Lisbon School of Economics & Management
Universidade de Lisboa

Master in Applied Econometrics and Forecasting

# Time Series Analysis and Forecasting

## Class #3: Stationarity, ACF and PACF

**Jorge Caiado, PhD**

CEMAPRE/REM/ISEG, University of Lisbon, jcaiado@iseg.ulisboa.pt

# Stationarity, ACF and PACF

**Stochastic Process and Stationarity**

A **stochastic process** is a family of time indexed random variables, $Z(\omega, t): t = 0, \pm 1, \pm 2, \ldots$ , where $\omega$ is the sample space and $t$ is the index set. A time series is a realization (or sample function) from a certain stochastic process, $X_t, t = 1, \ldots, T$.

A process $X_t, t = 1, \ldots, T$ is said to be **weak or covariance stationary** if it has constant mean,

$$E(X_t) = \mu$$

constant variance,

$$Var(X_t) = E(X_t - \mu)^2 = \sigma^2$$

and the covariance and the correlation between $X_t$ and $X_{t-k}$ depend only on time difference $k$,

$$\gamma_k = Cov(X_t, X_{t-k}) = E[(X_t - \mu)(X_{t-k} - \mu)]$$

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \frac{Cov(X_t, X_{t-k})}{\sqrt{X_t}\sqrt{X_{t-k}}}$$

where $\gamma_0 = Var(X_t) = \sigma^2$.

# Stationarity, ACF and PACF

## Autocorrelation Function (ACF)

The autocovariance function and the autocorrelation function have the following properties:

1) $\rho_0 = 1$

2) $|\gamma_k| \leq \gamma_0$; $|\rho_k| \leq 1$

3) $\gamma_k = \gamma_{-k}$; $\rho_k = \rho_{-k}$

4) $\gamma_k$ and $\rho_k$ are positive semidefinite

## Partial Autocorrelation Function (PACF)

The **partial autocorrelation function (PACF)** measures the correlation between $X_t$ and $X_{t-k}$, when the effects of intermediate variables $X_{t-1}$, $X_{t-2}$, …, $X_{t-k+1}$ are removed:

$$Cor(X_t, X_{t-k} | X_{t-1}, X_{t-2}, \dots, X_{t-k+1})$$

# Stationarity, ACF and PACF

## Partial Autocorrelation Function (PACF): The Yule-Walker Equations

The partial autocorrelation coefficient of order $k$ is denoted by $\emptyset_{kk}$ and can be derived by regressing $X_t$ against $X_{t-1}, X_{t-2}, \ldots, X_{t-k}$:

$$X_t = \emptyset_{k1}X_{t-1} + \emptyset_{k2}X_{t-2} + \cdots + \emptyset_{kk}X_{t-k} + a_t.$$

Multiplying $X_{t-j}$ on both sides of the equation and taking expected values, we get the so-called Yule-Walker equations. Solving for the last coefficient $\emptyset_{kk}$using Cramer's Rule successively for $k = 1,2,\ldots$, we have

$$\emptyset_{11} = \rho_1,\ \emptyset_{22} = \frac{\begin{vmatrix} 1 & \rho_1 \\ \rho_1 & \rho_2 \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{vmatrix}},\ \emptyset_{33} = \frac{\begin{vmatrix} 1 & \rho_1 & \rho_1 \\ \rho_1 & 1 & \rho_2 \\ \rho_2 & \rho_1 & \rho_3 \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_1 \\ \rho_2 & \rho_1 & 1 \end{vmatrix}},\ \ldots,$$

$$\emptyset_{kk} = \frac{\begin{vmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-2} & \rho_1 \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-3} & \rho_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & \rho_1 & \rho_k \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-2} & \rho_{k-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-3} & \rho_{k-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & \rho_1 & 1 \end{vmatrix}}$$

# Stationarity, ACF and PACF

**Problems**

**1.** Check the following properties for the ACF of a stationary process:

a) $\rho_1 = 1$

b) $|\rho_k| \leq 1$

c) $\rho_k = \rho_{-k}$

**2.** Given the time series: 65, 78, 75, 102, 91, 115, 105, 135, 121, 156.

a) Plot the series and describe their patterns

b) Compute the sample ACF for lags 1, 2 and 3

c) Compute the sample PACF for lags 1, 2 and 3