Master in Applied Econometrics and Forecasting

Time Series Analysis and Forecasting

Class #4: White Noise and Autoregressive (AR) Models

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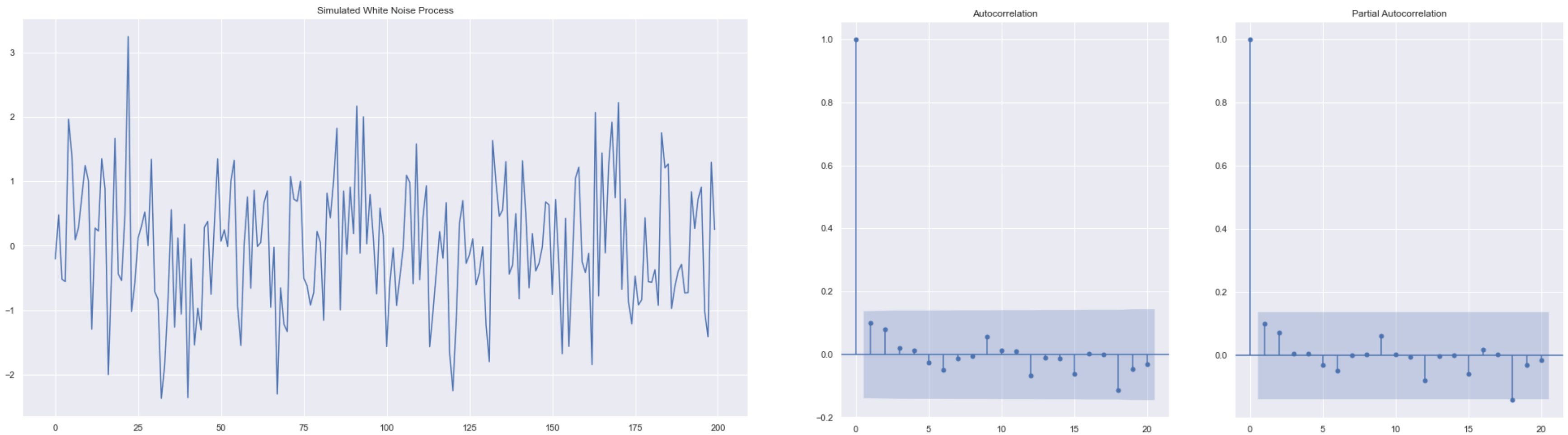
White Noise and Autoregressive (AR) Models

White Noise Process

A process is called a "white noise" process if it is a sequence of uncorrelated random variables:

$$X_t = a_t,$$

where a_t has constant mean $E(a_t) = \mu_a$ (usually assumed to be 0) , constant variance $Var(a_t) = \sigma_a^2$ and null covariance $Cov(a_t, a_{t-k}) = 0$ for all $k \neq 0$. The ACF and PACF of a white noise process are null for all $k \neq 0$.



Simulation of a white noise process with zero mean and unit variance

White Noise and Autoregressive (AR) Models

Autoregressive (AR) Representation

In time series analysis, there is a useful representation of a stationary time series or stochastic process X_t. The process X_t can be written in terms of an infinite autoregressive representation:

$$X_t = \pi_1 X_{t-1} + \pi_2 X_{t-2} + \cdots + a_t = \sum_{j=1}^{\infty} \pi_j X_{t-j} + a_t,$$

or, equivalently,

$$\pi(B) X_t = a_t,$$

where $\pi(B) = 1 - \pi_1 B - \pi_2 B^2 - \cdots = 1 - \sum_{j=1}^{\infty} \pi_j B^j$ and $1 + \sum_{j=1}^{\infty} \left|\pi_j\right| < \infty$.
Box-Jenkins (1976) call a “process invertible” if it can be written in this form.

White Noise and Autoregressive (AR) Models

AR(*p*) Model

The autoregressive model of order *p* is given by

$$X_t = \emptyset_1 X_{t-1} + \cdots + \emptyset_p X_{t-p} + a_t,$$

where a_t is a zero mean white noise series. Because $\sum_{j=1}^{\infty}|\pi_j| = \sum_{j=1}^{p}|\emptyset_j| < \infty$, the process is always invertible. To be stationary, the roots of $(1 - \emptyset_1 B - \cdots - \emptyset_p B^p) = 0$ must be outside of the unit circle.

