Time Series Analysis

Homework #2

Exercise 1. Let $\mathcal{H} = \mathbb{R}^n$. Show that for $\vec{x} = (x_1, ..., x_n)'$ and $\vec{y} = (y_1, ..., y_n)'$, the mapping $I(\vec{x}, \vec{y}) = \sum_{i=1}^{n} x_i y_i$ defines an inner product. What is the norm this inner product generates.

Exercise 2. Prove the Pythagoras theorem. If x and y are orthogonal elements of some inner product space $\mathcal{H}$, then $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.

Exercise 3. Let $\mathcal{H} = \mathcal{L}^2$. Let $\|\cdot\|$ be the norm induced by the inner product $\langle X, Y \rangle = \mathrm{E}[XY]$. (i.e. $\|X\| = \left(\mathrm{E}\left[X^2\right]\right)^{1/2}$) For a sequence of random variables in $\mathcal{H}$, $\{X_n\}_{n=1}^{\infty}$, show that if X_n converges to X, then $\mathrm{Var}[X_n] \longrightarrow \mathrm{Var}[X]$ as $n \longrightarrow \infty$.

Exercise 4. Let $\{X_t\}_{t \in \mathbb{Z}}$ be a time series process. Let $\mathcal{S} = \left\{T = f(X_n) : \mathrm{E}\left[T^2\right] < \infty\right\}$. It can be shown that $\mathcal{S}$ is a closed linear subspace of $\mathcal{L}^2$. Show that

$$\mathrm{E}\left[(X_{n+1} - \mathrm{E}[X_{n+1}|X_n])^2\right] \leqslant \mathrm{E}\left[(X_{n+1} - f(X_n))^2\right]$$

for any function f. i.e. $\mathrm{E}[X_{n+1}|X_n]$ is the projection of X_{n+1} onto $\mathcal{S}$. Hint: write

$$(X_{n+1} - f(X_n))^2 = (X_{n+1} - \mathrm{E}[X_{n+1}|X_n] + \mathrm{E}[X_{n+1}|X_n] - f(X_n))^2 .$$

Decompose the right hand side and use the law of iterated expectation.