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Exam in Fluid mechanics (MO5001)

Write the solution of each problem on a separate paper, and write your identification number on every paper.

Allowed aids: calculator, sheet with vector analysis relations.

Grading: A 90-100%, B 80-89%, C 65-79%, D 55-64%, E 50-54%, Fx 45-49%, F 0-44%

1. Answer the following short questions. You just need to write the final answer. Each question is worth 2 points.

   (a) A scalar function of three Cartesian coordinates, x, y, z is

   $$T(x, y, z) = \sin(x)\cos(y) + \cos(y)\sin(z) \tag{1}$$

   If $\vec{G} = \vec{\nabla} T$ then calculate $\vec{\nabla} \times \vec{G}$.

   (b) After deformation the displacement field in a material is given by the following expression

   $$u_x = \alpha[2x + \sin(y) + 5z^3] \tag{2}$$
   $$u_y = \alpha[e^{-x} - y + \cos(z)] \tag{3}$$
   $$u_z = \alpha[\sin(x) + \cos(y) - z] \tag{4}$$

   Here α is small such that you can apply the approximation of small deformation everywhere. Under this deformation calculate the change is volume of the material.

   (c) In which of the following cases can I write the velocity $\vec{v} = \nabla\Psi$ where Ψ is a scalar function without any loss of generality:
   (a) if the flow is incompressible, (b) if the flow is irrotational, or (c) if the flow is steady.

   (d) In a turbulent boundary layer very close to the wall how does the mean stream-wise velocity ($\langle v_x \rangle$) depends on the wall-normal coordinate (y) ?

   (e) The lubrication approximation, equations that describes a laminar boundary layer, and the shallow-water equations are all derived from the incompressible Navier–Stokes equations. They all assume that the Reynolds number is small. What is another crucial common aspect of all these derivations ?

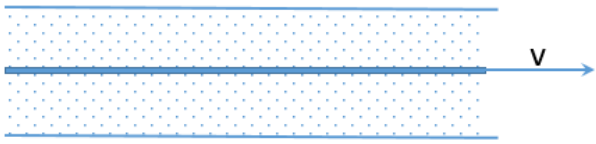

Figure 1: Problem 4

2. Two manometric tubes are mounted on a horizontal pipe of varying cross-section at the sections S_1 and S_2 (see left panel of figure 1). Find the flux (volume of water flowing across the pipe's cross section per unit time) if the difference in water columns is equal to Δh. (5 p)

3. A thin rectangular plate of dimension $L_x \times L_z$ is immersed in a fluid of kinematic viscosity ν and density ρ. The plate is being pulled by a force such that it moves with a velocity v along the x direction. Far away from the plate the fluid is at rest. Assume that the flow is laminar. Ignore the edge effects. Estimate the the power necessary to keep the plate moving with a constant velocity. (Hint : The power necessary is equal to the power dissipated by the viscous forces. You can estimate the viscous forces from the stress at the surface of the disc. You need the thickness of the boundary layer to estimate the stress.) (5p)

4. Same as in Problem 3 with an additional feature: the plate is confined vertically within a cavity. The clearance between the disk and the horizontal planes of the cavity is equal to h where $h \ll L_x$ as shown in figure 1. Ignore the edge effects. Calculate the the power necessary to keep the plate moving. (Hint : Use lubrication approximation) (5p)

5. The horizontal component of the momentum equation for a fluid in a rotating system is
$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + f\hat{\mathbf{z}} \times \mathbf{u} = -\frac{\nabla_z p}{\rho} + \nu \nabla^2 \mathbf{u},$$
where ∇_z denotes the horizontal components of ∇. We assume the velocity $\mathbf{u}(x, y.z, t)$ to be purely horizontal, and the density ρ to be constant.

   (a) Define the Rossby number and the Ekman number in terms of the time scale T, horizontal length scale L, vertical length scale H, and the coefficients of the equation, and explain how these numbers measure the ratio between various terms in the equation above.

   (b) Assume that the Rossby number and the Ekman number are both small, and derive an approximate expression for $\mathbf{u}$.

   (c) Assume that there is a solid boundary at $z = 0$, with fluid above it, and that $\mathbf{u}$ is stationary and horizontally homogeneous, i.e. does not depend on t, x or y. Simplify the equation above for this case, and give the appropriate boundary condition at $z = 0$.

   (d) Assume that the flow is constant far above the boundary, i.e. $\mathbf{u} = u_0\hat{\mathbf{x}}$, where $u_0 = const$. This flow does not satisfy the boundary condition at $z = 0$. Explain how the boundary condition can be satisfied. Also estimate the thickness of the layer in which the flow is not constant, by estimating the the magnitude of the terms in the equation obtained in (c). It is not necessary to solve the equation.

(10 p)

6. The non-rotating shallow-water equations are

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -g \nabla h,$$

$$\frac{\partial h}{\partial t} + \nabla \cdot (\mathbf{u} h) = 0,$$

where h is the depth and $\mathbf{u}$ the velocity.

(a) Linearise the equations around a suitable background state.

(b) Derive the dispersion relation for linear waves.

(c) How long time does it take for a wave of this kind to cross the Atlantic? Assume that it is 4000 km wide and 4000 m deep.

(10 p)

7. The rotating shallow-water equations are

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + f \hat{\mathbf{z}} \times \mathbf{u} = -g \nabla h,$$

$$\frac{\partial h}{\partial t} + \nabla \cdot (\mathbf{u} h) = 0,$$

where h is the depth and $\mathbf{u}$ the velocity. Assume that the water is surrounded by a solid boundary (a coast). Specify suitable boundary conditions, and show that the total mass is conserved.

(5 p)