Static Games - Mixed strategy

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2011

Matching Pennies

	Head	*Tail*
Head	$1, -1$	$-1, 1$
Tail	$-1, 1$	$1, -1$

- No Nash Equilibrium.
- The solution to such games must involve some uncertainty about what some players will do.

Mixed strategies

Definition

Let $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$ be a finite normal form game. A mixed strategy for player i is a probability distribution over the set of pure strategies S_i. One usually writes $\sigma_i \in \Delta(S_i)$, where

$$\Delta(S_i) = \left\{ \sigma_i \in [0,1]^{\# S_i} \mid \sum_{s_i \in S_i} \sigma_i(s_i) = 1 \right\}.$$

vNM utility

To get the full power of mixed strategies, we need to add a speficication of each player's preference relation over lotteries on S.

The standard practice is to assume that these preferences satisfy von Neumann and Morgenstein's axioms, and to interpret u_i as player i's vNM utility function.
Example: two players. Player 1's vMN utility from his pure strategy s_{1j} is

$$\tilde{u}_1(s_{1j}, \sigma_2) = \sigma_{21} u_1(s_{1j}, s_{21}) + ... + \sigma_{2k} u_1(s_{1j}, s_{2k}) + ... + \sigma_{2K} u_1(s_{1j}, s_{2K}).$$

Player 1's vMN utility from his mixed strategy σ_1 is hence

$$\tilde{u}_1(\sigma_1, \sigma_2) = \sigma_{11} \tilde{u}_1(s_{11}, \sigma_2) + ... + \sigma_{1j} \tilde{u}_1(s_{1j}, \sigma_2) + ... + \sigma_{2J} \tilde{u}_1(s_{1J}, \sigma_2).$$

vNM utility

Example: Matching pennies

Suppose $\sigma_1(h) = p$, $\sigma_1(t) = 1 - p$, $\sigma_2(h) = q$, $\sigma_1(t) = 1 - q$. Then, player 1's vNM utility from his pure strategy h is

$$\begin{aligned}\tilde{u}_1(h, \sigma_2) &= q * 1 + (1 - q) * (-1) \\ &= 2q - 1,\end{aligned}$$

and his vNM utility from his pure strategy t is

$$\begin{aligned}\tilde{u}_1(t, \sigma_2) &= q * (-1) + (1 - q) * 1 \\ &= 1 - 2q.\end{aligned}$$

Hence, his vNM utility from his mixed strategy σ_1 is

$$\begin{aligned}\tilde{u}_1(\sigma_1, \sigma_2) &= p * \tilde{u}_1(h, \sigma_2) + (1 - p) * \tilde{u}_1(t, \sigma_2) \\ &= (2q - 1)(2p - 1).\end{aligned}$$

Mixed extension

Definition

The mixed extension of the finite normal form game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$ is the infinite normal form game $\tilde{G} = \{\Delta(S_1), \ldots, \Delta(S_n); \tilde{u}_1, \ldots, \tilde{u}_n\}$, where for each $i \in I$, the payoff function $\tilde{u}_i : \Pi_{j \in I} \Delta(S_j) \to R$ assign to each $\sigma \in \Pi_{j \in I} \Delta(S_j)$ the expected value under u_i of the lottery over s induced by σ:

$$\tilde{u}_i(\sigma) = \Sigma_{s \in S} \Pi_{j \in I} \sigma_j(s_j) u_i(s).$$

Note that each $\tilde{u}_i$ is multilinear for each $\lambda \in [0, 1]$,

$$\begin{aligned} & \tilde{u}_i(\lambda \sigma_i + (1-\lambda)\sigma'_i, \sigma_{-i}) \\ = \;& \lambda \tilde{u}_i(\sigma_i, \sigma_{-i}) + (1-\lambda)\tilde{u}_i(\sigma'_i, \sigma_{-i}). \end{aligned}$$

One can also write

$$\tilde{u}_i(\sigma) = \sum_{s_i \in S_i} \sigma_i(s_i) \tilde{u}_i(s_i, \sigma_{-i})$$

Definition

A nash equilibrium in mixed strategies of G is a nash equilibrium in pure strategies of its mixed extension $\tilde{G}$.

It is each to check any pure strategy NE of G can be interpreted as a mixed strategy NE of G.

Mixed extension

Example

Matching Pennies. Mixed extension: Set of Players {Player 1, player 2}
Sets of strategies $\{\Delta(S_1) =$
$\left\{\sigma_1 \in [0,1]^2 \,|\sigma_1(h) + \sigma_1(t) = 1\right\}, \left\{\sigma_2 \in [0,1]^2 \,|\sigma_2(h) + \sigma_2(t) = 1\right\}\}$
Utilities:

$$\tilde{u}_1(\sigma_1, \sigma_2) = (\sigma_1(h) - \sigma_1(t))(\sigma_2(h) - \sigma_2(t)),$$

$$\tilde{u}_2(\sigma_1, \sigma_2) = -(\sigma_1(h) - \sigma_1(t))(\sigma_2(h) - \sigma_2(t)).$$

Theorem

(Nash 1950): Each finite normal form game has a mixed strategy nash equilibrium.

