EDQNM

$$\left[\partial_t + \nu(k_1^2 + k_2^2)\right]\langle \hat{u}(k_1)\,\hat{u}(k_2)\rangle = \langle \hat{u}\,\hat{u}\,\hat{u}\rangle$$

$$\left[\partial_t + \nu\left(k_1^2 + k_2^2 + k_3^2\right)\right]\langle \hat{u}(k_1)\,\hat{u}(k_2)\,\hat{u}(k_3)\rangle$$

$$= \langle \hat{u}\,\hat{u}\,\hat{u}\,\hat{u}\rangle$$

$$= \sum (\quad)\langle \hat{u}\,\hat{u}\rangle\langle \hat{u}\,\hat{u}\rangle$$

combinatorial factor

Assuming the PDF to be Gaussian. Quasi-Normal approximation

$\Rightarrow$ $\hat{u}(k_1)\,\hat{u}(k_2)\,\hat{u}(k_3)$

Notation

$$\left[\partial_t + \nu(k_1^2 + k_2^2)\right]\hat{U}_2(k_1, k_2) = \hat{U}_3(k_1, k_2; k_3)$$

$$\left[\partial_t + \nu(k_1^2 + k_2^2 + k_3^2)\right]\hat{U}_3(k_1, k_2, k_3)$$

$$= \hat{U}_4(k_1, k_2, k_3, k_4)$$

$$= \sum \hat{U}_2(k_1, k_2)\,U_2(k_3, k_4) + (\text{permutation})$$

$$\Rightarrow \quad \hat{U}_3(k_1, k_2, k_3) = \int \left[\hat{U}_2(k_1, k_2)\,\hat{U}_2(k_3, k_4) + \text{permutations}\right] e^{-\nu(k_1^2 + k_2^2 + k_3^2)\tau}\, d\tau$$

$$\Rightarrow \left[\partial_t + \nu(k_1^2+k_2^2)\right] \hat{U}_2(k_1, k_2)$$

$$= \int d\tau \; e^{-\nu\left(k_1^2+k_2^2+k_3^2\right)\tau} \; \delta(k_1+k_2-k_3)$$

$$\left[\hat{U}(k_1, k_2)\,\hat{U}(k_3, k_4) + \text{permutation}\right]$$

A closed equation at ~~second~~ third order. This is an example of closure.

Integrating over angular variables and simplifying one obtains

$$\left[\partial_t + 2\nu k^2\right] E(k, t) = \int_0^t d\tau \int e^{-\nu\left(k_1^2+k_2^2+k_3^2\right)(t-\tau)} S(k_1, k_2, k_3)\, dk_1\, dk_2$$

$$S(k, k_2, k_3) = \frac{k^3}{k_2 k_3}\, a(k, k_2, k_3)\, E(k_1)\, E(k_2)$$

$$- \text{other quadratic terms in } E$$

This equation can be solved numerically to obtain the spectrum.

It turns out that the numerical solutions have negative energy. So the closure gives unphysical answer.

Solution :

Add an "eddy damping" term

$$\left[\partial_t + \nu (k_1^2 + k_2^2 + k_3^2) + \mu_{k_1 k_2 k_3} \right] \hat{U}(k_1, k_2, k_3)$$

$$= \sum_{\text{permutations}} \hat{U}(k_1, k_2) \hat{U}(k_2, k_3)$$

$$\mu_{k_1 k_2 k_3} = \mu_{k_1} + \mu_{k_2} + \mu_{k_3}$$

$$\mu_q \simeq \left[q^3 E(q) \right]^{1/2}$$ for isotropic case.

But the positiveness of energy spectra is still not guaranteed.

Solution : Markovization :

$$\left(\partial_t + 2\nu k^2 \right) E(k, t) = \int \theta_{kpq} \sum \hat{U} \hat{U} \, dp \, dq$$

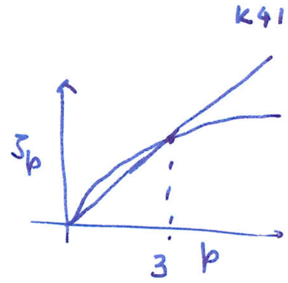
$$\theta_{kpq} = \int_0^t e^{-\mu_{kpq} + \nu (k^2 + p^2 + q^2)(t - \tau)} \, d\tau$$

μ_{kpq} $\downarrow$ not a function of time any more.

The problem of turbulence

- summary of known results: $S_3 = -\frac{4}{5}\varepsilon l$

  $S_p(l) \sim l^{\zeta_p}$



- How do we make a theory of intermittency?

Such a theory should start from the Navier-Stokes equation and give us the exponents ζ_p. This is the problem of turbulence.

- what pieces do we know?

(a) At small scales $S_p(l) \sim l^p$. i.e the structure functions are Taylor expandable.

