1.

$\because$ We're compounding continuously, our investment after time t can be expressed as:

$$P = P_0 e^{rt}$$

for an interest rate r and time t after the initial investment.

Sub $r = 3\%$, $t = 4$ years, $P_0 = \$2000$:

$$P = (2000)\, e^{(0.03)(4)}$$

$$= 2254.99$$

$\therefore$ An investment of \$2000 compounded continuously at 3% per annum for 4 years will be worth \$2254.99

2.

$y=e^x$ is unique because its derivative is equal to itself ($y'=e^x=y$). This is particularly powerful for applications where a function's instantaneous rate of change (its derivative) is dependent on the current value (or state) of the function (i.e. modelling biological systems, compound interest, etc.) It's also special that the derivative of e^x is itself - the function grows at the same rate as any of its derivatives. For $y=e^x$, $y'=e^x$ -> Points that satisfy $y=e^x$ will also satisfy derivatives of $y=e^x$, a property unique to $y=e^x$.

3.

a) $f(x) = x^2 + 5^x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 + 5^{x+h} - (x^2 + 5^x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} + \lim_{h \to 0} \frac{5^{x+h} - 5^x}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h} + \lim_{h \to 0} \frac{5^x(5^h - 1)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} + \left(\lim_{h \to 0} 5^x\right)\left(\lim_{h \to 0} \frac{5^h - 1}{h}\right)$$

$$= \lim_{h \to 0} 2x + h + \left(\lim_{h \to 0} 5^x\right)\left(\lim_{h \to 0} \frac{(e^{\ln 5})^h - 1}{h}\right)$$

$$= \lim_{h \to 0} 2x + h + 5^x \lim_{h \to 0} \frac{e^{h \ln 5} - 1}{h}$$

$\because \lim_{h \to 0} \frac{e^{h \ln 5} - 1}{h} = \ln 5,$

$= 2x + (\ln 5) 5^{x}$

$\therefore$ The derivative of $f(x)$ is $f'(x) = 2x + (\ln 5) 5^{x}$

b) $f(x) = \frac{1}{2^{x}} = 2^{-x}$

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{2^{-x-h} - 2^{-x}}{h}$

$= \lim_{h \to 0} \frac{2^{-x}(2^{-h} - 1)}{h}$

$= \left(\lim_{h \to 0} 2^{-x}\right)\left(\lim_{h \to 0} \frac{(e^{\ln 2})^{-h} - 1}{h}\right)$

$= 2^{-x} \lim_{h \to 0} \frac{e^{h(-\ln 2)} - 1}{h}$

$\because \lim_{h \to 0} \dfrac{e^{h(-\ln 2)} - 1}{h} = -\ln 2$

$= (-\ln 2) 2^{-x}$

$\therefore$ The derivative of $f(x)$ is $f'(x) = (-\ln 2) 2^{-x}$.

4.

a) The constant k can be found by solving for when $\frac{1}{2}$ of the isotope has decayed:

$$P = P_0 e^{kt}$$

$$\frac{P}{P_0} = e^{kt}$$

Sub $\frac{P}{P_0} = \frac{1}{2}$, $t = 30$

$$\frac{1}{2} = e^{30k}$$

$$\ln\left(\frac{1}{2}\right) = \ln\left(e^{30k}\right)$$

$$-\ln 2 = 30k$$

$$k = -\frac{\ln 2}{30}$$

$$k \simeq -0.023$$

Sub $P_0 = 10$, $k \simeq -0.023$,

$$\therefore P = 10e^{-0.023t}$$

$\therefore$ An expression for the amount of the isotope still active after t years is $P = 10e^{-0.023t}$.

b)

