1.a) $f(x) = x^3 \cos x$

Using the product rule,

Let $u = x^3$ $\quad g = \cos x$

$u' = 3x^2$ $\quad g' = -\sin x$

$f'(x) = u'g + g'u$

$= 3x^2 \cos x - x^3 \sin x$

$\therefore$ The derivative of $f(x) = x^3 \cos x$ is

$f'(x) = 3x^2 \cos x - x^3 \sin x$.

b) $h(x) = (x^2-1)^3 (x^2+1)^2$

Using product rule

Let $f = (x^2-1)^3$ $\quad g = (x^2+1)^2$

$f' = 6x(x^2-1)$ $\quad g' = 4x(x^2+1)$

$h'(x) = f'g + g'f$

$= 6x(x^2-1)^2(x^2+1)^2 + 4x(x^2+1)(x^2-1)^3$

$\therefore$ The derivative of $h(x) = (x^2-1)^3(x^2+1)^2$ is

$h'(x) = 6x(x^2-1)^2(x^2+1)^2 + 4x(x^2+1)(x^2-1)^3$

For f':

Use chain rule,

Let $u = x^2 - 1$, $\quad \frac{du}{dx} = \frac{d}{dx}(x^2-1)$

$= 2x$

$\therefore f(u) = u^3$

$f'(u) = 3u^2$

$\therefore f'(x) = 3(x^2-1)^2(2x)$

$= 6x(x^2-1)^2$

For g':

Use chain rule,

Let $u = x^2+1$ $\quad \frac{du}{dx} = \frac{d}{dx}(x^2+1)$

$= 2x$

$\therefore g(u) = u^2$

$g'(u) = 2u$

$\therefore g'(x) = 2(x^2+1)(2x)$

$= 4x(x^2+1)$

c)

$$g(x) = \frac{x^2}{e^x}$$

Use Quotient rule:

Let $f = x^2 \quad h = e^x$

$f' = 2x \quad h' = e^x$

$$g'(x) = \frac{f'h - h'f}{h^2}$$

$$= \frac{(2x)(e^x) - (e^x)x^2}{(e^x)^2}$$

$$= \frac{2xe^x - x^2e^x}{e^{2x}}$$

$\therefore$ The derivative of $g(x) = \frac{x^2}{e^x}$ is $g'(x) = \frac{2xe^x - x^2e^x}{e^{2x}}$.

2.

The tangent to the curve has slope $f'(1)$

For $f'(x)$:

$f(x) = (2x^2-1)^{-5}$

Use chain rule

Let $u = 2x^2-1$ $\qquad \frac{du}{dx} = \frac{d}{dx}(2x^2-1)$

$f(u) = u^{-5}$ $\qquad = 4x$

$f'(u) = -5u^{-6}$

$\therefore f'(x) = -5(2x^2-1)^{-6}(4x)$

$= -20x(2x^2-1)^{-6}$

At $x=1$,

$f'(1) = -20(1)(2(1)^2-1)^{-6}$

$= -20(1)^{-6}$

$= -20$ $\quad \therefore -20$ is the slope of the tangent line at $x=1$

Find a point, Sub $x=1$ into f

$$f(1) = (2(1)^2-1)^{-5}$$
$$= (1)^{-5}$$
$$= 1$$

$\therefore$ $(1,1)$ is a point on the curve

Sub $(1,1)$, $m=-20$

$$y = mx+b$$

$$1 = -20(1)+b$$

$$b = 21$$

$\therefore$ The slope of the tangent to the curve is $y = -20x+21$

3.

For y':

$y = e^{kx}$

Use chain rule,

Let $u = kx$ $\quad \frac{du}{dx} = \frac{d}{dx}(kx)$

$y = e^{u}$ $\quad = k$

$y' = e^{u}$

$\therefore y' = ke^{kx}$

For y'':

Use chain rule,

Let $u = kx$ $\quad \frac{du}{dx} = \frac{d}{dx}(kx)$

$y' = ke^{u}$ $\quad = k$

$y'' = ke^{u}$

$\therefore y'' = k^2 e^{kx}$

Solve:

$y + y' = y''$

$$\cancel{e^{kx}} + k\cancel{e^{kx}} = k^2\cancel{e^{kx}}$$

$$1+k=k^2$$

$$k^2-k-1=0$$

Use quadratic formula,

$$k=\frac{1\pm\sqrt{1+4}}{2}$$

$$=\frac{1\pm\sqrt{5}}{2}$$

$\therefore k=\frac{1-\sqrt{5}}{2}$ and $k=\frac{1+\sqrt{5}}{2}$ satisfies the equation $y+y'=y''$.

