Auction

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Auction

Many economic transactions are conducted through auctions

- mineral rights; airwave spectrum rights; art work; antiques; government contracts

Also can be thought of as auctions

- takeover battles; queues; wars of attrition; lobbying contests

Auction Formats

1, Open Bid Auction

- ascending-bid auction (English Auction)
  - price is raised until only one bidder remains, who wins and pays the final price
- descending-bid auction (Dutch auction)
  - price is lowered until someone accepts, who wins the object at the current price

Auction Formats

2, Sealed bid auctions

- first price auction
  - highest bidder wins; pays her bid
- second price auction (Vickrey auction)
  - highest bidder wins; pays the second highest bid

Strategically Equivalent Format

- English Auction $\Leftrightarrow$ Second price auction
- Dutch Auction $\Leftrightarrow$ First price auction

Private vs Common Value

Auctions also differ with respect to the valuation of the bidders

1 Private value auctions

- each bidder knows only her own value, which does not depends on others' valuation
  - consumer goods on eBay, a train ticket (queues), artwork, antiques

2 Common value auctions

- actual value of the object is the same for everyone
  - bidders have different private information about that value
  - oil field auctions, company takeovers

Independent Private Value

Each bidder knows only her own valuation
Valuations are independent across bidders
Bidders have beliefs over other bidders' values
Risk neutral bidders

- If the winner's value is v and pays p, her payoff is $v - p$

Auctions as a Bayesian Game

set of players $N = \{1, 2, ..., n\}$
type set $\Theta_i = [\underline{v}, \overline{v}]$, $v \geq 0$.
action set, $A_i = R_+$
beliefs

- opponents' valuations are independent draws from a distribution

function F, which is strictly increasing and continuous
payoff function

$$u_i(a, v) = \begin{cases} \frac{v_i - P(a)}{m}, \text{ if } a_j \leq a_i \text{ for all } j \neq i, \text{ and } |\{j : a_j = a_i\}| = m \\ 0, \text{ if } a_j > a_i \text{ for some } j \neq i. \end{cases}$$

- $P(a)$ is the price paid by the winner if the bid profile is a.

I. Bidding your value weakly dominates bidding higher

Suppose your value is \$10 but you bid \$15. Three cases:

1 There is a bid higher than \$15 (e.g. \$20)

- You loose either way: no difference

2 2nd highest bid is lower than \$10 (e.g. \$5)

- You win either way and pay \$5: no difference

3 2nd highest bid is between \$10 and \$15 (e.g. \$12)

- You loose with \$10: zero payoff
- You win with \$15: loose \$2

Second Price Auction

II. Bidding your value weakly dominates bidding lower

Suppose your value is \$10 but you bid \$5. Three cases:

1 There is a bid higher than \$10 (e.g. \$12)

- You loose either way: no difference

2 2nd highest bid is lower than \$5 (e.g. \$2)

- You win either way and pay \$2: no difference

3 2nd highest bid is between \$5 and \$10 (e.g. \$8)

- You loose with \$5: zero payoff
- You win with \$10: earn \$2

Assume two bidders, with valuation i.i.d on $U\left[0,1\right]$.
The expected revenue of the seller is The expected revenue is $1/3$. Why $1/3$? If we take two draws from a uniform distribution on [0; 1], the higher draw will be on average $2/3$ and the lower draw will be on average $1/3$.

$$ER = E\min\left(v_1, v_2\right) = \frac{1}{3}.$$

First Price Auction

Highest bidder wins and pays her bid
Would you bid your value?
What happens if you bid less than your value?
- You get a positive payoff if you win
- But your chances of winning are smaller
- Optimal bid reflects this tradeoff
Bidding less than your value is known as bid shading

Only 2 bidders
You are player 1 and your value is $v > 0$
You believe the other bidder's value is uniformly distributed over $[0, 1]$
You believe the other bidder uses strategy $b(v_2) = av_2$
Your expected payoff if you bid b is

$$\begin{aligned} (v-b)prob(\text{you win}) &= (v-b)prob(b > av_2) \\ &= (v-b)prob(v_2 < b/a) \\ &= (v-b)\frac{b}{a}. \end{aligned}$$

Bayesian Equilibrium of First Price Auctions

Maximizing implies first derivative equal to zero

$$-\frac{b}{a}+\frac{v-b}{a}=0.$$

Solving for b

$$b=\frac{v}{2}.$$

Bidding half the value is a Bayesian equilibrium

Bayesian Equilibrium of First Price Auctions

The expected revenue is again $1/3$. Why? On average the higher of the two values is $2/3$, and the higher of the two bids is $1/3$.

$$
\begin{aligned}
ER &= E\max\left(b_1, b_2\right) \\
&= E\max\left(\frac{v_1}{2}, \frac{v_2}{2}\right) \\
&= 1/3.
\end{aligned}
$$

Theorem

If there are two bidders with values drawn from U[0; 1], then any standard auction has an expected revenue $1/3$.

