Static Games of incomplete Information

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Games with Incomplete Information

- Some players have incomplete information about some components of the game
  - Firm does not know rival's cost
  - Bidder does not know valuations of other bidders in an auction
- We could also say some players have private information
- Suppose you make an offer to buy out a company
- If the value of the company is V it is worth 1.5V to you
- The seller accepts only if the offer is at least V
- If you know V what do you offer?
- You know only that V is uniformly distributed over [0, 100]. What
- should you offer?

■ \$100?
■ \$50?

Bayesian Games

- We will first look at incomplete information games where players move simultaneously—Bayesian games
- What is new in a Bayesian game?
- Each player has a type: summarizes a player's private information
- Type set for player i: Θ_i
  - A generic type: θ_i
  - Set of type profiles: $\Theta = \Pi_{i \in N} \Theta_i$
  - A generic type profile: $= \{\theta_i, \theta_{-i}\}$
- Each player has beliefs about others' types
  - $p_i : \Theta_i \to \Delta(\Theta_{-i})$
  - $p_i(\theta_{-i} | \theta_i)$
- Players' payoffs depend on types
  - $u_i : A \times \Theta \to R$
  - $u_i(a | \theta)$

- Different types of same player may play different strategies
  - $a_i : \Theta_i \to A_i$
  - $\alpha_i : \Theta_i \to \Delta A_i$
- The Harsanyi (1967) transformation
- An incomplete info. (Bayesian) game can be regarded to unfold as:

1 Nature draws a type according to initial beliefs (prior)
2 each player i observes his own type but not other players' type
3 each i simultaneously (or sequentially) choose actions from feasible set
4 payoff of each player is realized

Bayesian Games

- The intuition/theme of Harsanyi (1967/68): the construction transforms a game of incomplete info. to imperfect info. by the introduction of an imaginary player \Nature".

Definition

Bayesian equilibrium is a collection of strategies (one for each type of each player) such that each type best responds given her beliefs about other players' types and their strategies

Bank Runs

You (player 1) and another investor (player 2) have a deposit of \$100 each in a bank
If the bank manager is a good investor you will each get \$150 at the end of the year. If not you loose your money
You can try to withdraw your money now but the bank has only \$100 cash

- If only one tries to withdraw she gets \$100
- If both try to withdraw they each can get \$50

You believe that the manager is good with probability q
Player 2 knows whether the manager is good or bad
You and player 2 simultaneously decide whether to withdraw or not

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Bank Runs

The payoff can be summarized as follows

	W	N
W	$50, 50$	$100, 0$
N	$0, 100$	$150, 150$

Good q

	W	N
W	$50, 50$	$100, 0$
N	$0, 100$	$0, 0$

Bad $1 - q$

Bank Runs

Two Possible Types of Bayesian Equilibria

1 Separating Equilibria: Each type plays a different strategy

2 Pooling Equilibria: Each type plays the same strategy

How would you play if you were Player 2 who knew the banker was bad?

Player 2 always withdraws in bad state

Separating Equilibria

1 (Good: W, Bad: N)

- Not possible since W is a dominant strategy for Bad

2 (Good: N, Bad: W)
Player 1's expected payoffs
W: q *100 + (1 - q) *50
N: q *150 + (1 - q) *0
Two possibilities
1, $q < 1/2$: Player 1 chooses W. But then player 2 of Good type must play W, which contradicts our hypothesis that he plays N
2, $q \geq 1/2$: Player 1 chooses N. The best response of Player 2 of Good type is N, which is the same as our hypothesis

Separating Equilibria

1, q $<1/2$: No separating equilibrium

2, q $\geq 1/2$: Player 1: N. Player 2: (Good: N, Bad: W)

Pooling Equilibria

1 (Good: N, Bad: N)

- Not possible since W is a dominant strategy for Bad

2 (Good: W, Bad: W)
Player 1's expected payoffs
W: q *50 + (1 - q) *50
N: q *0+ (1 - q) *0
Player 1 chooses W. Player 2 of Good type's best response is W.
Therefore, for any value of q the following is the unique:

- Pooling Equilibrium: Player 1: W, Player 2: (Good: W, Bad: W)

