Master in Applied Econometrics and Forecasting

Time Series Analysis and Forecasting

Class #5: Moving Average (MA) and ARMA Models

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Moving Average (MA) and ARMA Models

Moving average (MA) Representation: Wold's decomposition

Another useful representation of a time series process is to write the process X_t as a linear combination of a sequence of uncorrelated random variables:

$$X \quad = \mu + a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \cdots = \sum_{j=0}^{\infty} \psi_j a_{t-j}$$

with $\psi_0 = 1$, and a_t is a zero mean white noise process, and $\sum_{j=0}^{\infty} \psi_j^2 < \infty$.

It can be shown that:

$$E(X_t) = 0,\ Var(X_t) = \sigma_a^2 \sum_{j=0}^{\infty} \psi_j^2,\ E(a_t X_{t-k}) = \begin{cases} \sigma_a^2, k = 0 \\ 0, k > 0, \end{cases}$$

and $\rho_k = \frac{\gamma_k}{\gamma_0} = \frac{E(X_t X_{t+k})}{Var(X_t)} = \frac{\sum_{j=0}^{\infty} \psi_j \psi_{j+k}}{\sum_{j=0}^{\infty} \psi_j^2}$

Moving Average (MA) and ARMA Models

MA*(q)* model

The moving average model of order q is given by

$$X_t = a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q},$$

where ε_t is a zero mean white noise series. Because $1 + \theta_1^2 + \cdots + \theta_q^2 < \infty$, the process is always stationary. To be invertible, the roots of $(1 - \theta_1 B - \cdots - \theta_q B^q) = 0$ must be outside of the unit circle.

MA(1) model

The first-order moving average model or MA(1) model is

$$X_t = a_t - \theta_1 a_{t-1},$$

or

$$X_t = \theta(B) a_t,$$

where $\theta(B) = 1 - \theta_1 B$ and a_t is white noise. To be invertible, the root of $\theta(L) = 0$ must lie outside the unit circle. Thus, we require $|\theta_1| < 1$.

Moving Average (MA) and ARMA Models

ACF and PACF of the MA(1) model

The ACF of the MA(1) process is

$$\rho_k = \begin{cases} \dfrac{-\theta_1}{1+\theta_1^2}, & k = 1 \\ 0, & k > 1 \end{cases}$$

The general expression of the PACF of the MA(1) process is more complicated to obtain, but it can be seen to be:

