Time Series Analysis

Homework #3

Exercise 1. Let Y_t be a stationary zero mean time series. Define $X_t = Y_t - 0.4Y_{t-1}$ and $W_t = Y_t - 2.5Y_{t-1}$.

(a) Express the autocovariance functions X_t and W_t in terms of the autocovariance function of Y_t.

(b) Show that X_t and W_t have the same autocorrelation functions.

(c) Show that the process

$$U_t = -\sum_{j=1}^{\infty} (0.4)^j X_{t+j}$$

satisfies the equation $U_t - 2.5U_{t-1} = X_t$.

Exercise 2. Let $\{Z_t\} \sim WN\left(0, \sigma^2\right)$. Determine which of the following ARMA processes are causal:

(a) $X_t + 0.2X_{t-1} - 0.48X_{t-2} = Z_t$;
(b) $X_t + 1.9X_{t-1} + 0.88X_{t-2} = Z_t + 0.2Z_{t-1} + 0.7Z_{t-2}$;
(c) $X_t + 0.6X_{t-2} = Z_t + 1.2Z_{t-1}$;
(d) $X_t + 1.8X_{t-1} + 0.81X_{t-2} = Z_t$;
(e) $X_t + 1.6X_{t-1} = Z_t - 0.4Z_{t-1} + 0.04Z_{t-2}$.

Problem 3. Let $\{X_t\}$ be an AR(2) process with AR polynomial $\phi(z) = 1 - \phi_1 z - \phi_2 z^2$. Show that the process is causal only if the parameters (ϕ_1, ϕ_2) lie in the region determined by the equations:

$$\begin{cases} \phi_1 + \phi_2 < 1 \\ \phi_2 - \phi_1 < 1 \\ |\phi_2| < 1. \end{cases}$$

Problem 4. Let $I_t \equiv \{\epsilon_t, \epsilon_{t-1}, ...\}$ to be all the random elements that are realized at or before time t. Suppose

$$Y_t = \phi_0 + \theta_1 \epsilon_{t-1} + \epsilon_t$$

where ϵ_t is a white noise process with the additional property that $\mathrm{E}\left[\epsilon_t | I_{t-1}\right] = 0$. Further assume that ϵ_t is homoskedastic: $\mathrm{E}\left[\epsilon_t^2 | I_{t-1}\right] = \sigma^2$, which is a constant.

(a). What is $\mathrm{E}\left[Y_t | I_{t-1}\right]$? What is $\mathrm{E}\left[Y_t\right]$?

(c). What is $\mathrm{Var}\left[Y_t | I_{t-1}\right]$? Recall that $\mathrm{Var}\left[Y_t | I_{t-1}\right] = \mathrm{E}\left[\left(Y_t - \mathrm{E}\left[Y_t | I_{t-1}\right]\right)^2 | I_{t-1}\right] = \mathrm{E}\left[Y_t^2 | I_{t-1}\right] - \mathrm{E}\left[Y_t | I_{t-1}\right]^2$. What is $\mathrm{Var}\left[Y_t\right]$?

Problem 5. Is the sum of two white noise processes necessarily a white noise process? If not, give a counter example.

Problem 6. Let $\{Z_t\} \sim WN\left(0, \sigma^2\right)$. For the MA(q) process

$$X_t = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q},$$

show the autocovariance satisfies

$$\operatorname{Cov}\left[X_t, X_{t+h}\right] = \begin{cases} \sigma^2 \left(\sum_{j=0}^{q-|h|} \theta_j \theta_{j+|h|}\right) & \text{if } |h| \leqslant q \\ 0 & \text{if } |h| > q. \end{cases}$$