Lecture XI DYNAMOS

11.1 Large scale features of MHD :

examples : large scale patterns,

(a) cyclones, trade winds, mean flow in the sea in terrestrial physics

(b) The dynamo cycle of the sun, differential rotation, etc.

(c) The distribution of mass in the galaxy

(d) The galactic magnetic field.

• How do we understand them:

Step 1 : write down the dynamical equations

Step 2 : average over the dynamical eqns. to write down effective equations at large scales.

Step 3 "close" the dynamical equations at large scales to obtain a complete dynamical theory.

Step 4 : Solve the dynamical theory to find out if the phenomenon you are trying to ~~describle~~ describe exists within the equations. This often involves numerical solutions.

If yes then good; if not start again from step 3.

An example :

The solar dynamo

STEP 1 Equations.

$$\partial_t \rho + \mathrm{div}(\rho v) = 0$$

$$\partial_t(\rho v_i) + \mathrm{div}\left[\rho v_i v_j + p\delta_{ij} - \sigma_{ij} - B_i B_j + \frac{B^2}{2}\delta_{ij}\right] = \rho g_i +$$

Gravity

$$\partial_t S + \mathrm{div}\left(v S + \frac{Q}{T}\right) = 0$$

$$Q = -K\nabla T - v_j \sigma_{ij} + \text{radiation} + \eta (J \times B)$$

$$\partial_t B = \nabla \times (v \times B - \eta J)$$

Approximations :

(a) Kinematic : the flow is given, consider only the induction equation.

$$\partial_t B = \nabla\times\left(v\times B - \eta J\right)$$

Step 2

$$B = \bar{B} + b$$
$$v = \bar{U} + u$$

substitute and expand to obtain:

$$\partial_t \bar{B} = \nabla\times\left(\bar{U}\times\bar{B} + \bar{\mathcal{E}} - \eta\bar{J}\right)$$

$$\mathcal{E} = u\times b$$

Step 3

$$\mathcal{E} = \alpha B - \eta_t \bar{J}$$

$$\Rightarrow \quad \partial_t \bar{B} = \nabla\times\left[\bar{U}\times\bar{B} + \alpha\bar{B} - (\eta+\eta_t)J\right]$$

Step 4 : consequence of this :

~~assu~~ assumptions:

(a) α is constant, non-zero, and a scalar

(b) η_t is constant, positive and large compared to η

Aside 1

Remember that in the simplest case:

$$\alpha \sim -\overline{\omega \cdot u}\,\tau_{cor} \sim -k_f\, u_{rms}^2\, \tau_{cor}$$

$$\omega \sim \nabla \times u \sim k_f u$$

k_f — large scale wave number

$$\eta \sim \overline{u^2}\,\tau_{cor}$$

$$\tau_{cor} \sim \frac{1}{k_f u_{rms}}$$

$$\Rightarrow \quad \alpha \sim u_{rms}, \qquad \eta \sim \frac{u_{rms}}{k_f}$$

Aside 2

α^2 dynamo

assume $\overline{U} = 0$

$$\Rightarrow \quad \partial_t \overline{B} = \nabla \times \left(\alpha \overline{B} - \eta_t \overline{J} \right)$$

$$= \alpha\, \nabla \times \overline{B} + \eta_T \nabla^2 \overline{B}$$

$$\overline{B} = \hat{B} \exp i\left(k \cdot x + \lambda t \right)$$

Dispersion relation:

$$\left(\lambda + \eta_T k^2 \right)\left[\left(\lambda + \eta_T k^2 \right)^2 - \alpha^2 k^2 \right] = 0$$

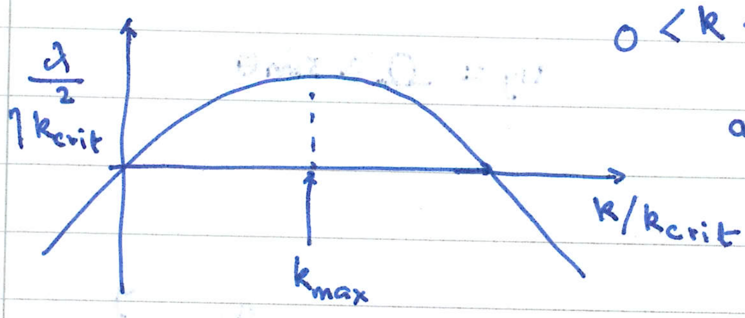
not compatiable with solenoidality

$$\Rightarrow \quad \lambda_0 = -\eta_T k^2 \qquad \lambda = -\eta_T k^2 \pm |\alpha k|$$

A growing solution for

$$0 < k < \frac{\alpha}{\eta_T} \equiv k_{crit}$$

a Dynamo.



