Part 3: ARMA Process

Instructor: Ma, Jun

School of Economics, Renmin University of China

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The Lag Operator

- Given any time series process $\{X_t\}_{t\in\mathbb{Z}}$, denote $\mathrm{L}X_t = X_{t-1}$ for all $t \in \mathbb{Z}$. Note that L is a linear operator.
- Applying n times:

$$\underbrace{\mathrm{L}\cdots\mathrm{L}}_{n \text{ times}} X_t = \mathrm{L}^n X_t = X_{t-n}.$$

- The inverse of the lag operator is the "forward" operator: $\mathrm{L}^{-1}X_t = X_{t+1}$. Note that $\mathrm{L}^{-1}\mathrm{L}X_t = X_t$.
- For any real numbers $a, b \in \mathbb{R}$, $m, n \in \mathbb{Z}$, any time series processes $\{X_t\}_{t\in\mathbb{Z}}$, $\{Y_t\}_{t\in\mathbb{Z}}$,

$$\left(a\mathrm{L}^m + b\mathrm{L}^n\right)\left(X_t + Y_t\right) = aX_{t-m} + bX_{t-n} + aY_{t-m} + bY_{t-n}.$$

- We define a "lag polynomial": $A\left(\mathrm{L}\right) := a_0 + a_1\mathrm{L} + \cdots + a_p\mathrm{L}^p$.
- Usual calculations apply: e.g., $A\left(\mathrm{L}\right) := 1 - 0.5\mathrm{L}$,
  $B\left(\mathrm{L}\right) := 1 + 4\mathrm{L}^2$ and
  $C\left(\mathrm{L}\right) := A\left(\mathrm{L}\right)B\left(\mathrm{L}\right) = 1 - 0.5\mathrm{L} + 4\mathrm{L}^2 - 2\mathrm{L}^3$. Then
  $A\left(\mathrm{L}\right)B\left(\mathrm{L}\right)X_t = C\left(\mathrm{L}\right)X_t$.

Linear Process

Definition 1
Let $\{Z_t\}_{t\in\mathbb{Z}}$ be any time series process and consider the time series given by

$$X_t := \sum_{k=-\infty}^{\infty} \psi_k Z_{t-k} = \cdots + \psi_{-1} Z_{t+1} + \psi_0 Z_t + \psi_1 Z_{t-1} + \cdots$$

with some sequence of real numbers $\{\psi_k\}_{k=-\infty}^{\infty}$ such that $\left\{\sum_{k=-n}^{n} \psi_k Z_{t-k}\right\}_{n=1}^{\infty}$ converges to some random variable in $\mathcal{L}^2(\Omega, \mathscr{F}, \mathrm{P})$ for all $t \in \mathbb{Z}$. This process is called a linear process with innovations $\{Z_t\}_{t\in\mathbb{Z}}$.

Remark 2
We will study linear processes with $\{Z_t\}_{t\in\mathbb{Z}} \sim WN\left(0, \sigma^2\right)$ or $\{Z_t\}_{t\in\mathbb{Z}} \sim IID\left(0, \sigma^2\right)$.

Linear Process

Remark 3

Let $\psi(z) := \sum_{k=-\infty}^{\infty} \psi_k z^k$ (an infinite order polynomial), then $\sum_{k=-\infty}^{\infty} \psi_k Z_{t-k} = \psi(\mathrm{L}) Z_t$.

Remark 4

We need to know under what conditions, the convergence of $\sum_{k=-\infty}^{\infty} \psi_k X_{t-k}$ is guaranteed, for some given process $\{X_t\}_{t\in\mathbb{Z}}$.

Proposition 5

Let $\{X_t\}_{t\in\mathbb{Z}}$ be a time series process (not necessarily stationary). Let $\{\psi_k\}_{k=-\infty}^{\infty}$ be a real sequence satisfying $\sum_{k=-\infty}^{\infty} |\psi_k| < \infty$ and define $\psi(z) := \sum_{k=-\infty}^{\infty} \psi_k z^k$. Suppose that $\left\{\mathrm{E}\left[X_n^2\right]\right\}_{n=-\infty}^{\infty}$ is bounded, i.e., there exists some $M > 0$ such that $\mathrm{E}\left[X_n^2\right] < M$ for all $n \in \mathbb{Z}$. Then, the series $\psi(\mathrm{L}) X_t = \sum_{k=-\infty}^{\infty} \psi_k X_{t-k}$ converges to some random variable in $\mathcal{L}^2(\Omega, \mathscr{F}, \mathrm{P})$.