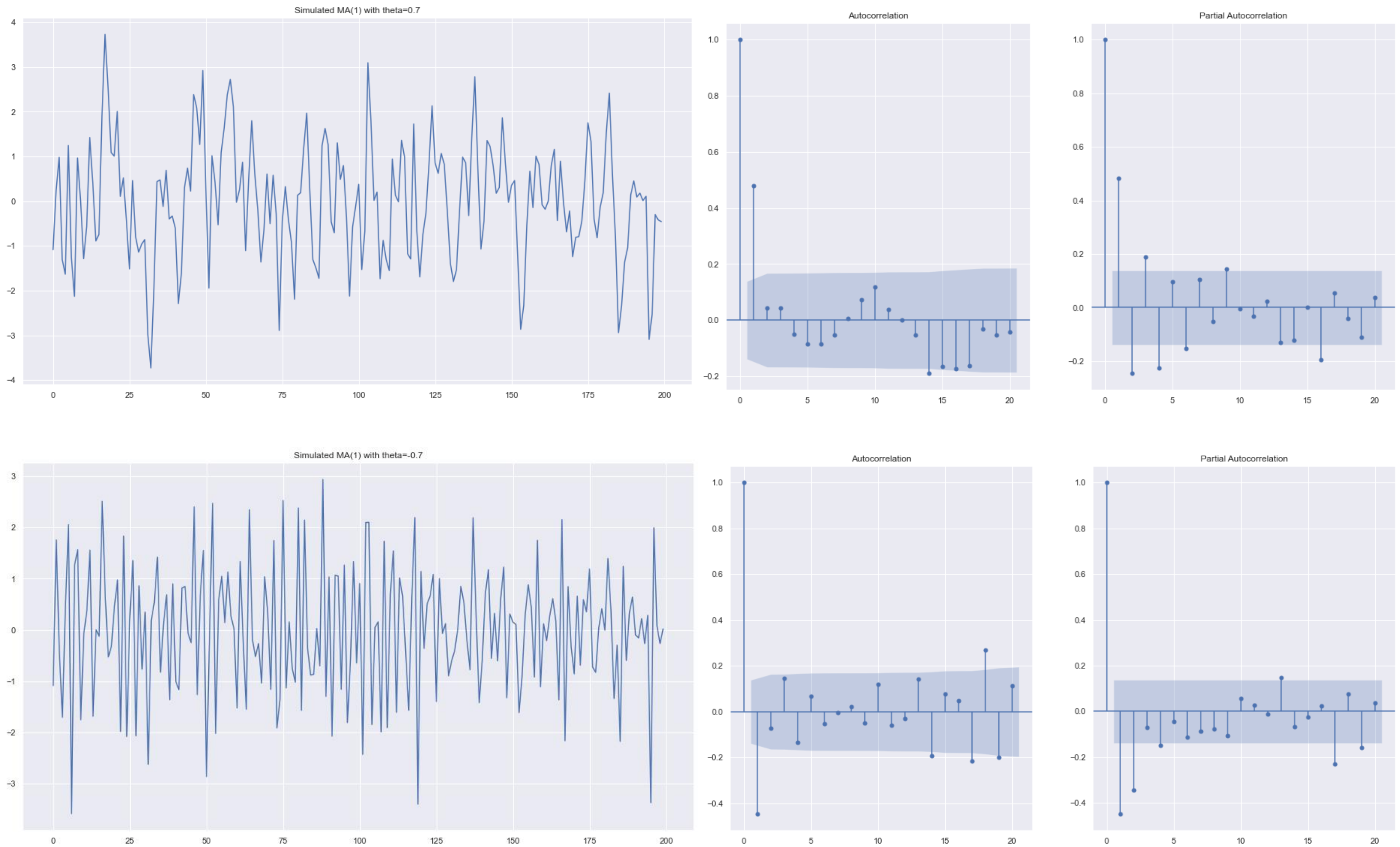
$$\emptyset_{11} = \rho_1 = \frac{-\theta_1}{1+\theta_1^2} = \frac{-\theta_1(1-\theta_1^2)}{1-\theta_1^4}$$

$$\emptyset_{22} = \frac{\rho_1^2}{1-\rho_1^2} = \frac{-\theta_1^2}{1+\theta_1^2+\theta_1^4} = \frac{-\theta_1^2(1-\theta_1^2)}{1-\theta_1^6}$$

(…)

$$\emptyset_{kk} = \frac{-\theta_1^k(1-\theta_1^2)}{1-\theta_1^{2(k+1)}}, \quad k \geq 1$$

Moving Average (MA) and ARMA Models



Simulated MA(1) models, θ=0.7 and θ= –0.7

Moving Average (MA) and ARMA Models

MA(2) model

The second-order moving average MA(2) model is given by

$$X_t = a - \theta_1 a_{t-1} - \theta_2 a_{t-2},$$

or

$$X_t = \theta(B)a_t,$$

Where $\theta(B) = 1 - \theta_1 B - \theta_2 B^2$ and a_t is white noise. To be invertible, the roots of $\theta(B) = 0$ must lie outside the unit circle. Hence, we have the following conditions:

$$\theta_2 + \theta_1 < 1,\ \theta_2 - \theta_1 < 1,\ -1 < \theta_2 < 1.$$

ACF of the MA(2) model cuts off after lag 2 and PACF tails off as an exponential decay or a damped sine wave depending on the roots of $\theta(B) = 0$.

MA model of order $q \geq 3$

More complicated conditions hold for MA(q) models with $q \geq 3$.

