Advanced Econometrics

Lecture 8: Restricted Estimation (Hansen Chapter 8)

有约束的估计

Instructor: Ma, Jun

Renmin University of China

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只讲前面四节.

Introduction

- The linear regression model:

$$\begin{aligned} Y_i &= \boldsymbol{X}_i'\boldsymbol{\beta} + e_i \\ \mathbb{E}\left(\boldsymbol{X}_i e_i\right) &= \mathbf{0}. \end{aligned}$$

  Partition $\boldsymbol{X}_i' = (\boldsymbol{X}_{1i}', \boldsymbol{X}_{2i}')$ and $\boldsymbol{\beta}' = \left(\boldsymbol{\beta}_1', \boldsymbol{\beta}_2'\right)$.
- "Exclusion restriction": $\boldsymbol{\beta}_2 = \mathbf{0}$.
- The constrained model:

$$\begin{aligned} Y_i &= \boldsymbol{X}_{1i}'\boldsymbol{\beta}_1 + e_i \\ \mathbb{E}\left(\boldsymbol{X}_i e_i\right) &= \mathbf{0}. \end{aligned}$$

- In the constrained model, e_i is uncorrelated with the entire regressor vector $\boldsymbol{X}_i' = (\boldsymbol{X}_{1i}', \boldsymbol{X}_{2i}')$.

Introduction

- A set of q linear constraints on $\boldsymbol{\beta}$ takes the form

$$\boldsymbol{R}'\boldsymbol{\beta} = \boldsymbol{c}.$$

  This is thought of as some restriction that the parameter satisfies we believe to be true, based on prior knowledge.
- $\boldsymbol{R}$ is $k \times q$ and $\operatorname{rank}(\boldsymbol{R}) = q < k$, $\boldsymbol{c}$ is $q \times 1$. The constraints are linearly independent.
- The restricted parameter space:

$R'\beta - c = 0$ 是一个约束.

$$\boldsymbol{B}_{\boldsymbol{R}} = \{\boldsymbol{\beta} : \boldsymbol{R}'\boldsymbol{\beta} = \boldsymbol{c}\}.$$

- The exclusion restriction $\boldsymbol{\beta}_2 = \mathbf{0}$ is a special case:

$$\boldsymbol{R} = \begin{pmatrix} \mathbf{0} \\ \boldsymbol{I} \end{pmatrix} \text{ and } \boldsymbol{c} = \mathbf{0}.$$

Constrained Least Squares

$$\min_{b \in B_R} \sum_{i=1}^{n} (Y_i - X_i' b)^2$$

有约束的最小二乘估计

- We estimate the constrained linear regression by minimizing the least-squares criterion subject to the constraint $\boldsymbol{R}'\boldsymbol{\beta} = \boldsymbol{c}$.
- The constrained LS estimator:

$$\widetilde{\boldsymbol{\beta}}_{\text{cls}} = \underset{\boldsymbol{R}'\boldsymbol{\beta}=\boldsymbol{c}}{\operatorname{argmin}} SSE(\boldsymbol{\beta})$$

$$SSE(\boldsymbol{\beta}) = \sum_{i=1}^{n} (Y_i - \boldsymbol{X}_i'\boldsymbol{\beta})^2 = \boldsymbol{Y}'\boldsymbol{Y} - 2Y'\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\beta}'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{\beta}.$$

- We find solution by using Lagrange multipliers. The Lagrangian function:

$$\boldsymbol{\mathcal{L}}(\boldsymbol{\beta}, \boldsymbol{\lambda}) = \frac{1}{2} SSE(\boldsymbol{\beta}) + \boldsymbol{\lambda}'(\boldsymbol{R}'\boldsymbol{\beta} - \boldsymbol{c}).$$

- The first-order conditions:

$$\frac{\partial}{\partial \boldsymbol{\beta}} \boldsymbol{\mathcal{L}}\left(\widetilde{\boldsymbol{\beta}}_{\text{cls}}, \widetilde{\boldsymbol{\lambda}}_{\text{cls}}\right) = -\boldsymbol{X}'\boldsymbol{Y} + \boldsymbol{X}'\boldsymbol{X}\widetilde{\boldsymbol{\beta}}_{\text{cls}} + \boldsymbol{R}\widetilde{\boldsymbol{\lambda}}_{\text{cls}} = \boldsymbol{0}$$

$$\frac{\partial}{\partial \boldsymbol{\lambda}} \boldsymbol{\mathcal{L}}\left(\widetilde{\boldsymbol{\beta}}_{\text{cls}}, \widetilde{\boldsymbol{\lambda}}_{\text{cls}}\right) = \boldsymbol{R}'\widetilde{\boldsymbol{\beta}}_{\text{cls}} - \boldsymbol{c} = \boldsymbol{0}.$$

