Advanced Econometrics

Lecture 2: An Introduction to Large Sample Asymptotics (Hansen Chapter 6)

Instructor: Ma, Jun

Renmin University of China

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依概率收敛

Convergence in Probability

②期望收敛 (均值收敛)　$X_n \xrightarrow{r\text{-th mean}} X \iff E|X_n - X|^r \to 0$

期望收敛一定是依概率收敛.

反之不成立.

Definition

A random variable $Z_n \in \mathbb{R}$ **converges in probability** to Z as $n \xrightarrow{p} \infty$, denoted $Z_n \xrightarrow{p} Z$, or alternatively $\operatorname{plim}_{n\to\infty} Z_n = Z$, if for all $\delta > 0$,

$$\lim_{n\to\infty} \Pr\left(|Z_n - Z| \leq \delta\right) = 1.$$

We call Z the **probability limit** (or **plim**) of Z_n.

一个数列 $\{a_n\}$

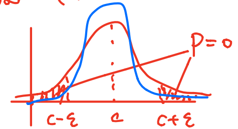
$a_n \to a \iff \forall \varepsilon > 0, \exists N_\varepsilon$. 当 $n \geq N_\varepsilon$, $|a_n - a| < \varepsilon$

一列随机变量 $\{X_n\}_{n=1}^{\infty}$. $X \in \mathbb{R}$

$X_n \to X \iff \forall \varepsilon > 0.\ \lim_{n\to\infty} Pr(|X_n - X| \geq \varepsilon) = 0$

①依概率收敛.

$X_n \to c \iff \forall \varepsilon > 0, \lim_{n\to\infty} Pr(|X_n - c| \geq \varepsilon) = 0$



大数定律

Weak Law of Large Numbers

马尔可夫不等式

$Pr(|X|>\varepsilon) \le \frac{E|X|^r}{\varepsilon^r}$

fix $\varepsilon>0$.

$0 \le Pr(|X_n - X| \ge \varepsilon) \le \frac{E|X_n-X|^r}{\varepsilon^r}$

if $E|X_n-X|^r \to 0$, $Pr(|X_n-X|\ge\varepsilon) \to 0$

Theorem (切比雪夫不等式)

Chebyshev's Inequality. *For any random variable Z_n and constant $\delta > 0$*

$$\Pr\left(|Z_n - \mathbb{E}Z_n| \geq \delta\right) \leq \frac{\operatorname{var}(Z_n)}{\delta^2}$$

Weak Law of Large Numbers

一个依概率收敛的例子

假设 $X_n \xrightarrow{p} a$, $Y_n \xrightarrow{p} b \neq 0$. $c \in \mathbb{R}$

(1) $cX_n \xrightarrow{p} c \cdot a$

(2) $X_n + Y_n \xrightarrow{p} a+b$

(3) $X_n Y_n \xrightarrow{p} a \cdot b$

(4) $\frac{X_n}{Y_n} \xrightarrow{p} \frac{a}{b}$

Theorem

Weak Law of Large Numbers (WLLN)

If Y_i are independent and identically distributed and $\mathbb{E}|Y| < \infty$, then as $n \to \infty$,

$$\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i \xrightarrow{p} \mathbb{E}(Y)$$

Definition

An Estimator $\hat{\theta}$ of a parameter θ is consistent if $\hat{\theta} \xrightarrow{p} \theta$ as $n \to \infty$.

一致的

Theorem

If Y_i are independent and identically distributed and $\mathbb{E}|Y| < \infty$, then $\hat{\mu} = \bar{Y}$ is consistent for the population mean μ.

prof. (2) fix $\varepsilon > 0$.

$Pr(|(X_n+Y_n) - (a+b)| \geq \varepsilon)$

$\leq Pr(|X_n - a| + |Y_n - b| \geq \varepsilon)$

$\leq Pr(|X_n - a| \geq \frac{\varepsilon}{2} \text{ or } |Y_n - b| \geq \frac{\varepsilon}{2})$

$\leq \underbrace{Pr(|X_n - a| \geq \frac{\varepsilon}{2})}_{\xrightarrow{p} 0} + \underbrace{Pr(|Y_n - b| \geq \frac{\varepsilon}{2})}_{\xrightarrow{p} 0} \to 0$

(有限样本性质,很难找到)

无偏性 $E\hat{\theta} = \theta$

一致性 $\hat{\theta}_n \xrightarrow{p} \theta$, $n \to \infty$

(大样本性质,容易找到)