Moving Average (MA) and ARMA Models

Duality between AR(*p*) and MA(*q*) models

The stationary AR(*p*) model $\emptyset_p(B)X_t = a_t$, where $\emptyset_p(B) = 1 - \emptyset_1 B - \cdots - \emptyset_p B^p$, can be written as

$$X_t = \frac{1}{\emptyset_p(B)} a_t = \psi(B) a_t ,$$

where $\psi(B) = 1 + \psi_1 B + \psi_2 B^2 + \cdots$ such that

$$\emptyset_p(B)\psi(B) = 1 .$$

The ψ weights can be obtained by equating coefficients of B^j on both sides of this equation. For example, given a stationary AR(2) model, it follows that

$$(1 - \emptyset_1 B - \emptyset_2 B^2)(1 + \psi_1 B + \psi_2 B^2 + \cdots) = 1$$

or

$$1 + (\psi_1 - \emptyset_1)B + (\psi_2 - \psi_1\emptyset_1 - \emptyset_2)B^2 + (\psi_3 - \psi_2\emptyset_1 - \psi_1\emptyset_2)B^3 + \cdots = 1$$

Moving Average (MA) and ARMA Models

Duality between AR(p) and MA(q) models

Thus, we get

$B^0: 1 = 1$

$B^1: (\psi_1 - \emptyset_1) = 0 \rightarrow \psi_1 = \emptyset_1$

B^2: $(\psi_2 - \psi_1\emptyset_1 - \emptyset_2) = 0 \rightarrow \psi_2 = \psi_1\emptyset_1 + \emptyset_2$

B^3: $(\psi_3 - \psi_2\emptyset_1 - \psi_1\emptyset_2) = 0 \rightarrow \psi_3 = \psi_2\emptyset_1 + \psi_1\emptyset_2$

In general, for $j \geq 2$, we have:

B^j: $\left(\psi_j - \psi_{j-1}\emptyset_1 - \psi_{j-2}\emptyset_2\right) = 0 \rightarrow \psi_j = \psi_{j-1}\emptyset_1 + \psi_{j-2}\emptyset_2$, where $\psi_0 = 1$.

In a special case, when $\emptyset_2 = 0$, we have $\psi_j = \psi_{j-1}\emptyset_1 = \emptyset_1^j$, for $j \geq 0$. Therefore,

$$X_t = \frac{1}{\emptyset_p(B)} a_t = \psi(B) a_t \leftrightarrow X_t = \frac{1}{1-\emptyset_1 B} a_t = (1 + \emptyset_1 B + \emptyset_2 B^2 + \cdots) a_t$$

This implies that the finite-order AR model is equivalent to an infinite-order MA model. It can be shown that the finite-order MA model is also equivalent to an infinite-order AR model.

Moving Average (MA) and ARMA Models

ARMA(1,1) model

The mixed autoregressive and moving average ARMA(1,1) model includes the autoregressive AR(1) and moving average MA(1) models as special cases:

$$X_t = \emptyset X_{t-1} + a_t - \theta a_{t-1},$$

or

$$\emptyset(B) X_t = \theta(B) a_t,$$

where $\emptyset(B) = 1 - \emptyset B$, $\theta(B) = 1 - \theta B$ and a_t is white noise.

To be stationary, the root of $\emptyset(B) = 0$ must lie outside the unit circle, i.e., $-1 < \emptyset < 1$. To be invertible, the root of $\theta(B) = 0$ must lie outside the unit circle, i.e., $-1 < \theta < 1$.

Both the ACF and PACF of a mixed ARMA(1,1) model tail off as *k* incresaes, with its shape depending on the signs and magnitudes of ϕ and θ.

Moving Average (MA) and ARMA Models

The ARMA(1,1) model can be written in a pure moving average representation as

$$X_t = \psi(B)a_t,$$

where

$$\psi(B) = (1 + \psi_1 B + \psi_2 B^2 + \cdots) = \frac{1 - \theta B}{1 - \emptyset B}.$$

The ψ weights can be obtained by equation coefficients of B^j in

$$(1 - \theta B)(1 + \psi_1 B + \psi_2 B^2 + \cdots) = (1 - \emptyset B)$$

The ARMA(1,1) model can also be written in a pure autoregressive representation as

$$\pi(B)X_t = a_t,$$

where

$$\pi(B) = 1 - \pi_1 B - \pi_2 B^2 - \cdots = \frac{1 - \emptyset B}{1 - \theta B}.$$

The π weights can be obtained by equation coefficients of B^j in

$$(1 - \theta B)(1 - \pi_1 B - \pi_2 B^2 - \cdots) = (1 - \emptyset B)$$

Both the ACF and PACF of a mixed ARMA(1,1) model tail off as k increases, with its shape depending on the signs and magnitudes of $\emptyset$ and θ.

