Dynamic Games of Complete Information

Subgame-Perfect Equilibrium

Repeated game

- A *repeated game* is a dynamic game of complete information in which a (*simultaneous-move*) game is played at least *twice*, and the previous plays are observed before the next play.

- We will find out the behavior of the players in a repeated game.

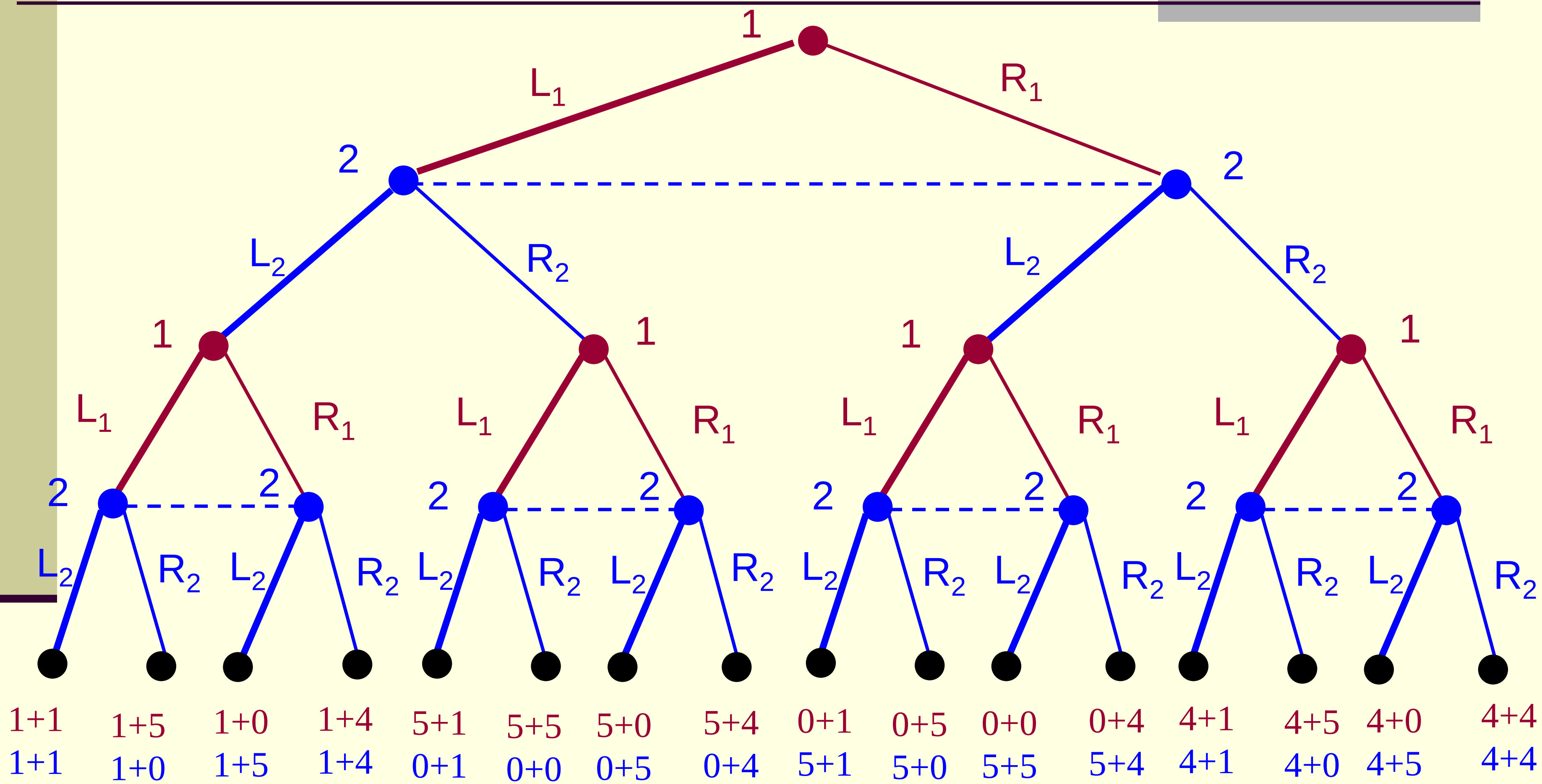
Two-stage repeated game

- Two-stage prisoners' dilemma
  - Two players play the following simultaneous move game twice
  - The outcome of the first play is observed before the second play begins
  - The payoff for the entire game is simply the sum of the payoffs from the two stages. That is, the discount factor is 1.

