Self-Organised Criticality of Sandpile Models

Project Report: Non-Equilibrium Statistical Mechanics (PHY-3620)

Dhruv Aryan

Instructor: Professor Somendra Bhattacharjee

Abstract

We study the Abelian Model (AM), a special case of the BTW sandpile model where the toppling condition depends on local height and not on gradients. We characterise the self-organised critical (SOC) state and determine its entropy for an arbitrary finite lattice in any number of dimensions. We calculate the two-point correlation function and use it to obtain an expression for the average number of topplings per added particle on a finite lattice. Additionally, we perform computational simulations, and show that the dynamics of the model give us the SOC state. We show that the avalanche size distribution follows a power law, and we then compute the avalanche size as a function of lattice length and compare this with the analytical result.

Introduction

Self-organized criticality (SOC) is a fascinating phenomenon observed in a wide range of complex systems, where they naturally evolve towards a critical state without the need for external fine-tuning or control parameters. At this critical state, the system exhibits emergent behavior characterized by scale invariance, long-range correlations, and the occurrence of spontaneous events known as avalanches. Understanding self-organized criticality is essential for unraveling the principles governing the behavior of diverse natural and artificial systems, ranging from earthquakes and forest fires to neuronal activity in the brain.

Sandpile models serve as prototypical examples of systems exhibiting self-organized criticality. Introduced by Per Bak, Chao Tang, and Kurt Wiesenfeld in 1987 [1], these models provide a simple yet powerful framework for studying the dynamics of complex systems. In a sandpile model, grains of sand are incrementally added to a lattice-based grid, and when a site exceeds a critical threshold, it topples, redistributing its grains to neighboring sites. This process of local perturbations and global rearrangements leads to the emergence of critical behavior, characterized by power-law distributions of avalanche sizes and durations.

The study of sandpile models and self-organized criticality has attracted considerable attention from researchers across various disciplines, including physics, mathematics, computer science, and complexity science. By simulating sandpile models and analyzing their behavior, researchers aim to uncover the underlying mechanisms driving self-organized criticality and elucidate its implications for the dynamics of complex systems.

In this project, we explore self-organized criticality in sandpile models, aiming to explore its fundamental principles, characterize its critical behavior, and investigate its implications for the dynamics of complex systems. Through analytical calculations and computational simulations, we seek to shed light on the mechanisms underlying self-organized criticality and deepen our understanding of its significance in the broader context of complex systems. The analytical and computational analysis in this paper is based on papers by Dhar [2], and Bak, Tang, and Wiesenfield [1].

Fractals in Nature

Fractals are geometric shapes or patterns that display self-similarity at different scales, meaning that they exhibit similar structures or patterns when viewed at magnifications. At first glance, such objects may seem purely confined to abstract geometrical territory that is far removed from the real world, with common examples of such mathematical fractals including the Sierpinski triangle and the Mandelbrot set. However, these intricate and irregular shapes are ubiquitous in nature, manifesting in various phenomena and structures across different scales and domains. The following are some natural examples:

1. **Coastlines**: Coastal landscapes exhibit fractal properties, with their jagged and irregular shapes repeating patterns at different scales. From rugged cliffs to intricate bays and coves, the coastline's self-similar structure is evident when viewed at varying levels of detail.
2. **Mountain Ranges**: Mountainous terrains display fractal characteristics, with peaks, ridges, and valleys forming self-similar patterns at different scales. The branching patterns of river networks and the undulating contours of mountain slopes reflect the fractal nature of these landscapes.
3. **Clouds and Weather Patterns**: Cloud formations and weather patterns often exhibit fractal geometry, with their complex and swirling shapes displaying self-similar structures at different levels of magnification. From wispy cirrus clouds to turbulent storm systems, the fractal nature of atmospheric phenomena is evident in their intricate patterns.
4. **Vegetation and Trees**: Natural vegetation, such as trees and plants, often display fractal branching patterns. The intricate network of branches, twigs, and leaves exhibits self-similarity, with smaller branches resembling larger ones in a repeating pattern.
5. **Snowflakes**: Snowflakes are classic examples of natural fractals, with their intricate crystalline structures displaying self-similar patterns at different scales. Each snowflake's unique and intricate geometry reflects the underlying fractal symmetry of its formation process.

