

There are 5 problems in this set. You need to do 3 problems the first week and 2 the second week. Instead of a sixth problem, **you are encouraged to work on your final project**. Feel free to work with other students, but make sure you write up the homework and code on your own (no copying homework *or* code; no pair programming). Feel free to ask students or instructors for help debugging code or whatever else, though. When implementing algorithms you may not use any library (such as sklearn) that already implements the algorithms but you may use any other library for data cleaning and numeric purposes (numpy or pandas). Use common sense. Problems are in no specific order.

1 (Gaussian Mixture Model) Consider the generative process for a Gaussian Mixture Model:

- (1) Draw $z_i \sim \text{Cat}(\pi)$ for $i = 1, 2, \dots, n$.
- (2) Draw $\mathbf{x}_i \sim \mathcal{N}(\mu_{z_i}, \Sigma_{z_i})$ for $i = 1, 2, \dots, n$.

Note that z_i is unobserved but \mathbf{x}_i is observed. Express this model as a directed graphical model, first ‘unrolled’ and then using Plate notation, before answering the following questions. Support all claims.

- (a) Is π independent of μ_{z_i} or Σ_{z_i} given your dataset $\mathcal{D} = \{\mathbf{x}_i\}$? Does the posterior distribution over $\{\mu, \Sigma\}$ and π factorize? How does this change what inference procedure we need to use for this model?
- (b) If z_i were observed, would this change? Would the posterior then factorize? *Hint:* what other model have we studied that corresponds to observing z_i ?
- (c) Find the maximum likelihood estimates for π , μ_k , and Σ_k if the latent variables z_i were observed.

2 (Linear Regression) Consider the Bayesian Linear Regression model with the following generative process:

- (1) Draw $\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{V}_0)$
- (2) Draw $\mathbf{y}_i \sim \mathcal{N}(\mathbf{w}^\top \mathbf{x}_i, \sigma^2)$ for $i = 1, 2, \dots, n$ where σ^2 is known.

Express this model as a directed graphical model using Plate notation. Is \mathbf{y}_i independent of \mathbf{w} ? Is \mathbf{y}_i independent of \mathbf{w} given $\mathcal{D} = \{\mathbf{x}_i\}$? Support these claims.

3 (Collaborative Filtering) Consider the ‘ratings’ matrix $\mathbf{R} \in \mathbb{R}^{n \times n}$ with the low rank approximation $\mathbf{R} = \mathbf{U}\mathbf{V}^\top$ where \mathbf{U} and \mathbf{V} live in $\mathbb{R}^{n \times k}$ where we have k latent factors. Define our optimization problem as

$$\text{minimize: } f(\mathbf{U}, \mathbf{V}) = \|\mathbf{R} - \mathbf{U}\mathbf{V}^\top\|_2^2 + \lambda\|\mathbf{U}\|_2^2 + \gamma\|\mathbf{V}\|_2^2$$

where $\|\cdot\|_2$ in this case is the Frobenius norm $\|\mathbf{R}\|_2^2 = \sum_{ij} \mathbf{R}_{ij}^2$. Derive the gradient of f with respect to \mathbf{U}_i and \mathbf{V}_j . Derive a stochastic approximation to this gradient where you consider a single data point at a time.

4 (Alternating Least Squares) Consider the same setup and objective

$$\text{minimize: } f(\mathbf{U}, \mathbf{V}) = \|\mathbf{R} - \mathbf{U}\mathbf{V}^\top\|_2^2 + \lambda\|\mathbf{U}\|_2^2 + \gamma\|\mathbf{V}\|_2^2$$

as above in problem (3).

- (a) Suppose we fix \mathbf{U} . Find the optimal \mathbf{V} .
- (b) Suppose we fix \mathbf{V} . Find the optimal \mathbf{U} .
- (c) Propose an EM-like (block coordinate ascent, to be exact) like algorithm for minimizing $f(\mathbf{U}, \mathbf{V})$ using (a) and (b).
- (d) Will the algorithm you propose in (c) necessarily converge to the global optimal?

5 (Non-Negative Matrix Factorization) Consider the dataset at <http://kdd.ics.uci.edu/databases/reuters21578/reuters21578.html>. Choosing an appropriate objective function and algorithm from Lee and Seung 2001¹ implement Non-Negative Matrix Factorization for topic modelling (choose an appropriate number of topics/latent features) and assert that the convergence properties proved in the paper hold. Display the 20 most relevant words for each of the topics you discover.

¹<https://papers.nips.cc/paper/1861-algorithms-for-non-negative-matrix-factorization.pdf>