Lecture 3

- Incompressible hydrodynamics.

The equations of dissipative hydrodynamis that we wrote down are:

continuity eqn $\quad \partial_t \rho + \mathrm{div}(\rho v) = 0$

momentum eqn $\quad \partial_t (\rho v) + \mathrm{div}\left(\rho v_i v_j + p\delta_{ij} + \sigma_{ij}\right) = 0$

with $\quad \sigma_{ij} = \eta\left(\partial_i v_j + \partial_j v_i - \frac{2}{3}\delta_{ij}\partial_k v_k\right) + \zeta\,\delta_{ij}\partial_k v_k$

~~energy~~
entropy eqn

$$\partial_t s + \mathrm{div}\left(v s + \frac{Q}{T}\right) = 0$$

with $\quad Q = -\kappa \nabla T - v_j \sigma_{ij}$

with an equation of state

This makes a complete dynamical theory. The picture becomes a lot simpler if we consider incompressible fluids.

The incompressible approximation:

$$\rho = \text{constant} = 1$$

$$\Rightarrow \quad \nabla \cdot v = 0$$

The dynamical theory ~~dets~~ gets the following simplified form:

$$\partial_t (\rho v) + \mathrm{div}\left(\rho v_i v_j + p\, \delta_{ij} \mp \sigma_{ij}\right) = 0$$

with $$\sigma_{ij} = \eta\left(\partial_i v_j + \partial_j v_i\right)$$

$$\Rightarrow \quad \rho\left[\partial_t v + (v \cdot \nabla) v\right] = -\nabla p + \eta\, \mathrm{div}\left(\partial_i v_j + \partial_j v_i\right)$$

$$\Rightarrow \quad D_t v = -\frac{\nabla p}{\rho} + \frac{\eta}{\rho} \nabla^2 v$$

$\frac{\eta}{\rho} \equiv \nu$ is the kinematic viscosity.

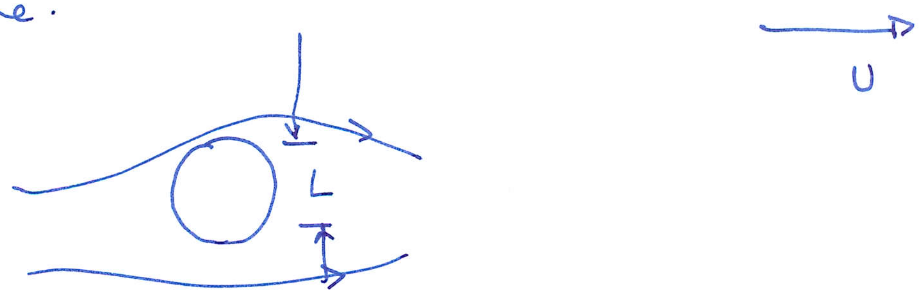
We obtain the famous Navier-Stokes equation

$$\boxed{\begin{aligned} \nabla \cdot v &= 0 \\ \partial_t v + (v \cdot \nabla) v &= -\nabla p + \nu \nabla^2 v \end{aligned}}$$

This makes a complete dynamical theory

3.2 Reynold's number :

consider flow of ~~in~~ incompressible fluid around a sphere.



The boundary conditions are that the velocity is a constant at infinity, and both tangential and vertical component of velocity are zero at the surface of the sphere.

Let us try to non-dimensionalize the governing equations.

velocity : U

length : L

time : $\frac{L}{U}$

$u \longrightarrow U \tilde{u}$

$\tilde{u}$ — non-dimensional velocity.

How should we non-dimensionalize pressure?

~~∇p has the same dimensi~~

Better take a curl of the governing equation

$\cancel{\Omega} = \cancel{\nabla \times} v$ $\quad \omega = \nabla \times v$

$$\partial_t \omega + \nabla \times \left[(v \cdot \nabla) v \right] = 0 + \nu \nabla^2 \omega$$

Using vector identities you can show that

$$\cancel{\nabla \times} (v \cdot \nabla) v = \cancel{\Omega \times v} \; \omega \times v$$

Hence:

$$\partial_t \omega + \nabla \times (\omega \times v) = \nu \nabla^2 \omega$$

Now, non-dimensionaliz:

$$[\omega] = \frac{1}{T}$$

$$[\partial_t \omega] = \frac{1}{T^2}$$

$$\frac{1}{T^2} \tilde{\partial}_t \tilde{\omega} + \cancel{\frac{\ }{L}}$$

$$\frac{1}{T}\frac{1}{T}\,\partial_t \omega + \frac{1}{L}\frac{1}{T}\frac{L}{T}\,\nabla\times(\omega\times v)$$

$$= \nu\frac{1}{L^2}\frac{1}{T}\,\nabla^2\omega$$

$$\Rightarrow \quad \partial_t \omega + \nabla\times(\omega\times v)$$

$$= \frac{\nu T}{L^2}\,\nabla^2\omega$$

$$= \frac{\nu}{L^2}\frac{L}{U}\,\nabla^2\omega = \frac{\nu}{UL}\,\nabla^2\omega.$$

$$\Rightarrow \quad \boxed{\partial_t \omega + \nabla\times(\omega\times v) = \frac{1}{Re}\nabla^2\omega}$$

The whole problem reduces to a single dimensionless parameter, the Reynold's no.