AR(1) Model

The first-order autoregressive model or AR(1) model is given by

$$X_t = \emptyset_1 X_{t-1} + a_t,$$

where a_t is a zero mean white noise series. As mentioned above, because $|\emptyset_1| < \infty$, the process is always invertible. To be stationary, the root of $(1 - \emptyset_1 B) = 0$, $B = 1/\emptyset_1$, must be outside of the unit circle. That is, $|\emptyset_1| < 1$ is the stationary condition.

White Noise and Autoregressive (AR) Models

ACF and PACF of an AR(1) Model

Assuming, without loss of generality, that $E(X_t) = 0$, the ACF of an AR(1) model can be deduced by multiplying both sides of AR(1) model by X_{t-k} and then taking expectations:

$$E(X_{t-k}X_t) = \emptyset_1 E(X_{t-k}X_{t-1}) + E(X_{t-k}a_t)$$

$$\gamma_k = \emptyset_1 \gamma_{k-1}, \qquad k \geq 1$$

Thus, the ACF becomes

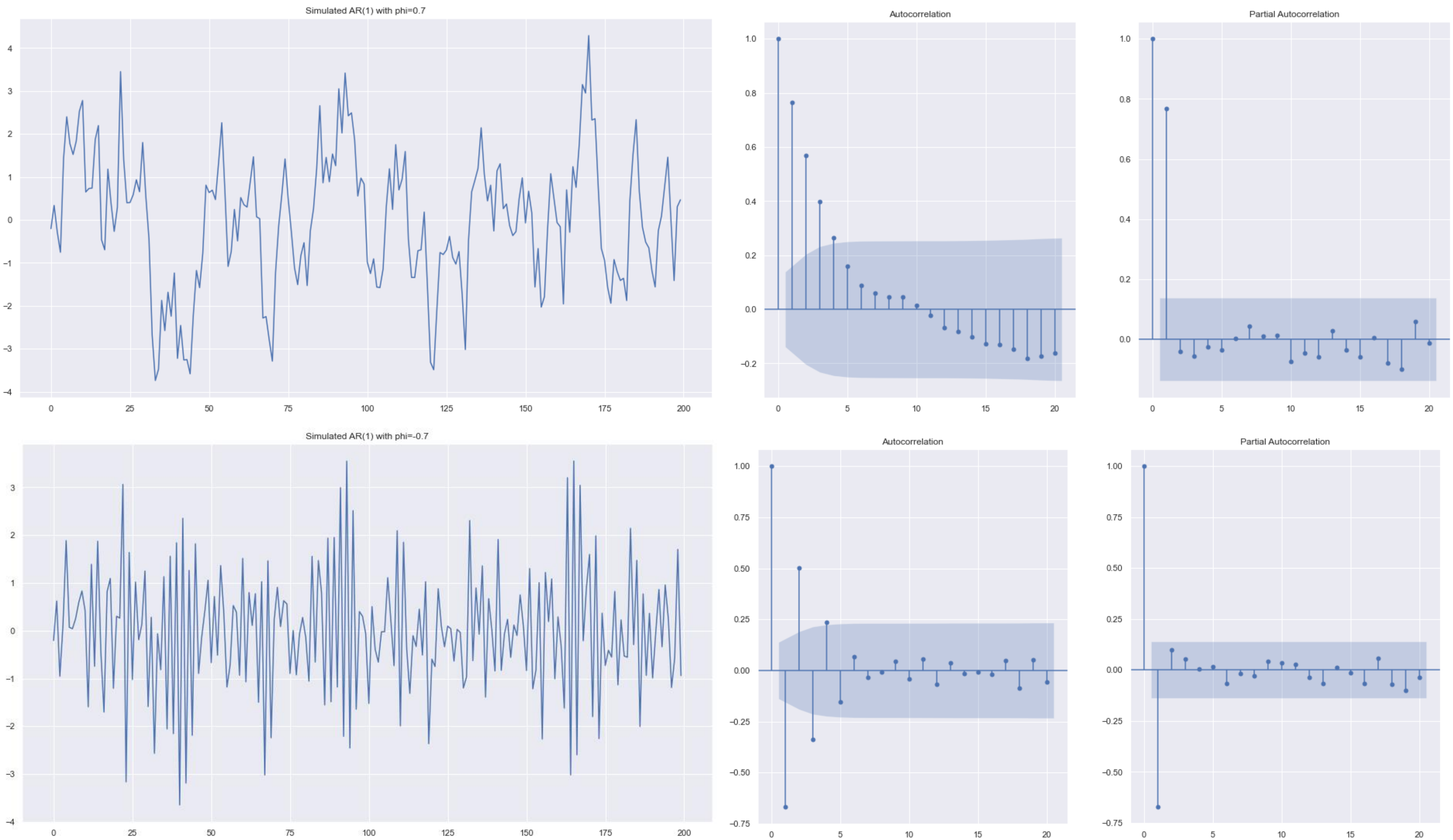
$$\rho_k = \emptyset_1 \rho_{k-1} = \emptyset_1^k, \qquad k \geq 1$$