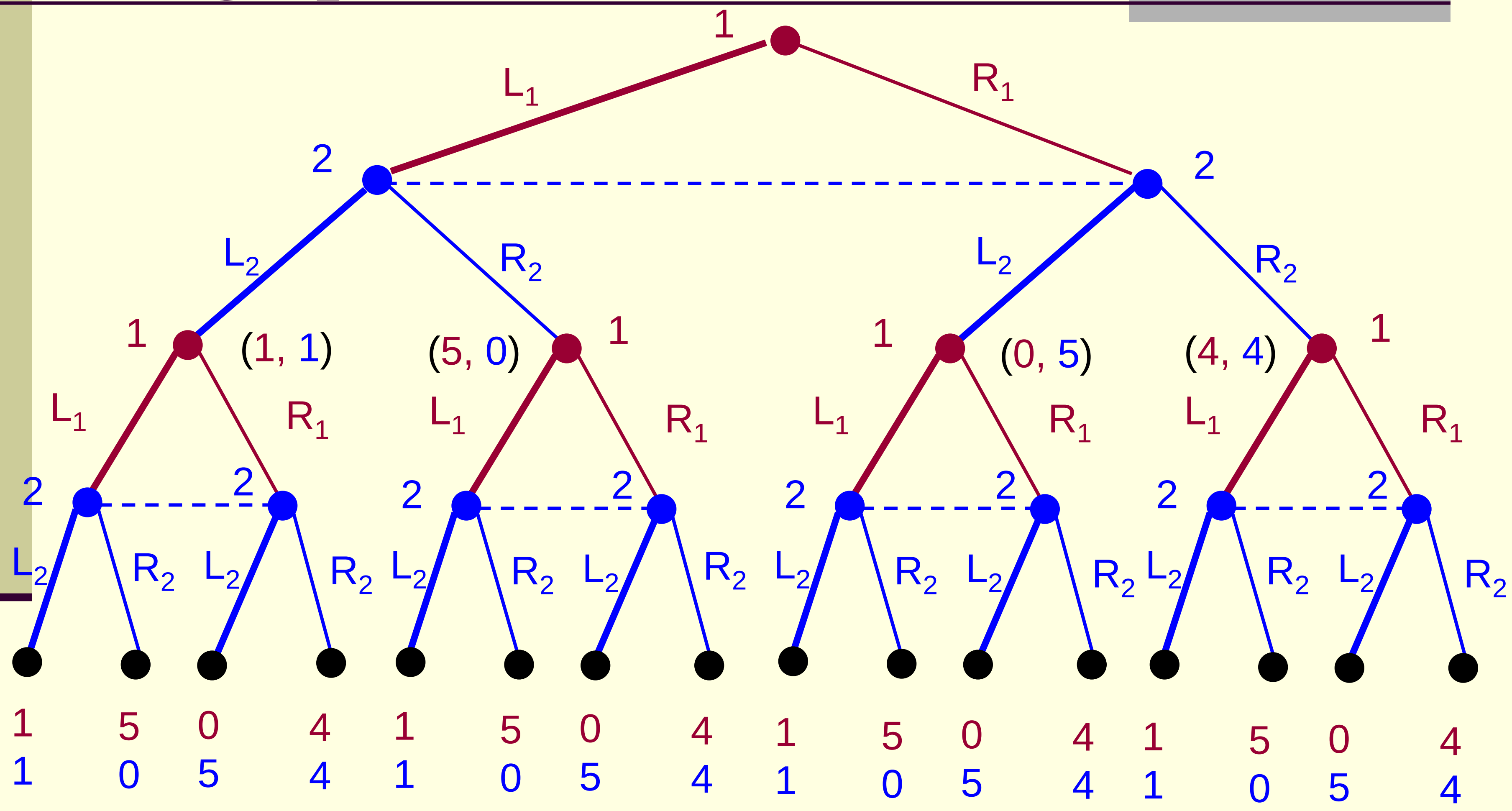
Player 2

Player 1		L_2	R_2
	L_1	1 , 1	5 , 0
	R_1	0 , 5	4 , 4

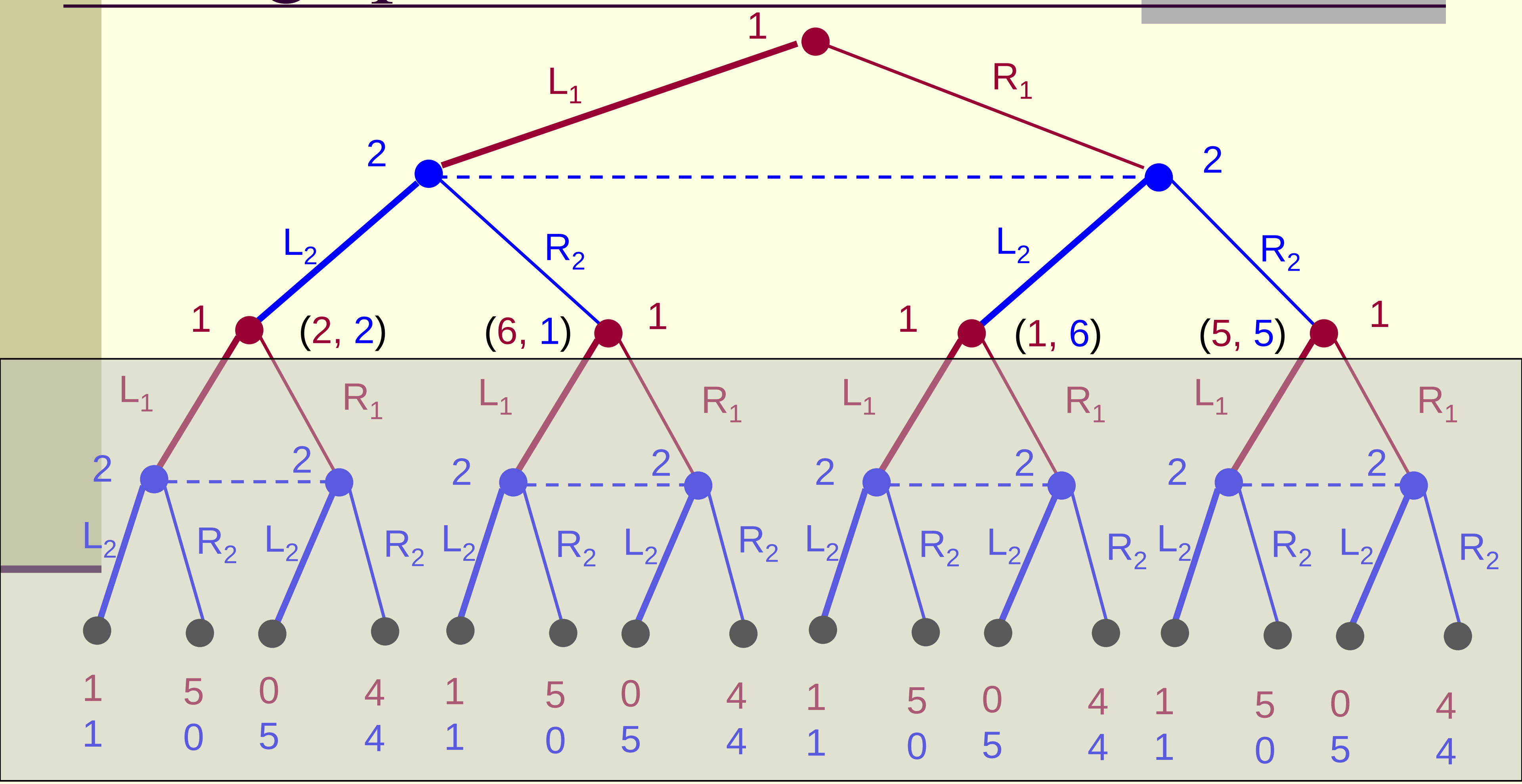
Game tree of the two-stage prisoners' dilemma



Informal game tree of the two-stage prisoners' dilemma



Informal game tree of the two-stage prisoners' dilemma



two-stage prisoners' dilemma

- The subgame-perfect Nash equilibrium
  (L_1 $L_1L_1L_1L_1$, L_2 $L_2L_2L_2L_2$)
  Player 1 plays L_1 at stage 1, and plays L_1 at stage 2 for any outcome of stage 1.
  Player 2 plays L_2 at stage 1, and plays L_2 at stage 2 for any outcome of stage 1.

