Static Games of Complete Information-Application

Nash Equilibrium-Pure Strategy

李三希, lisanxi@gmail.com

Renmin University

Cournot model of duopoly

- A product is produced by only two firms: firm 1 and firm 2. The quantities are denoted by q_1 and q_2, respectively. Each firm chooses the quantity without knowing the other firm has chosen.
- The market priced is $P(Q)=a-Q$, where a is a constant number and $Q=q_1+q_2$.
- The cost to firm i of producing quantity q_i is $C_i(q_i)=cq_i$.

Cournot model of duopoly

The normal-form representation:

- Set of players: **{ Firm 1, Firm 2}**
- Sets of strategies: $S_1=[0, +\infty)$, $S_2=[0, +\infty)$
- Payoff functions:

$u_1(q_1, q_2)=q_1(a-(q_1+q_2)-c)$

$u_2(q_1, q_2)=q_2(a-(q_1+q_2)-c)$

Cournot model of duopoly

- How to find a Nash equilibrium
  - Find the quantity pair (q_1*, q_2*) such that q_1* is firm 1's best response to Firm 2's quantity q_2* and q_2* is firm 2's best response to Firm 1's quantity q_1*
  - That is, q_1* solves
    Max $u_1(q_1, q_2^*)=q_1(a-(q_1+q_2^*)-c)$
    subject to $0 \le q_1 \le +\infty$

    and q_2* solves
    Max $u_2(q_1^*, q_2)=q_2(a-(q_1^*+q_2)-c)$
    subject to $0 \le q_2 \le +\infty$

Cournot model of duopoly

- How to find a Nash equilibrium
  - Solve

    Max $u_1(q_1, q_2^*) = q_1(a - (q_1 + q_2^*) - c)$

    subject to $0 \le q_1 \le +\infty$

    FOC: $a - 2q_1 - q_2^* - c = 0$

    $q_1 = (a - q_2^* - c)/2$

Cournot model of duopoly

- How to find a Nash equilibrium
  - Solve

    Max $u_2(q_1^*, q_2) = q_2(a - (q_1^* + q_2) - c)$

    subject to $0 \le q_2 \le +\infty$

    FOC: $a - 2q_2 - q_1^* - c = 0$

    $$q_2 = (a - q_1^* - c)/2$$

Cournot model of duopoly

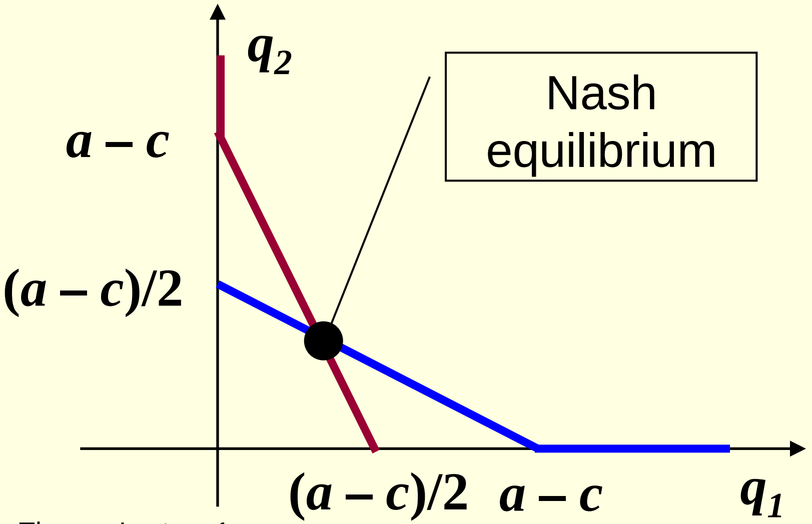
- How to find a Nash equilibrium
  - The quantity pair (q_1^*, q_2^*) is a Nash equilibrium if

$$q_1^* = (a - q_2^* - c)/2$$
$$q_2^* = (a - q_1^* - c)/2$$

  - Solving these two equations gives us
    $q_1^* = q_2^* = (a - c)/3$

Cournot model of duopoly

- Best response function
  - Firm 1's best function to firm 2's quantity q_2:
    $R_1(q_2) = (a - q_2 - c)/2$ if $q_2 < a - c$; 0, othwise
  - Firm 2's best function to firm 1's quantity q_1:
    $R_2(q_1) = (a - q_1 - c)/2$ if $q_1 < a - c$; 0, othwise



