Topology review

Douwe Hoekstra

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In this document we review some useful facts from topology.

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1 Topological properties

We start our review by recalling some topological properties one can use to study topological spaces.

Definition 1.1 (Hausdorff)**.** A topological space $(X, \mathcal{T})$ is *Hausdorff* if for every two distinct points $x, y \in X, x \neq y$ there exist open sets $U, V \in \mathcal{T}$ such that $x \in U$ and $y \in V$ and $U \cap V = \emptyset$. That is, U is an open neighbourhood of x, V is an open neighbourhood of V and these neighbourhoods are disjoint.

Definition 1.2 ((Dis)connectedness)**.** A topological space $(X, \mathcal{T})$ is *disconnected* if there exist disjoint non-empty open sets $U, V \in \mathcal{T}$ such that $X = U \cup V$.

A topological space is *connected* if it is not disconnected.

Remark 1.3. If we want to show that a topological space is connected, one has to show the following: Given two open sets $U, V \in \mathcal{T}$ such that $X = U \cup V$ and $U \cap V = \emptyset$ it follows that either $U = \emptyset$ or $V = \emptyset$.

Definition 1.4 (Path-connectedness)**.** A topological space $(X, \mathcal{T})$ is *path-connected* if for any two points $x, y \in X$ there exists a continuous path $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Lemma 1.5. *A path-connected topological space $(X, \mathcal{T})$ is connected.*

Proof. Suppose that X is disconnected. Then there exist non-empty disjoint open sets $U, V \in \mathcal{T}$ such that $X = U \cup V$. Pick $x \in U$ and $y \in V$. Then by path-connectedness there is a path $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Then by continuity $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$ are two disjoint open sets covering $[0,1]$. Moreover, $0 \in \gamma^{-1}(U)$ and $1 \in \gamma^{-1}(V)$. So the unit interval is disconnected. This is a contradiction, hence X is connected. □

Definition 1.6 (Compactness)**.** A topological space $(X, \mathcal{T})$ is *compact* if every open cover admits a finite subcover, i.e. given any open cover $\mathcal{U} = (U_i)_{i \in I}$ there is a finite set of indices $i_1, \ldots, i_N$ such that $\bigcup_{k=1}^{N} U_{i_k} = X$.

Note that we say that a subset $C \subset X$ has any of the properties above if it has this property as a topological space with the induced[1] topology.

Lemma 1.7. *Let $f\colon X \to Y$ be a continuous map.*

[1] subspace

(i) If $C \subset X$ is connected, then $f(C)$ is connected.
(ii) If $C \subset X$ is path-connected, then $f(C)$ is path-connected.
(iii) If $K \subset X$ is compact, then $f(K)$ is compact.

Proof. (i) Suppose that $f(C)$ is disconnected, then there are open subsets U, V of Y such that $U, V \neq \emptyset$, $U \cap V \cap f(C) = \emptyset$ and $f(C) \subset U \cup V$. Then $f^{-1}(U), f^{-1}(V)$ is a cover of C by two non-empty disjoint open sets of X, so C is disconnected. So if $f(C)$ is disconnected, then C is disconnected. The contrapositive of this implication is the desired statement.

(ii) Suppose that C is path-connected. Let $x, y \in f(C)$. Then there are points $x_0, y_0 \in C$ such that $f(x_0) = x$ and $f(y_0) = y$. Since C is path-connected, there is a path $\gamma\colon [0,1] \to C$ from x_0 to y_0. Then $f \circ \gamma$ is a continuous path in $f(C)$ and $f(\gamma(0)) = f(x_0) = x$ and $f(\gamma(1)) = f(y_0) = y$. So $f(C)$ is path-connected.

(iii) Let $\mathcal{U} = (U_i)_{i\in I}$ be an open cover of $f(K)$. Then $(f^{-1}(U_i))_{i\in I}$ is an open cover of K which is compact. So there is a finite subcover $(f^{-1}(U_{i_k}))_{k=1}^N$. So then we have that $(U_{i_k})_{k=1}^N$ is a finite subcover of $f(K)$. So $f(K)$ is compact. □

Lemma 1.8. *Let $(X, \mathcal{T})$ be a topological space.*
(i) If X is compact and $C \subset X$ is closed, then C is compact.
(ii) If X is Hausdorff and $C \subset X$ is compact, then C is closed.

