Dynamic Games of Complete Information

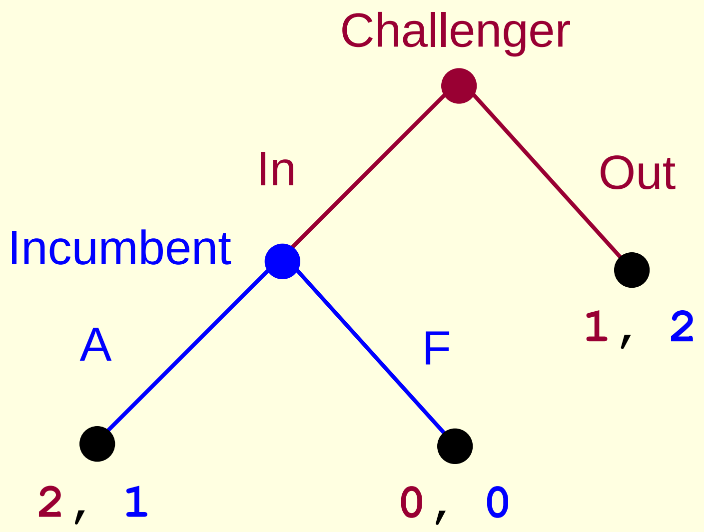
Subgame-Perfect Equilibrium

Outline of dynamic games of complete information

- *Dynamic games of complete information*
- *Extensive-form representation*
- *Dynamic games of complete and perfect information*
- *Game tree*
- Subgame-perfect Nash equilibrium
- Backward induction
- Applications
- Dynamic games of complete and imperfect information
- More applications
- Repeated games

Entry game

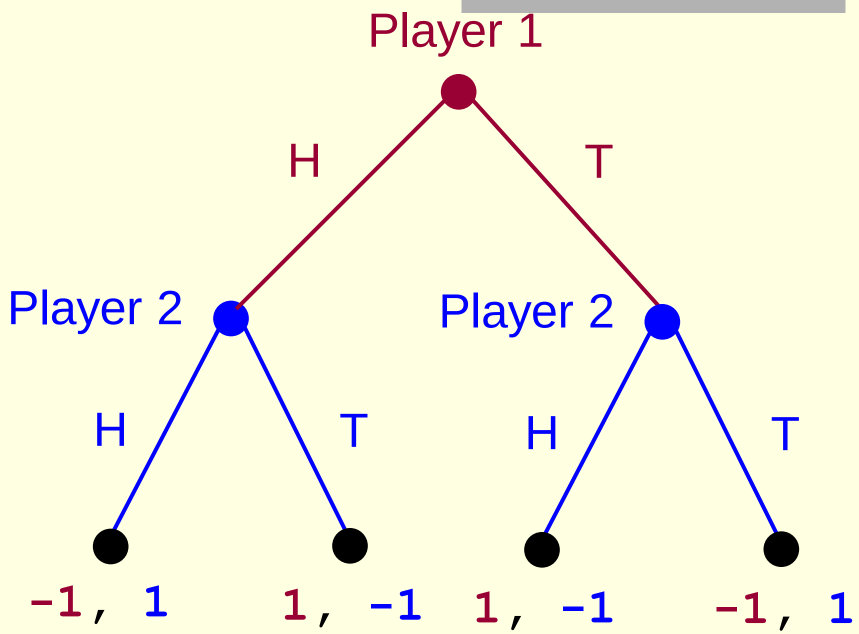
- An incumbent monopolist faces the possibility of entry by a challenger.
- The challenger may choose to enter or stay out.
- If the challenger enters, the incumbent can choose either to accommodate or to fight.
- The payoffs are common knowledge.



The first number is the payoff of the challenger. The second number is the payoff of the incumbent.

Sequential-move matching pennies

- Each of the two players has a penny.
- Player 1 first chooses whether to show the Head or the Tail.
- After observing player 1's choice, player 2 chooses to show Head or Tail
- Both players know the following rules:
  - If two pennies match (both heads or both tails) then player 2 wins player 1's penny.
  - Otherwise, player 1 wins player 2's penny.



Dynamic (or sequential-move) games of complete information

- A set of players
- Who moves when and what action choices are available?
- What do players know when they move?
- Players’ payoffs are determined by their choices.
- All these are common knowledge among the players.

Definition: extensive-form representation

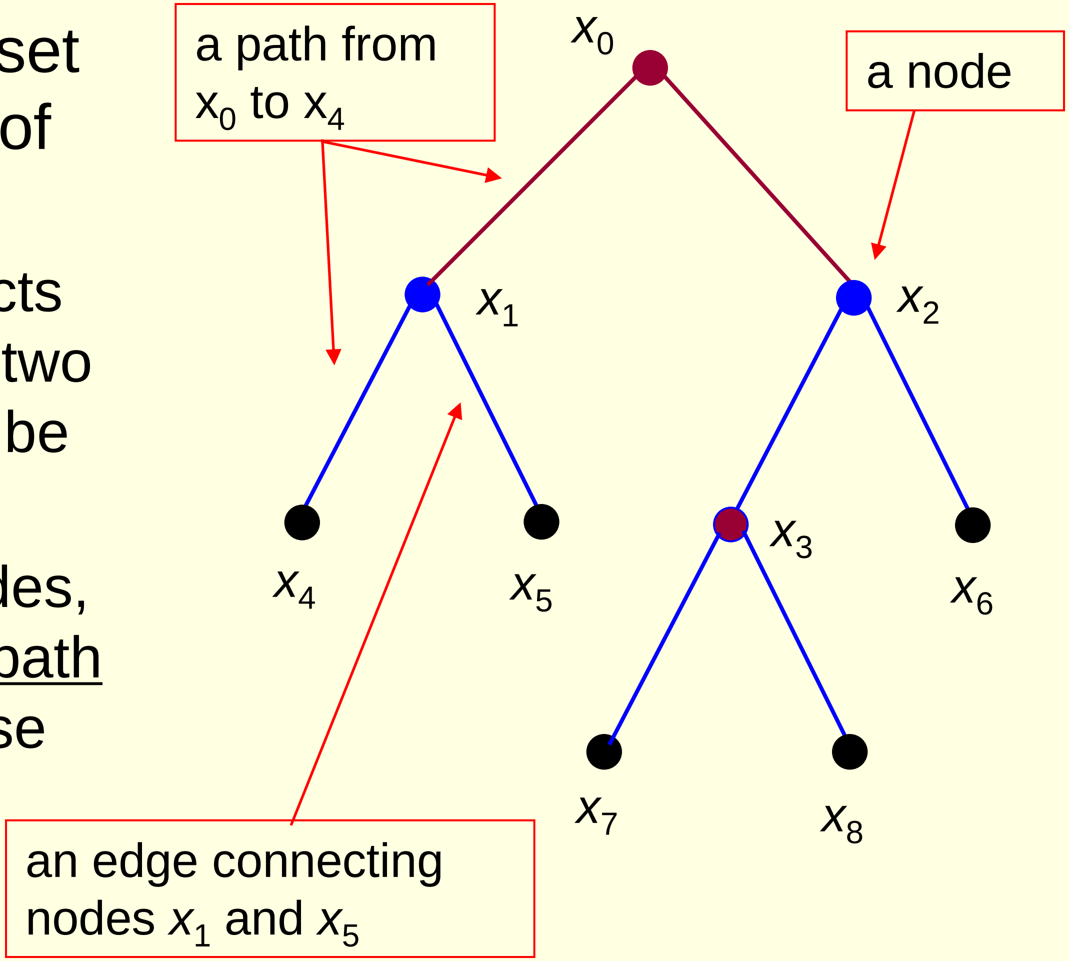
- The extensive-form representation of a game specifies:
  - the players in the game
  - when each player has the move
  - what each player can do at each of his or her opportunities to move
  - what each player knows at each of his or her opportunities to move
  - the payoff received by each player for each combination of moves that could be chosen by the players

Dynamic games of complete and perfect information

- Perfect information
  - All previous moves are observed before the next move is chosen.
  - A player knows **Who** has moved **What** before she makes a decision

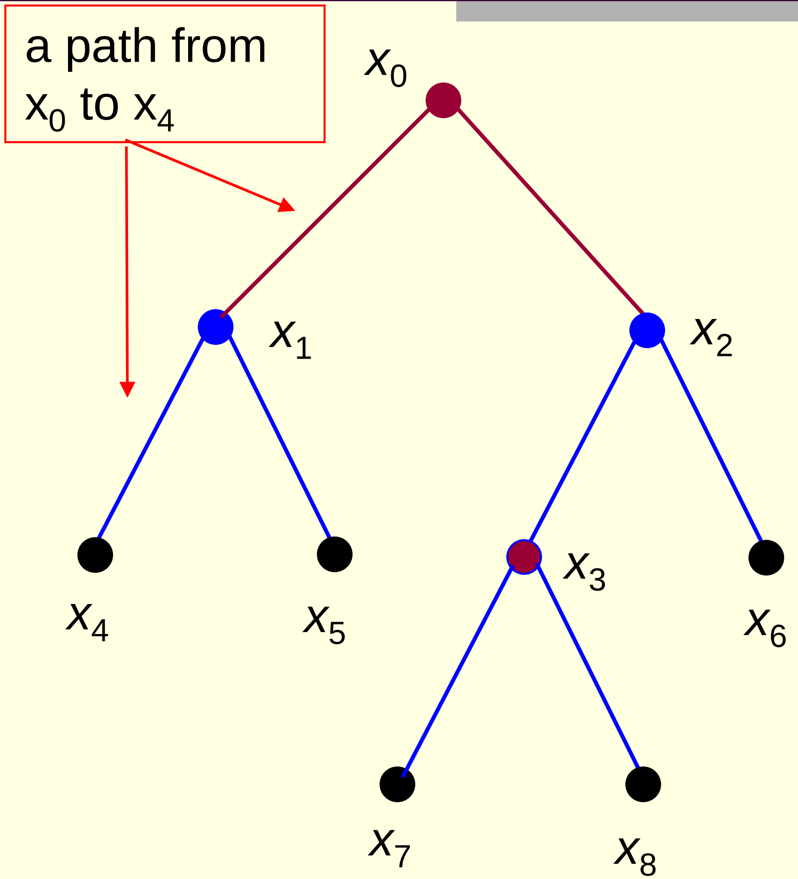
Game tree

- A game tree has a set of nodes and a set of edges such that
  - each edge connects two nodes (these two nodes are said to be *adjacent*)
  - for any pair of nodes, there is a unique path that connects these two nodes



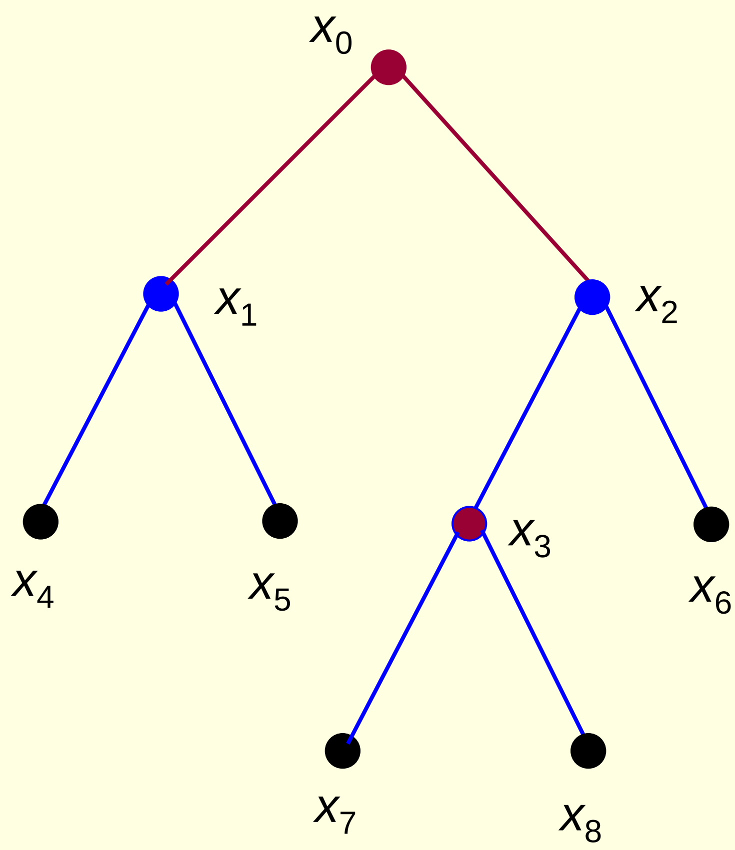
Game tree

- A *path* is a sequence of distinct nodes $y_1, y_2, y_3, \ldots, y_{n-1}, y_n$ such that y_i and y_{i+1} are adjacent, for $i=1, 2, \ldots, n-1$. We say that this path is from y_1 to y_n.
- We can also use the sequence of edges induced by these nodes to denote the path.
- The *length* of a path is the number of edges contained in the path.
- Example 1: x_0, x_2, x_3, x_7 is a path of length 3.
- Example 2: x_4, x_1, x_0, x_2, x_6 is a path of length 4



