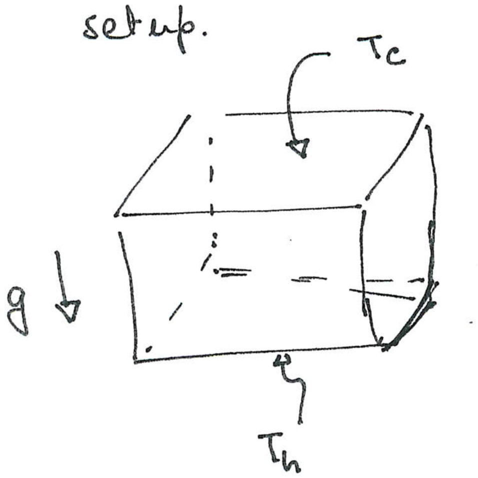
Convection

- This time we want to study something we avoided studying all through this course. That is the ~~tra~~ entropy equation. This controls two effects transport of heat by the flow and also flow that results due to temperature gradient.

- Convection is the most important effect by which turbulence is generated. We shall study astrophysical convection later. We shall first study convection in a laboratory setup.



$$\partial_t \rho + \mathrm{div}(\rho v) = 0$$

$$\partial_t(\rho v_i) + \mathrm{div}(\rho v_i v_j + p\delta_{ij} - \sigma_{ij}) = \rho g$$

$$\partial_t s + \mathrm{div}\left(v s + \frac{q}{T} - v_i \sigma_{ij} \right) = 0$$

$$q = -K \nabla T$$

For the moment we do not consider magnetic field.

- After spending so much time studying MHD we now know how to handle such problems.

- First we shall look for a steady (time independent) solution.

- Then we shall study linear stability of that solution. There will emerge some critical parameter ~~to~~ beyond which the steady solution would become unstable.

- Then we can study how this instability saturates and which secondary instabilities are built on them.

- Then we would study fully developed turbulence in this problem.

But before all this we shall first simplify ~~our~~ our problem further.

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- Incompressible convection or Bousinesq approximation.

~~In most of the~~

Remember that our derivation of the entropy eqn first started from dissipationless ~~by~~ ~~by~~ hydrodynamics where we had:

s : entropy per unit volume

$\tilde{s}$: entropy per unit mass.

s : deals with fluid problems

$\tilde{s}$: deals with thermodynamics

$$\frac{\text{entropy}}{\text{volume}} = \frac{\text{entropy}}{\text{mass}} \, \frac{\text{mass}}{\text{volume}}$$

$$\Rightarrow \qquad s = \rho \tilde{s}$$

The thermodynamics is given by:

$$T\,d\tilde{s} = d\tilde{e} - \frac{p}{\rho^2}\,d\rho$$

or

$$\rho T\,d\tilde{s} = \rho\,d\tilde{e} - \frac{p}{\rho}\,d\rho$$

Also the fluid equations must be supplemented by an equation of state which we always assumed is an ideal gas.

But liquids, for which most experiments are done are not ideal gas. A better equation of state is :

$$\rho = \rho_0 \left[1 - \alpha (T - T_0) \right]$$

α — volume expansion coefficient.

Also note that α is a very small number $\sim 10^{-3} - 10^{-4}$. So variation of temperature changes density by very little and this can be ignored. Then we can use the incompressible approximation:

$$\partial_t \rho + \operatorname{div}(\rho v) = 0 \longrightarrow \operatorname{div} v = 0$$

$$\partial_t (\rho v) + \operatorname{div}\left(\rho v_i v_j + \delta_{ij} p - \sigma_{ij}\right) = \rho g$$

The variation ~~is~~ in ρ <u>cannot</u> be ignored in the gravity term. Because that can be large.

$$\Rightarrow \partial_t v + \operatorname{div}\left(v_i v_j + \delta_{ij} \frac{p}{\rho} - \sigma_{ij}/\rho\right) = \left(1 + \frac{\delta\rho}{\rho_0}\right) g$$

$$\delta\rho = -\rho_0 \alpha (T - T_0)$$

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Now, first consider the ~~dissipation les~~ the entropy per unit mass

$$\tilde{s} = \tilde{s}(p, T)$$

$$\Rightarrow \quad d\tilde{s} = \left(\frac{\partial \tilde{s}}{\partial T}\right)_p dT + \left(\frac{\partial \tilde{s}}{\partial p}\right)_T dT$$