If q $<$ 1/2 the only equilibrium is a bank run

Cournot Duopoly with Incomplete Information

Two firms. They choose how much to produce $q_i \in R_+$
Firm 1 has high cost: c_H
Firm 2 has either low or high cost: c_L or c_H
Firm 1 believes that Firm 2 has low cost with probability μ ■ [0, 1]
payoff function of player i with cost c_j:

$$u_i(q_1, q_2, c_j) = (a - (q_1 + q_2))q_i - c_j q_i$$

Strategies:

$$q_1 \in R_+; \; q_2 : \{c_L, c_H\} \rightarrow R_+.$$

Complete Information

Firm 1

$$\max_{q_1} (a - (q_1 + q_2)) q_1 - c_H q_1$$

Best response function

$$BR_1(q_2) = \frac{a - q_2 - c_H}{2}.$$

Firm 2

$$\max_{q_2} (a - (q_1 + q_2)) q_2 - c_j q_2$$

Best response functions

$$BR_2(q_1, c_L) = \frac{a - q_1 - c_L}{2}$$

$$BR_2(q_1, c_H) = \frac{a - q_1 - c_H}{2}.$$

Nash Equilibrium

If firm 2's cost is c_H,

$$q_1 = q_2 = \frac{a - c_H}{3}.$$

If firm 2's cost is c_L,

$$q_1 = \frac{a - 2c_H + c_L}{3},$$

$$q_2 = \frac{a - 2c_L + c_H}{3}.$$

Inmplete Information

Firm 2

$$\max_{q_2}(a-(q_1+q_2))q_2 - c_j q_2$$

Best response functions

$$BR_2\left(q_1, c_L\right) = \frac{a-q_1-c_L}{2}$$
$$BR_2\left(q_1, c_H\right) = \frac{a-q_1-c_H}{2}.$$

Firm 1

$$\max_{q_1}\mu\left[(a-(q_1+q_2\left(c_L\right)))q_1 - c_H q_1\right]$$
$$+\left(1-\mu\right)\left[(a-(q_1+q_2\left(c_H\right)))q_1 - c_H q_1\right]$$

Best response function

$$BR_1\left(q_2\left(c_L\right), q_2\left(c_H\right)\right)$$
$$= \frac{a-\left[\mu q_2\left(c_L\right)+\left(1-\mu\right)q_2\left(c_H\right)\right]-c_H}{2}.$$

Incomplete Information

Bayesion Nash Equilibrium

$$q_1 = \frac{a - c_H - \mu\left(c_H - c_L\right)}{3}.$$

If firm 2's cost is c_L,

$$q_2\left(c_L\right) = \frac{a - c_L + \left(c_H - c_L\right)}{3} - \left(1 - \mu\right)\frac{c_H - c_L}{6},$$

If firm 2's cost is c_H,

$$q_2\left(c_H\right) = \frac{a - c_H}{3} + \mu\frac{c_H - c_L}{6}.$$

Is information good or bad for firm 2?
Does firm 1 want firm 2 to know its costs?

Mixed Strategies Revisited

The batter of sex

	Opera	*Fight*
Opera	2, 1	0, 0
Fight	0, 0	1, 2

Consider mixed strategy.

The crucial feature of a mixed strategy NE is not that player j chooses a strategy randomly, but rather that player i is uncertain about player j's choice.

This uncertainty can arise either because of randomization or because of a little incomplete information.

Mixed Strategies Revisited

Assume the payoff matrix is

$$\begin{array}{ccc} & \textit{Opera} & \textit{Fight} \\ \textit{Opera} & 2+t_c, 1 & 0, 0 \\ \textit{Fight} & 0, 0 & 1, 2+t_p \end{array},$$

with t_c, t_p independently drawn from uniform distribution $[0, 1]$.

A Bayesian Equilibrium

Consider the following strategies:
Chris goes to Opera if $t_c \geq c$; otherwise, goes to Fight
Pat chooses Fight if $t_p \geq p$; otherwise, chooses Opera.
Now we determine value of c and p such that these strategies are Bayesian Nash Equilibrium.

A Bayesian Equilibrium

Given Pat's strategy, Chris's expected payoffs from playing Opera and playing Fight are

$$\frac{p}{x}(2+t_c)+\left[1-\frac{p}{x}\right]*0=\frac{p}{x}(2+t_c),$$

and

$$\frac{p}{x}*0+\left[1-\frac{p}{x}\right]*1=1-\frac{p}{x}.$$

Thus, playing opera is optimal iff

$$t_c \geq \frac{x}{p}-3=c. \tag{1}$$

Similarly, Given Chris's strategy, Pat will play Fight iff

$$t_p \geq \frac{x}{c}-3=p. \tag{2}$$

A Bayesian Equilibrium

Conbining (1) and (2), we know the probability that Chris plays Opera and the probability that Pat Plays Fight equal to

$$\frac{x-c}{x} = \frac{x-p}{x} = 1 - \frac{-3+\sqrt{9+4x}}{2x},$$

which goes to $2/3$ as $x \to 0$.