4.5

~~The Sedov - Taylor problem~~:

Pressure ballanced plasma column

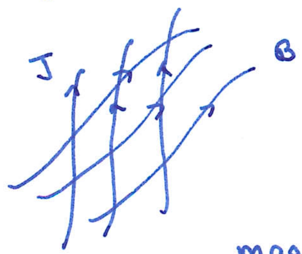
$$\operatorname{div} B = 0$$

$$\operatorname{div}(p\,\delta_{ij}) = J \times B \qquad v = 0, \quad \rho = \text{constant}$$

$$\nabla \times B = \mu_0 J$$

$$\Rightarrow \quad (B\cdot\nabla)\,p = 0 \quad \text{and} \quad (J\cdot\nabla)\,p = 0$$

pressure gradient is zero (pressure is constant) along lines of constant B and constant J



Hence $p(x, y, z) = \text{constant}$ are surfaces which contains magnetic lines of force and lines of J.

These are called magnetic surfaces. Every magnetic surface could be boundary of an equilibrium configuration.

$$\Rightarrow \quad \operatorname{div}\left(p\,\delta_{ij} + \frac{1}{\mu_0}\frac{B^2}{2}\delta_{ij} + \frac{1}{\mu_0}B_i B_j\right) = 0$$

$$\operatorname{div} \Pi_{ij} = 0$$

~~consider~~ $\cancel{x_j \Pi_{ij}\, dv}$

consider $\int x_i \partial_k \Pi_{ik} \, dv$

$$= \int \partial_k (\Pi_{ik} x_i) - (\Pi_{ik} \partial_k x_i) \, dv$$

$$= \int \partial_k (\Pi_{ik} x_i) \, dv - \int \Pi_{ii} \, dv$$

$= 0$

$$\Rightarrow \int \Pi_{ii} \, dv = \int \partial_k (\Pi_{ik} x_i) \, dv$$

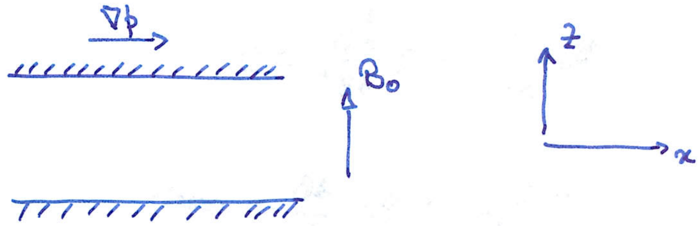
$$= \oint_S \Pi_{ik} x_i \hat{n}_k \, ds \qquad \text{- Gauss's theorem.}$$

$$\Rightarrow \int 3 \left(p + \frac{1}{\mu_0} \frac{B^2}{2} \right) dv = \oint_S \Pi_{ik} x_i \hat{n} \, ds$$

Let the plasma be confined by the surface $p = 0$ which does not extend to infinity. Then taking s at infinity the RHS $= 0$. But the LHS is never zero.

$\Rightarrow$ ~~Equi~~ steady state configuration of plasma in finite space is possible only if there are ~~currents at infinity~~. ~~So~~ external sources of currents.

Hartmann flow :



Pressure driven flow of plasma.

<u>symmetry</u> : v_x is non-zero. and function of z only.

B_x is non-zero and function of z only

p is function of x only.

<u>steady flow</u>

$$\partial_t v = 0$$
$$\partial_t \rho = 0$$
$$\partial_t B = 0$$

also $\operatorname{div} B = 0$

<u>Incompressible</u> : $\operatorname{div} v = 0$

$$(v \cdot \nabla) v = \nu \nabla^2 v - \nabla p + J \times B$$

$$J = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ B_x & 0 & 0 \end{vmatrix} = \partial_z B_x \hat{y}$$

LHS : $(v_x \partial_x) v_x = 0$

$$J \times B = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & \partial_z B_x & 0 \\ B_x & 0 & B_0 \end{vmatrix} = -\hat{z} B_x \partial_z B_x$$
$$\hat{x} B_0 \partial_z B_x$$

