A Brisk Overview of Convex Optimization

Looking at Figures That Took Too Long to Make.

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Convex Optimization

1. Core behind techniques in machine learning, signal processing, operations research, etc.
2. Can be very applied or very theoretical
3. If you need more resources check out *Convex Optimization* by Boyd and Vandenberghe[1].

[1] `http://stanford.edu/~boyd/cvxbook/`

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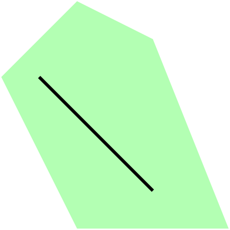
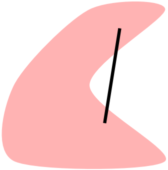
Convex Sets

Convex Set



Non-Convex Set



Convex Sets - an Intuitive Definition

A set C is *convex* if, given any two points in that set, every point on the line segment between those two points is also in C.

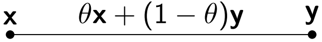
Convex Sets

Definition (Convex Combination / Line Segment)

A convex combination of two points **x** and **y** from an affine space is

$$\theta \mathbf{x} + (1-\theta)\mathbf{y},$$

where $0 \leq \theta \leq 1$.



Convex Sets - an Actual Definition

A set C is *convex* if, given any two points in that set, every point on the line segment between those two points is also in C. Mathematically, we have

Definition (Convex Set)

A set C is convex if, given $\mathbf{x}, \mathbf{y} \in C$, every convex combination of $\mathbf{x}$ and $\mathbf{y}$ still lies in C. That is,

$$\theta\mathbf{x} + (1-\theta)\mathbf{y} \in C. \qquad (0 \leq \theta \leq 1)$$

for all $\mathbf{x}$ and $\mathbf{y}$ in C.

Convex Sets

Example (The Space of Probability Distributions)

Let $\mathcal{P}$ be the space of continuous probability distributions over $\mathbb{R}^n$. That is, every element of $\mathcal{P}$ defines a unique probability density function $\mathbb{P}(x) \geq 0$ such that

$$\int_{\mathbb{R}^n} \mathbb{P}(x) dx = 1.$$

$\mathcal{P}$ is convex.

Convex Sets

Proof.

Let f and h be valid probability distributions from $\mathcal{P}$. That is,

$$f, h \in \left\{ \mathbb{P}(x) : \int_{\mathbb{R}^n} \mathbb{P}(x) dx = 1 \text{ and } \mathbb{P}(x) \geq 0 \right\}.$$

Now let $0 \leq \theta \leq 1$. Then

$$\theta f(x) + (1-\theta) h(x) \geq 0$$

as a positive combination of positive functions. Similarly,

$$\begin{aligned}\int_{\mathbb{R}^n} \left[\theta f(x) + (1-\theta) h(x)\right] dx &= \theta \int_{\mathbb{R}^n} f(x) dx + (1-\theta) \int_{\mathbb{R}^n} h(x) dx \\ &= \theta + (1-\theta) = 1.\end{aligned}$$

$\blacksquare$

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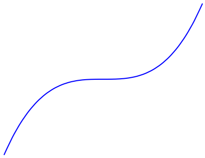
Convex Functions

Convex Function

Concave Function

Neither



Convex Function

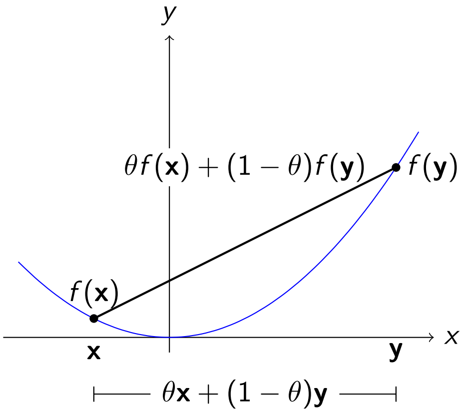
Definition (Convex Function)

A function $f : C \mapsto \mathbb{R}$ is *convex* if its domain C is convex and, for $0 \leq \theta \leq 1$, given any $\mathbf{x}, \mathbf{y} \in C$,

$$f(\theta \mathbf{x} + (1-\theta)\mathbf{y}) \leq \theta f(\mathbf{x}) + (1-\theta) f(\mathbf{y}).$$

That is, a line segment drawn between two function values lies above the function.