Proof. (i) Suppose that X is compact and $C \subset X$ is closed. Let $\mathcal{U} = (U_i)_{i\in I}$ be an open cover of C. Since C is closed, $X \setminus C$ is open and so $(X \setminus C) \cup \mathcal{U}$ is an open cover of X. Since X is compact, there is a finite subcover which in particular covers C. So C is compact.

(ii) Suppose that X is Hausdorff and that $C \subset X$ is compact. We will show that $X \setminus C$ is open. Let $x \in X \setminus C$. By Hausdorffness, for every $y \in C$ there exist disjoint open neighbourhoods U_y of x and V_y of y. Then the $(V_y)_{y\in C}$ form an open cover of C. So by compactness, there is a finite subcover $(V_{y_i})_{i=1}^N$. Then $U = \bigcap_{i=1}^N U_{y_i}$ is an open neighbourhood of x, and it is disjoint from $\bigcup_{i=1}^N V_{y_i}$. In particular, $U \cap C = \emptyset$. So $U \subset X \setminus C$. So $X \setminus C$ is open. □

Recall that a map is called *closed* if $f(C)$ is closed for every closed subset C.

Proposition 1.9. *Let $f\colon X \to Y$ be an injective continuous map. Suppose that X is compact and that Y is Hausdorff. Then f is an embedding.*

Proof. By the lemmas above any closed $C \subset X$ is compact and so $f(C)$ is compact and therefore closed. Consider the inverse $f^{-1}\colon f(X) \to X$. Then for a closed $C \subset X$ we have

$$(f^{-1})^{-1}(C) = f(C)$$

so by the above this is closed. Therefore, f^{-1} is continuous and so f is an embedding. □

Corollary 1.10. *Let $f\colon X \to Y$ be a bijective continuous map from a compact space X to a Hausdorff space Y. Then f is a homeomorphism.*

Proof. A bijective map is injective, so by the previous proposition f is an embedding. Since f is surjective this means that f is a homeomorphism between X and $f(X) = Y$. □

Definition 1.11 (Locally (path-)connected)**.** A topological space X is called *locally (path-)connected* if each point $x \in X$ has a neighbourhood U that is (path-)connected.

Definition 1.12 (Locally compact)**.** A topological space X is called *locally compact* if each point $x \in X$ has a neighbourhood K that is compact.

2 Quotient topology

The quotient topology formalizes the idea of gluing pieces of a topological space together. If you remember only one thing from this section let it be the following proposition.

Proposition 2.1 (Universal property of quotients)**.** *Let $\pi\colon X \to Y$ be a surjective function of sets from a topological space into a set Y. There is a unique topology on Y such that π is continuous, and the following* ***universal property*** *holds: Given a function $g\colon Y \to Z$ into any topological space Z, g is continuous if and only if $g \circ \pi\colon X \to Z$ is continuous.*

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "g\circ\pi"] & \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

Remark 2.2. There is a white lie going here. The actual universal property for the quotient topology is corollary 2.9. However, as the rest of this section will show, this corollary follows from this proposition.

Let us try to discover what this topology should be. First, we want $\pi\colon X \to Y$ to be continuous. This gives us the implication that if $V \subset Y$ is open, then $\pi^{-1}(V)$ is open in Y.

Secondly, we want the universal property to hold, so let us think about what this implies for the topology on Y. It tells us that if we have a function $g\colon Y \to Z$ such that $g \circ \pi\colon X \to Z$ is continuous, then g is continuous. So given an open $V \subset Z$ this means that $(g \circ \pi)^{-1}(V) = \pi^{-1}(g^{-1}(V))$ open implies that $g^{-1}(V)$ is open. So we want the implication $\pi^{-1}(V)$ implies that $V \subset Y$ is open for any subset $V \subset Y$.[2]

Definition 2.3 (Quotient topology)**.** Let X be a topological space and let $\pi\colon X \to Y$ be a surjective function of sets. Then the *quotient topology* on Y induced by π is given by

$$\mathcal{T}_\pi = \left\{V \subset Y : \pi^{-1}(V) \subset X \text{ is open}\right\}$$

So a subset of $V \subset Y$ is open if and only if $\pi^{-1}(V)$ is open.

