Part 2: Mathematical Preliminaries

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Complex Numbers

Definition 1 (Complex Numbers)

A complex number is an ordered pair of real numbers (x_1, x_2), where x_1 is called the real part and x_2 is called the imaginary part. We denote the field of complex numbers by $\mathbb{C}$. For $x = (x_1, x_2) \in \mathbb{C}$ and $y = (y_1, y_2) \in \mathbb{C}$, $x + y = (x_1 + y_1, x_2 + y_2)$, the same as "addition" in $\mathbb{R}^2$. $\mathbb{C}$ can be viewed as $\mathbb{R}^2$ endowed with a "multiplication" rule. A "multiplication" is a map from $\mathbb{C} \times \mathbb{C}$ to $\mathbb{C}$ that is commutative ($x \cdot y = y \cdot x$) and associative ($(x \cdot y) \cdot z = x \cdot (y \cdot z)$). There exists a unique multiplication rule that is both commutative and associative:

$$x \cdot y = (x_1 y_1 - x_2 y_2, x_1 y_2 + x_2 y_1) .$$

Notice that one can find that a division rule is also defined:

$$\frac{x}{y} = \frac{1}{y_1^2 + y_2^2} \begin{bmatrix} y_1 & y_2 \\ -y_2 & y_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{y_1^2 + y_2^2} \begin{bmatrix} x_1 y_1 + x_2 y_2 \\ x_2 y_1 - x_1 y_2 \end{bmatrix} .$$

Complex Numbers

Definition 2 (Modulus)

The "absolute value" or the "modulus" of a complex number x, denoted by $|x|$, is

$$|x| := \sqrt{x_1^2 + x_2^2}.$$

Notice that this is the same as the Euclidean norm of (x_1, x_2) as a pair of real numbers. The "distance" of two complex numbers is just $|x - y|$. So $\mathbb{C}$ is topologically equivalent to $\mathbb{R}^2$.

Remark 3

Note that complex numbers of the form $(x_1, 0)$ are called real numbers.

Imaginary Unit

Definition 4 (Imaginary Unit)

The complex number $(0, 1)$ is called the imaginary unit, denoted by i. Then we usually let $x_1 + \mathrm{i}x_2$ denote a complex number (x_1, x_2). Note that by the rule of multiplication $\mathrm{i}^2 = (0, 1)\,(0, 1) = (-1, 0)$.

Definition 5

The complex conjugate of $x = x_1 + \mathrm{i}x_2$ is $\overline{x} = x_1 - \mathrm{i}x_2$. Note that now we have $|x| = \sqrt{x \cdot \overline{x}}$ for all $x \in \mathbb{C}$.

Complex Exponential

Definition 6 (Complex Exponential)

For a complex number $x = x_1 + \mathrm{i}x_2$, its exponential, denoted by $\exp(x)$ or e^x, is defined as

$$\exp(x) = \exp(x_1 + \mathrm{i}x_2) = \exp(x_1)\left\{\cos(x_2) + \mathrm{i}\cdot\sin(x_2)\right\}.$$

Remark 7

It can be verified using trigonometric identities that

$$\exp(x)\exp(y) = \exp(x+y)\,,\ \forall(x,y) \in \mathbb{C}^2 \text{ and}$$
$$\exp(x)^n = \exp(nx)\,,\ \forall x \in \mathbb{C}.$$

If x is purely imaginary, i.e., x is of the form $(0, x_2)$,

$$|\exp(\mathrm{i}x_2)| = |\cos(x_2) + \mathrm{i}\cdot\sin(x_2)| = \sqrt{\cos(x_2)^2 + \sin(x_2)^2} = 1.$$

Complex Exponential

Remark 8 (Polar Representation of Complex Numbers)
For any $x_1 + \mathrm{i}x_2 \in \mathbb{C}$, let $r := |x_1 + \mathrm{i}x_2|$ and $\theta := \arctan\left(\frac{x_2}{x_1}\right)$.
Then $(x_1 + \mathrm{i}x_2) = r \cdot \exp(\mathrm{i}\theta)$.