We now say "NE" for "mixed strategy NE (including pure strategy NE and truly mixed trategy NE) and reserve the terminology "mixed strategy NE" for those equilibriua in which at least one player chooses a randomized (truly mixed) action.

Theorem

(Kakutani 1941): Let X *be a compact and convex subset of* R^n *and let* $f : X \to X$ *be a correspondence such that*
1) $\forall x \in X$, $f(x)$ *is nonempty and convex*
2) the graph of f *is closed (i.e., if* $y_n \in f(x_n)$ $\forall n$, *and if* $(x_n, y_n) \to (x, y)$, *then* $y \in f(x)$.
Then f *has a fixed point, i.e., their exists* $x^* \in X$, *such that* $x^* \in f(x^*)$.

Proof.

Let $BR : \Delta(S) \to \Delta(S)$ defined by $BR(\Delta(s)) = \Pi_{i \in I} BR_i(\Delta(s_{-i}))$. We just need to prove BR has a fixed point. $\forall i$, $\Delta(S_i)$ is nonempty compact convex subset of $[0,1]^{\#S_i}$. Since $\tilde{u}_i$ is linear in the probabilities, it is continuous on $\Delta(S)$ and quasi concave on $\Delta(S_i)$. Thus $\tilde{G}$ satisifies the conditions of our fixed point theorem, and the fixed point is a NE. □

Theorem

σ^* is a mixed strategy NE of a game G if and only if $\forall i$, any strategy s_i in the support of σ_i^* (i.e., $\sigma_i^*(s_i) > 0$) is a best response to σ_{-i}^*.

Proof.

If s_i is such that $\sigma_i^*(s_i) > 0$ and s_i is not a best response to σ_{-i}^*, then player i can increase his payoff by transferring probability from s_i to some best response to σ_{-i}^*. Therefore σ_i^* would not be a best response to σ_{-i}^*. Suppose next that σ^* is not a NE. Then there exists σ_i' which is a better respose to σ_{-i}^* than σ_i^*. Then some action in the support of σ_i' must give a strictly higher payoff than some action in the support of σ_i^*, so that not all actions in the support of σ_i^* are best response to σ_{-i}^*. □

A key implication is that every action in the support of any player's equilibrium mixed strategy yields that player has the same payoff. —key to calculate mixed strategy equilbrium.

Example

Maching Pennis. Suppose the equilibrium strategies are $\{(\sigma_1(h), 1-\sigma_1(h)), (\sigma_2(h), 1-\sigma_1(h))\}$. Then, player 1 must be indifferent between playing h and t:

$$\begin{aligned} & \sigma_2(h) * 1 + (1-\sigma_2(h)) * (-1) \\ = \; & \sigma_2(h) * (-1) + (1-\sigma_2(h)) * 1, \end{aligned}$$

which gives $\sigma_2(h) = 1/2$. Similarly, $\sigma_1(h) = 1/2$. Hence, $\{(1/2, 1/2), (1/2, 1/2)\}$ is a mixed strategy NE by our last Theorem.

Example

Paying tax. Player 1: auditor; player 2: tax payer

$$\begin{array}{ccc} & Honest & Cheat \\ Audit & 2,0 & 4,-10 \\ No & 4,0 & 0,4 \end{array}.$$

Denote p the probability of player 1 playing Audit and q the probability of player 2 playing honest. (show the mixed NE on the blackboard.)

Interpretation of mixed strategy: q can be explained as proportion of people being honest.
Policy: increase fine from 10 to 20.

$$\begin{array}{ccc} & Honest & Cheat \\ Audit & 2,0 & 4,-20 \\ No & 4,0 & 0,4 \end{array}.$$

What happen to the tax compliance q?
How can we increase tax compliance?

All pay auction (Competition for a girl)

- Two contestants(boys) compete for one object(girl).
- The prize valuations of the contestants are denoted by v_i, with $v_1 > v_2$. All the prize valuations are public information.
- Denote $p_i(b_1, b_2)$ the winning probability of contestant i given their bids(efforts) b_1 and b_2, with $b_i \geq 0$. When $b_i = 0$, we say that the contestant i does not enter contest.
- The expected net payoff of contestant i is:

$$Eu_i = p_i(b_1, b_2)\, v_i - b_i, (i = 1, 2.)$$

The function $p_i(b_1, b_2)$ is given by

$$p_1(b_1, b_2) = \begin{cases} 1, \text{ if } b_1 > b_2 \\ \frac{1}{2}, \text{ if } b_1 = b_2 \\ 0, \text{ if } b_1 < b_2 \end{cases}. \tag{1}$$

1. No pure strategy NE exists. Specially, there is no mass point that is strictly positive, i.e., no contestant will bid a positive amount with strictly positive probability.

Proof.