$\Rightarrow$ $$\hat{x} \nu \frac{d^2}{dz^2} v_x = \partial_x p - J \times B$$
$$= +\partial_x p \hat{x} + \hat{z} B_x \frac{d}{dz} \frac{B_x}{\mu_0} +$$
$$\partial_z p \hat{z} - \hat{x} B_0 \frac{d}{dz} \frac{B_x}{\mu_0}$$

$\Rightarrow$ $$p + \frac{1}{2} \frac{B_x^2}{\mu_0} = \text{function of } x \text{ only.}$$

Also : $$\nu \frac{d^2 v_x}{dz^2} + \frac{B_0}{\mu_0} \frac{dB_x}{dz} = \partial_x p = \text{const}$$

Induction eqn :

$$\nabla \times (v \times B - \eta J) = 0$$

$$\Rightarrow \quad \nabla \times (v \times B) + \frac{\eta}{\cancel{\mu_0}} \nabla^2 B = 0$$

$$\Rightarrow \quad B_0 \frac{d v_x}{d z} + \eta \frac{d^2 B}{d z^2} = 0$$

Boundary condition :

$v_x = 0$ at the boundary

$B_x = 0$

$J_y = 0$

At the end :

$$\frac{d^2 v_x}{d z^2} - \frac{1}{\delta^2} v_x + \Lambda = 0$$

with $\delta = \frac{\sqrt{\nu \eta \mu_0}}{B_0}$

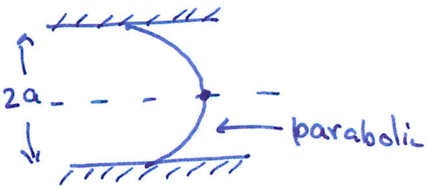
solution :

$$v = v_0 \frac{\cosh(a/\delta) - \cosh(z/\delta)}{\cosh(a/\delta) - 1}$$

The Hartmann number $G = \frac{a}{\delta}$. ← new length scale.

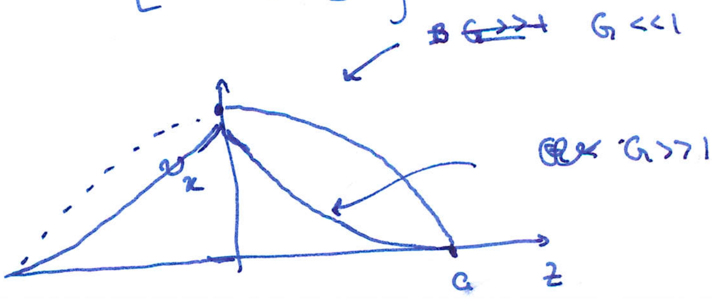
For $G \ll 1$, $\Rightarrow$ small magnetic field :

$$v = v_0 \left(1 - z^2/a^2\right)$$



For $G \gg 1$

$$v = v_0 \left\{ 1 - \exp\left[- \frac{a - |z|}{\delta} \right] \right\}$$



Lecture V

Linear stability

5.1 Consider a dynamical system with N degrees of freedom, $x_1, \ldots x_j, \ldots x_N$, & given by a state vector $|x\rangle$. Let the dynamical system be described by the following ~~partial~~ partial differential equation:

$$\partial_t |x\rangle = \mathcal{N}[|x\rangle]$$

where $\mathcal{N}$ is in general any non-linear function of $|x\rangle$, and its spatial derivatives.

Then assume that this system of equation have a solution $|x_0\rangle$ such that;

$$\partial_t |x_0\rangle = \mathcal{N}[|x_0\rangle]$$

But in general $|x_0\rangle$ may be just one of ~~ini~~ infinite number of possible solutions.

Is $|x_0\rangle$ stable?

