1\.

a) Use product rule: $(fg)' = f'g + g'f$,

$f = 3x^5+2x$, $g = 3\cos x$

$f' = 15x^4+2$ $g' = -3\sin x$

$k'(x) = f'(x)g(x) + g'(x)f(x)$

$k'(x) = (15x^4+2)(3\cos x) + (-3\sin x)(3x^5+2x)$

$= 45x^4\cos x + 6\cos x - 9x^5\sin x - 6x\sin x$

$\therefore$ The derivative of $k(x)$ is

$k'(x) = 45x^4\cos x + 6\cos x - 9x^5\sin x - 6x\sin x$

b) Use product rule, $(fg)' = f'g + g'f$

$f = -e^{2x}$ $g = \cos 3x$

$f' = -2e^{2x}$ $g' = -3\sin 3x$

$m'(x) = (-2e^{2x})(\cos 3x) + (-e^{2x})(-3\sin 3x)$

$= -2e^{2x}\cos 3x + 3e^{2x}\sin 3x$

$\therefore$ The derivative of $m(x)$ is

$m'(x) = -2e^{2x}\cos 3x + 3e^{2x}\sin 3x$

c)

Let $u = \sqrt{x^5}$

$u = x^{\frac{5}{2}}$

$\frac{du}{dx} = \frac{5}{2} x^{\frac{3}{2}}$

$h(x)$ is now a function of u

$$h(u) = \sqrt{7+u}$$

$$h'(u) = \frac{d}{du}(u+7)^{\frac{1}{2}}$$

$$= \frac{1}{2}(u+7)^{-\frac{1}{2}}$$

Apply chain rule:

$$\frac{d}{dx}(h(x)) = \left(\frac{d(h(u))}{du}\right)\left(\frac{du}{dx}\right)$$

$$h'(x) = \frac{1}{2}(u+7)^{-\frac{1}{2}}\left(\frac{5}{2}x^{\frac{3}{2}}\right)$$

Sub $u = \sqrt{x^5}$

$$h'(x) = \frac{5}{4}\left(\sqrt{x^5}+7\right)^{-\frac{1}{2}}\left(x^{\frac{3}{2}}\right)$$

$$= \frac{5x^{\frac{3}{2}}}{4\sqrt{x^{\frac{5}{2}}+7}}$$

d)

Let $\frac{h}{g} = \frac{(2x-1)^2}{\sqrt{x-1}}$

Use quotient rule: $\left(\frac{h}{g}\right)' = \frac{h'g - g'h}{g^2}$

Use the chain rule and power rule,

Let $u = 2x-1$

$\frac{du}{dx} = 2$

$h' = 2(2x-1)^1(2)$

$= 8x-4$

$g = \sqrt{x-1}$

Use the power rule,

$g' = \frac{1}{2}(x-1)^{-\frac{1}{2}}$

$$f' = \frac{(8x-4)(\sqrt{x-1}) - \left(\frac{1}{2}(x-1)^{-\frac{1}{2}}\right)(2x-1)^2}{(\sqrt{x-1})^2}$$

$$= \frac{(8x-4)\sqrt{x-1} - \frac{(2x-1)^2}{2\sqrt{x-1}}}{x-1}$$

$$= \frac{4(2x-1)\sqrt{x-1}}{x-1} - \frac{(2x-1)^2}{2\sqrt{x-1}(x-1)}$$

$$= \frac{4(2x-1)\sqrt{x-1}(2\sqrt{x-1})}{(x-1)(2\sqrt{x-1})} - \frac{(2x-1)^2}{2\sqrt{x-1}(x-1)}$$

$$= \frac{(2x-1)\,(8(x-1) - (2x-1))}{2\,(x-1)^{3/2}}$$

$$= \frac{(2x-1)(6x-7)}{2\,(x-1)^{\frac{3}{2}}}$$

$$= \frac{12x^2 - 20x + 7}{2\,(x-1)^{\frac{3}{2}}}$$

$\therefore$ The derivative of $f(x)$ is

$$f'(x) = \frac{12x^2 - 20x + 7}{2\,(x-1)^{\frac{3}{2}}}$$

$\therefore$ The derivative of $h(x)$ is

$$h'(x) = \frac{5x^{\frac{3}{2}}}{4\sqrt{x^{\frac{5}{2}}+7}}$$

2.

The rate of change at time t is given by $h'(t)$.

$h(t) = 5\sin\left(\frac{t}{2}+2\right)+6$ Let $u = \frac{t}{2}+2$

$h(u) = 5\sin(u)+6$ $\frac{du}{dt} = \frac{1}{2}$

$\frac{d(h(t))}{dt} = \left(\frac{dh}{du}\right)\left(\frac{du}{dt}\right)$

$h'(u) = 5\cos(u)$

$h'(t) = 5\cos(u)\left(\frac{1}{2}\right)$

$= \frac{5}{2}\cos\left(\frac{t}{2}+2\right)$

Sub $t=7$:

$h'(t) = \frac{5}{2}\cos\left(\frac{7}{2}+2\right)$

$= \frac{5}{2}\cos\left(\frac{11}{2}\right)$

$= 1.7717$ m/hour

$\therefore$ The rate of change at 07:00 h is 1.7717 meters/hour

3.

