Kolmogorov's theory of turbulence

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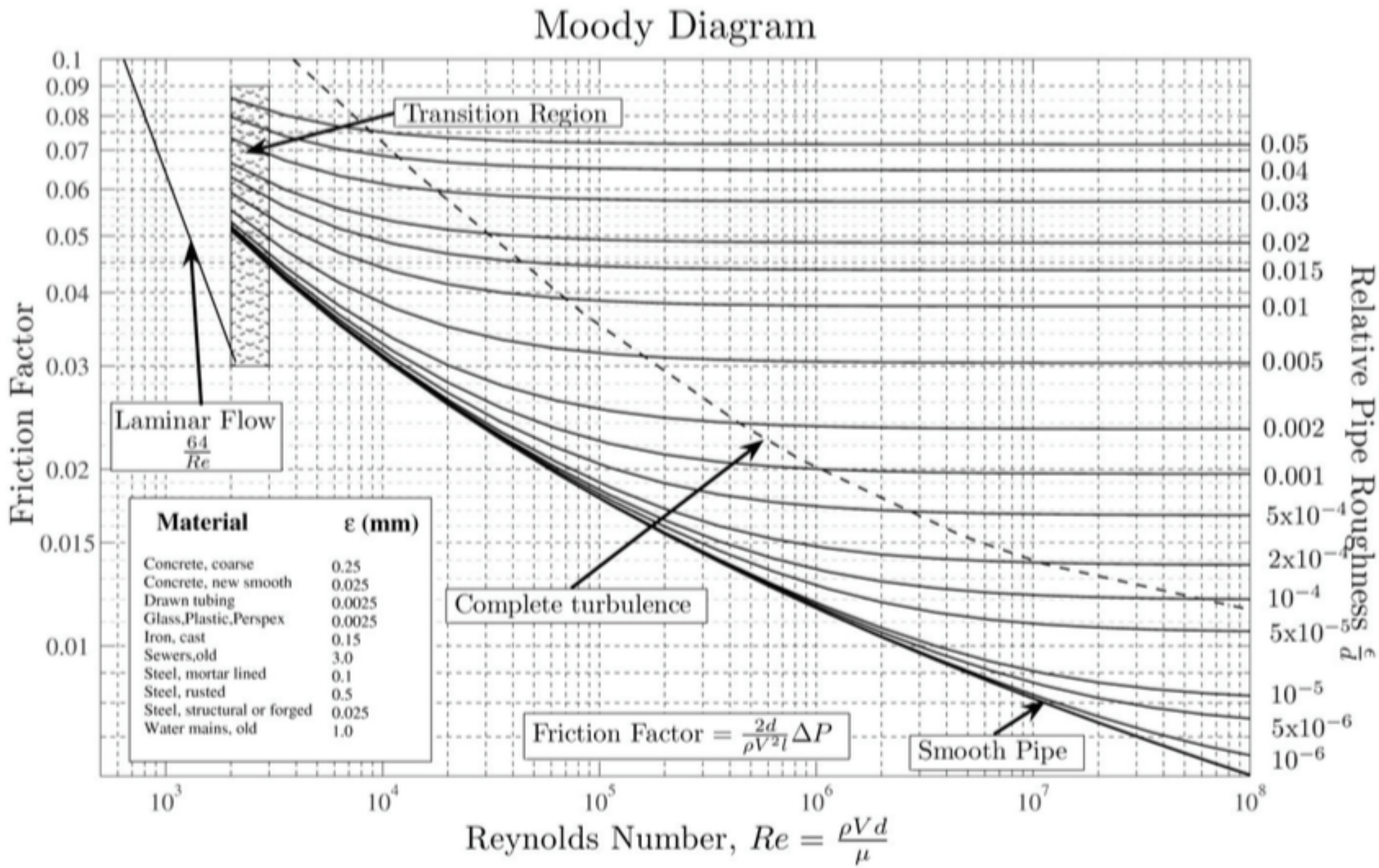
The energy dissipation law

$$\lim_{\nu \to 0} \varepsilon \equiv \lim_{\nu \to 0} \nu \langle \omega^2 \rangle \to \text{constant}$$

- In the limit of vanishing viscosity, or infinite Reynolds number the mean energy dissipation rate becomes a constant.
- Vorticity develops finer and finer structures as viscosity goes to zero.