Fractals are thus pervasive in nature, manifesting in diverse phenomena and structures across different scales and domains. Understanding the presence and significance of fractals in nature provides valuable insights into the underlying principles governing the organization and dynamics of natural systems.

Self-Organised Criticality

Self-organized criticality (SOC) is a concept in complex systems theory where a system evolves towards a critical state without any external driving or fine-tuning, resulting in emergent behavior characterized by scale invariance. In the context of sandpile models, self-organized criticality refers to the tendency of the sandpile to organize itself into a critical state where small perturbations can lead to cascading events, often termed as "avalanches," which can span various spatial and temporal scales.

Principles of Self-Organized Criticality

The concept of self-organized criticality was first proposed by Per Bak, Chao Tang, and Kurt Wiesenfeld in 1987 [1]. They introduced the idea through the sandpile model, demonstrating how simple local rules can lead to global behavior exhibiting criticality. The key principles of self-organized criticality include:

1. **Absence of External Driving**: Systems exhibiting SOC evolve towards criticality without any external driving forces or fine-tuning parameters. Instead, they reach criticality through the dynamics of internal interactions and feedback loops.
2. **Scale Invariance**: At criticality, the system displays scale invariance, meaning that the statistical properties of its behavior remain the same across different spatial and temporal scales. This property is often manifested through power-law distributions of various system observables.
3. **Avalanche Dynamics**: One of the hallmarks of SOC is the occurrence of avalanches, which are spontaneous and intermittent bursts of activity propagating through the system. These avalanches can vary widely in size and duration but follow power-law distributions, indicating scale invariance.
4. **Long-range Correlations**: Systems at criticality exhibit long-range correlations, where events at one scale can influence and trigger events at much larger scales. This leads to the emergence of coherent structures and patterns in the system's behavior.

The Abelian Sandpile Model

In 1987, Bak, Tang and Wiesenfield (BTW) introduced a simple cellular automaton model to describe the dynamics of sandpiles [1]. They showed that the model displayed the unique property that on starting with any arbitrary initial state, its stochastic evolution produces a critical state characterised by power law correlations in space and time in the long time limit. Therefore, the BTW sandpile model displays self-organised criticality. BTW argued that such models can describe diverse phenomena in non-equilibrium systems that involve dissipative, nonlinear transport in open systems, such as earthquakes and the distribution of matter in the universe. In 1990, Dhar showed that a BTW model in which the toppling condition depended only on the local heights, rather than on the gradients of the heights, satisfy a commutative algebra. He termed this model the Abelian model (AM). This is the model we will consider.

Definition of the Model

We now describe the general AM in any number of dimensions. We consider a set of N sites, labeled from 1 to N. We assign an integer z_i to each site i. We now have two rules:

1. **Adding a particle**: We select a site at random (with a probability of selecting site i being p_i) and increase z_i by 1. The other sites z_j $(j \neq i)$ are unchanged.
2. **Toppling**: We create an $N \times N$ integer matrix Δ and a set of N integers z_{ic}, where $i = 1, ..., N$. If $z_i > z_{ic}$ at any site, then that site topples. When this happens at site i, then

$$z_j \to z_j - \Delta_{ij}, \forall j \tag{1}$$

The matrix Δ satisfies the following conditions:

$$\Delta_{ii} > 0, \forall i \tag{2}$$

$$\Delta_{ij} \leq 0, \forall i \neq j \tag{3}$$

and

$$\sum_{j=1}^{N} \Delta_{ij} \leq 0, \forall i \tag{4}$$

These conditions tell us that when a toppling occurs at site i, z_i must decrease, z_j for $j \neq i$ cannot increase, and that there is no creation of particles during toppling. However, particles are allowed to leave the system at the boundaries, and in fact, no steady state is possible otherwise. Additionally, we assume that the matrix Δ is such that any configuration relaxes to a stable configuration in a finite number of steps. We do not assume that Δ is a symmetric matrix.

The Abelian Algebra of the AM

Without loss of generality, we assume that $z_{ic} = \Delta_{ii}$ for all i. Then, any configuration z_i in which $1 \leq z_i \leq \Delta_{ii}$ is a stable configuration. We now define N operators a_i $(i = 1, ..., N)$ on this space of stable configurations by defining $a_i C$ to be the stable configuration obtained by adding a particle at site i to the configuration C and allowing the system to evolve by toppling and relax.

