3a,) Assuming that this is a Homogenous MC:

$$\vec{p}(1) = \vec{p}(0)P^1, \vec{p}(2) = \vec{p}(0)P^2, \ldots \vec{p}(m) = \vec{p}(0)P^m, \ldots \vec{p}(m+N) = \vec{p}(0)P^{m+N}$$

It is given that $P^{N+m} = P^m$ for some $m \geq 0$ so:

$$\vec{p}(m) = \vec{p}(0)P^m = \vec{p}(0)P^{m+N} = \vec{p}(m+N)$$

$$\rightarrow \vec{p}(m) = \vec{p}(m+N)$$

Therefore $n_0 = m$, and so $n \geq m$ guarantees

$$\vec{p}(n) = \vec{p}(N+n)$$

3b.) Let P have $k \times k$ for some positive integer k. Then since P is a probability matrix, for each row of P:

$$\sum_{j=1}^{k} P_{ij} = 1, \text{ where } i = \text{row \# and } j = \text{column \#}$$

Since P is a probability matrix, then for any real value of n, P^n is a probability matrix as well, so $\sum_{j=1}^{k} P_{ij}^{(n)} = 1$ as well.

Let $Q = \frac{1}{N}(P + P^2 + \ldots P^N) = \frac{1}{N}P + \frac{1}{N}P^2 + \frac{1}{N}P^3 + \ldots \frac{1}{N}P^N$. Then:

$$\sum_{n=1}^{N} \sum_{j=1}^{k} \left(\frac{1}{N}\right) P_{ij}^{(n)} = \frac{1}{N} \sum_{n=1}^{N} \sum_{j=1}^{k} P_{ij}^{(n)} = \frac{1}{N}(\underbrace{1+1+\ldots}_{N \text{ times}}) = \frac{1}{N}(1 \cdot N) = 1$$

Therefore, each row of Q must add up to 1, and so Q is a probability matrix.

3c.) Since $\vec{p}(n_0+k)$ is a probability vector, where $1 \le k \le N$, the sum of all its elements will add up to 1. As a result, $\sum_{k=1}^{N} p(n_0+k)$ will result in a vector whose elements add up to N (based on a similar reasoning used in part b). Taking the average by multiplying this vector by $\frac{1}{N}$ will result in the sum of all the elements of the resulting vector to be 1 (also based on a similar reasoning used in part b). Since the sum of all the elements of this time-average vector is 1, this is a valid probability vector.

$$\frac{1}{N}\sum_{k=1}^{N}\vec{p}(n_0+k) = \frac{1}{N}\sum_{k=1}^{N}\vec{p}(0)P^{n_0+k}$$

$$= \frac{1}{N}\sum_{k=1}^{N}\vec{p}(0)P^{n_0}P^{k}$$

$$= \vec{p}(0)P^{n_0}\left(\frac{1}{N}\right)\left(\sum_{k=1}^{N}P^{k}\right)$$

$$= \vec{p}(0)P^{n_0}\left(\left(\frac{1}{N}\right)\left(P+P^2+P^3+\dots P^N\right)\right)$$

$$= \vec{p}(0)P^{n_0}Q$$

$$= \vec{p}(n_0)Q$$

3d.)