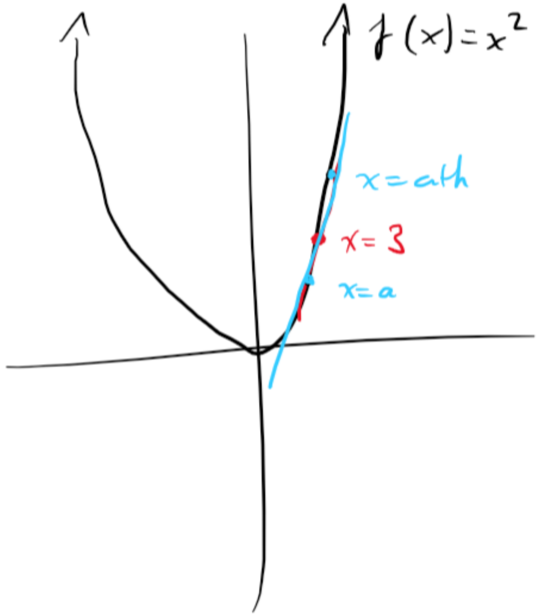
1.

Derivatives are calculated as the slope of a secant line that approaches a tangent line to the function.

Consider $f(x) = x^2$.



To find the instantaneous slope of $f(x)$ at $x=3$, we construct a secant line that intersects $f(x)$ at $x=a$ and $x=a+h$. This line approximates a tangent line (the derivative). By decreasing the distance between a and $a+h$ until it is infinitessimally close to $x=3$, the secant line will be a very good approximation of the tangent line. This is why the limit $\lim_{h \to 0}$ is taken.

Thus, the expression for the derivative is:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Finding the derivative is then just a matter of evaluating the limit when you can (manipulate the equations to cancel h and then evaluate the limit).

2.

a) Let $f(x) = x^2$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$= \lim_{h \to 0} 2x+h$$

$$\therefore f'(x) = 2x$$

Sub $x = -2$,

$$f'(-2) = 2(-2)$$
$$= -4$$

$\therefore$ The slope of the tangent of $f(x) = x^2$ at $x = -2$ is $f'(-2) = -4$.

b) Let $f(x) = 2x^2 - 4x + 3$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{2(x+h)^2 - 4(x+h) + 3 - (2x^2 - 4x + 3)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{2x^2} + 4xh + h^2 - \cancel{4x} - 4h + \cancel{3} - \cancel{2x^2} + \cancel{4x} - \cancel{3}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(4x + h - 4)}{\cancel{h}}$$

$$= \lim_{h \to 0} 4x + h - 4$$

$\therefore f'(x) = 4x - 4$

Sub $x = 4$

$f'(4) = 4(4) - 4$

$= 12$

$\therefore$ The slope of the tangent of $f(x)$ at $x = 4$ is $f'(4) = 12$.

c)

Let $f(x) = \sqrt{x-12}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{(x+h)-12} - \sqrt{x-12}}{h} \left(\frac{\sqrt{x+h-12} + \sqrt{x-12}}{\sqrt{x+h-12} + \sqrt{x-12}} \right)$$

$$= \lim_{h \to 0} \frac{\cancel{x} + h - \cancel{12} - \cancel{x} + \cancel{12}}{h\left(\sqrt{x+h-12} + \sqrt{x-12}\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h-12} + \sqrt{x-12}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+\cancel{h}^{\to 0}-12} + \sqrt{x-12}}$$

$$f'(x) = \frac{1}{\sqrt{x-12} + \sqrt{x-12}}$$

$$\therefore f'(x) = \frac{1}{2\sqrt{x-12}}$$

Sub $x = 37$,

$$f'(37) = \frac{1}{2\sqrt{37-12}}$$

$$= \frac{1}{10}$$

$\therefore$ The slope of the tangent of $f(x)$ at $x = 37$ is $f'(37) = \frac{1}{10}$.

3.

a) Let $f(x) = 4x^3 - 12x + 2$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{4(x+h)^3 - 12(x+h) + 2 - (4x^3 - 12x + 2)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{4x^3} + 12x^2h + 12xh^2 + 4h^3 \cancel{- 12x} - 12h \cancel{+ 2} \cancel{- 4x^3} \cancel{+ 12x} \cancel{- 2}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(12x^2 + 12xh + 4h^2 - 12)}{\cancel{h}}$$

$$= \lim_{h \to 0} 12x^2 + 12xh + 4h^2 - 12$$

$\therefore f'(x) = 12x^2 - 12$

$\therefore$ The derivative of $f(x)$ is $f'(x) = 12x^2 - 12$.

b)

Let $f(x) = \dfrac{4}{2-x}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{4}{2-(x+h)} - \frac{4}{2-x}}{h} \left(\frac{(2-x)(2-(x+h))}{(2-x)(2-(x+h))} \right)$$

$$= \lim_{h \to 0} \frac{4(2-x) - 4(2-(x+h))}{h(2-x)(2-(x+h))}$$

$$= \lim_{h \to 0} \frac{\cancel{8} - \cancel{4x} \cancel{- 8} + \cancel{4x} + 4h}{h(2-x)(2-(x+h))}$$

$$= \lim_{h \to 0} \frac{4\cancel{h}}{\cancel{h}(2-x)(2-(x+\cancel{h}^{0}))}$$

$$= \frac{4}{(2-x)(2-x)}$$

$$\therefore f'(x) = \frac{4}{(2-x)^2}$$

$\therefore$ The derivative of $f(x)$ is $f'(x) = \dfrac{4}{(2-x)^2}$.

