PERHAPS THE FUNDAMENTAL EQUATION THAT DESCRIBES THE SWIRLING NEBULAE AND THE CONDENSING, REVOLVING, AND EXPLODING STARS AND GALAXIES IS JUST THE SIMPLE EQUATION FOR HYDRODYNAMIC BEHAVIOR OF NEARLY PURE HYDROGEN GAS

R.P. FEYNMAN, *"flow of wet water"*

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INTRODUCTION TO FLUID MECHANICS

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1 Kinematics

1.1 Helmholtz's theorem

Let us start with an experiment commonly performed by six year olds in Sweden, we take a block of "slime" and deform it. Before deforming we mark two point, A and B, on this slime with a different color. After the deformation these two points have moved to two new positions. The vector that connected these two points has also changed to a new vector. Following Sommerfeld, [1] I state the following:

[1] A. Sommerfeld. Lectures on theoretical physics : Mechanics of deformable bodies. Levant Books, Kolkata, 2006

Theorem. *The most general motion of a sufficiently small element of a deformable (i.e., not rigid) body can be represented as the sum of*

1. *a translation*
2. *a rotation*
3. *an extension (contraction) in three mutually orthogonal directions.*

The original proof of this theorem is due to Helmholtz. The proof is based on Taylor series expansion as we describe below.

Let us use a Cartesian coordinate system, in which the coordinates of a point A are given by x, y, z. Under deformation every point in the "slime" as moved to new position. Thus to every point in the dough I can associate a vector $\boldsymbol{u}$ which denotes the displacement of that point under this deformation. This vector $\boldsymbol{u}$ itself can then be considered as a function of the three space coordinates x, y, z. $\boldsymbol{u}(x, y, z)$ is then a *v*ector field. You have encountered fields before, for example the temperature in this room is a *scalar* field. To imagine a scalar field think of a number at every point in space. To imagine a vector field think of an arrow associated with every point in space. A typical example of a vector field will be a gravitational field. Consider a spec of dirt. You take it to every point around the Earth and calculate the gravitational force the earth exerts on it. Then divide the force by the mass of the dirt to get force-unit-mass. This is a vector that at every point in space will point towards the center of the Earth (if Earth were a perfect sphere) and it magnitude will decrease as the spec of dirt is moved further and further away from Earth. Similarly we now consider another vector field, that of displacement of material points. Every point in the slime has been displaced: this displacement itself, in general, is a function of position given by the vector field $\boldsymbol{u}$.Then consider two neighboring points $A(x, y, z)$ and $B(x + \Delta x, y + \Delta y, z + \Delta z)$. Under deformation, A has moved to A' and B has moved to B'. The vector $\overrightarrow{AA'}$ is given by $\boldsymbol{u}(x, y, z)$ and the vector $\overrightarrow{BB'}$ is given by $\boldsymbol{u}(x + \Delta x, y + \Delta y, z + \Delta z)$. I can now expand each Cartesian component of the vector field $\boldsymbol{u}(x, y, z)$, given by $u_1(x, y, z)$, $u_2(x, y, z)$ and $u_3(x, y, z)$ in a Taylor series[2]

[2] This assumes that the field of displacement is "smooth" enough that the derivatives exists. This need not always be the case, for example, if you deform the slime in a way that there is hole inside it then the displacement field may no longer be Taylor expandable. No such case will be considered in these lectures.

$$u_1(x + \Delta x, y + \Delta y, z + \Delta z) = u_1(x, y, z) + \frac{\partial u_1}{\partial x}\Delta x + \frac{\partial u_1}{\partial y}\Delta y + \frac{\partial u_1}{\partial z}\Delta z + \dots \tag{1.1a}$$

$$u_2(x + \Delta x, y + \Delta y, z + \Delta z) = u_2(x, y, z) + \frac{\partial u_2}{\partial x}\Delta x + \frac{\partial u_2}{\partial y}\Delta y + \frac{\partial u_2}{\partial z}\Delta z + \dots \tag{1.1b}$$

$$u_3(x + \Delta x, y + \Delta y, z + \Delta z) = u_3(x, y, z) + \frac{\partial u_3}{\partial x}\Delta x + \frac{\partial u_3}{\partial y}\Delta y + \frac{\partial u_3}{\partial z}\Delta z + \dots \tag{1.1c}$$