Linear Process

Remark 6

If we assume $\{Z_t\}_{t\in\mathbb{Z}} \sim WN\left(0, \sigma^2\right)$, then we can relax the condition $\sum_{k=-\infty}^{\infty} |\psi_k| < \infty$ to $\sum_{k=-\infty}^{\infty} \psi_k^2 < \infty$.

Proposition 7

Assume that $\{Z_t\}_{t\in\mathbb{Z}}$ is stationary with $\mathrm{E}\left[Z_t\right] = \mu$ and ACF $\gamma_Z\left(\cdot\right)$. Let $\{\psi_k\}_{k=-\infty}^{\infty}$ be a real sequence satisfying $\sum_{k=-\infty}^{\infty} |\psi_k| < \infty$. Then $X_t := \sum_{k=-\infty}^{\infty} \psi_k Z_{t-k}$ is again stationary with (1). $\mathrm{E}\left[X_t\right] = \mu\left(\sum_{k=-\infty}^{\infty} \psi_k\right)$ and (2).

$$\gamma_X\left(h\right) = \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} \psi_j \psi_i \gamma_Z\left(h - i + j\right),\ h \in \mathbb{Z}.$$

Linear Process

Remark 8
If, in addition, we assume $\{Z_t\}_{t\in\mathbb{Z}} \sim WN\left(0, \sigma^2\right)$, then, (1). $\mathrm{E}\left[X_t\right] = 0$, for all $t \in \mathbb{Z}$. (2). Since

$$\gamma_Z\left(h - i + j\right) = \begin{cases} \sigma^2 & \text{if } h - i + j = 0 \\ 0 & \text{otherwise,} \end{cases}$$

we have $\gamma_X\left(h\right) = \sigma^2 \sum_{k=-\infty}^{\infty} \psi_k \psi_{k+h}$.

ARMA Process

Definition 9 (Autoregressive Moving Average (ARMA) Process)

A stationary process $\{X_t\}_{t\in\mathbb{Z}}$ is called ARMA (p, q) if it satisfies the equation

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} - \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q},$$

where $\{Z_t\}_{t\in\mathbb{Z}} \sim WN\left(0, \sigma^2\right)$, $\phi_1, ..., \phi_p$ and $\theta_1, ..., \theta_q$ are real numbers. If $\{X_t - \mu\}_{t\in\mathbb{Z}}$ is an ARMA (p, q) process, then $\{X_t\}_{t\in\mathbb{Z}}$ is called ARMA (p, q) with mean μ.

ARMA Process

Remark 10
The equation

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} - \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$$

is called ARMA equation. Let $\{X_t\}_{t\in\mathbb{Z}}$be a stationary solution with $\mathrm{E}[X_t] = \mu$. Taking expectations on both sides, we have

$$\mu(1 - \phi_1 - \cdots - \phi_p) = 0.$$

We will see that when $1 - \phi_1 - \cdots - \phi_p = 0$, there is no stationary solution. Hence, stationary solutions must have zero mean.

ARMA Process

Remark 11
We will show that if the "ARMA equation" has a stationary solution, it must be a linear process.

Remark 12
Define the autoregressive polynomial,
$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p$ and the moving average polynomial, $\theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q$. The ARMA equation can be written as $\phi(\mathrm{L}) X_t = \theta(\mathrm{L}) Z_t$.