① $E\hat{\alpha} = \alpha$. $E\hat{\beta} = \beta$

$E(\frac{\hat{\alpha}}{\hat{\beta}}) \neq \frac{\alpha}{\beta}$

② $\hat{\alpha} \xrightarrow{p} \alpha$. $\hat{\beta} \xrightarrow{p} \beta$

$\frac{\hat{\alpha}}{\hat{\beta}} \xrightarrow{p} \frac{\alpha}{\beta}$

Weak Law of Large Numbers

$E|Y| = \int_{-\infty}^{+\infty} |Y| f(y)\, dy$

$\uparrow$

$\frac{1}{\pi} \frac{1}{1+y^2}$ 柯西密度分布

$E\|Y\| = \left(\sum_{i=1}^{n} Y_i^2\right)^{\frac{1}{2}}$

① $0 \le X_n \le Y_n$, $Y_n \xrightarrow{p} 0 \Rightarrow X_n \xrightarrow{p} 0$

② $X_n \xrightarrow{p} 0 \Leftrightarrow |X_n| \xrightarrow{p} 0$

Theorem

WLLN for random vectors

If $\boldsymbol{Y}_i$ are independent and identically distributed and $\mathbb{E}\,\|\boldsymbol{Y}\| < \infty$, then as $n \to \infty$,

$$\bar{\boldsymbol{Y}} = \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{Y}_i \xrightarrow{p} \mathbb{E}(\boldsymbol{Y})$$

$\underset{k\times 1}{X_n} \xrightarrow{p} \underset{k\times 1}{X}$, 当 $X_{n,j} \xrightarrow{p} X_j \quad \forall j$

$\Updownarrow$

$\|X_n - X\| \xrightarrow{p} 0$

$\underset{k\times k}{\frac{1}{n}\sum_{i=1}^{n} X_i X_i'} \to_p E(X_i X_i')$

$\Updownarrow$

$\left\|\frac{1}{n}\sum_{i=1}^{n} X_i X_i' - E(X_i X_i')\right\| \to 0$

Theorem (continuous mapping theorem)

$X_n \xrightarrow{p} c \in \mathbb{R}$ $h(\cdot)$ 是连续的

$\Rightarrow h(X_i) \xrightarrow{p} h(c)$

Weak Law of Large Numbers

Theorem4 $X_1, \cdots X_n$. iid. $Var(X_1) < \infty$ $\mu = EX_1$ $(\Rightarrow E|X_1| < \infty)$

then $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{P} \mu$

Proof. $\Pr(|\bar{X}_n - \mu| \geq \varepsilon) = \Pr(|\sum_{i=1}^{n}(X_i - \mu)| \geq n\varepsilon)$

$$\leq \frac{1}{n^2\varepsilon^2} E|\sum_{i=1}^{n}(X_i-\mu)|^2$$

$$= \frac{1}{n^2\varepsilon^2} \underbrace{Var(\sum_{i=1}^{n} EX_i)}_{\sum_{i=1}^{n} \underbrace{Var(X_i)}_{n\sigma^2} + \underbrace{\sum cov(X_i, X_j)}_{=0}}$$

$$= \frac{n\sigma^2}{n^2\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2} \to 0$$

依分布收敛

Convergence in Distribution

指累积分布函数收敛

$X_n \in \mathbb{R}$. $X_n \xrightarrow{d} X$

$F_n(x) = \Pr(X \leq x)$. $\forall x$, $F_n(x) \to F(x)$,

$F(x) = \Pr(X \leq x)$.

Definition

Let $\boldsymbol{Z}_n$ be a random vector with distribution $F_n(\boldsymbol{u}) = \Pr(\boldsymbol{Z}_n \leq \boldsymbol{u})$. We say that $\boldsymbol{Z}_n$ **converges in distribution** to $\boldsymbol{Z}$ as $n \to \infty$, denoted $\boldsymbol{Z}_n \xrightarrow{d} \boldsymbol{Z}$, if for all $\boldsymbol{u}$ at which $F(\boldsymbol{u}) = \Pr(\boldsymbol{Z} \leq \boldsymbol{u})$ is continuous, $F_n(\boldsymbol{u}) \xrightarrow{d} F(\boldsymbol{u})$ as $n \to \infty$.