Convex Function



Second Order Conditions

Theorem

A twice differentiable function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is convex if and only if it's Hessian is positive semi-definite:

$$\nabla^2 f \succeq 0.$$

Concave and Convex Functions

Example

$f(x) = \log(x) : \mathbb{R}_{++} \mapsto \mathbb{R}$ is concave.

Proof.

$f''(x) = -\frac{1}{x^2} < 0$, so f is *strictly concave*. ■

Example

$f(x) = \exp(x) : \mathbb{R} \mapsto \mathbb{R}_{++}$ is convex.

Proof.

$f''(x) = e^x > 0$, so f is *strictly convex*. ■

Convex Function

Example

The function $f : \mathbb{R}^n \mapsto \mathbb{R}$ defined by

$$f(\mathbf{x}) = e^{\mathbf{x}^\top \mathbf{x}} = e^{\mathbf{x}_1^2 + \cdots + \mathbf{x}_n^2}$$

is convex.

Proof.

Each element of the Hessian

$$\nabla^2 f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j} = 4\mathbf{x}_i \mathbf{x}_j e^{\mathbf{x}^\top \mathbf{x}}$$
$$\nabla^2 f = 4\mathbf{x}\mathbf{x}^\top e^{\mathbf{x}^\top \mathbf{x}}.$$

Is this positive semi-definite? Consider $\mathbf{z} \in \mathbb{R}^n$. Then

$$\mathbf{z}^\top \nabla^2 f(x) \mathbf{z} = \mathbf{z}^\top 4\mathbf{x}\mathbf{x}^\top e^{\mathbf{x}^\top \mathbf{x}} \mathbf{z} = 4(\mathbf{z}^\top \mathbf{x})^2 e^{\mathbf{x}^\top \mathbf{x}} \geq 0$$

■

Convex Function

Example (Quadratic Functions)

The function $f : \mathbb{R}^n \mapsto \mathbb{R}$ defined by

$$f(\mathbf{x}) = \mathbf{x}^\top A\mathbf{x} + \mathbf{b}^\top \mathbf{x} + c$$

is convex if $A \succeq 0$.

Proof.

We know

$$\nabla^2 f = 2A.$$

Similarly we know that $2A \succeq 0$ if and only if $A \succeq 0$. Thus by the second order conditions f is convex if and only if $A \succeq 0$.
We can also show that f is concave if $A \preceq 0$. ∎

Affine Composition

Theorem

Given a convex $f : \mathbb{R}^m \mapsto \mathbb{R}$*, any matrix* $A \in \mathbb{R}^{m \times n}$, $\mathbf{x} \in \mathbb{R}^n$*, and* $\mathbf{b} \in \mathbb{R}^m$,

$$g(\mathbf{x}) = f(A\mathbf{x} + \mathbf{b})$$

is convex.

Proof.

$$\nabla^2 g = A^\top \nabla^2 f(A\mathbf{x} + \mathbf{b})A.$$

Then, for any $\mathbf{z} \in \mathbb{R}^n$,

$$\mathbf{z}^\top \nabla^2 g \mathbf{z} = \mathbf{z}^\top A^\top \nabla^2 f(A\mathbf{x} + \mathbf{b})A\mathbf{z} = (A\mathbf{z})^\top \nabla^2 f(A\mathbf{x} + \mathbf{b})(A\mathbf{z}) \geq 0$$

by the convexity assumption, so g is convex. ∎

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Optimization Problems

An optimization problem is a problem of the form

$$\begin{aligned} \text{minimize: } & f_0(\mathbf{x}) \\ \text{subj. to: } & f_i(\mathbf{x}) \leq 0, \quad i = 1, \ldots, m \\ & h_i(\mathbf{x}) = 0, \quad i = 1, \ldots, p \end{aligned}$$

The goal of an optimization problem, as you might be able to guess, is to optimize f_0 where $\mathbf{x}$ is in the problem domain domain

$$\mathcal{D} = \bigcap_{i=0}^{m} \textbf{dom } f_i \, \cap \, \bigcap_{i=1}^{p} \textbf{dom } h_i,$$

satisfying the problem constraints.