Theorem

If there are N bidders with values drawn from a continuous distribution (e.g. uniform on $[a, b]$*), then any standard auction leads to the same expected revenue.*

Bayesian Equilibrium of First Price Auctions

n bidders
You are player 1 and your value is $v > 0$
You believe the other bidders' values are independently uniformly distributed over $[0, 1]$
You believe the other bidders use strategy $b(v_2) = av_2$
Your expected payoff if you bid b is

$$
\begin{aligned}
(v-b)prob(\text{you win}) &= (v-b)prob(b > av_2, \ldots b > av_n) \\
&= (v-b)prob(v_2 < b/a \ldots v_n < b/a) \\
&= (v-b)prob(v_2 < b/a) \ldots prob(v_n < b/a) \\
&= (v-b)\left(\frac{b}{a}\right)^{n-1}.
\end{aligned}
$$

Maximizing implies first derivative equal to zero

$$-\left(\frac{b}{a}\right)^{n-1}+(n-1)\,\frac{v-b}{a}\left(\frac{b}{a}\right)^{n-2}=0.$$

Solving for b

$$b=\frac{(n-1)\,v}{n}.$$

Is b increasing or decreasing in n?

n bidders
You are player 1 and your value is $v > 0$
You believe the other bidders' values are independently distributed over $[\underline{v}, \overline{v}]$, with c.d.f F.
You believe the other bidders use strategy $B(v_2)$ (maybe non-linear).
Your expected payoff if you bid b is

$$\begin{aligned}(v-b)prob(\text{you win}) &= (v-b)prob(b > B(v_2), ..b > B(v_n)) \\ &= (v-b)prob(v_2 < B^{-1}(b) ... v_n < B^{-1}(b)) \\ &= (v-b)prob(v_2 < B^{-1}(b))...prob(v_n < B^{-1}(b) \\ &= (v-b)\left(F\left(B^{-1}(b)\right)\right)^{n-1}.\end{aligned}$$

Maximizing implies first derivative equal to zero

$$(n-1)(v-b) f\left(B^{-1}(b)\right) \frac{1}{B'(B^{-1}(b))} \left(\left(F\left(B^{-1}(b)\right)\right)^{n-1}\right)^{n-2} - \left(F\left(B^{-1}(b)\right)\right)^{n-1} = 0.$$

Assume symmetric solution, so that in equilibrium player 1's bidding strategy is $B(v_1)$. Then, the above equation becomes

$$-F(v_1) + (n-1)(v_1 - b) f(v_1) \frac{1}{B'(v_1)} = 0.$$

$$\Leftrightarrow \frac{B'(v_1)}{(v_1 - B(v_1))} = \frac{(n-1) f(v_1)}{F(v_1)}$$

Solving for $B(v_1)$, we get

$$B(v_1) = v_1 - \frac{\int_{\underline{v}}^{v_1} F(x)^{n-1}\, dx}{F(v_1)^{n-1}}.$$

Common Value and Winner's Curse

Suppose you are going to bid for an offshore oil lease
Value of the oil tract is the same for everybody
But nobody knows the true value
Each bidder obtains an independent and unbiased estimate of the value
Your estimate is $100 million
How much do you bid?
Suppose everybody, including you, bids their estimate and you are the winner
What did you just learn?
Your estimate must have been larger than the others'
The true value must be smaller than $100 million
You overpaid

Common Value and Winner's Curse

- If everybody bids her estimate, winning is "bad news"
- This is known as *Winner's Curse*
- Optimal strategies are complicated
- Bidders bid much less than their value to prevent winner's curse
- To prevent winner's curse

Base your bid on expected value conditional on winning

- Auction formats are not equivalent in common value auctions
- Open bid auctions provide information and ameliorates winner's curse
-
    - Bids are more aggressive
- Sealed bid auctions do not provide information
    - Bids are more conservative

Auction Design

Good design depends on objective
- Revenue
- Efficiency
- Other

One common objective is to maximize expected revenue

In the case of private independent values with the same number of risk neutral bidders format does not matter

Auction design is a challenge when
- values are correlated
- bidders are risk averse

