The Revelation Principal

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Definition

A direct revelation mechanism is a mechnism in which each player's only action is to submit a claim about his or her type.

Definition

A direct revelation mechanism is truthful if it is incentive compatible for each player to announce his true type for any type.

Theorem

Any Bayesian Nash Equilibrium of any Bayesian game can be represented by an incentive-compatible direct mechanism.

Proof.

Consider the Bayesian game $G = \{A_i, A_{-i}; \Theta_i, \Theta_{-i}; p_i, p_{-i}; u_i, u_{-i}\}$. Suppose the strategies $s^* = \left(s_i^*, s_{-i}^*\right)$ are a Bayesian Nash Equilibrium. Then, $s_i^*\left(\theta_i\right)$ solves

$$\max_{a_i \in A_i} \sum_{\theta_{-i} \in \Theta_{-i}} u_i\left(a_i, s_{-i}^*\left(\theta_{-i}\right); \theta\right) p_i\left(\theta_{-i} | \theta_i\right).$$

Now, consider direct mechanism Bayesian Game, $G' = \left\{A_i', A_{-i}'; \Theta_i, \Theta_{-i}; p_i, p_{-i}; u_i', u_{-i}'\right\}$, where $A_i' = \Theta_i$ and $u_i' = u_i\left(s_i^*\left(\theta_i\right), s_{-i}^*\left(\theta_{-i}\right); \theta\right)$. In this new direct mechnism, telling truth consists a Bayesian Nash Equilibrium. Moreover, each player's actions and payoffs are identical to those in the old equilibrium. □

Auction-optimal auction

Revelation principal $\Rightarrow$ Only need to consider the direct incentive compatible auction (Myerson 1981)

The auctioneer announce the direct auction $\{t_i(v_i, v_{-i}), t_{-i}(v_i, v_{-i}); p_i(v_i, v_{-i}), p_{-i}(v_i, v_{-i})\}$. All bidders anounce their own valuation simultaneously.

An auction is incentive compatible if

$$v_i E_{v_{-i}}\left[p_i(v_i, v_{-i}) - t_i(v_i, v_{-i})\right] \geq v_i E_{v_{-i}}\left[p_i\left(v_i', v_{-i}\right) - t_i\left(v_i', v_{-i}\right)\right].$$

Define $P_i(v_i) = E_{v_{-i}} p_i(v_i, v_{-i})$ and $T_i(v_i) = E_{v_{-i}} t_i(v_i, v_{-i})$, then the IC condition writes

$$v_i P_i(v_i) - T_i(v_i) \geq v_i P_i\left(v_i'\right) - T_i\left(v_i'\right).$$

Auction-optimal auction

Define $U_i(v_i) = v_i P_i(v_i) - T_i(v_i)$,
The auctioneer's problem is

$$\begin{aligned} &\max \textstyle\sum_i E_{v_i, v_{-i}} t_i(v_i, v_{-i}) \\ &s.t.\text{IC condition} \\ &\text{IR condition } U_i(v_i) \geq 0, .\forall v_i. \end{aligned}$$

Auction-optimal auction

Incentive-compatible condition means

$$U_i\left(v_i\right) = \max_{v_i'} v_i P_i\left(v_i'\right) - T_i\left(v_i'\right).$$

From envelope theorem, we have

$$\dot{U}_i\left(v_i\right) = P_i\left(v_i\right).$$

The auctioneer's problem can be rewritten as

$$\max \sum\nolimits_i E_{v_i}\left(v_i P_i\left(v_i\right) - U_i\left(v_i\right)\right)$$

$$\begin{aligned} s.t. \dot{U}_i\left(v_i\right) &= P_i\left(v_i\right) \\ U_i\left(v_i\right) &\geq 0, .\forall v_i. \end{aligned}$$

Auction-optimal auction

$$\dot{U}_i(v_i) = P_i(v_i).$$

$$\Rightarrow U_i(v_i) = \int_0^{v_i} P_i(x)\, dx.$$

Hence

$$\begin{aligned} E_{v_i} U_i(v_i) &= \int_0^{\overline{v}} \int_0^{v_i} P_i(x)\, dx d(1 - F(v_i)) \\ &= \int_0^{\overline{v}} (1 - F(v_i)) P_i(v_i)\, dv_i. \end{aligned}$$

Auction-optimal auction

The auctioneer's problem becomes

$$\begin{aligned}
& \max \sum\nolimits_i \left[E_{v_i} v_i P_i\left(v_i\right) - \int_0^{\overline{v}} \left(1 - F\left(v_i\right)\right) P_i\left(v_i\right) dv_i \right] \\
= \; & \max \sum\nolimits_i E_{v_i} \left\{ v_i P_i\left(v_i\right) - \frac{\left(1 - F\left(v_i\right)\right)}{f\left(v_i\right)} P_i\left(v_i\right) \right\} \\
= \; & \max \sum\nolimits_i E_{v_{i,} v_{-i}} \left\{ \left(v_i - \frac{\left(1 - F\left(v_i\right)\right)}{f\left(v_i\right)} \right) p_i\left(v_i, v_{-i}\right) \right\},
\end{aligned}$$

which gives

$$p_i\left(v_i, v_{-i}\right) = \begin{cases} 1, \text{ if } v_i - \frac{\left(1-F\left(v_i\right)\right)}{f\left(v_i\right)} \geq v_j - \frac{\left(1-F\left(v_j\right)\right)}{f\left(v_j\right)} \forall j \neq i \\ 0, \text{ otherwise.} \end{cases}$$

Auction-optimal auction

Assumption: $\frac{(1-F(v_i))}{f(v_i)}$ is decreasing. Then, the optimal auction rewards the object to the one with the highest valuation⇒The optimal auction is efficient.