Convex Optimization Problems

An optimization problem of the form

$$
\begin{aligned}
\text{minimize: } & f_0(\mathbf{x}) \\
\text{subj. to: } & f_i(\mathbf{x}) \leq 0, \quad i = 1, \dots, m \\
& A\mathbf{x} = \mathbf{b}
\end{aligned}
$$

is convex if f_i for all $i = 0, \dots, m$ are convex.

Global Optimality of Convex Optimization Problems

Theorem (Global Optimality)

Given a convex optimization problem, and local optimum $\mathbf{x}^\star$ *such that*

$$f_0(\mathbf{x}) = \inf\{f_0(\mathbf{z}) : \mathbf{z} \textit{ feasible and } ||\mathbf{z} - \mathbf{x}||_2 \leq R\}$$

for some $R > 0$*.* $\mathbf{x}^\star$ *is the global optimum.*

Proof.

(Boyd) Suppose local optimality but not global optimality, with some feasible $\mathbf{y}$ such that $f_0(\mathbf{y}) < f_0(\mathbf{x})$. Then $||\mathbf{y} - \mathbf{x}||_2 > R$ because otherwise $f_0(\mathbf{x}) \leq f_0(\mathbf{y})$. Consider a point $\mathbf{z}$ given by

$$\mathbf{z} = (1-\theta)\mathbf{x} + \theta\mathbf{y} \quad \text{and} \quad \theta = \frac{R}{2||\mathbf{y} - \mathbf{x}||_2}.$$

Then $||\mathbf{x} - \mathbf{z}||_2 = R/2 < R$. By convexity of the feasible set, $\mathbf{z}$ is feasible. By the convexity of the objective function f_0 we have

$$f_0(\mathbf{z}) \leq (1-\theta) f_0(\mathbf{x}) + \theta f_0(\mathbf{y}) < f_0(\mathbf{x})$$

Common Problems - Linear Programs

$$
\begin{aligned}
\text{minimize: } & \mathbf{c}^\top \mathbf{x} \\
\text{subj. to: } & G\mathbf{x} \preceq \mathbf{h}, \\
& A\mathbf{x} = \mathbf{b}
\end{aligned}
$$

Common Problems - Quadratic Programs

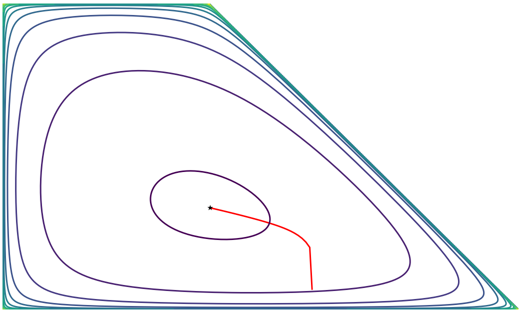
$$
\begin{aligned}
\text{minimize: } & \frac{1}{2}\mathbf{x}^\top P\mathbf{x} + \mathbf{q}^\top \mathbf{x} + \mathbf{r} \\
\text{subj. to: } & G\mathbf{x} \preceq \mathbf{h} \\
& A\mathbf{x} = \mathbf{b}
\end{aligned}
$$

with $P \in \mathbb{S}_+^n$

Example Problem - Analytic Centering

Want to find some sort of 'center' of a convex polygon (polytope) $A\mathbf{x} \preceq \mathbf{b}$:

$$\text{minimize: } f_0(\mathbf{x}) = -\sum_i \log(\mathbf{b}_i - a_i^\top \mathbf{x})$$



Example Problem - Least Squares Linear Regression

We have an overdetermined (tall A) system $A\mathbf{x} = \mathbf{b}$ we want to solve, and we want to find the "best" solution by minimizing