Let us consider an unstable configuration in which two sites α and β are both critical ($z_\alpha > \Delta_{\alpha\alpha}$ and $z_\beta > \Delta_{\beta\beta}$). First toppling α leaves β critical [Eq. (3)], and after toppling both α and β, we obtain a configuration in which z_i decreases by $\Delta_{\alpha i} + \Delta_{\alpha i}$. This is symmetrical under the exchange of α and β. Thus, regardless of whether α or β toppled first, we get the same resulting configuration. By a repeated use of this argument, we see that in an avalanche, the same final stable configuration is reached irrespective of the order in which unstable sites are toppled. Additionally, toppling at an unstable site α and then adding a particle at site β gives the same result as first adding a particle at β and then toppling at α. From these two properties, it follows that for all configurations C and all i and j from 1 to N, we have

$$a_i a_j C = a_j a_i C \tag{5}$$

In other words, the operators a_i commute with each other:

$$[a_i, a_j] = 0 \tag{6}$$

In other BTW models where the toppling condition depends on the gradients of heights rather than simply the heights, the inequality in Eq. (3) is not satisfied and the operators a_i do not commute.

Characterising the SOC State of the AM

From the general theory of Markov chains, the set of all stable configurations can be divided into two classes: recurrent and transient. We define a configuration to be recurrent if and only if there exist positive integers m_i $(i = 1, ..., N)$ such that

$$a_i^{m_i} C = C \tag{7}$$

for all i. Eq. (6) tells us that if C is recurrent then configurations $a_i C$ $(i = 1, ..., N)$ are also recurrent. Let us denote the set of all recurrent configurations by R. Thus, we can say that R is closed under multiplication by operators a_i. Under Markovian evolution, once the AM gets into a recurrent configuration, it can never get out of R. All nonrecurrent configurations are transient configurations, and have zero probability of occurrence in the SOC state.

For operators a_i restricted to R as their domains, inverses can be well-defined. For any configuration C satisfying Eq. (7), we define inverses as follows:

$$a_i^{-1} = a_i^{m_i - 1} C \tag{8}$$

for all $i = 1, ..., N$. It has been argued [3] that the existence of a unique inverse in the set of recurrent configurations implies that the state in which all recurrent configurations occur with equal probability is the invariant state of the Markovian evolution. Therefore, in the SOC state of the AM, only recurrent configurations have a nonzero probability of occurrence, and this nonzero probability is the same for all recurrent configurations.

Entropy of the SOC State

Let us now consider a configuration $C \in R$ to which we add Δ_{ii} particles one after another at some site i. $z_i > 0$ in C, and so, after these additions, the site will become unstable and topple, adding $-\Delta_{ij}$ at all other sites j $(i \neq j)$ in the process. Therefore, the operators a_i $(i = 1, ..., N)$ must satisfy the following equation:

$$a_i^{\Delta_{ii}} = \prod a_j^{-\Delta_{ij}} \tag{9}$$

where the product is over all $j \neq i$. This gives us

$$\prod_{j=1}^{n} a_j^{\Delta_{ij}} = 1 \tag{10}$$

for all $i = 1, ..., N$. Since the operators a_i commute with each other, all representations of the algebra given in the equation above are one-dimensional. We can write the operators as

$$a_j = \exp(i\phi_j) \tag{11}$$

where $j = 1, ..., N$ and ϕ_j are some real numbers. In terms of these ϕs, Eq. (10) can be rewritten as

$$\sum_{j=1}^{N} \Delta_{ij}\phi_j = 2\pi n_i \tag{12}$$

for all i, where n_i are some integers. Solving this equation, we get

$$\phi_i = 2\pi \sum_{j=1}^{N} [\Delta^{-1}]_{ij} n_j \tag{13}$$

for all i. This equation shows that the allowed values of $\{\phi_i\}$ form an N-dimensional periodic lattice. For each $\{n_i\}$, we have a set of values $\{\phi_i\}$, which gives a representation of the operator algebra of Eq. (10). However, the ϕ_is are phases. Hence, in Eq. (13), only points lying within the N-dimensional cube $0 \leq \phi_i < 2\pi$ $(i = 1, ..., N)$ give rise to distinct representations. The number of such representations is the ratio of the volume of the cube and the volume of the unit cell of the ϕ lattice. This number must also be equal to the number of distinct elements of the algebra. These distinct elements are products of the type $a_1^{m_1} a_2^{m_2} \cdots a_N^{m_N}$ $(m_1, m_2, ..., m_N$ are non-negative integers) which are not equal under Eq. (10). Each such element acting on a configuration C results in a distinct configuration. Therefore, this number must equal N_R, the number of distinct configurations in R. Therefore, we get