In this case,

$$P_i(v_i) = F(v_i)^{n-1},$$

$$U_i(v_i) = \int_0^{v_i} F(x)^{n-1}\, dx.$$

The auctioneer's expected revenue is

$$\begin{aligned} & \textstyle\sum_i \int_0^{\overline{v}} \left(v_i F(v_i)^{n-1} - \int_0^{\overline{v}} F(v_i)^{n-1}\, dv_i \right) f(v_i)\, dv_i \\ = \quad & E \max(v_i, v_{-i}) - n \int_0^{\overline{v}} (1 - F(v_i))\, F(v_i)^{n-1}\, dv_i. \end{aligned}$$

Revenue Equivalence Theorem

Revenue Equivalence Theorem (first proved by Vickrey in 1961 and generalized 20 years later by Myerson, and independently by Riley and Samuelson.),

Theorem

Any allocation mechanism/auction in which (i) the bidder with the highest type/signal/valuation always wins, (ii) the bidder with the lowest possible type/valuation/signal expects zero surplus, (iii) all bidders are risk neutral and (iv) all bidders are drawn from a strictly increasing and atomless distribution will lead to the same revenue for the seller (and player i of type v can expect the same surplus across auction types).

Revenue Equivalence Theorem

The winning bid should be epsilon above the second highest valuation. Relaxing these assumptions can provide valuable insights for auction design. Decision biases can also lead to predictable non-equivalencies. Additionally, if some bidders are known to have a higher valuation for the lot, techniques such as price discriminating against such bidders will yield higher returns. (In other words, if a bidder is known to value the lot at \$X more than the next highest bidder, the seller can increase their profits by charging that bidder \$X-delta more than any other bidder (or equivalently a special bidding fee of \$X - delta). This bidder will still win the lot, but will pay more than she would otherwise.

Bilateral Trade

Myerson-Satterthwaite (1981)
Suppose seller has an object and he want to sell it to the buyer. The seller's value of object is $v_s \in U\left[0,1\right]$,while the buyer's valuation is $v_b \in U\left[0,1\right]$. Assume v_s and v_b independent.
Question: can we design a mechanism which is BB, BIC and implement the efficiency allocation?

Theorem

There is no incentive compatible mechanism that satisfies individual rationality (voluntary participation), budget balance and implements the efficiency allocation.

Bilateral Trade

Proof.

Consider the direct mechanism $\{t(v_b, v_s), p(v_b, v_s)\}$, where $t(v_b, v_s)$ is the transfer from the buyer to the seller and $p(v_b, v_s)$ is the probability that the buyer gets the object.

As before, define $P_s(v_s) = E_{v_b} p(v_b, v_s)$, $P_b(v_b) = E_{v_s} p(v_b, v_s)$, $T_s(v_s) = E_{v_b} t(v_b, v_s)$, $T_b(v_b) = E_{v_s} t(v_b, v_s)$,

$$U_s(v_s) = T_s(v_s) - v_s P_s(v_s) \tag{1}$$

and

$$U_b(v_b) = v_b P_b(v_b) - T_b(v_b). \tag{2}$$

IC means

$$v_i P_i(v_i) - T_i(v_i) \geq v_i P_i(v_i') - T_i(v_i'), \, i = s, b.$$

□

Bilateral Trade

Proof.

The ICs give

$$U_b\left(v_b\right) = \int_0^{v_b} P_b\left(x\right) dx + U_b\left(0\right)$$

and

$$U_s\left(v_s\right) = \int_{v_s}^{1} P_s\left(x\right) dx + U_s\left(1\right).$$

$\square$

Bilateral Trade

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Proof.

Taking expectation and adding (1) and (2), we have

$$E_{v_s} U_s(v_s) + E_{v_b} U_b(v_b) = E_{v_b} v_b P_b(v_b) - E_{v_s} v_s P_s(v_s)$$

$$\begin{aligned} \Leftrightarrow \quad & E_{v_{b,v_s}} \left[\left(\frac{F(v_s)}{f(v_s)} + \frac{1 - F(v_b)}{f(v_b)} \right) p(v_b, v_s) \right] + U_b(0) + U_s(1) \\ = \quad & E_{v_{b,v_s}} (v_b - v_s) p(v_b, v_s) \end{aligned}$$

$$\Rightarrow E_{v_{b,v_s}} (v_b - v_s) p(v_b, v_s) \geq E_{v_{b,v_s}} \left[\left(\frac{F(v_s)}{f(v_s)} + \frac{1 - F(v_b)}{f(v_b)} \right) p(v_b, v_s) \right].$$

Efficient allocation means $p(v_b, v_s) = 1$ iff $v_b > v_s$. One can check the above inequality does not hold for efficient allocation. (LHS=1/6 and the RHS=1/3). □

Bilateral Trade

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Theorem

(Myerson-Satterthwaite Theorem). Suppose the buyer's valuation is distributed on $[\underline{v}_b, \overline{v}_b]$ *and the seller's valuation is distributed on* $[\underline{v}_s, \overline{v}_s]$. *Assume* $\underline{v}_s \leq \underline{v}_b < \overline{v}_s \leq \overline{v}_b$. *Then there does not exist any trading mechanism that is ex post effcient, expected budget balanced, and individually rational.*