ARMA Process

- MA (q) Process. Assume $\phi(z) = 1$ and $\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q$. Then the equation becomes

$$X_t = \theta(\mathrm{L}) Z_t = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}.$$

$(\theta_0 = 1)$ In this case, existence of a solution is obvious. We have to check if the process is stationary. (1). $\mathrm{Var}[X_t] < \infty$, since Z_t's have a finite variance. (2). $\mathrm{E}[X_t] = 0$ for all t. (3).

$$\mathrm{Cov}[X_t, X_{t+h}] = \begin{cases} \sigma^2 \sum_{j=0}^{q-|h|} \theta_j \theta_{j+|h|} & \text{if } |h| \leq q \\ 0 & \text{if } |h| > q, \end{cases}$$

which does not depend on t. (Homework Question)

ARMA Process

- AR(p) ProcessLet $\theta(z) = 1$, $\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$. The equation becomes:

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Z_t.$$

  A special case: AR(1) $X_t = \phi X_{t-1} + Z_t$.
- If $|\phi| < 1$, then we show the following three claims sequentially (1). A stationary solution to the ARMA equation $X_t = \phi X_{t-1} + Z_t$, if exists, must have the form $X_t = \lim_{n\to\infty} X_t^{(n)}$ for some fixed $\left\{X_t^{(n)}\right\}_{n=1}^{\infty}$ for each $t \in \mathbb{Z}$. Therefore, if the stationary solution exists, if must be unique. (2). The limit $X_t = \lim_{n\to\infty} X_t^{(n)}$ (as a vector in the $\mathcal{L}^2$ space) exists and is a solution of the ARMA equation. (3). $\{X_t\}_{t\in\mathbb{Z}}$ is weakly stationary.
- What happens if $|\phi| > 1$?
- What happens if $|\phi| = 1$?

Causality

Definition 13 (Causal Process)

Let $\{X_t\}_{t\in\mathbb{Z}}$ be a time series process. It is causal with respect to another time series $\{Z_t\}_{t\in\mathbb{Z}}$ if

$$X_t = f\left(\cdots, Z_{t-1}, Z_t\right),$$

i.e., X_t depends only on the past.

Remark 14

Causality is a relation between two processes: $\{X_t\}_{t\in\mathbb{Z}}$ is called the state process and $\{Z_t\}_{t\in\mathbb{Z}}$ is called the impulse process. If X_t can be represented as a function of the outcomes of current and past impulses, $Z_t, Z_{t-1}, \ldots$, we say that X_t is caused by past impulses.

ARMA Process

Proposition 15

Assume that $\{Z_t\}_{t\in\mathbb{Z}}$ *is stationary,* $\{\alpha_j\}_{j\in\mathbb{Z}}$ *and* $\{\beta_j\}_{j\in\mathbb{Z}}$ *are absolutely summable:* $\sum_{j=-\infty}^{\infty}|\alpha_j| < \infty$ *and* $\sum_{j=-\infty}^{\infty}|\beta_j| < \infty$ *.*
Denote $\alpha(z) := \sum_{j=-\infty}^{\infty}\alpha_j z^j$ *and* $\beta(z) := \sum_{j=-\infty}^{\infty}\beta_j z^j$*, then*
$(\alpha\beta)(z) := \alpha(z)\beta(z) = \sum_{j=-\infty}^{\infty}\psi_j z^j$ *with* $\psi_j = \sum_{i=-\infty}^{\infty}\alpha_i\beta_{j-i}$*,*
$X_t = \alpha(\mathrm{L})(\beta(\mathrm{L})Z_t)$ *is well-defined (the series converges in* $\mathcal{L}^2$*),*
$\{X_t\}_{t\in\mathbb{Z}}$ *is stationary and*

$$\alpha(\mathrm{L})(\beta(\mathrm{L})Z_t) = (\alpha\beta)(\mathrm{L})Z_t = \beta(\mathrm{L})(\alpha(\mathrm{L})Z_t).$$

Causal Solution of ARMA(p, q)

Theorem 16

Assume $\{Z_t\}_{t\in\mathbb{Z}} \sim WN\left(0, \sigma^2\right)$ *and let* $\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$ *and* $\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q$ *be two complex polynomials (*$\phi_1, ..., \phi_p, \theta_1, ..., \theta_q \in \mathbb{C}$*) which have no common root. (1). Let* $\mathbb{U} := \{z \in \mathbb{C} : |z| \leq 1\}$ *be the unit circle. If* $\phi(z) \neq 0$ *for all* $z \in \mathbb{U}$ *(i.e., all of the p roots of* $\phi(z) = 0$ *are outside of* $\mathbb{U}$*), the ARMA equation*