$f = ||A\mathbf{x} - \mathbf{b}||_2 = (A\mathbf{x} - \mathbf{b})^\top (A\mathbf{x} - \mathbf{b}) = \mathbf{x}^\top A^\top A\mathbf{x} - 2\mathbf{x}^\top A^\top \mathbf{b} + \mathbf{b}^\top \mathbf{b}.$

At optimality $\nabla f = 0$, so we can find

$$\nabla f = 2A^\top A\mathbf{x} - 2A^\top \mathbf{b} = \mathbf{0},$$

so

$$\mathbf{x}^\star = (A^\top A)^{-1} A^\top \mathbf{b}.$$

Example Problem - Logistic Regression

Have a dataset $\{\mathbf{x}_i, y_i\}_1^m$ with $y \in \{0, 1\}$. We want to predict

$$\hat{y} = \mathbb{P}(y_i = 1 | \mathbf{x}_i; \boldsymbol{\theta}) = \sigma(\boldsymbol{\theta}^\top \mathbf{x}) = \frac{1}{1 + \exp(-\boldsymbol{\theta}^\top \mathbf{x})}.$$

Optimization problem: maximize the (log) likelihood of our data given the parameters $\boldsymbol{\theta}$:

$$\begin{aligned} \text{maximize:} \;\; \log \mathbb{P}(\mathcal{D} | \boldsymbol{\theta}) &= \sum_{i=1}^{m} y_i \log \sigma(\boldsymbol{\theta}^\top \mathbf{x}) + (1 - y_i) \log(1 - \sigma(\boldsymbol{\theta}^\top \mathbf{x})) \\ &= \sum_{i=1}^{m} \ell_i \end{aligned}$$

Example Problem - Logistic Regression

Is this convex? Note $\sigma'(x) = \sigma(x)\,[1 - \sigma(x)]$. Then

$$\begin{aligned}
\nabla \ell &= y\mathbf{x}\frac{\sigma(\boldsymbol{\theta}^\top \mathbf{x})\left[1 - \sigma(\boldsymbol{\theta}^\top \mathbf{x})\right]}{\sigma(\boldsymbol{\theta}^\top \mathbf{x})} - (1-y)\mathbf{x}\frac{\sigma(\boldsymbol{\theta}^\top \mathbf{x})\left[1 - \sigma(\boldsymbol{\theta}^\top \mathbf{x})\right]}{1 - \sigma(\boldsymbol{\theta}^\top \mathbf{x})} \\
&= y\mathbf{x}\left[1 - \sigma(\boldsymbol{\theta}^\top \mathbf{x})\right] - (1-y)\mathbf{x}\sigma(\boldsymbol{\theta}^\top \mathbf{x}) = \left[y - \sigma(\boldsymbol{\theta}^T \mathbf{x})\right]\mathbf{x}
\end{aligned}$$

and then

$$\nabla^2 \ell = -\sigma(\boldsymbol{\theta}^\top \mathbf{x})\left[1 - \sigma(\boldsymbol{\theta}^\top \mathbf{x})\right]\mathbf{x}\mathbf{x}^\top \preceq 0$$

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Optimal Solutions

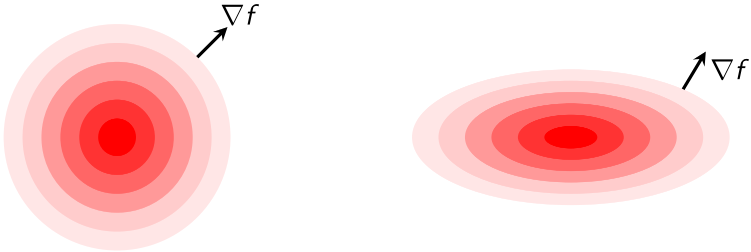
For convex unconstrained problems

$$\text{minimize: } f_0(\mathbf{x}) = f(\mathbf{x})$$

The optimal solution occurs when $\nabla f = \mathbf{0}$.

Gradient Descent

Dumb algorithm: start somewhere and take small steps down the hill.