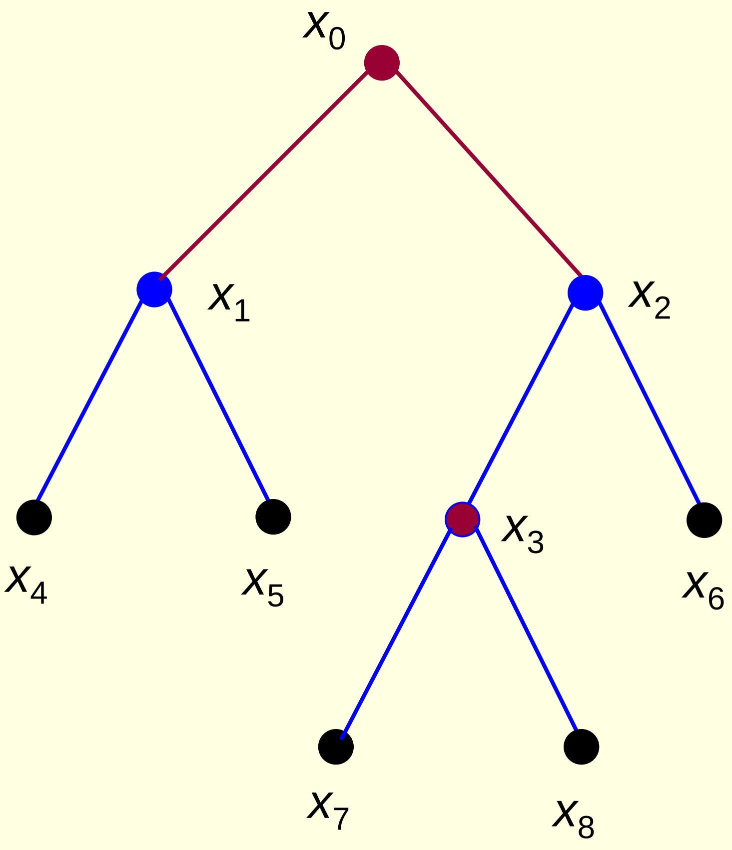
Game tree

- There is a special node x_0 called the *root* of the tree which is the beginning of the game
- The nodes adjacent to x_0 are successors of x_0. The successors of x_0 are x_1, x_2
- For any two **<u>adjacent</u>** nodes, the node that is connected to the root by a longer path is a *successor* of the other node.
- Example 3: x_7 is a successor of x_3 because they are adjacent and the path from x_7 to x_0 is longer than the path from x_3 to x_0



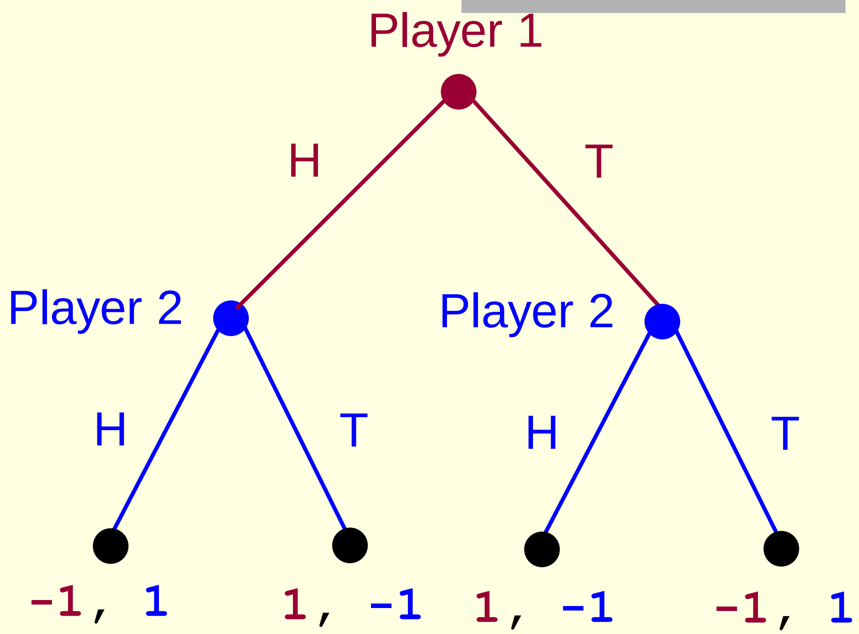
Game tree

- If a node *x* is a successor of another node *y* then *y* is called a *predecessor* of *x*.
- In a game tree, any node other than the root has a unique predecessor.
- Any node that has no successor is called a *terminal node* which is a possible end of the game
- Example 4: x_4, x_5, x_6, x_7, x_8 are terminal nodes



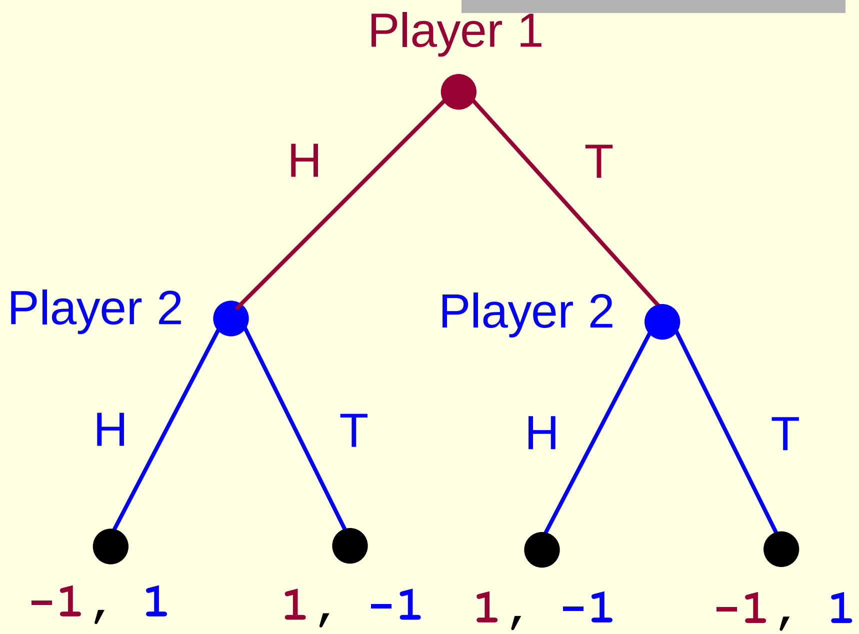
Game tree

- Any node other than a terminal node represents some player.
- For a node other than a terminal node, the edges that connect it with its successors represent the actions available to the player represented by the node



Game tree

- A path from the root to a terminal node represents a complete sequence of moves which determines the payoff at the terminal node



Strategy

- A strategy for a player is a complete plan of actions.
- It specifies a feasible action for the player in every contingency in which the player *might* be called on to act.
- What the players *can* possibly play, not what they *do* play.

Entry game

- Challenger's strategies
  - In
  - Out
- Incumbent's strategies
  - Accommodate (if challenger plays In)
  - Fight (if challenger plays In)
- Payoffs
- Normal-form representation

		Incumbent	
		Accommodate	Fight
Challenger	In	2 , 1	0 , 0
	Out	1 , 2	1 , 2

Strategy and payoff

- In a game tree, a strategy for a player is represented by a *set of edges*.
- A combination of strategies (sets of edges), one for each player, induce one path from the root to a terminal node, which determines the payoffs of all players

Sequential-move matching pennies

- Player 1's strategies
  - Head
  - Tail
- Player 2's strategies
  - H if player 1 plays H, H if player 1 plays T
  - H if player 1 plays H, T if player 1 plays T
  - T if player 1 plays H, H if player 1 plays T
  - T if player 1 plays H, T if player 1 plays T

Player 2's strategies are denoted by HH, HT, TH and TT, respectively.(n x m)

Sequential-move matching pennies

- Their payoffs
- Normal-form representation

		Player 2			
		HH	HT	TH	TT
Player 1	H	-1 , 1	-1 , 1	1 , -1	1 , -1
	T	1 , -1	-1 , 1	1 , -1	-1 , 1

Nash equilibrium

- The set of Nash equilibria in a dynamic game of complete information is the set of Nash equilibria of its normal-form.

Nash equilibrium in a dynamic game

- We can also use normal-form to represent a dynamic game
- The set of Nash equilibria in a dynamic game of complete information is the set of Nash equilibria of its normal-form
- How to find the Nash equilibria in a dynamic game of complete information
  - Construct the normal-form of the dynamic game of complete information
  - Find the Nash equilibria in the normal-form

Nash equilibria in entry game

- Two Nash equilibria
  - (In, Accommodate)
  - (Out, Fight)
- Does the second Nash equilibrium make sense?
- Non-creditable threats
  - Limitation to the normal form representation

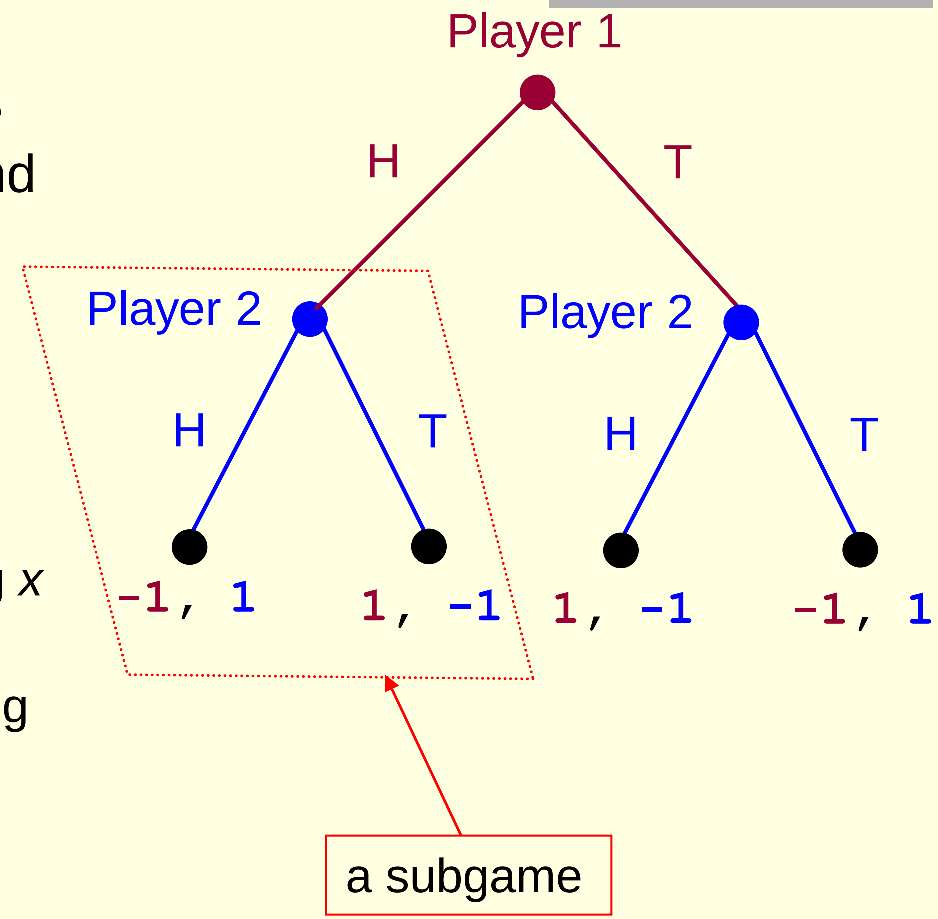
		Incumbent	
		Accommodate	Fight
Challenger	In	<u>2</u> , <u>1</u>	0 , 0
	Out	1 , <u>2</u>	<u>1</u> , <u>2</u>

Remove nonreasonable Nash equilibrium

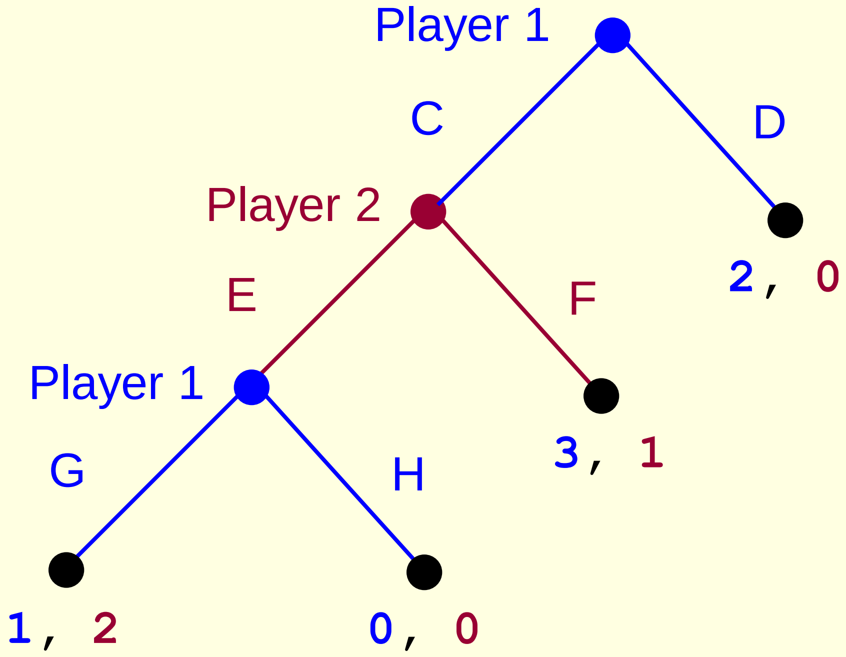
- Subgame perfect Nash equilibrium is a refinement of Nash equilibrium
- It can rule out nonreasonable Nash equilibria or non-creditable threats
- We first need to define subgame

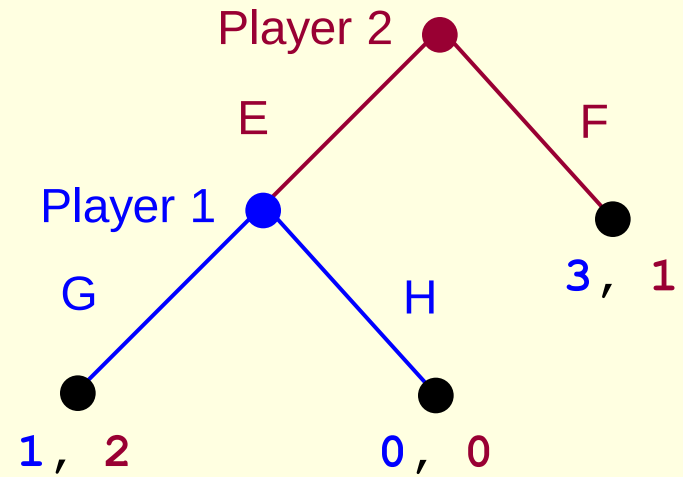
Subgame

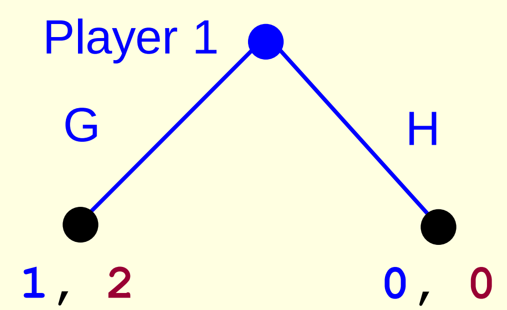
- A subgame of a game tree begins at a nonterminal node and includes all the nodes and edges following the nonterminal node
- A subgame beginning at a nonterminal node *x* can be obtained as follows:
  - remove the edge connecting *x* and its predecessor
  - the connected part containing *x* is the subgame