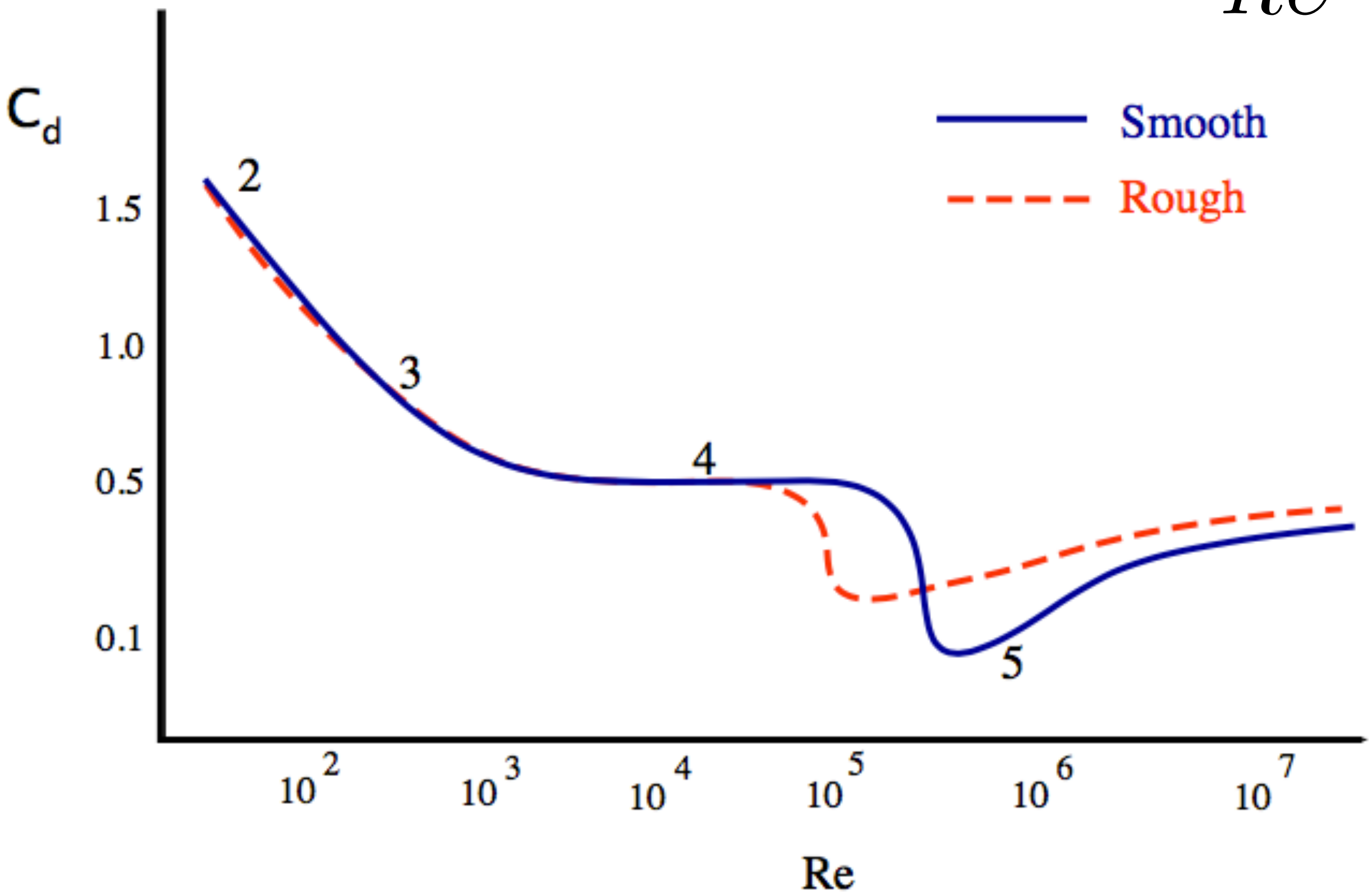
Friction factor for pipes

$$\Delta P = f_{\mathrm{D}} \frac{\rho U^2}{2} \frac{L}{D}$$



Drag law for smooth spheres

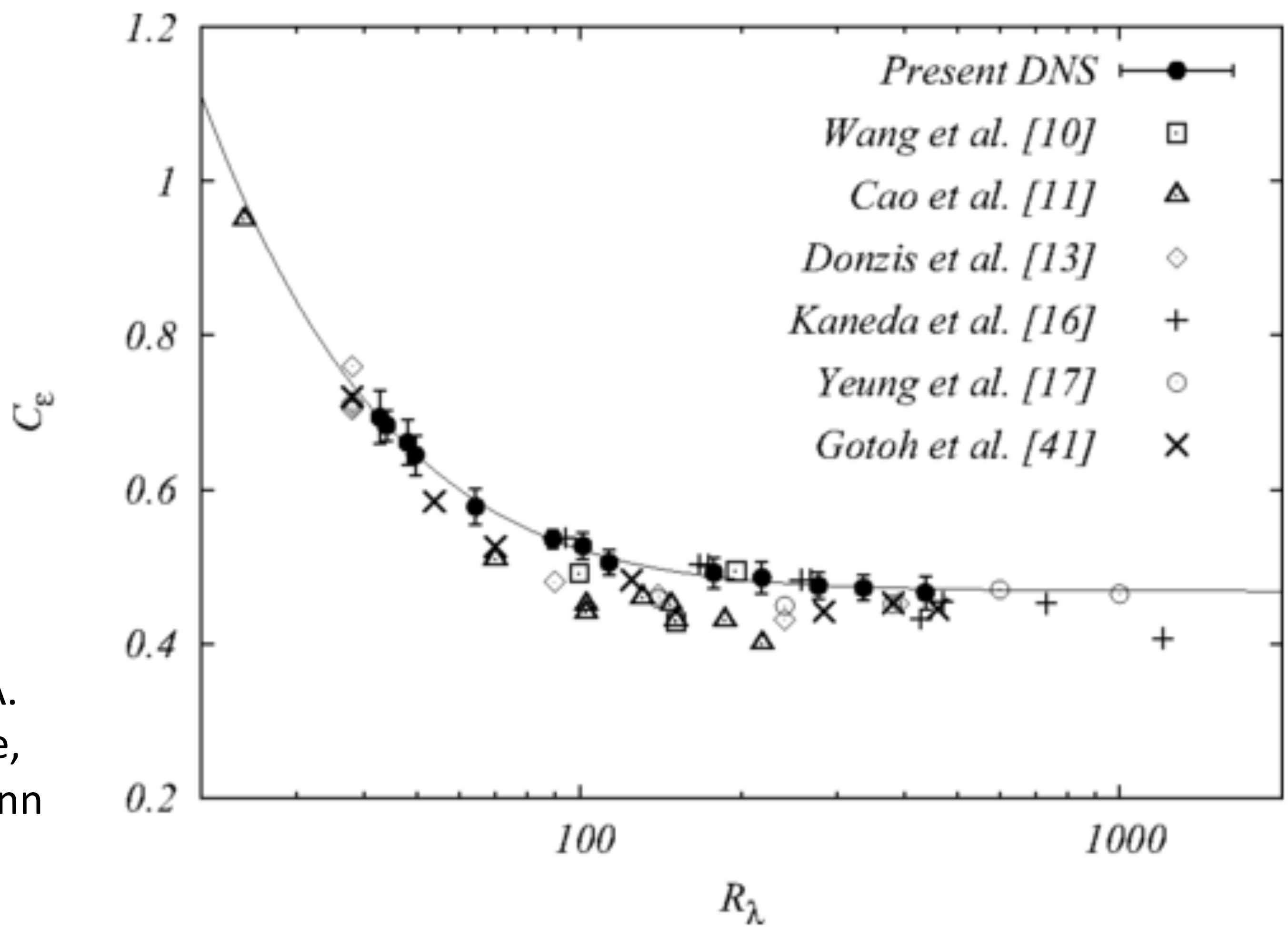
$$F_{\mathrm{drag}} \equiv C_{\mathrm{D}} \rho A U^2 \qquad C_{\mathrm{D}}^{\mathrm{Stokes}} \equiv 6\frac{\nu}{RU} \equiv \frac{6}{\mathrm{Re}}$$



By NASA - http://www.grc.nasa.gov/WWW/k-12/airplane/dragsphere.html

Simulations of HIT

$$\varepsilon \equiv C_\varepsilon \frac{U^3}{L}$$



W. D. McComb, A. Berera, S. R. Yoffe, and M. F. Linkmann Phys. Rev. E **91**, 043013, 2015

Kolmogorov's theory

- Correlation function and Structure functions.
- Inertial range.
- Dimensional argument.

$$\delta v(\ell) \sim (\varepsilon \ell)^{1/3}$$

- This implies that energy spectrum as a five-third law (After shell averaging)

$$E(k) \sim k^{-5/3}$$

Kolmogorov's theory

$$\partial_t u_\alpha + (u_\beta \partial_\beta) u_\alpha = \nu \partial_{\beta\beta} u_\alpha - \partial_\alpha p$$

$$\partial_\beta u_\beta = 0$$

It is useful to think in Fourier space:

$$u_\alpha(x) = \int \hat{u}_\alpha(k) e^{ik\cdot x} dk$$

$$\partial_\beta u_\alpha(x) = \int i k_\beta \hat{u}_\alpha(k) e^{ik\cdot x} dk$$

$$u_\beta \partial_\beta u_\alpha = \partial_\beta (u_\alpha u_\beta)$$

$$u_\alpha(x) = \int e^{ipx} \hat{u}(p) \, dp$$

$$u_\beta(x) = \int e^{iqx} \hat{u}(q) dq$$

$$\widehat{u_\beta \partial_\beta u_\alpha} = \int e^{-ikx} \partial_\beta (u_\alpha u_\beta) \, dx$$

