Lecture VII

- Magnetorotational Instability. (MRI)

~~In the~~

While studying the Taylor-Couette problem we already found that the stability criterion of Rayleigh is

$$\frac{1}{r^3}\frac{d}{dr}(r^2\Omega)^2 > 0$$

What happens if there is a constant magnetic field in the vertical direction?

We lineariz the equations with the following unperturbed state ~~$v = (0, 0, \Omega(r) r)$~~

~~$B = 0,$~~

$$v = (0, \Omega(r) r, 0)$$

$$B = (0, 0, B)$$

And look for perturbations that are functions of z only. Then we obtain

$$-i\omega\delta v_r - 2\Omega\delta v_\phi - \frac{ikB}{2\mu_0}\delta b_r = 0$$

$$-i\omega\delta v_\phi + \frac{\kappa^2}{2\Omega}\delta v_r - \frac{ikB}{2\mu_0}\delta b_\phi = 0$$

~~$-i\omega\delta b$~~

$$-\omega\delta b_r = kB\delta v_r$$

$$-i\omega\delta b_\phi = \delta b_r\frac{d\Omega}{d\ln r} + ikB\delta v_\phi$$

with epicyclic frequency $\kappa^2 = \frac{1}{r^3}\frac{d}{dr}(r^2\Omega)^2$

The dispersion relation:

$$\omega^4 - \omega^2\left[\kappa^2 + 2(k\,c_A)^2\right] + (k\,c_A)^2\left[(k\,c_A)^2 + \frac{d\Omega^2}{d\ln r}\right] = 0$$

where $c_A \equiv$ Alfven velocity $= \frac{B}{\sqrt{\mu_0 \rho_0}}$

~~can become negative when~~

stability criterion:

$$(k\,c_A)^2 > -\frac{d\Omega^2}{d\ln r}$$

The problem can always become unstable unless

$$\frac{d\Omega^2}{d\ln r} > 0$$

so the Rayleigh criterion

$$\frac{d}{dr}(r^2\Omega)^2 > 0$$

should be replaced by

$$\frac{d\,\Omega^2}{dr} > 0$$

- angular momentum must be replaced by angular velocity.

- The maximum growth rate of the instability:

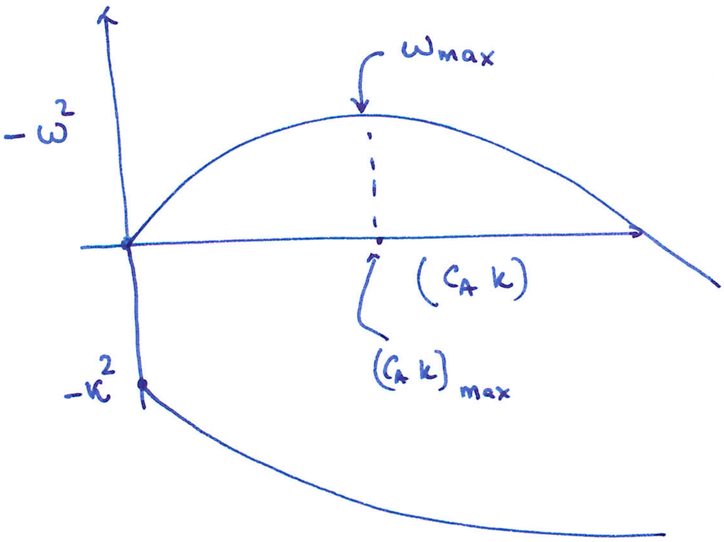
$$|\omega_{max}| = \frac{1}{2}\left|\frac{d\Omega}{d\ln r}\right|$$

when
$$(k\, c_A)^2_{max} = -\left(\frac{1}{4} + \frac{\kappa^2}{16\Omega^2}\right)\frac{d\Omega^2}{d\ln r}$$

- Keplerian rotation profile:

$$\Omega^2 r = \frac{GM_\odot}{r^2} \quad \Rightarrow \quad \Omega = \Omega_0 r^{-3/2}$$

$$\omega_{max} = \frac{3}{4}\Omega \qquad (c_A \cdot k)_{max} = \frac{\sqrt{15}}{4}\Omega$$



$$k_{max} \sim \frac{1}{c_A}$$

smaller the magnetic field larger k_{max} will be.

- It is the slow magnetosonic wave that becomes unstable.

This is the essence of the magnetorotational ~~is~~ instability.

