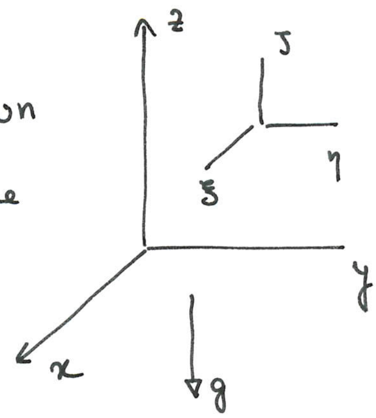
1. Wave equation from ~~as~~ a Lagrangian standpoint. :

Consider a vertically stratified compressible fluid at rest in gravity.



The coordinate system is as shown in the figure. (ξ, η, ζ) are the Lagrangian displacements.

The kinetic energy density

$$T = \frac{1}{2}\rho\left(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2\right)$$

The potential energy is the sum of several terms :

(a) Elastic energy : $V_{el} = \frac{1}{2}\lambda\epsilon^2$

with $\epsilon = \left(\frac{\partial\xi}{\partial x} + \frac{\partial\eta}{\partial y} + \frac{\partial\zeta}{\partial z}\right)$, $\lambda = \rho c^2$ is the bulk modulus

(b) If you move a fluid parcel by ζ upward, there is a difference in density between the ~~parcel~~ parcel and its new surrounding.

$$\Delta\rho = \rho_0(z+\zeta) - \rho_0(\zeta)$$

$$= \left(\frac{d\rho_0}{dz}\right)\zeta$$

The buoyancy force

$$F = -g\,\Delta\rho = -g\left(\frac{d\rho_0}{dz}\right)\zeta$$

The corresponding potential:

$$V_B = -\frac{1}{2}\,g\left(\frac{d\rho_0}{dz}\right)\zeta^2$$

(c) There is a third contribution:

As the particle is displaced upward; it is also compressed. This compression is due to the compressive field

$$\epsilon = \left(\partial_x\xi + \partial_y\eta + \partial_z\zeta\right)$$

The corresponding potential energy is

$$V_c = \zeta\,g\,\Delta\rho$$

$$= \zeta\,g\,(-\rho\epsilon)$$

$$= -\rho\,g\,\zeta\,\epsilon$$

The net Lagrangian

$$\mathcal{L} = T - V_{el} - V_B - V_c.$$

- Given this Lagrangian, show by taking functional derivatives that the corresponding Euler-Lagrange eqn. are:

$$\rho \ddot{\xi} - \frac{\partial}{\partial x} \lambda \epsilon + \rho g \frac{\partial \zeta}{\partial x} = 0$$

$$\rho \ddot{\eta} - \frac{\partial}{\partial y} \lambda \epsilon + \rho g \frac{\partial \zeta}{\partial y} = 0$$

$$\rho \ddot{\zeta} - \frac{\partial}{\partial z} \lambda \epsilon - g \rho \left(\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} \right) = 0$$

5 marks

2. Write down the linearized equations in a vertically stratified medium. Then use the Lagrangian displacement ~~to~~ to rewrite the equations.

show that

$$\tilde{\rho} + \rho_0 \partial_x \xi_1 + \rho_0 \partial_y \xi_2 + \frac{d}{dz} (\rho_0 \xi_3) = 0 \qquad -(1)$$

Here $\tilde{\rho}$ is the perturbed density and (ξ_1, ξ_2, ξ_3) are the three Lagrangian displacements.

Similarly, from the linearized momentum eqn. show that

$$\cancel{\rho_0 \partial_t^2 \xi_3 - \frac{d}{dz} \rho \xi_3}$$

$$\rho_0 \partial_t^2 \xi_3 + \partial_z \left(c^2 \tilde{\rho} \right) = - g \tilde{\rho} \qquad -(2)$$

In (1) ignore $\partial_x \xi_1$ and $\partial_y \xi_2$.

Substitute from (1) to (2) then do the following approximation

$$\frac{\partial \xi_3}{\partial z} \simeq 0$$

Under ~~this~~ these approximations show that the following holds:

5 marks

$$\partial_t^2 \xi_3 + N^2 \xi_3 = 0$$

where $$N^2 = -g\left(\frac{d}{dz}\ln \rho_0 + \frac{g}{c^2}\right)$$

N^2 is called the Brunt-Väissala frequency.

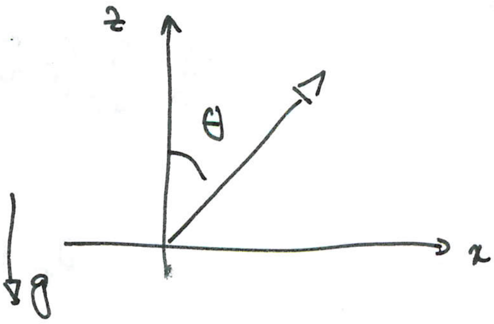
3. In a stratified fluid of uniform Brunt-Vaisalla frequency N^2 show that the equations

$$\tilde{p} = N\sin\theta \, \frac{q_1}{k} \exp\left[i\left(Nt\cos\theta - kx + kz\tan\theta\right)\right]$$

$$\rho_0 u = \left(\tan\theta, 0, 1\right) q_1 \exp\left[i\left(Nt\cos\theta - kx + kz\tan\theta\right)\right]$$

with $0 < \theta < \frac{\pi}{2}$ represents plane internal waves which transmit an energy flux

$$\frac{1}{2}\frac{q_1^2}{\rho_0 k_1} N\tan\theta$$ in a direction shown in the figure



5 marks

4. Stability of inviscid Couette flow:

Consider the inviscid Couette flow with

~~U_r~~ $U_r = U_z = 0$ and $U_\theta = V(r) = r\,\Omega(r)$

Here $V(r)$ [and $\Omega(r)$] is an arbitrary function of r.

(a) Write down the linearized equations for the perturbations, assuming axisymmetric perturbation (i.e. δv_r, δv_θ, δv_z, δp are ~~not~~ ~~cons~~ not functions of the angular variable θ)

(b) Assume the perturbations have the following dependence $\sim \exp i(pt + kz)$. Show that the eigenvalue problem is

$$\cancel{i\sigma\,\delta v_r}$$

$$ip\,\delta\hat{v}_r - 2\Omega\,\delta\hat{v}_\theta = -\frac{d}{dr}(\delta\hat{p})$$

$$ip\,\delta\hat{v}_\theta + \left[\Omega + \frac{d}{dr}(r\Omega)\right]\delta\hat{v}_r = 0$$

5 marks

$$ip\,\delta v_z = -ik(\delta\hat{p})$$

and

$$\frac{d\delta v_r}{dr} + \frac{\delta v_r}{r} + ik\,\delta v_z = 0$$

(c) Now use Lagrangian dispacements

$\delta u_r = ip\xi_r$, $\delta v_\theta = ip\xi_\theta - r\frac{d\Omega}{dr}\xi_r$, $\delta v_z = ip\xi_z$

to show that the following eqn. holds

$$\left[p^2 - \Phi(r)\right]\xi_r = \frac{d\delta\hat{p}}{dr}$$

$$\frac{1}{r}\frac{d}{dr}(r\xi_r) = \frac{k^2}{p^2}\delta\hat{p}$$

with $\Phi(r) = \frac{2\Omega}{r}\frac{d}{dr}(r^2\Omega)$