Assume pressure is constant

$$d\tilde{s} = \left(\frac{\partial \tilde{s}}{\partial T}\right) dT = \frac{c_p}{T} dT$$

because $\quad c_p = T\left(\frac{\partial \tilde{s}}{\partial T}\right)_p$

or $\quad T\, d\tilde{s} = c_p\, dT$

Dissipationless ~~by~~ hydrodynamics implies

$D_t s = 0 \qquad$ [s : entropy per unit volume]

$$\Rightarrow \quad D_t(\rho \tilde{s}) = 0 \qquad \left[\because D_t \rho = 0\right]$$

$$\Rightarrow \quad \rho\,(D_t \tilde{s}) = 0$$

In the presence of dissipation

$$T D_t s = -\operatorname{div}(\varphi)$$

heat flux

$$\Rightarrow \quad \rho T D_t \tilde{s} = -\operatorname{div}\varphi$$

$$\Rightarrow \quad \rho T c_p D_t T = -\operatorname{div}\varphi$$

$$\Rightarrow \quad \partial_t T + (v\cdot\nabla) T = -\frac{1}{\rho c_p} \operatorname{div}\varphi$$

$$Q = -K\nabla T \qquad \text{~~if we ignore dissip~~}$$

$$- v_i \sigma_{ij}$$

$$\Rightarrow \quad \partial_t T + (v\cdot\nabla)T = + \frac{K}{\rho c_p}\nabla^2 T + \frac{1}{\rho c_p} v_i \sigma_{ij}$$

$$\sigma_{ij} = \mu\left(\partial_i v_j + \partial_j v_i\right)$$

The last term; which denotes heating ~~by~~ due to viscous dissipation is also typically ignored. Then we have, the following eqn. for temperature:

$$\partial_t T + (v\cdot\nabla)T = \alpha \nabla^2 T \qquad \alpha = \frac{K}{\rho c_p}$$

together with the incompressible Navier-stokes eqn.

$$\partial_t v + (v\cdot\nabla)v = -\frac{\nabla p}{\rho} + \frac{\mu}{\rho}\,\mathrm{div}\,\left(\partial_i v_j + \partial_j v_i\right) + \left(1 + \frac{\delta\rho}{\rho}\right)g$$

and $\qquad \nabla\cdot v = 0$

This makes a complete dynamical theory with a constant ρ. This is the Bousinesq approximation.

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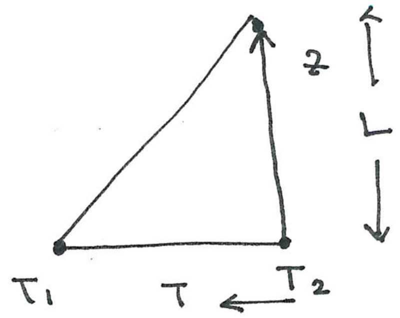
- steady state solution



~~T = constant~~

~~$v = 0$~~

$v = 0$, $\nabla^2 T = 0$

T is a function of one coordinate only with boundary condition

$T(0) = T_0$ $T(L) = T_2$

$$\frac{d^2T}{dz^2} = 0 \Rightarrow T = T_0 - \frac{(T_0 - T_2)}{L} z$$

$$= T_0 - \beta z$$

$$\frac{\partial p}{\partial z} = -\rho g$$

$$= -g\rho_0\left[1 + \alpha(T_0 - T)\right]$$

$$\Rightarrow p(z) = p_0 - g\rho_0\left(z + \frac{1}{2}\alpha\beta z^2\right)$$

-

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- <u>Perturbation</u> :

$$\delta v = u$$

$$\delta T = T_0 - \beta z + \theta$$

$$\delta p$$

The ~~pe~~ equations for the perturbed quantities :

$$\partial_t u_i = - \partial_i \left(\frac{\delta p}{\rho_0} \right) - g\alpha \, \theta \hat{z} + \nu \nabla^2 u_i$$

$$\partial_t \theta = - \beta u_z + \chi \nabla^2 \theta$$

$$\partial_j u_j = 0$$

Now eliminate pressure : $\omega = \nabla \times u$

$$\partial_t \omega_i = - g\alpha \, \nabla \times (\theta \hat{z}) + \nu \nabla^2 \omega_i$$

Taking curl again :

$$\partial_t (\nabla \times \omega) = - g\alpha \, \nabla \times \nabla \times (\theta \hat{z}) + \nu \nabla^2 (\nabla \times \omega)$$

Note : $\omega = \nabla \times u, \quad \nabla \times \omega = \nabla \times \nabla \times u = - \nabla^2 u$

$$\Rightarrow \quad \partial_t \nabla^2 u = g\alpha \left[\hat{z} \nabla^2 \theta - \nabla \nabla . (\hat{z} \theta) \right] + \nu \nabla^4 u$$

Projecting along z coordinate :

$\cancel{u_z = w}$, $\qquad w = u_z$ (not ω)

$\cancel{u_z = w}$ $\qquad \zeta = \omega_z$

$$\cancel{\partial_t \omega_z} \quad \partial_t \zeta = \nu \nabla^2 \zeta$$

$$\partial_t \nabla^2 w = g\alpha \left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right) + \nu \nabla^4 w$$

and $$\partial_t \theta = \beta w + \chi \nabla^2 \theta$$

Expand in the usual normal mode fashion

$$w = W(z) \exp\left[i(k_x x + k_y y) + pt \right]$$

$$\theta = \Theta(z) \exp [\qquad]$$

$$\zeta = Z(z) \exp [\qquad]$$

$$p(D^2 - k^2) W = -g\alpha k^2 \Theta + \nu (D^2 - k^2)^2 W$$

$$p \Theta = \beta W + \chi (D^2 - k^2) \Theta$$

$$p Z = \nu (D^2 - k^2) Z$$

Now: non-dimensionalize:

~~p~~ ~~d~~ ~~kg~~ ~~k~~

length by L,

time by L^2/ν

let $a = kL$, $\sigma = pL^2/\nu$

$$\Rightarrow \quad (D^2 - a^2)(D^2 - a^2 - \sigma)\, W = \left(\frac{g\alpha}{\nu} L^2\right) a^2\, \Theta$$

$$(D^2 - a^2 - \cancel{\phi}\, P_r\, \sigma)\, \Theta = -\left(\frac{\beta}{\chi} L^2\right) W$$

with $\quad P_r = \dfrac{\nu}{\chi}$.

Eliminating Θ, we obtain:

$$(D^2 - a^2)(D^2 - a^2 - \sigma)(D^2 - a^2 - P_r \sigma)\, W = -(Ra)\, a^2\, W$$

$$\boxed{Ra = \frac{g\alpha\beta}{\chi\nu} d^4}$$

A sixth order equation with a single dimensionless number.

- Finding the critical state:

- one can show that σ is real and $\sigma = 0$ corresponds to the critical state.

Putting $\sigma = 0$, we obtain for the critical state:

$$(D^2 - a^2)\, W = \left(\frac{g\alpha d^2}{\nu}\right) a^2\, \Theta$$

$$(D^2 - a^2)\, \Theta = -\left(\frac{\beta}{\chi} d^2\right) W$$

Eliminating Θ:

$$(D^2 - a^2)^3\, W = -(Ra)\, a^2\, W$$

This must be solved with the proper boundary conditions to solve for Ra; which would give the critical $Ra := Ra_{cr}$

For a given a^2 (a horizontal wave number) we must solve the above eqn. with boundary conditions. This can give many values of Ra but we must choose the minimum value as Ra_{cr}.

The easiest example is two free boundaries :

$$W = 0, \quad D^2 W = 0 \quad \text{for} \quad z = 0, \text{ and } 1.$$

$\downarrow$

vertical ~~velocity~~ ~~vorticity~~ velocity is zero at the boundary

on a free surface: $D^2 W = 0$

Because:

The stress tensor must vanish on a free surface:

$$\frac{\partial v_x}{\partial z} = \frac{\partial v_y}{\partial z} = 0$$

& continuity eqn. $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$

operate $\frac{\partial}{\partial z}$: $\frac{\partial}{\partial x}\frac{\partial v_x}{\partial z} + \frac{\partial}{\partial y}\frac{\partial v_y}{\partial z} + \frac{\partial^2 v_z}{\partial z^2} = 0$

$$\Rightarrow \quad \frac{\partial^2 v_z}{\partial z^2} = 0$$

$$\Rightarrow \quad D^2 W = 0$$

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$$(D^2 - a^2)^3 W = - Ra\, a^2 W$$

$\cancel{D^4 W} =$

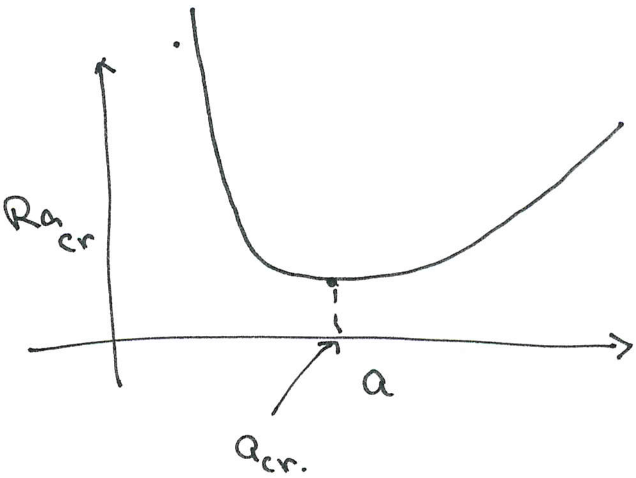
and $\quad (D^2 - a^2)^2 W = 0 \,, \quad W = 0, \quad D^2 W = 0$

at the boundary.

This implies $\qquad W = A \sin(n\pi z)$

$$Ra = \frac{(n^2\pi^2 + a^2)^3}{a^2}$$

$$Ra_{cr} = \frac{(\pi^2 + a^2)^3}{a^2}$$



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Notes :

1. The boundary condition of free surface is not physically appropriate. With a real free ~~surface~~ surface tension comes in and temperature dependence of viscosity plays a role. This is called Marangoni convection. This is what gives the hexagonal cells.

2. For real boundary condition the characteristic eqn. ~~is~~ must be solved numerically.

3. ~~Here we have~~

3. To see the structure that appears we must in principle do a finite amplitude calculation.

5. The convection in the sun is different, because radiation plays a major role.