Let us worry about the ∂_β later but look at the product first:

$$\widehat{u_\alpha u_\beta}(k) = \int u_\alpha(x) u_\beta(x) e^{-ikx} dx$$

$$= \int \hat{u}_\alpha(p) \hat{u}_\beta(q) e^{-i(k-p-q)} dx\, dp\, dq$$

$$= \int u_\alpha(p) u_\beta(q) \, dp\, dq \; \delta(k-p-q)$$

$$\widehat{\partial_\beta(u_\alpha u_\beta)} = i k_\beta \int u_\alpha(p) u_\beta(q) \, dp \, dq \, \delta(k-p-q)$$

Incompressibility can be imposed by a projection operator.

$$\cancel{P_{ij}} \quad P_{\alpha\beta}(k) = \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}$$

consider a vector function $u_\alpha(k)$, let $\cancel{v_\beta(x) =}$

$$v_\alpha(k) = P_{\alpha\beta}(k) \, u_\beta(k)$$

Then $\cancel{k_\alpha u_\alpha}$

$$k_\alpha v_\alpha(k) = k_\alpha P_{\alpha\beta}(k) \, u_\beta(k)$$

$$= k_\alpha \left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2} \right) u_\beta(k)$$

$$= \left(k_\beta - k_\alpha \frac{k_\alpha k_\beta}{k^2} \right) u_\beta(k) = 0$$

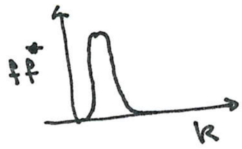
so the incompressible Navier-Stokes eqn. in Fourier space:

$$\boxed{\partial_t \hat{u}_\alpha(k) = P_{\alpha\beta}(k) \, i k_\gamma \int \hat{u}_\beta(p) \, \hat{u}_\gamma(q) \, \delta(p+q-k) \, dp \, dq - \nu k_\beta k_\beta \hat{u}_\alpha(k) + \hat{f}(k)}$$

peaks at high k

limited to small k.

There exists a range of scales (or Fourier modes) where the dynamics is dominated by the Non-linear term.



This is neither very small k, nor very large k, but intermediate k. This is called the inertial range.

Energy flux :

$$U_{ij}(k,t) \equiv \langle \hat{u}_i \hat{u}_j \rangle \leftarrow$$ averaging over the statistically stationary state of turbulence.

- comments on equilibrium, near-equilibrium and non-equilibrium. Non-equilibrium stationary state and general presence of flux. Analogies with heat conduct

$$\cancel{\partial_t U_{\alpha\alpha}(k,t)} =$$

$$\partial_t U_{\alpha\beta}(k,t) = \langle \hat{u}_\alpha \partial_t \hat{u}_\beta \rangle + \langle \hat{u}_\beta \partial_t \hat{u}_\alpha \rangle$$

Ultimately one obtains :

$$(\partial_t + 2\nu k^2)\, E(\vec{k},t) = - P_{\alpha\beta\gamma}(k) \int_{k+p+q=0} \mathcal{I}m \left[\langle \hat{u}_\alpha(k) u_\beta(p) u_\gamma(q) \rangle \right]$$

where $\quad P_{\alpha\beta\gamma}(k) = k_\alpha P_{\beta\gamma}(k) + k_\beta P_{\alpha\gamma}(k)$

The proof is left ~~an~~ as an exercise.

Hint : Use the dynamical equations, ~~than~~ but for the quantity $\quad U_{\alpha\beta}(k,k') = \langle \hat{u}_\alpha(k) \hat{u}_\beta(k') \rangle$. Then

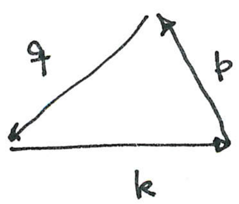
integrate over k' and take the trace.

Remember that $\quad E(k,t) = \langle \hat{u}(k)\, \hat{u}^*(k) \rangle$

and $\quad \hat{u}^*(k) = \hat{u}(-k)$; because $u(x)$ is real.

Triads of interaction

$$s(k,p,q) + s(k,q,p) + s(p,k,q) = 0$$



Each such triad conserves energy.

Energy conservation scale by scale:

$$\left(\partial_t + 2\nu k^2\right) E = \int s(k,p,q)\, \delta(k+p+q)\, dp\, dq$$

$$s(k,p,q) = s(k,q,p)$$

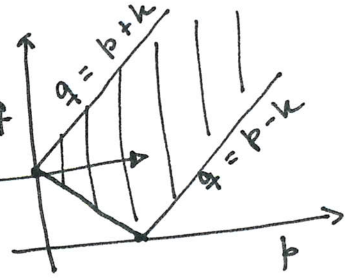
$$E(k,t) = E(\vec{k},t)\, 4\pi k^2$$

$$\Rightarrow \quad \left(\partial_t + 2\nu k^2\right) E = T(k,t)$$

with $\displaystyle T(k,t) = \int_{k+p+q=0} 2\pi k^2\, s(k,p,q)\, dp\, dq$

energy transfer

domain of integration.



~~The flux of energy through wave number k due to the non-linear term~~

~~$\partial_t \Pi(k,t) = T$~~

The energy contained in scales upto K

$$\mathcal{E}_K = \int_0^K E(k)\, dk$$

$$\partial_t \mathcal{E}_K = \int_0^K \partial_t E(k)\, dk$$

$$= -2\nu \int_0^K k^2 E(k)\, dk + \int_0^K T(k)\, dk$$

For a fixed K as $\nu \to 0$, $\quad \nu \int_0^K k^2 E(k)\, dk \to 0$

~~For~~ **Proof** :

~~As~~

$$\int_0^K k^2 E(k)\, dk < K^2 \underbrace{\int_0^K E(k)\, dk}_{\text{always finite}}$$

$$\Rightarrow \lim_{\nu \to 0} \partial_t \mathcal{E}_K = \int_0^K T(k)\, dk \equiv -\Pi(K)$$

flux of energy due to non linear term.

$$\cancel{T(k) = -\frac{\partial \Pi(k)}{\partial k}}$$

$$T(k) = -\frac{\partial \Pi(k)}{\partial k}$$

$$\partial_t \mathcal{E}_K = -2\nu \Omega_K - \Pi_K + \mathcal{F}_K$$

if there is an external force

(i) consider stationary state

(ii) $K \to \infty$ for fixed ν

$\Rightarrow \quad \Pi_K \to 0$ because the non-linear term conserves energy.

$$\Rightarrow \quad \underbrace{\mathcal{F}_\infty}_{\text{rate of energy injection}} = \underbrace{2\nu\Omega}_{\text{rate of energy dissipation}} = \varepsilon(\nu)$$

the equality is obvious.

• For a fixed K, with $K > K_{\text{injection}}$ (~~such that~~ such that $\mathcal{F}_K = \mathcal{F}_\infty$)

$$\partial_t \mathcal{E}_K = -2\nu\,\Omega_K - \Pi_K + \varepsilon(\nu)$$

Now consider stationarity and take limit $\nu \to 0$

$$\Rightarrow \quad \boxed{\lim_{\nu\to 0} \Pi_K = \varepsilon}$$

constant (By dissipative anomaly)

In the "turbulent" limit energy flux through K is constant and equal to ε.

