Lecture 2.

In this lecture we shall again derive the equations of MHD ~~but~~ but this time with slightly more care.

We start again by writing down equations for conserved quantities. [per unit volume; or density variables]

mass $$\partial_t \rho + \operatorname{div} g = 0 \qquad 1a$$

momentum $$\partial_t g + \operatorname{div} \pi_{ij} = 0 \qquad 1b$$

energy $$\partial_t \varepsilon + \operatorname{div} j_\varepsilon = 0 \qquad 1c$$

Without an expression for the ~~the~~ currents these equations are ~~so~~ useless.

The current for mass density is clearly

$$g = \rho v \quad \text{momentum.} \qquad 2a$$

The current for momentum density is clearly

$$\Pi_{ij} = \rho v_i v_j + \text{other forces.}$$

Let us assume that there are no external forces. Then the only force is pressure.

$$\Pi_{ij} = \rho v_i v_j + p\delta_{ij} \qquad 2b.$$

The derivation of the ~~heat~~ energy current is more involved. We need to remember thermodynamics.

The second law of thermodynamics for a fluid system with fixed mass is

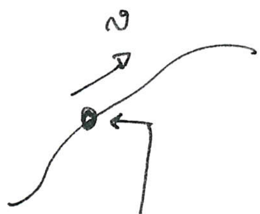
$$\underbrace{T}_{\text{temperature}}\, d\underbrace{S}_{\text{entropy}} = d\underbrace{E}_{\text{internal energy}} + \underbrace{p}_{\text{pressure}}\, dV \qquad (3)$$

Let us define the extensive quantities a per unit mass:

$$T\,d\tilde{s} = d\tilde{e} + p\, d\left(\frac{1}{\rho}\right)$$

$$= d\tilde{e} - \frac{p}{\rho^2}\, d\rho$$

$$\Rightarrow \quad \rho T\, d\tilde{s} = \rho\, d\tilde{e} - \frac{p}{\rho}\, d\rho$$



Now apply this equation to a "fluid parcel"
Here $\tilde{e}$ is the internal energy per unit mass, it does not contain the kinetic energy of the fluid parcel.

$$\rho T D_t \tilde{s} = \rho D_t \tilde{e} - \frac{p}{\rho} D_t \rho \qquad (4)$$

Now use 1a and 2a to write the continuity eqn.

$$\partial_t \rho + \mathrm{div}(\rho v) = 0$$

$$\Rightarrow \quad D_t \rho + \rho\, \mathrm{div}\, v = 0 \qquad (5)$$

Substitute (5) in (4) to obtain:

$$\rho T D_t \tilde{s} = \rho D_t \tilde{e} + p \,\mathrm{div}\, v \qquad - (6).$$

Consider dissipationless hydrodynamics:

$$D_t \tilde{s} = 0 \qquad - (7)$$

Now note the following identity

$$\rho D_t \psi = \partial_t(\rho\psi) + \mathrm{div}(v\psi) \qquad - (8)$$

true for any ~~quantity~~ ~~ψ~~ density variable ψ when ρ and v together satisfies the continuity eqn.

The equations for dissipationless hydrodynamics is then:

$$\partial_t \rho + \mathrm{div}(\rho v) = 0 \qquad 9a$$

$$\partial_t(\rho v) + \mathrm{div}\left(p\delta_{ij} + \rho v_i v_j\right) = 0 \qquad 9b$$

$$\partial_t s + \mathrm{div}(v s) = 0 \qquad 9c$$

where $s = \rho\tilde{s} \equiv$ entropy per unit volume.

The last equation is obtained by ~~ts~~ using (7) and (8)

2.2

An alternative formulation of the problem uses the energy equation and its current.

(7) implies:

$$\rho D_t \tilde{e} + p \,\mathrm{div}\, v = 0$$

$$\Rightarrow \quad \partial_t(\rho e) + \mathrm{div}(\rho e v) + p \,\mathrm{div}\, v = 0 \qquad (10)$$

~~Remember~~, ~~the~~ Here $e = \rho\tilde{e} \equiv$ internal energy per unit volume.