We find the derivative for y.

$y = e^{-x}$

$y' = -e^{-x}$

Sub $x=-1$,

$y' = -e^{-(-1)}$

$= -e$

Find a point on y using $x=-1$,

$y = e^{-(-1)}$

$= e$

$\therefore$ a point on the line is $(-1, e)$

Sub $(-1, e)$ and $m = -e$:

$y = mx + b$

$e = (-e)(-1) + b$

$b = 0$

$\therefore y = -ex$

$\therefore$ The equation of the tangent line to $y = e^{-x}$ at $x=-1$ is $y = -ex$.

4.

We model the ticket prices with an objective function of the revenue.

$R(d) = (12 - d)(11000 + 1000d)$, d is the number of dollars by which the ticket price is reduced, $0 \leq d \leq 4$

For $R'(d)$, use product rule.

$(fg)' = f'g + g'f$

$f = 12 - d \qquad g = 11000 + 1000d$

$f' = -1 \qquad g' = 1000$

$\therefore R'(d) = (-1)(11000 + 1000d) + (1000)(12 - d)$

$= -11000 - 1000d + 12000 - 1000d$

$= 1000 - 2000d$

Maximum Revenue occurs when $R'(d) = 0$.

Set $R'(d) = 0$,

$0 = 1000 - 2000d$

$2000d = 1000$

$d = \frac{1}{2}$

$\therefore$ The maximum revenue occurs when the ticket price is reduced by $\frac{1}{2}$ dollars, or 50 cents. The ticket price is $\$12 - \$0.50 = \$11.50$

$$R\left(\tfrac{1}{2}\right) = \left(12 - \tfrac{1}{2}\right)\left(11\,000 + 1000\left(\tfrac{1}{2}\right)\right)$$
$$= \left(\tfrac{23}{2}\right)(11500)$$
$$= 132\,250$$
$\therefore R\left(\tfrac{1}{2}\right) = \$132,250$

$\therefore$ The maximum revenue is $\$132,250$.

5.

$$N = N_0 e^{-\lambda t}$$

Sub $N_0 = 1$, $\lambda = 1.21 \times 10^{-4}$

$$N = (1)\, e^{-(1.21\times10^{-4})t}$$
$$N = e^{-(1.21\times10^{-4})t}$$

Let $u = -(1.21\times10^{-4})t$

$$\frac{du}{dt} = -(1.21\times10^{-4})$$

$$N = e^{u}$$
$$N' = e^{u}$$

Use chain rule,

$$N' = e^{u}(-1.21\times10^{-4})$$
$$= -(1.21\times10^{-4})e^{-(1.21\times10^{-4})t}$$

$\therefore$ The rate of change of an initial amount of 1 gm and the delay constant is

$$N' = -(1.21\times10^{-4})e^{-(1.21\times10^{-4})t}$$

b)
$$N = N_0 e^{-\lambda t}$$
$$\frac{N}{N_0} = e^{-\lambda t}$$
$$\ln\frac{N}{N_0} = -\lambda t$$
$$\lambda = \frac{-\ln\frac{N}{N_0}}{t}$$
$$= \frac{-\ln(0.79)}{5730}$$
$$= 0.0000411$$

$\therefore$ The decay rate is 4.11×10^{-5} gm/year.

6.

Find slope of the line:

$3x-y+6=0$

$y=3x+6$

$y'=3$

$\therefore$ the slope of the line is 3.

The point of the curve that is tangent and parallel to $3x-y+6=0$ is the point when the derivative of the curve is 3.

$y=x\sqrt{x}$

$y=x^{\frac{3}{2}}$

$y'=\frac{3}{2}x^{\frac{1}{2}}$

Sub $y'=3$,

$3=\frac{3}{2}x^{\frac{1}{2}}$

$2=x^{\frac{1}{2}}$

$x=4$

Sub $x=4$ into $y=x\sqrt{x}$:

$y=4\sqrt{4}$

$=8.$

$\therefore$ $(4,8)$ is the point on the curve that is tangent and parallel to $y=3x+6$.

7.

The average rate of change from $t=1$ to $t=4$ is given by

$$\Delta C_{avg}=\frac{C(4)-C(1)}{4-1}$$

$$=\frac{2\sqrt{4^3}+25-(2\sqrt{1^3}+25)}{4-1}$$

$$=\frac{2(8)\cancel{+25}-2\cancel{-25}}{4-1}$$

$$=\frac{14}{3}$$

$\therefore$ The average rate of change from $t=1$ to $t=4$ is $\frac{14}{3}$.