Lemma 2.4. *The quotient topology induced by $\pi\colon X \to Y$ is a topology.*

Proof. We have to show that $\mathcal{T}_\pi$ satisfies the axioms of a topology. We have that $\pi^{-1}(Y) = X$ and X is open in X. So $Y \in \mathcal{T}_\pi$. Similarly, $\pi^{-1}(\emptyset) = \emptyset$ which is open in X. So $\emptyset \in \mathcal{T}_\pi$.

Next, let $U, V \in \mathcal{T}_\pi$. Then we have

$$\pi^{-1}(U \cap V) = \pi^{-1}(U) \cap \pi^{-1}(V),$$

and so $\pi^{-1}(U \cap V)$ is open as the intersection of two subsets. Thus, $U \cap V \in \mathcal{T}_\pi$.

Let $(U_i)_{i\in I} \subset \mathcal{T}_\pi$ be a collection. Then

$$\pi^{-1}\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} \pi^{-1}(U_i),$$

which is open as the union of open sets. So $\bigcup_{i\in I} U_i \in \mathcal{T}_\pi$.

So $\mathcal{T}_\pi$ is a topology on Y. □

It turns out that this is precisely the unique topology that is meant in proposition 2.1.

[2] Or a priori only for any subset V that can appear as the preimage of a function $g\colon Y \to Z$ such that $g \circ \pi$ is continuous. (Can any open subset appear as the preimage of a continuous map? Yes!)

Proof of proposition 2.1. We will first show that $\mathcal{T}_\pi$ has the desired properties. We start with showing that π is continuous. So let $V \subset Y$ be open. Then by definition $\pi^{-1}(V)$ is open. So π is indeed continuous.

Next, we show that the universal property is satisfied. Let $g\colon Y \to Z$ be a function of sets into a topological space Z. The only if direction of the universal property is clear, since the composition of two continuous maps is again continuous. For the if direction, suppose that $g\circ\pi$ is continuous. Let $U \subset Z$ be open. Then we have by assumption that $(g\circ\pi)^{-1}(U) = \pi^{-1}(g^{-1}(U))$ is open. So by the definition of the quotient topology $g^{-1}(U)$ is open in Y. Thus, g is continuous.

Next, we need to show that this is the unique topology with this property. Suppose that $\mathcal{T}'$ is another topology with this property. Then we have the following commutative diagrams

$$\begin{tikzcd} X \arrow[d, "\pi"] \arrow[dr, "\pi"] & \\ (Y,\mathcal{T}_\pi) \arrow[r, "\mathrm{id}_Y"] & (Y,\mathcal{T}') \end{tikzcd} \qquad \begin{tikzcd} X \arrow[d, "\pi"] \arrow[dr, "\pi"] & \\ (Y,\mathcal{T}') \arrow[r, "\mathrm{id}_Y"] & (Y,\mathcal{T}_\pi) \end{tikzcd}$$

So using the universal property and the fact that π is continuous with respect to both topologies, we conclude that $\mathrm{id}_Y\colon (Y,\mathcal{T}_\pi) \to (Y,\mathcal{T}')$ is a homeomorphism. But this implies that $\mathcal{T}_\pi = \mathcal{T}'$. $\square$

Definition 2.5 (Quotient map)**.** Let $\pi\colon X \to Y$ be a surjective continuous map. We say that π is a *quotient map* if it satisfies the following condition: $V \subset Y$ is open if and only if $\pi^{-1}(V)$ is open.

Remark 2.6. The fact $\pi\colon X \to Y$ is a quotient map says precisely that the topology on Y is the quotient topology. In particular, π has the universal property from proposition 2.1.

In practice, quotient topologies arise when one identifies points of a topological space. This is formalized by defining an equivalence relation R on a topological space X and then modding out X by this relation R. Given an equivalence relation R on a set X, one obtains a surjective map $\pi\colon X \to X/R$ sending each element of X to its equivalence class under R (here X/R denotes the partition of X into its equivalence classes under R). Then we can endow X/R with the quotient topology to turn this into a topological space. The following proposition shows that this is the unique quotient space up to homeomorphism, that identifies points in X according to the equivalence relation R.