Remember

$$\Pi_K = -\int_0^K T(k)\,dk = \int_K^\infty T(k)\,dk$$

$$\left[\text{because} \quad \int_0^\infty T(k)\,dk = 0 \right]$$

$$= \int_0^K \left[\int_{-\infty}^{+\infty} \delta(\vec{k}+\vec{p}+\vec{q})\, 2\pi k^2\, s(k,p,q)\, d\vec{p}\, d\vec{q} \right] dk$$

$$s(k,p,q) = P_{\alpha\beta\gamma}(k)\, \mathrm{Im}\Big[\underbrace{\langle \hat{u}_\alpha(k)\, \hat{u}_\beta(p)\, u_\gamma(q) \rangle}_{\text{a third order quantity.}}$$

Count dimensions in the eqn:

$$\lim_{\nu\to 0} \Pi_K = \varepsilon$$

$$\int_0^K \underbrace{P_{\alpha\beta\gamma}(k)}_{k^1} \int \langle \hat{u}\,\hat{u}\,\hat{u} \rangle\, k^2 \, \underbrace{dp\,dq}_{k^6} \; \underbrace{dk}_{k^1} = \Bigg($$ is only a function of

because everything is integrated over

The third order structure function

$$S_3(\ell) = \langle [\delta u_{\parallel}(\ell)]^3 \rangle$$

$$\sim \tilde{F}FT\left[\langle \hat{u}\,\hat{u}\,\hat{u} \rangle\right]$$

$$\int \underbrace{\langle \hat{u}\,\hat{u}\,\hat{u} \rangle}_{u^3} \underbrace{dp\,dq\,dk}_{k^9}$$

Dimensionally

$$k\, S_3(\ell) \sim \varepsilon \qquad \longrightarrow k \sim K \sim \frac{1}{\ell}$$

$$\Rightarrow \quad \cancel{S_3(\ell)} \sim \cancel{\ell}$$

$$\Rightarrow \quad \boxed{S_3(\ell) \sim \varepsilon \ell.}$$

The constancy of flux in fourier space implies that the third order structure function is propor- to ℓ. This can be made into an exact (the only exact relation) in turbulence

$$\boxed{S_3(\ell) = -\frac{4}{5}\varepsilon \ell}$$ Kolmogorov's 4/5 th law.

negative.

$\Rightarrow$ energy goes from large to small scales small to large k

~~Kolm.~~

Phenomenology

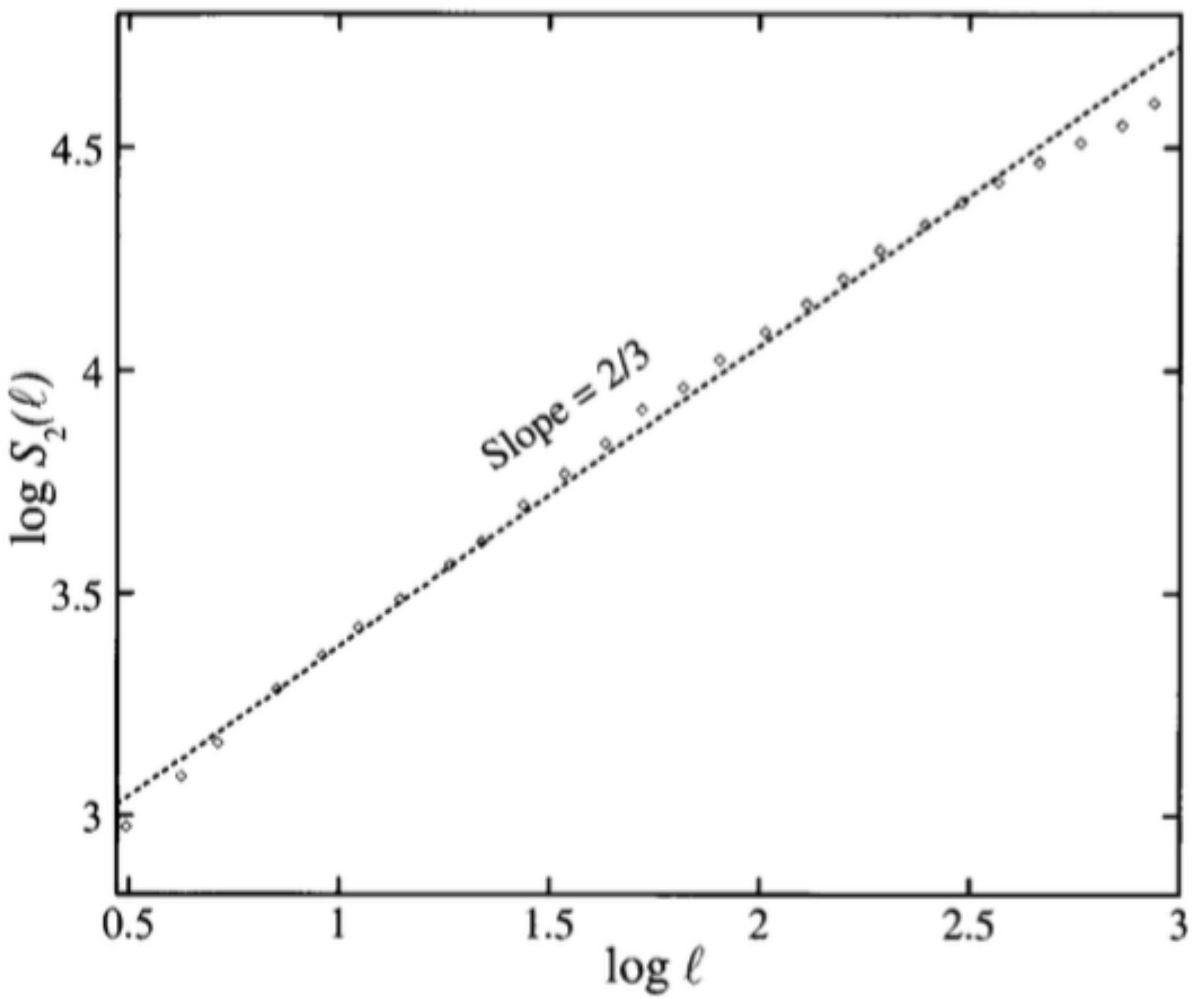
- Dissipation length $\eta \sim \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$
- Taylor microscale
- characteristic time scales, lifetime of eddies.
- Univasality, Kolmogorov constant, Landau's comment.
- Kolmogorov is not Gaussian. (Third order mom is not zero) But ~~all~~ ~~h~~ the scaling of higher moment's are determined by only moment.

Intermittency :

- When is a signal intermittent ?