$$\phi(\mathrm{L}) X_t = \theta(\mathrm{L}) Z_t$$

has a unique stationary and causal (with respect to $\{Z_t\}_{t\in\mathbb{Z}}$*) solution. Let* $\psi(z) := \frac{\theta(z)}{\phi(z)} = \sum_{j=0}^{\infty} \psi_j z^j$ *with* $\sum_{j=1}^{\infty} |\psi_j| < \infty$ *for all* $z \in \mathbb{U}$*. This solution can be represented as* $\psi(\mathrm{L}) Z_t$*.*
(2). If a stationary and causal solution exists, then $\phi(z) \neq 0$ *for all* $z \in \mathbb{U}$*.*

Causal Solution of ARMA(p, q)

Remark 17

If $\phi(z) = 0$ for some $z \in \mathbb{U}$, then no stationary solution exists.

Remark 18

If $\phi(z) = 0$ has roots in the interior of $\mathbb{U}$ (i.e., $\operatorname{int}(\mathbb{U}) := \{z \in \mathbb{C} : |z| < 1\}$) and $\mathbb{U}^c$, but no root on the circle, then a unique stationary of the form $X_t = \sum_{k \in \mathbb{Z}} \psi_k Z_{t-k}$ exists but $\{X_t\}_{t \in \mathbb{Z}}$ is not causal with respect to the impulses $\{Z_t\}_{t \in \mathbb{Z}}$.

Calculate the ACF of ARMA(p, q)

Assume that we have an ARMA process defined by

$$\phi\left(\mathrm{L}\right) X_t = \theta\left(\mathrm{L}\right) Z_t,\ \text{with}\ \{Z_t\} \sim WN\left(0, \sigma^2\right)$$

and ϕ,θ satisfy the conditions of Theorem 16 (so that there exists a unique causal stationary solution). How can we calculate its ACF?

Partial ACF

Definition 19 (Span)

Let $\mathcal{H}$ be a Hilbert space. Let $\{\boldsymbol{x}_t : t \in T\} \subseteq \mathcal{H}$ be a subset of $\mathcal{H}$. Then the span of $\{\boldsymbol{x}_t : t \in T\}$, denoted by $\mathrm{sp}\left(\{\boldsymbol{x}_t : t \in T\}\right)$, is defined to be the set of all finite linear combinations of the form

$$\alpha_1 \boldsymbol{x}_{t_1} + \cdots + \alpha_n \boldsymbol{x}_{t_n},$$

for $(t_1, ..., t_n) \in T^n$, $(\alpha_1, ..., \alpha_n) \in \mathbb{R}^n$, $n \in \mathbb{N}$. $\overline{\mathrm{sp}}\left(\{\boldsymbol{x}_t : t \in T\}\right)$ is the smallest closed subset of $\mathcal{H}$ that contains $\mathrm{sp}\left(\{\boldsymbol{x}_t : t \in T\}\right)$.

Remark 20

If $\{\boldsymbol{x}_t : t \in T\}$ has finitely many vectors, then $\mathrm{sp}\left(\{\boldsymbol{x}_t : t \in T\}\right)$ is closed and hence $\overline{\mathrm{sp}}\left(\{\boldsymbol{x}_t : t \in T\}\right) = \mathrm{sp}\left(\{\boldsymbol{x}_t : t \in T\}\right)$.

Partial ACF

Definition 21 (Partial autocorrelation function)

Let $\{X_t\}_{t\in\mathbb{Z}}$ be a stationary process. The PACF $\alpha : \mathbb{Z} \to \mathbb{R}$ is defined as:

$$\alpha(1) = \mathrm{Corr}(X_2, X_1) = \rho(1), \ldots,$$

$$\alpha(h) = \mathrm{Corr}\left(X_{h+1} - \Pi_{\mathrm{sp}\{1, X_2, \ldots, X_h\}} X_{h+1}, X_1 - \Pi_{\mathrm{sp}\{1, X_2, \ldots, X_h\}} X_1\right).$$

$\Pi_{\mathrm{sp}\{1, X_2, \ldots, X_h\}} X_{h+1}$ is the projection of X_{h+1} (as a vector in $\mathcal{L}^2$) to the subspace $\mathrm{sp}\{1, X_2, \ldots, X_h\}$ of $\mathcal{L}^2$.