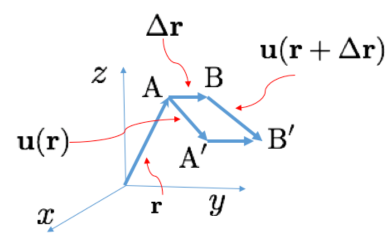


Figure 1.1: We use a Cartesian coordinate system. In this system, the point A has the position vector $\mathbf{r}$, and its neighboring point B has the position vector $\mathbf{r} + \Delta\mathbf{r}$. Under deformation A moves to A′ and B moves to B′. The deformation at A is $\boldsymbol{u}(\mathbf{r})$ and the deformation at B is $\boldsymbol{u}(\mathbf{r} + \Delta\mathbf{r})$.

Let us compactify our notation. I use u_α where $\alpha = 1, 2, 3$ for a component of the vector $\boldsymbol{u}$. Let $\mathbf{r} = (x, y, z)$ and $\Delta\mathbf{r} = (\Delta x, \Delta y, \Delta z)$. And I use the symbols

$$a_{11} = \frac{\partial u_1}{\partial x}, \quad a_{12} = \frac{\partial u_1}{\partial y}, \quad a_{13} = \frac{\partial u_1}{\partial z}, \tag{1.2a}$$

$$a_{21} = \frac{\partial u_2}{\partial x}, \quad a_{22} = \frac{\partial u_2}{\partial y}, \quad a_{23} = \frac{\partial u_2}{\partial z}, \tag{1.2b}$$

$$a_{31} = \frac{\partial u_3}{\partial x}, \quad a_{32} = \frac{\partial u_3}{\partial y}, \quad a_{33} = \frac{\partial u_3}{\partial z}, \tag{1.2c}$$

I can write the equation for deformation to be

$$u_\alpha(\mathbf{r}) = u_\alpha(\mathbf{r} + \Delta\mathbf{r}) + a_{\alpha\beta}\Delta r_\beta + \dots \tag{1.3}$$

Remember that the collection of number $a_{\alpha\beta}$ themselves are function of space, they depend on *where* we calculate the derivatives. As long as $\mid \Delta\mathbf{r} \mid$ is small enough I can ignore all the higher order terms that I have denoted by the triple-dots. This is assumed in the rest of this discussion.

By $a_{\alpha\beta}\Delta r_\beta$ I mean $\sum_{\beta=1,3} a_{\alpha\beta}\Delta r_\beta$. In these lectures any index, e.g., β here, that is repeated is assumed to be summed – this is known as the Einstein summation convention.

Now I am going to reorganize the collection of numbers $a_{\alpha\beta}$ in "symmetric" and "anti-symmetric" manner:

$$a_{\alpha\beta} = \frac{a_{\alpha\beta} - a_{\beta\alpha}}{2} + \frac{a_{\alpha\beta} + a_{\beta\alpha}}{2}. \tag{1.4}$$

Substituting back in Equation (1.1) I get

$$u_1(\mathbf{r}+\Delta\mathbf{r}) = u_1(\mathbf{r}) + 0 + \frac{a_{12}-a_{21}}{2}\Delta y + \frac{a_{13}-a_{31}}{2}\Delta z + a_{11}\Delta x + \frac{a_{12}+a_{21}}{2}\Delta y + \frac{a_{13}+a_{31}}{2}\Delta z \tag{1.5a}$$

$$u_2(\mathbf{r}+\Delta\mathbf{r}) = u_2(\mathbf{r}) + \frac{a_{21}-a_{12}}{2}\Delta x + 0 + \frac{a_{23}-a_{32}}{2}\Delta z + \frac{a_{12}+a_{21}}{2}\Delta x + a_{22}\Delta y + \frac{a_{23}+a_{32}}{2}\Delta z \tag{1.5b}$$

$$u_3(\mathbf{r}+\Delta\mathbf{r}) = u_3(\mathbf{r}) + \frac{a_{31}-a_{13}}{2}\Delta x + \frac{a_{32}-a_{23}}{2}\Delta y + 0 + \frac{a_{31}+a_{13}}{2}\Delta x + \frac{a_{23}+a_{32}}{2}\Delta y + a_{33}\Delta z \tag{1.5c}$$