Cournot model of oligopoly

- A product is produced by only *n* firms: firm 1 to firm *n*. Firm *i*'s quantity is denoted by q_i. Each firm chooses the quantity without knowing the other firms' choices.
- The market priced is $P(Q)=a-Q$, where a is a constant number and $Q=q_1+q_2+\ldots+q_n$.
- The cost to firm i of producing quantity q_i is $C_i(q_i)=cq_i$.

Cournot model of oligopoly

The normal-form representation:

- Set of players: **{ Firm 1, ... Firm n}**
- Sets of strategies: $S_i=[0, +\infty)$, **for** $i=1, 2, ..., n$
- Payoff functions:
  $u_i(q_1 ,..., q_n)=q_i(a-(q_1+q_2 +...+q_n)-c)$
  for $i=1, 2, ..., n$

Cournot model of oligopoly

- How to find a Nash equilibrium
  - Find the quantities (q_1*, ... q_n*) such that q_i* is firm *i*'s best response to other firms' quantities
  - That is, q_1* solves
    Max $u_1(q_1, q_2^*, ..., q_n^*)=q_1(a-(q_1+q_2^* +...+q_n^*)-c)$
    subject to $0 \leq q_1 \leq +\infty$

    and q_2* solves
    Max $u_2(q_1^*, q_2 , q_3^*, ..., q_n^*)=q_2(a-(q_1^*+q_2+q_3^*+ ...+ q_n^*)-c)$
    subject to $0 \leq q_2 \leq +\infty$

    ....

Cournot model of oligopoly

- Show that when n goes to infinity, the NE is the perfect competitive result, p=c.

Bertrand model of duopoly (homogeneous products)

- Two firms: firm 1 and firm 2.
- Each firm chooses the price for its product without knowing the other firm has chosen. The prices are denoted by p_1 and p_2, respectively.
- The quantity that consumers demand from firm 1: $q_1(p_1, p_2) = a - p_1$ if $p_1 < p_2$; $= (a - p_1)/2$ if $p_1 = p_2$; $=0$, ow.
- The quantity that consumers demand from firm 2: $q_2(p_1, p_2) = a - p_2$ if $p_2 < p_1$; $= (a - p_2)/2$ if $p_1 = p_2$; $=0$, ow.
- The cost to firm i of producing quantity q_i is $C_i(q_i)=cq_i$.

Bertrand model of duopoly (homogeneous products)

The normal-form representation:

- Set of players: **{ Firm 1, Firm 2}**
- Sets of strategies: $\boldsymbol{S_1=[0, +\infty), S_2=[0, +\infty)}$
- Payoff functions:

$$u_1(p_1, p_2) = \begin{cases} (p_1 - c)(a - p_1) & \text{if } p_1 < p_2 \\ (p_1 - c)(a - p_1)/2 & \text{if } p_1 = p_2 \\ 0 & \text{if } p_1 > p_2 \end{cases}$$

$$u_2(p_1, p_2) = \begin{cases} (p_2 - c)(a - p_2) & \text{if } p_2 < p_1 \\ (p_2 - c)(a - p_2)/2 & \text{if } p_1 = p_2 \\ 0 & \text{if } p_2 > p_1 \end{cases}$$

Bertrand model of duopoly (homogeneous products)