4.

Find T':

$T = A + (T_0 - A)e^{kt}$

$T' = \frac{d}{dt}(A) + \frac{d}{dt}(T_0 - A)e^{kt}$

$= 0 + (T_0 - A)\frac{d}{dt}(e^{k t})$

$= (T_0 - A)ke^{kt}$

$\therefore T' = (T_0 - A)ke^{kt}$

Sub $T_0 = 88$, $A = 27$, $k = 0.033$, $t = 4$,

$T' = (88 - 27)(0.033)e^{(0.033)4}$

$\simeq 2.297$ °C/min

$\therefore$ The temperature of the tea is cooling at 2.297 °C/min.

5. Let:

$(290 + 10d)$ describes the price of the ipods, d = # of price increases.

$(30-d)$ describes the # of ipods sold per week, d = # of price increases

$-110(30-d)$ describes the total cost of the ipods to the store, d = # of price increases.

Revenue is given as the product, Let R be revenue

Let $R = (290+10d)(30-d) - 110(30-d)$

$= (290 + 10d - 110)(30-d)$

$= (180 + 10d)(30-d)$

$= 5400 - 180d + 300d - 10d^2$

$= -10d^2 + 120d + 5400$

For R':

$R' = \frac{d}{d(d)}(-10d^2 + 120d + 5400)$

$= -20d + 120$

Set R' to 0:

$0 = -20d + 120$

$20d = 120$

$d = 6$ $\therefore$ $30 - 6 = \$24$ is the best price.

Sub $d = 6$ into R:

$R(6) = (240 + 10(6))(30-6) - 120(30-6)$

$= 5520$

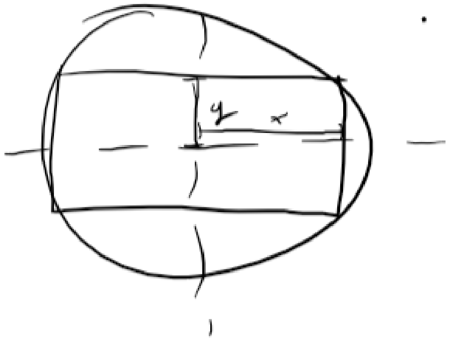
$\therefore$ A price of $30 - 6 = \$24$ yields the highest profit of $\$5520$.

6.

Rearrange

$x^2+y^2=25$

$y=\pm\sqrt{25-x^2}$



$A=(2x)(2y)$

$=4xy$

Total Area is given by

$A=(2x)(2y)$

$=4xy$

$=4x\sqrt{25-x^2}$ (Only consider top half of circle)

For A':

Use product rule

$f=4x \quad g=\sqrt{25-x^2}$

$f'=4 \quad g'=-x(25-x^2)^{-\frac{1}{2}}$

$A'=f'g+g'f$

$=4\sqrt{25-x^2}-4x^2(25-x^2)^{-\frac{1}{2}}$

Set A' to 0:

For g':

Use chainrule

Let $u=25-x^2$ $\quad \frac{du}{dx}=\frac{d}{dx}(25-x^2)$

$=-2x$

$g=\sqrt{u}$

$g'=\frac{1}{2}u^{-\frac{1}{2}}$

$g'=\frac{1}{2}(-2x)(25-x^2)^{-\frac{1}{2}}$

$\therefore g'=-x(25-x^2)^{-\frac{1}{2}}$

$$0 = 4\sqrt{25-x^2} - 4x^2(25-x^2)^{-\frac{1}{2}}$$

$$\frac{\cancel{4}x^2}{\sqrt{25-x^2}} = \cancel{4}\sqrt{25-x^2}$$

$$x^2 = 25-x^2$$

$$2x^2 = 25$$

$$x^2 = \frac{25}{2}$$

$$\therefore x = \frac{5}{\sqrt{2}}$$

Sub $x = \frac{5}{\sqrt{2}}$ into $y = \sqrt{25-x^2}$

$$y = \sqrt{25 - \frac{25}{2}}$$

$$= \frac{5}{\sqrt{2}}$$

Sub $x = \frac{5}{\sqrt{2}}$, $y = \frac{5}{\sqrt{2}}$ into A

$$A = 4\left(\frac{5}{\sqrt{2}}\right)\left(\frac{5}{\sqrt{2}}\right)$$

$$= 25$$

$\therefore$ The area of the largest rectangle in the circle is 25 sq. units

7.

