Lecture VII

Turbulence :

We shall essentially study homogeneous and isotropic turbulence. What is this strange beast?

- Demonstration of flow behind a circular cylinder at ~~vari~~ various Reynold's number from ~~the old~~ The Album of Fluid Motion (TAFM)

TAFM	Fig 1
TAFM	Fig 24
TAFM	Fig 40
TAFM	Fig 41, 42 43, 44
TAFM	Fig 152 Grid Turbulence , 153

Note the gradual loss of symmetry and statistical restoration of symmetry at very high Reynold's ~~nm~~ number.

- Navier-stokes with periodic boundary conditions

$$\partial_t u + (u \cdot \nabla) u = \nu \nabla^2 u - \nabla p + f$$

external force to reach a statistically stationary state.

- ~~Probabilistic~~ de

- Our problem now requires a statistical description.

We give up on writing down a solution given the initial condition and the force. We study only statistical quantities.

Assumptions

- ~~For a given f, we are limited to large length scales we would to obtain of f.~~

- An invariant measure exists.

Conservation laws

$$\langle f \rangle = \frac{1}{L^3} \int_V f(\vec{x})\, d\vec{x}$$

For periodic functions:

$$\langle \partial_j f \rangle = 0$$

$$\langle (\partial_j f)\, g \rangle = -\langle f\, \partial_j g \rangle$$

$$\langle (\nabla^2 f)\, g \rangle = -\langle (\partial_i f)(\partial_i g) \rangle$$

$$\langle u \cdot (\nabla \times v) \rangle = \langle (\nabla \times u) \cdot v \rangle$$

$$\langle u \cdot \nabla^2 v \rangle = -\langle (\nabla \times u) \cdot (\nabla \times v) \rangle \qquad \text{if } \nabla \cdot v = 0$$

- conservation of momentum

$$\frac{d}{dt} \langle v \rangle = 0$$

- conservation of energy

$$\frac{d}{dt} \langle \tfrac{1}{2} v^2 \rangle = -\frac{1}{2} \nu \sum_{i,j} \left(\partial_i v_j + \partial_j v_i \right)^2$$

$$= -\nu \langle |\omega|^2 \rangle$$

- conservation of helicity

$$\cancel{\frac{d}{dt} \langle \omega \cdot u \rangle} =$$

$$\frac{d}{dt} \left\langle \frac{\omega \cdot v}{2} \right\rangle = -\nu \langle \omega \cdot (\nabla \times \omega) \rangle$$

mean enstrophy

$$E \equiv \frac{1}{2}\langle v^2\rangle, \quad \Omega = \frac{1}{2}\langle \omega^2\rangle$$

$$H \equiv \frac{1}{2}\langle v\cdot\omega\rangle \qquad H_\omega \equiv \frac{1}{2}\langle \omega\cdot(\nabla\times\omega)\rangle$$

$$\frac{d}{dt}E = -2\nu\Omega \qquad \frac{dH}{dt} = -2\nu H_\omega$$

$\varepsilon = -\frac{d}{dt}E$ is the mean energy dissipation per unit mass.

$$\frac{d\Omega}{dt} = -2\nu P \quad \text{where} \quad P \equiv \left\langle \frac{1}{2}(\nabla\times\omega)^2 \right\rangle$$

$\rightarrow$ Palinstrophy (mean)

Probilistic tools and ideas:

- A random variable.
- probability distribution function

$$\begin{pmatrix} \text{Probability that } x \\ \text{lies between } X \text{ to } X+dX \end{pmatrix} = P(X)\,dX$$

$$\Rightarrow \quad \int P(x)\,dx = 1 \qquad \text{normalization.}$$

$P(x)$ is everywhere positive

- cumulative probability:

$$Q(X) = \int_{-\infty}^{X} P(x)\,dx$$

$$P(X) = \frac{dQ}{dX}$$

- moments

$$\cancel{\langle x^m \rangle} = \quad \cancel{p(x)}$$

$$\cancel{\langle v^2 \rangle} =$$

$$\langle v^m \rangle = \int x^m P(x)\, dx$$

may not always exist

$$\sigma^2 = \langle v^2 \rangle \quad \text{variance}$$

$$S = \frac{\langle v^3 \rangle}{(\langle v^2 \rangle)^{3/2}} \quad \text{skewness}$$

$$F = \frac{\langle v^4 \rangle}{(\langle v^2 \rangle)^2} \quad \text{Flatness}$$

- characteristic function

$$K(z) = \langle e^{izv} \rangle = \int e^{izx} P(x)\, dx$$

Characteristic function of a sum of two independent random variables is the product of their individual characteristic functions.

$$p(x) = \int K(z)\, e^{-izx}\, dz$$

They are Fourier transforms of each other

- centered random variable: $\langle v \rangle = 0$

- Gaussian random variable:

$$K(z) = \langle e^{izv} \rangle = e^{-\frac{1}{2}\sigma^2 z^2}$$

$$p(x) = \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-x^2/2\sigma^2}$$

- Multidimensional Gaussian:

$$\Gamma_{ij} \equiv \langle v_i v_j \rangle$$

$$K(z) \equiv \left\langle \exp i(\vec{z} \cdot \vec{v}) \right\rangle$$

$$= \cancel{\exp -\frac{1}{2} z_j z_k \Gamma_{jk}} \rangle$$

$$= \exp\left(-\frac{1}{2} z_j z_k \Gamma_{jk}\right)$$

- Gaussian integration by parts:

If v is a Gaussian random variable

$$\langle v f(v) \rangle = \langle v^2 \rangle \left\langle \frac{\partial f}{\partial v} \right\rangle$$

~~proof~~ $\cancel{\int_{-\infty}^{+\infty} v f(v) \frac{e^{-v^2/2\sigma^2}}{(2\pi\sigma^2)^{1/2}} dv}$

$$= \cancel{f(v) \int_{-\infty}^{+\infty} \frac{v\, e^{-v^2/2\sigma^2}}{(\ \ )} dv \Big|_{-\infty}^{+\infty}}$$

$$\cancel{- \int \left(\frac{\partial f}{\partial v}\right) \int v\, e^{-v^2/2\sigma^2} dv\, dv}$$

prove by integration by parts.