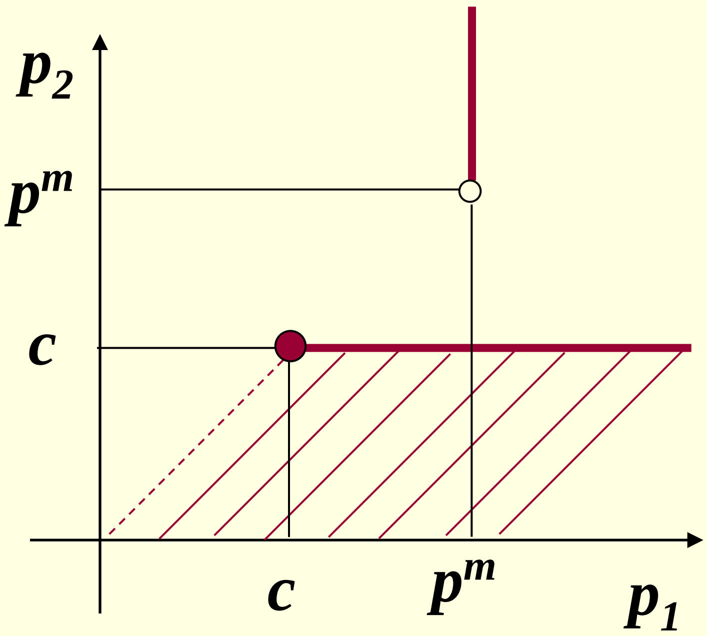
Best response functions: $\boldsymbol{p^m = (a + c)/2}$

$$B_1(p_2) = \begin{cases} \{p_1 : p_1 > p_2\} & \text{if } p_2 < c \\ \{p_1 : p_1 \geq p_2\} & \text{if } p_2 = c \\ \varnothing & \text{if } c < p_2 \leq p^m \\ p^m & \text{if } p^m < p_2 \end{cases}$$

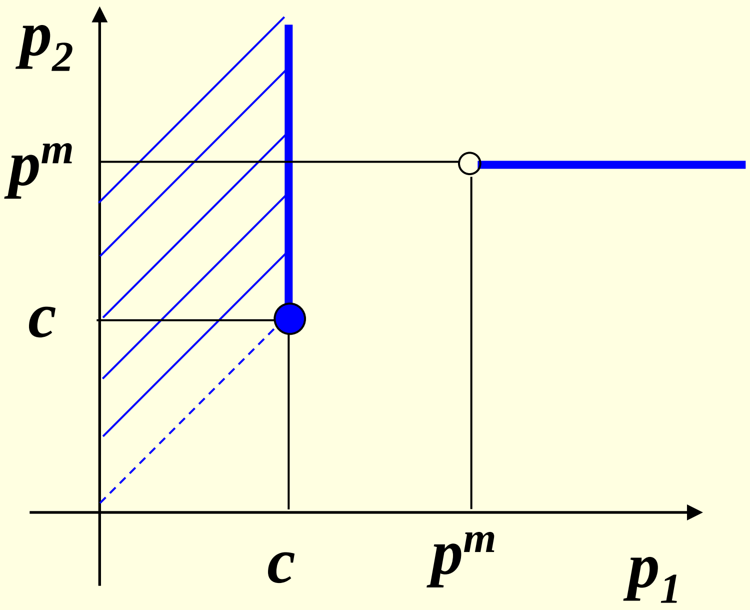
$$B_2(p_1) = \begin{cases} \{p_2 : p_2 > p_1\} & \text{if } p_1 < c \\ \{p_2 : p_2 \geq p_1\} & \text{if } p_1 = c \\ \varnothing & \text{if } c < p_1 \leq p^m \\ p^m & \text{if } p^m < p_1 \end{cases}$$

Bertrand model of duopoly (homogeneous products)

Best response functions:



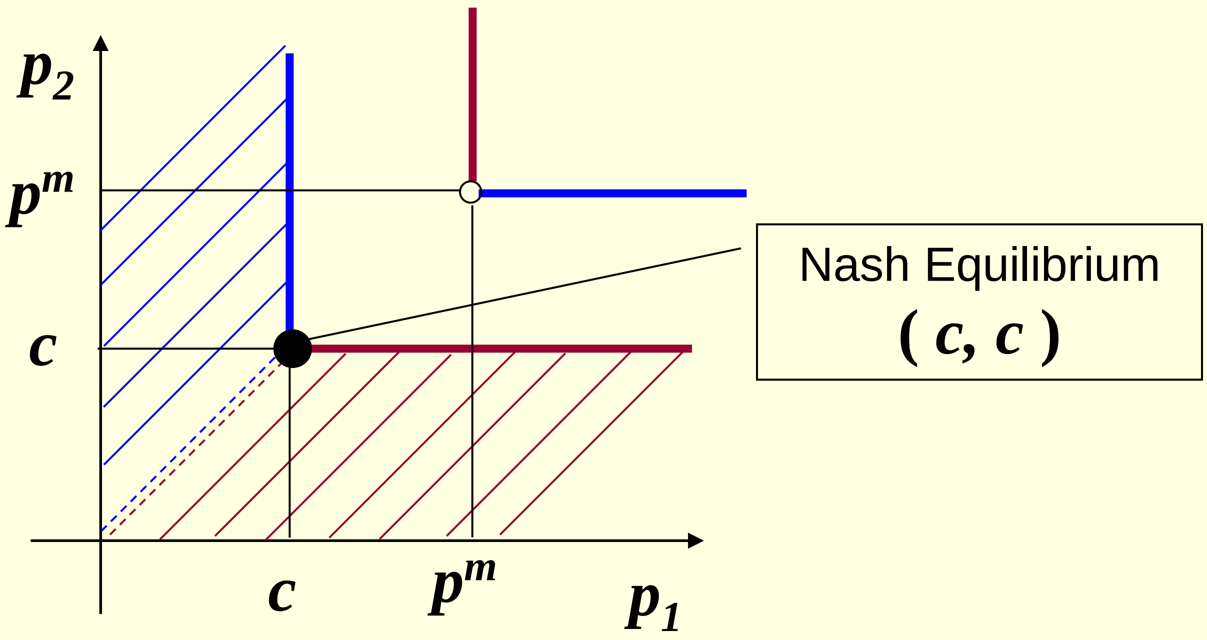
Firm 1's best response to Firm 2's p_2



Firm 2's best response to Firm 1's p_1

Bertrand model of duopoly (homogeneous products)

Best response functions:



Bertrand model of duopoly (differentiated products)

- Two firms: firm 1 and firm 2.
- Each firm chooses the price for its product without knowing the other firm has chosen. The prices are denoted by p_1 and p_2, respectively.
- The quantity that consumers demand from firm 1: $q_1(p_1, p_2) = a - p_1 + bp_2$.
- The quantity that consumers demand from firm 2: $q_2(p_1, p_2) = a - p_2 + bp_1$.
- The cost to firm i of producing quantity q_i is $C_i(q_i)=cq_i$.

Bertrand model of duopoly (differentiated products)

The normal-form representation:

- Set of players: **{ Firm 1, Firm 2}**
- Sets of strategies: $S_1=[0, +\infty)$, $S_2=[0, +\infty)$
- Payoff functions:

  $u_1(p_1, p_2)=(a - p_1 + bp_2)(p_1 - c)$

  $u_2(p_1, p_2)=(a - p_2 + bp_1)(p_2 - c)$

Bertrand model of duopoly (differentiated products)

- How to find a Nash equilibrium
  - Find the price pair (p_1*, p_2*) such that p_1* is firm 1's best response to Firm 2's price p_2* and p_2* is firm 2's best response to Firm 1's price p_1*
  - That is, p_1* solves
    Max $u_1(p_1, p_2^*) = (a - p_1 + bp_2^*)(p_1 - c)$
    subject to $0 \le p_1 \le +\infty$

    and p_2* solves
    Max $u_2(p_1^*, p_2) = (a - p_2 + bp_1^*)(p_2 - c)$
    subject to $0 \le p_2 \le +\infty$

Bertrand model of duopoly (differentiated products)