If $-1 < \emptyset_1 < 0$ the ACF decays exponentially to zero, while if $0 < \emptyset_1 < 1$ the ACF decays in an oscillating pattern (or damped sine wave) to zero, as shown in the next slide.

The PACF of the AR(1) model exhibits a positive or negative (depending on the sign of $\emptyset_1$) spike at lag 1 and then cuts off after lag 1:

$$\emptyset_{kk} = \begin{cases} \rho_1 = \emptyset_1, & k = 1 \\ 0, & k \geq 2 \end{cases}$$

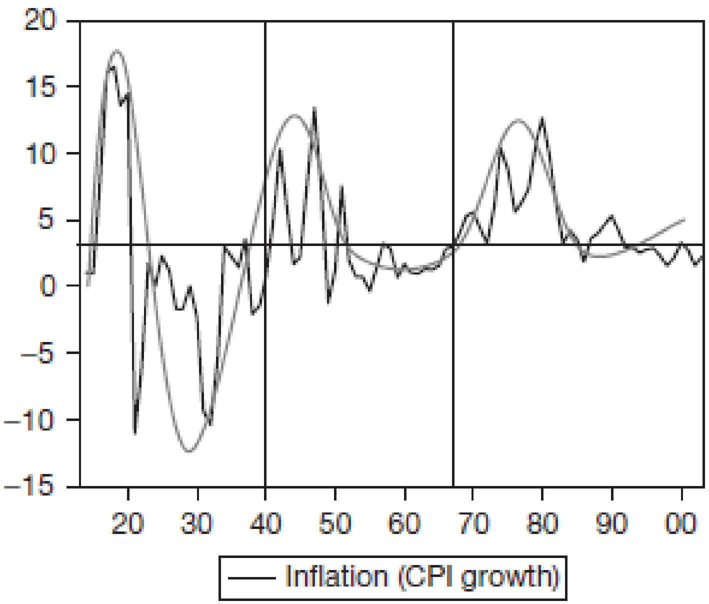
White Noise and Autoregressive (AR) Models

