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# Exam in Fluid mechanics (MO5001)

Write the solution of each problem on a separate paper, and write your identification number on every paper.

**Allowed aids:** calculator, sheet with vector analysis relations.

**Grading:** A 90-100%, B 80-89%, C 65-79%, D 55-64%, E 50-54%, Fx 45-49%, F 0-44%

1. Answer **any 5** of the 7 short questions. You just need to write the final answer. Each question is worth 2 points.

   (a) A vector function $\boldsymbol{A}$ is given by

   $$A_x = \cosh(y)\sinh(z) \tag{1a}$$
   $$A_y = J_0(x) + y \tag{1b}$$
   $$A_z = \cos(x^2 + y^2) \tag{1c}$$

   Calculate $\boldsymbol{\nabla} \cdot \boldsymbol{A}$.

   (b) A velocity field $\boldsymbol{u}$ in two-dimensions $(x, y)$ is given by the following expression

   $$u_x = x \tag{2a}$$
   $$u_y = Sx - y. \tag{2b}$$

   Calculate the gradient matrix $G_{\alpha\beta} \equiv \partial_\beta u_\alpha$, where $\partial_\beta$ denote spatial derivative, as a function of $x$ and $y$. Is this velocity field incompressible?

   (c) From Eq. (2) calculate vorticity and rate-of-strain as a function of the space coordinates $x, y$.

   (d) In which of the following cases can I write the velocity as $\vec{v} = \nabla\Psi$, where $\Psi$ is a scalar function, without any loss of generality:

   (i) if the flow is incompressible,
   (ii) if the flow is irrotational, or
   (iii) if the flow is steady.

   (e) In a turbulent boundary layer very close to the wall, how does the mean stream-wise velocity $\langle v_x \rangle$ depend on the wall-normal coordinate $y$?

(f) A vector field $\boldsymbol{u}$, with components $u_x$, $u_y$ and $u_z$, as a function of space (described by the $x$, $y$, and $z$ coordinates) is given by the following expression:

$$\begin{aligned} u_x &= \alpha[2x + \cos(y) + 5z^3] \\ u_y &= \alpha[e^{-x} - y + \sin(y)] \\ u_z &= \alpha[\sin(x) + \cos(y) - z] \end{aligned} \tag{3}$$

Let $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$. Calculate $\boldsymbol{\nabla} \cdot \boldsymbol{\omega}$.

(g) The lubrication approximation and the equations that describes a laminar boundary layer are both derived from the incompressible Navier–Stokes equations. They both assume that the Reynolds number is small. What is another crucial common aspect of these two derivations? What is a crucial difference?

2. A thin rectangular plate of dimension $L_x \times L_z$ (the $z$ axis is perpendicular to the plane of the paper) is immersed in a fluid of kinematic viscosity $\nu$ and density $\rho$. The plate is being pulled by a force such that it moves with a velocity $v$ along the $x$ direction. The plate is confined vertically within a cavity. The clearance between the disk and the horizontal planes of the cavity is equal to $h$ where $h \ll L_x$, as shown in figure 1. Ignore the edge effects. Calculate the power necessary to keep the plate moving. (Hint: Use the lubrication approximation.) (7p)

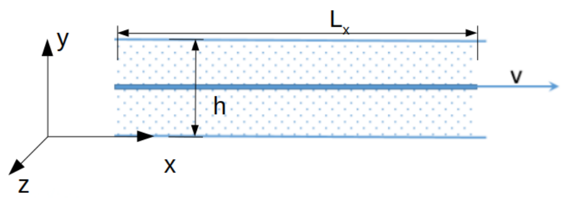
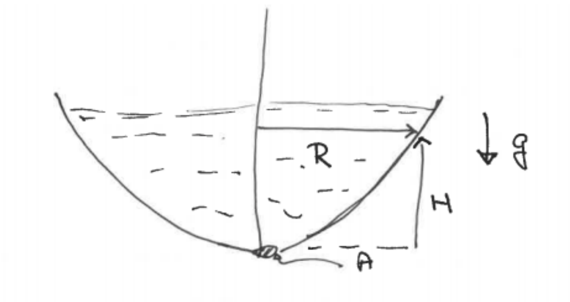



Figure 1: Problem 2

3. **Egyptian water clock:** An Egyptian water clock is a bowl with a small hole at the bottom. Time is measured by the fall of water level in this bowl as water flows out of the small hole in the bottom. To be able to function as a clock the rate of fall of the water level must be constant as a function of time for some interval of time. Wikipedia states: "The oldest documentation of the water clock is the tomb inscription of the 16th century BC Egyptian court official Amenemhet, which identifies himself as its inventor". Let us see if you can design one. The bowl is a surface of revolution whose cross-section is shown in figure 3. The surface is well-defined if we give the radius of the bowl, $R(H)$ as a function of its height $H$.

   (a) At time $t$, let the height of the water in the bowl be $H(t)$. What is the velocity, $v(t)$, of the water flowing out of the hole at that instant? Write clearly what assumptions you have made in arriving at the answer.




(b) If the area of the hole is $A$, the rate at which water flows out of the hole is $Q = vAC$ where $C$ is an empirically determined constant. In a small time $\Delta t$, the amount of water that flows out of this hole is given by $Q\Delta t$. If the fall in height in this small time interval is $\Delta H$, then

$$Q\Delta t = \pi R^2(H)\Delta H \tag{4}$$

where $R$ is the radius of the bowl at the height $H$. To be able to function as a clock the rate of fall of $H$ must be a constant. Find $H$ as a function of $R$.

(c) If an ancient Egyptian brought this clock from Egypt to Sweden, would it work equally well?

(8 p)