Numbers

Daniel Hong

April 29, 2018

Introduction

We have been using numbers since we learned how to count with our fingers. We use numbers all the time. So it is imperative that we understand what numbers are.

One way that we can define numbers is that they are objects that are used to represent quantity. While that definition is true, we are more interested in what we can do with numbers: add and multiply. We can use this fact to define what numbers are. **Numbers are objects that can be added and multiplied.**

Classifying Numbers

In this handout, we will be classifying numbers into the following categories:

- Real numbers
- Integers
- Rational numbers
- Natural numbers

Real Numbers

We denote real numbers with the symbol $\mathbf{R}$. The symbol is a bold R. When we handwrite it, we write it as $\mathbb{R}$. **We define real numbers as the set of all numbers that can be represented on a number line.** Here are some examples of real numbers:

- 3
- 0.5
- $\sqrt{2}$

Integers

We denote integers with the symbol $\mathbf{Z}$. The symbol is a bold Z. When we handwrite it, we write it as $\mathbb{Z}$. **We define integers as the set of all numbers that contains no decimal or fractional part to it.** Here are some examples of integers:

- -9
- 0
- 14

Rational Numbers

We denote rational numbers with the symbol **Q**. The symbol is a bold Q. When we handwrite it, we write it as $\mathbb{Q}$. **We define rational numbers as the set of all numbers that can be represented as a ratio of two integers.** Here are some examples of rational numbers:

- $\frac{1}{2}$
- 2.3
- $0.\overline{0.14728}$

Natural Numbers

We denote natural numbers with the symbol **N**. The symbol is a bold N. When we handwrite it, we write it as $\mathbb{N}$. **We define natural numbers as the set of numbers that we can count.** Here are some examples of natural numbers:

- 1
- 2
- 100

Properties of Numbers

Before we can use numbers, we must know how they work. Numbers come with the following basic properties:

- Commutative Property of Addition
- Commutative Property of Multiplication
- Associative Property of Addition
- Associative Property of Multiplication
- Additive Identity
- Multiplicative Identity
- Additive Inverse
- Multiplicative Inverse
- Distributive Property

Commutative Property of Addition

The Commutative Property of Addition says that it does not matter what order we add numbers. As an example, let's look at two numbers: 5 and 8. Then by the Commutative Property of Addition,

$$5 + 8 = 8 + 5.$$

Let's define two arbitrary numbers: a and b. Then by the Commutative Property of Addition, for all a and b:

$$a + b = b + a.$$

We can generalize this idea even further. The Commutative Property of Addition says that we add any number of numbers in any order we like. To illustrate this idea, the following statements are true:

- $2 + 7 + 1 = 1 + 7 + 2 = 7 + 1 + 2$
- $19 + 8 + 20 + 11 = 20 + 11 + 8 + 19 = 20 + 8 + 19 + 11$

To generalize, consider the numbers $a_0, a_1, \ldots, a_{n-1}, a_n$. Then by the Commutative Property of Addition, for all $a_0, a_1, \ldots, a_{n-1}, a_n$:

$$a_0 + a_1 + \cdots + a_{n-1} + a_n = a_1 + a_0 + \cdots + a_{n-1} + a_n = \cdots = a_n + a_{n-1} + \cdots + a_1 + a_0.$$

Commutative Property of Multiplication

The Commutative Property of Multiplication is very similar to the Commutative Property of Addition. In fact, it is *parallel* to the Commutative Property of Addition. The Commutative Property of Multiplication says that it does not matter what order we multiply numbers. As an example, consider two numbers: 6 and 7. Then by the Commutative Property of Multiplication,

$$6 \times 7 = 7 \times 6.$$

Let's define two arbitrary numbers: a and b. Then by the Commutative Property of Multiplication, for all a and b:

$$a \times b = b \times a.$$

We can generalize this idea even further. The Commutative Property of Multiplication says that it does not matter what order we multiply numbers. For illustration, the following statements are true:

- $12 \times 13 \times 14 = 13 \times 14 \times 12 = 12 \times 14 \times 13$
- $10 \times 8 \times 6 \times 12 = 6 \times 12 \times 10 \times 8 = 8 \times 10 \times 12 \times 6$

To generalize, consider the numbers $a_0, a_1, \ldots, a_{n-1}, a_n$. Then by the Commutative Property of Multiplication, for all $a_0, a_1, \ldots, a_{n-1}, a_n$:

$$a_0 \times a_1 \times \cdots \times a_{n-1} \times a_n = a_1 \times a_0 \times \cdots \times a_{n-1} \times a_n = \cdots = a_n \times a_{n-1} \times \cdots \times a_1 \times a_0.$$

Associative Property of Addition

The Associative Property of Addition is very much like the Commutative Property of Addition. The Associative Property of Addition says that we can change the order of precedence for addition if we have parentheses grouping numbers together. As an illustration, consider three numbers: 1, 3, and 5. Then by the Associative Property of Addition,

$$(1 + 3) + 5 = 1 + (3 + 5).$$

Let's define three arbitrary numbers: a, b, and c. Then by the Associative Property of Addition, for all a, b, and c:

$$(a + b) + c = a + (b + c).$$

We can generalize this idea even further. The Associative Property of Addition works on any number of numbers and any grouping of those numbers. For illustration, the following statements are true:

- $(2 + 3 + 4) + 7 = 2 + (3 + 4 + 7)$
- $(4 + 1) + 9 + 8 = 4 + (1 + 9) + 8 = 4 + 1 + (9 + 8)$

To further generalize, consider an arbitrary number of numbers: $a_0, a_1, \ldots, a_{n-1}, a_n$. Then by the Associative Property of Addition, for all $a_0, a_1, \ldots, a_{n-1}, a_n$:

$$\begin{gathered}
(a_0 + a_1) + \cdots + a_{n-1} + a_n = a_0 + (a_1 + a_2) + \cdots + a_{n-1} + a_n = \cdots = a_0 + a_1 + \cdots + (a_{n-1} + a_n), \\
(a_0 + a_1 + a_2) + \cdots + a_n = a_0 + (a_1 + a_2 + a_3) + \cdots + a_n = \cdots = a_0 + \ldots (a_{n-2} + a_{n-1} + a_n), \\
\vdots \\
(a_0 + a_1 + \cdots + a_{n-1}) + a_n = a_0 + (a_1 + \cdots + a_{n-1} + a_n).
\end{gathered}$$

Associative Property of Multiplication

The Associative Property of Multiplication is *parallel* to the Associative Property of Addition. The only difference between the two of them is that the Associative Property of Addition is for addition, whereas the Associative Property for Multiplication is for multiplication. The Associative Property of Multiplication says that we can change the order of precedence for multiplication if we have parentheses grouping numbers together. To illustrate, consider three numbers: 8, 9, and 12. Then by the Associative Property of Multiplication,

$$(8 \times 9) \times 12 = 8 \times (9 \times 12)$$

To generalize this idea, consider three arbitrary numbers: a, b, and c. Then by the Associative Property of Multiplication, for all a, b, and c.

$$(a \times b) \times c = a \times (b \times c)\,.$$

We can generalize this idea even further. The Associative Property of Multiplication works on any number of numbers and any grouping of those numbers. For illustration, the following statements are true:

- $(3 \times 4 \times 2) \times 5 = 3 \times (4 \times 2 \times 5)$
- $(11 \times 12) \times 9 \times 20 = 11 \times (12 \times 9) \times 20 = 11 \times 12 \times (9 \times 20)$