- Is the magnetic field ever too ~~smally~~ small to be dynamically ignored?

- A turbulent disk:

  - why do we need the MRI? Is not a Keplarian disk hydrodynamically unstable?

Rayleigh criterion implies, $\frac{d}{dr}(r^2\Omega)^2 > 0$

$$\Omega \sim r^{-3/2}, \qquad r^2\Omega \sim \sqrt{r}$$

$$\frac{d}{dr}(r^2\Omega)^2 = \frac{d}{dr} r = 0 .$$

Numerical simulations show that a Keplarian disk is ~~li~~ nonlinearly stable.

Although a shear flow in general is not. !

- The difference could be from boundary conditions.

- steady state Keplerian disk (ideal)

$$\partial_t \rho + \mathrm{div}(\rho v) = 0$$

$$\partial_t (\rho v) + \mathrm{div}\left(\rho v_i v_j + p\delta_{ij}\right)$$

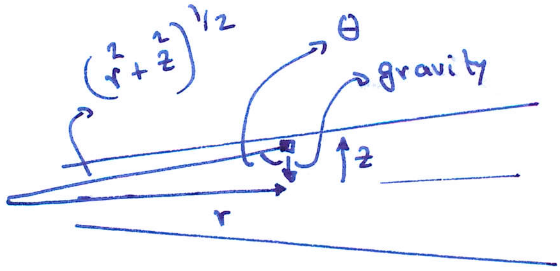
$$= J \times B + \nabla \Phi \leftarrow \text{Gravity.}$$

$$v_\phi = v_\phi + \Omega(r)\, r$$

$$c^2 = \gamma p/\rho$$

$$\partial_t B = \nabla \times (v \times B)$$

$$\Omega^2 = \frac{G M_\odot}{R^3}$$



The vertical structure of the disk

$$\frac{\partial p}{\partial z} = - \frac{G M_\odot}{(r^2+z^2)} \cos\theta\, \rho(z)$$

$$= - \frac{GM}{(r^2+z^2)} \frac{z}{(r^2+z^2)^{1/2}} \rho(z)$$

$$\simeq - \frac{GM}{r^3} z \rho = - \rho \Omega^2 z$$

$$\Rightarrow \quad \rho = \rho_0 \exp\left(-\frac{z^2}{H^2}\right) \qquad H = \frac{\sqrt{2}\, c_s}{\Omega}$$

For large r, $\Omega r > c_s$ is possible; actually quite common in astrophysical applications.

Then $\dfrac{H}{r} = \dfrac{\sqrt{2}\, c_s}{\Omega r} < 1$ thin disk approximation.

- Equation for angular momentum

$$\partial_t (\rho r v_\phi) + \nabla \cdot \Big[\rho r v_\phi \mathbf{v} - \frac{r B_\phi}{2\mu_0} \left(\cancel{B_\phi^2} + \cancel{B_z^2} \right) \left(\hat{r} B_r + \hat{\phi} B_\phi \right)$$

$$+ r \left(p + \frac{B_r^2 + B_z^2}{\mu_0} \right) \hat{e}_\phi \Big]$$

$$- \nabla \cdot \underbrace{[\quad]}_{\text{dissipative terms}} = 0$$

On averaging

$$\partial_t \langle l \rangle + \nabla \cdot [\underbrace{\langle \quad \rangle}_{\text{flux}}] = 0$$

$\mathbf{u} = \mathbf{v} - \Omega r$ ← fluctuating part

The radial component of the flux

$$W_{r\phi} = \left\langle r \left[\rho \{ u_r (\Omega r + u_\phi) - \frac{B_r B_\phi}{\mu_0} \right] \right\rangle$$

$$= \Sigma r \left[r \Omega \langle u_r \rangle_\rho + \underbrace{\langle u_r u_\phi - c_{Ar} c_{A\phi} \rangle_\rho}_{\text{turbulent transport.}} \right]$$

where $\Sigma = \int_{-\infty}^{+\infty} \rho \, dz$ $\qquad c_A = \dfrac{B}{\sqrt{\mu_0 \rho_0}}$

$$\langle X \rangle_\rho = \frac{1}{2\pi \Sigma \Delta r \, r} \int_{\substack{r \in \Delta r \\ \phi \in 2\pi}} X \rho \, d\phi \, dr \, dz$$

If there is a non-vanishing angular momentum flux outward, then matter slowly looses angular momentum and ~~spi~~ accrete inward. So the disk looses mass at the following rate:

$$\dot{M} = - 2\pi r \Sigma \langle u_r \rangle_\rho$$

It can be shown that there is a flux of energy inward, contributing to the Luminosity of the disk. Typical boundary conditions are stress-free at the inner boundary.