- How to find a Nash equilibrium
  - Solve firm 1's maximization problem

    Max $u_1(p_1, p_2^*) = (a - p_1 + bp_2^*)(p_1 - c)$

    subject to $0 \leq p_1 \leq +\infty$

    FOC: $a + c - 2p_1 + bp_2^* = 0$

    $p_1 = (a + c + bp_2^*)/2$

Bertrand model of duopoly (differentiated products)

- How to find a Nash equilibrium
  - Solve firm 2's maximization problem

    Max $u_2(p_1^*, p_2)=(a - p_2 + bp_1^*)(p_2 - c)$

    subject to $0 \le p_2 \le +\infty$

    FOC: $a + c - 2p_2 + bp_1^* = 0$

    $p_2 = (a + c + bp_1^*)/2$

Bertrand model of duopoly (differentiated products)

- How to find a Nash equilibrium
  - The price pair (p_1^*, p_2^*) is a Nash equilibrium if

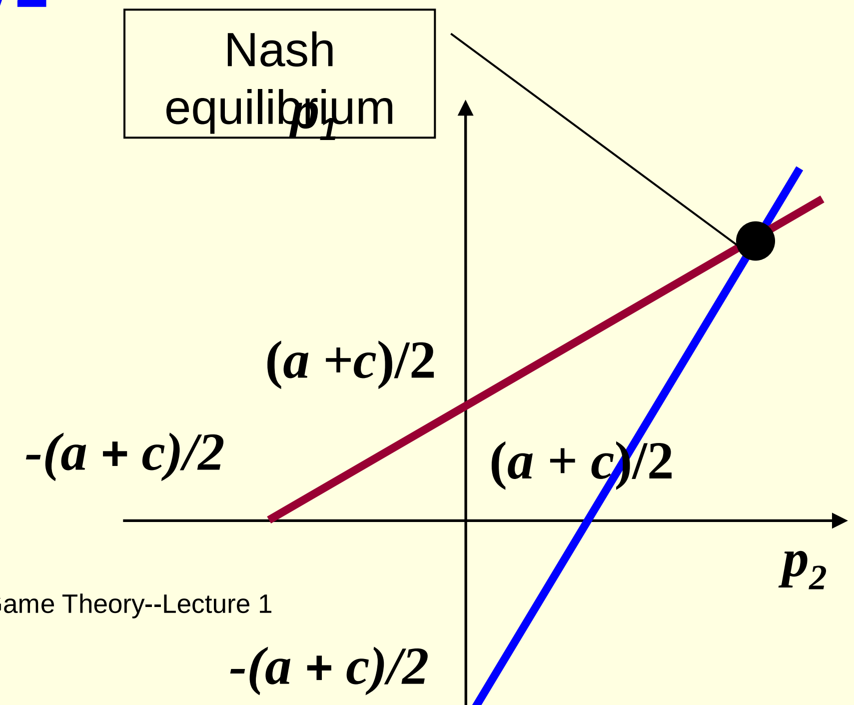
$$p_1^* = (a + c + bp_2^*)/2$$
$$p_2^* = (a + c + bp_1^*)/2$$

  - Solving these two equations gives us

$$p_1^* = p_2^* = (a + c)/(2 - b)$$

Bertrand model of duopoly

- Best response function
  - Firm 1's best function to firm 2's quantity q_2:
    $R_1(p_2) = (a + c + bp_2)/2$
  - Firm 2's best function to firm 1's quantity q_1:
    $R_2(p_1) = (a + c + bp_1)/2$



Cournot vs. Bertrand

- Strategic substitutes vs. complements
  - Quantity vs. Price
- Downward-sloping vs. upward-sloping response curve
- Connection between Cournot and Bertrand?