Let us qualify the question. If we make a small change to $|x_0\rangle$; $|\delta x\rangle$, then we can write

$$\partial_t\left[|x_0\rangle + |\delta x\rangle\right] = \mathcal{N}\left[|x_0\rangle + |\delta x\rangle\right]$$

If $|\delta x\rangle$ is small, and the function $\mathcal{N}$ can be Taylor expanded at $|x_0\rangle$, we can write

$$\cancel{\partial_t |x_0\rangle} + \partial_t |\delta x\rangle = \cancel{\mathcal{N}[|x_0\rangle]} + \left.\frac{\delta \mathcal{N}}{\delta x}\right|_{|x_0\rangle} |\delta x\rangle + \text{h.o.t.}$$

~~Let~~ The operator $\left.\frac{\delta \mathcal{N}}{\delta x}\right|_{|x_0\rangle} |\delta x\rangle$ is linear in $|\delta x\rangle$. So, upto leading order in $|\delta x\rangle$ we can write

$$\boxed{\partial_t |\delta x\rangle = \mathcal{L}[x_0] |\delta x\rangle}$$

where $\mathcal{L}[x_0]$ is the linearized operator of $\mathcal{N}$ about $|x_0\rangle$.

This equation is a linear equation; subject to the same boundary conditions as the original equations. Hence can often be solved in N-dimensional vector space. This problem is far easier to solve than the original non-linear problem.

- It may not always be tractable analytically but can often be solved numerically.

- By solving this problem we can find out whether ~~to~~ the perturbation $|\delta x\rangle$ shall grow in time or not. Often time-dependence of $|\delta x\rangle$ is written as $\exp(i\omega t)$.

Let $\omega = \omega_R + i\omega_I$

$\Rightarrow \exp(i\omega t) \sim e^{i\omega_R t} e^{-\omega_I t}$

If
$\omega_I > 0$, $|\delta x\rangle$ decays in time
$\omega_I < 0$ $|\delta x\rangle$ grows in time
$\omega_I = 0$ $|\delta x\rangle$ has wave-like behavior.

This solves the stability problem, BUT only upto leading order in perturbation theory; hence only applies to infinitismal perturbations which may not be physically relevant. We shall come back to this point later.

5.2 A simple case :

Linear stability step I : find a solution to the dynamical eqns.

Finding a solution itself may be non-trivial. Also, if the solution itself is time-dependent then its stability analysis can be quite complicated.

So let us start with a simple time-independent solution.

Equations of isothermal MHD :

$$\partial_t(\rho \mathbf{v}) + \operatorname{div}(\rho v_i v_j + p\delta_{ij} + \sigma_{ij}) = J \times B$$

$$-\frac{1}{2}\nabla B^2 \frac{1}{\mu_0}$$

$$(B\cdot\nabla)B\frac{1}{\mu_0}$$

$$\partial_t \rho + \operatorname{div}(\rho v) = 0 \qquad c^2 =$$

$$\partial_t B = \nabla \times (v \times B - \eta J)$$

<u>solution</u> $\quad v = 0, \quad \rho = \text{constant} = \rho_0, \quad B = \hat{z} B_0 = \text{constant}$

<u>Linear stability step II</u> : Add an infinitismal perturbation and linearize the equations :

$$v = v_0 + \delta v, \qquad \rho_0 \partial_t \delta v = -c^2 \nabla \delta\rho - \nabla\left(\frac{B_0 \cdot \delta b}{\mu_0}\right) + \frac{1}{\mu_0}(B_0\cdot\nabla)\delta b$$

$$\rho = \rho_0 + \delta\rho,$$

$$B = B_0 + \delta b, \qquad \partial_t \delta\rho = -\rho_0(\nabla\cdot\delta v)$$

$$\partial_t \delta b = (B_0\cdot\nabla)\delta v - B_0(\nabla\cdot\delta v)$$

(ignoring dissipative terms which are linear in δv, and δb)

This is a matrix equation if we use Fourier transforms:

$$\cancel{\delta u} \quad \delta v = \tilde{v} \exp(i\omega t + i\, k\cdot x)$$

$$\delta \rho = \tilde{\rho} \exp(i\omega t + i\, k\cdot x)$$

$$\delta b = \tilde{b} \exp(i\omega t + i\, k\cdot x)$$

substituting we get the form:

$$M|\psi\rangle = 0$$

where $|\psi\rangle = (\tilde{v}_x, \tilde{v}_y, \tilde{v}_z, \tilde{\rho}, \tilde{b}_x, \tilde{b}_y, \tilde{b}_z)$

$$M = \begin{pmatrix} i\omega & x & x & x & & & \\ & i\omega & x & x & & & \\ & & \cdot & & & & \\ x & x & & \cdot & & & \\ x & x & & & \cdot & & \\ & & & & & i\omega & \\ x & x & & & & & \end{pmatrix}$$

A solution exists only when $\det M = 0$

That would be $f(\omega, k) = 0$ a function polynomial in

which is ~~an~~ ~~ed~~ a 7th order polynomial; in other

ω, k; with

words there are 7 branches of the (ω, k) relationship. These are called dispersion relations.

