Danny Hong ECE 402 PSET 5

1a) $P = \begin{bmatrix} \frac{1}{8} & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix}$

1b) Let $e = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, then if $Pe = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, 1 is an simple eigenvalue

$$Pe = \begin{bmatrix} \frac{1}{8} & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \\ 0 + \frac{1}{4} + \frac{1}{2} + \frac{1}{4} \\ 0 + 0 + \frac{1}{2} + \frac{1}{2} \\ \frac{1}{2} + \frac{1}{2} + 0 + 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \checkmark$$

Therefore, 1 is a simple eigenvalue. Another explanation would be that the elements of each row of P all sum up to 1, so 1 has to be an eigenvalue.

See attached Matlab code for rest of 1

2a) $\Lambda = \begin{bmatrix} -\alpha & \alpha & 0 & 0 \\ 0 & -\alpha & \alpha & 0 \\ 0 & 0 & -\alpha & \alpha \\ \alpha & 0 & 0 & -\alpha \end{bmatrix}$

See attached Matlab code for rest of 2.

2b) $\Lambda_0 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix}$

The eigenvalues of Λ is just the eigenvalues of Λ_0 multiplied by a factor of α.

$\phi(t) = e^{\Lambda t} = e^{\Lambda_0(\alpha t)} = \phi_0(\alpha t)$

Therefore, $\pi = \pi_0$

2c.) Let $e = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, then if $\Lambda e = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, then

0 is a simple eigenvector

$$\Lambda e = \begin{bmatrix} -\alpha & \alpha & 0 & 0 \\ 0 & -\alpha & \alpha & 0 \\ 0 & 0 & -\alpha & \alpha \\ \alpha & 0 & 0 & -\alpha \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\alpha+\alpha+0+0 \\ 0-\alpha+\alpha+0 \\ 0+0-\alpha+\alpha \\ \alpha+0+0-\alpha \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

Therefore, 0 is a simple eigenvalue. Another explanation would be that the elements of each row of Λ all sum up to 0, so 0 has to be a simple eigenvalue.

2d.) Since $\vec{\pi} = [\frac{1}{4}\ \frac{1}{4}\ \frac{1}{4}\ \frac{1}{4}]$ and does not depend on the alpha value, and that each row of the limiting matrix should converge to $\vec{\pi}$, then the limiting matrix does not depend on the alpha value either.

3a.) Assuming that this is a Homogenous MC:

$$\vec{p}(1) = \vec{p}(0)P^1,\ \vec{p}(2) = \vec{p}(0)P^2, \ldots \vec{p}(m) = \vec{p}(0)P^m, \ldots \vec{p}(m+N) = \vec{p}(0)P^{m+N}$$

It is given that $P^{N+m} = P^m$ for some $m \geq 0$ so:

$$\vec{p}(m) = \vec{p}(0)P^m = \vec{p}(0)P^{m+N} = \vec{p}(m+N)$$

$$\rightarrow \vec{p}(m) = \vec{p}(m+N)$$

Therefore $n_0 = m$, and so $n \geq m$ guarantees $\vec{p}(n) = \vec{p}(N+n)$

3b.) Let P have $K \times K$ for some positive integer k. Then since P is a probability matrix, for each row of P:

$$\sum_{j=1}^{k} p_{ij} = 1, \text{ where } i = \text{row \#} \text{ and } j = \text{column \#}$$

Since P is a probability matrix, then for any real value of n, P^n is a probability matrix as well, so $\sum_{j=1}^{k} p_{ij}^{(n)} = 1$ as well.

Let $Q = \frac{1}{N}(P + P^2 + \ldots P^N) = \frac{1}{N}P + \frac{1}{N}P^2 + \frac{1}{N}P^3 + \ldots \frac{1}{N}P^N$. Then:

$$\sum_{n=1}^{N}\sum_{j=1}^{k}\left(\frac{1}{N}\right)p_{ij}^{(n)} = \frac{1}{N}\sum_{n=1}^{N}\sum_{j=1}^{k} p_{ij}^{(n)} = \frac{1}{N}(\underbrace{1+1+\ldots}_{N \text{ times}}) = \frac{1}{N}(1 \cdot N) = 1$$

Therefore, each row of Q must add up to 1, and so Q is a probability matrix.

3c.) Since $\vec{p}(n_0+k)$ is a probability vector, where $1 \le k \le N$, the sum of all its elements will add up to 1. As a result, $\sum_{k=1}^{N} p(n_0+k)$ will result in a vector whose elements add up to N (based on a similar reasoning used in part b). Taking the average by multiplying this vector by $\frac{1}{N}$ will result in the sum of all the elements of the resulting vector to be 1 (also based on a similar reasoning used in part b). Since the sum of all the elements of this time-average vector is 1, this is a valid probability vector.

$$\frac{1}{N}\sum_{k=1}^{N}\vec{p}(n_0+k) = \frac{1}{N}\sum_{k=1}^{N}\vec{p}(0)P^{n_0+k}$$

$$= \frac{1}{N}\sum_{k=1}^{N}\vec{p}(0)P^{n_0}P^{k}$$

$$= \vec{p}(0)P^{n_0}\left(\frac{1}{N}\right)\left(\sum_{k=1}^{N}P^{k}\right)$$

$$= \vec{p}(0)P^{n_0}\left(\left(\frac{1}{N}\right)\left(P+P^2+P^3+\ldots P^N\right)\right)$$

$$= \vec{p}(0)P^{n_0}Q$$

$$= \vec{p}(n_0)Q$$

3d.) See Matlab code for 3d.)