Theorem

Lévy's Continuity Theorem. *$\boldsymbol{Z}_n \xrightarrow{d} \boldsymbol{Z}$ if and only if $\mathbb{E}(\exp(\boldsymbol{t}'\boldsymbol{Z}_n)) \to \mathbb{E}(\exp(\boldsymbol{t}'\boldsymbol{Z}))$ for every $\boldsymbol{t} \in \mathbb{R}^k$.*

为了简单，认为这个定理是对的.

Theorem

Cramér-Wold Device. *$\boldsymbol{Z}_n \xrightarrow{d} \boldsymbol{Z}$ if and only if $\boldsymbol{\lambda}'\boldsymbol{Z}_n \xrightarrow{d} \boldsymbol{\lambda}'\boldsymbol{Z}$ for every $\boldsymbol{\lambda} \in \mathbb{R}^k$ with $\boldsymbol{\lambda}'\boldsymbol{\lambda} = 1$.*

中心极限定理 Central Limit Theorem — 依分布收敛的一个特例.

Theorem Y是随机变量

Lindeberg—Lévy Central Limit Theorem. *If* Y_i *are independent and identically distributed and* $\mathbb{E}\left|Y_i^2\right| < \infty$, *then as* $n \to \infty$

$$\sqrt{n}\left(\bar{Y} - \mu\right) \xrightarrow{d} \mathrm{N}\left(0, \sigma^2\right)$$

where $\mu = \mathbb{E}\left(Y\right)$ *and* $\sigma^2 = \mathbb{E}\left(Y_i - \mu\right)^2$.

$X_n - \mu \xrightarrow{p} 0$

$\sqrt{n}(X_n - \mu) \xrightarrow{d} N(0, \sigma^2)$

$\sqrt{n}(X_n - \mu) \underset{n\to\infty}{\overset{a}{\sim}} N(0, \sigma^2)$

$\Leftrightarrow \bar{X}_n \overset{a}{\sim} N(\mu, \frac{\sigma^2}{n})$

$E|Y|^2 < \infty \Rightarrow E|Y| < \infty$

若Y是正态分布, $E|Y|^k < \infty$

Multivariate Central Limit Theorem

多元中心极限定理

Theorem Y是随机向量

Multivariate Lindeberg—Lévy Central Limit Theorem. *If* $\boldsymbol{Y}_i \in \mathbb{R}^k$ *are independent and identically distributed and* $\mathbb{E}\|\boldsymbol{Y}_i\|^2 < \infty$ *then as* $n \to \infty$

$$\sqrt{n}\left(\bar{\boldsymbol{Y}} - \boldsymbol{\mu}\right) \xrightarrow{d} \mathrm{N}\left(\mathbf{0}, \boldsymbol{V}\right)$$

where $\boldsymbol{\mu} = \mathbb{E}\left(\boldsymbol{Y}\right)$ *and* $\boldsymbol{V} = \mathbb{E}\left(\left(\boldsymbol{Y} - \boldsymbol{\mu}\right)\left(\boldsymbol{Y} - \boldsymbol{\mu}\right)'\right)$.

极限是一个多维正态.

Function of Moments

$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} Z \quad Z \sim N(0, \sigma^2)$

$\frac{1}{\sqrt{n}(\bar{X}_n - \mu)} \xrightarrow{d} \frac{1}{Z}$

$Pr(Z=0)=0$

依概率版本

Theorem

Continuous Mapping Theorem (CMT). *If $\boldsymbol{Z}_n \xrightarrow{p} \boldsymbol{c}$ as $n \to \infty$ and $\boldsymbol{g}(\cdot)$ is continuous at $\boldsymbol{c}$, then $\boldsymbol{g}(\boldsymbol{Z}_n) \xrightarrow{p} \boldsymbol{g}(\boldsymbol{c})$ as $n \to \infty$.*

Theorem

If $\boldsymbol{Y}_i$ are independent and identically distributed, $\boldsymbol{\beta} = \boldsymbol{g}(\mathbb{E}(\boldsymbol{h}(\boldsymbol{Y})))$, $\mathbb{E}\|\boldsymbol{h}(\boldsymbol{Y})\| < \infty$, and $\boldsymbol{g}(\boldsymbol{u})$ is continuous at $\boldsymbol{u} = \boldsymbol{\mu}$, then for $\hat{\boldsymbol{\beta}} = \boldsymbol{g}\left(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{h}(\boldsymbol{Y}_i)\right)$, as $n \to \infty$, $\hat{\boldsymbol{\beta}} \xrightarrow{p} \boldsymbol{\beta}$.