Let us make life particularly simple:

space in 1-dimensional, along z; and $B_0 = B_0 \hat{z}$

$$\Rightarrow \quad \partial_x = 0, \quad \partial_y = 0$$

Then we obtain the following solution for ω.

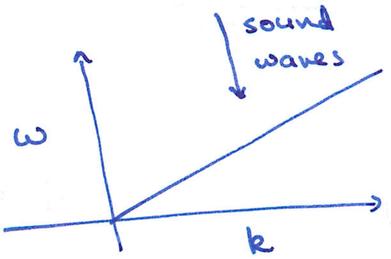
$\omega = 0$, (one branch)

$\omega^2 - c^2 k^2 = 0$ (two degenerate branches)

$\omega^4 - 2 v_A^2 k^2 \omega^2 + v_A^4 k^4 = 0$

$\left(\omega^2 - v_A^2 k^2\right)^2 = 0$

four degenerate branches. $v_A = \dfrac{B_0}{(\rho_0 \mu_0)^{1/2}}$



Alfven waves

- Furthermore the Alfvenic modes are perpendicular to the direction of propagation $\hat{z}$. They are transverse waves. Magnetic field behaves like a string.

- If we look at incompressible approximation; then $\Longleftrightarrow$ div $\delta v = 0$
  $\Rightarrow$ $i k_\mu \tilde{v}_\mu = 0$ $\Rightarrow$ $k_z v_z = 0$
  $\Rightarrow$ $\tilde{v}_z = 0$ and $\tilde{\rho} = 0$, But the Alfven wave survives. Although sound waves do not.

- How large can the amplitude of the waves be? There is no limit from linear theory, they can be anything.

5.3 The more general case of 3-d space :
(But keep the equations isentropic)

The linearized equations look like :

$$\omega \vec{b} = \vec{k} \times (\vec{v} \times \vec{B})$$

$\Rightarrow$ The perturbations in $\vec{b}$ are perpendicular to the direction of propagation of the wave.
They are transverse waves.
Take $\vec{k}$ along the x direction.
Also ~~noth~~ note that $\vec{k} \cdot \vec{b} = 0$ is automatically satisfied.
and define phase velocity $u = \frac{\omega}{k}$
The the x-y plane as the plane containing $\vec{k}$ and $\vec{B}$.
Then we have :

$$u b_z = - v_z B_x \qquad u v_z = - \frac{B_x b_z}{\mu_0 \rho_0}$$

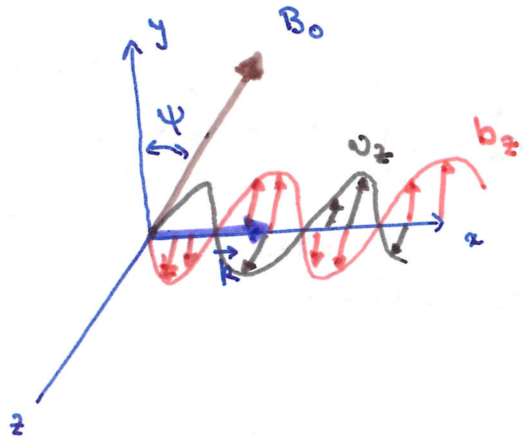
This two forms a pair.
For both of them to be true : $u^2 = \frac{B_x^2}{\mu_0 \rho_0}$

$\Rightarrow$ $\boxed{\omega = c_A k}$. $c_A = \frac{B_x}{\sqrt{\mu_0 \rho_0}}$

We get the Alfven result back again.