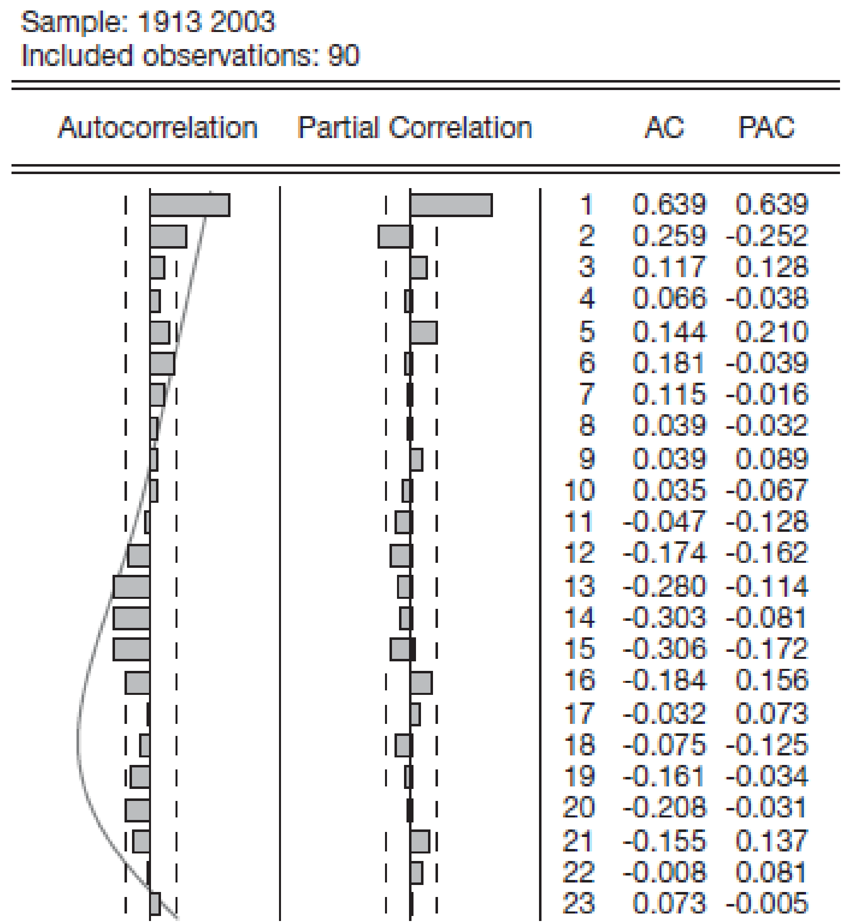


Time Series, ACF and PACF of the simulated AR(1) models with ϕ=0.7 and ϕ=–0.7

White Noise and Autoregressive (AR) Models

Example: Inflation Rate in US, 1913-2003





	AC	PAC
1	0.639	0.639
2	0.259	-0.252
3	0.117	0.128
4	0.066	-0.038
5	0.144	0.210
6	0.181	-0.039
7	0.115	-0.016
8	0.039	-0.032
9	0.039	0.089
10	0.035	-0.067
11	-0.047	-0.128
12	-0.174	-0.162
13	-0.280	-0.114
14	-0.303	-0.081
15	-0.306	-0.172
16	-0.184	0.156
17	-0.032	0.073
18	-0.075	-0.125
19	-0.161	-0.034
20	-0.208	-0.031
21	-0.155	0.137
22	-0.008	0.081
23	0.073	-0.005

Source: Gloria Gonzalez-Rivera, *Forecasting for Economics and Business*, Pearson, 2013

White Noise and Autoregressive (AR) Models

AR(2) model

The second-order autoregressive AR(2) models is

$$X_t = \emptyset_1 X_{t-1} + \emptyset_2 X_{t-2} + a_t,$$

or

$$\emptyset(B) X_t = a_t,$$

where ε_t is a zero mean white noise series. To be stationary, the roots of $\emptyset(B) = 1 - \emptyset_1 B - \emptyset_2 B^2 = 0$ must be outside of the unit circle. Thus, we have the following necessary and sufficient conditions for stationarity:

$$\emptyset_1 + \emptyset_2 < 1,\ \emptyset_2 - \emptyset_1 < 1, -1 < \emptyset_2 < 1.$$

