S^2 \to M$ is a map defined by attaching $\tilde{x}$ and $\tilde{x}_1$ along their boundaries. Our assumption that $\langle \omega, \pi_2(M) \rangle = 0$ assures that the map $\mathcal{A}_H$ is well-defined.

Caution 1. Here's the first point; the definition of $\mathcal{A}_H$ depends on the assumption. Otherwise, one might define $\mathcal{A}_H$ as S^1-valued map. See [MS] for example.

With Morse-point of view, we want to see the properties that the critical points of $\mathcal{A}_H$. There is a quite nice description on it. Recall that the tangent vector of a loop x is a vector field along x. To use the *calculus of variation*, we need to define an appropriate variation for x, and here for $\tilde{x}$ too. Let $\alpha : S^1 \times (-\varepsilon, \varepsilon) \to M$ be a variation so that

$$\alpha(t,0) = x(t), \ \frac{\partial \alpha}{\partial s}(t,0) = X_t$$

for some vector field X_t along x. We extend α to $\tilde{\alpha} : D^2 \times (-\varepsilon, \varepsilon) \to M$ so that

$$\tilde{\alpha}(z,0) = \tilde{x}(z), \ \tilde{\alpha}(e^{2\pi i t}, s) = \alpha(t,s), \ \frac{\partial \tilde{\alpha}}{\partial s}(z,0) = \tilde{X}_t$$

where $\tilde{X}_t$ is an extension of X_t to D^2. We have the formula.

$$d_x\mathcal{A}_H(X) = \frac{\partial}{\partial s}\mathcal{A}_H(\alpha)\bigg|_{s=0} = -\frac{\partial}{\partial s}\int_{D^2}\tilde{\alpha}^*\omega + \frac{\partial}{\partial s}\int_0^1 H_t(\alpha(t,s))dt\bigg|_{s=0}$$

Let's compute this term by term. For the first term, we have

$$\begin{aligned}\frac{\partial}{\partial s}\int_{D^2}\tilde{\alpha}^*\omega &= \int_{D^2}\left(\frac{d}{ds}\tilde{\alpha}^*\omega\right)|_{s=0} = \int_{D^2}\alpha^*(\mathcal{L}_X\omega) \\ &= \int_{D^2}\alpha^*(di_X\omega) = \int_{S^1}x^*(i_X\omega) = \int_0^1\omega(X_t,\dot{x}(t))dt\end{aligned}$$

For the second term, we have

$$\begin{aligned}\frac{\partial}{\partial s}\int_0^1 H_t(\alpha(t,s))dt\bigg|_{s=0} &= \int_0^1\frac{\partial}{\partial s}H_t(\alpha(t,s))|_{s=0}dt \\ &= \int_0^1 dH_t(X_t)dt = \int_0^1\omega(X_t,X_H(x(t)))dt\end{aligned}$$

Here, X_H is Hamiltonian vector field associated to Hamiltonian H_t. Put these together, we have

$$d_x\mathcal{A}_H(X) = \int_0^1\omega(X_H(x(t)) - \dot{x}(t), X_t)dt.$$

Since $x \in \mathrm{Crit}(\mathcal{A}_H)$ iff $d_x\mathcal{A}_H(X) = 0$ for any X, we have following description of the critical points of $\mathcal{A}_H$.

Proposition 2.1. $x \in \mathrm{Crit}(\mathcal{A}_H)$ iff $\dot{x} = X_H(x)$. In other words, x is critical loop of symplectic action functional $\mathcal{A}_H$ iff x is a 1-periodic Hamiltonian orbit of H.

2.3 Maslov Index: Gradings for $\mathrm{Crit}(\mathcal{A}_H)$

2.4 Floer Equation and Floer Trajectories

2.5 Transversality: Smoothness of $\mathcal{M}(x_+, x_-)$

2.6 Gromov Compactness: Compactness of $\mathcal{M}(x_+, x_-)$

2.7 Gluing: $\partial_X^2 = 0$

2.8 Invariance of Floer Homology on (H, J)

2.9 Floer Homology and Morse Homology

References

[A] V.Arnold: *Mathematical Methods of Classical Mechanics*, Springer, Second Edition (1989)

[AD] M.Audin, M.Damian: *Morse Theory and Floer Homology*, Springer (2010)

[dS] C.da Silva: *Lectures on Symplectic Geometry*, Springer (2008)

[Mat] Y.Mastumoto: *An Introduction to Morse Theory*, Iwanami Series in Modern Mathematics (2002)

[Mil] J.Milnor: *Morse Theory*, Princeton University Press (1963)

[MS] D.McDuff, D.Salamon: *Introduction to Symplectic Topology*, Oxford (1998)

[S] D.Salamon: *Lectures on Floer Homology*, University of Warwick (1997)