Exercise 10 — 25 marks

Please return on 22nd March

1. Consider a case where an incompressible fluid of variable density ~~is given~~ under gravity has a stationary solution to the equation of motion with $\rho_0 = \rho(z)$ and $\vec{v}_0 = 0$. Let the three components of velocity be (u, v, w). Linearize the equation of motion and show that the linearized equation of motion is

$$\rho_0 \partial_t u = -\frac{\partial}{\partial x}\delta p + \mu \nabla^2 u \cancel{+} \boxed{\cancel{\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)}}$$

$$\rho_0 \partial_t v = -\frac{\partial}{\partial y}\delta p + \mu \nabla^2 v$$

$$\rho_0 \partial_t w = -\frac{\partial}{\partial z}\delta p + \mu \nabla^2 w - g\,\delta\rho$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\partial_t \delta\rho = -w\frac{d\rho_0}{dz}$$

Here we have assumed that the perturbations of velocity are (u, v, w) and the perturbations of density and pressure are $\delta\rho$ and δp. The dynamical viscosity μ is constant. g is the gravitational acceleration.

Seek solutions of the form

$$\exp i(k_x x + k_y y + nt)$$

Then show that

$$\cancel{\boxed{D\left\{\left[\rho - \frac{\mu}{n}(D^2-k^2)\right] D w - \frac{1}{n}(D\mu) D\right.}}$$

$$D\left[\rho_0 - \frac{\mu}{n}(D^2-k^2)\right] D\hat{\omega}$$

$$= k^2\left\{-\frac{g}{n^2}(D\rho)\hat{\omega} + \left[\rho - \frac{\mu}{n}(D^2-k^2)\right]\hat{\omega}\right\} \quad -(1)$$

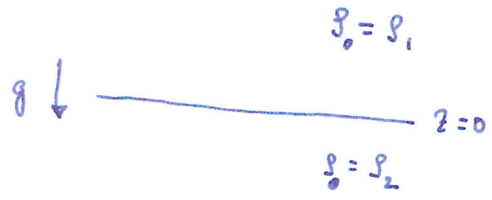
where $D = \dfrac{d}{dz}$, $k^2 = k_x^2 + k_y^2$

and $\vec{\omega} = \hat{\omega}(z) \exp i(k_x x + k_y y + nt)$

Now consider the inviscid case : $\mu = 0$
~~wrote~~ ~~down~~ ~~the~~ and show that the above relation simplifies to

$$D(\rho D\hat{\omega}) - \rho k^2 \hat{\omega} = -\frac{k^2}{n^2} g (D\rho)\hat{\omega} \quad -(2)$$

Now further simplify the problem to the case of two fluids of density ρ_1 and ρ_2 separated by a boundary at $z=0$. Ignore surface tension such that the above equations apply.



The way to solve this problem is to apply Eq. (2) ~~&~~ separately to $z>0$ and $z<0$. And then match the solutions at $z=0$.

Show that for $z>0$ (or $z<0$) Eq. (2) reduces to:

$$\left(D^2 - k^2\right)\hat{\omega} = 0$$

solve this with the boundary condition $\hat{\omega} \to 0$ as $z \to +\infty$

similarly for $z<0$, solve the same eqn. with boundary condition $\hat{\omega} \to 0$ as $z \to -\infty$. Then assume $\hat{\omega}$ should be continuous at $z=0$.

Then $$w = A e^{kz} \quad (z<0)$$
$$= A e^{-kz} \quad (z>0)$$

2. The equation obeyed by a passive scalar in a flow is

$$\partial_t \theta + (u.\nabla)\theta = \kappa \nabla^2 \theta$$

Assuming $$\theta(x) = \int \hat{\theta}(k) e^{ikx} dk$$
$$u(x) = \int \hat{u}(k) e^{ikx} dk$$

write down the equation satisfied by $\hat{\theta}(k)$.

3. Consider the passive scalar equation

$$\partial_t \theta + (\mathbf{U}.\nabla)\theta = \kappa \nabla^2 \theta$$

- assume $\mathbf{U}$ is incompressible
- Next do mean-field decomposition

$$U = \bar{U} + u$$
$$\theta = \bar{\theta} + \phi$$

Demand that the equation at large scale will be given by the closure:

$$\overline{u_j \phi} = K_{ij} \partial_j \bar{\theta}$$

Then show, using FOSA as described in class that

$$K_{ij} = \int \overline{u_i(t)\, u_j(t)}\, dt$$

and

$$\partial_t \bar{\Theta} = \text{div}\left(\kappa \nabla \bar{\Theta} + K_{ij}\, \partial_j \bar{\Theta}\right)$$

comment on why there is no alpha effect here?

(• Assume $\bar{U} = 0$)

4. Consider the dynamo problem in a case where its axisymmetric. In spherical coordinates

$$B = B_\phi(r,\theta)\, \hat{e}_\phi + B_p$$

Where B_ϕ is the toroidal component and B_p is the poloidal component. Write

$$B_p = \nabla \times \left[A_\phi(r,\theta)\, e_\phi \right]$$

and write the velocity field as

$$v = \Omega(r,\theta)\, r \sin\theta\, \hat{e}_\phi$$

Show that the ~~gover~~ ~~inducti~~ ~~d~~ mean field dynamo eqn.

$$\partial_t B = \nabla \times \left(\alpha B - \eta_T J \right)$$

with constant α and η_T reduces to:

$$\partial_t B_\phi = r \sin\theta \left(B_p \cdot \nabla \right) \Omega + \hat{e}_\phi \cdot \left[\nabla \times \left(\alpha\, B_p \right) \right] + \eta_T \left(\nabla^2 - \frac{1}{r^2 \sin^2\theta} \right) B_\phi$$

~~&~~ $$\partial_t A_\phi = \alpha B_\phi + \eta \left(\nabla^2 - \frac{1}{r^2 \sin^2\theta} \right) A_\phi$$

5. The solar dynamo has a period of 22 years and its half wavelength corresponds to about 40° in lattitude. Assuming the dynamo to be an alpha-shear dynamo make a rough estimate of the quentity (αG) [where G is the shear] and turbulent diffusion coefficient η_T. Assume the dynamo to be maginally stable.