Moving Average (MA) and ARMA Models

ACF of the ARMA(1,1) Model

Assuming, without loss of generality, that $E(X_t) = 0$, the ACF of the ARMA(1,1) model can be derived by multiplying by X_{t-k} on both sides of the equation and then taking expected values:

$$E(X_{t-k}X_t) = \emptyset_1 E(X_{t-k}X_{t-1}) + E(X_{t-k}a_t) - \theta_1 E(X_{t-k}a_{t-1})$$
$$\gamma_k = \emptyset_1 \gamma_{k-1} + E(X_{t-k}a_t) - \theta_1 E(X_{t-k}a_{t-1})$$

For $k = 0$, we have

$$\gamma_0 = \emptyset_1 \gamma_1 + E(X_t a_t) - \theta_1 E(X_t a_{t-1}) = \emptyset_1 \gamma_1 + \sigma_a^2 - \theta_1(\emptyset_1 - \theta_1)\sigma_a^2$$

For $k = 1$, we have

$$\gamma_1 = \emptyset_1 \gamma_0 + E(X_{t-1} a_t) - \theta_1 E(X_{t-1} a_{t-1}) = \emptyset_1 \gamma_0 - \theta_1 \sigma_a^2$$

For $k \geq 2$, we have

$$\gamma_k = \emptyset_1 \gamma_{k-1}$$

Hence, after some algebraic manipulations, we obtain

$$\rho_k = \begin{cases} \dfrac{(\emptyset_1 - \theta_1)(1 - \emptyset_1\theta_1)}{1 + \theta_1^2 - 2\emptyset_1\theta_1}, & k = 1 \\ \emptyset_1 \rho_{k-1}, & k > 1 \end{cases}$$

Moving Average (MA) and ARMA Models

PACF of the ARMA(1,1) Model

The PACF of the ARMA(1,1) model is complicated to obtain, but it can be noted that, as the ARMA(1,1) model contains the MA(1) model as a special case, the PACF of the ARMA(1,1) decays exponentially to zero in one of two forms depending on the signs and magnitudes of $\emptyset$ and θ.

ARMA(p,q) Model

The general mixed autoregressive and moving average ARMA(*p*,*q*) model is given by

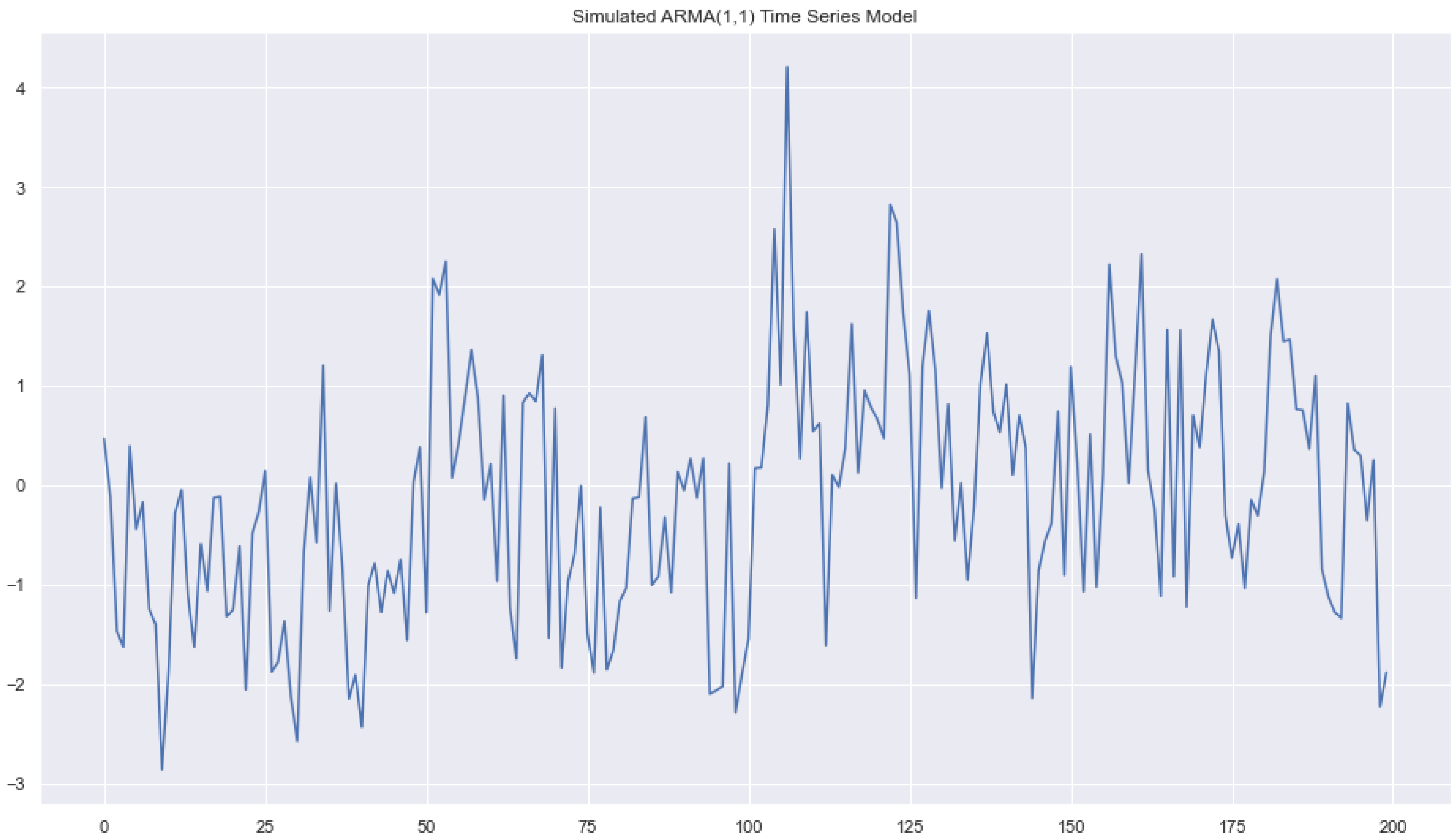
$$X_t = \emptyset_1 X_{t-1} + \cdots + \emptyset_p X_{t-p} + a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q},$$

