Time Series Analysis HW#1
Due: March 07

Exercise 1. Let $X_1, ..., X_n$ satisfy:

$$\begin{cases} X_i = \mu + e_i & i = 1, ..., n \\ e_i = \beta e_{i-1} + \epsilon_i & i = 1, ..., n, \;\; e_0 = 0, \end{cases}$$

where $\epsilon_1, \epsilon_2, \epsilon_3, ...$ are independent, identically distributed, $\mathrm{E}[\epsilon_i] = 0$ and $\mathrm{Var}[\epsilon_i] = \sigma^2$. (This is the standard time series AR(1) model (autoregressive of order 1)).

(a) Use $\mathrm{E}[X_i]$ to give a method of moments estimate of μ.

(b) Suppose $\mu = \mu_0$ and $\beta = b$ are fixed. Use $\mathrm{E}\left[U_i^2\right]$ where U_i

$$U_i \equiv \frac{X_i - \mu_0}{\left(\sum_{j=0}^{i-1} b^{2j}\right)^{1/2}}$$

to give a method of moments estimate of σ^2.

(c) If μ and σ^2 are fixed, can you give a method of moments estimate of β?

Exercise 2. Let $\{Z_t\}$ be a sequence of i.i.d. $\mathrm{N}\left(0, \sigma^2\right)$ random variables. Which of the following processes are weakly stationary? c is a constant. Compute the mean $\mathrm{E}[X_t]$ and the autocovariance function $\gamma_X(h) = \mathrm{Cov}[X_t, X_{t+h}]$.

(a) $X_t = a + bZ_t + cZ_{t-1}$;

(b) $X_t = Z_1 \cos(ct) + Z_2 \sin(ct)$;

(c) $X_t = Z_t \cos(ct) + Z_{t-1} \sin(ct)$;

(d) $X_t = a + bZ_0$.

Exercise 3. Let $\{X_t\}$ be the AR(1) process defined by $X_t = \theta X_{t-1} + Z_t$ where $\{Z_t\}$ is a sequence of i.i.d. $\mathrm{N}\left(0, \sigma^2\right)$ random variables. Compute the variance of the sample mean $\frac{1}{n}\sum_{j=1}^{n} X_j$.

Exercise 4. Let $\{S_t\}_{t \in \mathbb{Z}_+}$ be the random walk with constant drift μ, defined by

$$\begin{cases} S_0 = 0 \\ S_t = \mu + S_{t-1} + X_t & t = 1, 2, 3, ... \end{cases}$$

where $\{X_t\} \sim IID(0, \sigma^2)$. Show that the first difference of S_t, i.e. $S_t - S_{t-1}$, for $t = 1, 2, 3, ...$ is strictly stationary and compute its mean and autocovariance function.