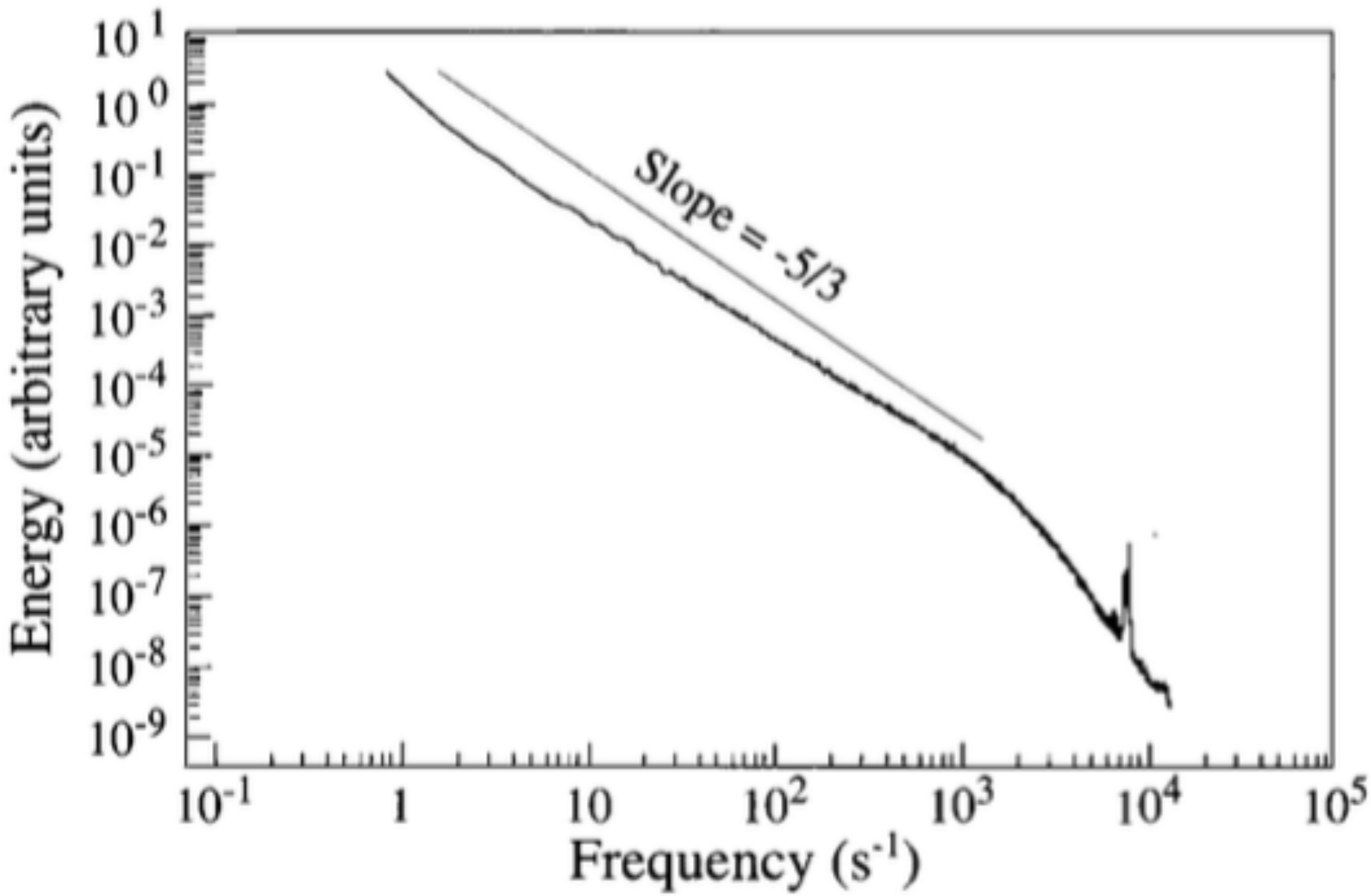
Closure and EDQNM

Experimental evidence



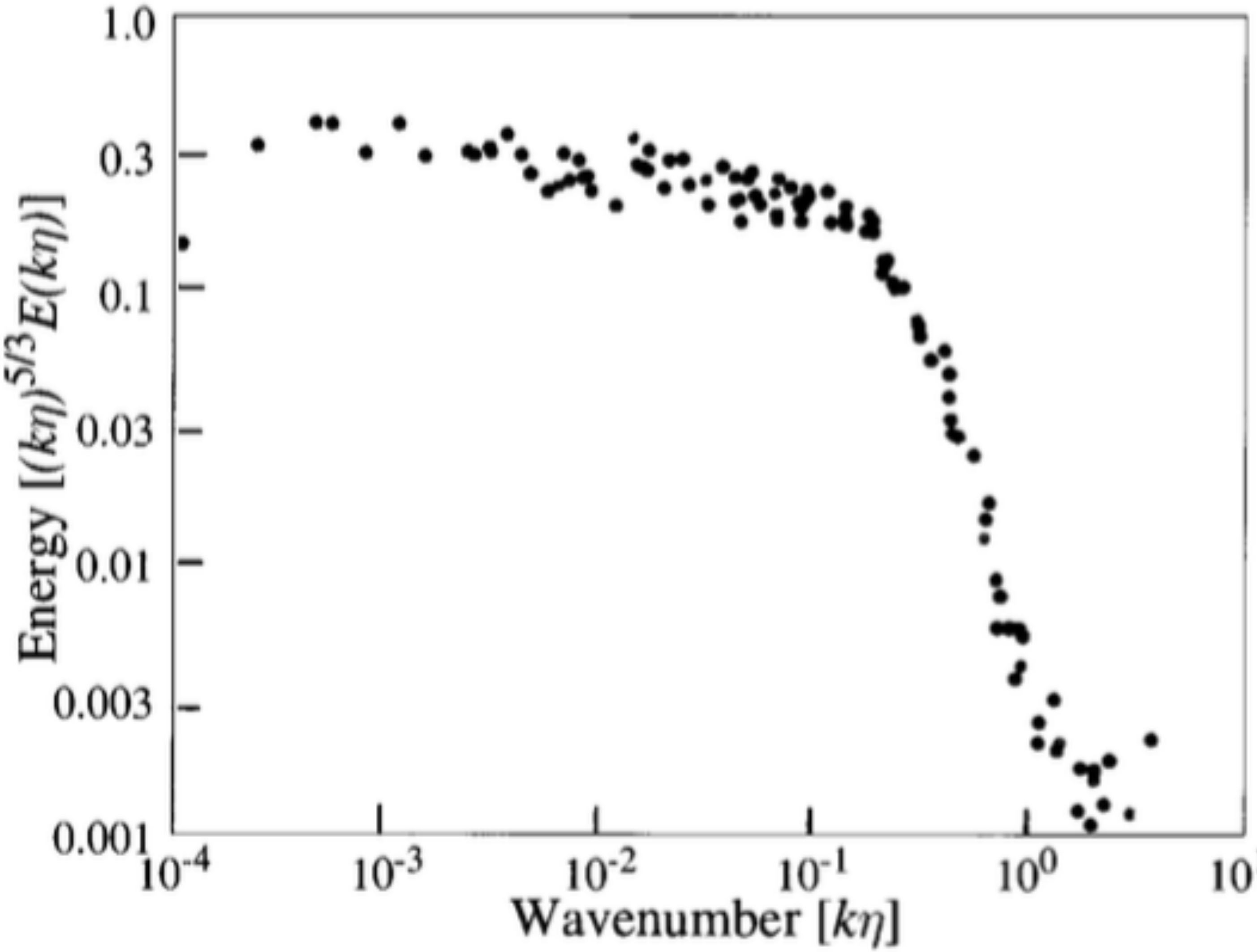