$$N_R = \det(\Delta) \tag{14}$$

Since all these configurations occur with equal probability in the SOC state, the entropy S of the SOC state is given by

$$\boxed{S = \ln\det(\Delta)} \tag{15}$$

Two-Point Correlation Function

Let us now calculate the two-point correlation function in the SOC state. We define G_{ij} to be the expected number of topplings at site j due to the avalanche caused by adding a particle at site i. Then, the total average flux of particles out of the site j is $G_{ij}\Delta_{jj}$. The total average flux of particles into the site j is $\sum_k G_{ik}(-\Delta_{kj})$, where k is summed over all sites $\neq j$. In the SOC state, the average influx must equal the average outflux, and thus we get

$$\sum_k G_{ik}\Delta_{kj} = \delta_{ij} \tag{16}$$

Solving this equation, we get

$$\boxed{G_{ij} = [\Delta^{-1}]_{ij}} \tag{17}$$

for all i, j. The argument leading to this equation is general, and therefore, it holds even in a non-Abelian case when the toppling condition depends on gradients.

Let us now use this equation to calculate $\langle T \rangle$, the average number of topplings per added particle on a finite $L \times L$ square lattice for the nearest-neighbour AM. Writing down the matrix Δ, we get

$$\boxed{\langle T \rangle = \frac{1}{L^2(L+1)^2} \sum_{m,n} \cot^2 \frac{\pi m}{2(L+1)} \cot^2 \frac{\pi n}{2(L+1)} \left(\sin^2 \frac{\pi m}{2(L+1)} \sin^2 \frac{\pi n}{2(L+1)} \right)^{-1}} \tag{18}$$

where the summation over m, n extends over all odd integers from 1 to L. For large L, we get

$$\boxed{\langle T \rangle \sim L^2} \tag{19}$$

Relaxation Time

Now, we determine the spectrum of relaxation times to the SOC state. Let $P_t(C)$ be the probability that the stable configuration obtained after adding the tth particle is C. Then, this probability must satisfy the master equation:

$$P_{t+1}(C) = \sum_{C'} W(C, C') P_t(C') \tag{20}$$

The rates $W(C, C')$ can be written in terms of an N_R-dimensional matrix W. In terms of the operators $\{a_i\}$, this matrix is given by

$$W = \sum_{i=1}^{N} p_i a_i \tag{21}$$

We showed that the operators a_i commute with each other, and therefore they are simultaneously diagonalisable. Their eigenvalues are given by Eqs. (11) and (13). Therefore, the eigenvalues of W are completely determined.

$n_i = 0$ for all i in Eq. (13) corresponds to the steady state with eigenvalue 1. The next largest eigenvalue determines the relaxation time of the slowest decaying fluctuations in the SOC state. For the undirected d-dimensional AM with nearest-neighbour toppling only, we find that the relaxation time varies as L^d, where L is the linear extent of the system. For large L, this is much larger than the average duration τ of an avalanche, which satisfies $\tau \leq \langle T \rangle \sim L^2$.

Simulation

I have performed a simple simulation of the 2D square lattice nearest-neighbour AM in order to obtain results computationally and test some of these against the analytical solutions given by Dhar. These simulations were performed on Python.

Parameters

1. **Model**: 2D square lattice nearest-neighbour Abelian Model
2. **Boundary Conditions**: Free (particles can leave at the boundary)
3. **Lattice Size**: 50×50
4. **Number of Grains Added**: 10000 (for small lattices), 100000 (for large lattices)
5. **Critical Value (z_{ic})**: 3 for all sites i

Results

We first plot the average height per site for each iteration, since we know that in the transient state, this average height will keep rising until the system reaches the steady state, in which this average height will plateau at a particular value. From Fig. 1, we see that in our case, we reach a steady state after around 5000 iterations.