Proposition 2.7. *Let R be an equivalence relation on a topological space X and let $\pi\colon X \to Y$ be a quotient map whose fibres $\pi^{-1}(y)$ are the equivalence classes of R. Then Y is homeomorphic to X/R with the quotient topology.*

Proof. Let $q\colon X \to X/R$ denote the quotient map. Since π is constant on the equivalence classes of R we may define $\bar\pi\colon X/R \to Y$ by sending $[x] \mapsto \pi(x)$. This is well-defined because of the constancy assumption. We then have the following diagram

$$\begin{tikzcd} X \arrow[d, "q"] \arrow[dr, "\pi"] & \\ X/R \arrow[r, "\bar\pi"] & Y \end{tikzcd}$$

so since q is a quotient map, $\bar\pi$ is continuous.

Similarly, let us define $\bar q\colon Y \to X/R$ by sending $y \mapsto \pi^{-1}(y)$ which is well-defined by assumption. This gives another diagram

$$\begin{tikzcd} X \arrow[d, "\pi"] \arrow[dr, "q"] & \\ Y \arrow[r, "\bar q"] & X/R \end{tikzcd}$$

from which we conclude that $\bar{q}$ is also continuous.

Finally, we have

$$\bar{\pi} \circ \bar{q}(y) = \bar{\pi}(\pi^{-1}(y)) = y \quad \text{and} \quad \bar{q} \circ \bar{\pi}([x]) = \bar{q}(\pi(x)) = \pi^{-1}(x) = [x].$$

So $\bar{\pi}$ and $\bar{q}$ are inverses, and so $\bar{\pi}$ is a homeomorphism. $\square$

Remark 2.8. In fact, every quotient topology arises in this way: Given a quotient map $\pi\colon X \to Y$, the statement above implies that $Y \cong X/R_\pi$ where R_π is the equivalence relation $xR_\pi y$ if and only if $\pi(x) = \pi(y)$.

Corollary 2.9 (Descent to the quotient)**.** *Let $\pi\colon X \to Y$ be a quotient map, and suppose that $f\colon X \to Z$ is a continuous map which is constant on the fibres of π. Then there is a unique continuous $\bar{f}\colon Y \to Z$ such that $\bar{f} \circ \pi = f$. We say that f "descends to the quotient".*

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "f"] & \\ Y \arrow[r, "\bar{f}"] & Z \end{tikzcd}$$

Proof. Using the remark above, we see that $Y \cong X/R_\pi$, so it suffices to show that there is a unique function $\bar{f}\colon X/R_\pi \to Z$ such that $\bar{f} \circ q = f$, where $q\colon X \to X/R_\pi$ is the quotient map.

Define $\bar{f}\colon X/R_\pi \to Z$ by $[x] \mapsto f(x)$. This is well-defined, since f is constant on the fibres of π. Moreover, we have $\bar{f}(q(x)) = \bar{f}([x]) = f(x)$, for all $x \in X$, so $\bar{f} \circ q = f$. Then using the universal property for the quotient, we conclude that $\bar{f}$ is continuous.

Assume that $g\colon X/R_\pi \to Z$ is another such map. Then we have $g \circ q = f = \bar{f} \circ q$, and q is surjective, so $g = \bar{f}$, proving the uniqueness. $\square$

3 Product topology

Another useful construction in topology which satisfies a universal property is the product of two spaces. Recall the definition of the product topology.

Definition 3.1. Let X, Y be topological spaces. The *product topology* on $X \times Y$ is defined as the topology generated by the basis

$$\mathcal{B} = \{U \times V : U \subset X, V \subset Y \text{ open}\}.$$

Just like the quotient topology, the product topology is precisely the topology that makes a universal property (for the product) hold true.

Proposition 3.2 (Universal property of products)**.** *Let X, Y be topological spaces. Then the product $X \times Y$ endowed with the product topology together with the continuous projection maps is the unique space (up to unique homeomorphism) satisfying the following **universal property:** Given two continuous maps $f\colon Z \to X$ and $g\colon Z \to Y$ there is a unique continuous map $h\colon Z \to X \times Y$ such that* $\mathrm{pr}_X \circ h = f$ *and* $\mathrm{pr}_Y \circ h = g$.

$$\begin{tikzcd} & & X \\ Z \arrow[rru, bend left, "f"] \arrow[rrd, bend right, "g"'] \arrow[r, dashed, "\exists! h"] & X \times Y \arrow[ru, "\mathrm{pr}_X"] \arrow[rd, "\mathrm{pr}_Y"'] & \\ & & Y \end{tikzcd}$$