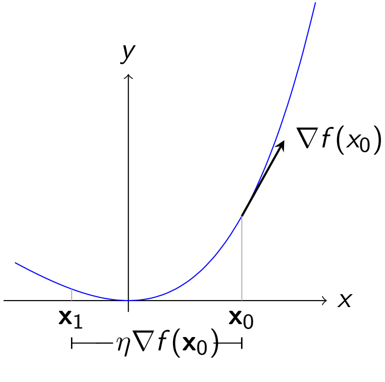


Gradient Descent

Gradient Descent
input : f, ∇f, $\eta(t)$, starting point $\mathbf{x}_0$, tolerance ϵ
output: optimal point $\mathbf{x}^\star$
$t \leftarrow 0$
while $||\nabla f||_2 \geq \epsilon$ **do**
    $\mathbf{x}_{t+1} \leftarrow \mathbf{x}_t - \eta(t)\nabla f(\mathbf{x}_t)$
    $t \leftarrow t + 1$
end
return $\mathbf{x}^\star = \mathbf{x}_{t+1}$

Newton's Method

Good algorithm: start somewhere, approximate f as a quadratic, and optimize that quadratic at every step



Newton's Method

Approximating f as a quadratic with it's Taylor Expansion:

$$f_{\mathbf{x}_0}(\mathbf{x}) \approx f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0)^\top (\mathbf{x} - \mathbf{x}_0) + \frac{1}{2}(\mathbf{x} - \mathbf{x}_0)^\top \nabla^2 f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0).$$

At optimality we have

$$\nabla f = \nabla f + \nabla^2 f(\mathbf{x} - \mathbf{x}_0) = 0,$$

so

$$\mathbf{x}^\star = \mathbf{x}_0 - \nabla^2 f^{-1} \nabla f.$$

Newton's Method

Newton's Method

input : f, ∇f, $\nabla^2 f$, starting point $\mathbf{x}_0$, tolerance ϵ

output: optimal point $\mathbf{x}^\star$

$t \leftarrow 0$

while $||\nabla f||_2 \geq \epsilon$ **do**

  solve $\nabla^2 f(\mathbf{x}_t)\mathbf{d}_t = \nabla f(\mathbf{x}_t)$ for $\mathbf{d}_t$

  $\mathbf{x}_{t+1} \leftarrow \mathbf{x}_t - \mathbf{d}_t$

  $t \leftarrow t + 1$

end

return $\mathbf{x}^\star = \mathbf{x}_{t+1}$

Newton's Method is Good

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\text{minimize: } f(\mathbf{x}) \arrow[r, "\text{Linear Transformation}"] \arrow[d, "\text{Newton}"'] & \text{minimize: } f(T\mathbf{y}) \arrow[d] \\
\mathbf{x}_0, \mathbf{x}_1, \ldots \arrow[r] & T\mathbf{y}_0, T\mathbf{y}_1, \ldots
\end{tikzcd}
$$

Newton's Method is Scale Invariant

Proof.

(Induction) Assume $\mathbf{x}_t = A\mathbf{y}_t$. Recall at each step

$$\mathbf{x}_{t+1} = \mathbf{x}_t - \nabla^2 f^{-1} \nabla f.$$

If we have $g(\mathbf{y}) = f(A\mathbf{y})$, then as before we have

$$\nabla g = A^T \nabla f(A\mathbf{y}) \qquad \text{and} \qquad \nabla^2 g = A^T \nabla^2 f(A\mathbf{y}) A$$

and then

$$\begin{aligned}
\mathbf{y}_{t+1} &= A^{-1}\mathbf{x}_t - (A^T \nabla^2 f(A\mathbf{x}_t) A)^{-1} A^T \nabla f(\mathbf{x}_t) \\
&= A^{-1}\mathbf{x}_t - A^{-1} \nabla^2 f(\mathbf{x}_t)^{-1} A^{-\top} A^T \nabla f(\mathbf{x}_t) \\
&= A^{-1} \left[\mathbf{x}_t - \nabla^2 f(\mathbf{x}_t)^{-1} \nabla f(\mathbf{x}_t)\right] \\
A\mathbf{y}_{t+1} &= \mathbf{x}_{t+1}.
\end{aligned}$$

∎

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