Module 3
Integration :
Review And Some New Techniques

Sub-Module 3.1

Beta functions with properties

Example:8

- **Evaluate $\int_0^1 x^4 \cos^{-1} x \, dx$**
- **Solution:** Integrating by Parts,
- $I = \cos^{-1}x \, \frac{x^5}{5} \Big|_0^1 - \int_0^1 \left(\frac{-1}{\sqrt{1-x^2}}\right) \frac{x^5}{5} dx$
- $= 0 + \frac{1}{5} \int_0^1 \frac{x^5}{\sqrt{1-x^2}} dx$
- Put $x = \sin\theta$
- $dx = \cos\theta \; d\theta$
- $I = \frac{1}{5} \int_0^{\frac{\pi}{2}} \frac{\sin^5\theta}{\cos\theta} \cos\theta \; d\theta$
- $= \frac{1}{5} \times \frac{1}{2} \times \beta\left(\frac{5+1}{2}, \frac{1}{2}\right) = \frac{1}{10} \times \frac{\left|\overline{3}\right.\left|\overline{\frac{1}{2}}\right.}{\left|\overline{\frac{7}{2}}\right.}$
- $= \frac{1}{10} \times \frac{2 \times \sqrt{\pi}}{\frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi}} = \frac{8}{75}$

x	0	1
θ	0	$\frac{\pi}{2}$

Example:9

- **Evaluate** $\int_0^{\infty} \frac{x^2}{(1+x^6)^3} dx$
- **Solution:** Put $x^3 = t$
- $3x^2 dx = dt$
- $I = \frac{1}{3} \int_0^{\infty} \frac{dt}{(1+t^2)^3}$
- Now Put $t = \tan\theta$
- $dt = \sec^2\theta \, d\theta$
- $I = \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{\sec^2\theta \, d\theta}{(\sec^2\theta)^3}$

x	0	∞
t	0	∞

x	0	1
θ	0	$\frac{\pi}{2}$

- $I = \frac{1}{3}\int_0^{\frac{\pi}{2}} \frac{1}{\sec^4\theta}\, d\theta$
- $= \frac{1}{3}\int_0^{\frac{\pi}{2}} \sin^0\theta \ \cos^4\theta\, d\theta$
- $= \frac{1}{3} \times \frac{1}{2}\beta\left(\frac{1}{2}, \frac{5}{2}\right)$
- $= \frac{1}{6}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(3)}$
- $= \frac{1}{6} \times \sqrt{\pi} \times \frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi} \times \frac{1}{2}$
- $= \frac{\pi}{16}$

Example:10

- **Evaluate $\int_0^{\frac{\pi}{6}} \cos^3 3\theta \sin^2 6\theta \; d\theta$**
- **Solution:** Put $3\theta = t \quad \therefore d\theta = \frac{dt}{3}$
- When $\theta = 0, \; t = 0$
- When $\theta = \frac{\pi}{6}, \; t = \frac{\pi}{2}$
- $I = \int_0^{\frac{\pi}{2}} \cos^3 t \sin^2 2t \frac{dt}{3}$
- $= \frac{1}{3}\int_0^{\frac{\pi}{2}} \cos^3 t \; (2\sin t \cos t)^2 \; dt$
- $= \frac{4}{3}\int_0^{\frac{\pi}{2}} \cos^3 t \; \sin^2 t \cos^2 t \; dt$
- $= \frac{4}{3}\int_0^{\frac{\pi}{2}} \sin^2 t \; \cos^5 t \; dt$
- $= \frac{4}{3} \times \frac{1}{2}\beta\left(\frac{3}{2}, \frac{6}{2}\right)$
- $= \frac{2}{3}\frac{\left\lceil\frac{3}{2}\right.\left\lceil 3\right.}{\left\lceil\frac{9}{2}\right.} = \frac{\frac{2}{3}\times\frac{1}{2}\sqrt{\pi}\times 2}{\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}} = \frac{32}{315}$

x	0	$\frac{\pi}{6}$
θ	0	$\frac{\pi}{2}$

Example:11

- **Evaluate** $\int_0^{2\pi} \sin^2\theta\,(1+\cos\theta)^4\,d\theta$
- **Solution:** $I = \int_0^{2\pi} \left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2 \left(2\cos^2\frac{\theta}{2}\right)^4 \,d\theta$

$$= 64\int_0^{2\pi} \sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}\cos^8\frac{\theta}{2}\,d\theta$$