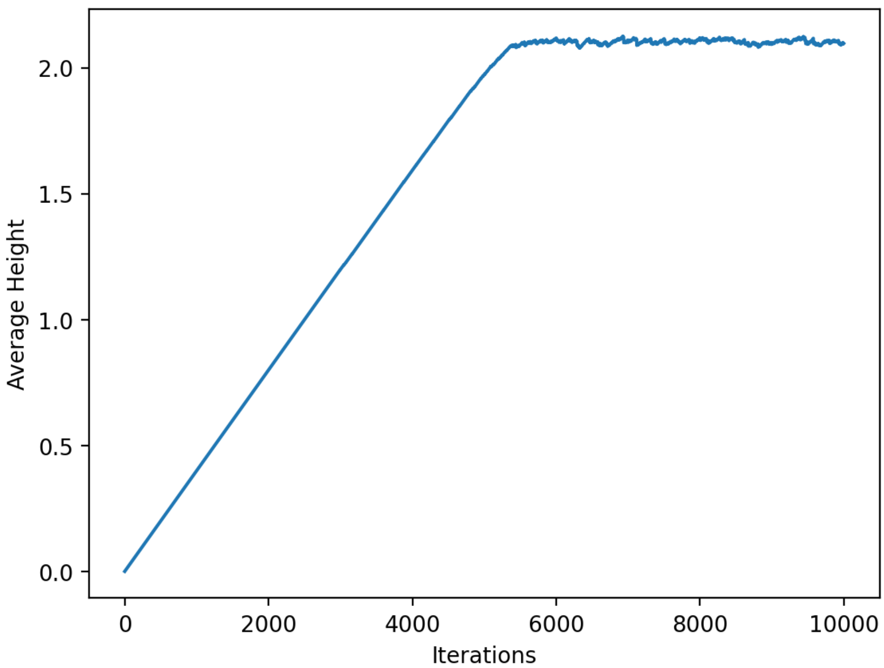
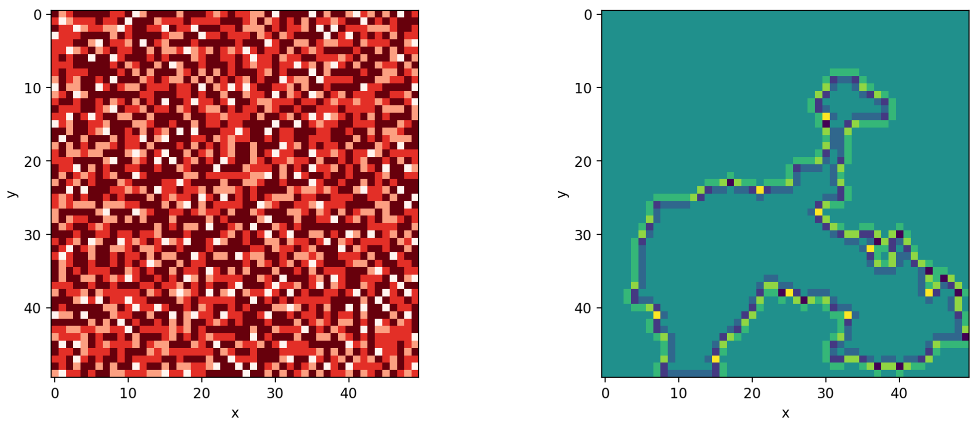


Figure 1: Evolution of average height per site over time. We see that the system reaches steady state at around 5000 iterations.

On plotting the configurations, we cannot make out much, since the image is dominated by noise (see Fig. 2a). Hence, in order to see the important features, we plot the difference between each configuration and the preceding configuration. This directly shows us the avalanche, since only the addition of the grain and the toppling would be visible in this image. On looking at such snapshots after adding each grain, we observe that initially, there is no large-scale change in the state of the system, and that the small-scale perturbations of adding a grain result in small-scale responses. Therefore, the system seems to be linear in this phase. However, as we keep adding particles, we start to observe small-scale changes, and the system then quickly begins to show large-scale response. These large-scale responses are the avalanches. The system begins to show such behaviour at around 5000 iteratons, which is when we showed that it reaches a steady state. Fig. 2b shows us one snapshot, taken in the SOC state of the AM. On observing the image closely, we see that the avalanche pattern visible in the image displays fractal-like behaviour, strongly suggesting that the state in which we start to see these large-scale avalanches is the critical state of the system. We also see that in this state, sometimes we see almost no response to the addition of a particle, whereas we sometimes see large-scale avalanches, displaying another characteristic property of criticality. This is seen in Fig. 3, where we plot the avalanche size T against the iterations, showing the variation in the avalanches over time. The presence of spikes in this graph suggests that the response of the system to small perturbations can be small or big, and thus there is nonlinear response. Therefore, these properties being present in the steady state of the system seems to suggest what we would expect: that this is the SOC state of the AM.