Assume $\bar{U} = (0, sx, 0)$

with a constant s.

consider the axisymmetric problem:

$$k_y = 0$$

The dispersion relation is

$$(\lambda + \eta_T k^2)\left[(\lambda + \eta_T k^2) + i\alpha s k_z - \alpha^2 k^2\right] = 0$$

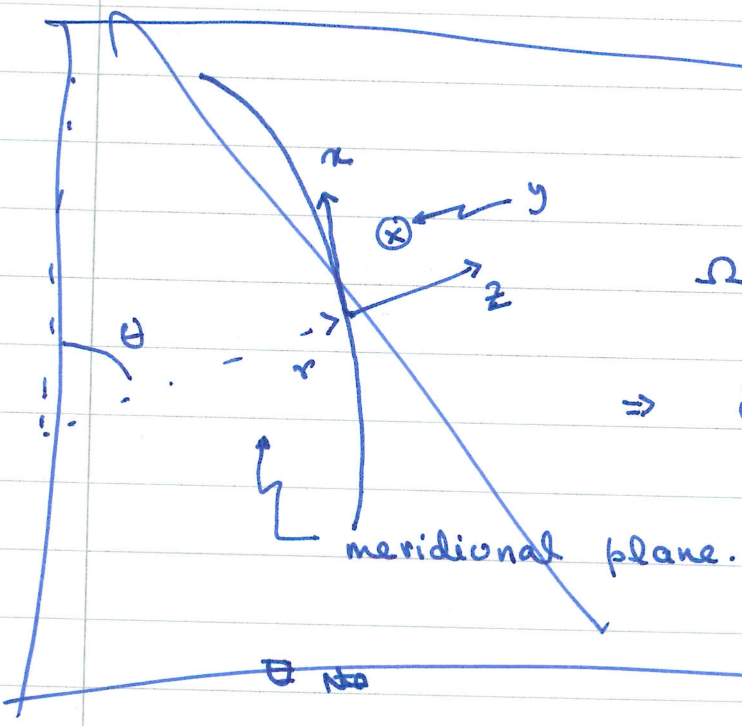
$$\mathrm{Re}\,\lambda \approx -\eta_T k^2 \pm \tfrac{1}{2}\left(\frac{\alpha s k_z}{2}\right)^{1/2}$$

$$\mathrm{Im}\,\lambda \equiv -\omega_{cyc} \approx \pm\left(\frac{\alpha s k_z}{2}\right)^{1/2}$$

this is a travelling wave solution.

$\Rightarrow$ with shear we can get a growing and travelling wave dynamo solution.

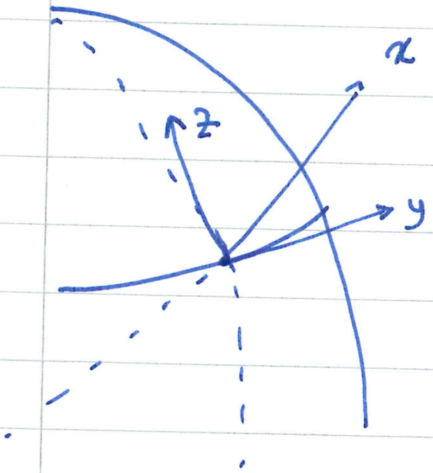
what does that mean for the ~~sa~~ Sun?



$$\bar{U} = (0, 0, \bar{U}_\phi)$$ ← in spherical coordinate

$$\Omega = \frac{\bar{U}_\phi}{r \sin\theta}$$

$$\Rightarrow \quad \bar{U} = \Omega r \sin\theta$$



choose a local coordinate.