Subgame: example





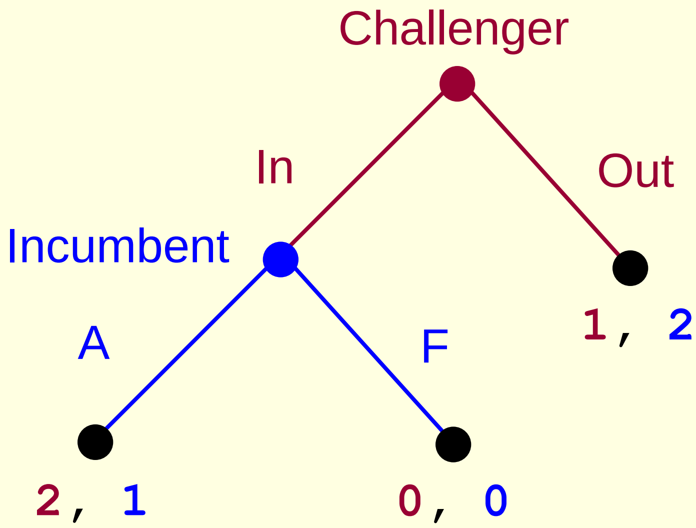


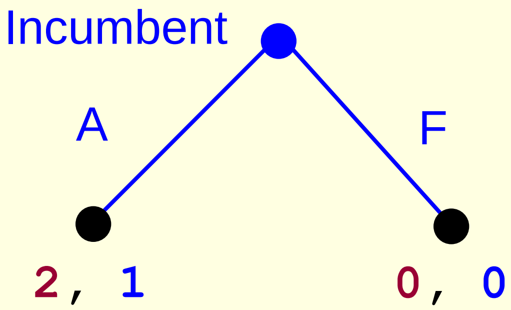
Subgame-perfect Nash equilibrium

- A Nash equilibrium of a dynamic game is subgame-perfect if the strategies of the Nash equilibrium constitute a Nash equilibrium in every subgame of the game.
- Subgame-perfect Nash equilibrium is a Nash equilibrium.

Entry game

- Two Nash equilibria
  - (In, Accommodate) is subgame-perfect.
  - (Out, Fight) is not subgame-perfect because it does not induce a Nash equilibrium in the subgame beginning at Incumbent.

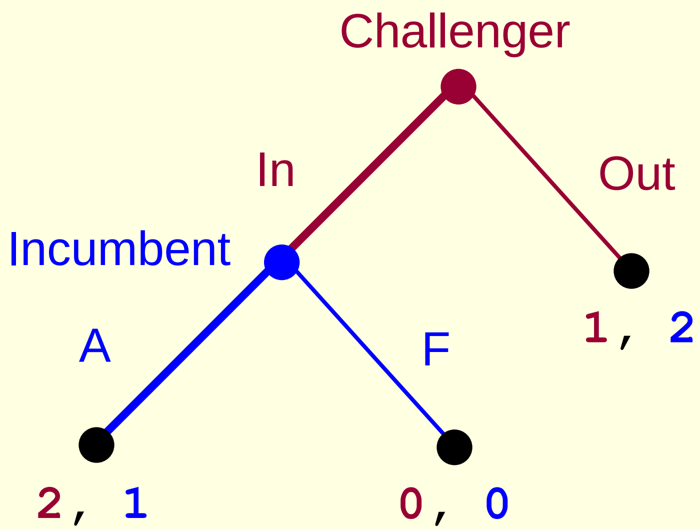




Accommodate is the Nash equilibrium in this subgame.

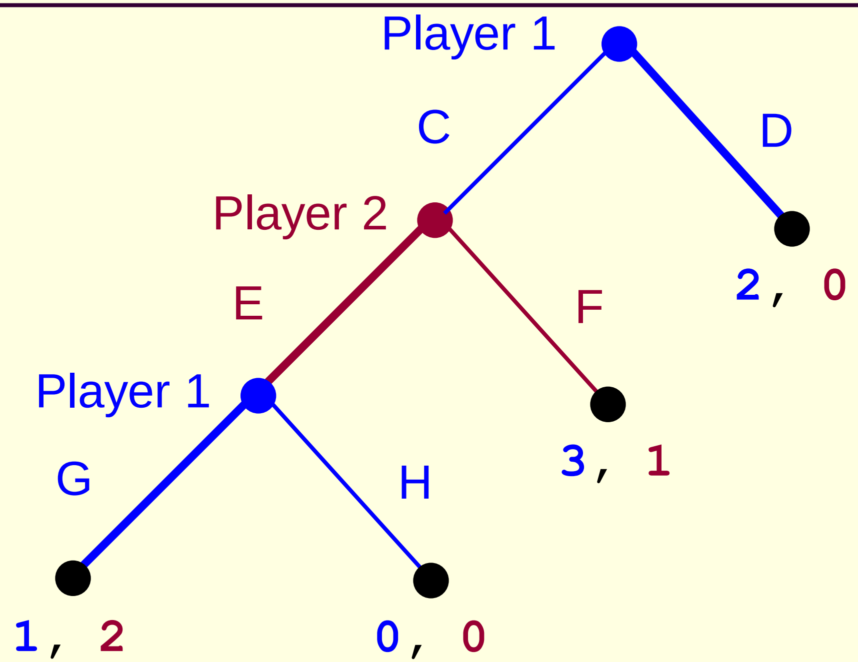
Find subgame perfect Nash equilibria: backward induction

- Starting with those smallest subgames
- Then move backward until the root is reached



The first number is the payoff of the challenger. The second number is the payoff of the incumbent.

Find subgame perfect Nash equilibria: backward induction

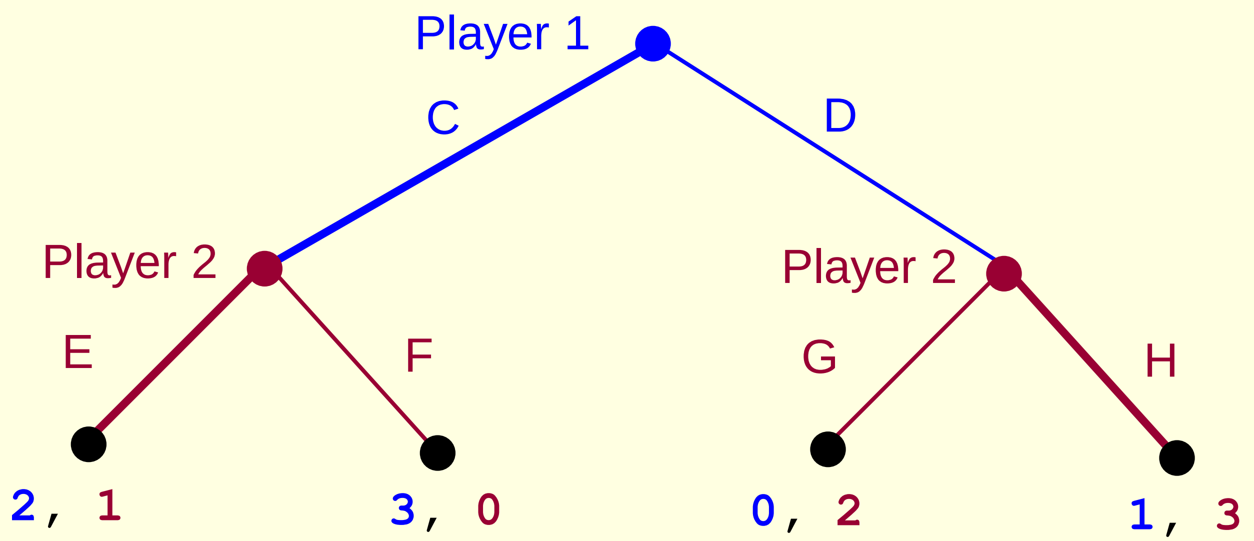


- Subgame perfect Nash equilibrium (DG, E)
  - Player 1 plays D, and plays G if player 2 plays E
  - Player 2 plays E if player 1 plays C

Existence of subgame-perfect Nash equilibrium

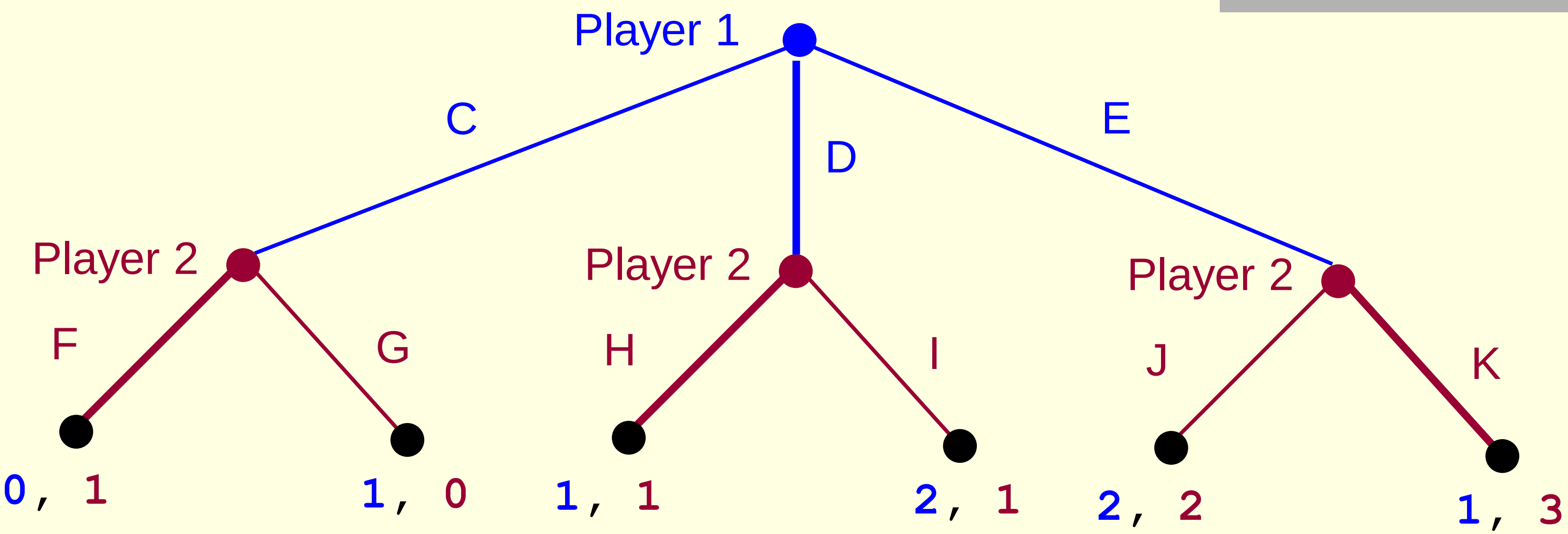
- Every finite dynamic game of complete and perfect information has a subgame-perfect Nash equilibrium that can be found by backward induction.

Backward induction: illustration



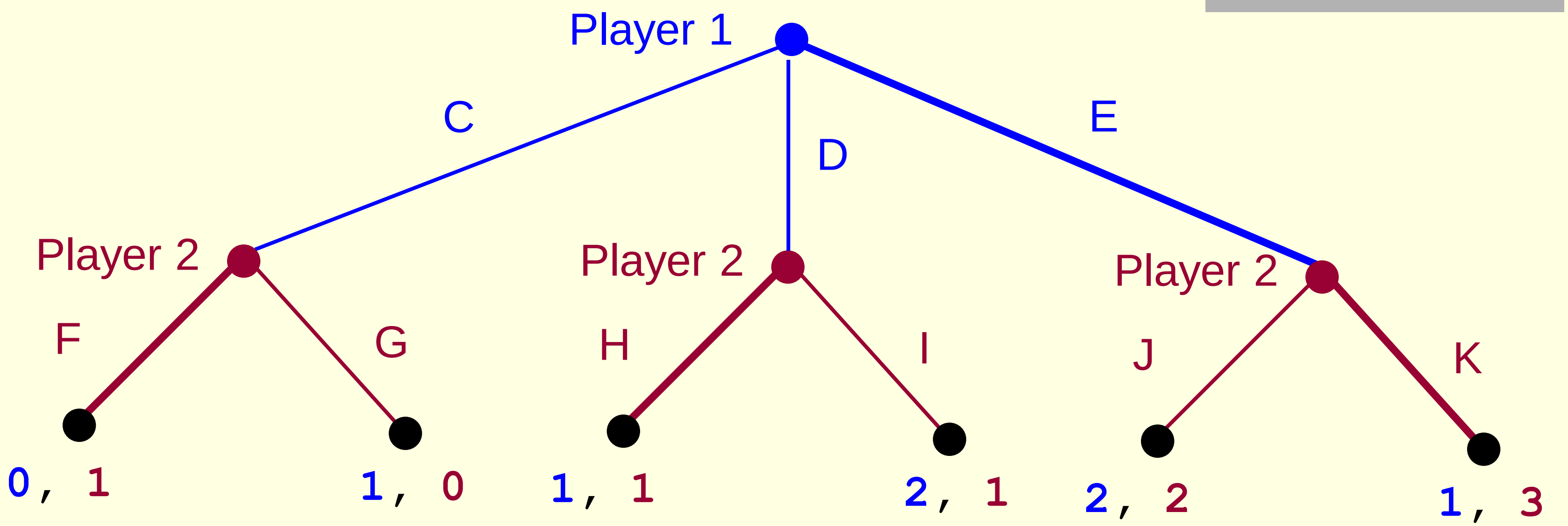
- Subgame-perfect Nash equilibrium (C, EH).
  - player 1 plays C;
  - player 2 plays E if player 1 plays C, plays H if player 1 plays D.