PACF for AR(p)

- Suppose $\{X_t\}_{t\in\mathbb{Z}}$ is the unique stationary and causal solution to the $\mathrm{AR}\,(p)$ equation

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + Z_t,$$

  where $\{Z_t\}_{t\in\mathbb{Z}} \sim WN$ and $\{\phi_1, ..., \phi_p\}$ satisfy the condition of Theorem 16.
- We can show: $\alpha_X\,(h) = 0$, if $h > p$.
- We can determine the order (p) of an $\mathrm{AR}\,(p)$ process using $\widehat{\alpha}_X$, an estimate of α_X.
- We know that if $\{X_t\}_{t\in\mathbb{Z}}$ is $\mathrm{MA}\,(q)$, then $\gamma_X\,(h) = 0$, if $h > q$. We can determine the order using $\widehat{\gamma}_X$, an estimate of γ_X.

More on Projections

Proposition 22

Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \mathcal{H}$ be linear subspaces. Then for any $\boldsymbol{x}$, $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots \in \mathcal{H}$, (1). $\Pi_{\mathcal{S}_1}\boldsymbol{x} = \Pi_{\mathcal{S}_1}\Pi_{\mathcal{S}_2}\boldsymbol{x}$; (2). $\|\Pi_{\mathcal{S}_1}\boldsymbol{x}\| \leq \|\boldsymbol{x}\|$; (3). If $\boldsymbol{x} \in \overline{\mathrm{sp}}\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots\}$, then $\lim_{n\to\infty} \Pi_{\mathrm{sp}\{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n\}}\boldsymbol{x} = \boldsymbol{x}$ (or equivalently, $\lim_{n\to\infty} \left\|\Pi_{\mathrm{sp}\{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n\}}\boldsymbol{x} - \boldsymbol{x}\right\| = 0$).

The ARMA Identification Theorem

Proposition 23

If $\{X_t\}_{t\in\mathbb{Z}}$ *is a stationary process with* $\mathrm{E}\left[X_1\right]=0$ *and* $\gamma\left(h\right)=0$ *if* $|h|>q$*, then* $\{X_t\}_{t\in\mathbb{Z}}$ *is an* $\mathrm{MA}\left(q\right)$ *process.*

The ARMA Identification Theorem

Theorem 24

Suppose that $\{Y_t\}_{t\in\mathbb{Z}}$ *is an* $\mathrm{ARMA}\,(p,q)$ *process with ACF* γ_Y*. Let* $\{X_t\}_{t\in\mathbb{Z}}$ *be a stationary process with* $\mathrm{E}\,[X_1] = 0$ *and ACF* γ_X*. If* $\gamma_X = \gamma_Y$*, then* $\{X_t\}_{t\in\mathbb{Z}}$ *is also an* $\mathrm{ARMA}\,(p,q)$ *process. (I.e., no other stationary process could have the same covariance structure as ARMA.)*

Invertiblity of ARMA

Definition 25 (Invertibility)

Let $\{X_t\}_{t\in\mathbb{Z}}$ be a linear process with respect to $\{Z_t\}_{t\in\mathbb{Z}}$. It is invertible if there are coefficients $\{\pi_k\}_{k=0}^{\infty}$ such that $Z_t = \sum_{k=0}^{\infty} \pi_k X_{t-k}$, with $\sum_{k=0}^{\infty} |\pi_k| < \infty$, i.e., we can recover innovations from the observations.

Invertiblity of ARMA

Theorem 26

Assume $\{Z_t\}_{t\in\mathbb{Z}} \sim WN\left(0,\sigma^2\right)$ *and let* $\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$ *and* $\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q$ *be two complex polynomials (*$\phi_1, ..., \phi_p, \theta_1, ..., \theta_q \in \mathbb{C}$*) which have no common root. Let the ARMA process* $\{X_t\}_{t\in\mathbb{Z}}$ *be defined by the ARMA equation*

$$\phi(\mathrm{L})\, X_t = \theta(\mathrm{L})\, Z_t.$$

Let $\mathbb{U} := \{z \in \mathbb{C} : |z| \leq 1\}$ *be the unit circle. If* $\theta(z) \neq 0$ *for all* $z \in \mathbb{U}$ *(i.e., all of the p roots of* $\theta(z) = 0$ *are outside of* $\mathbb{U}$*), then* $\{X_t\}_{t\in\mathbb{Z}}$ *is invertible.*

Invertiblity of ARMA

- The $\mathrm{AR}\,(p)$ process

$$X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + Z_t$$

is trivially invertible.