Write the dynamo eqn in this local coordinate with the following large scale velocity

$$\bar{U} = (0, U_\phi, 0)$$

where $U_\phi = \Omega(r)\, r \sin\theta$

Differential rotation implies

$$G = \frac{d\Omega}{dr} \quad \text{is non-zero.}$$

Next assume axi-symmetry $\left(\frac{\partial}{\partial y} \equiv 0 \right)$

A general theorem:

Any axisymmetric B can be written as

$$\cancel{B = B_\phi + \nabla \times A_\phi}$$

~~where~~ $\cancel{B_t \equiv \text{toroidal component}}$

$$\cancel{\equiv \nabla \times A_\phi.}$$

$$\cancel{B_\phi}$$

$$\vec{B} = \underset{\substack{\uparrow \\ \text{poloidal}}}{B_p} + \underset{\substack{\uparrow \\ \text{toroidal.}}}{\hat{\phi}\, B_\phi} \qquad B_p = \nabla \times A_\phi$$

So, we can reduce the dynamo equations to two couples equations for the fields, B_ϕ and A_ϕ. Then assume

$$\cancel{B_\phi = \hat{B} \exp i}$$

$$B_\phi = \hat{B} \exp(\sigma t + ikz)$$

$$A_\phi = \hat{A} \exp(\sigma t + ikz)$$

The solution of the dynamo eqn. is a travelling wave of the form

growing part

$$A_\phi , B_\phi \sim \exp\left[- i \left(\frac{k \alpha G}{2} \right)^{1/2} t + i k z \right] (\quad)$$

If $\alpha G > 0$ the wave propagates poleward
$\alpha G < 0$ the wave propages equatorward.

Parker - Yoshimura rule.

So the solar dynamo may have wave like solution with propagating magnetic field.

<u>But:</u>

(i) This was a kinematic analysis. Clearly the dynamo must saturate. When it saturates does the imaginary part remain the same?

Ignoring such questions we obtain that the sign of

$$\alpha G < 0$$

in the northern hemisphere.

But $\quad \alpha \sim - \overline{\omega \cdot u}$

In the north $\quad \overline{\omega \cdot u} \sim \vec{g} \cdot \vec{\Omega} \cdot \nabla \rho$

$\sim$ negative

$\Rightarrow \quad \alpha \sim$ positive

$\Rightarrow \quad G \sim$ negative

$\frac{d\Omega}{dr} < 0$ in the north.

The sun should rotate slower as we go radially outward.

BUT; helioseismology shows that this is not the case. The solar differential rotation has an opposite sign in most of the convection zone.

$$\frac{d\Omega}{dr} \gtrsim 0$$

(except a thin layer near the surface)

How do we understand the solar dynamo then? The simple answer is; as yet we ~~dar~~ do not have a proper model. There are ~~some ideas~~ · two ideas that seem ~~as~~ to be competing

(i) flux transport dynamo

(ii) near surface dynamo

But that would take us too far.

Notes for further reading :

(i) Other than mean-field theory; similar large scale equations can also be obtained from the method of multiple scales.

(ii) We obtain a new transport coefficient α, and "renormalize" an existing transport coefficient η to $(\eta + \eta_t) \equiv \eta_T$

This is somewhat typical of the large-scale averaging procedure.
For example; how should we average Reynold's Stress ?

$$R_{ij} = \overline{\rho v_i v_j}$$

• assume ρ has no fluctuating part (valid only for incompressible flows)

$$R_{ij} = \rho \overline{v_i v_j}$$

$$= \bar{\rho} \overline{v_i v_j} = \dots + \nu_t \frac{\partial \bar{V}_i}{\partial x_j}$$

ν_t : turbulent viscosity.

But also:

$$\cancel{\overline{v_i v_j} = \Lambda \text{ from } \Omega}$$

$$\overline{v_i v_j} = \Lambda_{ijk} \Omega_k$$ in presence of rotation.

Lambda effect.