* The limit of $Re \to 0$ is not the same as $Re = 0$ because this is a problem of singular perturbation theory. In other words $Re \to \infty$, which implies $\nu \to 0$ is ~~not~~ the problem does not reduce to ideal problem.

The vorticity eqn :

- $$\partial_t \omega + \nabla \times (\omega \times v) = \frac{1}{Re} \nabla^2 \omega$$

$$\Rightarrow \quad \partial_t \omega = \nabla \times (v \times \omega) + \frac{1}{Re} \nabla^2 \omega$$

Is exactly the same as the eqn. for the magnetic field

$$\partial_t B = \nabla \times (v \times B) + \frac{1}{Re_M} \nabla^2 B .$$

where $Re_M \equiv$ magnetic Reynold's number

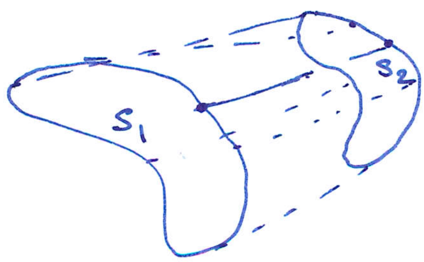
$$= \frac{UL}{\eta}$$

- In the ideal case, $\nu = 0$

$$\partial_t \omega + \nabla \times (\omega \times v) = 0$$

Kelvin's vorticity theorem :

Consider a open surface in a fluid, ~~that~~ The surface is made out of fluid parcels.



With time the parcels will move.

The flux of vorticity through this surface will remain a constant in time

$$\int_{S_1} \omega . \hat{n}\, ds = \int_{S_2} \omega . \hat{n}\, ds$$

or
$$D_t \int_S \omega . \hat{n}\, ds = 0$$

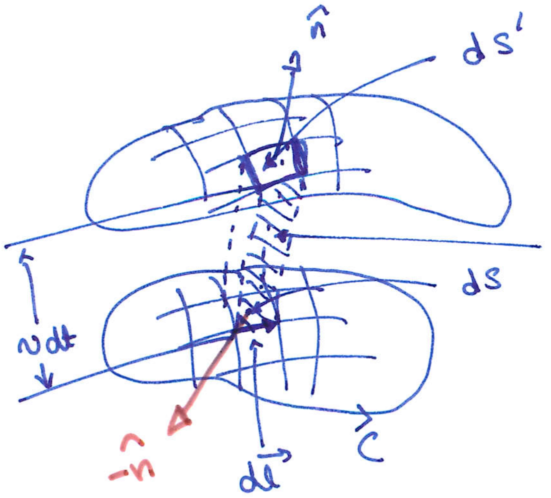
<u>proof</u>

$$D_t \int_S \omega . \hat{n}\, ds$$

$$= \int \frac{\partial \omega}{\partial t} . \hat{n}\, ds + \int \omega . D_t (\hat{n}\, ds)$$

$$= - \int \nabla \times (\omega \times v) . \hat{n}\, ds + \text{2nd term}$$

$$= - \oint_C (\omega \times v) . dl + (\text{2nd term})$$



$\cancel{d}\; d\vec{l} \times \vec{v}\, dt$

clearly :

$$\hat{n}'\, ds' - \hat{n}\, ds + \begin{pmatrix}\text{vertical}\\ \text{area}\end{pmatrix} = 0$$

$$\omega \cdot \frac{d}{dt}(\hat{n}\, ds) \qquad \hat{n}'\, ds' - ds\,\hat{n} - dt \oint \vec{v} \times d\vec{l} = 0$$

$$D_t(\hat{n}\, ds) = \lim_{dt \to 0} \frac{\hat{n}'\, ds' - \hat{n}\, ds}{dt}$$

$$= \oint \vec{v} \times d\vec{l}$$

$$\int \omega \cdot D_t(\hat{n}\, ds) = \int \oint \omega \cdot (\vec{v} \times d\vec{l})$$

$$= \int \oint (\omega \times v) \cdot dl$$

$$= \oint_C (\omega \times v) \cdot dl$$

(Because all the inner contributions cancel out)

~~AD~~ $\cancel{D_t \oint \omega \cdot \hat{n}}\, ds$

$$\Rightarrow \quad D_t \oint \omega \cdot \hat{n}\, ds = 0$$

In other words

$$\int_S \omega \cdot \hat{n}\, ds = \int_S (\nabla \times v) \cdot \hat{n}\, ds$$

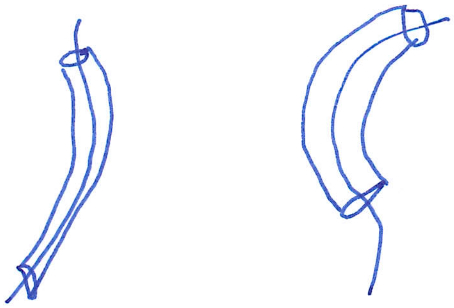
$$= \oint_C v \cdot dl \equiv \text{circulation}$$

Circulation is a conserved quantity in ideal hydrodynamics.