or

$$\emptyset_p(B) X_t = \theta_q(B) a_t,$$

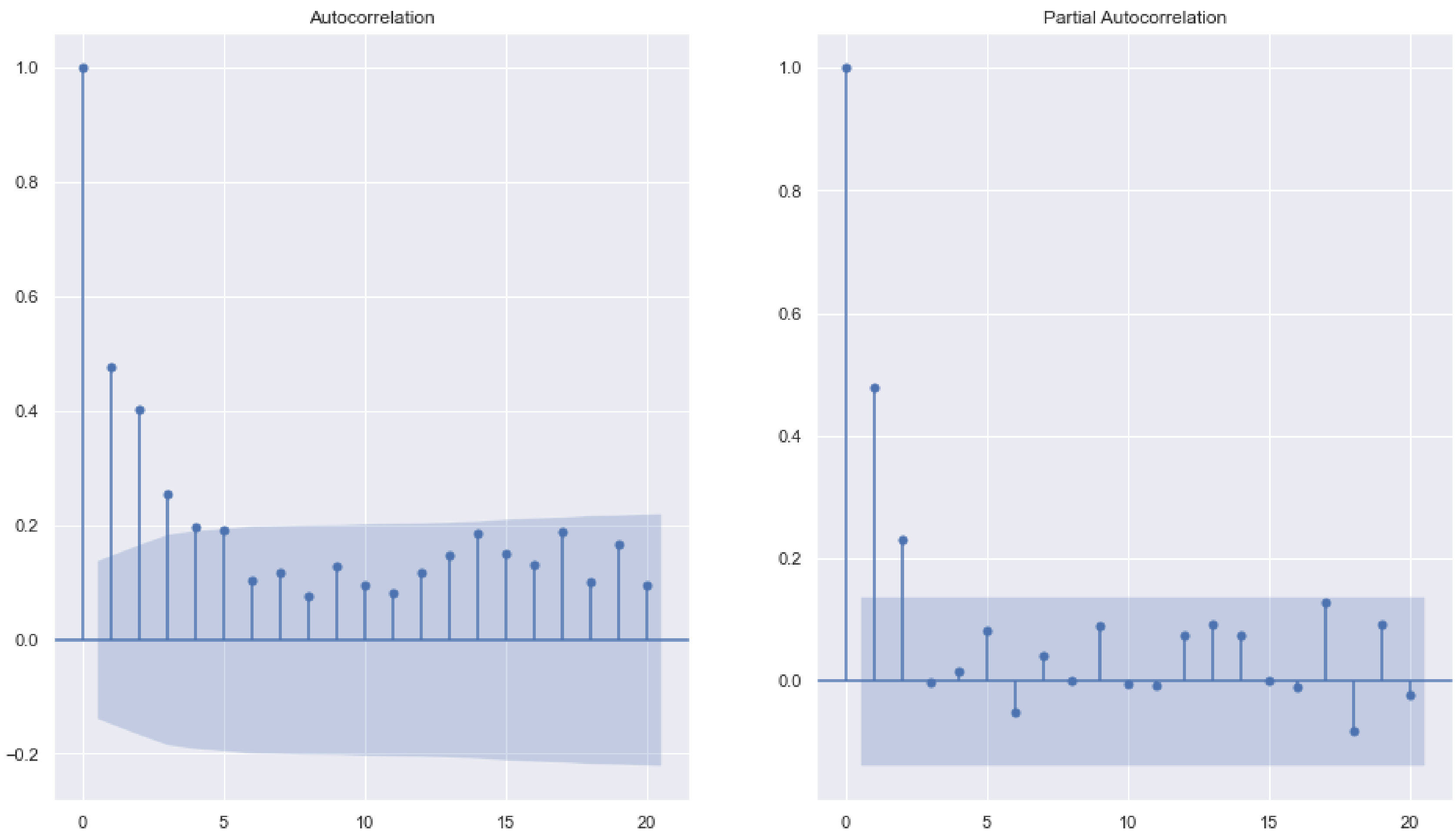
where $\emptyset_p(B) = 1 - \emptyset_1 B - \cdots - \emptyset_p B^p$, $\theta_q(B) = 1 - \theta_1 B - \cdots - \theta_q B^q$ and a_t is white noise. To be stationary, the roots of $\emptyset_p(B) = 0$ must lie outside the unit circle. To be invertible, the roots of $\theta_q(B) = 0$ must lie outside the unit circle.