Remark 9 (Power Series Expansion)
For all $x \in \mathbb{C}$,

$$\exp(x) = \sum_{j=0}^{\infty} \frac{x^j}{j!},$$

where the right hand side converges absolutely and is equal to the left hand side.

Remark 10
For all $x \in \mathbb{C}$,

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = \exp(x).$$

Fundamental Theorem of Algebra

Theorem 11

Let $c_0, c_1, ..., c_n$ *be* n *complex numbers. The polynomial*

$$c_0 + c_1 x + \cdots + c_n x^n$$

has n *roots in* $\mathbb{C}$.

Characteristic Function

Definition 12 (Characteristic Function)

For a $d-$dimensional random vector $\boldsymbol{X}$, its characteristic function, usually denoted by $\phi_{\boldsymbol{X}}$, is

$$\phi_{\boldsymbol{X}}(t) := \mathrm{E}\left[\exp\left(\mathrm{i}t^{\mathrm{T}}\boldsymbol{X}\right)\right] = \mathrm{E}\left[\cos\left(t^{\mathrm{T}}\boldsymbol{X}\right)\right] + \mathrm{i}\cdot\mathrm{E}\left[\sin\left(t^{\mathrm{T}}\boldsymbol{X}\right)\right].$$

Note that $\phi_{\boldsymbol{X}}$ is $\mathbb{C}$-valued.

Remark 13

Result 1: For two random variables, $\phi_X = \phi_Y$ if and only if $F_X = F_Y$. Result 2: Let $X_1, X_2, ...$ be a sequence of random variables. Let X another random variable. If for all $t \in \mathbb{R}$, $\phi_{X_n}(t) \overset{n\to\infty}{\longrightarrow} \phi_X(t)$, then $X_n \to_d X$.

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Inner Product Space and Hilbert Space

- A Hilbert space has a similar geometry like Euclidean spaces $(\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, ...)$.
- A Hilbert space could have more complicated elements than real numbers/vectors, but also has concepts of orthogonality and projection.

Inner Product Space and Hilbert Space

Definition 1 (Real Vector Space)

A real vector space is a set $\mathcal{V}$, endowed with an "addition" operation, $+ : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$, i.e., for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$, $\boldsymbol{x} + \boldsymbol{y} \in \mathcal{V}$, and a "scalar multiplication", i.e. $\cdot : \mathbb{R} \times \mathcal{V} \to \mathcal{V}$, for all $c \in \mathbb{R}$, $\boldsymbol{x} \in \mathcal{V}$, $c \cdot \boldsymbol{x} \in \mathcal{V}$.

Definition 2 (Inner Product Space)

A real vector space $\mathcal{H}$ is called an inner product space if it is endowed with a map $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$, which satisfies (1). $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \langle \boldsymbol{y}, \boldsymbol{x} \rangle$, for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{H}$, (2). $\langle \boldsymbol{x} + \boldsymbol{z}, \boldsymbol{y} \rangle = \langle \boldsymbol{x}, \boldsymbol{y} \rangle + \langle \boldsymbol{z}, \boldsymbol{y} \rangle$, for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathcal{H}$, (3). $\langle \alpha \boldsymbol{x}, \boldsymbol{y} \rangle = \alpha \langle \boldsymbol{x}, \boldsymbol{y} \rangle$for all $\alpha \in \mathbb{R}$, $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{H}$, (4). $\langle \boldsymbol{x}, \boldsymbol{x} \rangle \geq 0$ and $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ if and only if $\boldsymbol{x} = \boldsymbol{0}$.