CMT: g 是连续的

$\Rightarrow$ ① $Z_n \to_p c$, $g(Z_n) \to_p g(c)$

② $Z_n \to_d Z$, $g(Z_n) \to_d g(Z)$

Delta Method

$X_n = Z_n + \varepsilon_n \quad Z_n \xrightarrow{d} N(0, \sigma^2)$

$\varepsilon_n \xrightarrow{p} 0$

$\Rightarrow X_n \xrightarrow{d} N(0, \sigma^2)$

依分布版本

Theorem

Continuous Mapping Theorem

If $\boldsymbol{Z}_n \xrightarrow{d} \boldsymbol{Z}$ *as* $n \to \infty$ *and* $\boldsymbol{g} : \mathbb{R}^m \to \mathbb{R}^k$ *has the set of discontinuity points* D_g *such that* $\Pr(\boldsymbol{Z} \in D_g) = 0$, *then* $\boldsymbol{g}(\boldsymbol{Z}_n) \xrightarrow{d} \boldsymbol{g}(\boldsymbol{Z})$ *as* $n \to \infty$.

① $X_n \xrightarrow{d} X$. then $X_n^2 \xrightarrow{d} X^2$

if $X_n \sim N(0,1)$ $X^2 \sim \chi^2_1$

② if $(X_n, Y_n) \xrightarrow{d} (X, Y)$

then $X_n^2 + Y_n^2 \xrightarrow{d} X_n + Y_n$

if $\begin{pmatrix} X \\ Y \end{pmatrix} \sim N(0, I)$ 联合正态且独立

then $X^2 + Y^2 \sim \chi^2_2$

记住

Theorem

Slutsky's Theorem

If $Z_n \xrightarrow{d} Z$ *and* $C_n \xrightarrow{p} c$ *as* $n \to \infty$, *then*

1. $Z_n + C_n \xrightarrow{d} Z + c$
2. $Z_n C_n \xrightarrow{d} Zc$
3. $\frac{Z_n}{C_n} \xrightarrow{d} \frac{Z}{c}$ *if* $c \neq 0$

hw: $X_n \xrightarrow{d} X,\ Y_n \xrightarrow{d} Y \not\Rightarrow X_n + Y_n \xrightarrow{d} X + Y$

$(X_n, Y_n) \xrightarrow{d} (X, Y) \Rightarrow X_n + Y_n \xrightarrow{d} X + Y$

Delta Method

$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} Y$
$k\times 1 \quad k\times 1 \quad k\times 1$

$h: \mathbb{R}^k \to \mathbb{R}$. 连续可导.

$\frac{\partial h(x)}{\partial x} = \nabla h(x) = \begin{pmatrix} \frac{\partial h(x)}{\partial x_1} \\ \vdots \\ \frac{\partial h(x)}{\partial x_k} \end{pmatrix}$

$\frac{\partial h(x)}{\partial x}\Big|_{x=\theta}$

then $\sqrt{n}(h(\hat{\theta}_n) - h(\theta)) \xrightarrow{d} \frac{\partial h(x)}{\partial x'}\Big|_{x=\theta} Y$

$\sqrt{n}(\bar{X} - \mu) \xrightarrow{d} N(0, \sigma^2)$

$h(\cdot)$ 可导

$\bar{X} \xrightarrow{d} N(\mu, \frac{\sigma^2}{n})$

$h(\sqrt{n}(\bar{X}_n - \mu)) \xrightarrow{d} h(N(0,\sigma^2))$

$\boxed{h(\bar{X}_n)}$

Theorem

Delta Method:

If $\sqrt{n}(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu}) \xrightarrow{d} \boldsymbol{\xi}$, *where* $\boldsymbol{g}(\boldsymbol{u})$ *is continuously differentiable in a neighborhood of* $\boldsymbol{\mu}$ *then as* $n \to \infty$

$$\sqrt{n}(\boldsymbol{g}(\hat{\boldsymbol{\mu}}) - \boldsymbol{g}(\boldsymbol{\mu})) \xrightarrow{d} \boldsymbol{G}'\boldsymbol{\xi}$$

where $\boldsymbol{G}(\boldsymbol{u}) = \frac{\partial}{\partial \boldsymbol{u}}\boldsymbol{g}(\boldsymbol{u})'$ *and* $\boldsymbol{G} = \boldsymbol{G}(\boldsymbol{\mu})$. *In particular, if* $\boldsymbol{\xi} \sim \mathrm{N}(\mathbf{0}, \boldsymbol{V})$ *then as* $n \to \infty$

$$\sqrt{n}(\boldsymbol{g}(\hat{\boldsymbol{\mu}}) - \boldsymbol{g}(\boldsymbol{\mu})) \xrightarrow{d} \mathrm{N}\left(\mathbf{0}, \boldsymbol{G}'\boldsymbol{V}\boldsymbol{G}\right)$$