Proof. We will first show that $X \times Y$ with the product topology satisfies the universal property. So let $f\colon Z \to X$ and $g\colon Z \to Y$ be continuous maps. We define $h\colon Z \to X \times Y$ by $z \mapsto (f(z), g(z))$. This function satisfies $\mathrm{pr}_X \circ h = f$ and $\mathrm{pr}_Y \circ h = g$, so it remains to show that h is continuous. For this it suffices to show that the preimage of basis elements are open in Z. So let $U \subset X$ and $V \subset Y$ and consider

$$h^{-1}(U \times V) = \{z \in Z : f(z) \in U \text{ and } g(z) \in V\} = f^{-1}(U) \cap g^{-1}(V),$$

which is open by the continuity of f and g. So $X \times Y$ satisfies the universal property.

Suppose that P is a topological space together with continuous maps $\pi_X\colon P \to X$ and $\pi_Y\colon P \to Y$ that satisfies the universal property as well. Then we have the following diagram

$$\begin{tikzcd}
 & & X \\
P \arrow[r, dashed, "\exists!\phi"] \arrow[urr, bend left, "\pi_X"] \arrow[drr, bend right, "\pi_Y"'] & X \times Y \arrow[ur, "\mathrm{pr}_X"] \arrow[dr, "\mathrm{pr}_Y"] & \\
 & & Y
\end{tikzcd}$$

so by the universal property of the product topology $X \times Y$ we have a continuous map $\phi\colon P \to X \times Y$. Similarly, we have

$$\begin{tikzcd}
 & & X \\
X \times Y \arrow[r, dashed, "\exists!\psi"] \arrow[urr, bend left, "\mathrm{pr}_X"] \arrow[drr, bend right, "\mathrm{pr}_Y"'] & P \arrow[ur, "\pi_X"] \arrow[dr, "\pi_Y"] & \\
 & & Y
\end{tikzcd}$$

so by the universal property of P we have a continuous map $\psi\colon X \times Y \to P$. Then $\psi \circ \phi$ is a map from P to itself, and $\pi_X \circ \psi \circ \phi = \mathrm{pr}_X \circ \phi = \pi_X$ by construction. Similarly, $\pi_Y \circ \psi \circ \phi = \pi_Y$. Then by the uniqueness assertion from the universal property, it follows that $\psi \circ \phi = \mathrm{id}_P$. Applying the same reasoning, we obtain $\phi \circ \psi = \mathrm{id}_{X \times Y}$. Thus, P is homeomorphic to $X \times Y$. □

Remark 3.3. This proposition essentially tells us that giving a map into a product is the same thing as giving a map into each of the two factors of the product.

Proposition 3.4 (Product of compact spaces, Tychonoff's theorem)**.** *Let X and Y be compact topological spaces. Then $X \times Y$ endowed with the product topology is compact.*

For the proof of this proposition, we need the following lemma.

Lemma 3.5 (Tube lemma)**.** *Let X be a topological space and let Y be a compact topological space. Let $W \subset X \times Y$ be open (in the product topology) and assume that $\{x\} \times Y \subset W$ for some $x \in X$. Then there is an open neighbourhood $U \subset X$ of x such that $U \times Y \subset W$.*

Proof. Let $y \in Y$. Then $(x, y) \in W$, so by definition of the product topology, there exist open subsets $U_y \subset X$ and $V_y \subset Y$ such that $x \in U_y$ and $y \in V_y$ and $U_y \times V_y \subset W$. This yields an open cover $(V_y)_{y \in Y}$ of Y, so by compactness, there is a finite subcover indexed by $y_1, \dots, y_N$. Let $U = \bigcap_{i=1}^N U_{y_i}$, which is an open neighbourhood of x. For each i we have $U \times V_{y_i} \subset U_{y_i} \times V_{y_i} \subset W$, so

