Module 1
Differential Equations
of
First Order and First Degree

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Syllabus

Module No.	Unit No.	Details	Hrs.	CO
1	**Differential Equation of First Order and First Degree**		13	CO 1
	1.1	Differential Equation of first order and first degree- Exact differential equations, Equations reducible to exact equations by integrating factors.		
	1.2	Linear differential equations (Review), Equation reducible to linear form. Applications of Differential Equation of first order and first degree		
	1.3	Linear Differential Equation with constant coefficients: Complimentary function, particular integrals of differential equation of the type f(D)y=X, where X is e^{ax}, $\sin(ax+b)$, $\cos(ax+b)$, x^n, $e^{ax}V$		
	1.4	Cauchy's homogeneous linear differential equation		
	1.5	Method of variation of parameters		
		# **Self-learning topic:** Bernoulli's equation. Equation reducible to Bernoulli's equation.		

Expected Course Outcome

- At the end of the module students can successfully

CO1. Identify and solve different types of ordinary differential equations using various methods.

What Do you know about DE?

Differential Equation

- An equation which involves dependent variable, independent variable & differential coefficient of dependent variable with respect to the independent variable is called differential equation.
- Examples:
- $\frac{dy}{dx} = c$
- $\frac{d^3y}{dx^3} + 3x\frac{dy}{dx} = e^y$
- $\left(\frac{d^2y}{dx^2}\right)^4 + \frac{dy}{dx} = 3$

Order of Differential Equation(D.E.)

- Differential Equations are classified on the basis of the order. Order of a differential equation is the order of the highest derivative (also known as differential coefficient) present in the equation.
- Examples:

(i) $\frac{d^3y}{dx^3} + 3x\frac{dy}{dx} = e^y$

In this equation, the order of the highest derivative is 3 hence, this is a third order differential equation.

(ii) $\left(\frac{d^2y}{dx^2}\right)^4 + \frac{dy}{dx} = 3$

This equation represents a second order differential equation.

Degree of DE

- The degree of the differential equation is represented by the power of the highest order derivative in the given differential equation.
- Examples:

(i) $\frac{d^4y}{dx^4} + 4x\frac{dy}{dx} + 10y = x^2$

In this equation, the order of the highest derivative is 4 hence, this is a Forth order differential equation. The exponent of the highest order derivative is one. Hence, the degree of this equation is 1.

(ii) $\left(\frac{d^2y}{dx^2}\right)^4 + \frac{dy}{dx} = 3$

- Order : 2
- Degree: 4

DE of First order and First Degree

- $(e^y + 1)\cos x\, dx + e^y \sin x\; dy = 0$
- $e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right) dy + \left(1 + e^{\frac{x}{y}}\right) dx = 0$
- $3y^2 \frac{dy}{dx} + 2y^3 x = 4x^3 e^{x^2}$
- $\frac{du}{dy} + \frac{1}{y} u = \log y$
- $\frac{dy}{dx} + \frac{1}{x} \tan y = \frac{1}{x^2} \tan y \sin y$

Exact Differential Equation

- An exact differential equation is a differential equation which is obtained from its primitive (solution) by differentiation only, without adopting any multiplication, elimination or any reduction process.
- The necessary and sufficient condition for an equation of the form $Mdx + Ndy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

❖ Steps to solve exact differential equation:

i. Check for the condition $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ to verify the exactness of the given equation.

ii. Integrate M with respect to x keeping y constant.

iii. Integrate N with respect to y. (only those terms of N which do not contain x.)

iv. The final solution is of the form

$$\int_{(y\ constant)} M\,dx + \int_{(terms\ of\ N\ free\ from\ x\ )} N\,dy = c$$

Example:1

- **Solve $(e^y+1)\cos x\,dx + e^y \sin x\;dy = 0$**
- Here $M = (e^y+1)\cos x$ and $N = e^y \sin x$

$$\frac{\partial M}{\partial y} = e^y \cos x \qquad \frac{\partial N}{\partial x} = e^y \cos x$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$; Given equation is exact.