The Shakura Sunayev model

$$W_{r\phi} \sim \alpha c_s^2$$

- Can the MRI produce sustained turbulence? How does it saturate?

- Can we calculate α from first principles?

How does instabilities saturate?

Consider the problem in an abstract manner.

There is a critical number, the Reynold's number, the Taylor number, or the Richardson number, which when exceeds a critical value an ~~is~~ instability develops. This instability is typically of the following form:

$$u(x,t) \sim A(t)\, f(x)$$

whee $f(x)$ is an appropriate eigenfunction that satisfies the boundary conditions. $A(t)$ is typically complex. In the simplest case it obeys the following equation:

$$A(t) = e^{-i\omega t} = e^{\gamma t - i\omega_1 t} \qquad \frac{dA}{dt} = (\gamma - i\omega_1)\, e^{\gamma t - i\omega_1 t}$$

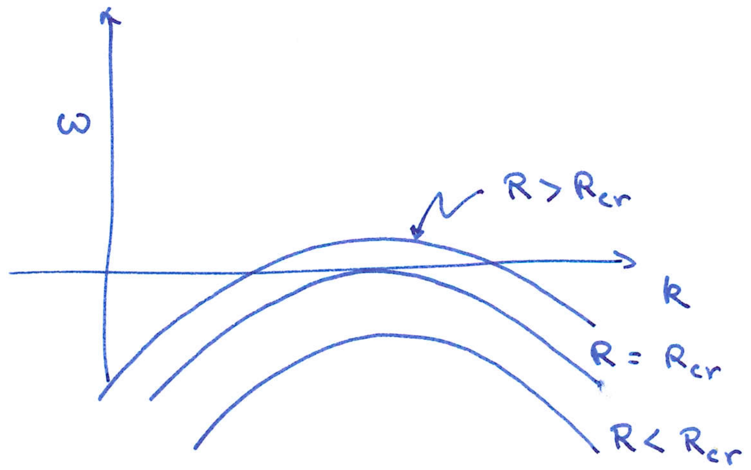
$$\frac{d}{dt}(A A^*) = A^* \frac{dA}{dt} + A \frac{dA^*}{dt}$$

$$= e^{\gamma t}\, \cancel{e^{i\omega_1 t}}\, (\gamma - i\omega_1)\, e^{\gamma t - \cancel{i\omega_1 t}} + \text{c.c.}$$

$$= 2\gamma\, (A A^*)$$

This is merely a restatement of the fact that the amplitude of the eigenfunctions are complex and their magnitude grows exponentially.

Now consider the problem near the critical point



Near $R = R_{cr}$ there are only a few (maybe even one) unstable mode that grows expontically with time with a rate γ and oscillates with a frequency ω_1. So near this point we can ignore all the other possible modes. So our dynamical equation becomes

$$\frac{d|A|^2}{dt} = 2\gamma |A|^2 + \left(\begin{array}{l} \text{functions of} \\ |A|^2 \text{ that} \\ \text{makes } |A|^2 \\ \text{saturate} \end{array} \right)$$

Next possible term:

$$A^2 A^* \quad , \quad (A^*)^2 A$$

* They are not allowed by two arguments:

(i) they contain terms $e^{-i\omega_1 t}$ which when averaged over time scales longer than $1/\omega_1$ becomes zero.

(ii) they are not real.

So the first non-zero term is

$$\boxed{\frac{d}{dt}|A|^2 = 2\gamma |A|^2 \mp \mu |A|^4}$$

First example of an amplitude equation.