$$U \times Y = \bigcup_{i=1}^{N} U \times V_{y_i} \subset W.$$

□

Proof of proposition 3.4. Let $(W_i)_{i \in I}$ be an open cover of $X \times Y$. Since Y is compact $\{x\} \times Y$ is compact for each $x \in X$. In particular $(W_i)_{i \in I}$ is a cover of $\{x\} \times Y$, so compactness yields a finite subcover $(W_{x,k})_{k=1}^{N_x}$ for each $x \in X$. Then for each $x \in X$, we have $\{x\} \times Y \subset \bigcup_{x,k=1}^{N_x} W_{x,k}$, so by the tube lemma, there is an open neighbourhood U_x of x, such that $U_x \times Y \subset \bigcup_{x,k=1}^{N_x} W_{x,k}$. This yields an open cover $(U_x)_{x \in X}$ of X, so by compactness, there is a finite subcover $(U_{x_l})_l^M$. Then we have

$$\bigcup_{l=1}^{M} \bigcup_{k=1}^{N_{x_l}} W_{x_l,k} \supset \bigcup_{l=1}^{M} U_{x_l} \times Y = X \times Y,$$

so the cover admits a finite subcover. □

Remark 3.6 (Universal properties)*.* Propositions 2.1, 2.7, and 3.2 are examples of a more general principle, namely that universal properties uniquely characterize spaces (more strictly speaking, maps between spaces) up to unique isomorphism. This can be formalized using the language of category theory, which will lead one eventually into the world of (co)limits and special types of functors.

Let us discuss one more example of a construction characterized by a universal property, which is dual to the construction of the product of two spaces. A product of two spaces X, Y is a space $X \times Y$ together with two continuous maps $\mathrm{pr}_X \colon X \times Y \to X$ and $\mathrm{pr}_Y \colon X \times Y \to Y$ such that we have the universal property from proposition 3.2. If we now reverse the arrows in the diagram, we may ask if there is a space U together with two maps $j_X \colon X \to U$ and $j_Y \colon Y \to U$ such that we have the following diagram

$$\begin{tikzcd}
& & X \arrow[dl, "j_X"'] \arrow[dll, bend right, "f"'] \\
Z & U \arrow[l, dashed, "\exists! h"'] & \\
& & Y \arrow[ul, "j_Y"] \arrow[ull, bend left, "g"]
\end{tikzcd}$$

That is, given continuous maps $f \colon X \to Z$ and $g \colon Y \to Z$, there is a unique map $h \colon U \to Z$ such that $h \circ j_X = f$ and $h \circ j_Y = g$. It turns out that such a space exists (and is unique up to homeomorphism, as it is characterized by a universal property).

Definition 3.7 (Disjoint union)**.** Let X, Y be topological spaces. The *disjoint union* of X and Y is given as a set as the disjoint union of the sets X and Y, i.e.

$$X \coprod Y = \{(x, X) : x \in X\} \cup \{(y, Y) : y \in Y\}.$$

This set has natural inclusion maps $j_X \colon X \to X \coprod Y$ and $j_Y \colon X \to X \coprod Y$. The topology on $X \coprod Y$ is defined as

$$\mathcal{T} = \left\{ U \subset X \coprod Y : j_X^{-1}(U) \subset X \text{ and } j_Y^{-1}(U) \subset Y \text{ are open} \right\}.$$

Remark 3.8. One can view the disjoint union as taking all the elements of X and Y together into one set, while remembering if each of the elements came from X or Y.

4 Some examples

Let us now apply this knowledge to two concrete examples to see how one uses these results in practice

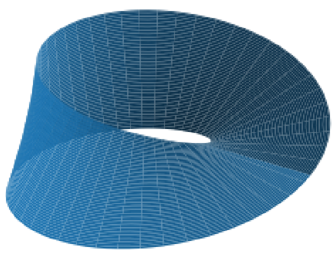
Example 4.1 (Möbius band). The Möbius band can be constructed by identifying two sides of a square. Alternatively, considering the construction using a piece of paper, we see that we should be able to embed the Möbius band in three-dimensional space. Using the theory we have developed so far, we can prove that this is indeed the case, and we can give an explicit embedding from the abstract identification of sides of a square to the concrete model in $\mathbb{R}^3$.

Let $X = [0,1]\times[0,1]$. We say that two pairs $(x,y),(x',y') \in X$ are equivalent, $(x,y) \sim (x',y')$ if $(x,y) = (x',y')$ or if $\{x,x'\} = \{0,1\}$ and $y = 1-y'$. The Möbius band then is defined as the quotient space $M = X/\sim$.