Multiple subgame-perfect Nash equilibria: illustration



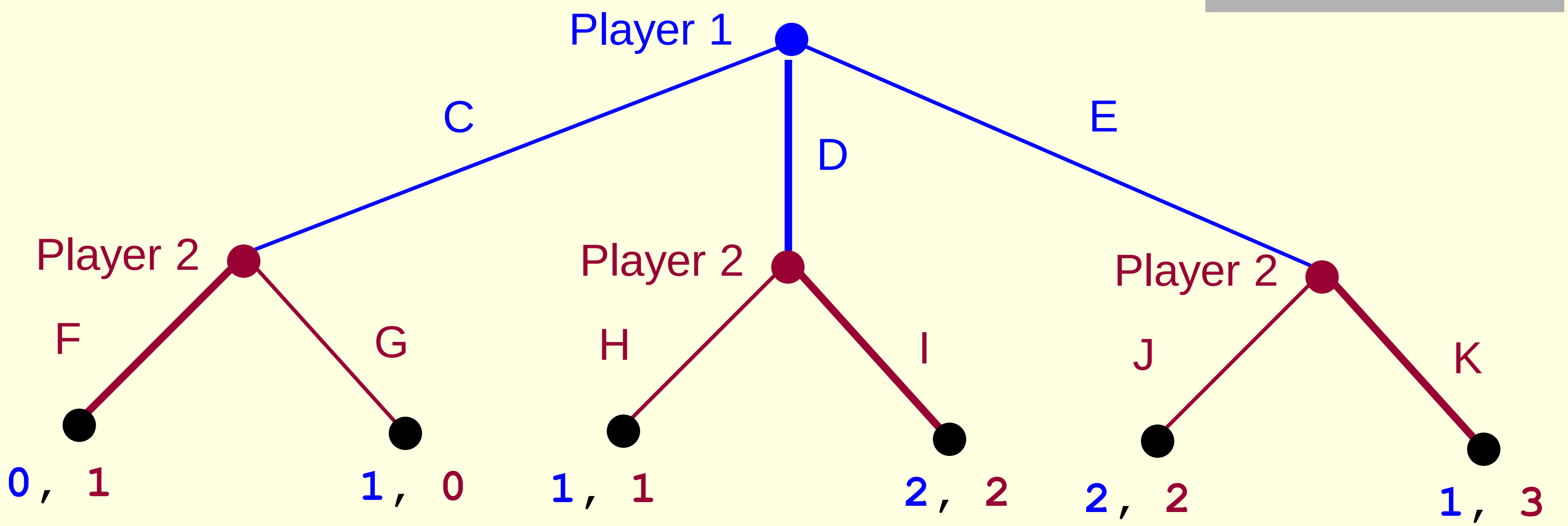
- Subgame-perfect Nash equilibrium (D, FHK).
  - player 1 plays D
  - player 2 plays F if player 1 plays C, plays H if player 1 plays D, plays K if player 1 plays E.

Multiple subgame-perfect Nash equilibria



- Subgame-perfect Nash equilibrium (E, FHK).
  - player 1 plays E;
  - player 2 plays F if player 1 plays C, plays H if player 1 plays D, plays K if player 1 plays E.

Multiple subgame-perfect Nash equilibria

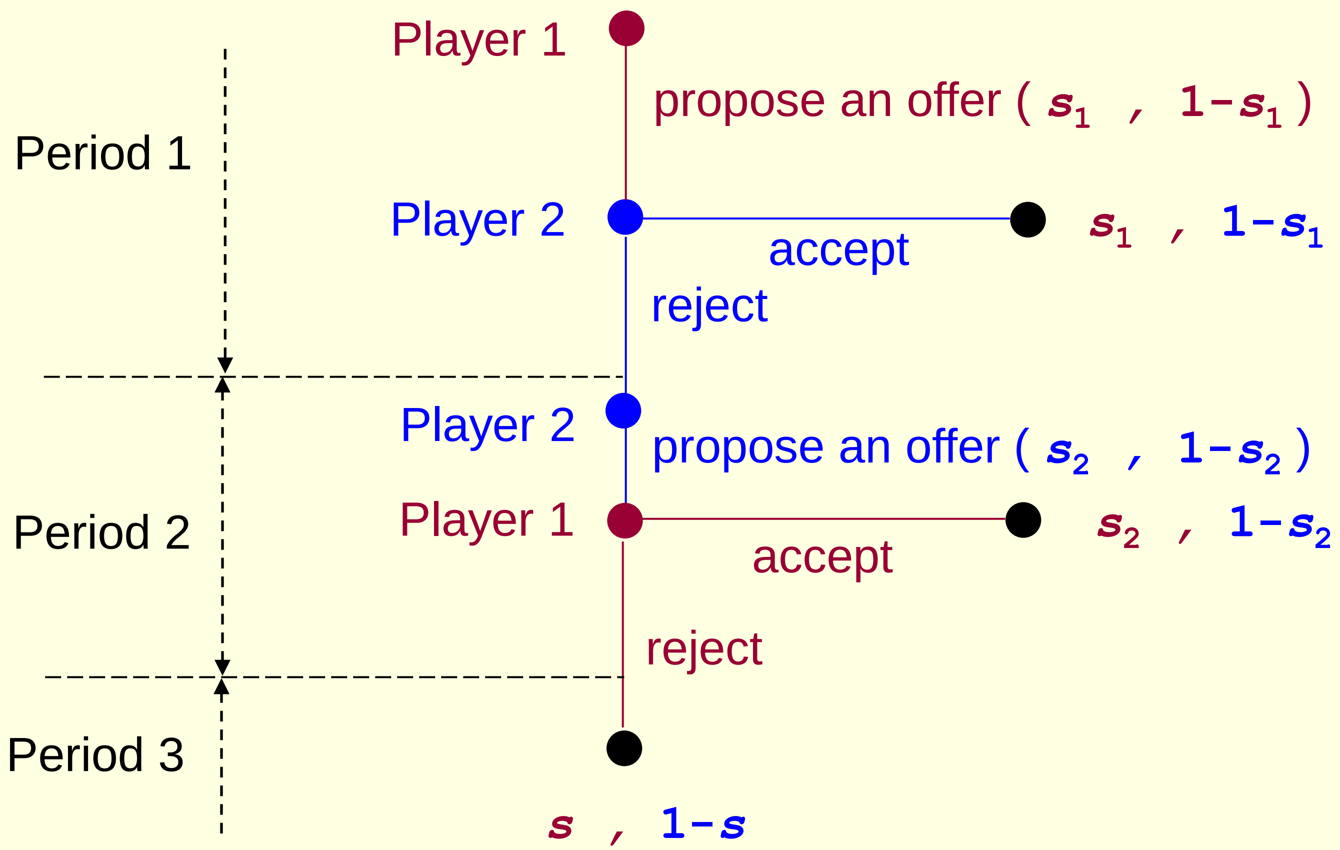


- Subgame-perfect Nash equilibrium (D, FIK).
  - player 1 plays D;
  - player 2 plays F if player 1 plays C, plays I if player 1 plays D, plays K if player 1 plays E.

Sequential bargaining (2.1.D of Gibbons)

- Player 1 and 2 are bargaining over one dollar. The timing is as follows:
- At the beginning of the first period, player 1 proposes to take a share s_1 of the dollar, leaving 1-s_1 to player 2.
- Player 2 either accepts the offer or rejects the offer (in which case play continues to the second period)
- At the beginning of the second period, player 2 proposes that player 1 take a share s_2 of the dollar, leaving 1-s_2 to player 2.
- Player 1 either accepts the offer or rejects the offer (in which case play continues to the third period)
- At the beginning of third period, player 1 receives a share s of the dollar, leaving 1-s for player 2, where $0<s<1$.
- The players are impatient. They discount the payoff by a fact δ, where $0< \delta <1$

Sequential bargaining (2.1.D of Gibbons)



Solve sequential bargaining by backward induction

- Period 2:
  - Player 1 accepts s_2 if and only if $s_2 \geq \delta s$. (We assume that each player will accept an offer if indifferent between accepting and rejecting)
  - Player 2 faces the following two options:
    (1) offers $s_2 = \delta s$ to player 1, leaving $1\text{-}s_2 = 1\text{-}\delta s$ for herself at this period, or
    (2) offers $s_2 < \delta s$ to player 1 (player 1 will reject it), and receives $1\text{-}s$ next period. Its discounted value is $\delta(1\text{-}s)$
  - Since $\delta(1\text{-}s)<1\text{-}\delta s$, player 2 should propose an offer $(s_2^*,\ 1\text{-}s_2^*)$, where $s_2^* = \delta s$. Player 1 will accept it.

Stackelberg model of duopoly

- A homogeneous product is produced by only two firms: firm 1 and firm 2. The quantities are denoted by q_1 and q_2, respectively.
- The timing of this game is as follows:
  - Firm 1 chooses a quantity $q_1 \geq 0$.
  - Firm 2 observes q_1 and then chooses a quantity $q_2 \geq 0$.
- The market priced is $P(Q)=a-Q$, where a is a constant number and $Q=q_1+q_2$.
- The cost to firm i of producing quantity q_i is $C_i(q_i)=cq_i$.
- Payoff functions:

$$u_1(q_1, q_2)=q_1(a-(q_1+q_2)-c)$$
$$u_2(q_1, q_2)=q_2(a-(q_1+q_2)-c)$$

Stackelberg model of duopoly

- Find the subgame-perfect Nash equilibrium by backward induction
  - We first solve firm 2's problem for any $q_1 \geq 0$ to get firm 2's best response to q_1 . That is, we first solve all the subgames beginning at firm 2.
  - Then we solve firm 1' s problem. That is, solve the subgame beginning at firm 1

Stackelberg model of duopoly

- Solve firm 2's problem for any $q_1 \geq 0$ to get firm 2's best response to q_1.
  - Max $u_2(q_1, q_2)=q_2(a-(q_1+q_2)-c)$
    subject to $0 \leq q_2 \leq +\infty$

    FOC: $a - 2q_2 - q_1 - c = 0$

  - Firm 2's best response,
  - $R_2(q_1) = (a - q_1 - c)/2$ if $q_1 \leq a - c$
    $= 0$ if $q_1 > a - c$

Stackelberg model of duopoly

- Solve firm 1's problem. Note firm 1 can also solve firm 2's problem. That is, firm 1 knows firm 2's best response to any q_1. Hence, firm 1's problem is
  - Max $u_1(q_1, R_2(q_1))=q_1(a-(q_1+R_2(q_1))-c)$
    subject to $0 \le q_1 \le +\infty$

    That is,
    Max $u_1(q_1, R_2(q_1))=q_1(a-q_1-c)/2$
    subject to $0 \le q_1 \le +\infty$

    FOC: $(a - 2q_1 - c)/2 = 0$
    $q_1 = (a - c)/2$

Stackelberg model of duopoly

- *Subgame-perfect Nash equilibrium*
  - $((a-c)/2, R_2(q_1))$, where
    $$R_2(q_1) = (a - q_1 - c)/2 \text{ if } q_1 \le a - c$$
    $$= 0 \text{ if } q_1 > a - c$$
  - That is, firm 1 chooses a quantity $(a-c)/2$, firm 2 chooses a quantity $R_2(q_1)$ if firm 1 chooses a quantity q_1.
  - The *backward induction outcome* is $((a-c)/2, (a-c)/4)$.
  - Firm 1 chooses a quantity $(a-c)/2$, firm 2 chooses a quantity $(a-c)/4$.

Stackelberg model of duopoly

- Firm 1 produces
  $q_1=(a-c)/2$ and its profit
  $q_1(a-(q_1+q_2)-c)=(a-c)^2/8$

- Firm 2 produces
  $q_2=(a-c)/4$ and its profit
  $q_2(a-(q_1+q_2)-c)=(a-c)^2/16$

- The aggregate quantity is $3(a-c)/4$.

Cournot model of duopoly

- Firm 1 produces
  $q_1=(a-c)/3$ and its profit
  $q_1(a-(q_1+q_2)-c)=(a-c)^2/9$

- Firm 2 produces
  $q_2=(a-c)/3$ and its profit
  $q_2(a-(q_1+q_2)-c)=(a-c)^2/9$

- The aggregate quantity is $2(a-c)/3$.

Monopoly

- Suppose that only one firm, a monopoly, produces the product. The monopoly solves the following problem to determine the quantity q_m.
- Max $\quad q_m (a - q_m - c)$
  subject to $0 \leq q_m \leq +\infty$

  FOC: $a - 2q_m - c = 0$
  $q_m = (a - c)/2$
- Monopoly produces
  $q_m = (a - c)/2$ and its profit
  $q_m(a - q_m - c) = (a - c)^2/4$

Discusssion

- The first-mover advantage
  - Strategic substitutes and commitment (threat)
    - Stackelberg model
- The curse of knowledge
  - More knowlege is not always good (business spy)

Sequential-move Bertrand model of duopoly (differentiated products)

- Two firms: firm 1 and firm 2. (partial substitutes)
- Each firm chooses the price for its product. The prices are denoted by p_1 and p_2, respectively.
- The timing of this game as follows.
  - Firm 1 chooses a price $p_1 \geq 0$.
  - Firm 2 observes p_1 and then chooses a price $p_2 \geq 0$.
- The quantity that consumers demand from firm 1: $q_1(p_1, p_2) = a - p_1 + bp_2$.
- The quantity that consumers demand from firm 2: $q_2(p_1, p_2) = a - p_2 + bp_1$.
- The cost to firm i of producing quantity q_i is $C_i(q_i)=cq_i$.