I can now interpret the right-hand-side(RHS) of Equation (1.5) as a sum of three displacements,

$$\boldsymbol{u}(\mathbf{r}+\Delta\mathbf{r}) = \boldsymbol{d}_1 + \boldsymbol{d}_2 + \boldsymbol{d}_3 \tag{1.6}$$

The black ones show that two neighboring points are displaced by exactly the same amount,

$$\boldsymbol{u}(\mathbf{r}+\Delta\mathbf{r}) = \boldsymbol{u}(\mathbf{r}), \tag{1.7}$$

this is translation.

Now organize the red ones

$$\boldsymbol{d}_2 = \begin{pmatrix} 0 & \frac{a_{12}-a_{21}}{2} & \frac{a_{13}-a_{31}}{2} \\ \frac{a_{21}-a_{12}}{2} & 0 & \frac{a_{23}-a_{32}}{2} \\ \frac{a_{31}-a_{13}}{2} & \frac{a_{32}-a_{23}}{2} & 0 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} \tag{1.8}$$

The matrix the appears is an anti-symmetric matrix. A different but equivalent way to interpreting $\boldsymbol{d}_2$ is

$$\boldsymbol{d}_2 = \frac{1}{2}\Delta\boldsymbol{\phi} \times \Delta\mathbf{r} \tag{1.9}$$

where the vector $\Delta\boldsymbol{\phi} \equiv (\Delta\phi_1, \Delta\phi_2, \Delta\phi_3)$ is an infinitismal rotation vector which points along the axis of rotation with a magnitude equal to the angle of rotation, with

$$\Delta\phi_1 = a_{23} - a_{32}, \tag{1.10a}$$

$$\Delta\phi_2 = a_{13} - a_{31}, \tag{1.10b}$$

$$\Delta\phi_3 = a_{12} - a_{21}. \tag{1.10c}$$

Let us now consider the blue terms in Equation (1.5)

$$\boldsymbol{d}_3 = \begin{pmatrix} a_{11} & \frac{a_{12}+a_{21}}{2} & \frac{a_{13}+a_{31}}{2} \\ \frac{a_{21}+a_{12}}{2} & a_{22} & \frac{a_{23}+a_{32}}{2} \\ \frac{a_{31}+a_{13}}{2} & \frac{a_{32}+a_{23}}{2} & a_{33} \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} \tag{1.11}$$

An infinitismal rotation can be represented by a vector, although not an "usual" vector but a *polar* vector. Rotation by a finite amount cannot in general be represented by a vector but can be represented by an anti-symmetric matrix as in Equation (1.8)

This is a real symmetric matrix. Hence it can always be diagonalized by an orthogonal transformation. In other words, by a suitable rotation of coordinates I can always find a new coordinate system in which this matrix is diagonal. In that coordinate the displacement $\boldsymbol{d}_3$ will have the form

$$\boldsymbol{d}_3 = \begin{pmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & \Lambda_3 \end{pmatrix} \begin{pmatrix} \Delta X_1 \\ \Delta X_2 \\ \Delta X_3 \end{pmatrix} \tag{1.12}$$

You may be already familiar with real symmetric matrices and their properties, in particular their eigenvalues and eigenvectors from a course on quantum mechanics where they appear as Hamiltonian or from a course in classical mechanics where they appear during the discussions of moment of inertia of rigid objects. Proofs can be found in any book on matrices or linear algebra, e.g., Arfken and Weber[3].