Examples

1. $\mathcal{H} = \mathbb{R}^n$, $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{i=1}^{n} x_i y_i$, where $\boldsymbol{x} = (x_1, ..., x_n)$ and $\boldsymbol{y} = (y_1, ..., y_n)$.
2. Given a probability space $(\Omega, \mathscr{F}, \mathrm{P})$, $\mathcal{H} = \left\{X : \Omega \to \mathbb{R} : \mathrm{E}\left[X^2\right] < \infty\right\}$. i.e. the random variables with finite variances, $\langle X, Y \rangle = \mathrm{E}\left[XY\right]$. We use $\mathcal{L}^2\left(\Omega, \mathscr{F}, \mathrm{P}\right)$ to denote such a space.
3. Let $\mathcal{L}_0^2\left(\Omega, \mathscr{F}, \mathrm{P}\right)$ denote the subset of $\mathcal{L}^2\left(\Omega, \mathscr{F}, \mathrm{P}\right)$ including the random variables X with $\mathrm{E}\left[X\right] = 0$. $\mathcal{L}_0^2\left(\Omega, \mathscr{F}, \mathrm{P}\right)$ is subspace of $\mathcal{L}^2\left(\Omega, \mathscr{F}, \mathrm{P}\right)$.

Inner Product Space and Hilbert Space

Proposition 3 (Cauchy-Schwarz Inequality)

Let $\mathcal{H}$ be an inner product space, the for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{H}$, $|\langle \boldsymbol{x}, \boldsymbol{y} \rangle| \leq \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}\sqrt{\langle \boldsymbol{y}, \boldsymbol{y} \rangle}$. A special case is the Cauchy-Schwarz inequality from probability theory: $|\mathrm{E}[XY]| \leq \sqrt{\mathrm{E}[X^2]}\sqrt{\mathrm{E}[Y^2]}$, for $\mathcal{H} = \mathcal{L}^2(\Omega, \mathscr{F}, \mathrm{P})$.

Definition 4 (Norm)

An inner product $\langle \cdot, \cdot \rangle$ induces a "norm" for $\mathcal{H}$, i.e., measure of the "length" of a vector. Let $\|\boldsymbol{x}\| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}$. Then $\|\cdot\|$ satisfies (1). $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$, i.e., triangle inequality. (2). $\|\alpha \cdot \boldsymbol{x}\| = |\alpha| \|\boldsymbol{x}\|$, for all $\alpha \in \mathbb{R}$, for all $\boldsymbol{x} \in \mathcal{H}$. (3). $\|\boldsymbol{x}\| \geq 0$ and $\|\boldsymbol{x}\| = 0$ if and only if $\boldsymbol{x} = \mathbf{0}$.

Definition 5 (Convergence)

Let $\{\boldsymbol{x}_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{H}$. We say that $\boldsymbol{x}_n$ converges to $\boldsymbol{x}$ if $\|\boldsymbol{x}_n - \boldsymbol{x}\| \to 0$, as $n \to \infty$. $\|\boldsymbol{x} - \boldsymbol{y}\|$ measures the distance between $\boldsymbol{x}$ and $\boldsymbol{y}$.

Inner Product Space and Hilbert Space

Proposition 6 (Continuity of Norm and Inner Product)

If $\{\boldsymbol{x}_n\}_{n=1}^{\infty}$ and $\{\boldsymbol{y}_n\}_{n=1}^{\infty}$ are sequences in $\mathcal{H}$, and $\boldsymbol{x}_n \to \boldsymbol{x}$ and $\boldsymbol{y}_n \to \boldsymbol{y}$ as $n \to \infty$, then (1). $\|\boldsymbol{x}_n\| \to \boldsymbol{x}$ as $n \to \infty$. (2). $\langle \boldsymbol{x}_n, \boldsymbol{y}_m \rangle \to \langle \boldsymbol{x}, \boldsymbol{y} \rangle$, as $n \to \infty$ and $m \to \infty$.

Definition 7 (Cauchy Sequence)

If $\{\boldsymbol{x}_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{H}$, then $\{\boldsymbol{x}_n\}_{n=1}^{\infty}$ is called Cauchy if $\|\boldsymbol{x}_n - \boldsymbol{x}_m\| \to 0$, as $n \to \infty$ and $m \to \infty$. I.e., For all $\epsilon > 0$, there exists some $N_\epsilon \in \mathbb{N}$, such that $\|\boldsymbol{x}_n - \boldsymbol{x}_m\| < \epsilon$ if $n \geq N_\epsilon$ and $m \geq N_\epsilon$.