Delta Method

$\hat{\theta}$ 1×1

θ 1×1

$\sqrt{n}(\hat{\theta}-\theta) \to_d Y$

$h: \mathbb{R} \to \mathbb{R}$, h'连续

$\hat{\theta}$变成多维

h'变成梯度

$$\Rightarrow \sqrt{n}\left(h(\hat{\theta})-h(\theta)\right) \to_d h'(\theta)\cdot Y$$

proof: ① $\sqrt{n}(\hat{\theta}-\theta) \to_d Y \Rightarrow \hat{\theta} \to_p \theta$

$$\hat{\theta}-\theta = \underbrace{\frac{1}{\sqrt{n}}}_{\to 0} \underbrace{\sqrt{n}(\hat{\theta}-\theta)}_{\to_d Y} \to_d 0\cdot Y = 0$$

$$\boxed{Z_n \to_d C \Rightarrow Z_n \to_p C}$$

$$\Rightarrow \hat{\theta}-\theta \to_p 0 \Rightarrow \hat{\theta} \to_p \theta$$

② $h(\hat{\theta})-h(\theta) = h'(\theta^*)(\hat{\theta}-\theta)$

$\theta \quad \theta^* \quad \hat{\theta}$

θ^*一定是θ与$\hat{\theta}$之间的一个点.

θ^*是随机的

$$|\theta^*-\theta| \le |\hat{\theta}-\theta|$$

$$\to_p 0 \Leftarrow \to_p 0$$

$$\Rightarrow h'(\theta^*) \to_p h'(\theta)$$

③ $\sqrt{n}\left(h(\hat{\theta})-h(\theta)\right) = \underbrace{h'(\theta^*)}_{\to_p h'(\theta)} \underbrace{\sqrt{n}(\hat{\theta}-\theta)}_{\to_d Y}$

$$\Rightarrow \sqrt{n}\left(h(\hat{\theta})-h(\theta)\right) \to_d h'(\theta) Y$$

Delta Method

example1. $\sqrt{n}(\bar{X}-\mu) \xrightarrow{d} Z$. $Z \sim N(0,\sigma^2)$

关注 $\frac{1}{\bar{X}_n}$ 的分布.

$h(x)=\frac{1}{x}$ $\quad h'(x)=-\frac{1}{x^2}$

$h'(-\mu)=-\frac{1}{\mu^2}$

$\sqrt{n}\left(\frac{1}{\bar{X}_n}-\frac{1}{\mu}\right) \xrightarrow{d} h'(\mu)\cdot Z \sim N\left(0,\frac{\sigma^2}{\mu^4}\right)$

example 2.

$\sqrt{n}(\bar{X}_n^3-\mu^3) \xrightarrow{d} (3\mu^2)\cdot Z \sim N(0, 9\mu^4\sigma^2)$

$h(x)=x^3$ $\quad h'(x)=3x^2$

$h'(\mu)=3\mu^2$

Theorem

If $\boldsymbol{Y}_i$ are independent and identically distributed, $\boldsymbol{\mu}=\mathbb{E}(\boldsymbol{h}(\boldsymbol{Y}))$, $\boldsymbol{\beta}=\boldsymbol{g}(\boldsymbol{\mu})$, $\mathbb{E}\|\boldsymbol{h}(\boldsymbol{Y})\|^2<\infty$, and $\boldsymbol{G}(\boldsymbol{u})=\frac{\partial}{\partial \boldsymbol{u}}\boldsymbol{g}(\boldsymbol{u})'$ is continuous in a neighborhood of $\boldsymbol{\mu}$, then for $\hat{\boldsymbol{\beta}}=\boldsymbol{g}\left(\frac{1}{n}\sum_{i=1}^n \boldsymbol{h}(\boldsymbol{Y}_i)\right)$, as $n\to\infty$

$$\sqrt{n}\left(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta}\right) \xrightarrow{d} \mathrm{N}\left(\mathbf{0}, \boldsymbol{G}'\boldsymbol{V}\boldsymbol{G}\right)$$

where $\boldsymbol{V}=\mathbb{E}\left((\boldsymbol{h}(\boldsymbol{Y})-\boldsymbol{\mu})(\boldsymbol{h}(\boldsymbol{Y})-\boldsymbol{\mu})'\right)$ and $\boldsymbol{G}=\boldsymbol{G}(\boldsymbol{\mu})$.

