Danny Hong ECE-411 PSET 3

4.) $dV(t) = \alpha(t, V(t))dt + \sigma(t, V(t))dW(t),\ V(t) > 0,\ X = \log V$

$$\frac{\partial X}{\partial V} = \frac{1}{V} \qquad \frac{\partial X}{\partial t} = 0 \qquad \frac{\partial^2 X}{\partial V^2} = \frac{-1}{V^2}$$

Using Ito's Lemma $\rightarrow$ $dX = \left(\frac{\partial X}{\partial t}\right)dt + \left(\frac{\partial X}{\partial V}\right)dV + \frac{1}{2}\left(\frac{\partial^2 X}{\partial V^2}\right)dV^2$

$$= (0)dt + \left(\frac{1}{V}\right)dV + \frac{1}{2}\left(\frac{-1}{V^2}\right)dV^2$$

$$= \frac{1}{V}dV - \frac{1}{2V^2}dV^2$$

substituting using given SDE

$$\rightarrow dX = \frac{1}{V}\left[\alpha(t, V(t))dt + \sigma(t, V(t))dW(t)\right] - \frac{1}{2V^2}\left[\sigma(t, V(t))dt + \sigma(t, V(t))dW(t)\right]^2$$

$$= \frac{1}{V}\alpha(t, V(t))dt + \frac{1}{V}\sigma(t, V(t))dW(t) - \frac{1}{2V^2}(\sigma(t, V(t)))^2 dt^2 - \frac{1}{2V^2}2\sigma(t, V(t))^2 dt\, dW(t) - \frac{1}{2V^2}(\sigma(t, V(t)))^2 \underbrace{dW(t)^2}_{dt}$$

$$= \left[\frac{\sigma(t, V(t))}{V}dt + \frac{\sigma(t, V(t))}{V}dW(t)\right] - \left(\frac{1}{2}\right)\left(\frac{\sigma^2(t, V(t))}{V^2}\right)$$

Ito's Isometry $\left[\cancel{dt^2} + \cancel{2\,dt\,dW(t)} + \underbrace{dW(t)^2}_{dt}\right]$

$$= \frac{\sigma(t,v(t))}{V} dW(t) + \left[\frac{\sigma(t,v(t))}{V} - \frac{1}{2}\left(\frac{\sigma(t,v(t))}{V}\right)^2\right] dt$$

plug in $V = e^X$

$$\boxed{dX = \frac{\sigma(t,e^X)}{e^X} dW(t) + \left[\frac{\sigma(t,e^X)}{e^X} - \frac{1}{2}\left(\frac{\sigma^2(t,e^X)}{e^{2X}}\right)\right] dt}$$

2.) $V(t) = X^2(t) e^{2Y(t)}$, $dX(t) = a(t)dt + b(t)dW(t)$,
$dY(t) = c(t)dt + d(t)dW(t)$

Ito's Lemma:

$$dV = \left(\frac{\partial V}{\partial t}\right) dt + \left(\frac{\partial V}{\partial X}\right) dX + \left(\frac{\partial V}{\partial Y}\right) dY$$
$$+ \left(\frac{1}{2}\right)\left(\frac{\partial^2 V}{\partial X^2}\right) dX^2 + \left(\frac{\partial^2 V}{\partial X \partial Y}\right) dX dY + \left(\frac{1}{2}\right)\left(\frac{\partial^2 V}{\partial Y^2}\right) dY^2$$

$$\frac{\partial V}{\partial t} = 0, \quad \frac{\partial V}{\partial X} = 2Xe^{2Y}, \quad \frac{\partial V}{\partial Y} = 2X^2 e^{2Y},$$

$$\frac{\partial^2 V}{\partial X^2} = 2e^{2Y}, \quad \frac{\partial^2 V}{\partial X \partial Y} = 4Xe^{2Y}, \quad \frac{\partial^2 V}{\partial Y^2} = 4X^2 e^{2Y}$$

substituting: $dV = (2Xe^{2Y})dX + (2X^2e^{2Y})dY + \frac{1}{2}(\cancel{2}e^{2Y})dX^2$
$+ (4Xe^{2Y})dXdY + \frac{1}{2}(\overset{2}{\cancel{4}}X^2 e^{2Y})dY^2$

substituting $dX(t)$ and $dY(t)$:

$$dV = (2Xe^{2Y})(a\,dt + b\,dW) + (2X^2e^{2Y})(c\,dt + d\,dW)$$
$$+ e^{2Y}(a^2\cancel{dt^2} + 2ab\,\cancel{dt\,dW} + b^2\underbrace{dW^2}_{dt})$$
$$+ (4Xe^{2Y})(ac\,\cancel{dt^2} + (ad+bc)\,dt\,dW + bd^2\underbrace{dW^2}_{dt})$$
$$+ (2X^2e^{2Y})(c^2\cancel{dt^2} + 2cd\,\cancel{dt\,dW} + d^2\underbrace{dW^2}_{dt})$$