(a) A single configuration of sites. The darker the colour of the site, the greater the number of particles at that site.

(b) The difference between the configuration shown on the left and the previous configuration.

Figure 2: Snapshots of the 50×50 square lattice after 9903 iterations.

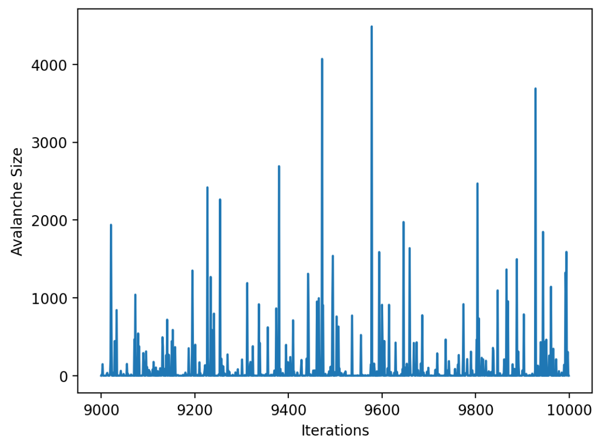


Figure 3: Variation of avalanche size over time. Only the last 1000 iterations have been taken.

We now want to check if this state displays the power law scaling seen in SOC systems. To do this, we want to find the distribution of avalanche size $D(T)$ as a function of the avalanche size T. Here, the avalanche size is the number of topplings that occur as a result of adding a particle. On plotting this, we obtain the log-log graph shown in Fig. 4. We thus see that we obtain a straight line. While the curve obtained does seem to deviate from the straight line for large avalanche sizes, this is largely a finite lattice size effect. Hence, we get a power law distribution:

$$D(T) = T^{-\tau}. \tag{22}$$

We obtain the value of τ from the slope of the line obtained in Fig. Therefore, we get

$$\boxed{\tau = 0.90} \tag{23}$$

This value of τ almost matches the value obtained by Bak, Tang and Wiesenfeld, which is $\tau = 0.98$ [1].

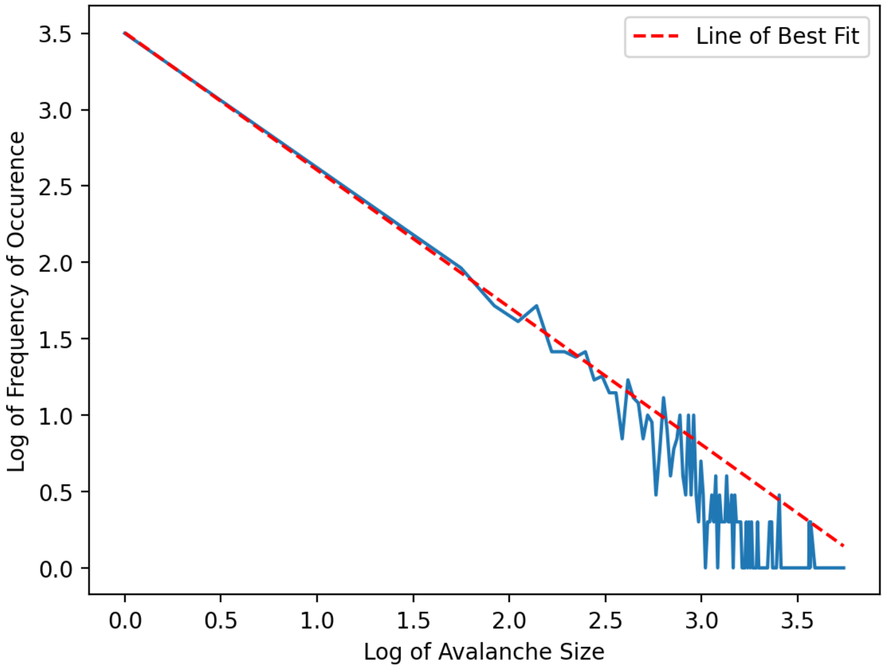


Figure 4: Log-log plot of distribution of avalanche sizes, plotted along with a line of best fit.