4. A)

$$\frac{f(a+h)-f(a)}{h}$$

$$= \frac{2(a+h)^2 - 8(a+h) + 3 - (2a^2 - 8a + 3)}{h}$$

$$= \frac{\cancel{2a^2} + 4ah + 2h^2 - \cancel{8a} - 8h \cancel{+3} - \cancel{2a^2} + \cancel{8a} \cancel{-3}}{h}$$

$$= \frac{4ah + 2h^2 - 8h}{h}$$

$$= 4a + 2h - 8$$

$\therefore \frac{f(a+h)-f(a)}{h}$ for $f(x) = 2x^2 - 8x + 3$ is $\frac{f(a+h)-f(a)}{h} = 4a + 2h - 8$.

a)

a) i) Sub $a=3, h=2$.

h

$4a+2h-8$

$=4(3)+2(2)-8$

$=8$

$\therefore \frac{f(a+h)-f(a)}{h} = 8$ for $f(x)=2x^2-8x+3$ at $a=3, h=2$

b)

Sub $a=-2, h=0.5$:

$4a+2h-8$

$=4(-2)+2(0.5)-8$

$=-15$

$\therefore \frac{f(a+h)-f(a)}{h} = -15$ for $f(x)=2x^2-8x+3$ at $a=-2, h=0.5$

5.

a)

$$s'(t) = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h}$$

$$v(t) = \lim_{h \to 0} \frac{15(t+h) - 1.5(t+h)^2 - (15t - 1.5t^2)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{15t} + 15h - \cancel{1.5t^2} - 3th - 1.5h^2 - \cancel{15t} + \cancel{1.5t^2}}{h}$$

$$= \lim_{h \to 0} \frac{15\cancel{h} - 3t\cancel{h} - 1.5h^{\cancel{2}}}{\cancel{h}}$$

$$= \lim_{h \to 0} 15 - 3t - 1.5h$$

$\therefore v(t) = 15 - 3t$

$\therefore s'(t)$ (and thus the velocity at time t) is given by $v(t) = 15 - 3t$.

b)

i) Sub $t=2$

$v(t)=15-3t$

$v(t)=15-3(2)$

$\therefore v(t)=9$ km/h

$\therefore v(t)$ (the instantaneous velocity) at $t=2$ hours is $v(2)=9$ km/h.

ii) Sub $t=7$

$v(t)=15-3t$

$v(t)=15-3(7)$

$\therefore v(t)=-6$ km/h

$\therefore v(t)$ (the instantaneous velocity) at $t=7$ hours is $v(7)=-6$ km/h.

c)

The max height occurs when the upward velocity of the balloon is 0. Using $s(t) = 15t - 1.5t^2$ as the function of the balloon's distance above the ground, $s'(t)$ or $v(t) = 15 - 3t$ describes the balloon's upward velocity at any time t. When $v(t) = 0$, the balloon reaches its max height. Since $s(t)$ has a negative leading coefficient, the value at $v(t)=0$ is a maximum.

$v(t) = 15 - 3t$

Set $v(t) = 0$.

$0 = 15 - 3t$

$t = 5h.$

Sub $t=5$ into $s(t)$:

$$\begin{aligned} s(t) &= 15t - 1.5t^2 \\ &= 15(5) - 1.5(5^2) \\ &= 37.5 \text{ km.} \end{aligned}$$

$\therefore$ The balloon reaches its max height of 37.5 km at $t=5$ hours.

d)

t	$-\infty$		$t=5$		$+\infty$
$s'(t)$		$+$	0	$-$	
$s(t)$		increasing	max	decreasing	

$, t \in \mathbb{R}$

Test $t < 5$, choose $t=0$.

$s'(t) = 15 - 3t$

$s'(0) = 15 - 3(0)$

$= 15$

$\because s'(0)$ is positive,

$\therefore s(t)$ is increasing on $t \in (-\infty, 5)$

Test $t > 5$, choose $t = 6$.

$s'(t) = 15 - 3t$

$s'(6) = 15 - 3(6)$

$= -3$

$\because s'(6)$ is negative,

$\therefore s(t)$ is decreasing on $t \in (5, \infty)$

$\therefore s(t)$ is increasing on $t \in (-\infty, 5), t \in \mathbb{R}$, reaches a max at $t = 5$, and decreasing on $t \in (5, \infty), t \in \mathbb{R}$.