Sequential-move Bertrand model of duopoly (differentiated products)

- Solve firm 2's problem for any $p_1 \geq 0$ to get firm 2's best response to p_1.
  - Max $u_2(p_1, p_2)=(a - p_2 + bp_1)(p_2 - c)$
    subject to $0 \leq p_2 \leq +\infty$

    FOC: $a + c - 2p_2 + bp_1 = 0$
    $p_2 = (a + c + bp_1)/2$
  - Firm 2's best response,
  - $R_2(p_1) = (a + c + bp_1)/2$

Sequential-move Bertrand model of duopoly (differentiated products)

- Solve firm 1's problem. Note firm 1 can also solve firm 2's problem. Firm 1 knows firm 2's best response to $\boldsymbol{p_1}$. Hence, firm 1's problem is
  - Max $\boldsymbol{u_1(p_1, R_2(p_1))=(a - p_1 + b\times R_2(p_1)\ )(p_1 - c)}$
    subject to $0 \le \boldsymbol{p_1} \le +\infty$

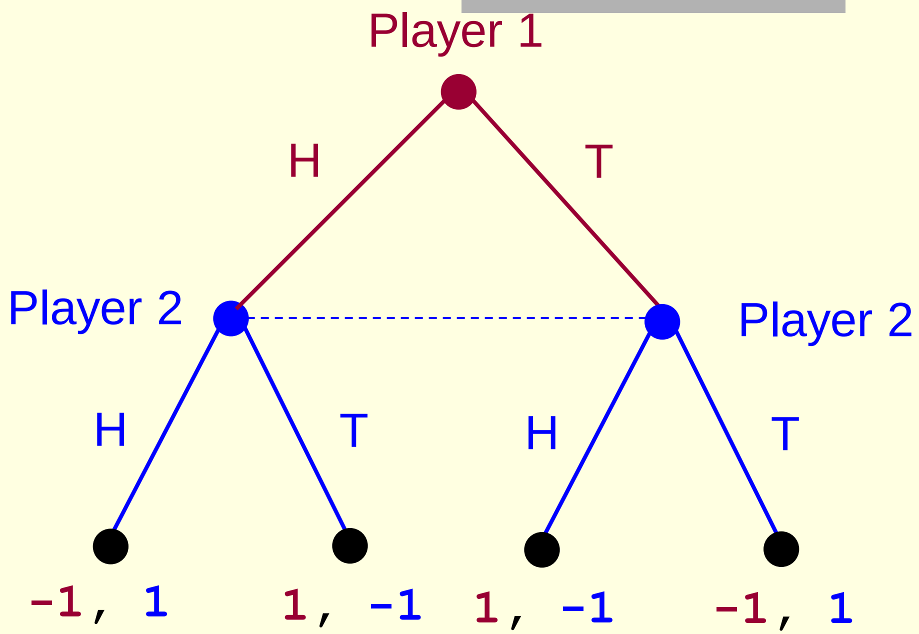
    That is,
    Max $\boldsymbol{u_1(p_1, R_2(p_1))=(a - p_1 + b\times(a + c + bp_1)/2\ )(p_1 - c)}$
    subject to $0 \le \boldsymbol{p_1} \le +\infty$
  - FOC: $\boldsymbol{a - p_1 + b\times(a + c + bp_1)/2+(-1+b^2/2)\ (p_1 - c) = 0}$
    $\boldsymbol{p_1 = (a+c+(ab+bc-b^2c)/2)/(2-b^2)}$

Sequential-move Bertrand model of duopoly (differentiated products)

- *Subgame-perfect Nash equilibrium*
  - $((a+c+(ab+bc-b^2c)/2)/(2-b^2), R_2(p_1))$,
    where $R_2(p_1) = (a + c + bp_1)/2$
  - Firm 1 chooses a price
    $(a+c+(ab+bc-b^2c)/2)/(2-b^2)$,
    firm 2 chooses a price $R_2(p_1)$ if firm 1 chooses a price p_1.

Imperfect information: illustration

- Each of the two players has a penny.
- Player 1 first chooses whether to show the Head or the Tail.
- Then player 2 chooses to show Head or Tail without knowing player 1's choice,
- Both players know the following rules:
  - If two pennies match (both heads or both tails) then player 2 wins player 1's penny.
  - Otherwise, player 1 wins player 2's penny.

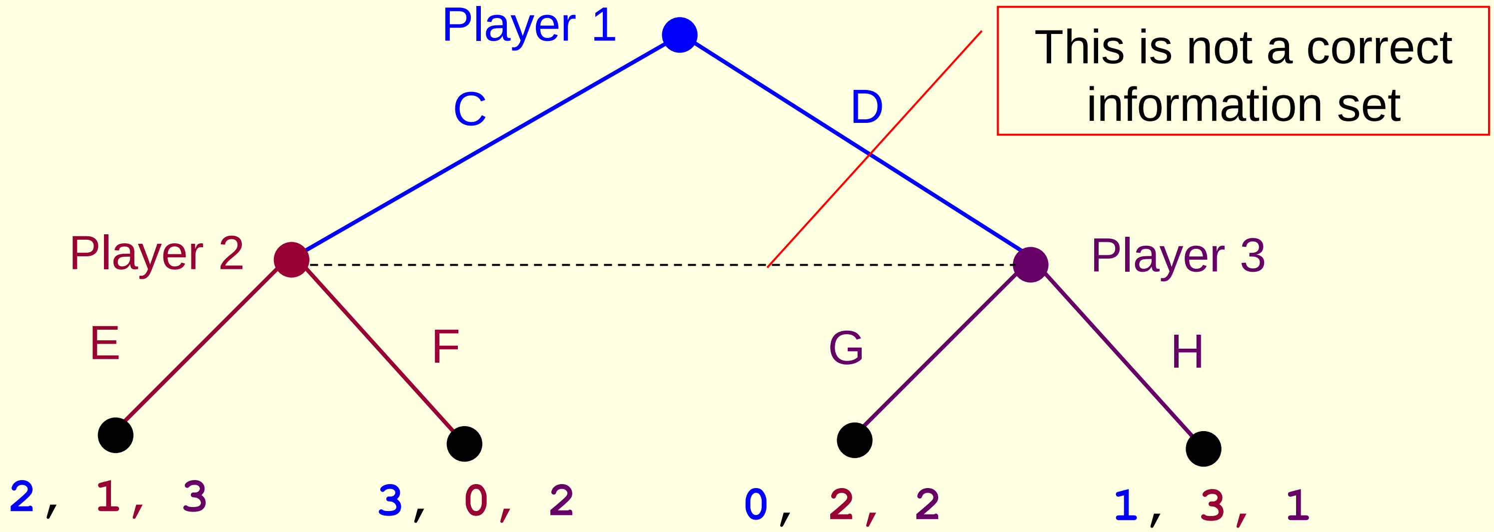


Information set

- Gibbons' definition: An information set for a player is a collection of nodes satisfying:
  - the player has the move at every node in the information set, and
  - when the play of the game reaches a node in the information set, the player with the move does not know which node in the information set has (or has not) been reached.
- All the nodes in an information set belong to the same player
- The player must have the same set of feasible actions at each node in the information set.

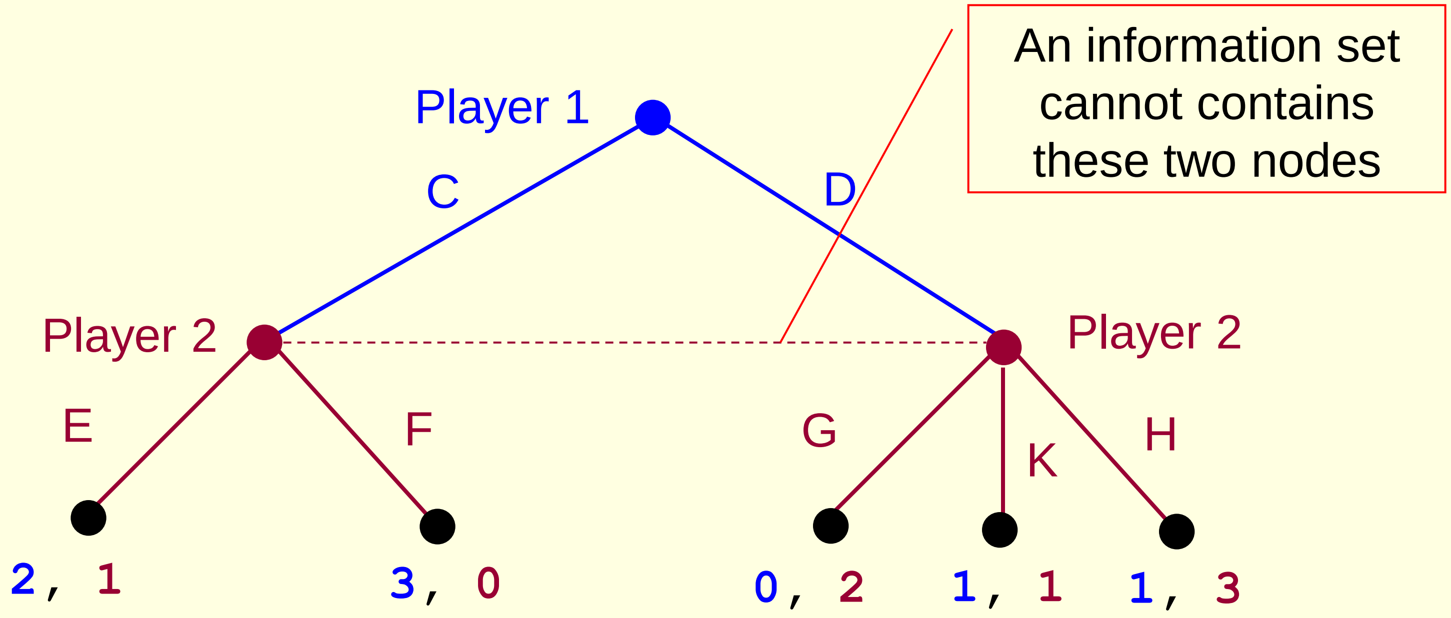
Information set: illustration

- All the nodes in an information set belong to the same player



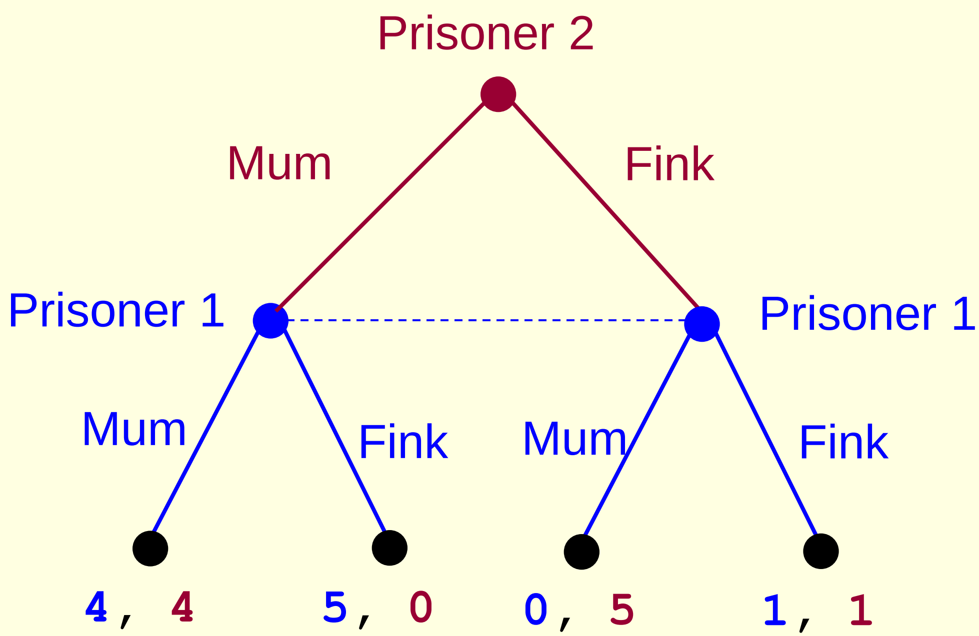
Information set: illustration

- The player must have the same set of feasible actions at each node in the information set.