Player 2

		L_2	R_2
Player 1	L_1	1 , 1	5 , 0
	R_1	0 , 5	4 , 4

Finitely repeated game

- A *finitely repeated game* is a dynamic game of complete information in which a (simultaneous-move) game is played a finite number of times, and the previous plays are observed before the next play.
- The finitely repeated game has a unique subgame perfect Nash equilibrium if the stage game (the simultaneous-move game) has a unique Nash equilibrium. *The Nash equilibrium of the stage game is played in every stage*.

What happens if the stage game has more than one Nash equilibrium?

- Two players play the following simultaneous move game twice
- The outcome of the first play is observed before the second play begins
- The payoff for the entire game is simply the sum of the payoffs from the two stages. That is, the discount factor is 1.
- Question: can we find a subgame perfect Nash equilibrium in which M_1, M_2 are played? Or, can the two players cooperate in a subgame perfect Nash equilibrium?

Player 2

		L_2	M_2	R_2
	L_1	$\underline{1}$, $\underline{1}$	5 , 0	0 , 0
Player 1	M_1	0 , 5	4 , 4	0 , 0
	R_1	0 , 0	0 , 0	$\underline{3}$, $\underline{3}$

Two-stage repeated game

- Subgame perfect Nash equilibrium:
  - player 1 plays M_1 at stage 1, and at stage 2, plays R_1 if the first stage outcome is (M_1,M_2), or plays L_1 if the first stage outcome is not (M_1,M_2)
  - player 2 plays M_2 at stage 1, and at stage 2, plays R_2 if the first stage outcome is (M_1, M_2), or plays L_2 if the first stage outcome is not (M_1,M_2)

Player 2

Player 1		L_2	M_2	R_2
	L_1	<u>1</u> , <u>1</u>	5 , 0	0 , 0
	M_1	0 , 5	4 , 4	0 , 0
	R_1	0 , 0	0 , 0	<u>3</u> , <u>3</u>

Two-stage repeated game

- Subgame perfect Nash equilibrium:
  - At stage 1, player 1 plays M_1, and player 2 plays M_2.
  - At stage 2,
    - player 1 plays R_1 if the first stage outcome is (M_1, M_2), or plays L_1 if the first stage outcome is not (M_1, M_2)
    - player 2 plays R_2 if the first stage outcome is (M_1, M_2), or plays L_2 if the first stage outcome is not (M_1, M_2)

The payoffs of the 2nd stage has been added to the first stage game.

Player 2

		L_2	M_2	R_2
	L_1	<u>2</u> , <u>2</u>	6 , 1	1 , 1
Player 1	M_1	1 , 6	7 , 7	1 , 1
	R_1	1 , 1	1 , 1	<u>4</u> , <u>4</u>

Caveat of Backward Induction and Subgame Perfection

- The chain-store paradox (Selton 1978): (Entry game repeated)
- A chain-store has branches in K cities. In each city k there is a single potential competitor, player k.
- In each period k, player k decides to compete or not with the chain-store, and then the chain-store can decide to fight or acquiesce.
- Assume that at all players know all the actions previously taken(perfect info.) and that the payoff the chain-store is the sum of its payoffs in the K cities.
- Intuitively, it's in the myopic interest of the chain-store to acquiesce,
- It may be in its long-term interest to build a reputation for aggressivebehavior, in order to deter future competition.)
- How long is the construction \reputation needed

Infinitely repeated game

- A *infinitely repeated game* is a dynamic game of complete information in which a (simultaneous-move) game called the *stage game* is played infinitely, and the outcomes of all previous plays are observed before the next play.
- Precisely, the simultaneous-move game is played at stage 1, 2, 3, ..., t-1, t, t+1, ... The outcomes of all previous t-1 stages are observed before the play at the t^{th} stage.
- Each player discounts her payoff by a factor δ, where 0< δ < 1.
- A player's payoff in the repeated game is the present value of the player's payoffs from the stage games.

Present value

Definition: Given a discount factor δ, the present value of an infinite sequence of payoffs $\pi_1,\ \pi_2,\ \pi_3,\ \pi_4, ...$ is

$$\pi_1 + \delta\pi_2 + \delta^2\pi_3 + \delta^3\pi_4 + ... = \sum_{t=1}^{\infty} \delta^{t-1}\pi_t$$

Example 1: The present value of an infinite sequence of payoffs 1, 1, 1, ... ($\pi_t = 1$, for all t) is $\frac{1}{1-\delta}$.