For L':

$$L' = \frac{d}{dn}(6) - \frac{d}{dn}\left(\frac{20.4n}{n^2+49}\right)$$

$$= -\frac{d}{dn}\left(\frac{20.4n}{n^2+49}\right)$$

Use Quotient rule,

Let $f = 20.4n \qquad g = n^2+49$

$f' = 20.4 \qquad g' = 2n$

$$L' = -\left(\frac{f'g - g'f}{g^2}\right)$$

$$= -\left(\frac{20.4(n^2+49) - 2n(20.4n)}{(n^2+49)^2}\right)$$

$$= -\frac{20.4n^2 + 999.6 - 40.8n^2}{(n^2+49)^2}$$

$$= -\frac{-20.4n^2 + 999.6}{(n^2+49)^2}$$

$$= \frac{20.4n^2 - 999.6}{(n^2+49)^2}$$

Sub $n = 4$,

$$L'(4) = \frac{20.4(4^2) - 999.6}{(4^2+49)^2}$$

$$= -0.159 \text{ pH levels/min}$$

$\therefore$ The instantaneous rate of change in the pH of your mouth is changing at $-0.159 \ \frac{\text{pH levels}}{\text{min}}$. This value is the instantaneous rate at which the pH of your mouth changes, this most closely matchs the rate at which the concentration of H^+ ions changes in your mouth on a log scale.

g.

Let h be the altitude, Let b be the length of the base.

$h' = 1$

Let A be the area,

$A' = 2$

$A = \frac{bh}{2}$

$A+$ $A = 100$ cm², $h = 10$cm,

$A = \frac{bh}{2}$

$100 = \frac{10}{2} b$

$b = 20$ cm

Use product rule,

Let $f = b$, $g = h$

$f' = b'$, $g' = h'$

$A' = \frac{1}{2}(f'g + g'f)$

$= \frac{1}{2}(b'h + h'b)$

Sub $h = 10$, $h' = 1$, $A' = 2$, $b = 20$:

$2 = \frac{1}{2}(b'(10) + (1)(20))$

$4 = 10b' + 20$

$\frac{-16}{10} = b'$

$\therefore b' = -\frac{8}{5}$ cm/min

$\therefore$ The base of the triangle is changing at $-\frac{8}{5}$ cm/min (it's shrinking by $-\frac{8}{5}$ cm/min) when the altitude is 10cm and the area is 100 cm^2.

9.

For $\frac{dV}{dS}$,

$$\frac{dV}{dt} = \frac{dV}{dr}\left(\frac{4}{3}\pi r^3\right)\left(\frac{dr}{dt}\right) \quad (\text{Chain rule})$$

$$= (4\pi r^2)(r')$$

$$\frac{dS}{dt} = \frac{dS}{dr}\left(4\pi r^2\right)\left(\frac{dr}{dt}\right) \quad (\text{Chain rule})$$

$$= (8\pi r)r'$$

$$\frac{dV}{dS} = \frac{\frac{dV}{dt}}{\frac{dS}{dt}}$$

$$= \frac{4\pi r^2 (r')}{(8\pi r)r'}$$

$$= \frac{r}{2}$$

Sub $r = \frac{5\sqrt{3}}{2\sqrt{\pi}}$

When $S = 75$,

$$S = 4\pi r^2$$

$$75 = 4\pi r^2$$

$$r^2 = \frac{75}{4\pi}$$

$$r = \frac{5\sqrt{3}}{2\sqrt{\pi}}$$

$$\frac{dV}{dS} = \frac{1}{2}\left(\frac{5\sqrt{3}}{2\sqrt{\pi}}\right)$$

$$= \frac{5\sqrt{3}}{4\sqrt{\pi}} \text{ cm}^3/\text{min} \simeq 1.222 \text{ cm}$$

$\therefore$ The balloon's volume is changing at a rate of $\frac{5\sqrt{3}}{4\sqrt{\pi}}$ cm^3/min or about 1.222 cm^3/min with respect to its surface area when its surface area measures 75 cm^2.