Data from wind tunnel ONERA

Experimental evidence



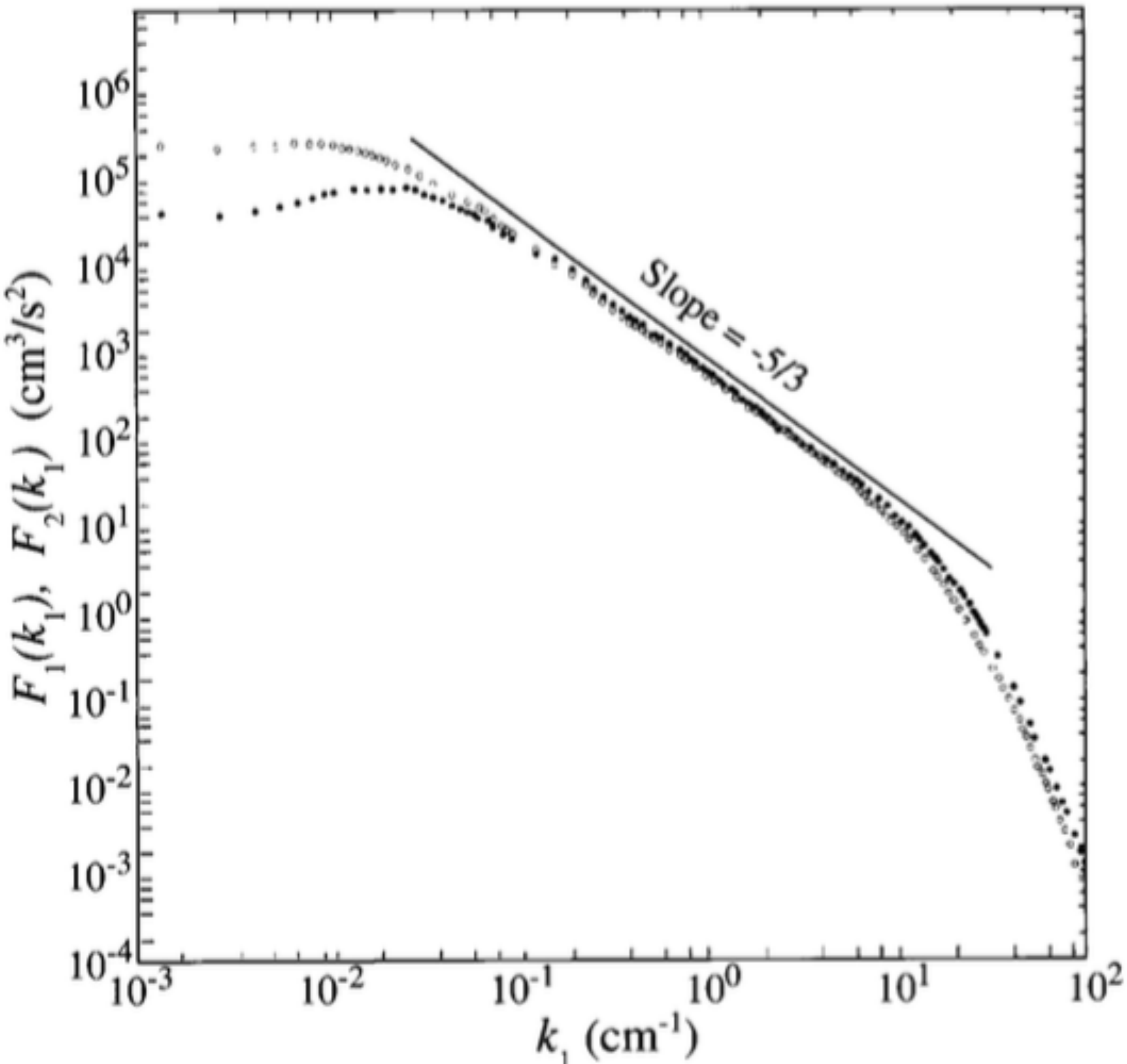
Data from wind tunnel ONERA, energy spectrum