Definition 8 (Completeness)

An inner product space $\mathcal{H}$ is said to be complete if every Cauchy sequence has a limit in $\mathcal{H}$, i.e., there exists some $\boldsymbol{x} \in \mathcal{H}$ such that $\|\boldsymbol{x}_n - \boldsymbol{x}\| \to 0$ as $n \to \infty$.

Definition 9 (Hilbert Space)

A complete inner product space is called a Hilbert space.

Examples

- $\mathbb{R}^n$ with $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{i=1}^{n} x_i y_i$ is complete.
- $\mathcal{L}^2 (\Omega, \mathscr{F}, \mathrm{P})$ is complete.
- $(0, 1)$ with $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{i=1}^{n} x_i y_i$ is not complete. $\{1/n\}_{n=1}^{\infty}$ is a Cauchy sequence but has no limit in $(0, 1)$.

Hilbert Space

Proposition 10 (Cauchy Criterion)

Let $\mathcal{H}$ be a Hilbert space and $\{\boldsymbol{x}_n\}_{n=1}^{\infty}$ a sequence in $\mathcal{H}$, then $\{\boldsymbol{x}_n\}_{n=1}^{\infty}$ converges to some $\boldsymbol{x} \in \mathcal{H}$ if and only if $\{\boldsymbol{x}_n\}_{n=1}^{\infty}$ is Cauchy.

Definition 11 (Orthogonality)

Let $\mathcal{H}$ be an inner product space. Then $\boldsymbol{x}$ is orthogonal to $\boldsymbol{y}$ (denoted as $\boldsymbol{x} \perp \boldsymbol{y}$) if $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$.

Proposition 12 (Pythagoras' Theorem)

Let $\mathcal{H}$ be an inner product space, $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{H}$ satisfying $\boldsymbol{x} \perp \boldsymbol{y}$, then $\|\boldsymbol{x} + \boldsymbol{y}\|^2 = \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2$.

Proposition 13 (Parallelogram Rule)

Let $\mathcal{H}$ be an inner product space, $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{H}$, $\|\boldsymbol{x} - \boldsymbol{y}\|^2 = 2\|\boldsymbol{x}\|^2 + 2\|\boldsymbol{y}\|^2 - \|\boldsymbol{x} + \boldsymbol{y}\|^2$.

Hilbert Space Projection Theorem

Theorem 14

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{S}$ be a closed linear subspace. I.e., $\alpha \boldsymbol{x} + \beta \boldsymbol{y} \in \mathcal{S}$, for all $\alpha, \beta \in \mathbb{R}$ and for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}$ and $\mathcal{S}$ is a closed subset of $\mathcal{H}$. Then, (1). For all $\boldsymbol{x} \in \mathcal{H}$, there is a unique $\widehat{\boldsymbol{x}} \in \mathcal{S}$ such that $\|\boldsymbol{x} - \widehat{\boldsymbol{x}}\| = \inf\{\|\boldsymbol{y} - \boldsymbol{x}\| : \boldsymbol{y} \in \mathcal{S}\}$, i.e., $\|\boldsymbol{x} - \widehat{\boldsymbol{x}}\| \leq \|\boldsymbol{y} - \boldsymbol{x}\|$, for all $\boldsymbol{y} \in \mathcal{S}$. (2). $\boldsymbol{x} - \widehat{\boldsymbol{x}} \perp \boldsymbol{y}$ for all $\boldsymbol{y} \in \mathcal{S}$. (3). If $\boldsymbol{z} \in \mathcal{S}$ satisfies $\boldsymbol{x} - \boldsymbol{z} \perp \boldsymbol{y}$ for all $\boldsymbol{y} \in \mathcal{S}$, then $\boldsymbol{z} = \widehat{\boldsymbol{x}}$. I.e., $\widehat{\boldsymbol{x}}$ is the only element in $\mathcal{S}$ that satisfies $\boldsymbol{x} - \widehat{\boldsymbol{x}} \perp \boldsymbol{y}$, for all $\boldsymbol{y} \in \mathcal{S}$.