[3] George B Arfken and Hans J Weber. Mathematical methods for physicists. AAPT, 1999

The three Λs are the three eigenvalues of the strain matrix $s_{\alpha\beta}$ given in Equation (1.11). They must be real, but they can be either positive or negative. A positive(negative) Λ denotes extension (compression) along the corresponding eigenvector X. The three eigenvectors are orthogonal to each other, together they for a Cartesian triad. This proves Helmholtz's theorem.

1.2 *Strain ellipsoid*

One way to visualize the strain matrix is to construct what is called its "quadratic form" , $f(x,y,z) \equiv r_\alpha s_{\alpha\beta} r_\beta$ where the vector $\mathbf{r} = (x,y,z)$ has components r_α with $\alpha = 1,2,3$. Written explicitly as

$$f(x,y,z) = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} a_{11} & \frac{a_{12}+a_{21}}{2} & \frac{a_{13}+a_{31}}{2} \\ \frac{a_{21}+a_{12}}{2} & a_{22} & \frac{a_{23}+a_{32}}{2} \\ \frac{a_{31}+a_{13}}{2} & \frac{a_{32}+a_{23}}{2} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{1.13}$$

The function $f(x,y,z)$ is in general a quadratic in all its arguments. Setting it equal to unity (or any other constant) defines a surface in three dimensional space. This surface is called a strain ellipsoid. Geometrically, it is an actual ellipsoid only if all the three eigenvalues of the strain matrix are positive.

A complete classification of quadratic surfaces can be found at `http://mathworld.wolfram.com/QuadraticSurface.html`. You should use a visualization software to play with quadratic surface. The three simplest examples given in the margin are plotted using Mathematica.

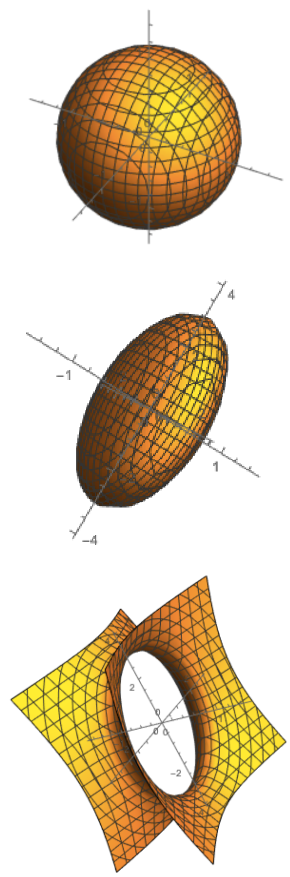


Figure 1.2: Visualization of quadratic surfaces in three dimensions. The top one is a sphere with the equation $x^2+y^2+z^2=1$. The middle one is an ellipsoid $4x^2+\frac{y^2}{4}+\frac{z^2}{9}=1$. The bottom one is $-4x^2+\frac{y^2}{4}+\frac{z^2}{9}=1$. These are the equations of these surfaces in terms of their eigencoordinates which are the axes shown in the figure.

1.3 *Vorticity*

Let us now pass from displacement to velocity. We write the total displacement $\boldsymbol{u}$ as $\boldsymbol{u} = v\Delta t$ and the infinitismal rotation vector as

$\Delta\boldsymbol{\phi} = (1/2)\boldsymbol{\omega}\Delta t$. In terms of components this becomes

$$u_1 = v_1\Delta t \quad u_2 = v_2\Delta t \quad u_3 = v_3\Delta t \tag{1.14a}$$

$$\Delta\phi_1 = (1/2)\omega_1\Delta t \quad \Delta\phi_2 = (1/2)\omega_2\Delta t \quad \Delta\phi_3 = (1/2)\omega_3\Delta t \tag{1.14b}$$

The vector $\boldsymbol{\omega}$ is called the vorticity vector. You should explicitly check that $\boldsymbol{\omega}$ and $\boldsymbol{v}$ are related by

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{v} \tag{1.15}$$

The vorticity vector plays a crucial role in understanding the behavior of fluids.