Sub $t = 12$,

$$P = 10e^{-0.023(12)}$$

$$= 7.588 \text{ mg}$$

$\therefore$ The present amount of isotope is 7.588 mg

c)

$$P' = \lim_{h \to 0} \frac{P(t+h) - P(t)}{h}$$

$$= \lim_{h \to 0} \frac{10e^{-0.023(t+h)} - 10e^{-0.023t}}{h}$$

$$= \lim_{h \to 0} \frac{10e^{-0.023t}(e^{-0.023h} - 1)}{h}$$

$$= (10e^{-0.023t}) \lim_{h \to 0} \frac{e^{-0.023h} - 1}{h}$$

$$\because \lim_{h \to 0} \frac{e^{-0.023h} - 1}{h} = -0.023$$

$$= 10e^{-0.023t}(-0.023)$$

$$P' = -0.23e^{-0.023t}$$

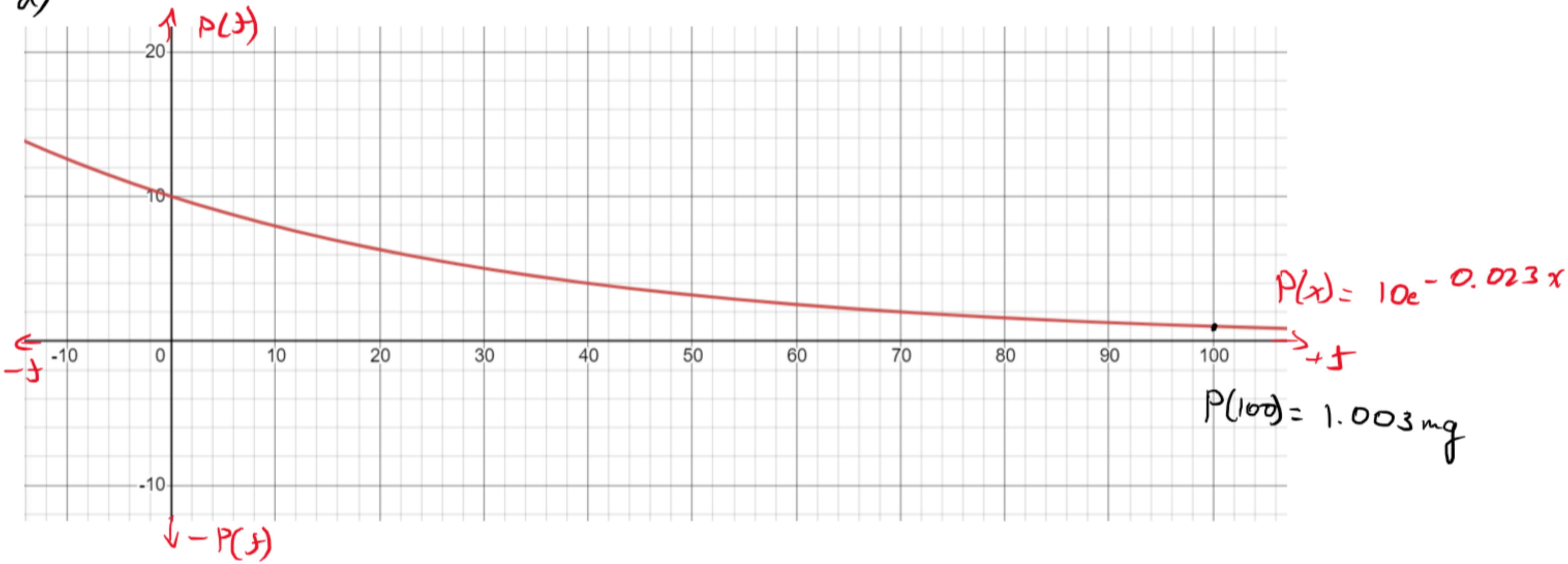
Sub $t = 12$:

$P'(12) = -0.23e^{-0.023(12)}$

$= -0.175$ mg/year

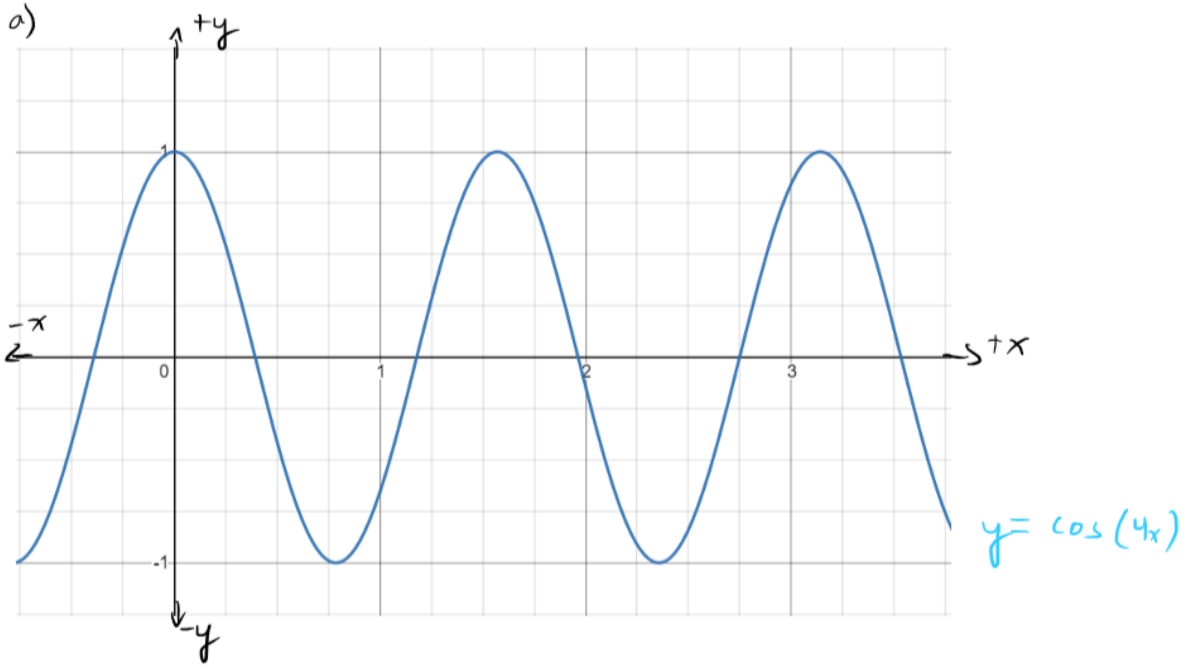
$\therefore$ The present instantaneous rate of decay is -0.175 mg/year

d)



5.

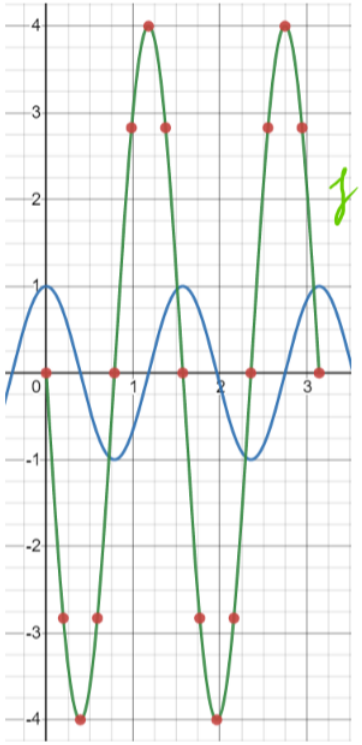
a)



b)

x	$f'(x)$
0	1
$\frac{\pi}{16}$	-2.828
$\frac{\pi}{8}$	-4
$\frac{3\pi}{16}$	-2.828
$\frac{\pi}{4}$	0
$\frac{5\pi}{8}$	4
$\frac{11\pi}{16}$	-2.828
$\frac{3\pi}{4}$	0
$\frac{13\pi}{16}$	2.828
$\frac{7\pi}{8}$	4
$\frac{15\pi}{16}$	2.828
π	0

c)



Points from b) have been plotted in red.

$f'(x) = -4\sin(4x)$

d)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cos(4x+4h) - \cos(4x)}{h}$$

$$= \lim_{h \to 0} \frac{\cos(4x)\cos(4h) - \sin(4x)\sin(4h) - \cos(4x)}{h}$$

$$= \lim_{h \to 0} \frac{\cos(4x)(\cos(4h) - 1)}{h} - \lim_{h \to 0} \frac{\sin 4x \sin 4h}{h}$$

Use squeeze theorem: $\lim_{h \to 0} \frac{\cos 4h - 1}{h} = 0$, $\lim_{h \to 0} \frac{\sin h}{h} = 1$

$$\therefore f'(x) = \lim_{h \to 0} \frac{\cos(4x)\,\cancel{(\cos 4h - 1)}^{\to 0}}{\cancel{h}} - \lim_{h \to 0} \frac{4 \sin 4x \,\cancel{\sin 4h}^{\to 1}}{\cancel{4h}}$$