$C=2\sqrt{t^3}+25$

$C=2(t^{\frac{3}{2}})+25$

$C'=3(t^{\frac{1}{2}})$

Set $C'=\frac{14}{3}$,

$$\frac{14}{3} = 3(t^{\frac{1}{2}})$$

$$t^{\frac{1}{2}} = \frac{14}{9}$$

$$t = \frac{196}{81}$$

$\therefore$ The time t at which the instantaneous rate of change of C is equal to the average rate of change from $t=1$ to $t=4$ is $t = \frac{196}{81}$ hours or $t \simeq 2.42$ hours.

8.

$\because f(x)$ has a horizontal tangent at $(-1, 8)$, $f'(-1) = 0$.

$f(x) = ax^2 + bx + c$

$f'(x) = 2ax + b$

$f'(-1) = 2a(-1) + b$

$0 = -2a + b$

$b = 2a$

Sub $b = 2a$,

$f(x) = ax^2 + 2ax + c$

Sub $(-1, 8)$ and $(2, 19)$,

$f(-1) = a(-1)^2 + 2a(-1) + c$

$8 = a - 2a + c$

$8 = c - a \quad (1)$

$f(2) = a(2^2) + 2a(2) + c$

$19 = 4a + 4a + c$

$19 = 8a + c \quad (2)$

② − ①:

$$\begin{array}{r} 8a + c = 19 \\ -(-a + c = 8) \\ \hline 9a = 11 \end{array}$$

$a = \frac{11}{9}$

Sub $a = \frac{11}{9}$ into ②:

$c - \left(\frac{11}{9}\right) = 8$

$c = 8 + \frac{11}{9}$

$c = \frac{83}{9}$

Sub $a = \frac{11}{9}$ into $b = 2a$,

$b = 2a$

$= 2\left(\frac{11}{9}\right)$

$= \frac{22}{9}$

$\therefore f(x) = \frac{11}{9}x^2 + \frac{22}{9}x + \frac{83}{9}$ is a quadratic function that has a horizontal tangent at $(-1, 8)$ and passes through $(2, 19)$.

9.

We attempt to apply power rule in reverse.

Let $y = x^n$.

From the power rule,

$$y' = nx^{n-1}.$$

The "reverse" power rule for y' will transform y' into y.

We observe:

→ The exponent n for y is 1 more than the exponent for y';

→ The value of the exponent of y becomes a coefficient for y'. Reversing this would require dividing this coefficient out.

From our observations, we suggest that the reverse power rule for x^n is:

$$\text{RevPower}(x^n) = \frac{x^{n+1}}{n+1}, \quad n \neq -1.$$

If this is correct, its derivative should be x^n.

$$\frac{d}{dx}\left(\text{RevPower}(x^n)\right) = \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right), \quad n \neq -1$$

Constants come out of the derivative,

$$= \left(\frac{1}{n+1}\right) \frac{d}{dx}(x^{n+1})$$

Using the power rule,

$$= \left(\frac{1}{n+1}\right)(n+1)x^{n+1-1}$$

$$= x^n$$

$\because$ The derivative of $\frac{x^{n+1}}{n+1}, n \neq -1$, is x^n, we can transform y' into y using this "reverse" power rule.

$$\text{RevPower}(y) = \text{RevPower}(n x^{n-1})$$

$$= \frac{n x^{n-1+1}}{n-1+1}$$

$$= \frac{\cancel{n} x^n}{\cancel{n}}$$

$$= x^n$$

$\therefore$ The reverse power rule for x^n is $\frac{x^{n+1}}{n+1}$. However, differentiating a constant will yield a value of 0, which would be lost in the derivative but kept in the original function. Thus, we add a C to represent an arbitrary constant in the reversed function.

$\therefore \mathrm{RevPower}(x^n) = \dfrac{x^{n+1}}{n+1} + c,\ n \neq -1.$

Applying this rule to $f'(x) = 12x^2 + 4x - 10$, we have

$$\mathrm{RevPower}(f'(x)) = \mathrm{RevPower}(12x^2 + 4x - 10)$$

$$f(x) = \mathrm{RevPower}(12x^2) + \mathrm{RevPower}(4x) + \mathrm{RevPower}(-10)$$

$\because$ Addition doesn't affect the reversed power of individual parts.

$$= \frac{12x^{2+1}}{2+1} + C + \frac{4x^{1+1}}{1+1} + C + \frac{(-10)x^{0+1}}{0+1} + C$$

$$= \frac{12x^3}{3} + \frac{4x^2}{2} + \frac{-10x}{1} + C$$ ← We represent all constants as C

$$= 4x^3 + 2x^2 - 10x + C.$$

For a specific function, let $C = 0$.

$\therefore f(x) = 4x^3 + 2x^2 - 10x + 0$

$\therefore$ A function with a derivative of $f'(x) = 12x^2 + 4x - 10$ could be

$f(x) = 4x^3 + 2x^2 - 10x.$