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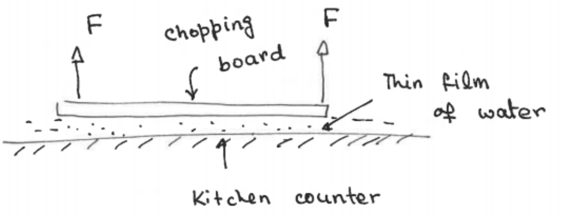
Exam in Fluid mechanics (MO5001)

Write the solution of each problem on a separate paper, and write your identification number on every paper.

Allowed aids: calculator, sheet with vector analysis relations.

Grading: A 90-100%, B 80-89%, C 65-79%, D 55-64%, E 50-54%, Fx 45-49%, F 0-44%

1. While cooking last night, I put my chopping board on the kitchen counter. I had not noticed that there was a small amount of water on the kitchen counter already. After using the board I tried to lift it up in the way I sketch in figure 1. I realised that I had to exert a force much larger than the weight of the board itself.



   (a) Explain qualitatively (and concisely) why ? (3 p)

   (b) Assume that the shape of my chopping board is a circular disk of radius a. The thickness of the film of water trapped between the board and the kitchen counter is h. Calculate the force necessary to pull the board by a distance Δh in time Δt. State clearly all the assumption you need to make to solve this problem. (7 p)

2. A thin rectangular plate of dimension $L_x \times L_z$ is immersed in a fluid of kinematic viscosity ν and density ρ. The plate is being pulled by a force such that it moves with a velocity v along the x direction. Far away from the plate the fluid is at rest. Assume that the flow is laminar. Ignore the edge effects. Estimate the the power necessary to keep the plate moving with a constant velocity. (Hint : The power necessary is equal to the power dissipated by the viscous forces. You can estimate the viscous forces from the stress at the surface of the disc. You need the thickness of the boundary layer to estimate the stress.) (5p)

3. Answer the following short questions. You just need to write the final answer. Each question is worth 2 points.

   (a) A scalar function of three Cartesian coordinates, x, y, z is

   $$T(x, y, z) = \sin(x)\cos(y) + \cos(y)\sin(z) \tag{1}$$

   If $\vec{G} = \vec{\nabla} T$ then calculate $\vec{\nabla} \times \vec{G}$.

   (b) After deformation the displacement field in a material is given by the following expression

   $$u_x = \alpha[2x + \sin(y) + 5z^3] \tag{2}$$
   $$u_y = \alpha[e^{-x} - y + \cos(z)] \tag{3}$$
   $$u_z = \alpha[\sin(x) + \cos(y) - z] \tag{4}$$

   Here α is small such that you can apply the approximation of small deformation everywhere. Under this deformation calculate the change is volume of the material.

   (c) In which of the following cases can I write the velocity $\vec{v} = \nabla\Psi$ where Ψ is a scalar function without any loss of generality:
   (a) if the flow is incompressible, (b) if the flow is irrotational, or (c) if the flow is steady.

   (d) In a turbulent boundary layer very close to the wall how does the mean stream-wise velocity ($\langle v_x \rangle$) depends on the wall-normal coordinate (y) ?

   (e) Which of the following are true ? (More than one may be true.) Viscosity of a Newtonian fluid is

      i. a scalar.
      ii. can be described by two scalar quantities.
      iii. is a fourth rank tensor.

   (10 p)

4. The horizontal component of the Navier-Stokes equations in a rotating system is

   $$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + f\hat{\mathbf{z}} \times \boldsymbol{u} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \boldsymbol{u},$$

   where the velocity $\boldsymbol{u}$ is assumed to be purely horizontal.

   (a) Define the Rossby number and the Ekman number in terms of the time scale T, horizontal length scale L and vertical length scale H, and explain how these numbers measure the ratio between various terms in the equation above. Also give the thickness of the Ekman layer.

   (b) Simplify the equation above for the case that $\boldsymbol{u}$ is stationary and horizontally homogeneous, i.e. does not depend on t, x or y.

   (c) A constant wind blows over the ocean and exerts the wind stress $\boldsymbol{\tau}$ on the surface of the ocean. Give the appropriate boundary condition for the flow $\boldsymbol{u}$ in the ocean.

(d) Assume that there is no geostrophic flow, and determine the volume flux in the ocean Ekman layer.

(10 p)

5. The rotating shallow-water equations are

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + f\hat{\mathbf{z}} \times \boldsymbol{u} = -g\nabla h,$$

$$\frac{\partial h}{\partial t} + \nabla \cdot (\boldsymbol{u}h) = 0,$$

where h is the depth and $\boldsymbol{u}$ the velocity.

(a) Mention two different kinds of waves that can be described by these equations.

(b) The equations can describe a slow mode and a fast mode. What assumption must be made to derive an equation that only describes the slow mode? Motivate the answer.

(c) The potential vorticity for the shallow-water equations is defined by

$$q = \frac{f(y) + \zeta}{h},$$

where $\zeta = \partial v/\partial x - \partial u/\partial y$ is the relative vorticity. Describe the conservation law for potential vorticity in words, and write it as an equation.

(5 p)

6. An ocean current flows northeastward at the latitude 20°N. The current is barotropic, i.e. the velocity is uniform from the bottom to the surface, and the local depth of the ocean is 3000 m. The Rossby number of the flow is very small, much smaller than 0.1.

(a) When the current reaches the latitude 30°N the Rossby number is still very small. What is the local depth of the ocean?

(b) Suppose that the current instead flowed along contours of constant depth. What would the Rossby number and the relative vorticity be when it reached 30°N?

(5 p)

7. A fluid layer is bounded by vertical walls at $x = 0$ and $x = L$. The bottom is flat, and the layer thickness is h_0 at $x = 0$ and h_L at $x = L$. Calculate the northward volume transport between the walls. Assume that the flow is governed by the rotating shallow-water equations:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + f\hat{\mathbf{z}} \times \boldsymbol{u} = -g\nabla h,$$

$$\frac{\partial h}{\partial t} + \nabla \cdot (\boldsymbol{u} h) = 0,$$

where h is the depth and $\boldsymbol{u}$ the velocity. The flow is steady, and $\boldsymbol{u} = v(x)\hat{\mathbf{y}}$. (5 p)