1.

$\vec{AB} = \vec{B} - \vec{A}$

$= (0,3,4) - (2,1,1)$

$= (-2, 2, 3)$

$\therefore \vec{AB} = (-2,2,3)$

$\vec{AC} = \vec{C} - \vec{A}$

$= (1,7,-2) - (2,1,1)$

$= (-1, 6, -3)$

$\therefore \vec{AC} = (-1,6,-3)$

Use $A(2,1,1)$,

$(x,y,z) = (2,1,1) + s(-2,2,3) + t(-1,6,-3)$

$\therefore x = 2 - 2s - t$

$\therefore y = 1 + 2s + 6t$

$\therefore z = 1 + 3s - 3t$

$\therefore$ The parametric equations for the plane are $x = 2-2s-t$, $y = 1+2s+6t$, $z = 1+3s-3t$

2.

If the plane is perpendicular to $2x+3y-z+1=0$, a direction vector for the plane is the normal vector of $2x+3y-z+1=0$.

$\because \vec{n}_{plane\,2} = (2,3,-1)$ Normal vector of $2x+3y-z+1=0$

$\therefore \vec{m}_{plane} = (2,3,-1)$

$\overrightarrow{MN} = \vec{N} - \vec{M}$

$= (3,2,-1) - (1,2,3)$

$= (2,0,-4)$

$\vec{n} = \overrightarrow{MN} \times \vec{m}$

$= (2,0,-4) \times (2,3,-1)$

$$\begin{array}{cccc} 0 & -4 & 2 & 0 \\ \times & \times & \times & \\ 3 & -1 & 2 & 3 \end{array}$$

$= (3(-4) - (-1)(0), (-1)(2) - (2)(-4), 2(0) - 3(2))$

$= (12, -6, 6)$

$\therefore \vec{n} = (2, -1, 1)$

$2x - y + z + D = 0$

Sub $M(1,2,3)$:

$2(1) - (2) + (3) + D = 0$

$D = -3$

$\therefore 2x - y + z - 3 = 0$ is a scalar equation for the plane,

3.

$x + 3y + 2z = 6$ (1)

$2x + 4y - 3z = 20$ (2)

$3x - y + 7z = -14$ (3) (Using elimination)

(2) $-$ 2 (1):

$-2y - 7z = 8$

$2y + 7z = -8$ (4)

6 (1) $-$ 2 (3):

$20y - 2z = 64$ (5)

10 (4) $-$ (5):

$72z = -144$

$z = -2$

Sub $z=-2$ into 4:

$2y + 7(-2) = -8$

$y = 3$

Sub $z=-2$, $y=3$ into 1:

$x + 3(3) + 2(-2) = 6$

$x = 1$

$\therefore (x, y, z) = (1, 3, -2)$

$\therefore$ There exists a unique solution for the system of equations, the 3 planes intersect at $(1, 3, -2)$.

4.

a)

$\overrightarrow{AB} = \vec{B} - \vec{A}$

$= (3, -2, -4) - (4, -2, 5)$

$= (-1, 0, -9)$

$\therefore$ A vector equation for the line is $(x, y, z) = (4, -2, 5) + t(-1, 0, -9)$

b)

$(x, y, z) = (4, -2, 5) + (-t, 0, -9t)$

$= (4 - t, -2 - 0, 5 - 9t)$

$= (4 - t, -2, 5 - 9t)$

$\therefore$ The parametric equations for the vector equation in a) are:

$\therefore x = 4 - t$

$\therefore y = -2$

$\therefore z = 5 - 9t$

c)

There are no symmetric equations for this line because the direction vector for the line has a y-component that is 0. The corresponding term in the symmetric equation would have a denominator equal to 0, which is not defined.

5.

Since there are infinitely many points on a line and infinitely many scalar multiples of a direction vector, there are infinitely many vector and parametric equations for a line.

In the vector equation of a line, there are infinitely many points on the line that could be used to define the vector equation. (i.e. $\vec{r} = (0,0,0) + t(1,1,1)$ and $\vec{r} = (1,1,1) + t(1,1,1)$)

In the parametric equation of a line, there are infinitely many points on the line and scalar multiples of the direction vector that could be used to define the parametric equation.

$$\left(\text{i.e. } \begin{aligned} x &= 1 + t \\ y &= 1 + t \\ z &= 1 + t \end{aligned} \quad \text{and} \quad \begin{aligned} x &= 2 + 2t \\ y &= 2 + 2t \\ z &= 2 + 2t \end{aligned} \right)$$

6. The two lines are perpendicular if their direction vectors are perpendicular.

$\vec{m}$ of the line defined by $\dfrac{x-2}{3k+1} = \dfrac{y-3}{k} = \dfrac{z+9}{1}$ is $\vec{m}_1 = (3k+1, k, 1)$

$\vec{m}$ of the line defined by $\dfrac{x-4}{2} = \dfrac{y+3}{-3k} = \dfrac{z-2}{7}$ is $\vec{m}_2 = (2, -3k, 7)$

Two vectors are perpendicular if their dot product is 0.