Represent a static game as a game tree: illustration

- Prisoners' dilemma (another representation of the game in Figure 2.4.3 of Gibbons. The first number is the payoff for player 1, and the second number is the payoff for player 2)
- Static game as a game of imperfect information

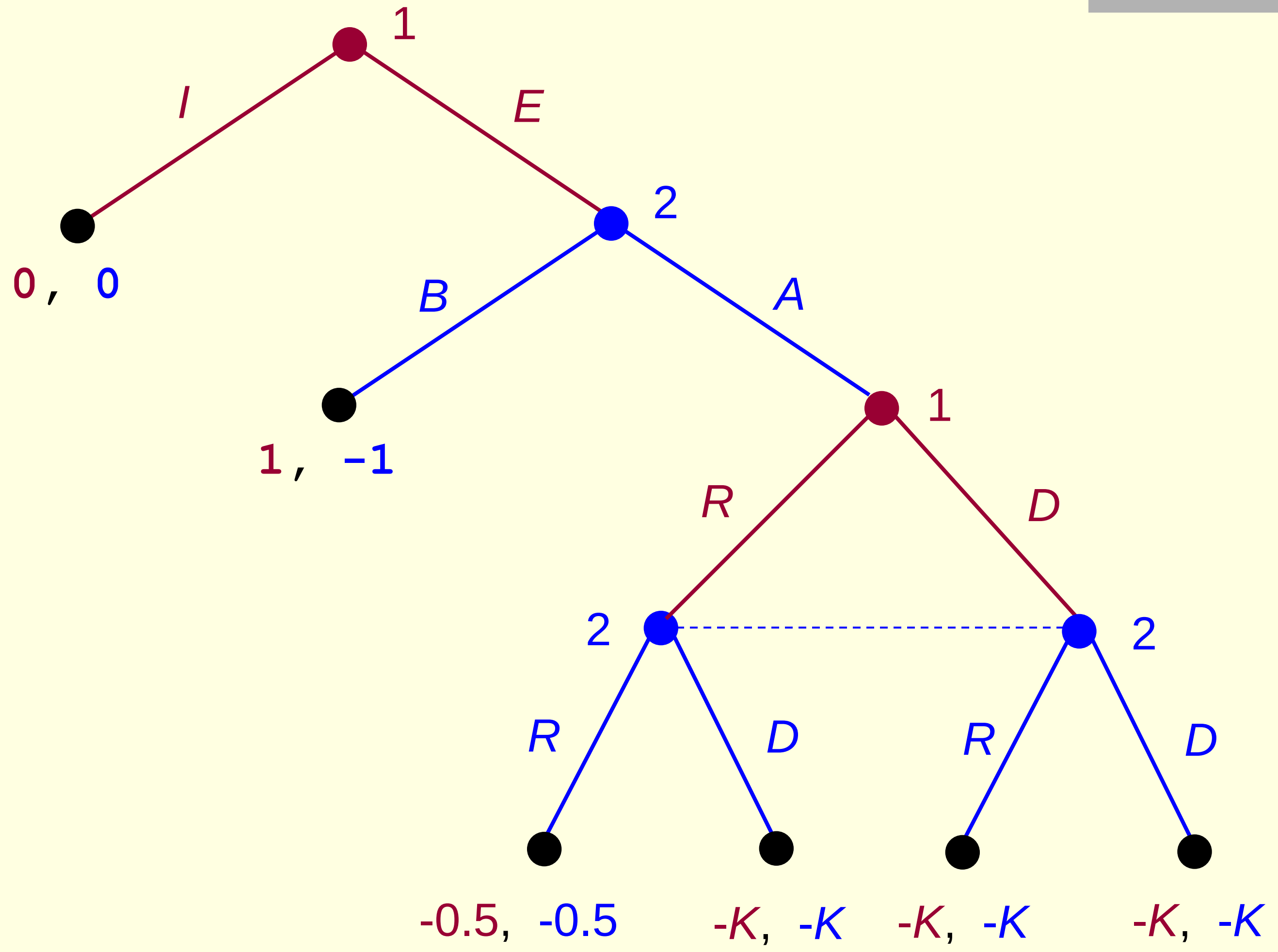


Example: mutually assured destruction

- Two superpowers, 1 and 2, have engaged in a provocative incident. The timing is as follows.
- The game starts with superpower 1's choice: either ignore the incident (*I*), resulting in the payoffs (0, 0), or to escalate the situation (*E*).
- Following escalation by superpower 1, superpower 2 can back down (*B*), causing it to lose face and result in the payoffs (1, -1), or it can choose to proceed to an atomic confrontation situation (*A*). Upon this choice, the two superpowers play the following simultaneous move game.
- They can either retreat (*R*) or choose to doomsday (*D*) in which the world is destroyed. If both choose to retreat then they suffer a small loss and payoffs are (-0.5, -0.5). If either chooses doomsday then the world is destroyed and payoffs are ($-K$, $-K$), where K is very large number.

Example: mutually assured destruction

(think of Cuba crisis)



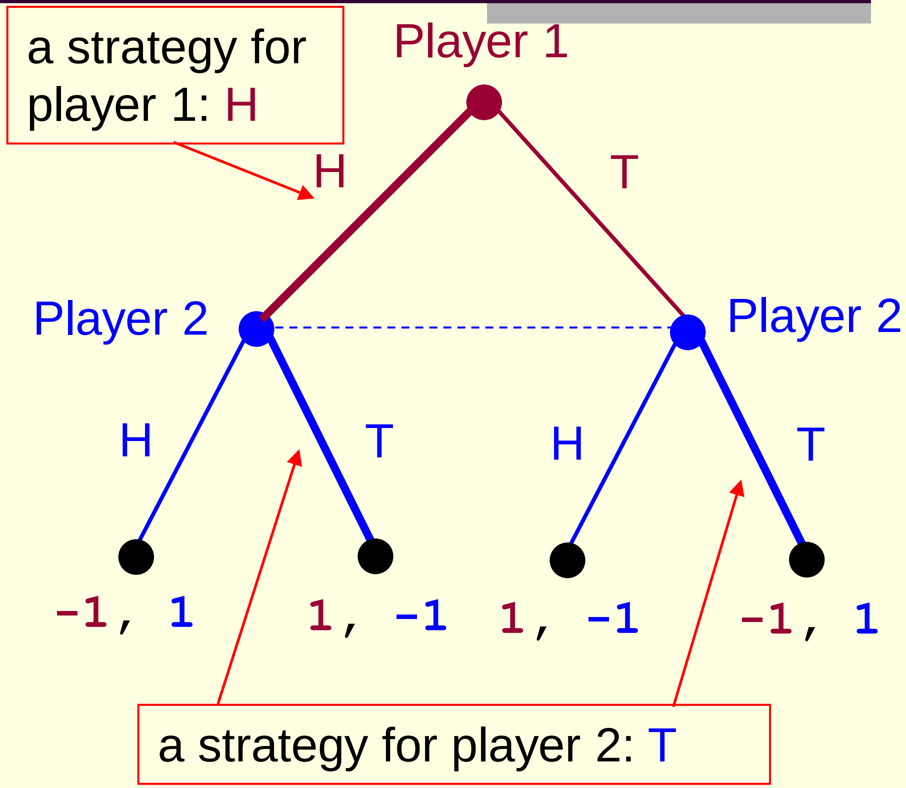
Perfect information and imperfect information

- A dynamic game *in which every information set contains exactly one node* is called a game of *perfect information*.

- A dynamic game *in which some information sets contain more than one node* is called a game of *imperfect information*.

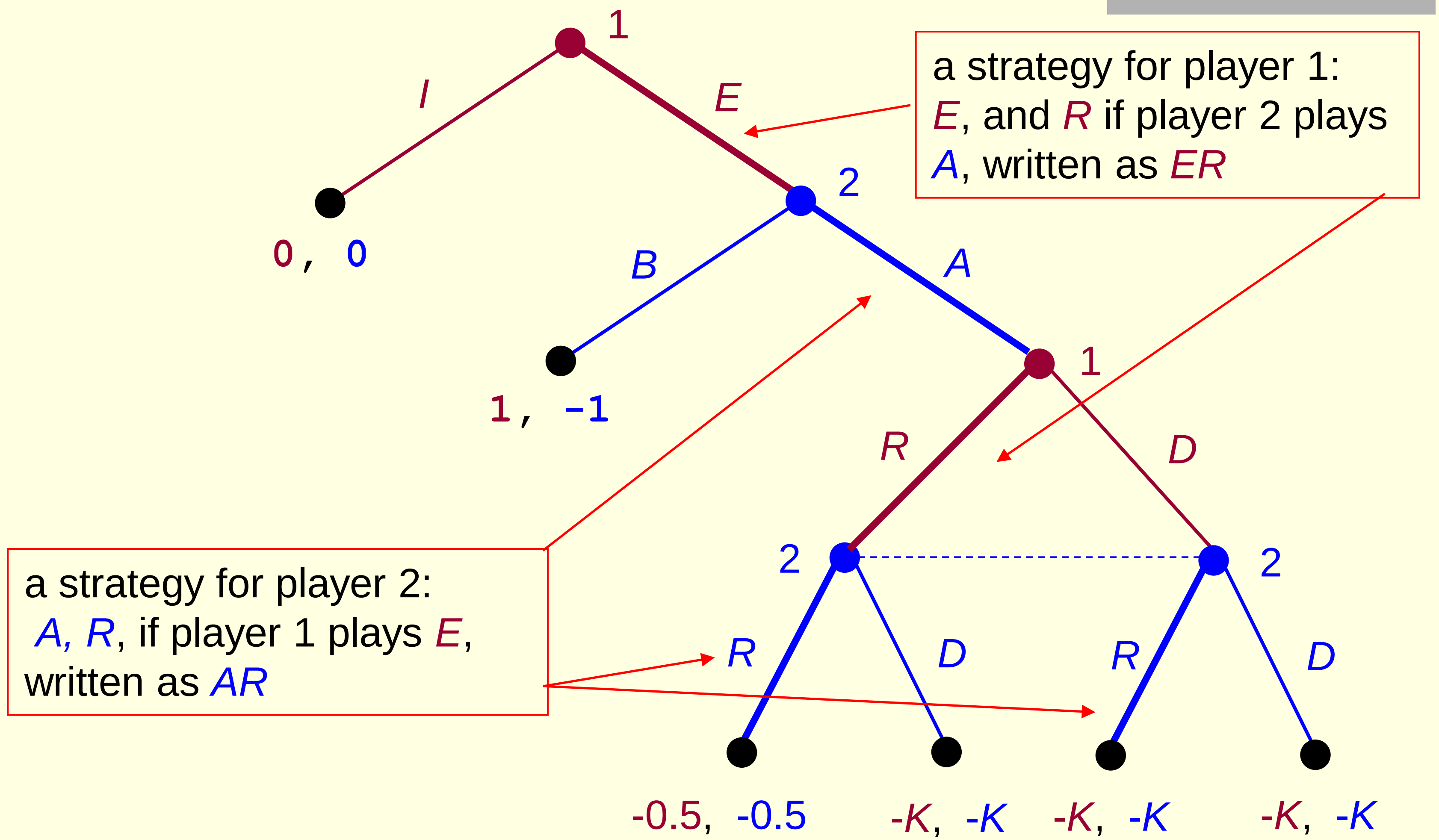
Strategy and payoff

- A strategy for a player is a complete plan of actions.
- It specifies a feasible action for the player in every contingency in which the player might be called on to act.
- It specifies what the player does at each of *her information sets*



Player 1’s payoff is 1 and player 2’s payoff is -1 if player 1 plays H and player 2 plays T

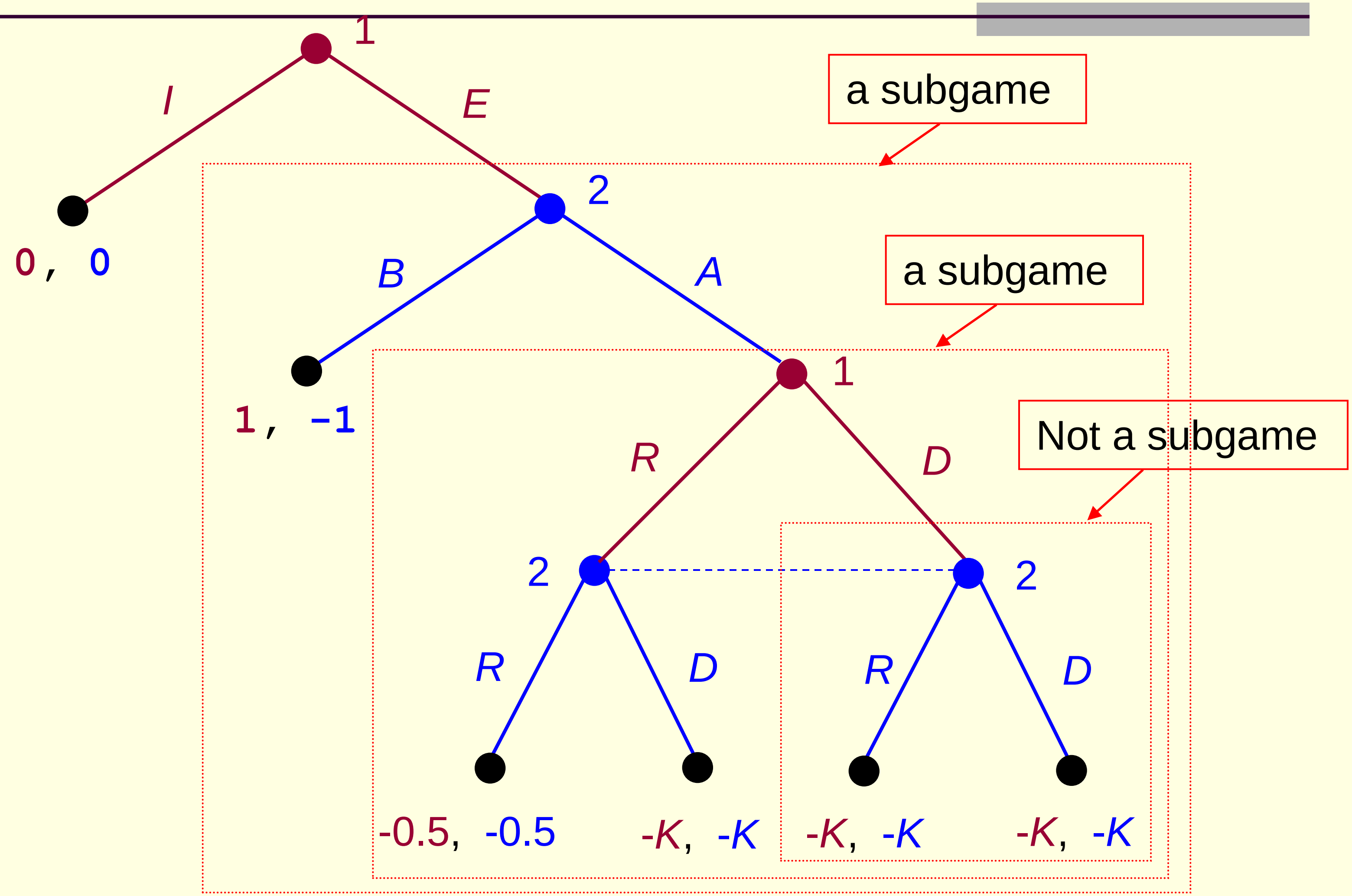
Strategy and payoff: illustration



Subgame

- A subgame of a dynamic game tree
  - begins at a singleton information set (an information set contains a single node), and
  - includes all the nodes and edges following the singleton information set, and
  - does not cut any information set; that is, if a node of an information set belongs to this subgame then all the nodes of the information set also belong to the subgame.