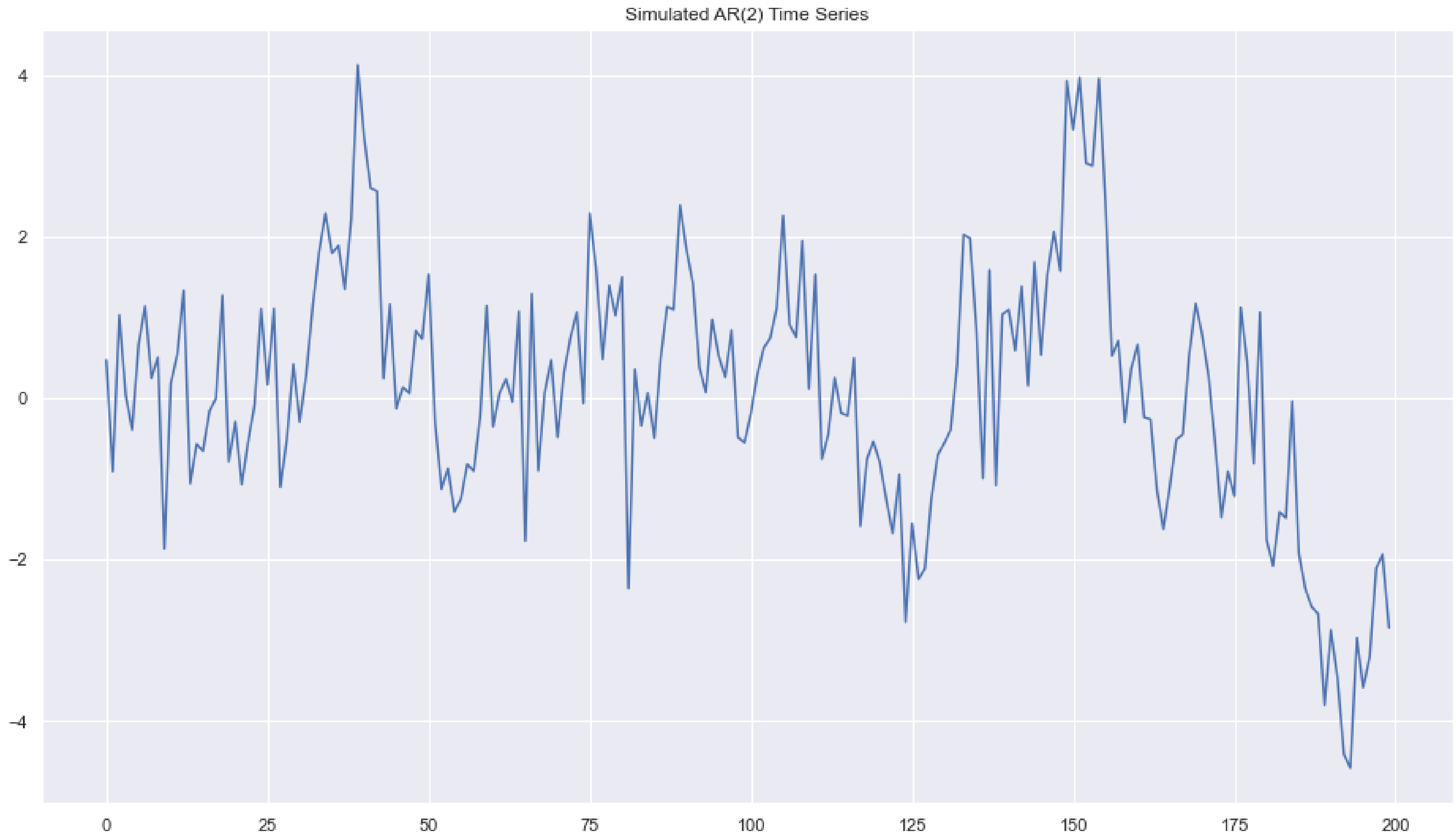
The ACF tails off as an exponential decay or a damped sine waves depending on the roots of $\emptyset(B) = 0$, and the PACF cuts off after lag 2, $\emptyset_{kk} = 0$ for $k \geq 3$.

AR models of order $p \geq 3$

More complicated conditions hold for AR(*p*) models with $p \geq 3$.