Moving Average (MA) and ARMA Models



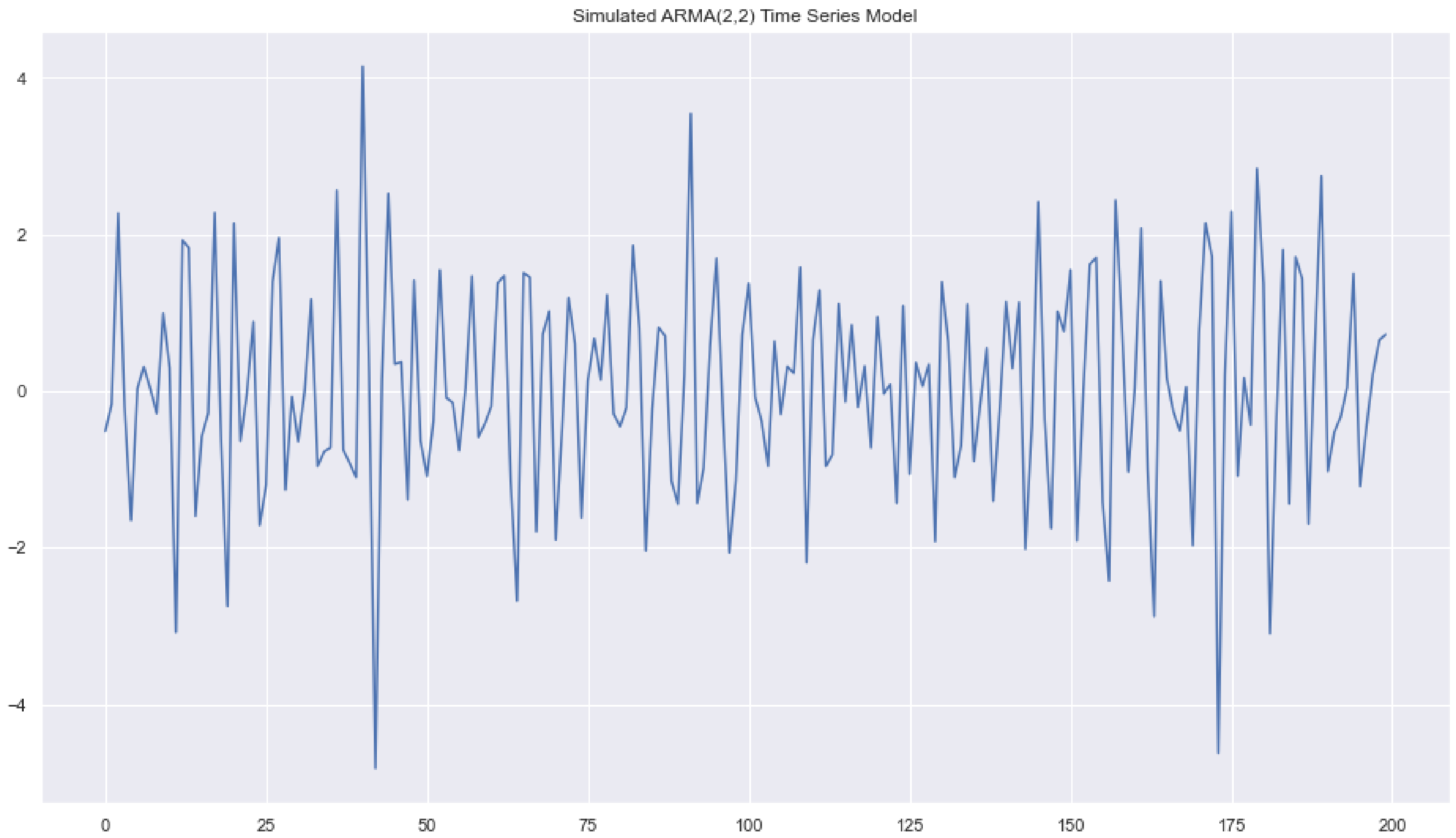
Simulation of an ARMA(1,1) model: $X_t = 0.85X_{t-1} + a_t + 0.5a_{t-1}$