The total energy per unit volume:

$$\mathcal{E} = e + \underbrace{\tfrac{1}{2}\rho v^2}_{\text{kinetic energy.}}$$

From (9b)

$$\partial_t(\rho v_i) + \partial_j(\rho v_i v_j) + \partial_i p = 0$$

multiply by v_i and sum over i to obtain

$$\partial_t\left(\frac{\rho v_i^2}{2}\right) + \partial_j\left(v \rho \frac{v^2}{2}\right) + v\cdot\nabla p = 0 \qquad -(11)$$

Add (10) and (11) to obtain

$$\partial_t \mathcal{E} + \mathrm{div}\left[v(\mathcal{E} + p)\right] = 0 \qquad -(12)$$

$\Rightarrow$ The heat flux: $\quad j_{\mathcal{E}} = v(\mathcal{E} + p) \qquad$ 13a

with $\quad \mathcal{E} = e + \frac{1}{2}\rho v^2 \qquad$ 13b

2.3 Dissipation:

Once we allow for dissipation the entropy equation will change to:

$$\partial_t s + \mathrm{div}\left(v s + \frac{Q}{T}\right) = 0 \qquad - 14.$$

where Q is the heat flux

There can be several contributions to this heat flux.

i)

(i) heat transport due to gradient of temperature

$$\varphi = -K \nabla T$$

K — thermal conductivity

(a) K must be a scalar for an isotropic fluid
(b) K must be positive for entropy to ~~always~~ to be non-decreasing.

(ii) heat transport due to viscous heating.

In a fluid there can be momentum transport because of viscous stresses.

$$\Pi_{ij} = \rho v_i v_j + p\delta_{ij} - \sigma_{ij}$$

Then $$Q = -v_j \sigma_{ij}$$

In general $$\sigma_{ij} = \eta_{ijkl}\, \partial_k v_l$$

η_{ijkl} — viscosity tensor

For an isotropic fluid there are only ~~to~~ two independent contribution to a 4th rank tensor

$$\sigma_{ij} = \eta\left(\partial_i v_j + \partial_j v_i - \frac{2}{3}\delta_{ij}\,\partial_k v_k\right) + \zeta\,\delta_{ij}\,\partial_k v_k$$

η — shear viscosity

ζ — bulk viscosity.

The equations of viscous hydrodynamics : ⑥

$$\partial_t \rho + \operatorname{div}(\rho v) = 0$$

$$\partial_t(\rho v) + \operatorname{div}\left(p\,\delta_{ij} + \rho v_i v_j - \sigma_{ij}\right) = 0$$

$$\sigma_{ij} = \eta\left(\partial_i v_j + \partial_j v_i - \frac{2}{3}\delta_{ij}\,\partial_k v_k\right) + J\,\delta_{ij}\,\partial_k v_k$$

$$\partial_t s + \operatorname{div}\left(v s + \frac{q}{T}\right) = 0$$

$$q = -k\dot{\nabla} T - v_j \sigma_{ij} + \text{radiation}$$

× In addition we need an equation of state often the ideal gas equation $\cancel{c_s^2 = \frac{\partial p}{\partial \rho}}$

$$p = \rho T$$

× Incompressible approximation :

$$\rho : \text{constant} \equiv 1.$$

$$\Rightarrow \qquad \operatorname{div} v = 0$$

$$\partial_t v + \operatorname{div}(v_i v_j) = -\nabla p + \nu \nabla^2 v \qquad \nu = \frac{\eta}{\rho}$$

— The Navier-Stokes equation.

× Isothermal approximation.

3.1 How to include the magnetic field.

* We consider a plasma that is a very good conductor. The charge separation is negligible. Electrostatic field is almost zero.

The force on a current density J is

$$J \times B.$$

Hence the contribution of magnetic field to the momentum eqn. is

$$\partial_t (\rho v) + \operatorname{div}\left(p\delta_{ij} + \rho v_i v_j - \sigma_{ij}\right) = J \times B.$$

Maxwell's eqn

$$\nabla \times B = \mu_0 J - \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

Assume that all time dependence is much slower compared to speed of light; hence ignore the displacement current.

$$\Rightarrow \quad \partial_t (\rho v) + \operatorname{div}\left(p\delta_{ij} + \rho v_i v_j - \sigma_{ij}\right) = \frac{1}{\mu_0}(\nabla \times B) \times B$$

Using vector identities one can write this as

$$\partial_t (\rho v) + \operatorname{div}\left[p\delta_{ij} + \rho v_i v_j - \sigma_{ij} - B_i B_j + \delta_{ij}\frac{B^2}{2}\right] = 0$$