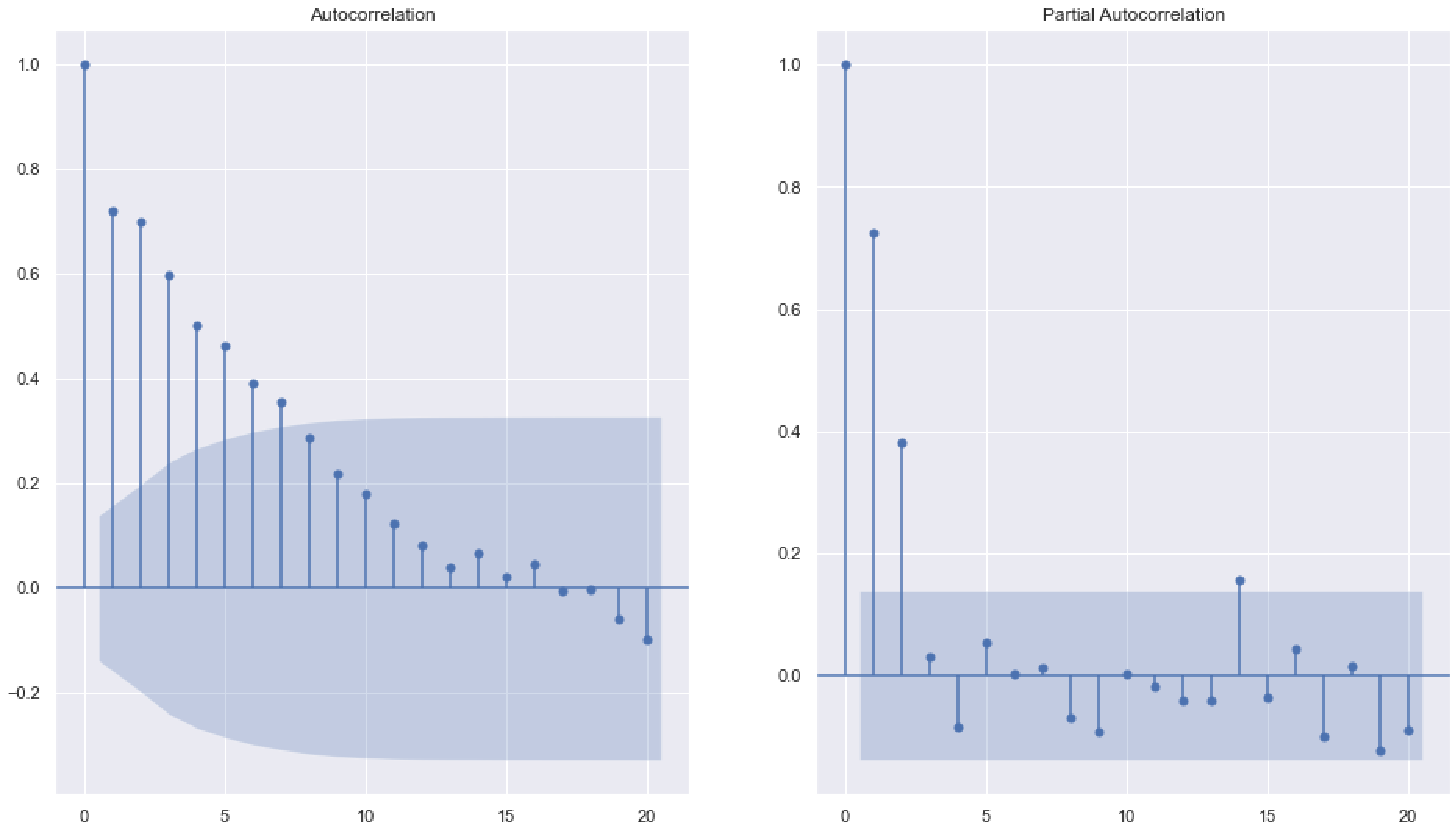
Econometric software (EViews, Python, Stata, among others) takes care of this.