Thus

$$R_{ij} = \bar{\rho} \left[\Lambda_{ijk} \Omega_k + \eta_{ijkl} \frac{\partial \bar{U}_k}{\partial x_l} \right].$$

~~Suc~~ one new effect.

one renormalized coefficient.

~~Such~~ Similar contributions shoud appear in the entropy equation.

The resultant equations must be solved to find the differential rotation of the sun.

(iii) For compressible flows one must take care of the fluctuating density.

(iv) Note that we have not actually calculated the turbulent transport coefficients themselves. For example we have not made first principle calculations of $\alpha, \eta_t, \Lambda, \nu_t$

~~But just also~~
We have calculated α and η_t for low Reynold's number using FOSA.
(First order Smoothing Approximation)
How do we calculate them for ~~higher~~ turbulence? ~~As~~ Analytically; it is not clear how. Numerically it is possible ~~but~~ but numerical work is also limited to moderate Re.

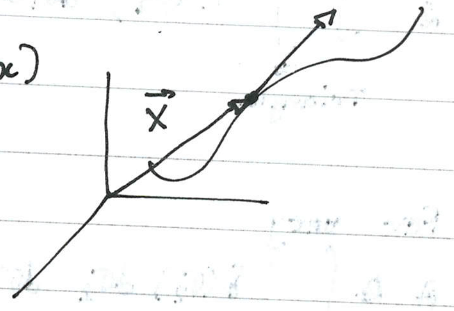
The use of symmetry is a very useful tool in this business.

(v) A very important observation is that the turbulent transport coefficients are typically determined by large-scale quantities, hence they remain order unity even in the limit of infinite Reynolds number (zero molecular ~~di~~· transport coefficients). Hence we can simulate the Large scale quantities with low resolution.

11.2 Turbulence with particles : Lagrangian viewpoint.

Consider a Lagrangian particle in a turbulent flow
The eqn. of this particle is

$$\frac{dX}{dt} = u(x)\,\delta(X - x)$$



A complete description of the flow can be given in the following way :

Every fluid particle is described at all times by their position $X(t;\, x_0, t_0)$

Now let us ask the following question :

Let $S(t) = X(t) - X(0)$
what is the average value of $\langle S(t) \rangle$ in the stationary state of turbulence ?

$$S(T) = X(t) - X(0)$$
$$= \int_0^T u(x)\,\delta(X - x)\, dt \quad \to \cancel{X(0)})$$

$$\langle S(T) \rangle = \int_0^T \langle u(x) \rangle \delta(X - x)\, dt = 0$$

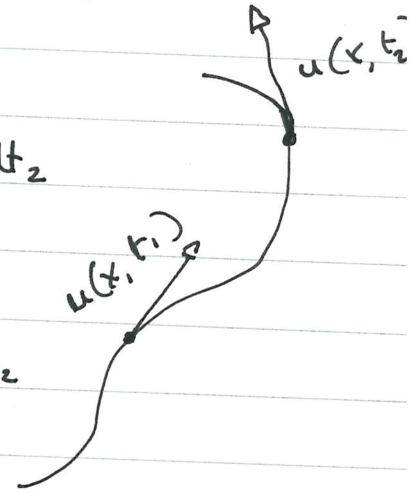
Because we assume homogeneous turbulence.

What about $\langle s^2(T) \rangle$

$$s^2(T) = \left[\int_0^T u(x,t)\, dt \right]^2$$

$$= \int_0^T u(x,t_1)\, dt_1 \int_0^T u(x,t_2)\, dt_2$$

$$= \int_0^T \int_0^T u(x,t_1)\, u(x,t_2)\, dt_1\, dt_2$$



$$\langle s^2(T) \rangle = \int_0^T \int_0^T \langle u(x,t_1)\, u(x,t_2) \rangle\, dt_1\, dt_2$$