Example 2: The present value of an infinite sequence of payoffs 4, 1, 4, 1, 4, 1 ....(4 in every odd stage, 1 in every even stage) is $\frac{4}{1-\delta^2} + \frac{\delta}{1-\delta^2}$.

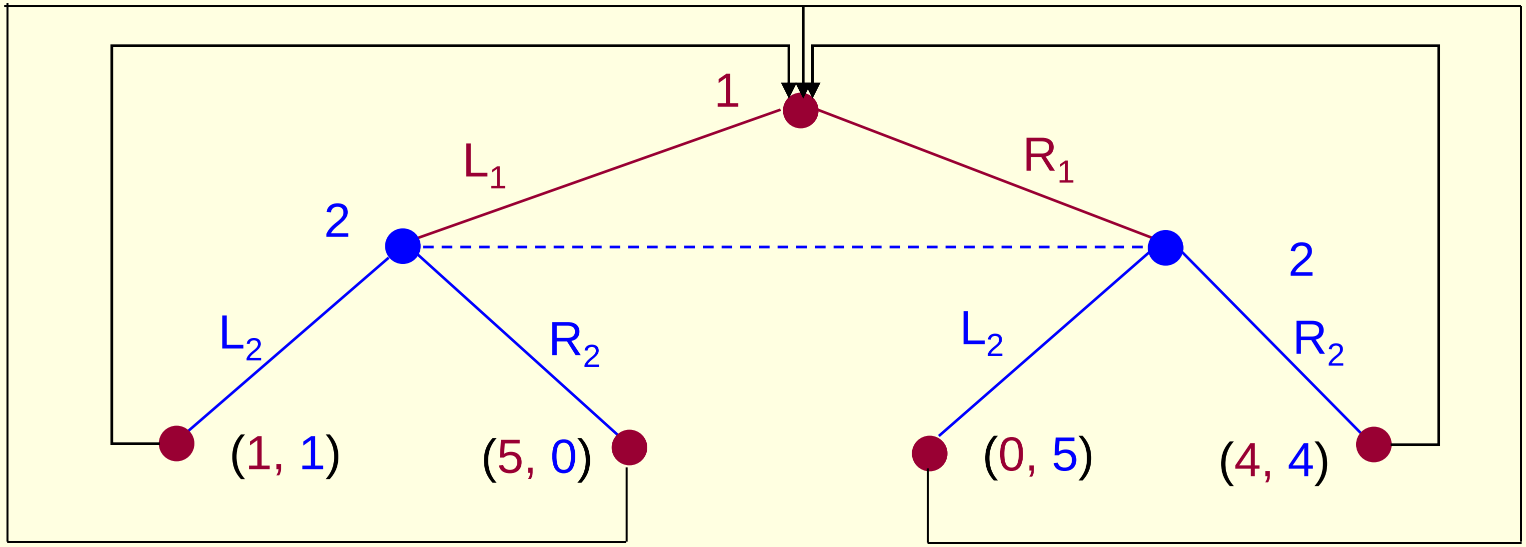
Infinitely repeated game: example

- The following simultaneous-move game is repeated infinitely
- The outcomes of all previous plays are observed before the next play begins
- Each player's payoff for the infinitely repeated game is present value of the payoffs received at all stages.
- Question: what is the subgame perfect Nash equilibrium?

Player 2

Player 1		L_2	R_2
	L_1	$\underline{1}$, $\underline{1}$	5 , 0
	R_1	0 , 5	4 , 4

Example: subgame



- Every subgame of an infinitely repeated game is identical to the game as a whole.

Example: strategy

- A strategy for a player is a complete plan. It can depend on the history of the play.
- One strategy for player i: play $\boldsymbol{L_i}$ at every stage (or at each of her information sets)
- An other stategy called *trigger strategy* for player $\boldsymbol{i}$: play $\boldsymbol{R_i}$ at stage 1, and at the t^{th} stage, if the outcome of each of all t-1 previous stages is ($\boldsymbol{R_1}$, $\boldsymbol{R_2}$) then play $\boldsymbol{R_i}$; otherwise, play $\boldsymbol{L_i}$.