White Noise and Autoregressive (AR) Models



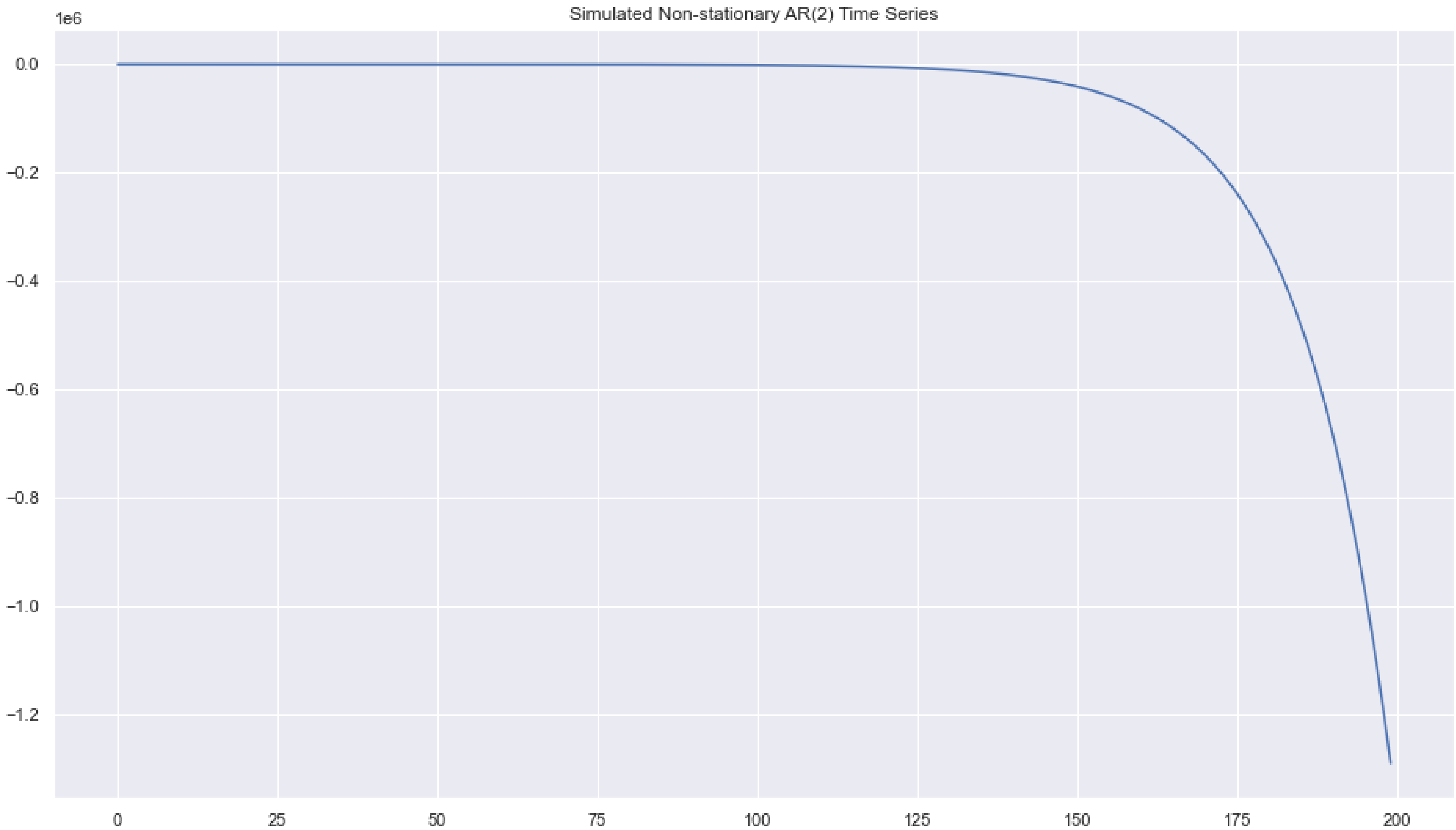
Simulated stationary AR(2) model: $X_t = 0.6X_{t-1} + 0.3X_{t-2} + a_t$

White Noise and Autoregressive (AR) Models



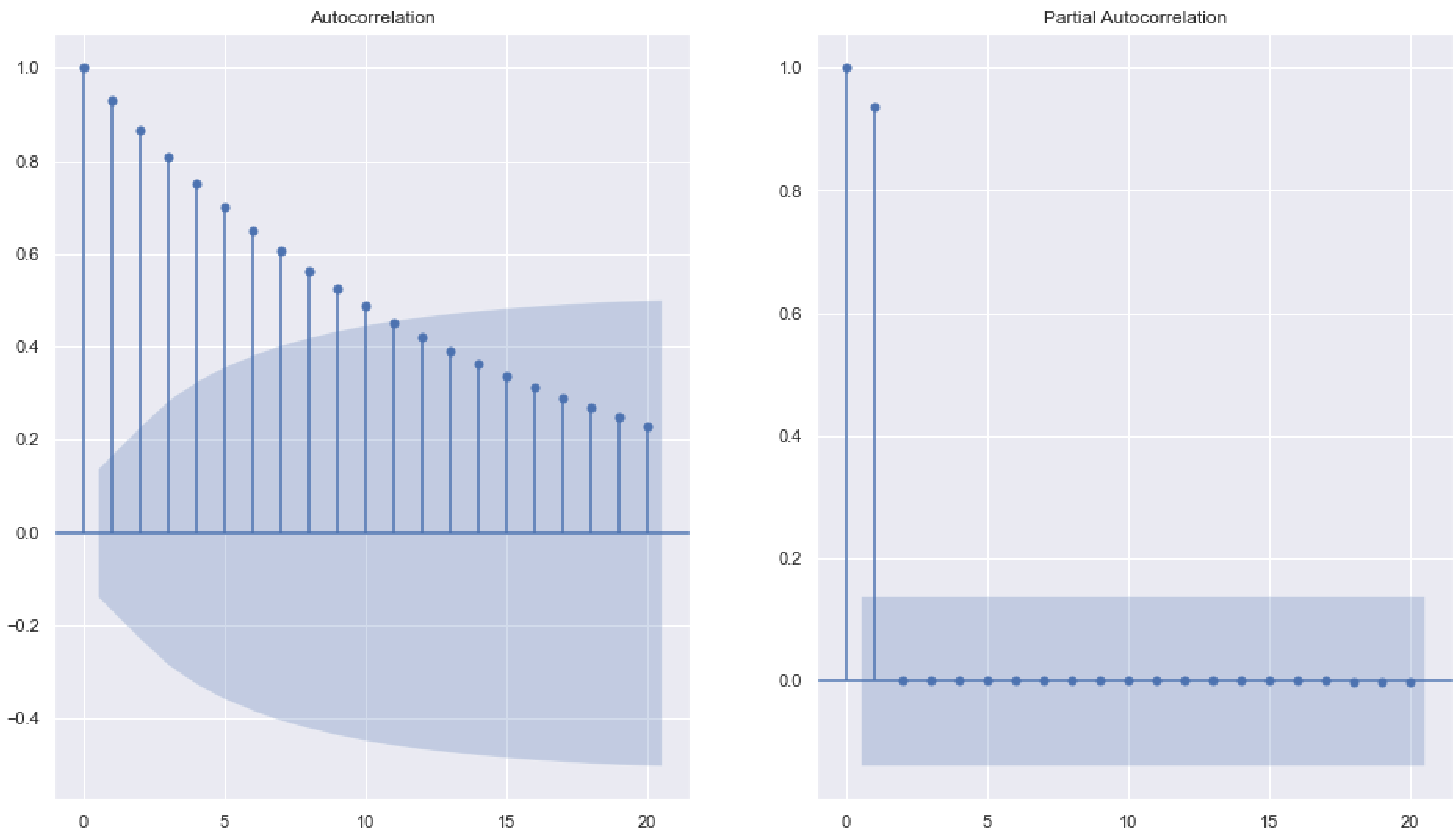
ACF and PACF of a simulated stationary AR(2) model: $X_t = 0.6X_{t-1} + 0.3X_{t-2} + a_t$