中心极限定理的证明

$X_1, \cdots, X_n$ i.i.d mean μ . variance σ^2

CLT: $\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} Z \sim N(0, \sigma^2)$

$E e^{tZ} = e^{\frac{\sigma^2 t^2}{2}}$

MGF of $\sqrt{n}(\bar{X}_n - \mu) \longrightarrow$ MGF of Z

$$E \exp(t \cdot \underbrace{\sqrt{n}(\bar{X}_n - \mu)}_{= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \mu)})$$

$$= E \exp\left(\sum_{i=1}^{n} t \frac{1}{\sqrt{n}} (X_i - \mu)\right)$$

$$= E \prod_{i=1}^{n} \exp\left(\frac{t}{\sqrt{n}} (X_i - \mu)\right)$$

$$= \prod_{i=1}^{n} E \exp\left(\frac{t}{\sqrt{n}} (X_i - \mu)\right)$$

$$= \left(E \exp\left(\frac{t}{\sqrt{n}} (X_i - \mu)\right)\right)^n \quad \boxed{X \perp\!\!\!\perp Y \Rightarrow E(XY) = E(X)E(Y)}$$

$$= \left(E\left(1 + \frac{t}{\sqrt{n}} (X_1 - \mu) + \frac{1}{2} \frac{t^2}{n} (X_1 - \mu)^2 + o(\quad)\right)\right)^n$$

$$\approx \left(E\left(1 + \frac{t}{\sqrt{n}} (X_1 - \mu) + \frac{t^2}{2n} (X_1 - \mu)^2\right)\right)^n$$

$$= \left(1 + 0 + \frac{1}{n} \frac{t^2 \sigma^2}{2}\right)^n = \left(1 + \frac{1}{n} \frac{t^2 \sigma^2}{2}\right)^n \to e^{\frac{t^2 \sigma^2}{2}}$$

$$\boxed{\left(1 + \frac{a}{n}\right)^n \to e^a}$$

补充（不考）6.13节

$a_n \in \mathbb{R}$, $b_n \in \mathbb{R}$

① $\lim\limits_{n\to\infty} a_n = 0$: $a_n = o(1)$

（代表 a_n 收敛到 0 的一个数列）

$\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$: $a_n = o(b_n)$

→ 收敛

② $|a_n| \le M < \infty, \forall n$: $a_n = O(1)$

$\left|\frac{a_n}{b_n}\right| \le M, \forall n$: $a_n = O(b_n)$

→ 有界

$a_n = o(1) \Rightarrow a_n = O(1)$ 一个收敛数列一定有界

$o(1)\, O(1) = o(1)$

$o(1) + o(1) = o(1)$

$o(1) + O(1) = O(1)$

$X_n \to_p 0$: $X_n = o_p(1)$

$\frac{X_n}{Y_n} \to_p 0$: $X_n = o_p(Y_n)$ X_n 的阶数小于 Y_n 的阶数

Theorem: $X_n \to_d Z \Rightarrow X_n$ 有界: $X_n = O_p(1)$

$X_n = O_p(a_n) \Leftrightarrow \frac{X_n}{a_n} = O_p(1)$

$\sqrt{n}(\bar{X}_n - \mu) = O_p(1)$

$\bar{X}_n - \mu = O_p(\frac{1}{\sqrt{n}})$

$\underset{\text{估计量}}{\bar{X}_n} = \underset{\text{参数}}{\mu} + \underset{\text{误差项}}{O_p(\frac{1}{\sqrt{n}})}$ 阶数是 $\frac{1}{\sqrt{n}}$

$\bar{X}_n$ 收敛到 μ 的速度是 $\frac{1}{\sqrt{n}}$

$O_p(1) \cdot O_p(1) = O_p(1)$

$O_p(1) + O_p(1) = O_p(1)$

$o_p(1) \cdot O_p(1) = o_p(1)$