Experimental evidence



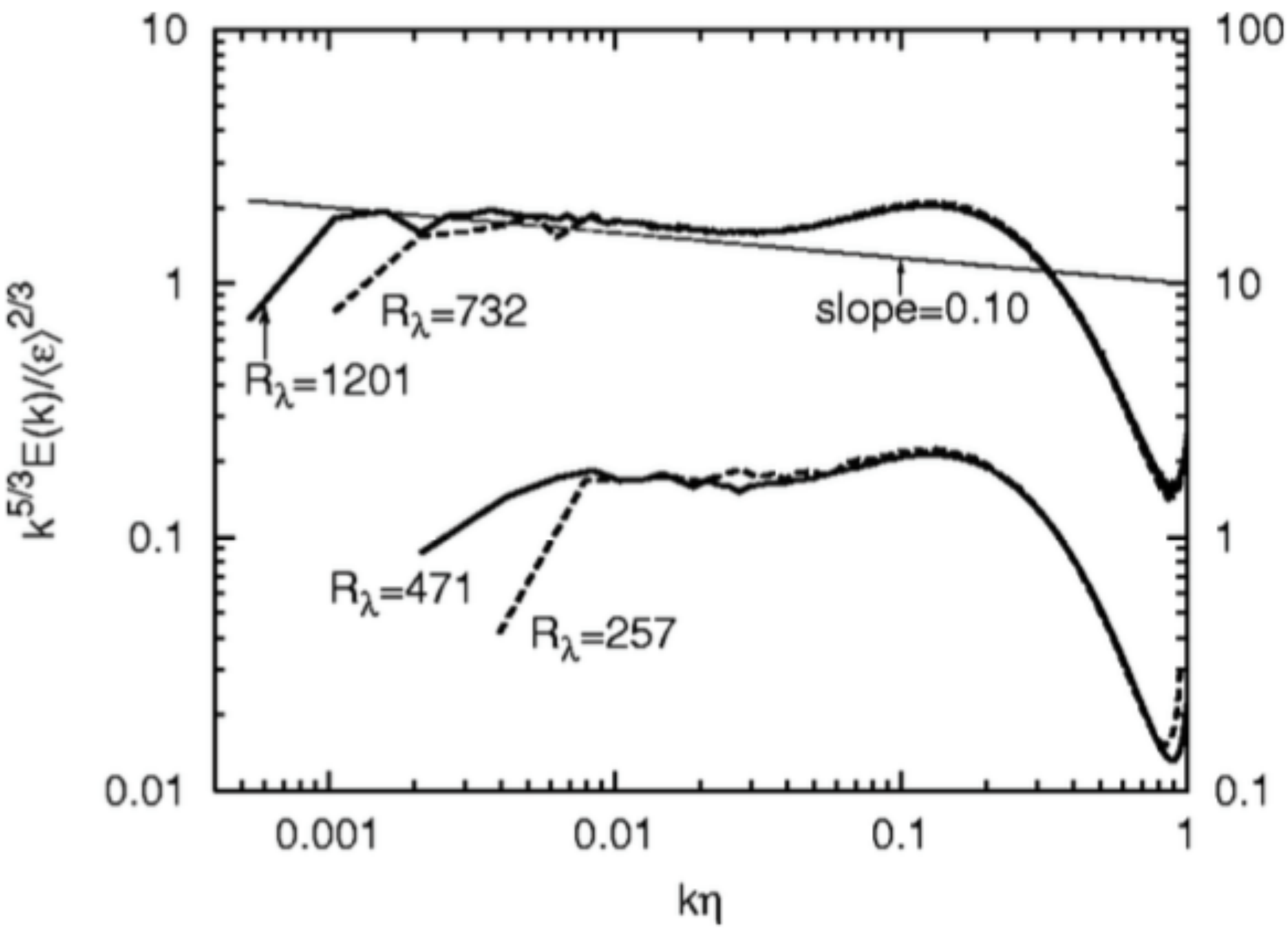
Compensated spectra from tidal channel

Experimental evidence



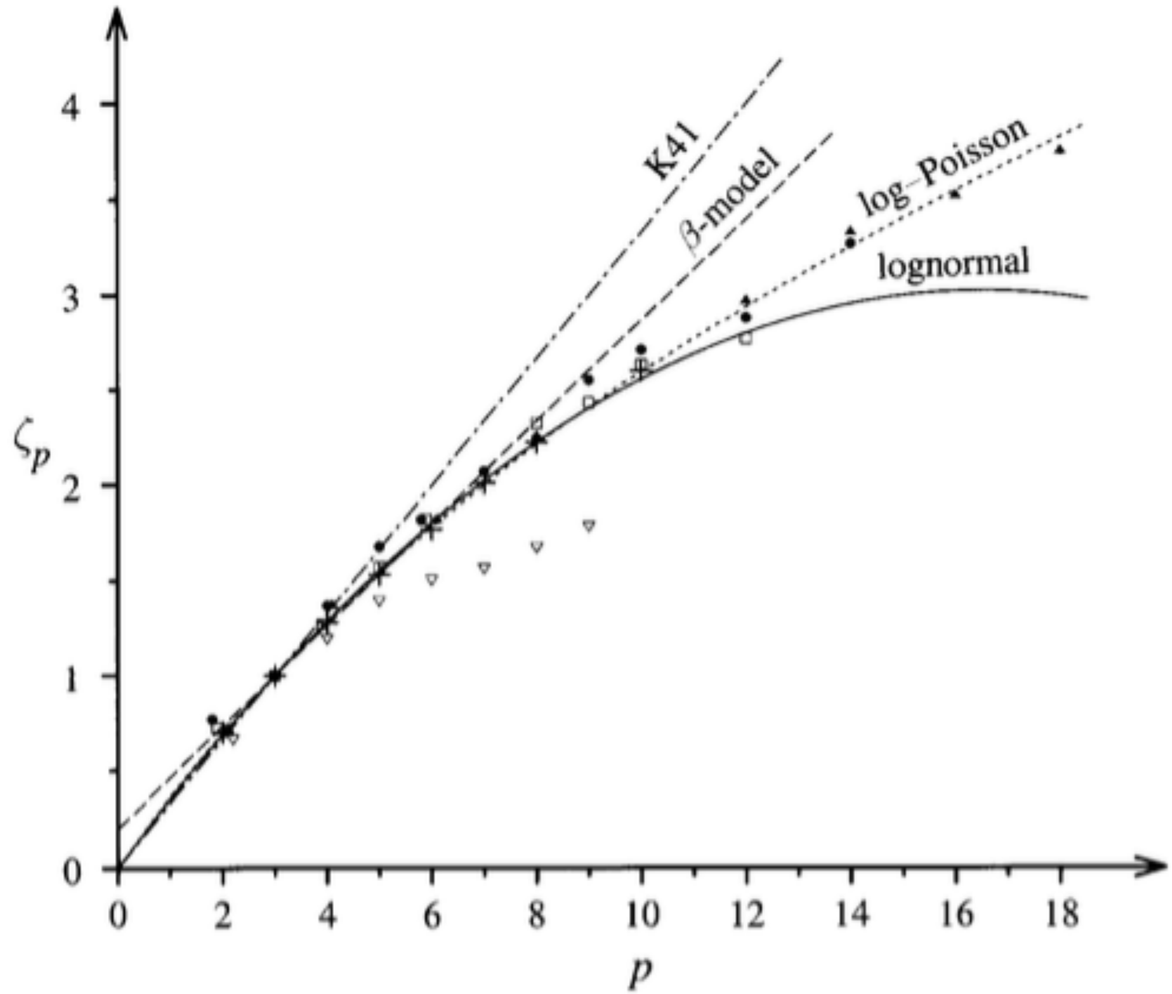
Velocity fluctuations (in time domain) in a jet (Champagne 1978)