To further generalize, consider an arbitrary number of numbers: $a_0, a_1, \ldots, a_{n-1}, a_n$. Then by the Associative Property of Multiplication, for all $a_0, a_1, \ldots, a_{n-1}, a_n$:

$$\begin{gathered}(a_0 \times a_1) \times \cdots \times a_{n-1} \times a_n = a_0 \times (a_1 \times a_2) \times \cdots \times a_{n-1} \times a_n = \cdots = a_0 \times a_1 \times \cdots \times (a_{n-1} \times a_n)\,, \\ (a_0 \times a_1 \times a_2) \times \cdots \times a_n = a_0 \times (a_1 \times a_2 \times a_3) \times \cdots \times a_n = \cdots = a_0 \times \ldots (a_{n-2} \times a_{n-1} \times a_n)\,, \\ \vdots \\ (a_0 \times a_1 \times \cdots \times a_{n-1}) \times a_n = a_0 \times (a_1 \times \cdots \times a_{n-1} \times a_n)\,.\end{gathered}$$

Additive Identity

The Additive Identity says that there exists a number such that when it is added to any number, it will *not* change that number's value. Let's look at an example. If we have a number 3, there exists a number such that when it is added to 3, we will still have 3. *That number is* 0 because

$$3 + 0 = 3.$$

We can generalize this idea by arbitrarily defining a number a. **The Additive Identity says that there exists a number 0 called "zero" such that for all a:**

$$a + 0 = a.$$

Multiplicative Identity

The Multiplicative Identity goes along with the Additive Identity. In other words, it is *parallel* to the Additive Identity. The Multiplicative Identity says that there exists a number such that when it is multiplied to any number, it will *not* change that number's value. Let's look at an example. If we have a number 7, there exists a number such that when it is multiplied to 7, we will still have 7. *That number is* 1 because

$$7 \times 1 = 7.$$

We can generalize this idea by arbitrarily defining a number a. **The Multiplicative Identity says that there exists a number 1 called "one" such that for all a:**

$$a \times 1 = a.$$

Additive Inverse

The Additive Inverse says that for every number there exists a number such that when it is added to any number, we will get 0. For example, if we have a number 2, there exists a number such that when added to 2, we will get 0. *That number is* -2 because

$$2 + (-2) = 0.$$

What we essentially did was subtract the number from itself, which is how we undo addition. To undo an arbitrary number a, we subtract a from it or add $-a$ to it. **The Additive Inverse says that for every number a there exists a number $-a$ such that:**

$$a + (-a) = 0.$$

Multiplicative Inverse

The Multiplicative Inverse follows the similar idea as the Additive Inverse. Rather than undoing addition, we are undoing multiplication by division. The Multiplicative Inverse says that for every number there exists a number such that when multiplied to it, we will get 1. For example, if we have a number 6, there exists a number such that when multiplied to 6, we will get 1. *That number is* $\frac{1}{6}$ because

$$6 \times \frac{1}{6} = 1.$$

Note that if we divide 6 by itself, we also get 1. To undo an arbitrary number a, we divide a or multiply it by $\frac{1}{a}$ or divide it by a. **The Multiplicative Inverse says that for every number a there exists a number $\frac{1}{a}$ such that:**

$$a \times \frac{1}{a} = 1.$$

Distributive Property

The Distributive Property uses both addition and multiplication. The Distributive Property says that if we have a number k and multiply it over a sum of several numbers, we can multiply each of those numbers by k first and then add them together. For illustration, consider the following expression:

$$5 \times (10 + 2).$$

When we apply the Distributive Property, we multiply each term in the parentheses by 5 to get

$$5 \times 10 + 5 \times 2.$$

We can use the Distributive Property on any number of numbers being added together. For example:

$$2 \times (7 + 4 + 3 + 9) = 2 \times 7 + 2 \times 4 + 2 \times 3 + 2 \times 9.$$

To generalize, suppose we have a set of numbers that we want to add up $a_0, \ldots, a_n$,and we want to multiply their sum by k.

$$k \times (a_0 + \cdots + a_n)$$

Then by the Distributive Property, we multiply each term inside the parentheses by k to get

$$k \times a_0 + \cdots + k \times a_n.$$

Usages of Symbols

Throughout this handout, there were symbols (namely letters) that were being used. Those symbols were only used when we move from concrete numbers to abstract symbols. The symbols that you see were used to represent numbers. Later, you will find that we commonly use symbols for variables, which is our first level of abstraction in Algebra!