Suppose agent i does spend $\beta > 0$ with strictly positive probability. Then the probability that agent j beats i rises discontinuously as a function of b_j at $b_j = \beta$. Therefore there is some $\varepsilon > 0$ such that agent j will bid on the interval $[\beta - \varepsilon, \beta]$ with zero probability. But then agent i is better off spending $\beta - \varepsilon$ rather than β since his probability of winning is the same, contradicting the hypothesis that $b_i = \beta$ is an equilibrium strategy.
Next, $(0, 0)$ is not NE, because player i can be strictly better off by playing ε. □

Thus, the equilibrium strategies are mixed strategies and continuous over the interval $[0, +\infty)$. The mixed strategies are described by a pair (F_1, F_2). F_i is a c.d.f over $[0, +\infty)$.

2. Denote $\bar{b}_1$ and $\bar{b}_2$ the upper bounds of bidder 1 and bidder 2 separately. Then, it follows immediately that $\bar{b}_1 = \bar{b}_2$.

Proof.

For if $\bar{b}_1 > \bar{b}_2$, contestant 1 wins with probability 1 by spending $\bar{b}_2 + \varepsilon < \bar{b}_1$; while if $\bar{b}_1 < \bar{b}_2$, contestant 2 wins with probability 1 by spending $\bar{b}_1 + \varepsilon < \bar{b}_2$. $\square$

3. $\underline{b}_1$ and $\underline{b}_2$ the infimum of positive bids for contestant 1 and contestant 2 separately. Then, we must have $\underline{b}_1 = \underline{b}_2 = 0$.

Proof.

Suppose $\underline{b}_2 > 0$ so that contestant 2 bid with zero probability on $(0, \underline{b}_2)$. Then, for contestant 1, any bidding between 0 and $\underline{b}_2$ yields a negative payoff since the probability of winning is zero. Since contestant 1 can always bid zero, it follows that he will never bid on $(0, \underline{b}_2)$. But then contestant 2 could reduce his bidding below $\underline{b}_2$ without altering the probability of winning, contradicting the hypothesis that contestant 2 could do no better than take $\underline{b}_2$ as him minimum bidding level. Using the similar logic, we can argue $\underline{b}_1 > 0$ is impossible and hence $\underline{b}_1 = 0$. □

4. At most one agent spend zero with strictly positive probability

Proof.

Were this happen, then one agent can be strictly better off by transferring probability from zero to an arbitrary small bid, because it increases his win probability discontinuously and hence his expected payoff. □

Define $1 - F_i(b_i)$ the probability that contestant i spends more than b_i. From the analysis above, $F_i(b_i)$ is continuous over $(0, +\infty)$. If $0 < F_i(b_i) < 1$, then agent i spends a strictly possitive amount with probability less than 1. His remaining alternatives are to spend zero. Contestant 1's expected payoff is

$$u_1(b_1) = F_2(b_1) v_1 - b_1, \tag{2}$$

And contestant 2's expected payoff is

$$u_2(b_2) = F_1(b_2) v_2 - b_2. \tag{3}$$

- By remaining inactive (bidding zero), an agent gets 0. Therefore, agent i will enter with probability whenever his equilibrium payoff is strictly positive.
- Agent 2 has the option of remaining inactive, so he will never spend more than his valuation.
- Then agent 1 always enter because, by spending $v_2 + \varepsilon$, he can get $v_1 - (v_2 + \varepsilon) > 0$.
- It follows that the equilibrium payoff $u_1 \geq v_1 - (v_2 + \varepsilon) > 0$. Setting $b_1 = 0$ in (2), we know $F_2(0) > 0$.
- This also gives $F_1(0) = 0$, since at most one agent spend zero with strictly positive probability. From (3), we know $u_2 = u_2(0) = 0$.

- From (3), we know

$$F_1(b_1) = \frac{b_1}{v_2}, b_1 \in [0, v_2].$$

- Since both agents have the same maximum spending level v_2, we know that v_2 is in the support of agent 1's bid distribution. Moreover, $F_2(v_2) = 1$.Hence

$$u_1 = F_2(b_1) v_1 - b_1 = v_1 - v_2,$$

which gives

$$F_2(b_2) = \frac{v_1 - v_2}{v_1} + \frac{b_2}{v_1}.$$

Natural explanation: agent 2 stay out with probability $\frac{v_1 - v_2}{v_1}$. With probability $\frac{v_2}{v_1}$, agent 2 enters. Conditional upon entering the contest, agent 2 also adopt a uniform mixed strategy over $[0, 1]$.

Theorem

(Hillman and Riley 1989) *In all pay auction, there exists an unique mixed strategy equilibrium. contestant 1 always enters the contest while contestant 2 enters with probability* $\frac{v_2}{v_1}$*. Conditional upon entry, each contestant bids according to a uniform mixed strategy over the interval* $[0, v_2]$.

Assume the seller (girl)'s utility is equal to $E\left(b_1 + b_2\right)$.
Q: will the seller be better off as $v_i \uparrow$? What is the intuition?