$\rightarrow dV = 2xe^{2Y}(a\,dt + b\,dW) + 2x^2e^{2Y}(c\,dt + d\,dW)$
$\quad + e^{2Y}(b^2 dt) + (4xe^{2Y})(bd^2 dt) + (2x^2e^{2Y})(d^2 dt)$

$dV = (2xe^{2Y}a + 2x^2e^{2Y}c + e^{2Y}b^2 + 4xe^{2Y}bd^2 + 2x^2e^{2Y}d^2)dt$
$\quad + (2xe^{2Y}b + 2x^2e^{2Y}d)\,dW$

$$\boxed{\begin{aligned} dV = {} & e^{2Y(t)}\left[2aX(t) + b^2 + 2cX^2(t) + 4bd^2X(t) + 2d^2X^2(t)\right]dt \\ & + 2X(t)e^{2Y(t)}\left[b + dX(t)\right]dW(t) \end{aligned}}$$

1.) (a) $V(t) = \frac{1}{D(t)}\widetilde{E}(D(T)V(T)\mid \mathcal{F}_t)$, $V(T) > 0$, $D(T) = e^{-rT} > 0$

$\rightarrow D(t)V(t) = \widetilde{E}(D(T)V(T)\mid \mathcal{F}_t)$

If $V(T) > 0$ and $D(T) > 0$, then $D(T)V(T) > 0$ and so $\widetilde{E}(D(T)V(T)\mid \mathcal{F}_t) > 0$ which means that $D(t)V(t) > 0$

and since $D(t) = e^{-\int_0^t R(s)ds} > 0$, therefore $V(t) > 0$ ✓

(b) $dV(t) = R(t)V(t)dt + \frac{\widetilde{\Gamma}(t)}{D(t)}d\widetilde{W}(t)$

$d(D(t)V(t)) = \widetilde{\Gamma}(t)d\widetilde{W}(t)$ by martingale representation thm.

$d(D(t)V(t)) = D(t)dV(t) + dD(t)V(t)$
$\quad + \frac{1}{\cancel{2}}(\cancel{dD(t)dD(t)}dV(t) + \cancel{2}dD(t)dV(t) + dD(t)\cancel{dV(t)}dV(t))$

Since $= D(t)dV(t) + dD(t)V(t) + dD(t)dV(t)$

from Ito-Doeblin formula

Since $dD(t) = -R(t)D(t)dt$, then substituting in

$\rightarrow d(D(t)V(t)) = -V(t)R(t)D(t)dt + D(t)dV(t)$
$\quad - R(t)D(t)\underbrace{dV(t)dt}_{\text{since } dV(t)dt = 0}$

$\rightarrow d(D(t)V(t)) = -V(t)R(t)D(t)dt + D(t)dV(t)$

$\rightarrow$ Set martingale expression equal to the Ito product rule expression

$\rightarrow -V(t)R(t)D(t)dt + D(t)dV(t) = \tilde{\Gamma}(t)d\tilde{W}(t)$

$\rightarrow -V(t)R(t)dt + dV(t) = \dfrac{\tilde{\Gamma}(t)d\tilde{W}(t)}{D(t)}$ ✓

$\rightarrow dV(t) = V(t)R(t)dt + \dfrac{\tilde{\Gamma}(t)}{D(t)}d\tilde{W}(t)$ ✓

(c) From the result obtained from part b, it can be deduced that $\sigma(t)V(t) = \dfrac{\tilde{\Gamma}(t)}{D(t)}$

$\rightarrow \boxed{\sigma(t) = \dfrac{\tilde{\Gamma}(t)}{V(t)D(t)}}$

given that $V(t) > 0$ and can't be 0, then for $\sigma(t) > 0$, $\tilde{\Gamma}(t) > 0$ as well, which is true since if $\tilde{\Gamma}(t) < 0$, then $dV(t)$ would have a negative diffusion coefficient (coefficient of dW term), which is not possible for a continuous stochastic process starting from a positive value a.s.. In addition, $\sigma(t) \neq 0$ since $\tilde{\Gamma}(t) \neq 0$ as it is a martingale process that is continuous.

