Repeated Game

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The Friedman's Theorem

Definition

We call the payoffs (x_i, x_{-i}) feasible in the stage game if they are a convex combination of the pure-stategy payoffs.

Definition

The average payoff of the infinite sequance of payoffs $(\pi_1, \pi_2, \ldots)$ is defined by

$$\pi = (1-\delta)\sum_{t=1}^{+\infty} \delta^t \pi_t.$$

The Friedman's Theorem

Theorem

Let G be a finite, static game of complete information. Let (e_i, e_{-i}) denote the payoffs from a NE of G, and let (x_i, x_{-i}) be any feasible payoffs from G. If $x_i > e_i$ for every player i and if δ is sufficiently close to one, then there exists a subgame-perfect Nash equilibrium of the infinitely repeated game that achieves (x_i, x_{-i}) as the average payoff.

Application 1: Tacit collusion-Bertrand Competition

- Repeat the Bertrand competition infinitely
- Consider the following strategies:
    - Each firm charge p^m in period 0, and continue to charge p^m as long as both firm have charged p^m in every previous period.
    - Otherwise, charge the marginal cost c forever.

Application 1: Tacit collusion-Bertrand Competition

- Check the strategies are NE. No deviation if

$$\frac{\Pi^m}{2}\left(1+\delta+\delta^2+...\right) \geq \Pi^m,$$

$$\Leftrightarrow \delta \geq 1/2.$$

- Check they are subgame-perfect. 1) No deviation before. 2) Deviation happened before.
- In fact, any payoffs $\left(\Pi^1, \Pi^2\right)$ such that $\Pi^1 > 0, \Pi^2 > 0$ and $\Pi^1 + \Pi^2 \leq \Pi^m$ is an equilibrium payoff for δ close to 1.

Application 2: Tacit collusion-Cournot Competition

- Repeat the Cournot competition infinitely
- Consider the following strategies:
  - Each firm produce $\frac{q^m}{2}$ in period 0, and continue to charge $\frac{q^m}{2}$ as long as both firm have charged $\frac{q^m}{2}$ in every previous period.
  - Otherwise, produce the cournot quantity q_C forever.
- Check the strategies are NE. No deviation if

$$\frac{1}{1-\delta}\frac{1}{2}\pi_m \geq \pi_d + \frac{\delta}{1-\delta}\pi_C,$$

where $\pi_m = \frac{(a-c)^2}{4}$ is the monopoly profit, $\pi_C = \frac{(a-c)^2}{4}$ is the cournot profit, and $\pi_d = \frac{9(a-c)^2}{64}$ is the profit from deviation, i.e.,

$$\pi_d = \max_{q_j} \left(a - q_j - \frac{q^m}{2} - c \right) q_j.$$

- No deviation requires $\delta \geq 9/17$.
- Check the strategies are subgame perfect.

Application 2: Tacit collusion-Cournot Competition

What the firms can achieve if $\delta < 9/17$?
- Consider the trigger strategies what switch for ever to the stage-game NE, and see what average payoffs can be achieved.

- Consider the following strategies:
  - Each firm produce q^* in period 0, and continue to charge q^* as long as both firm have charged q^* in every previous period.
  - Otherwise, produce the cournot quantity q_C forever.

Application 2: Tacit collusion-Cournot Competition

No deviation gives profit $\pi^* = (a - 2q^* - c)\, q^*$.
If deviate, then a firm will choose q_i to maximize his profit,

$$\pi_d = \max_{q_j} (a - q_j - q^* - c)\, q_j,$$

which gives $\pi_d = \frac{(a-q^*-c)^2}{4}$.
A firm will not deviate if

$$\frac{1}{1-\delta}\pi^* \geq \pi_d + \frac{\delta}{1-\delta}\pi_C.$$

The lowest value of q^* for which the trigger strategies are a subgame perfect equilibrium is

$$q^* = \frac{9-5\delta}{3(9-\delta)}(a-c).$$

Application 3: Relational Contract

Two players: the principal and the agent.
In each period, the agent chooses an unobservable action, a, that stochastically determines the agent's total contribution, y.
Assume $y \in \{y_H, y_L\}$.

$$\Pr(y = y_H) = a.$$

Assume agent's total contribution cannot be objectively measured, but can be subjectively assessed and used in a relational contract.

Application 3: Relational Contract

Compensation contract: s (base salary) and b (bonus if $y = y_H$).
Timing: 1, P offers the relational contract $\{s, b\}$
2, The agent either accept or reject
3, The agent chooses the effort a at cost $c(a)$.
4, y realized and both the P and the A oberve y. If $y = y_H$, then P decides whether pay the bonus or not.

Application 3: Relational Contract

Stage game: P will not pay the bonus and A will exert no effort.
Repeated game.
Benchmark: first best

$$\max y_L + a(y_H - y_L) - c(a),$$

which gives

$$c'(a^*) = y_H - y_L.$$

Application 3: Relational Contract

- Consider the trigger strategies: the parties begin by cooperating and then continue to cooperate unless one side defects, in which case they refuse to cooperate forever after.
- Given a relational-contract bonus b, if the agent believes the principal will honor the relational contract, then the agent's problem is

$$\max_a s + ab - c(a),$$

which gives

$$c'(a) = b.$$

- A will accept P's offer iff

$$s + a^*(b)\,b - c(a^*(b)) \geq 0.$$

Application 3: Relational Contract

P's expected payoff per period is

$$V(b) = y_L + a^*(b)(y_H - y_L) - c(a^*(b)).$$

Should the agent believe P's offer?
Yes if

$$y_H - s - b + \frac{\delta}{1-\delta} V(b) \geq y_H - s,$$

which is equivalent to

$$b \leq \frac{\delta}{1-\delta} V(b).$$

Application 3: Relational Contract

Assume $c(a) = \frac{ca^2}{2}$. Hence $a^*(b) = \frac{b}{c}$, $V(b) = y_L + \frac{b}{c}(y_H - y_L) - \frac{1}{2}\frac{b^2}{c}$. Then the condition becomes

$$\frac{\delta}{1-\delta} f(b) \geq 1,$$

where $f(b) = \frac{y_L}{b} + \frac{1}{c}(y_H - y_L) - \frac{1}{2}\frac{b}{c}$ is decreasing in b.

Application 3: Relational Contract

Define $\widehat{\delta}$ given by the equation

$$\frac{\widehat{\delta}}{1-\widehat{\delta}} f\left(y_H - y_L\right) = 1.$$

Then, for $\delta > \widehat{\delta}$, first best can be created by setting $b = y_H - y_L$.
For $\delta < \widehat{\delta}$, the P will set b such that

$$\frac{\delta}{1-\delta} f\left(b^*\right) = 1.$$

Notice that b^* is increasing in δ: an impatient [high interest rates] principal can only offer a small bonus and induce weak incentive.