The Cooper Union
Department of Electrical Engineering
ECE402 Probability Models & Stochastic Processes
Problem Set V: Markov Processes
April 19, 2022

Feel free to use MATLAB to help with the matrix computations. **Note:** When you use *eig* to compute eigenvectors, recall that MATLAB assume eigenvectors on the *right.* You should call *eig(A')* and the eigenvectors will appear as column vectors, which you can then transpose to get the row (left) eigenvectors.

1. Figure 1 shows the transition probabilities for a discrete-time Markov chain. In this problem, use MATLAB to perform calculations.

   (a) Find the state transition matrix P.

   (b) Show that 1 is a simple eigenvalue, other eigenvalues have magnitude less than one, and there is a unique eigenvector $\vec{\pi}$ associated with eigenvalue 1 that is a probability vector. In particular, find $\vec{\pi}$.

   (c) We expect every row of $P^{\infty} = \lim_{n\to\infty} P^n$ to converge to $\vec{\pi}$. Compute P^n for n large enough that every entry matches the steady-state value to 3 decimal places.

   (d) Assume the initial condition is being in state 1 with probability 1, at time 0. Let $\vec{p}\,(n)$ denote the occupancy distribution at time n. Graph $\|\vec{p}\,(n) - \vec{\pi}\|$ (this means Euclidean distance between vectors) versus n, up to the time found above.

2. Figure 2 shows the transition rates for a continuous-time Markov chain ($\alpha > 0$ is a fixed constant).

   (a) Find the transition rate matrix Λ.

   (b) If Λ_0 denotes the result for the case $\alpha = 1$, how are the eigenvalues of Λ related to the eigenvalues of Λ_0? How is $\Phi(t) = e^{\Lambda t}$ related to $\Phi_0(t) = e^{\Lambda_0 t}$? **[Hint:** $\Phi(t) = \Phi_0(\xi)$ for some ξ**].** If $\Pi = \lim_{t\to\infty} \Phi(t)$ and $\Pi_0 = \lim_{t\to\infty} \Phi_0(t)$, how are Π and Π_0 related? You will need these results to answer the following for general α (i.e., you can work for the case $\alpha = 1$ and then generalize).

   (c) Show that 0 is a simple eigenvalue, other eigenvalues are in the LHP, and there is a unique eigenvector $\vec{\pi}$ associated with eigenvalue 0 that is a probability vector. In particular, find $\vec{\pi}$. Does it depend on α?

   (d) We expect every row of $\Pi = \lim_{t\to\infty} e^{\Lambda t}$ to converge to $\vec{\pi}$. How does the limiting matrix depend on α?

   (e) Now turn your attention to computing Π for the case $\alpha = 1$. We can compute Π via the final value theorem (since $\mathcal{L}\left\{e^{\Lambda t}\right\} = (sI - \Lambda)^{-1}$):

$$\Pi = \lim_{s\to 0} s\,(sI - \Lambda)^{-1}$$

Use symbolic tools in MATLAB:

$$\begin{aligned} s &= sym\,('s') \\ B &= \text{simplify}(s * inv(s * eye(4) - \Lambda)) \\ answer &= subs(B, 0) \end{aligned}$$

Note the *simplify* will cancel out the s so you should not get a 0/0 error when you substitute in $s = 0$.

(f) Check your answer by computing $\exp m\,(\Lambda t)$ in MATLAB for a large value of t, say $t = 100$.

3. Let P denote the probability transition matrix for a discrete-time Markov chain, which may have either a finite or inifnite number of states. We say the Markov chain is periodic, with period N, if $P^{N+m} = P^m$, for some $m \geq 0$. Note that this does NOT imply that $P^N = I$ since P is not necessarily invertible!

(a) Let $\vec{p}\,(0)$ denote the initial occupancy distribution. Show that $\vec{p}\,(n)$ becomes periodic "eventually," that is, there is some n_0 such that $n \geq n_0$ guarantees $\vec{p}\,(n) = \vec{p}\,(N+n)$. Find n_0.

(b) Let $Q = \frac{1}{N}\left(P + P^2 + \cdots + P^N\right)$. Show that Q is a probability matrix. You can use the notation $p_{ij}^{(n)}$ to denote the (i,j) element of P^n.

(c) Given $\vec{p}\,(0)$, the time-average occupancy distribution is $\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \vec{p}\,(n)$; in the general case, this limit, even if it exists, may not be a valid probability vector (e.g., for an infinite Markov chain, we could imagine the process marching out to infinity and thus $\vec{p}\,(n)$ could collapse to $\vec{0}$).. However, in the periodic case, it can be shown that this sum converges to:

$$\frac{1}{N} \sum_{k=1}^{N} \vec{p}\,(n_0 + k)$$

where n_0 is as found above. Thus, intuitively, the time-average vector is the average over one period. You do not have to prove this. However, first explain why this is guaranteed to be a valid probability vector, and next show that this is $\vec{p}\,(n_0)\,Q$, which can also be written as $\vec{p}\,(0)\,P^{n_0} Q$.

(d) Consider first:

$$P = \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix}$$

Find smallest N and $m \geq 1$ such that $P^{N+m} = P^m$. Find the associated Q, defined above. Does the time-average occupancy distribution depend on the initial vector $\vec{p}\,(0)$? If not, what is the UNIVERSAL steady-state time-average distribution?

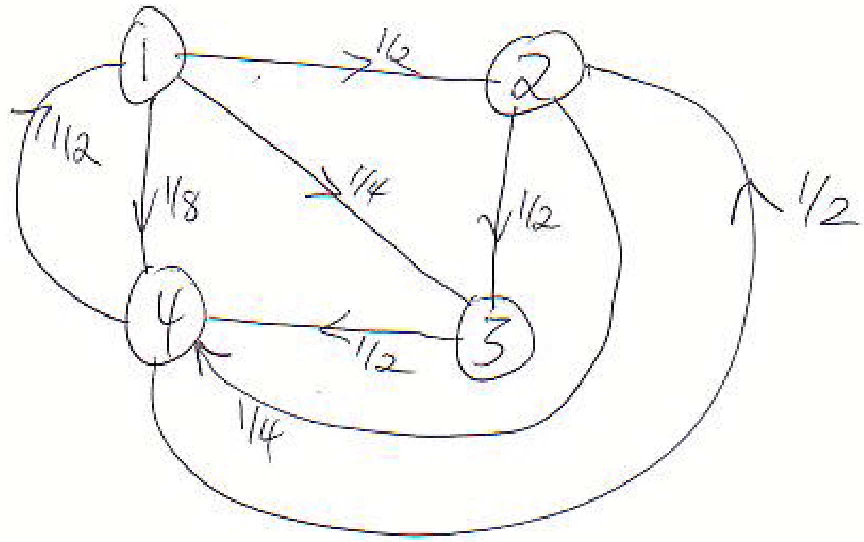

FIGURE 1

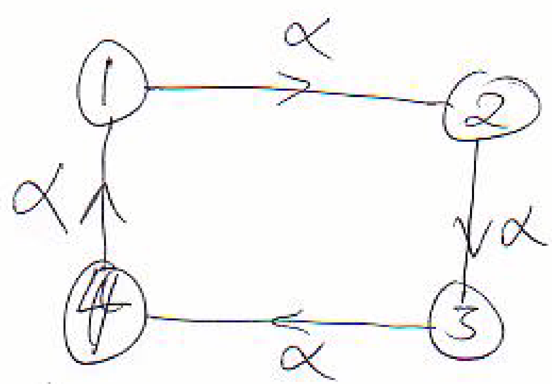

FIGURE 2