$\Rightarrow$ At large scales

$$E(k) \sim e^{-(\delta k)^2} \text{ at large } k$$

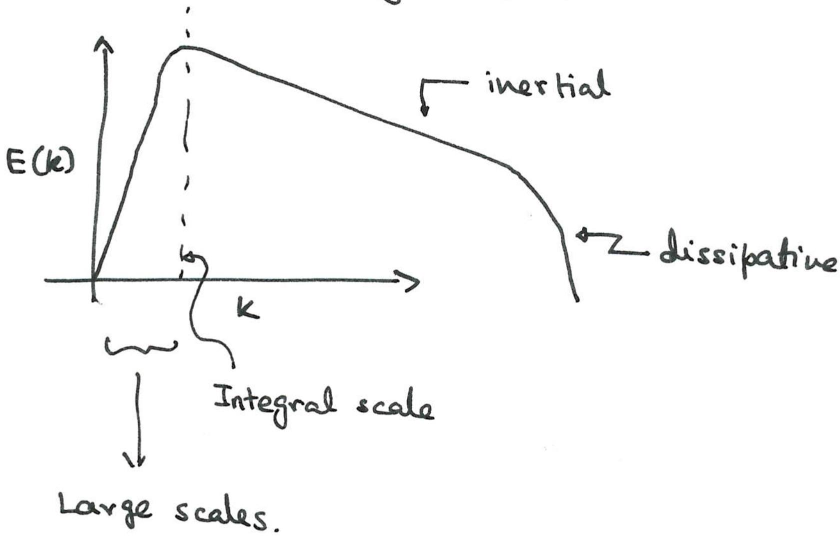
where δ is the distance to the nearest singularity in the complex plane.

(b) In a model of passive scalar, — a model that is linear but stochastic — one can carry out the program of calculation of the anomalous exponents.

- How is this related to the famous problem of singularities of the Navier-Stokes eqn?

Not in a very obvious way. The intermittency does not imply singular structures but singular behaviour on "average".

Large scale turbulent dynamos :



Question :
How can we extract large scale patterns from turbulence?

Answer By averaging the equations of MHD to write an effective equation for large scale behaviour. Such equations may be ~~more~~ nastier than the MHD equations themselves and may have to be solved numerically.

- ~~Ave~~
- Reynold's Averaging :

$$\overline{U_1+U_2} = \overline{U}_1+\overline{U}_2 \quad ; \quad \overline{\overline{U}} = \overline{U}$$

$$\overline{\partial U/\partial t} = \partial \overline{U}/\partial t \quad , \quad \overline{\nabla U} = \nabla \overline{U}$$

$$\overline{u} = 0 . \quad \overline{b} = 0$$

~~They~~ ' The Reynold's rules are satisfied by ~~the~~ averaging over one or more coordinate direction but not Fourier filtering. To see how it works let us apply this to the induction equation

$$\partial_t B = \nabla \times (U \times B) + \nabla \times (-\eta J)$$

$$B = \bar{B} + b \qquad \Rightarrow \qquad J = \bar{J} + j$$

$$U = \bar{U} + u$$

with $\bar{J} = \nabla \times \bar{B}$

and $j = \nabla \times b$

Average the whole eqn:

$$\partial_t \bar{B} = \nabla \times \overline{U \times B} + \nabla \times (-\eta \bar{J})$$

$$U \times B = (\bar{U} + u) \times (\bar{B} + b)$$

$$= \bar{U} \times \bar{B} + u \times \bar{B} + \bar{U} \times b + u \times b$$

$$\Rightarrow \quad \overline{U \times B} = \bar{U} \times \bar{B} + \overline{u \times b}$$

$$\Rightarrow \quad \underbrace{\partial_t \overline{B} = \nabla \times \left(\overline{U} \times \overline{B} - \eta \overline{J} \right)}_{\text{closed}} + \underbrace{\nabla \times \overline{u \times b}}_{\text{not closed.}}$$

$$\overline{\mathcal{E}} \equiv \underbrace{\overline{u \times b}}$$

A term we need to model.

We demand closure; i.e.

$$\mathcal{E}_i = \alpha_{ij} \overline{B}_j + \beta_{ijk} \partial_j \overline{B}_k + \left(\text{higher order terms} \right)$$

when $\overline{U} = 0$.

Otherwise there would be other coefficients multiplying $\overline{U}$. α_{ij}, β_{ijk} are "turbulent transport coefficients."