We now want to find T as a function of the lattice length L and compare it with the analytical solution given by Dhar [Eq. (18)]. On computing and plotting T against L, we obtain the graph shown in Fig. 5. The computed quantity is plotted alongside the analytical values of T given by Eq. (18). We see that the numerical values are close to the analytical ones, although they do not match exactly. This may be due to the limited number of samples that were used to compute T.

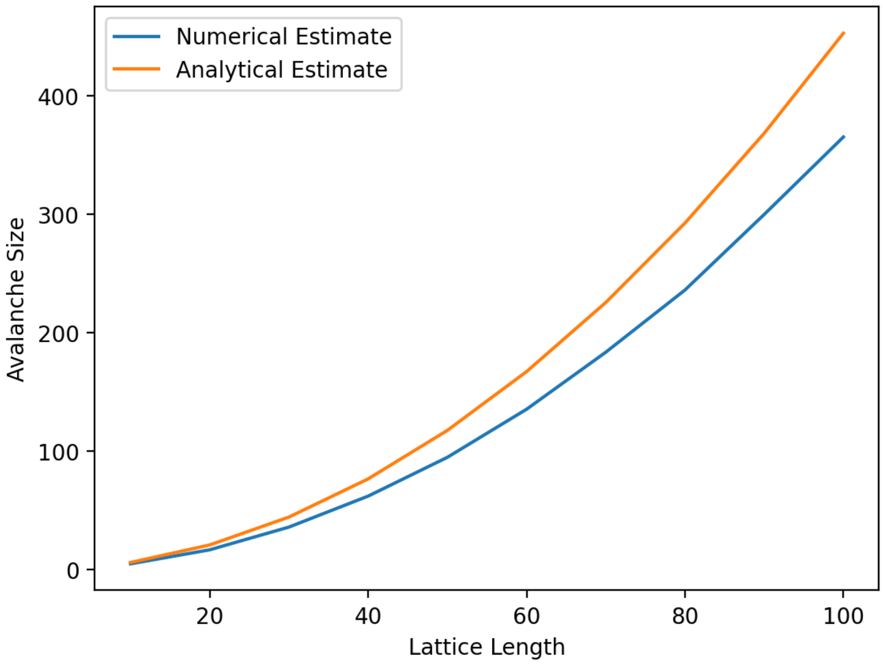


Figure 5: Plot of avalanche size against lattice length, with lengths ranging from 1 to 100. Both the numerical and analytical estimates have been plotted.

Conclusions

We studied the Abelian Model, which is a BTW-type sandpile model in which the toppling condition depends on local height, but not on its gradient. We characterised the critical state and determined its entropy for an arbitrary finite lattice in any number of dimensions. We then showed that the two-point correlation function satisfies a linear equation, and we obtained an expression for the average number of topplings per added particle on a finite lattice. We also performed computational simulations, and showed that the resulting stochastic dynamics give us the SOC state. We showed that the avalanche size distribution follows a power law $D(T) = T^{-\tau}$, with $\tau = 0.90$, which is close to the result obtained by Bak, Tang and Wiesenfield. We then computed the avalanche size as a function of the lattice length and compared the result to the analytical result obtained by Dhar [Eq. (18)] to find that the numerical and analytical estimates seems to match broadly, but that there is a slight difference in values. These results further show that we obtain self-organised criticality in Abelian sandpile models. Thus, such models can be used to further understand the dynamics of systems displaying self-organised criticality.

Acknowledgments

I would like to thank Professor Somendra Bhattacharjee for the opportunity to work on this project and for his guidance.

References

[1] Per Bak, Chao Tang, and Kurt Wiesenfeld. "Self-organized criticality". In: *Physical review A* 38.1 (1988), p. 364.

[2] Deepak Dhar. "Self-organized critical state of sandpile automaton models". In: *Physical Review Letters* 64.14 (1990), p. 1613.

[3] Terence Hwa and Mehran Kardar. "Dissipative transport in open systems: An investigation of self-organized criticality". In: *Physical review letters* 62.16 (1989), p. 1813.