Solution is $\int M\,dx + \int N\,dy = c$

$$\therefore \int (e^y+1)\cos x\,dx + \int 0\,dy = c$$

Keeping y constant

No terms free from x

$$\therefore\ (e^y+1)\sin x = c$$

Example:2

- **Solve** $\left(x^2 - x\tan^2 y + \sec^2 y\right)dy = (\tan y - 2xy - y)dx$
- $M = \tan y - 2xy - y$ $\qquad$ $N = -x^2 + x\tan^2 y - \sec^2 y$

$\frac{\partial M}{\partial y} = \sec^2 y - 2x - 1$ $\qquad$ $\frac{\partial N}{\partial x} = -2x + \tan^2 y$

$= \tan^2 y - 2x$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$; The given equation is exact.

- Solution is $\int M\,dx + \int(\text{Term in}'N'\text{free from x})\,dy = c$

$$\therefore \int(\tan y - 2xy - y)dx + \int -\sec^2 y\ dy = c$$

$$\Rightarrow x\tan y - 2y\frac{x^2}{2} - xy - \tan y = c$$

$$\therefore\ x\tan y - x^2 y - xy - \tan y = c$$

Example:3

❖ **Solve** $\frac{dy}{dx} = \frac{y+1}{(y+2)e^y - x}$

❖ Write in $Mdx + Ndy = 0$ form

$\therefore\ (y+1)dx - [(y+2)e^y - x]dy = 0$

$M =$ $\qquad$ $N =$

$\therefore \frac{\partial M}{\partial y} =$ $\qquad$ $\therefore \frac{\partial N}{\partial x} =$

Is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$?

given equation is exact?

❖ Solution is

Example:3

- Solve $\frac{dy}{dx} = \frac{y+1}{(y+2)e^y - x}$
- $\frac{dy}{dx}[(y+2)e^y - x] = y+1$ $\quad \therefore\ (y+1)dx - [(y+2)e^y - x]dy = 0$

$M = (y+1)$ $\quad N = -[(y+2)e^y - x]$

$\therefore \frac{\partial M}{\partial y} = 1,$ $\quad \therefore \frac{\partial N}{\partial x} = 1$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ the given equation is exact.

- Solution is $\int M\,dx + \int (\text{Term in}'N'\text{free from x})\,dy = c$

$$\therefore \int (y+1)dx + \int -(y+2)e^y\,dy = c$$

integrating by parts

$$\therefore x(y+1) - \int (y+2)e^y\ dy = c$$

$$\therefore\ x(y+1) - [(y+2)e^y - (1)e^y] = c$$

$$\therefore\ x(y+1) - (y+1)e^y = c$$

$$\therefore\ (y+1)(e^y - x) = c$$

Example:4

- **Solve** $\boldsymbol{e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy+\left(1+e^{\frac{x}{y}}\right)dx=0}$

- Write in $Mdx + Ndy = 0$ form

$M =$ $\qquad$ $N=$

$\therefore \frac{\partial M}{\partial y} =$ $\qquad$ $\therefore \frac{\partial N}{\partial x} =$

Is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$?

given equation is exact?

- Solution is

Example:4

- **Solve** $\boldsymbol{e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy+\left(1+e^{\frac{x}{y}}\right)dx=0}$

- **Solution:** Here $M = 1 + e^{\frac{x}{y}}$ $\qquad N = e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)$

$$\therefore \frac{\partial M}{\partial y} = e^{\frac{x}{y}}\left(-\frac{x}{y^2}\right)$$

$$\therefore \frac{\partial N}{\partial x} = e^{\frac{x}{y}}\left(-\frac{1}{y}\right) + \left(1-\frac{x}{y}\right)e^{\frac{x}{y}}\frac{1}{y}$$

$$= -\frac{e^{\frac{x}{y}}}{y} + \frac{1}{y}e^{\frac{x}{y}} - \frac{x}{y^2}e^{\frac{x}{y}}$$

$$\therefore \frac{\partial N}{\partial x} = -\frac{x}{y^2}e^{\frac{x}{y}}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Given equation is exact.