$$= 64\int_0^{2\pi} \sin^2\frac{\theta}{2}\cos^{10}\frac{\theta}{2}\,d\theta$$

θ	0	2π
t	0	π

Put $\frac{\theta}{2} = t \therefore d\theta = 2dt$

$$I = 64\times 2\int_0^{\pi} \sin^2 t\cos^{10} t\;dt$$

$$If\ f(\pi - t) = f(t) then\ \int_0^{a} f(t)dt = 2\int_0^{a/2} f(t)dt$$

$$I = 128\times 2\int_0^{\frac{\pi}{2}} \sin^2 t\cos^{10} t\;dt$$

$$= 128\times 2\times\frac{1}{2}\beta\left(\frac{3}{2},\frac{11}{2}\right)$$

$$= 128\times\frac{\left|\frac{3}{2}\right.\left|\frac{11}{2}\right.}{|7} = \frac{128\times\frac{1}{2}\times\sqrt{\pi}\times\frac{9}{2}\times\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}}{6\times5\times4\times3\times2} = \frac{21\pi}{8}$$

Example:12

- **Evaluate :** $\int_0^{\pi} x\, sin^7 x\ cos^4 x\, dx$
- Hint: $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Example:13

- **Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\, d\theta$ and prove that the result $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\, d\theta \int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\, d\theta = \frac{\pi^2}{2}$**

- **Solution:** Let $I_1 = \int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\, d\theta$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{\sin\theta}{\cos\theta}\right]^{\frac{1}{2}} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^{1/2}\theta \cos^{-1/2}\theta \; d\theta$$

$$= \frac{1}{2}\beta\left(\frac{\frac{1}{2}+1}{2}, \frac{-\frac{1}{2}+1}{2}\right) = \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{4}\right)$$

$$= \frac{1}{2} \times \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)} = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) = \frac{1}{2}\frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{2\times\frac{1}{\sqrt{2}}} = \frac{\pi}{\sqrt{2}}$$

Let $I_2 = \int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\, d\theta$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{\cos\theta}{\sin\theta}\right)^{\frac{1}{2}} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos^{\frac{1}{2}}\theta \sin^{-\frac{1}{2}}\theta \ \ d\theta$$

$$= \frac{1}{2}\beta\left(\frac{-\frac{1}{2}+1}{2}, \frac{\frac{1}{2}+1}{2}\right)$$

$$= \frac{1}{2}\beta\left(\frac{1}{4}, \frac{3}{4}\right)$$

$$= \frac{1}{2}\left|\overline{\frac{1}{4}}\right|\overline{\frac{3}{4}}$$

$$= \frac{1}{2}\frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{\sqrt{2}}$$

Hence, $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\, d\theta \int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\, d\theta = \frac{\pi}{\sqrt{2}} \cdot \frac{\pi}{\sqrt{2}} = \frac{\pi^2}{2}$

Example 14

- **Express $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\sin\theta + \cos\theta)^{\frac{1}{3}}\, d\theta$ as a Gamma function.**

Solution: Consider $I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\sin\theta + \cos\theta)^{\frac{1}{3}}\, d\theta$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{2}^{\frac{1}{3}} \left[\sin\theta \frac{1}{\sqrt{2}} + \cos\theta \frac{1}{\sqrt{2}}\right]^{\frac{1}{3}} d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2^{\frac{1}{6}} \left[\sin\theta \cos\frac{\pi}{4} + \cos\theta \sin\frac{\pi}{4}\right]^{\frac{1}{3}} d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2^{\frac{1}{6}} \left[\sin\left(\theta + \frac{\pi}{4}\right)\right]^{\frac{1}{3}} d\theta$$

Put $\frac{\pi}{4} + \theta = t \rightarrow d\theta = dt$

When $\theta = -\frac{\pi}{4},\ t = 0$

When $\theta = \frac{\pi}{4},\ t = \frac{\pi}{2}$

$$= \int_0^{\frac{\pi}{2}} 2^{\frac{1}{6}} [\sin t]^{\frac{1}{3}} dt$$

$$= 2^{\frac{1}{6}} \int_0^{\frac{\pi}{2}} \sin^{\frac{1}{3}} t \cos^0 t \ \ dt$$

$$= 2^{\frac{1}{6}} \frac{1}{2} \beta\left(\frac{\frac{1}{3}+1}{2}, \frac{1}{2}\right)$$

$$= 2^{\frac{1}{6}-1} \beta\left(\frac{2}{3}, \frac{1}{2}\right) = 2^{-\frac{5}{6}} \frac{\left|\overline{\tfrac{2}{3}}\right. \left|\overline{\tfrac{1}{2}}\right.}{\left|\overline{\tfrac{7}{6}}\right.}$$

$$= 2^{\frac{1}{5/6}} \frac{\left|\overline{\tfrac{2}{3}}\right. \sqrt{\pi}}{\left|\overline{\tfrac{7}{6}}\right.}$$