Subgame: illustration

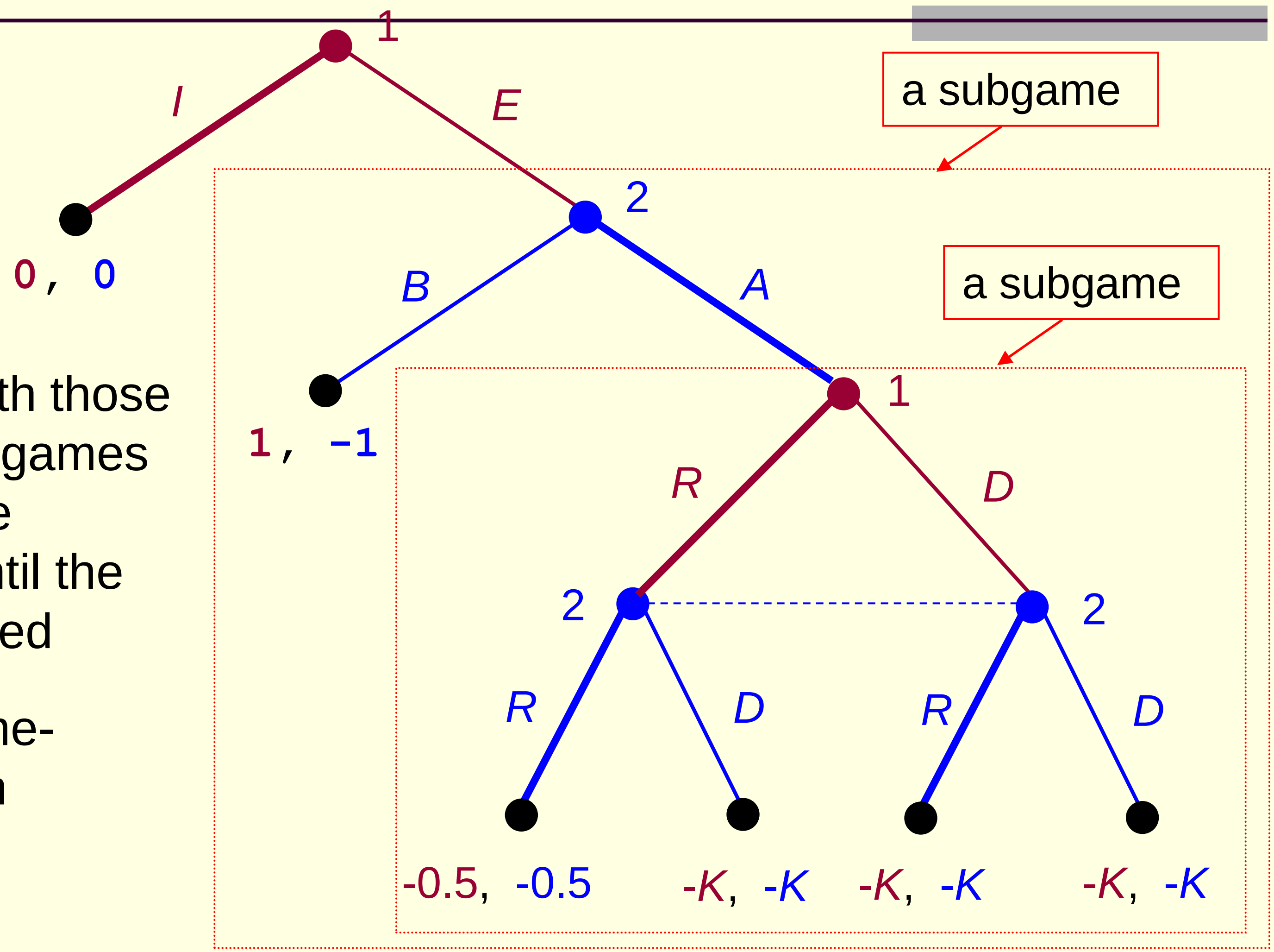


Subgame-perfect Nash equilibrium

- A Nash equilibrium of a dynamic game is subgame-perfect if the strategies of the Nash equilibrium constitute or induce a Nash equilibrium in every subgame of the game.

- Subgame-perfect Nash equilibrium is a Nash equilibrium.

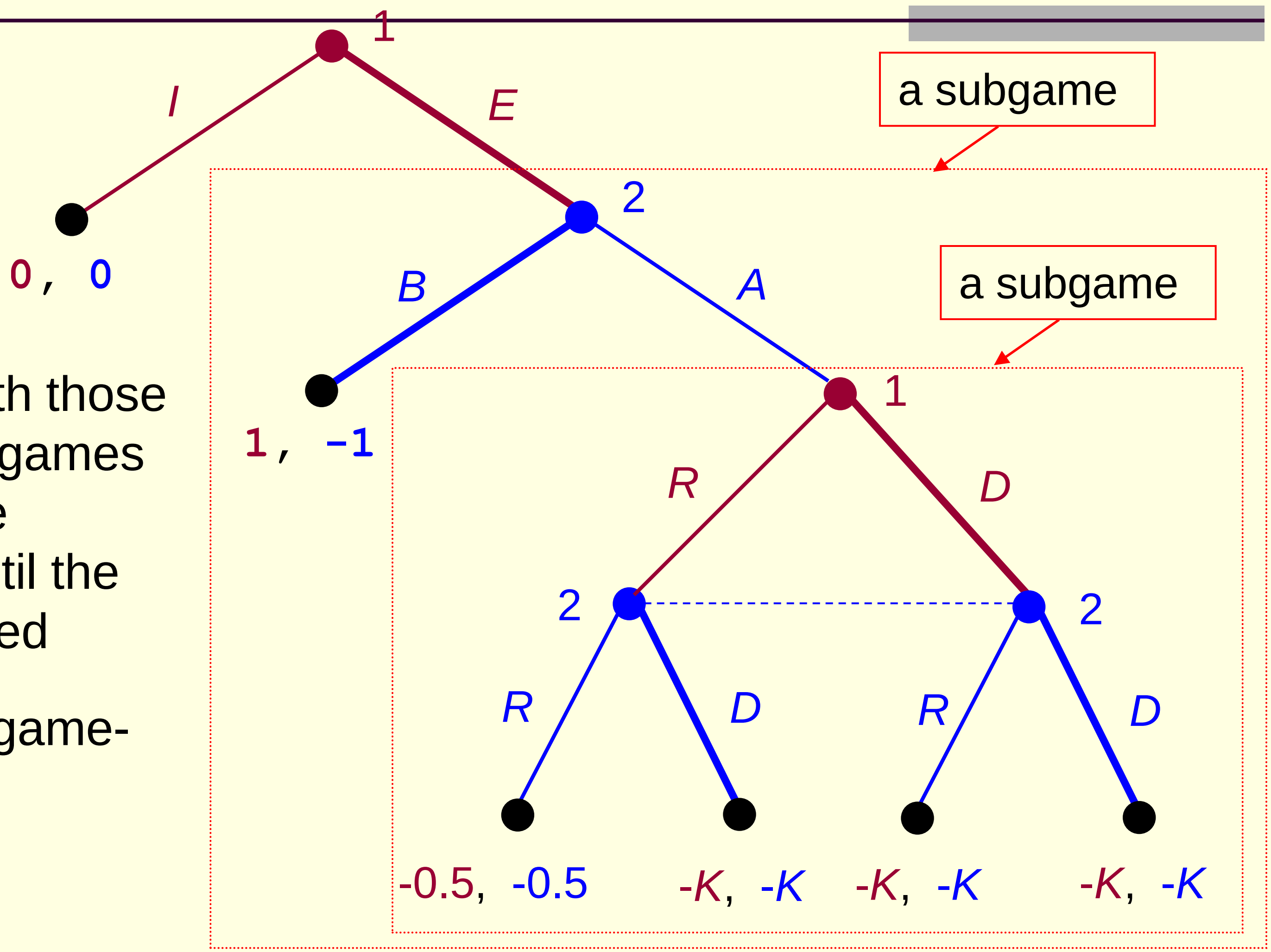
Find subgame perfect Nash equilibria: backward induction



- ➢Starting with those smallest subgames
- ➢Then move backward until the root is reached

One subgame-perfect Nash equilibrium
(*IR*, *AR*)

Find subgame perfect Nash equilibria: backward induction



- Starting with those smallest subgames
- Then move backward until the root is reached

Another subgame-perfect Nash equilibrium (*ED*, *BD*)

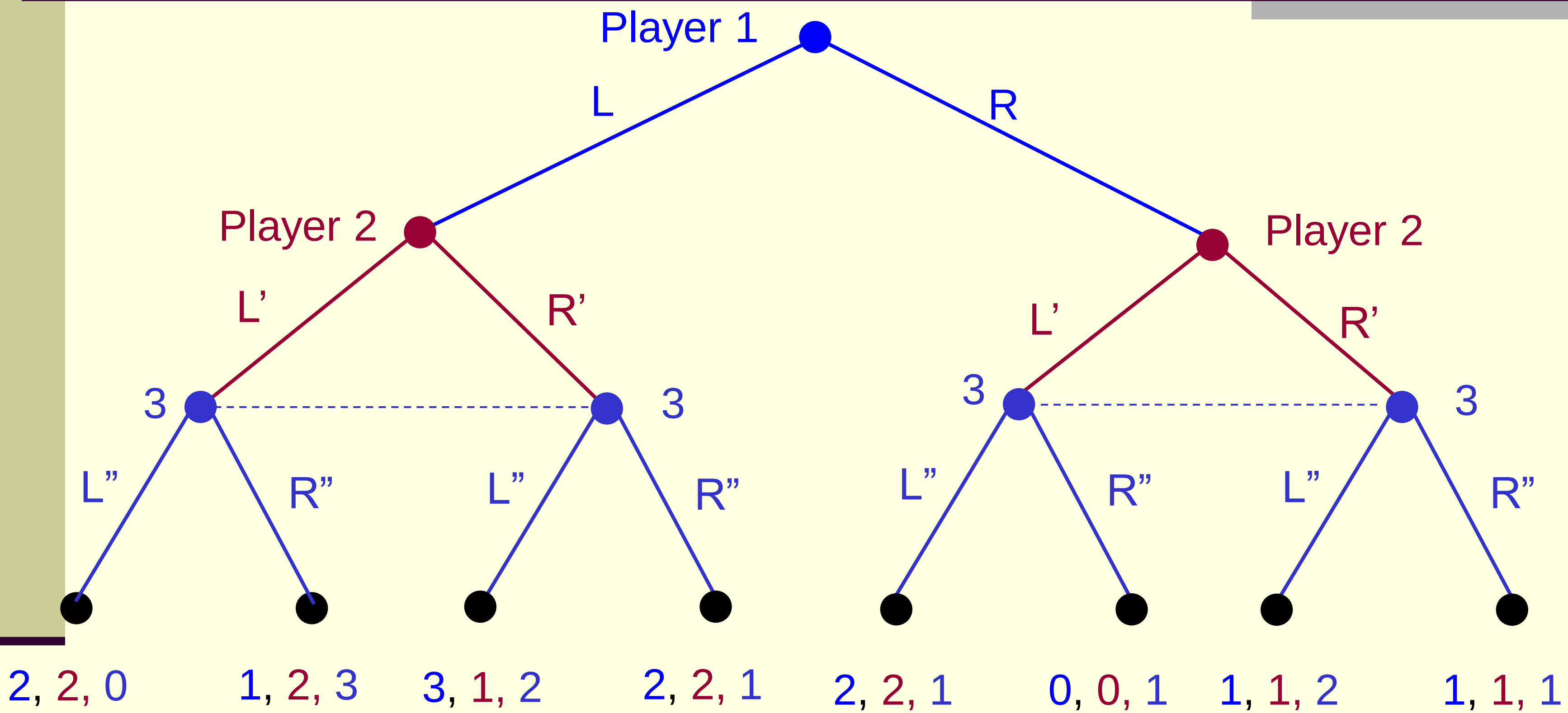
Dynamic games of complete information

- Perfect information
  - A player knows **Who** has made **What** choices when she has an opportunity to make a choice
- Imperfect information
  - A player may not know exactly **Who** has made **What** choices when she has an opportunity to make a choice.

Subgame-perfect Nash equilibrium

- A Nash equilibrium of a dynamic game is subgame-perfect if the strategies of the Nash equilibrium constitute or induce a Nash equilibrium in every subgame of the game.
- A subgame of a game tree
  - begins at a singleton information set (an information set containing a single node), and
  - includes all the nodes and edges following the singleton information set, and
  - does not cut any information set; that is, if a node of an information set belongs to this subgame then all the nodes of the information set also belong to the subgame.

Find subgame perfect Nash equilibria: backward induction



- What is the subgame perfect Nash equilibrium?