Moving Average (MA) and ARMA Models



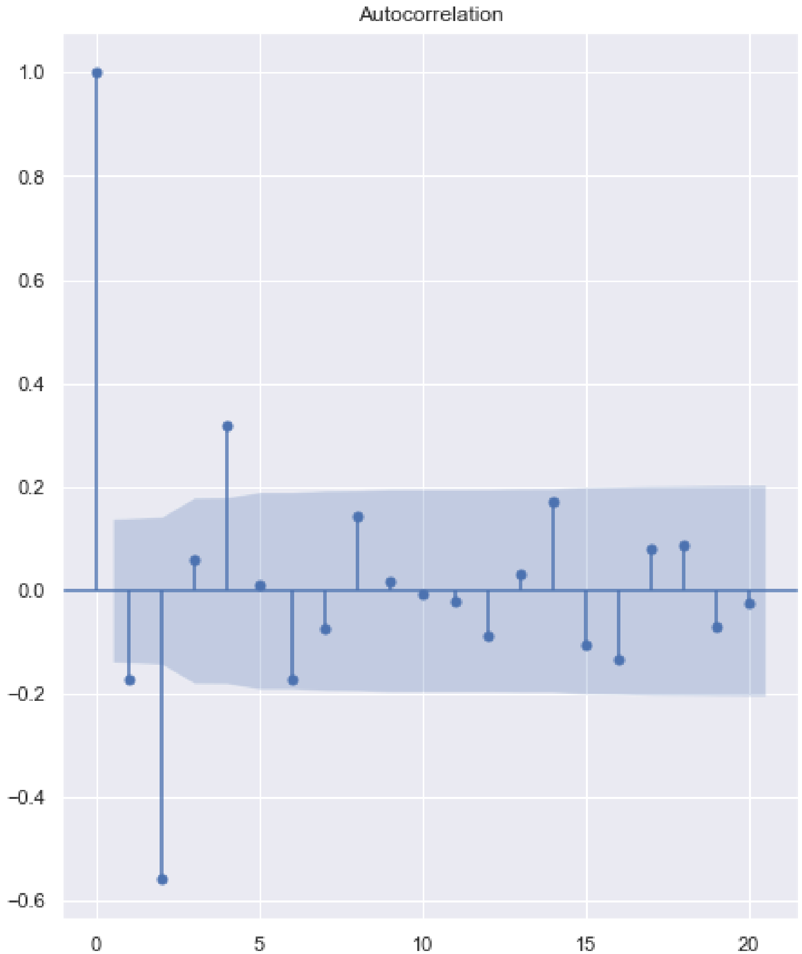
Sample ACF and PACF of an ARMA(1,1) model: $X_t = 0.85X_{t-1} + a_t + 0.5a_{t-1}$

