Lecture 4

Solutions of equations of MHD (contd.)

- comments of the Stokes solution.

To solve

$$\left.\begin{aligned} \nabla\cdot v &= 0 \\ \nu\nabla^2 v - \nabla p &= 0 \end{aligned}\right\}$$

we used a prescription of

$$v_i = U_{ij} a_j, \quad p = \Pi_j a_j$$

with $U_{ij} = \delta_{ij}\nabla^2\chi - \frac{\partial^2\chi}{\partial x_i \partial x_j}$

with χ ultimately obtained by solving ~~d~~ $\boxed{\nabla^4\chi = 0}$

where did this choice come from?

~~If we had conside~~

The idea is to solve the original pde we want to turn it into a pde which has less variables. The original PDE was for three components of velocity with one constraint. ~~No~~ If we ~~are~~ were in ~~three dimensi~~ two dimensions we could write: $\nu\nabla^2(\nabla\times v) = 0 \Rightarrow \nabla^2\omega = 0$ [which would still be one PDE with two variables]

using $\omega = \nabla^2\psi$ where ψ is the streamfunction we could write $\nabla^2\psi = 0$. But in 3-d this would not ~~help~~ ~~as~~ ~~ψ~~ ~~should~~ help; because ψ would also be a vector function. But the idea is somewhat similar. We write ~~$v \sim \nabla\times\nabla\times\psi$~~ ~~$\omega \sim \nabla\times\psi$~~

~~and~~ ~~$\omega =$~~ $v \sim \nabla\times\nabla\times\psi, \sim \nabla\nabla\psi - \nabla^2\psi$

which is roughly what U_{ij} is such that $\nabla\cdot v = 0$

Force - free Solution

4. 1. Isothermal ~~Incompressible~~, ideal MHD eqn and some of it's solutions :

consider :

$$\partial_t \rho + \mathrm{div}\,(\rho v) = 0$$

$$\partial_t (\rho v) + \mathrm{div}\,(\rho v_i v_j + p\,\delta_{ij}) = J \times B$$

$$\partial_t B = \nabla \times (v \times B) \qquad c_s^2 = \frac{\gamma p}{\rho} = \text{constant}$$

Eqns. of isothermal MHD.

Clearly : $v = 0$ is a solution ~~even~~ (hydrostatics) which ~~also~~ would still remain a solution if we consider viscosity. If we have

$$\boxed{J \times B = 0}$$ force - free magnetic field.

One solution is $B =$ constant. Which is also a solution if we include η. But there are other solutions

If $\underbrace{\nabla \times B = \lambda B}_{B \text{ is an eigenfunction of the curl operator.}}$, $\Rightarrow J \times B = 0$.

In ~~an~~ Cartesian coordinates, the solutions are called Beltrami solution. Remember that in the incompressible case the flow eqn can be written as $\partial_t v + \omega \times v = \nu \nabla^2 v - \nabla p$

if $\omega = \lambda v$, then the non-linear term is zero! if B are both Beltrami, $J \times B = 0$, $\omega \times v = 0$

so if v and B are both Beltrami, ~~both~~ the nonlinear terms are zero.

i.e. ~~both~~ both the nonlinear terms are zero.

Taking another curl : $\nabla \times \nabla \times B = -\nabla^2 B = \lambda^2 B$

$$\Rightarrow \eta \nabla^2 B = -\lambda^2 B$$

Also : $$\boxed{\nabla^2 B + \lambda^2 B = 0}$$ vector Helmholtz Eqn.

The Beltrami fields in Cartesian:

$$u_x = A \sin \Lambda z + C \cos \Lambda z$$
$$u_y = B \sin \Lambda x + A \cos \Lambda z$$
$$u_z = C \sin \Lambda y + B \cos \Lambda x$$

- Arnold - Beltrami - Childress is a solution of the incompressible Navier - Stokes equations.

$$\partial_t u + \cancel{} \ \omega \times u = \nu \nabla^2 u - \nabla p$$
$$\nabla \cdot u = 0$$

The solution in spherical polar coordinates is called Chandrasekhar - Kendall functions

To solve the vector Helmholtz eqn, write down first the solutions to the scalar Helmholtz eqn.