Bank runs (2.2.B of Gibbons)

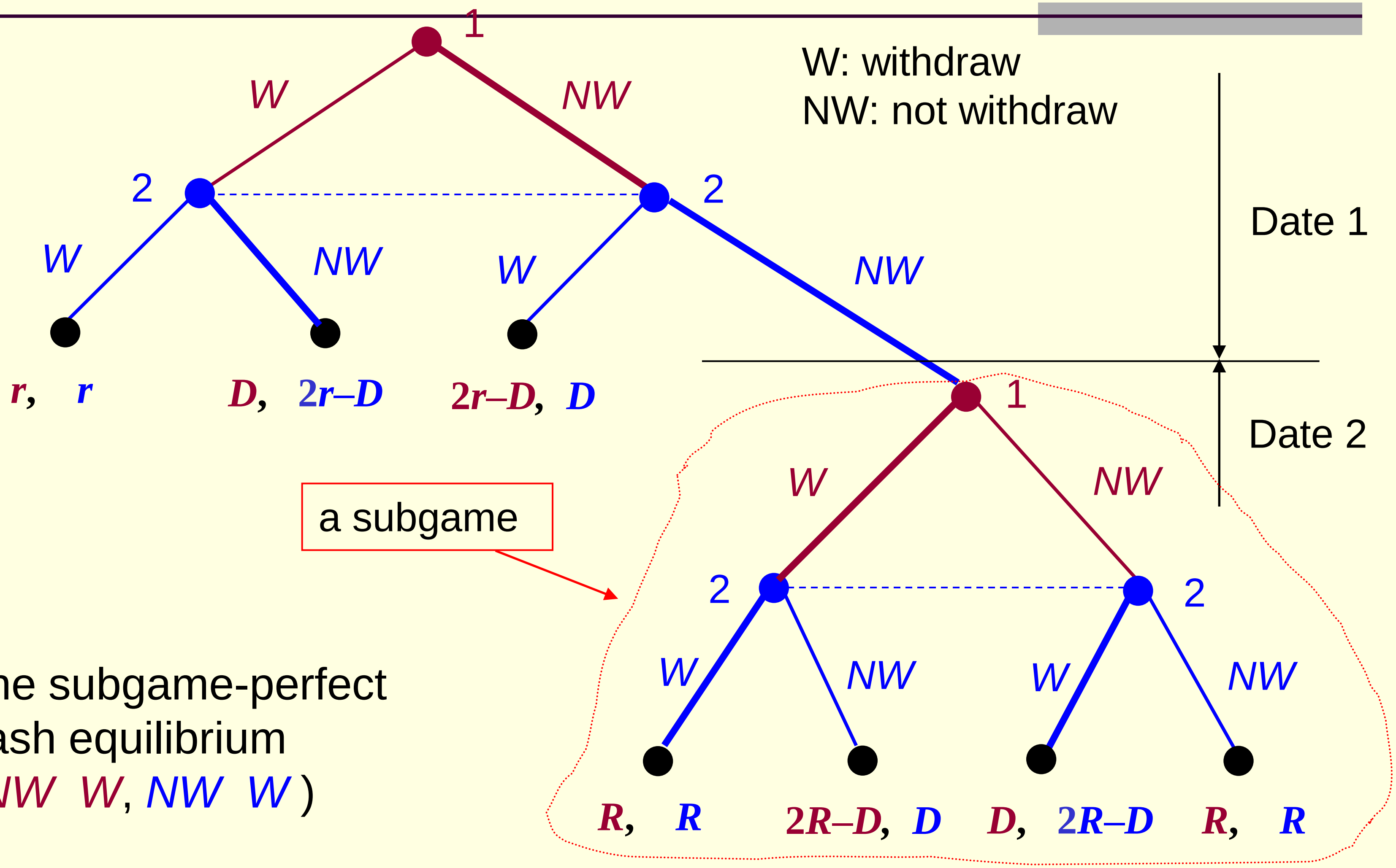
- Two investors, 1 and 2, have each deposited D with a bank.
- The bank has invested these deposits in a long-term project. If the bank liquidates its investment before the project matures, a total of $2r$ can be recovered, where $D > r > D/2$.
- If bank's investment matures, the project will pay out a total of $2R$, where $R>D$.
- Two dates at which the investors can make withdrawals from the bank.

Bank runs: timing of the game

- The timing of this game is as follows
- Date 1 (before the bank's investment matures)
  - Two investors play a simultaneous move game
  - If both make withdrawals then each receives r and the game ends
  - If only one makes a withdrawal then she receives D, the other receives $2r$-D, and the game ends
  - If neither makes a withdrawal then the project matures and the game continues to Date 2.

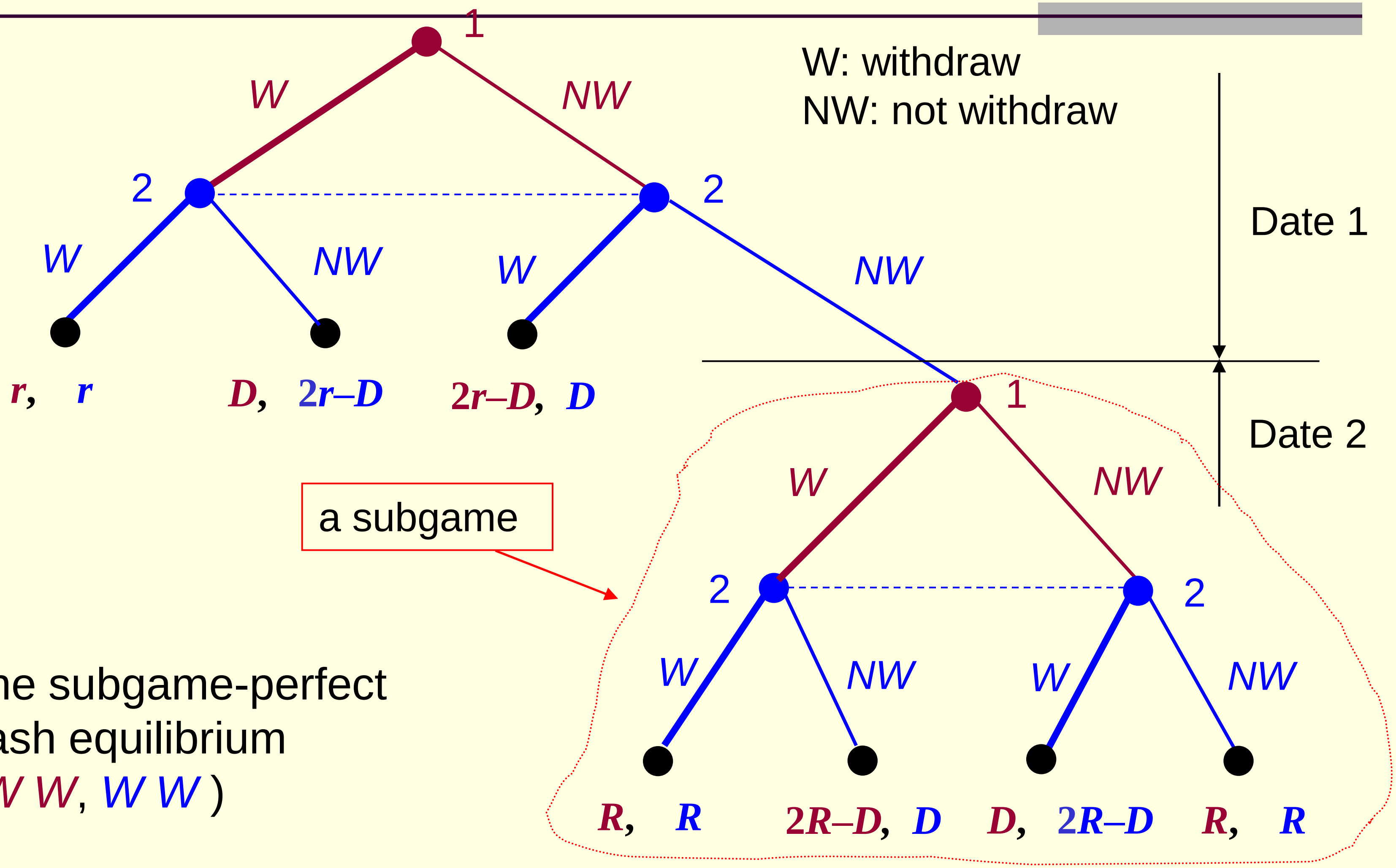
- Date 2 (after the bank's investment matures)
  - Two investors play a simultaneous move game
  - If both make withdrawals then each receives R and the game ends
  - If only one makes a withdrawal then she receives $2R$-D, the other receives D, and the game ends
  - If neither makes a withdrawal then the bank returns R to each investor and the game ends.

Bank runs: game tree



One subgame-perfect Nash equilibrium
(*NW W*, *NW W*)

Bank runs: game tree



One subgame-perfect Nash equilibrium
(*W W*, *W W*)

Tariffs and imperfect international competition (2.2.C of Gibbons)

- Two identical countries, 1 and 2, simultaneously choose their tariff rates, denoted t_1, t_2, respectively.
- Firm 1 from country 1 and firm 2 from country 2 produce a homogeneous product for both home consumption and export.
- After observing the tariff rates chosen by the two countries, firm 1 and 2 simultaneously chooses quantities for home consumption and for export, denoted by (h_1, e_1) and (h_2, e_2), respectively.
- Market price in two countries $P_i(Q_i)=a-Q_i$, for i=1, 2.
- $Q_1=h_1+e_2$, $Q_2=h_2+e_1$.
- Both firms have a constant marginal cost c.
- Each firm pays tariff on export to the other country.

Tariffs and imperfect international competition

Firm 1's payoff is its profit:

$$\pi_1(t_1, t_2, h_1, e_1, h_2, e_2) = [a - (h_1 + e_2)]h_1 + [a - (e_1 + h_2)]e_1 - c(h_1 + e_1) - t_2 e_1$$

Firm 2's payoff is its profit:

$$\pi_2(t_1, t_2, h_1, e_1, h_2, e_2) = [a - (h_2 + e_1)]h_2 + [a - (e_2 + h_1)]e_2 - c(h_2 + e_2) - t_1 e_2$$

Tariffs and imperfect international competition

Country 1's payoff is its total welfare: sum of the consumers' surplus enjoyed by the consumers of country 1, firm 1's profit and the tariff revenue

$$W_1(t_1, t_2, h_1, e_1, h_2, e_2) = \frac{1}{2}Q_1^2 + \pi_1(t_1, t_2, h_1, e_1, h_2, e_2) + t_1 e_2$$

where $Q_1 = h_1 + e_2$.

Country 2's payoff is its total welfare: sum of the consumers' surplus enjoyed by the consumers of country 2, firm 2's profit and the tariff revenue

$$W_2(t_1, t_2, h_1, e_1, h_2, e_2) = \frac{1}{2}Q_2^2 + \pi_2(t_1, t_2, h_1, e_1, h_2, e_2) + t_2 e_1$$

where $Q_2 = h_2 + e_1$.

Backward induction: subgame between the two firms

Here we will find the Nash equilibrium of the subgame between the two firms for any given pair of (t_1, t_2).

Firm 1 maximizes

$$\pi_1(t_1,t_2,h_1,e_1,h_2,e_2)=[a-(h_1+e_2)]h_1+[a-(e_1+h_2)]e_1-c(h_1+e_1)-t_2e_1$$

FOC:

$$a-2h_1-e_2-c=0 \quad\Leftrightarrow\quad h_1=\frac{1}{2}(a-e_2-c)$$

$$a-2e_1-h_2-c-t_2=0 \quad\Leftrightarrow\quad e_1=\frac{1}{2}(a-h_2-c-t_2)$$

Firm 2 maximizes

$$\pi_2(t_1,t_2,h_1,e_1,h_2,e_2)=[a-(h_2+e_1)]h_2+[a-(e_2+h_1)]e_2-c(h_2+e_2)-t_1e_2$$

FOC:

$$a-2h_2-e_1-c=0 \quad\Leftrightarrow\quad h_2=\frac{1}{2}(a-e_1-c)$$

$$a-2e_2-h_1-c-t_1=0 \quad\Leftrightarrow\quad e_2=\frac{1}{2}(a-h_1-c-t_1)$$

Backward induction: subgame between the two firms

Here we will find the Nash equilibrium of the subgame between the two firms for any given pair of (t_1, t_2).

Given (t_1, t_2), a Nash equilibrium $((h_1^*, e_1^*), (h_2^*, e_2^*))$ of the subgame should satisfy these equations.

$$h_1 = \frac{1}{2}(a - e_2 - c) \qquad h_2 = \frac{1}{2}(a - e_1 - c)$$

$$e_1 = \frac{1}{2}(a - h_2 - c - t_2) \qquad e_2 = \frac{1}{2}(a - h_1 - c - t_1)$$

Solving these equations gives us

$$h_1^* = \frac{1}{3}(a - c + t_1) \qquad e_1^* = \frac{1}{3}(a - c - 2t_2)$$

$$h_2^* = \frac{1}{3}(a - c + t_2) \qquad e_2^* = \frac{1}{3}(a - c - 2t_1)$$

Backward induction: whole game

Both countries know that two firms' best response for any pair (t_1, t_2)

Country 1 maximizes $(Q_1 = h_1 + e_2)$

$$W_1(t_1,t_2,h_1,e_1,h_2,e_2) = \frac{1}{2}Q_1^2 + \pi_1(t_1,t_2,h_1,e_1,h_2,e_2) + t_1 e_2$$

Plugging what we got into country 1's objective function

$$\frac{1}{18}(2(a-c)-t_1)^2 + (a-\frac{2}{3}(a-c)+\frac{1}{3}t_1)\times\frac{1}{3}(a-c+t_1) + (a-\frac{2}{3}(a-c)-\frac{1}{3}t_2)\times\frac{1}{3}(a-c-2t_2)$$

$$-c(\frac{2}{3}(a-c)+\frac{1}{3}(t_1-2t_2)) - t_2\times\frac{1}{3}(a-c-2t_2) + t_1\times\frac{1}{3}(a-c-2t_1)$$

FOC:

$$t_1 = \frac{1}{3}(a-c)$$

By symmetry, we also get $$t_2 = \frac{1}{3}(a-c)$$

Tariffs and imperfect international competition

The subgame-perfect Nash equilibrium

$$\left(t_1^* = \frac{1}{3}(a-c),\ t_2^* = \frac{1}{3}(a-c),\ \begin{pmatrix} h_1 = \frac{1}{3}(a-c+t_1) \\ e_1 = \frac{1}{3}(a-c-2t_2) \end{pmatrix}, \begin{pmatrix} h_2 = \frac{1}{3}(a-c+t_2) \\ e_2 = \frac{1}{3}(a-c-2t_1) \end{pmatrix} \right)$$

The subgame-perfect outcome

$$\left(t_1^* = \frac{1}{3}(a-c),\ t_2^* = \frac{1}{3}(a-c),\ \begin{pmatrix} h_1^* = \frac{4}{9}(a-c) \\ e_1^* = \frac{1}{9}(a-c) \end{pmatrix}, \begin{pmatrix} h_2^* = \frac{4}{9}(a-c) \\ e_2^* = \frac{1}{9}(a-c) \end{pmatrix} \right)$$