- Consequences:

(a) The instability saturates when $\frac{d}{dt}|A|^2 = 0$

$$\Rightarrow \quad 2\gamma |A|^2 = + \mu |A|^4$$

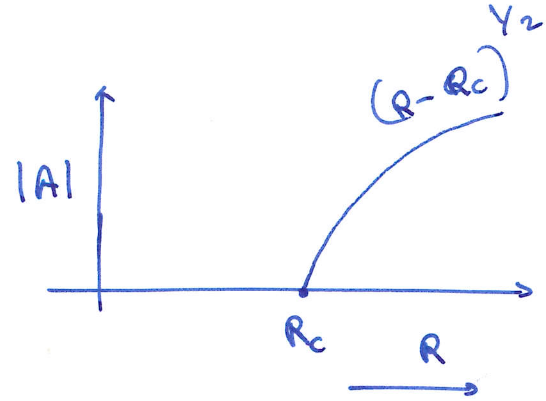
$$\Rightarrow \quad |A|^2 = \left(\frac{2\gamma}{\mu}\right)$$

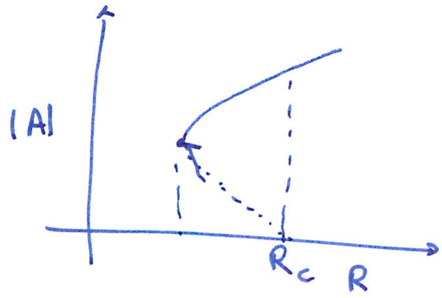
Note that γ is also a function of $R - R_c$ and goes to zero at $R = R_c$.

$$\Rightarrow \quad \gamma \sim (R - R_c) \quad \text{for } R - R_c \text{ small}$$

$$\Rightarrow \quad |A|^2 \sim R - R_c$$

$$\boxed{|A| \sim (R - R_c)^{1/2}}$$





very similar to Landau's theory of phase transition.

- If $\mu < 0$ Then $|A|$ will grow fast and very soon the amplitude eqn will not remain applicable.

But we can still apply the eqn. to study fluctuations below $R = R_c$.

Then γ is negative; but

$$\frac{d|A|^2}{dt} = 2\gamma\, |A|^2 - \mu\, |A|^4$$

can become positive for large enough $|A|$.

$\Rightarrow$ The motion will become unstable even for $R < R_{cr}$ but not for infinitismal ~~ampl~~ perturbations but for finite perturbations for which

$$\mu\, |A|^4 > 2\gamma\, |A|^2$$

$$\Rightarrow \quad |A|^2 > \frac{2\gamma}{\mu}$$

$$\Rightarrow \quad A > \left(\frac{2\gamma}{\mu}\right)^{1/2}$$

- The above arguments apply only when the instability selects a few unstable mode.
  If there are whole ranges of unstable modes then we ~~i~~ immediately land up in a more complicated situation.

- Furthermore, e.g for shear flow ~~t~~ instabilities, it is not clear how this mechanism may work, as the linear theory does not give any instability in many cases.

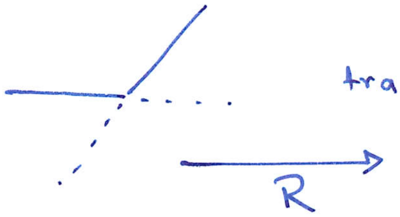
- γ can be calculated from linearized equations but not μ. How do we obtain μ from the dynamical equations ?. We have to average our equations of ω, and write an effective theory. This has been done in certain cases by using the method of ~~small~~ multiple scales.

Bifurcations

- $\partial_t A = R - A^2$

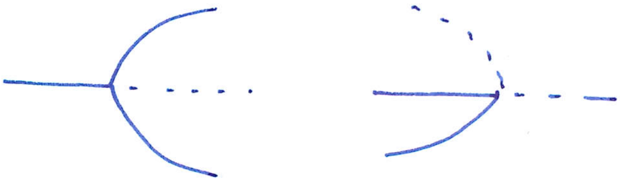
saddle node

- $\partial_t A = RA - A^2$



transcritical.

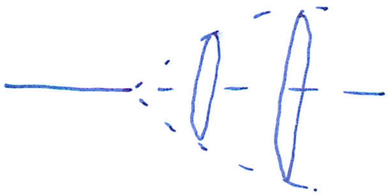
- $\partial_t A = RA - g A^3$



pitchfork

super critical   subcritical.

- $$\partial_t A_1 = -A_2 + RA_1 - (A_1^2 + A_2^2) A_1$$
$$\partial_t A_2 = -A_1 + RA_2 - (A_1^2 + A_2^2) A_2$$



Hopf bifercation.

Nonlinear stability

If ~~the~~ a point in the neighbourhood of a fixed point remains close to it ~~to~~ for all times then ~~the point is stable~~. fixed point in nonlineary stable. If the point asymtotically tends to the fixed point then it is asymptotically stable.