* The same equation is obeyed by the magnetic field hence in ideal MHD the flux of a magnetic field through a surface is conserved. This is called the theorem of "flux freezing". (Alfven ~~194~~ 1942)

× Possible consequences of flux freezing.

The magnetic field is slaved to the flow. If you know how the flow goes ~~you~~ you can predict the field.



Two fluid parcels on the same field line remain on it.

* If a structure collapses under gravity its magnetic field can become very intense.

The vorticity eqn:

$$\partial_t \omega + \nabla \times (\omega \times v) = \nu \nabla^2 \omega$$

$$\nabla \cdot v = 0$$

$$\omega = \nabla \times v$$

makes a complete dynamical theory.

* We obtain v by the Biot-Savant law.

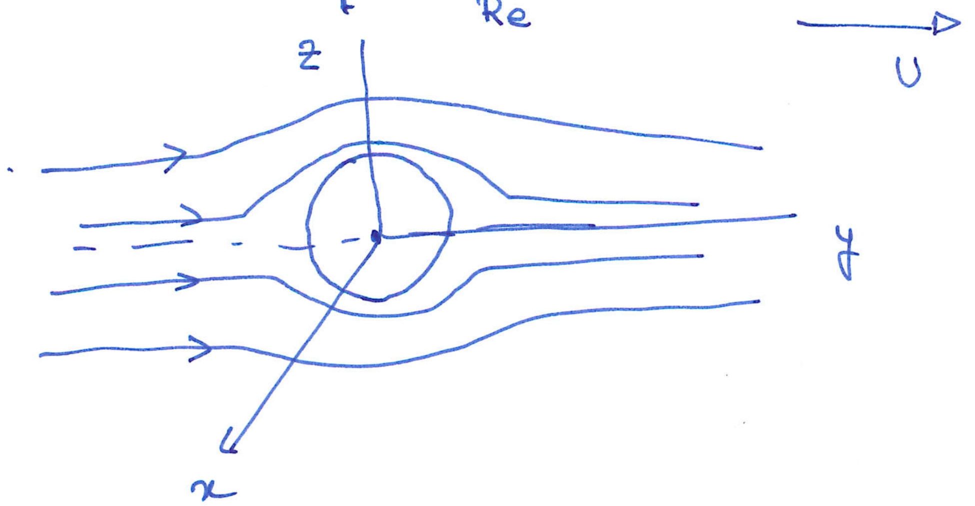
* In the non-ideal case ~~vo~~ vorticity diffuses.

Some solutions of the Navier-Stokes eqn.

1. Stokes eqn.

$$\nabla \cdot v = 0$$

$$\partial_t v = \frac{1}{Re} \nabla^2 v - \nabla p$$



Look for stationary solutions:

$$\nabla \cdot v = 0$$

$$\frac{1}{Re} \nabla^2 v - \nabla p = 0$$

Look for solutions in the following form:

$$v_i = U_{ij} a_j \quad , \qquad p = \Pi_j a_j$$

where $\vec{a}$ is a constant vector.

$$U_{ij} = \delta_{ij} \nabla^2 \chi - \frac{\partial^2}{\partial x_i \partial x_j} \chi$$

$$\cancel{\nabla\cdot v} = \cancel{\frac{\partial}{\partial x_j}} \cancel{\delta_{ij} \nabla^2 \chi}\; a_i$$

$$\nabla\cdot v = \frac{\partial}{\partial x_i}\left[\delta_{ij} \nabla^2 \chi \, a_j - \frac{\partial^2 \chi}{\partial x_i \partial x_j} a_j \right]$$

$$= \left[\delta_{ij} \nabla^2 \frac{\partial \chi}{\partial x_i} a_j - \cancel{\partial^2} \frac{\partial}{\partial x_j} \nabla^2 \chi \, a_j \right] = 0$$

$$\nabla^2 v = \left[\delta_{ij} \nabla^4 \chi - \frac{\partial^2}{\partial x_i \partial x_j} \nabla^2 \chi \right] a_j$$

Choose $\Pi_j = \frac{\partial}{\partial x_j} \nabla^2$

Then we obtain

$$\boxed{\nabla^4 \chi = 0}$$

a biharmonic eqn.