$\vec{m}_1 \cdot \vec{m}_2 = 0$

$2(3k+1) + (-3k)(k) + 7(1) = 0$

$6k+2 - 3k^2 + 7 = 0$

$-3k^2 + 6k + 9 = 0$

$k^2 - 2k - 3 = 0$

$(k-3)(k+1) = 0$

if $k-3=0$ if $k+1=0$

$k=3$, $k=-1$

if $k=3$,

$$\frac{x-2}{3k+1} = \frac{y-3}{k} = \frac{z+9}{1} \rightarrow \frac{x-2}{10} = \frac{y-3}{3} = \frac{z+9}{1}$$

$$\frac{x-4}{2} = \frac{y+3}{-3k} = \frac{z-2}{7} \rightarrow \frac{x-4}{2} = \frac{y+3}{-9} = \frac{z-2}{7}$$

if $k=-1$,

$$\frac{x-2}{3k+1} = \frac{y-3}{k} = \frac{z+9}{1} \rightarrow \frac{x-2}{-2} = \frac{y-3}{-1} = \frac{z+9}{1}$$

$$\frac{x-4}{2} = \frac{y+3}{-3k} = \frac{z-2}{7} \rightarrow \frac{x-4}{2} = \frac{y+3}{3} = \frac{z-2}{7}$$

$\because$ $k=3$ and $k=-1$ can produce valid symmetric equations, $k=3$ or $k=-1$ will make the lines perpendicular.

7. If there exists a unique v, s, t that satisfy the parametric equations for the line and the plane, the line intersects the plane and does not lie on the plane.

Find parametric equations for the line and the plane.

$$(x,y,z) = (10,4,-4) + v(1,2,3), \quad (x,y,z) = (3,0,1) + s(1,2,-1) + t(3,1,-2)$$

<u>Line</u>

$x = 10 + v$

$y = 4 + 2v$

$z = -4 + 3v$

<u>Plane</u>

$x = 3 + s + 3t$

$y = 2s + t$

$z = 1 - s - 2t$

Equate and Simplify.

$10 + v = 3 + s + 3t$

$4 + 2v = 2s + t$

$-4 + 3v = 1 - s - 2t$

$u - s - 3t + 7 = 0 \quad (1)$

$2u - 2s - t + 4 = 0 \quad (2)$

$3u + s + 2t - 5 = 0 \quad (3)$

$2(1) - (2)$:

$2u - 2s - 6t + 14 = 0$

$-(2u - 2s - t + 4 = 0)$

$-5t + 10 = 0$

$\therefore t = 2$

$6(1) - 2(3)$:

$6u - 6s - 18t + 42 = 0$

$-(6u + 2s + 4t - 10 = 0)$

$-8s - 22t + 52 = 0$

$4s + 11t - 26 = 0$

Sub $t = 2$,

$$4s + 11(2) - 26 = 0$$
$$\therefore s = 1$$

Sub $s = 1$, $t = 2$ into (1):

$$u - (1) - 3(2) + 7 = 0$$
$$\therefore u = 0$$

Test

Line:

$(x, y, z) = (10, 4, -4) + u(1, 2, 3)$

Sub $u = 0$:

$(x, y, z) = (10, 4, -4)$

Plane:

$(x, y, z) = (3, 0, 1) + s(1, 2, -1) + t(3, 1, -2)$

Sub $s = 1$, $t = 2$:

$(x, y, z) = (3, 0, 1) + (1, 2, -1) + (6, 2, -4)$
$= (10, 4, -4)$

∴ There exists a unique v, s, and t for the intersection of the line and the plane. The line does not lie on the plane but intersects the plane at $(10, 4, -4)$.

b)

If the direction vector of the line is coplanar with the direction vectors of the plane, the line lies on the plane.

We find the scalar triple product of the line's direction vector and the plane's direction vectors. If this is 0, the vectors are coplanar and the line lies on the plane.

$$S = \vec{a} \cdot (\vec{b} \times \vec{c})$$

$$= (1,2,3) \cdot ((1,2,-1) \times (3,1,-2))$$

$$\begin{array}{cccc} 2 & -1 & 1 & 2 \\ 1 & -2 & 3 & 1 \end{array}$$

$$= (1,2,3) \cdot (2(-2) - (-1)(1), (-1)(3) - (-2)(1), 1 - 6)$$

$$= (1,2,3) \cdot (-3, -1, -5)$$

$$= (1)(-3) + (-1)(2) + (3)(-5)$$

$$= -20$$

$\because \vec{a} \cdot (\vec{b} \times \vec{c}) \neq 0$, the direction vectors of the line and the plane are not coplanar.

∴ The line does not lie on the plane.