$$\nabla^2 \psi + \Lambda^2 \psi = 0$$

Then define:
$$B = T + S$$

with $T = \nabla \times (\hat{e}\, \psi)$ and $S = \frac{1}{\Lambda} \nabla \times T$

any constant unit vector

Also note that the force-free solutions are helical; depending on what you choose as Λ.

$$\nabla \times B = \Lambda B \quad \Rightarrow \quad \cancel{\text{if}} \quad \cancel{A =} \ \cancel{\nabla \times B};$$
$$\cancel{\nabla \times A} = \cancel{\nabla \times} \ \cancel{\nabla \times}$$

$\Rightarrow$ if $B = \nabla \times A$, $A = \frac{1}{\Lambda} B$

such that $\nabla \times A = \frac{1}{\Lambda} \nabla \times B = \frac{1}{\Lambda} \Lambda \cancel{\nabla\times} B = \cancel{} B$

<u>Helicity</u>: $\mathcal{H} = \int \vec{A} \cdot \vec{B} \, dV = \frac{1}{\Lambda} \int \vec{B} \cdot \vec{B} \, dV = \frac{1}{\Lambda} E_M$

magnetic energy; always +ve.

~~4.2 Pressure balanced plasma column~~ :

4.2 Taylor's theory of decay in plasma :

How should plasma decay from an initial arbitrary configuration ?

Taylor's hypothesis : In the limit of high Rem or small η plasma shall decay to a state with minimum energy but constant helicity.

[What ? Is not that strange ? One conserved quantity gets minimized and the other one remains constant ?]

Let us see what the consequence will be.

Energy $\equiv E_M = \int \frac{B^2}{2}\, dV$ Helicity $\equiv \mathcal{H} = \int A \cdot B \, dV$

Then the plasma will decay to a state given by :

$$\delta \left[E_M + \Lambda \mathcal{H} \right] = 0$$

Λ : Lagrange's multiplier.

$$\Rightarrow \delta \int \left[\frac{B^2}{2} + (A \cdot B)\Lambda \right] dV = 0$$

$$\Rightarrow \int \left[B \cdot \delta B + \Lambda \left(B \cdot \delta A + A \cdot \delta B \right) \right] dV = 0$$

~~consider~~ : $\cancel{\int (B \cdot \delta A)\, dV = \int \delta A \cdot (\nabla \times A)\, dV}$

$\cancel{= \nabla \cdot (\delta A \times A) + \delta A \cdot (\nabla \times A)}$

consider $\int (A \cdot \delta B)\, dV = \int A \cdot \delta (\nabla \times A)\, dV$

$$= \int A \cdot \nabla \times \delta A \, dV$$

$$= \int \Big[\underbrace{\nabla \cdot (\delta A \times A)}_{\text{zero.}} + \delta A \cdot (\nabla \times A) \Big] dV = \int \delta A \cdot B \, dV$$

$$\Rightarrow \int \left(B\cdot\delta B + 2\Lambda\, B\cdot\delta A \right) dV = 0$$

$$\Rightarrow \qquad \delta B = \Lambda\,\delta A$$

The variation in B should be proportional to variation in A which would imply: $B = \Lambda A$

or $\qquad \nabla\times B = \Lambda\,\nabla\times A = \Lambda B$ — force-free eqn.

So, under Taylor's hypothesis the ~~magno~~ relaxed state of the magnetic field is obtained by the force-fee eqn.

Mathematical aside:

The above proof is mathematically speaking not quite satisfactory. Also it does not make immediate connection to the Euler-Lagrange eqns. which are typically used in ~~min~~ functional minimisation problems. This connection is better ~~mege~~ made in the appendix on functional ~~methods~~ derivatives.