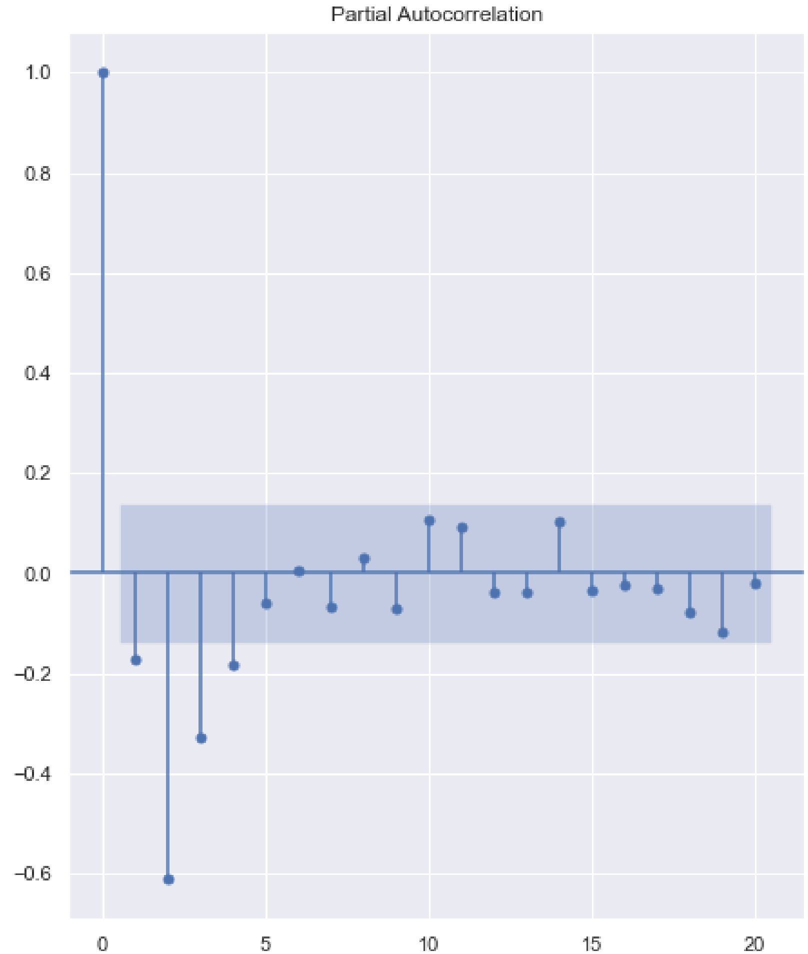
Moving Average (MA) and ARMA Models



Simulation of an ARMA(2,2) model: $X_t = 0.2X_{t-1} - 0.7X_{t-2} + a_t - 0.75a_{t-1} + 0.25a_{t-2}$

Moving Average (MA) and ARMA Models





Sample ACF and PACF of an ARMA(2,2): $X_t = 0.2X_{t-1} - 0.7X_{t-2} + a_t - 0.75a_{t-1} + 0.25a_{t-2}$

Moving Average (MA) and ARMA Models

Problems

1. Consider the MA(1) processes: $X_t = (1 - 0.4B)a_t$ and $Y_t = (1 - 2.5B)a_t$.
Show that both processes have the same autocorrelation function

2. Consider the MA(2) model: $X_t = a_t - 0.1a_{t-1} - 0.21a_{t-2}$

a) Is the model stationary? Why?
b) Is the model invertible? Why?
c) Find the ACF for the model

3. Consider the ARMA(1,1) model: $(1 - B)X_t = (1 - 0.5B)a_t$

a) Is it stationary? Is it invertible?
b) Express the model in na AR representation if it exists
c) Express the model in na MA representation if it exists

4. Consider the ARMA(1,1) model: $(1 - 0.9B)X_t = (1 - 0.3B)a_t$

a) Is it stationary? Is it invertible?
b) Express the model in na AR representation if it exists
c) Express the model in na MA representation if it exists