The correlation function

$$C(t_1, t_2) = \langle u(x,t_1)\, u(x,t_2) \rangle$$

must be calculated at the ~~position~~ along a particle trajectory

In the statistically stationary state

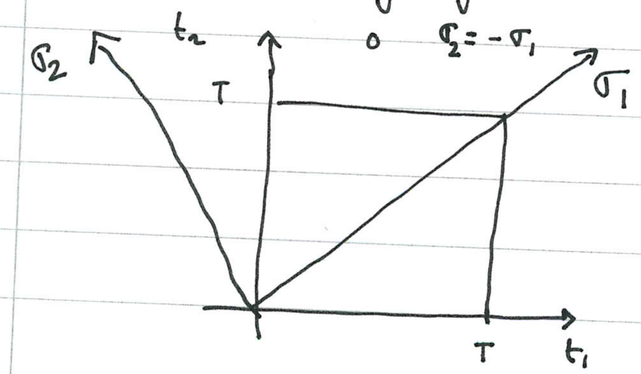
$$C(t_1, t_2) = C(t_2 - t_1)$$

function of the difference only.

$$\langle s^2(T) \rangle = \int_0^{\sqrt{2}T} \int_{\sigma_2 = -\sigma_1}^{+\sigma_1} C(\sigma_2)\, d\sigma_2\, d\sigma_1$$

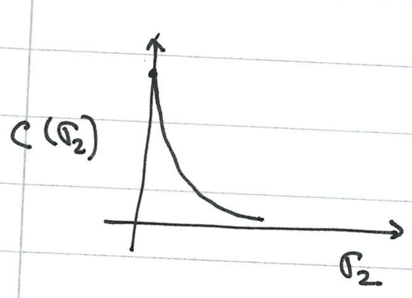
$\sigma_1 = t_1 + t_2$

$\sigma_2 = t_1 - t_2$



- First assume that T is ~~veri)~~ very large:

and $C(\sigma_2)$ has a finite correlation time



Then we can approximate $C(\sigma_2)$ by

$$C(\sigma_2) = C_0 \delta(\sigma_2)$$ ← delta correlated

$$\Rightarrow \quad \langle s^2(T) \rangle = \int_0^{\sqrt{2}T} d\sigma_1 \int_{-\sigma_1}^{\sigma_1} C_0 \, \delta(\sigma_2) \, d\sigma_2$$

$$= \int_0^{\sqrt{2}T} d\sigma_1 \, C_0 \quad \sim (\,)\, T$$

$$\Rightarrow \quad \left[\langle s^2(T) \rangle \right]^{1/2} = \sqrt{T} \quad - \text{ diffusion.}$$

But we did not consider any dissipation.
This diffusive property comes from ~~rand~~ advection by random (short correlated) velocity!
" Diffusion by random motion "

- In the limit where T is very small:

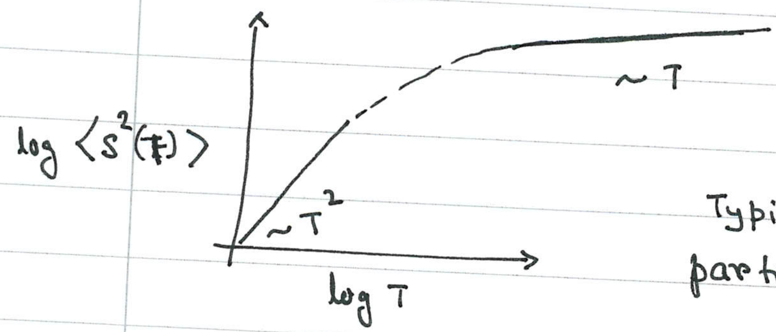
$$u(t+T) = u(t) + (D_t u) T + \text{h.o.t}$$

$$\Rightarrow \quad \langle u(t)\, u(t+T) \rangle = \langle u^2(t) \rangle + \langle u(t)\, D_t(u) \rangle T + \cdots$$

$$\Rightarrow \quad \langle s^2(T) \rangle = \int_0^T \int_0^T \langle u^2(t) \rangle \, dt_1 \, dt_2 \; + \quad h.o.t$$

$$\sim \quad T^2$$

ballistic at very ~~shrot~~ short times.



Typical behavior of Lagrangian particles in turbulent flows.

So if we add a patch of color to a turbulent fluid it will diffuse but not due to molecular diffusion but by "turbulent" diffusion.