- Solution is, $\int M\,dx + \int (\text{Term in}'N'\text{free from x})dy = c$

$$\therefore \int\left(1+e^{\frac{x}{y}}\right)dx + \int 0\,dy = c$$

$$\therefore x + \frac{e^{\frac{x}{y}}}{\frac{1}{y}} = c$$

$$\therefore x + y\,e^{\frac{x}{y}} = c$$

Example:5

❖ **Solve :**

$$\left[y\sin(xy) + xy^2\cos(xy)\right]dx + \left[x\sin(xy) + x^2y\cos(xy)\right]dy = 0$$

❖ **Solution:** $M =$

$$\frac{\partial M}{\partial y} =$$

$$=$$

$$= 3\,xy\cos(xy) + (1 - x^2y^2)\sin(xy)$$

❖ $N =$

$$\frac{\partial N}{\partial x} =$$

$$=$$

$$= 3xy\cos(xy) + (1 - x^2y^2)\sin(xy)$$

Example:5

❖**Solve :**
$$\left[y\sin(xy)+xy^2\cos(xy)\right]dx+\left[x\sin(xy)+x^2y\cos(xy)\right]dy=0$$

❖ **Solution:** $M = y\sin(xy) + xy^2\cos(xy)$

$$\frac{\partial M}{\partial y} = xy\cos xy + \sin(xy) + x[y^2(-\sin xy)x + \cos(xy)\,(2y)]$$
$$= xy\cos xy + \sin(xy) - x^2y^2\sin(xy) + 2xy\cos(xy)$$
$$= 3\,xy\cos(xy) + (1 - x^2y^2)\sin(xy)$$

❖ $N = x\sin(xy) + x^2y\cos(xy)$

$$\frac{\partial N}{\partial x} = yx\cos xy + \sin(xy) + y[x^2(-\sin xy)y + \cos(xy)\,(2x)]$$
$$= xy\cos(xy) + \sin(xy) - x^2y^2\sin(xy) + 2xy\cos(xy)$$
$$= 3xy\cos(xy) + (1 - x^2y^2)\sin(xy)$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ the given equation is exact.

Solution is, $\int M\,dx + \int (\text{Term in}'N'\text{free from x})dy = c$

$\therefore \int [y\sin(xy) + xy^2\cos(xy)]dx + \int 0\,dy = c$

$\therefore \frac{-y\cos(xy)}{y} + y^2\left[\frac{x}{y}(\sin xy) - (1)\frac{(-\cos xy)}{y^2}\right] = c$

$\therefore -\cos(xy) + xy\sin(xy) + \cos(xy) = c$

$\therefore xy\sin(xy) = c$

Example:6

- **Solve $[1+\log(xy)]dx+\left[1+\frac{x}{y}\right]dy=0$**
- Here $M=1+\log(xy)$, $\quad N=1+\frac{x}{y}$

$$\frac{\partial M}{\partial y}=\frac{1}{xy}\times x=\frac{1}{y}, \qquad \frac{\partial N}{\partial x}=\frac{1}{y}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ the given equation is exact.

- Solution is, $\int M\,dx+\int(Term\ in\ N\ free\ from\ x)dy=c$

$$\therefore \int 1+\log(xy)\,dx+\int 1\,dy=c$$

$$\therefore x+\int \log(xy)\cdot 1\,dx+y=c$$

integrating by parts

$$\therefore x+\left[\log(xy)\cdot x-\int x\frac{1}{xy}\times y\ dx\right]+y=c$$

$$\therefore x+x\log(xy)-x+y=c$$

$$\therefore y+x\log(xy)=c$$

Practice Problems

❖ **Solve the following:**

1. $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$
2. $\left(y^2 e^{xy^2} + 4x^3\right)dx + \left(2xy\, e^{xy^2} + 3y^2\right)dy = 0$
3. $\frac{dy}{dx} = \frac{y}{2y\log y + y - x}$
4. $x\,dx + y\,dy = \frac{a^2(x\,dx - y\,dy)}{x^2 + y^2}$
5. $(y^3 - 3x^3 y)dy + (x^3 - 3xy^3)dx = 0$
6. $\left(x\sqrt{x^2 + y^2} - y\right)dx + \left(y\sqrt{x^2 + y^2} - x\right)dy = 0$
7. $\left[y\left(1 + \frac{1}{x}\right) + \cos y\right]dx + (x + \log x - x\sin y)dy = 0$