Using the ideas in the appendix we prove the same mathematical theorem again:

$$S[A,B] = \int \mathcal{L}[A,B]\, dV$$

with $\quad \mathcal{L}[A,B] = \dfrac{B^2}{2} + \Lambda A\cdot B = \frac{1}{2} B_k B_k + \Lambda A_k B_k$

$$\frac{\delta S}{\delta A_j(y)} = 0 \qquad \text{— minimisation principle.}$$

$$\frac{\delta S}{\delta A_j(y)} = \int \frac{\partial \mathcal{L}}{\partial A_j}\frac{\delta A_j(x)}{\delta A_j(y)}\, d^d x + \cancel{\frac{\partial \mathcal{L}}{\partial B_j}} \int \frac{\partial \mathcal{L}}{\partial B_i}\frac{\delta B_i(x)}{\delta A_j(y)}\, d^d x$$

$$\frac{\delta A_j(x)}{\delta A_j(y)} = \delta(x-y)$$

$$\frac{\delta B_i(x)}{\delta A_j(y)} = \frac{\delta}{\delta A_j} \epsilon_{imn} \partial_m A_n(x)$$

$$= \epsilon_{imn} \partial_m \delta_{jn} \delta(x-y)$$

$$\Rightarrow \quad \frac{\delta S}{\delta A_j} = \frac{\partial \mathcal{L}}{\partial A_j} + \int \frac{\partial \mathcal{L}}{\partial B_i} \epsilon_{imn} \delta_{jn} \partial_m \delta(x-y)\, d^d x$$

$$= \frac{\partial \mathcal{L}}{\partial A_j} - \int \epsilon_{imn} \delta_{jn} \delta(x-y) \partial_m \left(\frac{\partial \mathcal{L}}{\partial B_i} \right) d^d x$$

$$= \frac{\partial \mathcal{L}}{\partial A_j} - \epsilon_{imj} \partial_m \left(\frac{\partial \mathcal{L}}{\partial B_i} \right) = 0$$

$$\frac{\partial \mathcal{L}}{\partial A_j} = \Lambda B_j \qquad \frac{\partial \mathcal{L}}{\partial B_i} = \frac{1}{2} B_i + \Lambda A_i$$

$$\Rightarrow \quad \Lambda B_j - \epsilon_{imj} \partial_m \left(\frac{1}{2} B_i + \Lambda A_i \right) = 0$$

$$\epsilon_{imj} \partial_m A_i = - \epsilon_{jmi} \partial_m A_i = -(\nabla \times A)_j = - B_j$$

$$\epsilon_{imj} \partial_m B_i = - \epsilon_{jmi} \partial_m B_i = -(\nabla \times B)_j$$

$$\Rightarrow \quad 2 \Lambda B = \nabla \times B, \qquad \Rightarrow \quad \boxed{\nabla \times B = \Lambda B}$$

The force-free eqn.

J. Taylor PRL 33 1139 (1974)

- Although the mathematical problem is easily treatable the physical applicability of this theorem is not clear. It may be the case that the energy may decay over a time scale much faster than magnetic helicity. Then, ~~the~~ this ~~formulation~~ can apply to the case ~~where~~ in intermediate time scales.

- It may also be that the problem requires completely different formulation. Typically, minimum energy principle ~~may~~ do not hold for dissipative systems. ~~The case~~ A better principle could be maximisation of some kind of entropy. A surrogate for entropy could be the rate of energy dissipation

$$\epsilon_M = \eta \int J^2 \, dV$$

So, the new extreemisation principle could be :

$$\delta \left(\epsilon_M + \Lambda \mathcal{H} \right) = 0$$

This can give the following expression :

$$\nabla \times \nabla \times \nabla \times B = \Lambda B$$

– B. Dasgupta et al. PRL 81 3144 (1988)

A subset of which are the force-free equations.

- What is the experimental / numerical evidence ?

In certain experimental situations Taylor's hypothesis gives pretty good fit but not in all cases. There are now known to be many exceptions. See Section 15.4 of PFP.

- There is another way often use to circumvent the disagreement of the numerical data.

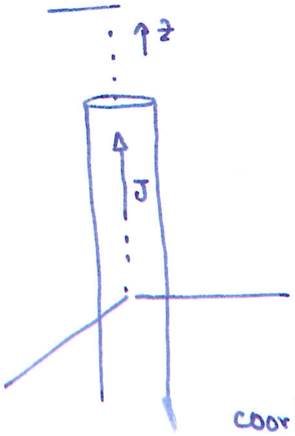
  To propose non-linear force-free fields:

$$\nabla \times B = \Lambda B$$

where Λ is no longer a constant but a function of coordinate, often only one.