Example: subgame perfect Nash equilibrium

- Check whether there is a <u>subgame perfect Nash equilibrium</u> in which player i plays L_i at every stage (or at each of her information sets).
- This can be done by the following two steps.
- Step 1: check whether the combination of strategies is a Nash equilibrium of the infinitely repeated game.
  - If player 1 plays L_1 at every stage, the best response for player 2 is to play L_2 at every stage.
  - If player 2 plays L_2 at every stage, the best response for player 1 is to play L_1 at every stage.
  - Hence, it is a Nash equilibrium of the infinitely repeated game.

Example: subgame perfect Nash equilibrium cont'd

- Step 2: check whether the Nash equilibrium of the infinitely repeated game induces a Nash equilibrium in every subgame of the infinitely repeated game.
  - Recall that every subgame of the infinitely repeated game is identical to the infinitely repeated game as a whole
  - Obviously, it induces a Nash equilibrium in every subgame
- Hence, it is a subgame perfect Nash equilibrium.

Trigger strategy: step 1

Stage 1: (R_1, R_2)
Stage 2: (R_1, R_2)
Stage t-1: (R_1, R_2)
Stage t: (R_1, L_2)
Stage t+1: (L_1, L_2)
Stage t+2: (L_1, L_2)

- Suppose that player 1 plays the trigger strategy.
- Can player 2 be better-off if she deviates from the trigger strategy at stage *t*?
- If she continues to play the trigger strategy at stage *t* and after, then she will get a sequence of payoffs 4, 4, 4, ... (from stage *t* to stage +∞). Discounting these payoffs to stage *t* gives us

$$4 + 4\delta + 4\delta^2 + 4\delta^3 + \ldots\ldots = \frac{4}{1-\delta}$$

- If she deviates from the trigger strategy at stage *t* then she will trigger noncooperation. Player 1 will play L_1 after stage *t* forever. Player 2' best response is L_2. So player 2 will get a sequence of payoffs 5, 1, 1, 1 ... (from stage *t* to stage +∞). Discounting these payoffs to stage *t* gives us

$$5 + 1\delta + 1\delta^2 + 1\delta^3 + \ldots\ldots = 5 + \frac{\delta}{1-\delta}$$

Trigger strategy: step 1 cont'd

Stage 1: (R_1, R_2)
Stage 2: (R_1, R_2)
⋮
Stage t-1: (R_1, R_2)
Stage t: (R_1, L_2)
Stage t+1: (L_1, L_2)
Stage t+2: (L_1, L_2)
⋮

$$\frac{4}{1-\delta} \geq 5 + \frac{\delta}{1-\delta} \iff \delta \geq \frac{1}{4}$$

- Hence, if $\delta \geq \frac{1}{4}$, player 2 cannot be better off if she deviates from the trigger strategy.
- This implies that if player 1 plays the trigger strategy the player 2's best response is the trigger strategy for $\delta \geq \frac{1}{4}$.
- By symmetry, if player 2 plays the trigger strategy then player 1's best response is the trigger strategy.
- Hence, there is a Nash equilibrium in which both players play the trigger strategy if $\delta \geq \frac{1}{4}$.

Trigger strategy: step 2

Stage 1: (R_1, R_2)

Stage 2: (R_1, R_2)

Stage t-1: (R_1, R_2)

Stage t: (R_1, R_2)

Stage t+1: (R_1, R_2)

Stage t+2: (R_1, R_2)

- Step 2: check whether the Nash equilibrium induces a Nash equilibrium in every subgame of the infinitely repeated game.
  - Recall that every subgame of the infinitely repeated game is identical to the infinitely repeated game as a whole

Trigger strategy: step 2 cont'd

- We have two classes of subgames:
  - subgame following a history in which the stage outcomes are all (R_1, R_2)
  - subgame following a history in which at least one stage outcome is not (R_1, R_2)

- The Nash equilibrium of the infinitely repeated game induces a Nash equilibrium in which each player still plays trigger strategy for the first class of subgames
- The Nash equilibrium of the infinitely repeated game induces a Nash equilibrium in which (L_1, L_2) is played forever for the second class of subgames.

Discussion

- Multiple equilibrium [Friedman's Theorem]
- Social norm
  - Coordination on certain equilibrium