Constrained Least Squares

- Premultiplying by $\boldsymbol{R}'\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}$,

$$-\boldsymbol{R}'\widehat{\boldsymbol{\beta}} + \boldsymbol{R}'\widetilde{\boldsymbol{\beta}}_{\text{cls}} + \boldsymbol{R}'\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{R}\widetilde{\boldsymbol{\lambda}}_{\text{cls}} = \mathbf{0},$$

  where $\widehat{\boldsymbol{\beta}} = \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{X}'\boldsymbol{Y}$ is the unrestricted LS.
- Solving for $\widetilde{\boldsymbol{\lambda}}_{\text{cls}}$:

$$\widetilde{\boldsymbol{\lambda}}_{\text{cls}} = \left[\boldsymbol{R}'\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{R}\right]^{-1}\left(\boldsymbol{R}'\widehat{\boldsymbol{\beta}} - \boldsymbol{c}\right).$$

- Solving for $\widetilde{\boldsymbol{\beta}}_{\text{cls}}$:

$$\widetilde{\boldsymbol{\beta}}_{\text{cls}} = \widehat{\boldsymbol{\beta}} - \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{R}\left[\boldsymbol{R}'\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{R}\right]^{-1}\left(\boldsymbol{R}'\widehat{\boldsymbol{\beta}} - \boldsymbol{c}\right).$$

if $R'\widehat{\beta} - c = 0$, $\widetilde{\beta}_{cls} = \widehat{\beta}$

如果无约束的极值点满足约束，那么无约束与有约束是一样的.

Exclusion Restriction

- The unconstrained model:

$$Y_i = \boldsymbol{X}_{1i}'\boldsymbol{\beta}_1 + \boldsymbol{X}_{2i}'\boldsymbol{\beta}_2 + e_i.$$

- The exclusion restriction: $\boldsymbol{\beta}_2 = \mathbf{0}$. The constrained equation is

$$Y_i = \boldsymbol{X}_{1i}'\boldsymbol{\beta}_1 + e_i.$$

- The constrained LS is

$$\widetilde{\boldsymbol{\beta}}_{\text{cls}} = \widehat{\boldsymbol{\beta}} - (\boldsymbol{X}'\boldsymbol{X})^{-1}\begin{pmatrix} \mathbf{0} \\ \boldsymbol{I} \end{pmatrix}\left[\begin{pmatrix} \mathbf{0} & \boldsymbol{I} \end{pmatrix}(\boldsymbol{X}'\boldsymbol{X})^{-1}\begin{pmatrix} \mathbf{0} \\ \boldsymbol{I} \end{pmatrix}\right]^{-1}\begin{pmatrix} \mathbf{0} & \boldsymbol{I} \end{pmatrix}\widehat{\boldsymbol{\beta}}.$$

- Easy to check:

$$\widetilde{\boldsymbol{\beta}}_{\text{cls}} = \begin{pmatrix} \left(\sum_{i=1}^n \boldsymbol{X}_{1i}\boldsymbol{X}_{1i}'\right)^{-1}\left(\sum_{i=1}^n \boldsymbol{X}_{1i}Y_i\right) \\ \mathbf{0} \end{pmatrix}.$$

无约束最小二乘估计

$Y_i = X_i'\beta + e_i$

$H_0: \beta_1 = 0, \ \beta_2 = 0$

$H_1: \beta_1 \neq 0$ 或 $\beta_2 \neq 0$

用 F test (有限样本)

Wald test (大样本)

$$\tilde\beta = \hat\beta - (X'X)^{-1}R'(R(X'X)^{-1}R')^{-1}(R\hat\beta - c)$$

$$\tilde e = Y - X\tilde\beta = \hat e + (X'X)^{-1}R'(R(X'X)^{-1}R')^{-1}(R\hat\beta - c)$$

$$\underbrace{\tilde e'\tilde e}_{SSR_r} = \hat e'\hat e + (R\hat\beta-c)'(R(X'X)^{-1}R')^{-1}(R\hat\beta-c)$$

$$+2\underbrace{\hat e'X}_{0}(X'X)^{-1}R'(R(X'X)^{-1}R')^{-1}(R\hat\beta-c)$$

$$= \underbrace{\hat e'\hat e}_{SSR_{ur}} + \underbrace{(R\hat\beta-c)'(R(X'X)^{-1}R')^{-1}(R\hat\beta-c)}_{>0 \text{ 二次型}} \Rightarrow \boxed{SSR_r \geq SSR_{ur}}$$

$$F = \frac{(SSR_r - SSR_{ur})/p}{SSR_{ur}/(n-k)} \qquad \frac{SSR_{ur}}{n-k} = s^2$$

$$= \frac{(R\hat\beta-c)'(s^2R(X'X)^{-1}R')^{-1}(R\hat\beta-c)}{p} \qquad F\cdot p = \text{Wald (同方差下)}$$

$R\, s^2(X'X)^{-1}R'$ — 同方差假设下, $R\hat V_{\hat\beta}R'$.