White Noise and Autoregressive (AR) Models



Simulated non-stationary AR(2) model: $X_t = 0.7X_{t-1} + 0.4X_{t-2} + a_t$

White Noise and Autoregressive (AR) Models



ACF and PACF of a simulated non-stationary AR(2) model: $X_t = 0.7X_{t-1} + 0.4X_{t-2} + a_t$

White Noise and Autoregressive (AR) Models

AR(2) Model Estimation Output (Python)

```
                              ARMA Model Results
==============================================================================
Dep. Variable:                      y   No. Observations:                  200
Model:                     ARMA(2, 0)   Log Likelihood                -279.390
Method:                       css-mle   S.D. of innovations              0.976
Date:                Sun, 13 Dec 2020   AIC                            566.779
Time:                        18:08:25   BIC                            579.973
Sample:                             0   HQIC                           572.118

==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
const         -0.0114      0.408     -0.028      0.978      -0.810       0.787
ar.L1.y        0.4505      0.065      6.894      0.000       0.322       0.579
ar.L2.y        0.3857      0.066      5.874      0.000       0.257       0.514
                                    Roots
=============================================================================
                  Real          Imaginary           Modulus         Frequency
-----------------------------------------------------------------------------
AR.1            1.1288           +0.0000j            1.1288            0.0000
AR.2           -2.2968           +0.0000j            2.2968            0.5000
-----------------------------------------------------------------------------
```

AR(2) Model: $(1 - \emptyset_1 B - \emptyset_2 B^2)(X_t - c) = \varepsilon_t$

$$(1 - 0.45B - 0.39B^2)(X_t + 0.0114) = \varepsilon_t$$

White Noise and Autoregressive (AR) Models

Problems

1. Consider the AR(2) models:

(i) $(1 - 1.4B + 0.6X_t B^2)\, X_t = a_t$

(ii) $(1 - 0.6B - 0.3X_t B^2)\, X_t = a_t$

a) Simulate a series of 100 observations from each of the models with σ_a^2 =1. Plot the series.

b) For each simulated series, calculate and study the sample ACF and PACF for the lags 1, 2, …, 20.

2. Consider the AR(2) model: $(1 - 0.3B - 0.6X_t B^2)\, X_t = a_t$.

a) Find the ACF

b) Find the PACF