The problems of commons

- n farmers in a village. Each summer, all the farmers graze their goats on the village green.
- Let g_i denote the number of goats owned by farmer i.
- The cost of buying and caring for a goat is c, independent of how many goats a farmer owns.
- The value of a goat is $v(G)$ per goat, where $G = g_1 + g_2 + \ldots + g_n$
- There is a maximum number of goats that can be grazed on the green. That is, $v(G)>0$ if $G < G_{max}$, and $v(G)=0$ if $G \geq G_{max}$.
- Assumptions on $v(G)$: $v'(G) < 0$ and $v''(G) < 0$.
- Each spring, all the farmers simultaneously choose how many goats to own.

The problems of commons

The normal-form representation:

- Set of players: **{ Farmer 1, ... Farmer n}**
- Sets of strategies: $S_i=[0, G_{max})$, **for** $i=1, 2,..., n$
- Payoff functions:

  $u_i(g_1, ..., g_n)=g_i\, v(g_1 + ...+ g_n) - c\, g_i$

  for $i = 1, 2, ..., n$.

The problems of commons

- How to find a Nash equilibrium
  - Find (g_1*, g_2*, ..., g_n*) such that g_i* is farmer i's best response to other farmers' choices.
  - That is, g_1* solves
    Max $u_1(g_1, g_2^*, ..., g_n^*)= g_1\, v(g_1 + g_2^* ...+ g_n^*) - c\, g_1$
    subject to $0 \leq g_1 < G_{max}$

    and g_2* solves
    Max $u_2(g_1^*, g_2, g_3^*, ..., g_n^*)= g_2 v(g_1^*+g_2+g_3^*+ ...+ g_n^*) - c g_2$
    subject to $0 \leq g_2 < G_{max}$

    ....

The problems of commons

- How to find a Nash equilibrium
  - and g_n* solves
    Max $u_n(g_1^*, ..., g_{n-1}^*, g_n) = g_n v(g_1^* + ... + g_{n-1}^* + g_n) - c g_n$
    subject to $0 \leq g_n < G_{max}$

    ....

The problems of commons

- FOCs:

$$v(g_1 + g_2 * + ... + g_n *) + g_1 v'(g_1 + g_2 * + ... + g_n *) - c = 0$$

$$v(g_1 * + g_2 + g_3 * + ... + g_n *) + g_2 v'(g_1 * + g_2 + g_3 * + ... + g_n *) - c = 0$$

....

$$v(g_1 * + ... + g_{n-1} * + g_n) + g_n v'(g_1 * + ... + g_{n-1} * + g_n) - c = 0$$

The problems of commons

- How to find a Nash equilibrium
  - $(g_1^*, g_2^*, ..., g_n^*)$ is a Nash equilibrium if

$$v(g_1^*+g_2^*+...+g_n^*)+g_1v'(g_1^*+g_2^*+...+g_n^*)-c=0$$
$$v(g_1^*+g_2^*+g_3^*+...+g_n^*)+g_2v'(g_1^*+g_2^*+g_3^*+...+g_n^*)-c=0$$
$$.........$$
$$v(g_1^*+...+g_{n-1}^*+g_n^*)+g_nv'(g_1^*+...+g_{n-1}^*+g_n^*)-c=0$$

The problems of commons

- Summing over all *n* farmers' FOCs and then dividing by *n* yields

$$v(G^*) + \frac{1}{n} G^* v'(G^*) - c = 0$$

$$\text{where } G^* = g_1^* + g_2^* + \ldots + g_n^*$$

The problems of commons

- The social problem

$$\text{Max} \quad Gv(G) - Gc$$
$$\text{s.t.} \quad 0 \le G < G_{\max}$$

$$\text{FOC}:$$
$$v(G) + Gv'(G) - c = 0$$
Hence, the optimal solution G^{**} satisfies
$$v(G^{**}) + G^{**}v'(G^{**}) - c = 0$$

The problems of commons

$$v(G^*) + \frac{1}{n} G^* v'(G^*) - c = 0$$

$$v(G^{**}) + G^{**} v'(G^{**}) - c = 0$$

$$G^* > G^{**}?$$

The problems of commons

- The Moral of the story
  - Externality and Property rights
  - Global governance