- See also the illuminating comments on decay of a magnetic field in section 15.1 of PFP.

4.3 Pressure ballanced plasma column:



$$\partial_t (\rho v) + \mathrm{div}(\rho v_i v_j + p\delta_{ij}) = J \times B$$

A steady state can be obtained iff

$$v = 0, \qquad \nabla p = J \times B$$

Consider a column of plasma, use cylindrical coordinates. Assume all quantities are function of r only. We set up a current $J_z(r)$.

$$\mu_0 J = \nabla \times B = \frac{1}{r}\left(\frac{\partial}{\partial r} r B_\theta - \frac{\partial B_r}{\partial \theta}\right)\hat{z}$$

$$= \frac{1}{r}\frac{\partial}{\partial r} r B_\theta \,\hat{z}$$

$$B = (0, B_\theta(r), 0)$$

$$J \times B = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{z} \\ 0 & 0 & J_z \\ 0 & B_\theta & 0 \end{vmatrix} = -\hat{r} J_z B_\theta = -\frac{\hat{r}}{\mu_0}\frac{1}{r}\frac{d}{dr}(r B_\theta) B_\theta$$

$$\Rightarrow \quad \frac{dp}{dr} = -\frac{1}{r} B_\theta \frac{d}{dr}(r B_\theta)\frac{1}{\mu_0}$$

$$= -\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2 B_\theta^2}{2}\right)\frac{1}{\mu_0}$$

The solution of this can give a ~~pres~~ stationary solution.

If we assume J = constant $= J_0 \hat{z}$

$$\Rightarrow \quad \frac{1}{r}\frac{d}{dr}\left(r B_\theta\right) = J_0 \mu_0 \Rightarrow \quad B_\theta(r) = \frac{r J_0 \mu_0}{2}$$

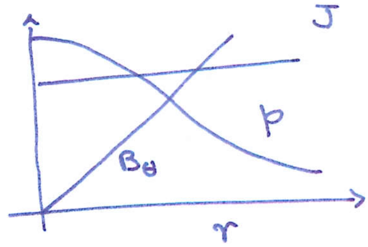
$$\Rightarrow \quad \frac{dp}{dr} = -\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{2}\cdot\frac{r^2}{4}J_0^2\mu_0^2\right)\frac{1}{\mu_0}$$

$$\cancel{p(r) - p(a) = -\frac{1}{8}\frac{J_0^2 r^5}{r^2\, 5} = -\frac{J_0^2 r^3}{40}}$$

$$= -\frac{1}{r^2}\frac{1}{\cancel{\mu_0}}\frac{d}{dr}\left(\frac{r^4}{8}J_0^2\mu_0^{\cancel{2}}\right)J_0^2$$

$$= -\frac{\mu_0}{\cancel{8}\,2}\frac{1}{\cancel{r^2}}\cancel{4}\cdot r^{\cancel{3}} J_0^2$$

$$\Rightarrow \quad p(r) - p(0) = -\frac{\mu_0 J_0^2 r^2}{4}$$



pressure decrease outward!

At a distance $r = a$, $p = 0$

$$\Rightarrow \quad p(0) = \frac{\mu_0 J_0^2}{4}a^2$$

Given a J_0, and a ~~...~~ pressure at $r=0$, The pressure can become 0 at a radius $r = a$. For $r > a$ the pressure can become negative? This is ~~clarly~~ clearly unphysical. But this shows what a "pinch" is.

4.4 The solar wind.

Consider a central star. We look for a spherically symmetric solution of the equations of hydrodynamics (not MHD)

$$\vec{v} = (v_r(r), 0, 0) \quad ; \quad \rho = \rho(r)$$

$$c_s^2 = \frac{p}{\rho} = \text{constant}$$

Also, assume that the solutions is stationary.

$$\partial_t \rho = 0, \qquad \partial_t (\rho v) = 0$$

In spherical polar coordinate system:

$$\underbrace{\cancel{\partial_t \rho}}_{0} + \operatorname{div}(\rho v_r) = 0$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \rho v_r\right) = 0 \quad \Rightarrow \quad \underbrace{r^2 \rho v_r}_{\text{rate of mass injection}} = \dot{M} = \text{constant}$$

$$\underbrace{\cancel{\partial_t(\rho v_r)}}_{0} + \rho v_r \frac{dv_r}{dr} = -\frac{dp}{dr} - \frac{GM}{r^2}\rho$$

gravity force.