Numerical evidence



Biggest numerical simulations so far (4096 cubed), Kaneda et al 2003

Intermittency



Biggest numerical simulations so far (4096 cubed), Kaneda et al 2003

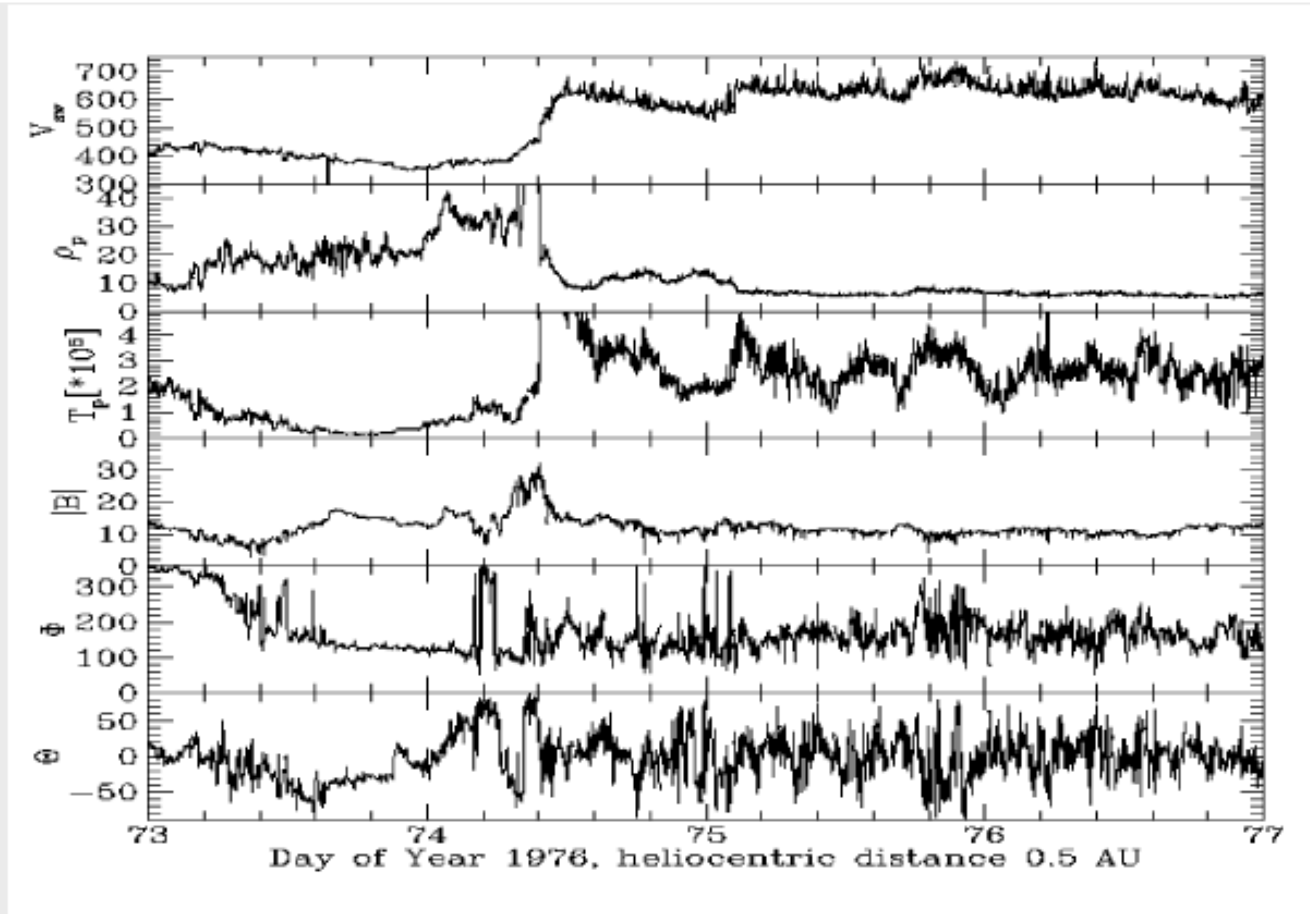
Turbulence in solar wind

- Compressible and with magnetic field. This implies that Kolmogorov's theory, as we have described, does not apply. It must be extended.
- We need new relations to replace the Karman-Howarth-Monin relation.

$$S_3(\ell) = -\frac{4}{5}\varepsilon\ell$$

- This is not trivial. In incompressible MHD this was first worked out by Chandrasekhar. A similar relation was worked out by Politano and Pouquet which is possibly wrong. We shall not get into this at the moment.
- In compressible turbulence (not MHD) an equivalent relation has been worked out by Banerjee and Galtier.

Solar wind turbulence (experiments)



- There is fast and slow wind. The wind is turbulent, and also "intermittent". (wind during solar minima, Bruno and Carbone, living reviews in solar physics)

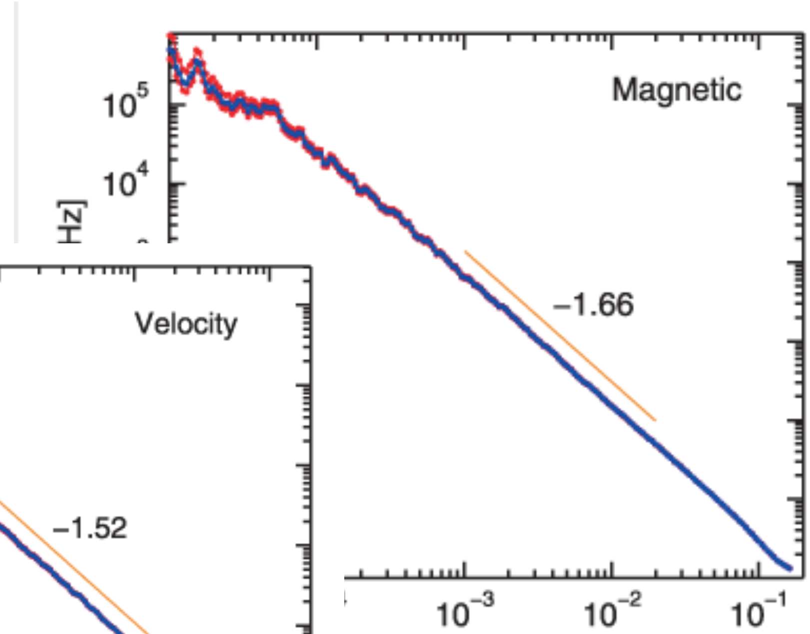
spectra

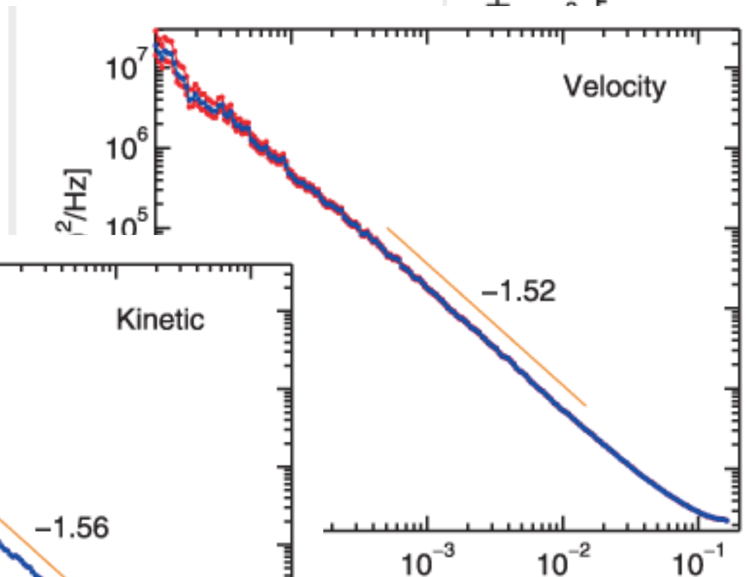
$$E(k) \sim k^{-3/2}$$

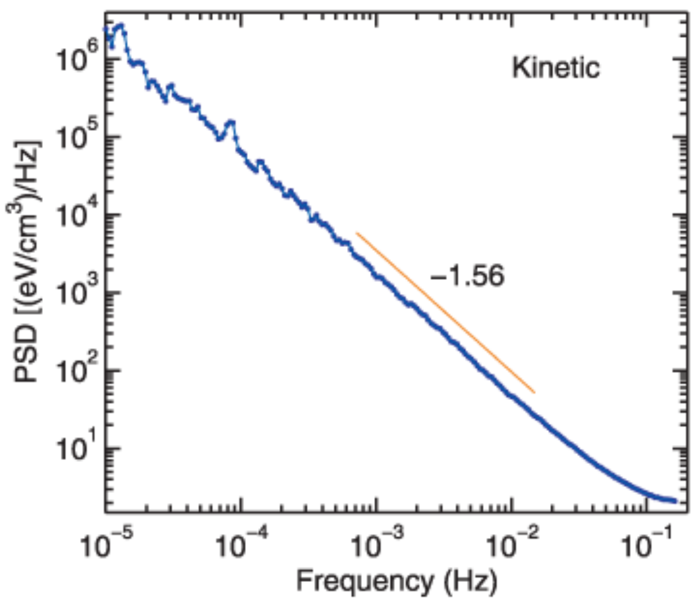
Irishnikov-Kraichnan

$$E(k) \sim k^{-5/3}$$

Kolmogorov







$$E_\perp(k) \sim k_\perp^{-2}$$

- We have no theory of this.

- Modes perpendicular to mean magnetic field.
- Weak turbulence theory, Goldrich-Sridhar scaling.
- Galtier et al, Journal of Plasma Physics, 2000