3.) (a) $dXdY = \left(\alpha_1(t)dt + \sigma_{11}(t)dW_1(t)\right)\left(\alpha_2(t)dt + \sigma_{21}(t)dW_1(t) + \sigma_{22}(t)dW_2(t)\right)$

$= \cancel{\alpha_1(t)dt\alpha_2(t)dt} + \cancel{\alpha_1(t)dt\sigma_{21}(t)dW_1(t)} + \cancel{\alpha_1(t)dt\sigma_{22}(t)dW_2(t)}$
$+ \cancel{\sigma_{11}(t)dW_1(t)\alpha_2(t)dt} + \underbrace{\cancel{\sigma_{11}(t)dW_1(t)\sigma_{21}(t)dW_1(t)}}$
$+ \cancel{\sigma_{11}(t)dW_1(t)\sigma_{22}(t)dW_2(t)}$

since $dW_1(t)$ and $dW_2(t)$ are independent since they are 2-D so $dW_1(t)dW_2(t)=0$

$= \sigma_{11}(t)\sigma_{21}(t)dt = \rho(t)dt$

$\rightarrow \boxed{\rho(t) = \sigma_{11}(t)\sigma_{21}(t)}$

(b) mean: $E[dW'(t)] = E[a(t)dW_1(t) + b(t)dW_2(t)]$

$= a(t)\underbrace{E[dW_1(t)]}_{\rightarrow 0 \text{ mean}} + b(t)\underbrace{E[dW_2(t)]}_{\rightarrow 0 \text{ mean}} = 0$

variance: $E[dW'(t)^2] - \underbrace{\left(E[dW'(t)]\right)^2}_{0^2} = E[dW'(t)^2]$

$= E\left[\left(a(t)dW_1(t) + b(t)dW_2(t)\right)^2\right] - 0^2$

$= E\left[a(t)^2 dW_1(t)^2 + 2a(t)b(t)\underbrace{dW_1(t)dW_2(t)}_{0 \text{ since } dW_1(t)dW_2(t)=0} + b(t)^2 dW_2(t)^2\right]$

$= a^2(t)\underbrace{E[dW_1^2(t)]}_{dt} + b^2(t)\underbrace{E[dW_2^2(t)]}_{dt}$

$= a^2(t)dt + b^2(t)dt$

The constraint that results in $dW'(t)$ being a Wiener process is that $E[dW'(t)^2] = dt$, or that the variance of $dW'(t)$ must equal dt. Therefore:

$E[dW'(t)^2] = a^2(t)dt + b^2(t)dt = dt$

$\rightarrow \left(a^2(t) + b^2(t)\right)dt = dt$

$\rightarrow \boxed{a^2(t) + b^2(t) = 1}$

(c) $\sigma_{21}(t)\,dW_1(t) + \sigma_{22}(t)\,dW_2(t) = \sigma_{22}'(t)\,dW_2'(t)$

$\rightarrow \sigma_{21}(t)\,\underbrace{dW_1(t)\,dW_1(t)}_{dt} + \cancel{\sigma_{22}(t)\,dW_2(t)\,dW_1(t)} = \sigma_{22}'(t)\,dW_2'(t)\,dW_1(t)$

$\rightarrow \dfrac{\sigma_{21}(t)\,dt}{\sigma_{22}'(t)} = \dfrac{\sigma_{22}'(t)\,dW_2'(t)\,dW_1(t)}{\sigma_{22}'(t)}$

$= \dfrac{\sigma_{21}(t)}{\sigma_{22}'(t)}\,dt = dW_2'(t)\,dW_1(t)$

$\rightarrow \rho_0(t) = \dfrac{\sigma_{21}(t)}{\sigma_{22}'(t)}$

$\left(\sigma_{21}(t)\,dW_1(t) + \sigma_{22}(t)\,dW_2(t)\right)^2 = \sigma_{22}'(t)^2\,dW_2'(t)^2$

$= \sigma_{21}(t)^2\,dW_1(t)^2 + \cancel{2\sigma_{21}(t)\sigma_{22}(t)\,dW_1(t)\,dW_2(t)} + \sigma_{22}(t)^2\,dW_2(t)^2$

$= \sigma_{22}'(t)^2\,dt$

$\sigma_{21}(t)^2\,dt + \sigma_{22}(t)^2\,dt = \sigma_{22}'(t)^2\,dt$

$\rightarrow \sigma_{21}(t)^2 + \sigma_{22}(t)^2 = \sigma_{22}'(t)^2$

$$\boxed{\begin{aligned} &\rightarrow \sigma_{22}'(t) = \sqrt{\sigma_{21}^2(t) + \sigma_{22}^2(t)} \\ &\rightarrow \rho_0(t) = \frac{\sigma_{21}(t)}{\sqrt{\sigma_{21}^2(t) + \sigma_{22}^2(t)}} \end{aligned}}$$