$$\Rightarrow \quad \frac{d}{dr}\left(\frac{v_r^2}{2} + c^2 \ln \rho - \frac{GM}{r}\right) = 0$$

$$\Rightarrow \quad \frac{v_r^2}{2} + c^2 \ln \rho - \frac{GM}{r} = \text{constant} \equiv \mathcal{E}$$

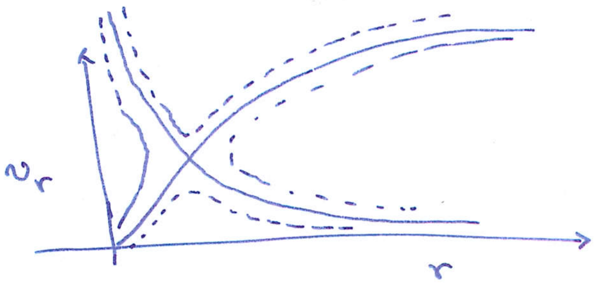
Energy conservation

combining:

$$\boxed{-\frac{GM}{r} + \frac{v_r^2}{2} - c^2 \ln v_r - 2c^2 \ln r = \mathcal{E}'}$$

The Parker solution.

- To find the actual ~~numbe~~ solution we have to numerically solve the transcendental eqn. This shows that for a given ~~bou~~ parameters there are four branches



- The solution can be negative, i.e. accretion. or positive : i.e. wind. Numerically each branch is found separately and then connected by hand.

- It is remarkable how little assumptions we need to obtain the wind. The pressure goes to zero at infinity.

- A different way to write the same eqn.

$$\left(v_r^2 - c^2\right) \frac{d}{dr} \left| \ln v_r \right| = \frac{2c^2}{r} - \frac{GM}{r^2}$$

Assume that at $r = r_*$, $v_r = c$

$$\Rightarrow \quad \frac{2c^2}{r_*} = \frac{GM}{r_*^2} \quad \Rightarrow \quad r_* = \frac{GM}{2c^2}$$

The radius at which the wind becomes supersonic.

- The problem can also be easily extended to adiabatic wind.

- This is a steady wind, in the sense that $v_r(r)$ is not a function of time ~~but~~ the ~~solar~~ ~~wind~~ actual solar wind is a highly turbulent process with huge fluctuations.

- The mass loss due to the wind is quite small.

- Also consider the magnetized wind. As we move ~~aga~~ away from the star the velocity of the wind increases, and in the present solution the energy is conserved. If we include magnetic field then total energy (magnetic + kinetic + gravitational + ··) will be conserved. Far away from the star the magnetic energy must decrease. Hence the kinetic energy must increase. The point upto which the (magnetic energy) > kinetic energy is called Alfven radius.

Appendix 4A

Functional Calculus

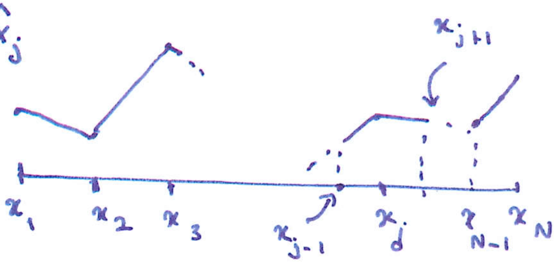
~~consider a function of many variable~~ :

$$f(\{x_i\})$$

~~How to find the minim minima/maxima of the function~~ ?

$$\cancel{\nabla f = \frac{\partial f}{\partial x_j}\hat{x}_j}$$

consider a disreete space



At each x_i, consider a height function $h(x_j)$.