The eqn. of a colar advected by ~~the~~ a flow:

$$\partial_t \theta + (u \cdot \nabla)\theta = \kappa \nabla^2 \theta + J_\theta$$

source

This is the Eulerian picture. The Lagrangian picture is

$$\theta(x; x_0, t_0) = \theta(x_0, t_0)$$

[- color is the same along particle trajectories]

if we ignore source and diffusion.

- ~~Two par~~
- How does distance between two particles change in turbulence?

$$\partial_t X_1 = u(X_1)$$
$$\partial_t X_2 = u(X_2)$$

$$\partial_t R = \partial_t (X_1 - X_2) = u(X_1) - u(X_2)$$

If $X_1 - X_2$ is small then

$$u(X_1) = u(X_2) + (\nabla u)\cdot(X_1 - X_2)$$

$$= u(X_2) + (\nabla u)\,.\,R$$

$$\Rightarrow \quad \partial_t R = (\nabla u)\,.\,R$$

to be more specific

$$\partial_t R_i = G_{ij} R_j$$

where $G_{ij} = \dfrac{\partial u_i}{\partial x_j}$ the velocity gradient matrix.

The matrix G_{ij} has ~~three~~ at every instance has three eigenvalues; such that

$$\lambda_1 + \lambda_2 + \lambda_3 = 0$$

so, if every instant I go to the eigensystem of G I should see ~~one~~ ~~in~~ at least one increasing direction.

~~$\Rightarrow$~~ ~~R(t)~~) ~~satisfies~~ ~~an~~ .

$\Rightarrow$ ~~one~~ can ~~roughly~~ ~~write~~

$\Rightarrow$ R^2 satisfies an eqn. with

$$\partial_t R^2 \sim \lambda R^2$$

λ +ve

$\Rightarrow$ R^2 grows exponentially with time.

Two particles in turbulence, who are initially very close to each other separate exponentially fast.

Now consider the eqn. for magnetic field without dissipation

$$\partial_t B = \nabla \times (u \times B)$$

$$= (B \cdot \nabla) u - B(\nabla \cdot u)$$

Assume incompressibility: $\nabla \cdot u = 0$

$$\Rightarrow \quad \partial_t B_i = \frac{\partial u_i}{\partial x_j} B_j = G_{ij} B_j$$

Exactly the same eqn. as R

(This is the same statement as ~~flux~~ ~~eqn~~ flux-freezing)

So, in turbulence B satisfies the same eqn. as a vector connecting two tracer particles.

- For small B, the vector R is also small,

$\Rightarrow$
$$B^2 \sim \exp(\lambda t)$$

exponential growth; a dynamo.

$\Rightarrow$ Small magnetic field always undergoes a dynamo when advected by an incompressible turbulent flow.

$\Rightarrow$ what happens for large magnetic fields?

The above argument works only when B is so small that

$$u(x+R) = u(x) + \frac{\partial u}{\partial x} R$$

is a reasonable approximation.

This is true when $R \ll \eta_K \longleftarrow$ Kolmogorov scale.

As soon as B becomes larger such a simple prescription stops working; does the dynamo then saturate?

This is the "fluctuation dynamo" as opposed to the ~~large~~ large scale dynamo.

If R is in inertial range then two particles separated by R also go away from each other but not exponentially.

$$R(t) \sim t^{3/2} \leftarrow \text{Richardson diffusion.}$$

which is, technically speaking, super diffusion.

This law has no rigorous proof so far. Anyhow, this implies that the dynamo stops as the growth is not exponential.

Summary :

1. Large scale magnetic fields can be understood using mean-field theory or method of multiple scales. ~~But~~ We generally require helical flows to create large scale magnetic fields. ~~Fi~~ But the details of how the solar dynamo ~~Z~~ works is still a mystery.

2. Fluctuating, small scale, magnetic field is typical generated by random flows, not necessarily helical.

Questions I have not addressed:

- How does the large-scale dynamo saturate?
- What is the dependence of small scale dynamo on magnetic Prandtl number?
- How is conservation of magnetic helicity connected with the saturation of the large scale dynamo.

- There can be ~~no~~ no large-scale dynamo in two dimensions (Zeldovich anti-dynamo theorem)