- Generalization to multidimensional Gaussian variables:

$$\langle v_j f(v) \rangle = \Gamma_{ji} \left\langle \frac{\partial f}{\partial v_i} \right\rangle$$

- To Gaussian random functions

$$\langle u(x) \mathcal{F}[u] \rangle = \langle u(x) u(x') \rangle \left\langle \frac{\delta \mathcal{F}}{\delta u(x')} \right\rangle$$

A very useful theorem as we shall see soon.

- Moments of Gaussian random variable:

$$\langle v \rangle = 0$$

$$\langle v^{2m+1} \rangle = 0$$

$$\langle v^{2m} \rangle = \int x^{2m} \frac{e^{-x^2/2\sigma^2}}{(\ \ )} dx$$

$$= \begin{pmatrix} \text{combinatorial} \\ \text{factor} \end{pmatrix} \left[\langle v^2 \rangle \right]^m$$

<u>proof</u>

clearly it holds for $m = 1$.

~~For~~ Assume it holds for $m - 1$

$$\langle \underbrace{v^2 \cdot v^2 \cdots v^2}_{2(m-1)} \rangle = (\ \ ) \underbrace{\langle v^2 \rangle \langle v^2 \rangle \cdots \langle v^2 \rangle}_{m-1}$$

Now $$\langle \underbrace{v^2 \cdot v^2 \cdots v^2}_{2m} \rangle = (\,)\langle v^2 \rangle \left\langle \frac{\partial}{\partial v} v^{2m-1} \right\rangle$$

$$\langle v \quad v^{2m-1} \rangle$$

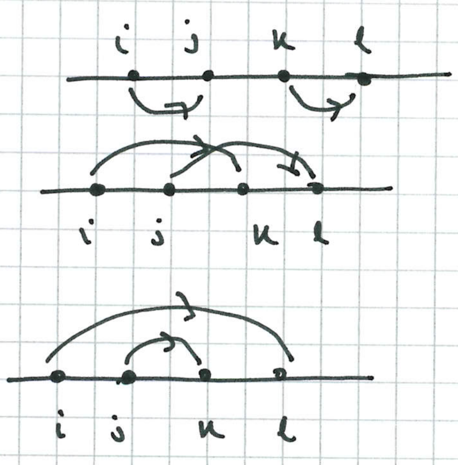
$$= (\ \ ) \langle v^2 \rangle \langle v^{2m-2} \rangle (2m-1)$$

$$\Rightarrow \quad \langle v^{2m} \rangle = \underbrace{\left[(2m-1) \cdots 1 \right]}_{\text{combinatorial factor.}} \underbrace{\langle v^2 \rangle \cdots \langle v^2 \rangle}_{m}$$

Now for vector valued Gaussians:

$$\langle v_i v_j v_k v_l \rangle = \Gamma_{ij} \Gamma_{kl} + \Gamma_{ik} \Gamma_{jl} + \Gamma_{il} \Gamma_{jk}$$

– Wick's theorem.



- Random functions:

A random variable that is function of space or time or both. When it is a function of time ~~its~~ it is often called a stochastic process.

$$\Gamma_{ij}(x, x', t, t') = \langle v_i(x,t)\, v_j(x', t') \rangle$$

is called the correlation function.

- characteristic functional

$$K[z] = \left\langle \exp\left[i \int dt \; \underbrace{z(t)}_{\text{non-random test function}} v(t) \right] \right\rangle$$

A ~~ran~~ random function is called Gaussian ~~when~~ if for all test functions $z(t)$

$$\int z(t)\, v(t)\, dt$$

is a ~~g~~ Gaussian random variable.

$$K[z] = \exp - \frac{1}{2} \int dt\, dt' \left[z(t)\, z(t')\, \Gamma(t,t') \right]$$

- Statistical symmetry.

A random function is said to be stationary (time translation invariant) if

$$v(t+h) \overset{law}{=} v(t)$$

which implies that all statistical properties (including pdf and moments) of $v(t+h)$ are same as $v(t)$.

consequently

$$\Gamma(t,t') \equiv \langle v(t)\, v(t') \rangle$$
$$= \Gamma(t-t')$$

- spectrum of stationary random functions

$$v(t) = \int e^{i\omega t}\, \hat{v}\, d\omega$$

$$v_F^{<}(t) = \int_{|\omega|<F} e^{i\omega t}\, \hat{v}(\omega)\, d\omega \qquad F \geq 0$$

low-pass filtered

$$\hat{v}(\omega) = \int e^{-i\omega t}\, v(t)\, dt$$

The energy in Fourier space

$$E(\omega) \equiv \langle \hat{v}(\omega)\, \hat{v}(-\omega) \rangle = \int e^{i\omega s}\, \Gamma(s)\, ds$$

for stationary random functions.

— Weiner Khinchin formula.