Other design problems
- collusion
- entry deterrence
- reserve price

Auction Design

Correlated values: Ascending bid auction is better
Risk averse bidders
- Second price auction: risk aversion does not matter
- First price auction: higher bids

Collusion: Sealed bid auctions are better to prevent collusion
Entry deterrence: Sealed bid auctions are better to promote entry
A hybrid format, such as Anglo-Dutch Auction, could be better.
Anglo-Dutch auction has two stages:
1 Ascending bid auction until only two bidders remain
2 Two remaining bidders make offers in a first price sealed bid auction

A double auction

Both the buyer and the seller have private information
Buyer's valuation $v_b \in U[0,1]$; seller's valuation $v_s \in U[0,1]$
Game: the buyer names a price p_b and the seller names a price p_s simultaneously.
If $p_b \geq p_s$, the trade occurs at $\frac{p_b+p_s}{2}$. Otherwise, no trade!
Utilities. In the case of trade,

$$u_b = v_b - p,$$

and the seller gets

$$u_s = p - v_s.$$

In the case of no trade, both get zero (change in utilities).

A double auction

Strategies. Buyer: $p_b\left(v_b\right)$; seller: $p_s\left(v_s\right)$.
A pair of strategies $\left\{p_b^*\left(v_b\right), p_s^*\left(v_s\right)\right\}$ is a Bayesian Nash Equilibrium, if $p_b^*\left(v_b\right)$ solves

$$\int_{p_s^*\left(v_s\right)\leq p_b}\left[v_b-\frac{p_b+p_s^*\left(v_s\right)}{2}\right]dF\left(v_s\right),$$

$$\Leftrightarrow \max_{p_b}\left[v_b-\frac{p_b+E\left[p_s^*\left(v_s\right)|p_b\geq p_s^*\left(v_s\right)\right]}{2}\right]\Pr\left(p_b\geq p_s^*\left(v_s\right)\right),$$

and $p_s^*\left(v_s\right)$ sloves

$$\max_{p_s}\left[\frac{p_s+E\left[p_b^*\left(v_b\right)|p_s\leq p_b^*\left(v_b\right)\right]}{2}-v_s\right]\Pr\left(p_s\leq p_b^*\left(v_b\right)\right)$$

A double auction

Strategies. Buyer: $p_b(v_b)$; seller: $p_s(v_s)$.
A pair of strategies $\{p_b^*(v_b), p_s^*(v_s)\}$ is a Bayesian Nash Equilibrium, if $p_b^*(v_b)$ solves

$$\int_{p_s^*(v_s) \leq p_b} \left[v_b - \frac{p_b + p_s^*(v_s)}{2} \right] dF(v_s),$$

$$\Leftrightarrow \max_{p_b} \left[v_b - \frac{p_b + E\left[p_s^*(v_s) \,|\, p_b \geq p_s^*(v_s)\right]}{2} \right] \Pr\left(p_b \geq p_s^*(v_s)\right),$$

and $p_s^*(v_s)$ sloves

$$\max_{p_s} \left[\frac{p_s + E\left[p_b^*(v_b) \,|\, p_s \leq p_b^*(v_b)\right]}{2} - v_s \right] \Pr\left(p_s \leq p_b^*(v_b)\right)$$

A double auction

There are many BNE!
One price equilibrium:

- Buyer offers x if $v_b \geq x$; 0 otherwise
- Seller offers x if $v_s \leq x$; 1 othereise

Linear equilibrium:

- Seller offers

$$p_s^* \left(v_s \right) = a_s + c_s v_s$$

Then the buyer

$$\max_{p_b} \left[v_b - \frac{1}{2} \left(p_b + \frac{a_s + p_b}{2} \right) \right] \frac{p_b - a_s}{c_s},$$

The f.o.c yields

$$p_b = \frac{2}{3} v_b + \frac{1}{3} a_s.$$

A double auction

Similarly:
 - Buyer offers

$$p_b^*(v_b) = a_b + c_b v_b$$

Then the seller

$$\max_{p_s} \left[\frac{1}{2} \left(p_s + \frac{a_b + c_b + p_s}{2} \right) - v_s \right] \frac{a_b + c_b - p_s}{c_b},$$

The f.o.c yields

$$p_b = \frac{2}{3} v_s + \frac{1}{3} (a_b + c_b).$$

A double auction

If the player's linear strategies are to be best response to each other, we have

$$p_b^*\left(v_b\right) = \frac{2}{3}v_b + \frac{1}{12}$$

$$p_s^*\left(v_s\right) = \frac{2}{3}v_s + \frac{1}{4}.$$