Exercise 1

1. To reinforce the idea that the displacement $\boldsymbol{d}_2$ is indeed a rotation show that under this displacement the length of the vector $\Delta\mathbf{r}$ remains unchanged.
2. Consider the quadratic form in two variables

$$2x^2 + 5xy + 3y^2 = 1 \tag{1.16}$$

   Write this in a form similar to the RHS of Equation (1.13), you should get a 2×2 matrix. Diagonalise this matrix, find its eigenvalues and eigenvectors. Sketch the two eigenvectors in the $x - y$ coordinate system.

1.A Who is afraid of Cartesian Tensors?

I have so far interpreted the strain as a 3×3 matrix at every point in space. It is also viewed as a second rank tensor field. Any old collection of nine numbers at every point in space does not make a tensor field, just like any collection of three numbers does not make a vector field. They must satisfy the correct *transformation laws*. A scalar field, which a single number at every point is space must be invariant under coordinate transformation. Any vector quantity does not remain invariant, but must change exactly like (remain *covariant*) distance between two points under coordinate transformation. A vector is a tensor of rank one, the strain is a tensor of rank two. The vorticity can be either thought of as a tensor of rank one, i.e., a vector but it is a *polar* vector, one that changes sign under the coordinate transformation $x \to -x$, $y \to -y$ and $z \to -z$. Or it can be interpreted as a second rank anti-symmetric tensor. In this course I assume you know what a vector is, but not what a second or higher rank tensor is – second or higher rank tensor are common occurrence in fluid mechanics, I enthusiastically recommend a wonderful book

Here, by coordinate transformation I mean only rotation of coordinates. The transformation properties under more general, nonlinear coordinate transformation gives rise to more general non-Cartesian tensors – they become useful while studying the general theory of relativity.

by Rutherford Aris [4] for a comprehensive introduction to this subject. During this course we shall learn tensors as and when we need them, starting now.

Consider a coordinate system $\{x_1, x_2, x_3\}$. We do a linear transformation and obtain a new coordinate system $\{y_1, y_2, y_3\}$. The linear tranformation that relates them is give by

$$y_\alpha = a_{\alpha\beta} x_\beta, \tag{1.17}$$

where α, β runs from 1 to 3 and we have again assumed the Einstein summation convention. A typical example is rotation about an axis, as shown in Equation (1.18). The magnitude of the position vector remains unchanged under rotation; hence

$$y_\alpha y_\alpha = a_{\alpha\beta} a_{\alpha\mu} x_\beta x_\mu = x_\mu x_\mu. \tag{1.19}$$

Then we must have

$$a_{\alpha\beta} a_{\alpha\mu} = \delta_{\beta\mu} \tag{1.20}$$

where $\delta_{\beta\mu}$

$$\delta_{\beta\mu} = 1 \qquad \text{for} \beta = \mu \tag{1.21}$$

$$= 0 \qquad \text{otherwise} \tag{1.22}$$

is called the "Kronerker delta". You also know it as the identity matrix

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.23}$$

Let us check Equation (1.20) explicitly with the matrix $a_{\alpha\beta}$ given in Equation (1.18):

$$\beta = 1, \mu = 1 \qquad a_{11}a_{11} + a_{21}a_{21} = \cos^2\theta + \sin^2\theta = 1 \tag{1.24a}$$

$$\beta = 1, \mu = 2 \qquad a_{11}a_{12} + a_{21}a_2 = \cos\theta\sin\theta - \cos\theta\sin\theta = 0 \tag{1.24b}$$