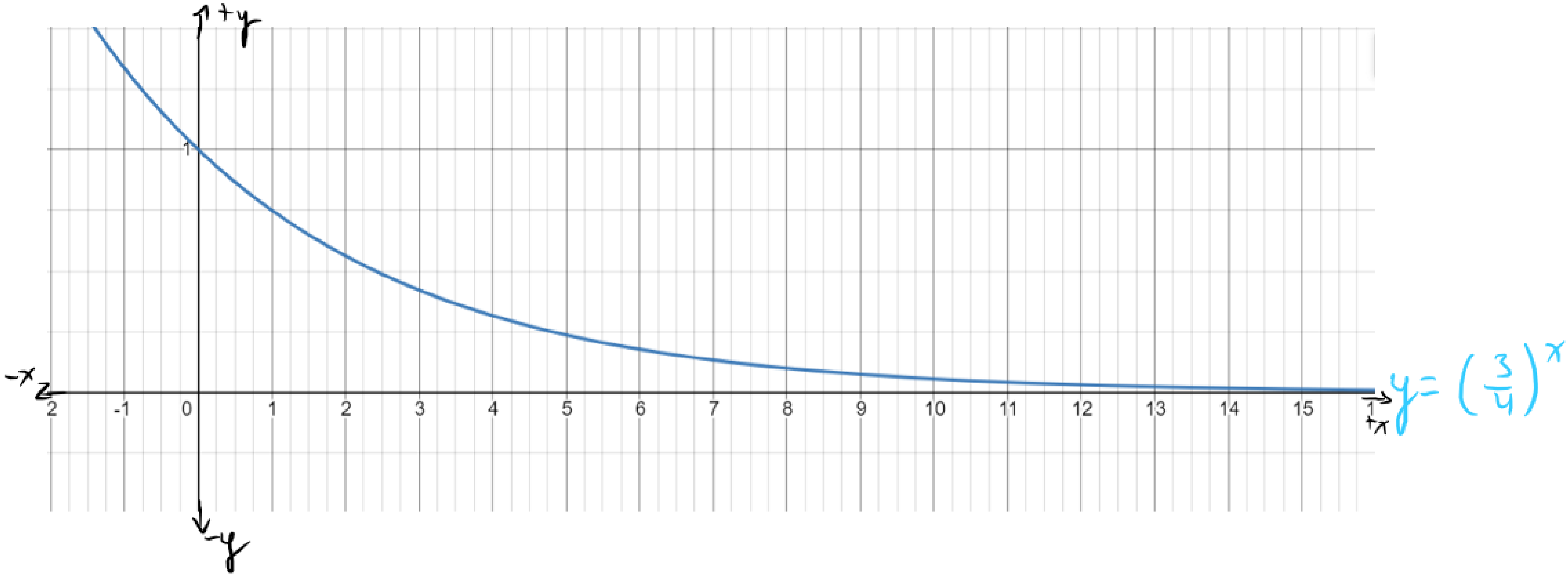
$$= -4\sin(4x)$$

$\therefore$ The derivative of $f(x)$ is $f'(x) = -4\sin(4x)$.

e)

For $y = (0.75)^x$:

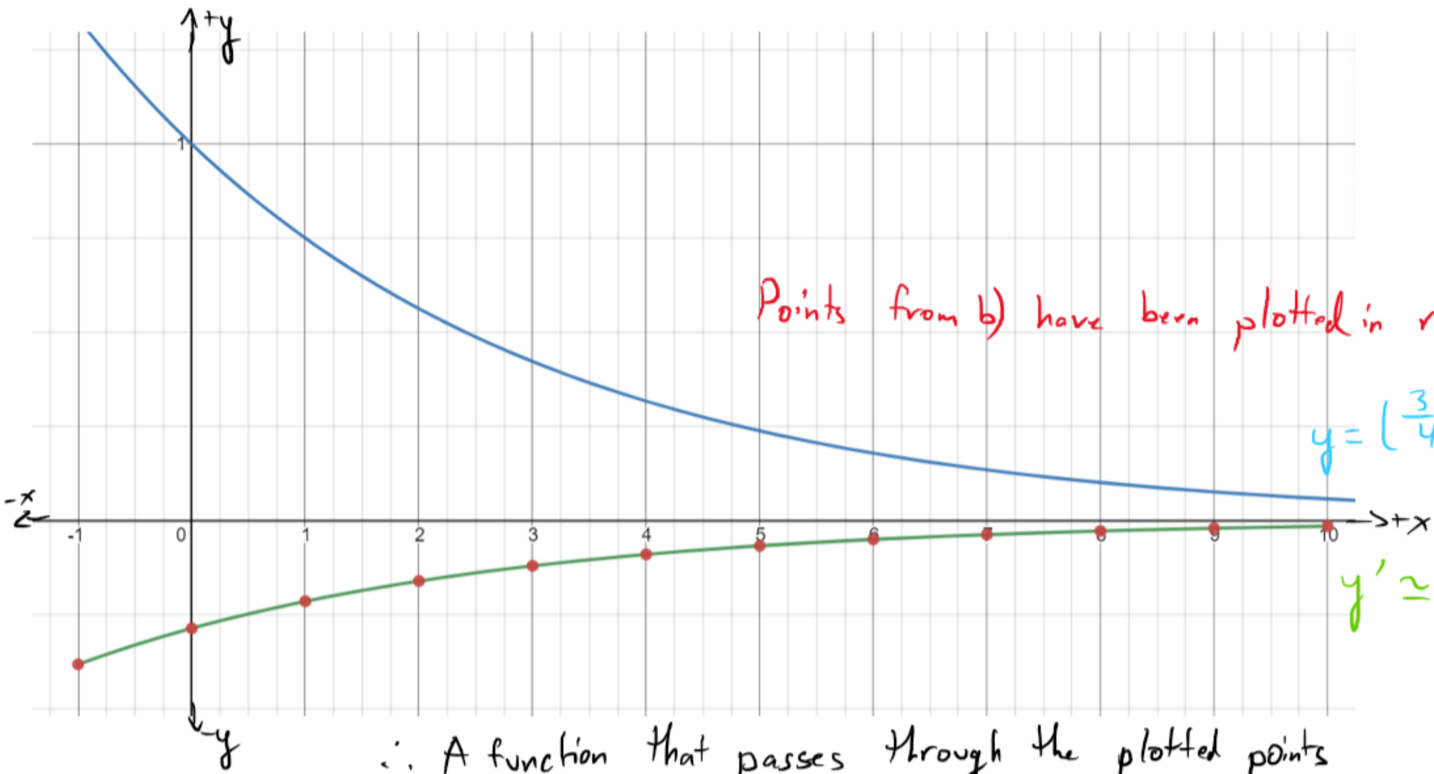
e.a) (sports. It's in the game)



e.b)

x	$f'(x)$
-1	-0.3836
0	-0.2877
1	-0.2158
2	-0.1618
3	-0.1214
4	-0.091
5	-0.0683
6	-0.0512
7	-0.0384
8	-0.0288
9	-0.0216
10	-0.0162

e.c)



$\therefore$ A function that passes through the plotted points of the derivative of y is $y' = -0.288\left(\frac{3}{4}\right)^x$ or $y = \ln\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)^x$

e.d)

$f(x) = \left(\frac{3}{4}\right)^x$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{3}{4}\right)^{x+h} - \left(\frac{3}{4}\right)^{x}}{h}$$

$$= \lim_{h\to 0} \left(\frac{3}{4}\right)^{x} \left(\frac{\left(\frac{3}{4}\right)^{h} - 1}{h}\right)$$

$$= \left(\frac{3}{4}\right)^{x} \lim_{h\to 0} \left(\frac{e^{\ln\left(\frac{3}{4}\right)h} - 1}{h}\right)$$

Use $\lim_{h\to 0} \frac{e^{\ln\left(\frac{3}{4}\right)h} - 1}{h} = \ln\left(\frac{3}{4}\right)$,

$\therefore f'(x) = \ln\left(\frac{3}{4}\right) \left(\frac{3}{4}\right)^{x}$

$\therefore$ The derivative of $y = 0.75^{x}$ is $y' = \ln\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)^{x}$.