- Magnetic field contributes to pressure:

$$\delta_{ij}\left(p + \frac{B^2}{2}\right)$$

- Maxwell's stress: $B_i B_j$

The eqn describing the evolution of magnetic field

$$\nabla \times E = -\frac{\partial B}{\partial t} \qquad \text{Faraday's law.}$$

Ohm's law

$$J = \sigma\left(E + v \times B\right)$$

~~σ: thermal conductive~~

σ: electrical conductivity.

$$\Rightarrow \quad \frac{\partial B}{\partial t} = \nabla \times \left[v \times B - \frac{1}{\sigma} J\right]$$

$$= \nabla \times \left(v \times B - \frac{1}{\mu_0 \sigma} \nabla \times B\right)$$

$$= \nabla \times \left(v \times B\right) + \eta \nabla^2 B \qquad \eta = \frac{1}{\mu_0 \sigma}$$

$\downarrow$ magnetic diffusivity

The magnetic field would also contribute to energy:

$$\mathcal{E} = e + \frac{1}{2}\rho v^2 + B^2$$

The energy eqn.

$$\partial_t \mathcal{E} + \operatorname{div} j_{\mathcal{E}} = 0$$

The magnetic contribution to the energy flux must be the Poynting flux

$$
\begin{aligned}
S &= E \times B \\
&= \left(- v \times B + \tfrac{1}{\sigma} J\right) \times B \\
&= B \times (v \times B) + \tfrac{1}{\sigma} J \times B
\end{aligned}
$$

$$
j_\varepsilon = \rho v \left(e + \tfrac{1}{2} \rho v^2 + p\right) + B \times (v \times B) + \eta (\nabla \times B) \times B
$$
$$
- v \sigma_{ij} - k \nabla T
$$

The ~~contribute~~ magnetic contribution to the entropy eqn. must be the Joule heating

$$
\partial_t s + \operatorname{div}\left(v s + \frac{q}{T}\right) = 0
$$

where
$$
q = - k \nabla T - v \sigma_{ij} + \cancel{B \times v - \eta B^2}
$$
$$
+ \eta (\nabla \times B) \times B
$$

- Incompressible MHD equations
- Isothermal MHD equations.

The magnetic part always includes the constraint

$$
\nabla \cdot B = 0
$$

This can be always satisfied by solving for the vector potential instead of B

$$
B = \nabla \times A
$$

The evoluction equation for the vector potential is :

$$\partial_t \mathbf{A} = v \times B - \frac{1}{\sigma} J$$

× conservation laws :

The MHD equations have two conserved quantities in the ideal (dissipation less case)

1. The total energy $\mathcal{E} = e + \frac{1}{2}\rho v^2 + B^2$ integrated over all volume:

$$E = \int_V \mathcal{E}(\vec{x})\, dV = \text{constant}$$

2. The magnetic helicity

$$\mathcal{H} = \int_V \vec{A} \cdot \vec{B}\, dV$$

$$\partial_t \mathcal{H} = \int \left[\partial_t A \cdot B + A \cdot \partial_t B \right] dV$$

$$\int \partial_t A \cdot B\, dV = \int (v \times B) \cdot B\, dV = 0$$

$$\int A \cdot \partial_t B\, dV = \int A \cdot \nabla \times (v \times B)\, dV$$

$$\nabla \cdot (P \times Q) = P \cdot (\nabla \times Q) + Q \cdot (\nabla \times P)$$

$$\vec{A} \cdot \nabla \times (v \times B) = \nabla \cdot (A \times v \times B) - (v \times B) \cdot \nabla \times A$$

$$= \nabla \cdot (A \times v \times B) - (v \times B) \cdot B$$

$$\Rightarrow \quad \partial_t \mathcal{H} = \int \mathrm{div}\,(\dot{A} \times \nabla \times B)\, dV$$

$$= \oint_S (A \times \nabla \times B) \cdot \hat{n}\, ds$$

Assuming velocities, and magnetic field has zero normal component at surfaces at ~~infinit~~ at the boundary we have

$$\partial_t \mathcal{H} = 0$$

$\Rightarrow$ $\mathcal{H}$ is a conserved quantity.

Also, when $\mathcal{H}$ is conserved it is gauge independent.