Let us define a map

$$\begin{aligned}&\varphi\colon X \to \mathbb{R}^3\\ &(t,s) \mapsto \left(\cos(2\pi t)\left(1+\left(s-\frac{1}{2}\right)\cos(\pi t)\right), \sin(2\pi t)\left(1+\left(s-\frac{1}{2}\right)\cos(\pi t)\right), \left(s-\frac{1}{2}\right)\sin(\pi t)\right)\end{aligned}$$

This map has the following image.



A short computation shows that $\varphi(1,1-s) = \varphi(0,s)$ for all $s \in [0,1]$, so this map descends to a map $\bar{\varphi}\colon M \to \mathbb{R}^3$ and one can show that this map $\bar{\varphi}$ is injective. Moreover, since X is compact, so is M. Finally, since $\mathbb{R}^3$ is Hausdorff, we conclude that $\bar{\varphi}$ is an embedding of the Möbius band in $\mathbb{R}^3$.

Example 4.2 (Sphere from a disk). As a second example, we will use the theory to prove that $\mathbb{D}^n/\mathbb{S}^{n-1}$ is homeomorphic to $\mathbb{S}^n$, where $\mathbb{D}^n$ is the closed unit n-disk and $\mathbb{S}^n$ is the unit n-sphere. The geometric idea is that we put the centre of $\mathbb{D}^n$ at the north pole of $\mathbb{S}^n$ and stretch the disk over the sphere, such that the boundary $\partial\mathbb{D}^n = \mathbb{S}^{n-1}$ is mapped to the south pole. With this procedure each concentric sphere in $\mathbb{D}^n$ is mapped to a level set of $\pi\colon \mathbb{S}^n \to \mathbb{R}$, mapping the point on the sphere to the first coordinate.

Let us view $\mathbb{S}^n \subset \mathbb{R}\times\mathbb{R}^n$ and let us construct a map $\varphi\colon \mathbb{D}^n \to \mathbb{S}^n$ by sending $0 \mapsto (1,0)$ and any non-zero element

$$x \mapsto \left(1 - 2\|x\|, \frac{x}{\|x\|}\sqrt{1-(1-2\|x\|)^2}\right)$$

(Here $\|x\|$ is the Euclidean norm of x in $\mathbb{R}^n$). Away from 0 this map is continuous, so it remains to check that φ is continuous at 0. This follows from the fact that

$$\left\|\frac{x}{\|x\|}\sqrt{1-(1-2\|x\|)^2}\right\| = \frac{\|x\|}{\|x\|}\sqrt{1-(1-2\|x\|)^2} = \sqrt{1-(1-2\|x\|)^2} \longrightarrow 0$$

as $x \to 0$ and thus

$$\varphi(x) \longrightarrow (1,0) \quad \text{as } x \to 0.$$

Now, suppose that $x \in \partial\mathbb{D}^n$, then $\|x\| = 1$, and so $\varphi(x) = (-1,0)$. So φ descends to a continuous map $\bar{\varphi}\colon \mathbb{D}^n/\mathbb{S}^{n-1} \to \mathbb{S}^n$. We claim that this map is bijective.

For injectivity, suppose that $\bar{\varphi}([x]) = \bar{\varphi}([y])$. Then from the first component, we see that $\|x\| = \|y\|$. If $\|x\| = 0$, then $[x] = [y] = [0]$. If $\|x\| = 1$ then $x,y \in \partial\mathbb{D}^n$ and so $[x] = [y]$. Finally, if $0 < \|x\| < 1$, then $\sqrt{1-(1-2\|x\|)^2} \neq 0$ and so the second component implies that $x = y$. So $\bar{\varphi}$ is injective.

For surjectivity, let $z \in \mathbb{S}^n$. We have already seen that the north pole $(1, 0)$ and south pole $(-1, 0)$ are in the image, so let us assume that z is not one of the poles. Write $z = (h, v)$. Then $|h| < 1$. Let $r = \frac{1-h}{2}$. Then $x = \frac{rv}{\sqrt{1-h^2}} \in \mathbb{D}^n$ and $\bar{\varphi}\left(\frac{rv}{\sqrt{1-h^2}}\right) = z$.

Since $\mathbb{S}^n$ is Hausdorff and $\mathbb{D}^n$ is compact, we conclude that $\mathbb{D}^n/\mathbb{S}^{n-1}$ is compact and therefore $\bar{\varphi}$ is a homeomorphism.