- How to calculate the transport coefficients?

$$\partial_t b = \nabla \times \left(\underbrace{\overline{U} \times b}_{\substack{\text{assume } \overline{U}=0 \\ \text{(simple case)}}} + u \times \overline{B} + u \times b - \overline{\mathcal{E}} - \eta j \right)$$

Now calculate :

$$\partial_t \overline{\mathcal{E}} = \partial_t \overline{u \times b}$$
$$= \overline{\partial_t u \times b} + \overline{u \times \partial_t b}$$

First work in the kinematic approximation:

$$\partial_t U = -U \cdot \nabla U - \nabla p + F_{visc} + f + \underbrace{(J \times B)}_{\text{ignored.}}$$

obtain : $\partial_t u = \partial_t \overline{U} - \partial_t U$

After substitution and simplifications :

$$\partial_t \overline{\mathcal{E}} = \overline{u \times \nabla \times (u \times \overline{B})} + \left(\begin{array}{c}\text{triple}\\ \text{correlations}\end{array}\right)$$

Note that terms like

$$\overline{u \times \nabla \times \overline{\mathcal{E}}} = \overline{u} \times \overline{\nabla \times \overline{\mathcal{E}}} = 0 \quad (\text{as } \overline{u} = 0)$$

on simplification :

$$\partial_t \overline{\mathcal{E}} = \tilde{\alpha} \overline{B} - \overline{\eta}_t \overline{J} - \underbrace{\frac{\overline{\mathcal{E}}}{\tau}}_{\text{effect of all the triple correlations}}$$

The last term is a closure hypothesis.

Next we assume $\partial_t \overline{\mathcal{E}} = 0$ is in the stationary state. This assumption is also tricky.

where we obtain:

$$\left.\begin{aligned} \tilde{\alpha} &= -\frac{1}{3}\,\overline{\omega\cdot u} \\ \tilde{\eta}_t &= \frac{1}{3}\,\overline{u^2} \end{aligned}\right\}$$ turbulent transport coefficients.

<u>Exercise</u>: show this.

- <u>comments</u>

- For a non-zero $\tilde{\alpha}$

$$\partial_t \bar{B} = \nabla \times \left(\tilde{\alpha} B - \eta_t \bar{J} \right)$$

we can obtain exponential growth of $\bar{B}$.

This is the dynamo effect.

- But $\tilde{\alpha}$ is non-zero only when kinetic helicity $\mathcal{H} = \overline{\omega \cdot u}$ is non-zero.

We need helical flows to generate large scale magnetic fields.

There is another way to derive the same result:

Start with
$$\partial_t b = \nabla\times\left(u\times B - u\times b - \overline{\mathcal{E}} - \eta j\right)$$
$$\simeq \nabla\times(u\times B)$$

ignore the others because:

(a) $\overline{u\times b}$ is one order higher in fluctuations

(b) ~~so~~ Lator $\overline{u\times\overline{\mathcal{E}}} = 0$

(c) η is very small.

Then
$$b(t) = \int_0^t \nabla\times(u\times B)\, d\tau$$

$$\Rightarrow \quad \overline{\mathcal{E}} = \overline{u\times b}$$
$$= \int_0^t \overline{u\times\nabla\times(u\times B)}\, d\tau$$
$$= \alpha B - \eta_t J$$

This is called the first order smoothing approximation and the earlier one is called the minimal tau approximation. They are not exactly the same. But quite similar.

Mean field theory in general:

$$\partial_t(\rho U) + \mathrm{div}\left(\rho U_i U_j - \delta_{ij} p + \sigma_{ij} - \delta_{ij}\frac{B^2}{\mu_0} + \frac{B_i B_j}{\mu_0}\right) = 0$$

• How does one do Reynold's averaging with density:

$$\overline{\rho U} = \bar{\rho}\,\bar{U} \;?$$

For the moment consider incompressible flows, $\Rightarrow$ $\rho = \mathrm{const}$

Then

$$\partial_t \overline{\rho U} + \mathrm{div}\left(\rho\,\overline{U_i U_j} - \delta_{ij}\bar{p} + \bar{\sigma}_{ij} - \delta_{ij}\frac{\overline{B^2}}{\mu} + \frac{\overline{B_i B_j}}{\mu_0}\right) = 0$$

$\overline{U_i U_j}$ $\equiv$ Reynold's stress

$\frac{1}{\mu_0}\overline{B_i B_j}$ $\simeq$ Maxwell's stress.

In the sense of closure:

$$\overline{U_i U_j} = (\;)\,\bar{U} + (\;)\,\nabla U + \left(\text{higher order term}\right)$$

Can be a problem because can violate Galilean invariance.

turbulent diffusivity.

$$= \eta^t_{ijkl}\,\partial_k \overline{U_l}$$

- The heart of the problem is to calculate the turbulent-transport coefficient, α_{ij}, η^t_{ij}, ν^t_{ijkl} and many others. There is no systematic way to calculate them at high Reynold's number. They can either be calculated analytically by uncontrolled closure. Or calculated numerically.