- The $\mathrm{MA}\,(1)$ process

$$X_t = Z_t + \theta Z_{t-1}$$

is invertible if $|\theta| < 1$ and $Z_t = \sum_{k=0}^{\infty} (-\theta)^k X_{t-k}$.

Invertibility of ARMA

Remark 27

Consider two $\mathrm{MA}(1)$ processes with $|\theta| < 1$,
$\{Z_t\}_{t\in\mathbb{Z}} \sim WN\left(0, \sigma^2\right)$,

$$Y_t = Z_t + \theta Z_{t-1}$$

and

$$W_t = Z_t + \frac{1}{\theta} Z_{t-1}.$$

$\{Y_t\}_{t\in\mathbb{Z}}$ and $\{W_t\}_{t\in\mathbb{Z}}$ have the same ACF. $\{Y_t\}_{t\in\mathbb{Z}}$ is invertible but $\{W_t\}_{t\in\mathbb{Z}}$ is not. In fact, every invertible MA process has a non-invertible representation that gives the same ACF.

The CLT for Linear Process

Theorem 28 (CLT for Linear Processes)

Let $\{X_t\}_{t\in\mathbb{Z}}$ be a linear process defined by $X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}$ with $\{Z_t\}_{t\in\mathbb{Z}} \sim IID\left(0, \sigma^2\right)$, $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$. Denote $\overline{X}_n = n^{-1} \sum_{j=1}^{n} X_j$. Then

$$\sqrt{n}\left(\overline{X}_n - \mu\right) \to_d \mathrm{N}\left(0, \tau^2\right),$$

where $\tau^2 = \sum_{h=-\infty}^{\infty} \gamma_X(h)$ and γ_X is the ACF of $\{X_t\}_{t\in\mathbb{Z}}$. Notice that $\gamma_X(h) = \sigma^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h}$ so that $\sum_{h=-\infty}^{\infty} \gamma_X(h) = \sigma^2 \left(\sum_{j=-\infty}^{\infty} \psi_j\right)^2$.

The CLT for Linear Process

Lemma 29

For $m = 1, 2, \ldots$, let $\{X_{n,m}\}_{n=1}^{\infty}$ be a sequence of random variables such that $X_{n,m} \to_d X_m$ as $n \to \infty$ and $X_m \to_d X$ as $m \to \infty$. Then there exists a sequence $m_n \to \infty$ as $n \to \infty$ such that $X_{n,m_n} \to_d X$. Suppose $\{Z_n\}_{n=1}^{\infty}$ is another sequence of random variables such that

$$\limsup_{m \to \infty} \limsup_{n \to \infty} \mathrm{P}\left[|X_{n,m} - Z_n| > \epsilon\right] = 0$$

for every $\epsilon > 0$. Then $Z_n \to_d X$.

The CLT for Linear Process

Definition 30

A strictly stationary process $\{X_t\}_{t\in\mathbb{Z}}$ is called m-dependent ($m > 0$ is a fixed integer), if the two sets of random variables $\{X_s\}_{s\leq t}$ and $\{X_s\}_{s\geq t+m+1}$ are independent for all t.

Remark 31

An $\mathrm{MA}\,(q)$ process is q-dependent if $\{Z_t\}_{t\in\mathbb{Z}} \sim IID\left(0,\sigma^2\right)$.

Theorem 32 (CLT for m-dependent Process)

Let $\{X_t\}_{t\in\mathbb{Z}}$ be a strictly stationary m-dependent process with mean zero. Then

$$\sqrt{n}\cdot\overline{X}_n \to_d \mathrm{N}\left(0,\sum_{h=-m}^{m}\gamma_X\left(h\right)\right).$$