But with B_x

and $b_z = - \sqrt{\mu_0 \rho_0}\, v_z$



Now consider the other two components

$$u b_y = v_x B_y - v_y B_x$$

$$u v_y = -\frac{B_x b_y}{\mu_0 \rho_0}$$

$$v_x \left(u - \frac{c^2}{u} \right) = B_y b_y / \mu_0 \rho_0$$

By setting the determinant to zero we have:

$$u_{f,s}^2 = \frac{1}{2} \left\{ \frac{B^2}{\rho_0 \mu_0} + c^2 \pm \left[\left(\frac{B^2}{\mu_0 \rho_0} + c^2 \right)^2 - \frac{4 B_x^2}{\mu_0 \rho_0} c^2 \right]^{1/2} \right\}$$

fast and slow magnetosonic waves.

<u>Case 1</u> $\quad \frac{B^2}{\mu_0 \rho_0} << c^2$ $\quad$ small magnetic energy compared to the energy in sound modes.

$u_f \simeq c \quad \Rightarrow \quad v_y << v_x$

we essentially get back usual sound waves.

$u_s \simeq c_A \quad \Rightarrow \quad v_x << 1,$

$\Rightarrow \quad c_A b_y \simeq - v_y B_x$

$\Rightarrow \quad \cancel{v_y \simeq} \; \frac{b_y}{\;}$

$\Rightarrow \quad b_y \simeq - \sqrt{\mu_0 \rho_0} \; v_y$

Alfven like. But different polarization.

<u>case 2</u> $\quad \dfrac{B^2}{\mu_0 \rho_0} >> c^2$

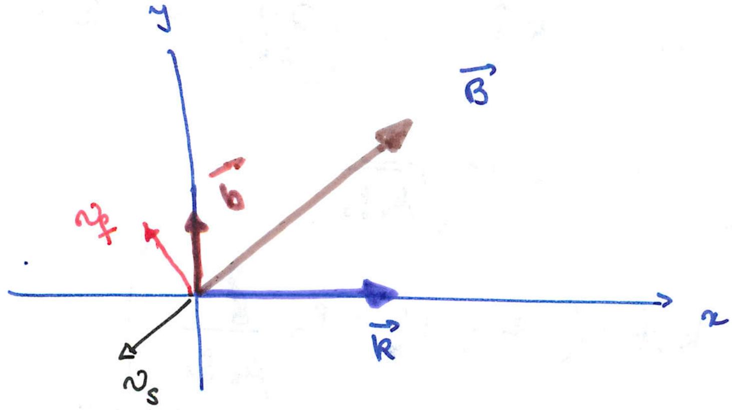
$\Rightarrow \quad u_f \sim \dfrac{B}{\sqrt{\mu_0 \rho_0}}$

and $\quad (v_x B_x + v_y B_y) \sim 0$

$\Rightarrow \quad \vec{v}$ is perpendicular to $\vec{B}$.

$u_s \sim c \dfrac{B_x}{B}$

and $\vec{v}$ is anti-parallel to $\vec{B}$



$u_s \leq c_A \leq u_f \quad , \; u_f \geq c, \quad u_s \leq c$

If $\vec{k}$ and $\vec{B}$ are parallel: we have, c_A and c

for $\vec{k}$ and $\vec{B}$ are perpendicular: only fast magnetosonic waves exists.

5.4 Effects of dissipative terms :

clearly the solution will loose energy and the waves will damp as they progress.

Let $\langle Q \rangle \equiv$ average energy dissipation rate

$$\simeq \eta J^2 + \nu \omega^2 \rho_0$$

$$\simeq \eta \left(\frac{\partial b}{\partial x}\right)^2 + \nu \left(\frac{\partial v}{\partial x}\right)^2 \rho_0$$

$\langle q \rangle$ = average energy flux

$$\simeq - B_x \, b . v$$

Both of them are quadratic in ~~the~~ fluctuations which is the first non-zero correction.

As the wave progresses

$$(\text{wave energy}) \sim e^{-\langle Q \rangle / \langle q \rangle \, x}$$

$$(\text{wave amplitude}) \sim e^{-\langle Q \rangle / 2\langle q \rangle \, x}$$

From Alfven waves $\sim e^{-\gamma x}$

with $$\gamma = \frac{\omega^2}{2 C_A^2} \left(\eta + \frac{\nu}{\mu_0}\right)$$