Then Consider a function of $h(x_j)$; for example:

$$\tilde{\Phi}[h_j] \equiv \tilde{\Phi}[h(x_j)]$$

This is a function of N variables; h_j not a functional.

specifically this has a Taylor series expansion

$$\tilde{\Phi}(h_j + \delta h_j) = \tilde{\Phi}(h_j) + \sum \frac{\partial \tilde{\Phi}}{\partial h_j}\delta h_j + \frac{1}{2}\sum \frac{\partial^2 \tilde{\Phi}}{\partial h_j \partial h_k}\delta h_j \delta h_k$$

and

$$\frac{\partial \tilde{\Phi}}{\partial h_j} = \lim_{\delta h_j \to 0} \frac{\tilde{\Phi}(h_1, \dots, h_j + \delta h_j, \dots h_N) - \tilde{\Phi}(h_1, \dots h_N)}{\delta h}$$

Now to calculate the functional take the continuum limit $\lim_{N\to\infty}$, $\lim_{\delta x \to 0}$ and replace the sum

$$\sum \to \sum \Delta x \to \int dx$$

$$\Phi[h+\delta h] = \Phi[h] + \int dx \frac{\delta \Phi}{\delta x} + \frac{1}{2}\int dx\,dx' \frac{\delta^2 \Phi}{\delta h(x)\,\delta h(x')}$$

$+ \dots$

In particular

$$\frac{\delta h(x)}{\delta h(y)} = \delta(x-y)$$

$$\frac{\delta f(h(x))}{\delta h(y)} = \frac{df(z)}{dz} \cdot \frac{\delta h(x)}{\delta h(y)} = f' \delta(x-y)$$

— usual chain rule of derivatives.

Now consider

$$\Phi[h] = \int f\left(h, \frac{dh}{dx}\right) dx = \int f(h, h')\, dx$$

is a function of h and it's derivative.

$$\Rightarrow \frac{\delta \Phi}{\delta h(y)} = \int \frac{\partial f}{\partial h} \frac{\delta h(x)}{\delta h(y)} dx + \int \frac{\partial f}{\partial h'} \frac{\delta h'(x)}{\delta h(y)} dx$$

$$= \int \frac{\partial f}{\partial h} \delta(x-y)\, dx + \int \frac{\partial f}{\partial h'} \cdot \frac{\delta}{\delta h(y)} \left(\frac{dh}{dx}\right) dx$$

$$= \frac{\partial f}{\partial h} + \int \frac{\partial f}{\partial h'} \frac{d}{dx} \delta(x-y)\, dx$$

$$= \frac{\partial f}{\partial h} - \int \delta(x-y) \frac{d}{dx} \frac{\partial f}{\partial h'} dx + \text{(terms zero in integration by parts)}$$

$$= \frac{\partial f}{\partial h(y)} - \frac{d}{dy} \frac{\partial f}{\partial h'}$$

$\frac{\delta \Phi}{\delta h} = 0$ typically gives us the Euler-Lagrange eqn.

$$\boxed{\frac{\partial f}{\partial h} - \frac{d}{dy} \frac{\partial f}{\partial h'} = 0}$$

In higher dimensions, we get $\boxed{\frac{\partial f}{\partial h} - \nabla \cdot \frac{\partial f}{\partial \nabla h} = 0}$

Examples

1. Simplest from Lagrangian mechanics.

A particle in a one dimensional potential.

The Newton's eqn gives the eqn. of motion to be:

$$\cancel{\boxed{m\frac{dv}{dt} = -\frac{\partial V}{\partial x}}} \qquad \frac{d}{dt}\left(m\dot{x}\right) = -\frac{\partial V}{\partial x}$$

The Lagrangian mechanics ~~th~~ states that the action

$$S = \int_{t_1}^{t_2} \mathcal{L}(x, \dot{x})\, dt$$

will reach ~~a~~ a minima.

$$\Rightarrow \quad \cancel{\frac{\delta S}{\delta x}} = 0 = \int_{t_1}^{t_2} \mathcal{L}(x, \dot{x})\, dt \,, \quad \frac{\delta S}{\delta x} = 0$$

The corresponding Euler-Lagrange eqn. is:

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0$$

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 - V(x)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$$

$$\frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial V}{\partial x}$$

$$\frac{\partial \mathcal{L}}{\partial t} = 0$$

$$\boxed{\frac{d}{dt}\left(m\dot{x}\right) = -\frac{\partial V}{\partial x}}$$

↑

True for all ~~x(t)~~ $x(t)$