A vector is defined to be a quantity that transforms just like the position vector under the same coordinate transformation, i.e., if any vector $\boldsymbol{A} = (A_1, A_2, A_3)$ in the coordinate system (x_1, x_2) transforms to $\boldsymbol{A}' = (A'_1, A'_2, A'_3)$ under the coordinate transformation given in Equation (1.17) then

$$A'_\alpha = a_{\alpha\beta} A_\beta \tag{1.25}$$

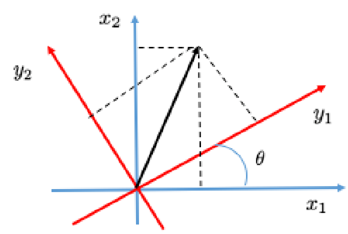
Any scalar quantity, e.g., the length of a vector, must remain unchanged. Now multiply both sides by $a_{\sigma\alpha}$ and sum over the repeated index α to get

$$a_{\alpha\sigma} A'_\alpha = a_{\alpha\beta} a_{\alpha\sigma} A_\beta \tag{1.26}$$

$$\implies a_{\alpha\sigma} A'_\alpha = \delta_{\beta\sigma} A_\beta = A_\sigma \tag{1.27}$$

$$\implies A_\alpha = a_{\beta\alpha} A'_\beta. \tag{1.28}$$

[4] Rutherford Aris. Vectors, tensors and the basic equations of fluid mechanics. Courier Corporation, 2012



An example of linear transformation of coordinates. The two Cartesian coordinate systems are related to each other by a rotation by the angle θ along an axis perpendicular to the plane. The same point in space are now labelled by two different coordinate systems (x_1, x_2), and (y_1, y_2) related to each other by

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{1.18}$$

In other words, the inverse of the matrix $a_{\alpha\beta}$ is its transpose. This is a property of the rotation matrix that you may already know. This is a consequence of demanding that the length of vectors remain unchanged under the transformation Equation (1.17).

Now let us try to find out how $s_{\alpha\beta}$ transforms under the same coordinate transformation. Start with Equation (1.11) which we now write in compact notation as

$$d_\alpha = s_{\alpha\beta} r_\beta \tag{1.29}$$

Here for notational simplicity we have used $\mathbf{r}$ instead of the symbol $\Delta\mathbf{r}$. As $\boldsymbol{d}$ is a vector, its transform $\boldsymbol{d}'$ will satisfy an equation like Equation (1.28). Hence

$$a_{\mu\alpha} d'_\mu = s_{\alpha\beta} r_\beta \tag{1.30}$$

Multiplying both sides by $a_{\kappa\alpha}$ and summing over the repeated index α and subsequent simplification we obtain

$$d'_\mu = a_{\kappa\alpha} a_{\gamma\beta} s_{\alpha\beta} r'_\gamma \tag{1.31}$$

In the transformed (primed) coordinate system we should have

$$d'_\mu = s'_{\mu\gamma} r'_\gamma \tag{1.32}$$

The intermediate steps works out in the following manner:

$$\begin{aligned} d_\alpha &= s_{\alpha\beta} r_\beta \\ \implies a_{\mu\alpha} d'_\mu &= s_{\alpha\beta} r_\beta \\ \implies a_{\kappa\alpha} a_{\mu\alpha} d'_\mu &= a_{\kappa\alpha} s_{\alpha\beta} r_\beta \\ \implies \delta_{\kappa\mu} d'_\mu &= a_{\kappa\alpha} s_{\alpha\beta} r_\beta \\ \implies d'_\mu &= a_{\kappa\alpha} s_{\alpha\beta} r_\beta \\ \implies d'_\mu &= a_{\kappa\alpha} a_{\gamma\beta} s_{\alpha\beta} r'_\gamma \end{aligned}$$

Comparing Equation (1.31) and Equation (1.32) we obtain the transformtion law for the strain

$$\boxed{s'_{\mu\gamma} = a_{\kappa\alpha} a_{\gamma\beta} s_{\alpha\beta}} \tag{1.33}$$

We have now proved that the strain is a second rank tensor. In general, any linear relationship between two vectors of the form

$$A_\mu = T_{\mu\nu} B_\nu \tag{1.34}$$

implies that the quantity $T_{\mu\nu}$ is a second rank tensor. Other examples are the moment of inertia tensor, or the polarizability tensor of a medium.

2 Hydrostatics

3 Flow of dry water: The Euler equation

4 Viscosity

Bibliography

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