Module 3
Integration :
Review And Some New Techniques

Sub-Module 3.2

DUIS

(Differentiation Under Integral Sign)

Syllabus

3	**Integration : Review And Some New Techniques**		**7**	**CO 3**
	3.1	Beta and Gamma functions with properties		
	3.2	Differentiation under integral sign with constant limits of integration.(without proof)		
		# Self-learning topic: Differentiation under integral sign with variable limits of integration.		

DUIS

If $f(x,\alpha)$ is a continuous function of $x, and\ \alpha$ is a parameter and if $\frac{\partial f}{\partial \alpha}$ is a continuous function of $x\ and\ \alpha$ together throughout the interval $[a,b]$ where a, b are constants and independent of α, and if

$$I\ (\alpha) = \int_a^b f\ (x,\alpha)dx$$

then,

$$\frac{dI}{d\alpha} = \int_a^b \frac{\partial}{\partial \alpha} f(x,\alpha)dx$$

One Parameter

- **Ex.1 Prove that** $\int_0^\infty \frac{e^{-\alpha x}\sin x}{x}dx = cot^{-1}\alpha$ **Deduce that,** $\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$
- **Solution:** Let $I(\alpha) = \int_0^\infty \frac{e^{-\alpha x}\sin x}{x}dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(1)$
- Step :1 Using Differentiation under Integral Sign w. r. t. α
- $\frac{dI}{d\alpha} = \int_0^\infty \frac{\partial}{\partial\alpha}\frac{e^{-\alpha x}\sin x}{x}dx$

$$= \int_0^\infty \frac{-xe^{-\alpha x}\sin x}{x}dx = -\int_0^\infty e^{-\alpha x}\sin x\ dx$$

$$= -\frac{e^{-\alpha x}}{\alpha^2+1}[-\alpha\sin x - \cos x]\Big|_0^\infty = \frac{e^{-\alpha x}}{\alpha^2+1}[\alpha\sin x + \cos x]\Big|_0^\infty$$

$$= 0 - \frac{e^0}{\alpha^2+1}\ [\alpha\sin 0 + \cos 0]$$

$$\frac{dI}{d\alpha} = -\frac{1}{\alpha^2+1}$$

- Step 2: Integrating w.r.to a we get,
- $I(\alpha) = cot^{-1}\alpha + c \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$
- $Step\ 3: Find\ c$

$Put\ \alpha = \infty$ we get,

$I(\infty) = cot^{-1}(\infty) + c$

But from (1), $I(\infty) = 0$

$$\therefore C = -cot^{-1}(\infty) = 0$$

- $\therefore$ from (2) $\quad I(\alpha) = \int_o^{\infty} \frac{e^{-\alpha x} \sin x}{x} dx = cot^{-1}\alpha$
- Put, $\alpha = 0$ we get,

$$\int_o^{\infty} \frac{\sin x}{x} dx = cot^{-1}(0) = \frac{\pi}{2}$$

Example 2

- **Evaluate,** $\int_0^\infty \frac{e^{-x}}{x}(1-e^{-ax})\,dx, \quad a > -1$
- **Solution:** Let, $I(a) = \int_0^\infty \frac{e^{-x}}{x}(1-e^{-ax})\,dx \ldots\ldots\ldots\ldots\ldots\ldots(1)$
- Using differentiation under Integral Sign w. r. t. ‘a’
- $\frac{dI}{da} = \int_0^\infty \frac{\partial}{\partial a}\frac{e^{-x}}{x}(1-e^{-ax})\,dx$

$$= \int_0^\infty \frac{e^{-x}}{x}\left(0 - e^{-ax}(-x)\right)dx$$

$$= \int_0^\infty e^{-(1+a)x}\,dx = \left.\frac{e^{-(1+a)x}}{-(1+a)}\right|_0^\infty$$

$$= -\frac{1}{1+a}\left[e^{-\infty} - e^{0}\right]$$

$$= -\frac{1}{1+a}[0-1]$$

$$\therefore \frac{dI}{da} = \frac{1}{1+a}$$

- Integrating both sides w.r.to a we get,
- $I(a) = \log(1+a) + c \ldots\ldots\ldots\ldots\ldots.(2)$
- To find c put $a = 0$ we get,
- $I(0) = \log 1 + c$
- But, from (1) $I(0) = 0$
- $\therefore c = 0$
- $\therefore$ From (2)
- $I(\alpha) = \int_0^\infty \frac{e^{-x}}{x}(1 - e^{-ax})\, dx = \log(1+a)$

Example 3

- **Prove that** $\int_0^\infty \frac{\log(1+ax^2)}{x^2}dx = \pi\sqrt{a}, \quad a > 0$
- **Solution:** Let, $I(a) = \int_0^\infty \frac{\log(1+ax^2)}{x^2}dx \ldots\ldots\ldots\ldots\ldots\ldots (1)$
- Using Differentiation under integral sign w. r. t. 'a'

$$\frac{dI}{da} = \int_0^\infty \frac{\partial}{\partial a}\frac{\log(1+ax^2)}{x^2}dx$$

$$= \int_0^\infty \frac{1}{x^2}\cdot\frac{1}{1+ax^2}\cdot x^2\,dx$$

$$= \int_0^\infty \frac{1}{1+ax^2}dx = \int_0^\infty \frac{1}{1+(\sqrt{a}\cdot x)^2}dx$$

$$= \frac{1}{\sqrt{a}}tan^{-1}\sqrt{a}\cdot x\,\Big|_0^\infty$$

$$= \frac{1}{\sqrt{a}}[tan^{-1}\infty - tan^{-1}0]$$

$$\frac{dI}{da} = \frac{1}{\sqrt{a}}\ \frac{\pi}{2}$$

- Integrating both sides w.r.to a we get,
- $I(a) = \frac{\pi}{2} \cdot \frac{\sqrt{a}}{\frac{1}{2}} + c$
- $I(a) = \pi\sqrt{a} + c \ldots \ldots \ldots (2)$
- To find c put $a = 0$, we get $I(0) = c$
- But from (1) $I(0) = 0 \;\; \therefore c = 0$
- $\therefore$ From (2)
- $I(a) = \int_0^\infty \frac{\log(1+ax^2)}{x^2} dx = \pi\sqrt{a}, \quad a > 0$

Example 4

- **HW. Prove that** $\int_0^\infty \frac{e^{-x}-e^{-ax}}{x\sec x}dx = \frac{1}{2}\log\left(\frac{a^2+1}{2}\right)$
- **Solution:** Let $I(a) = \int_0^\infty \frac{e^{-x}-e^{-ax}}{x\sec x}dx \ldots\ldots\ldots\ldots(1)$
- Using differentiation under integral sign w. r. t. 'a' we get,
- $\frac{dI}{da} = \int_0^\infty \frac{\partial}{\partial a}\frac{e^{-x}-e^{-ax}}{x\sec x}dx$
- $= \int_0^\infty \frac{1}{x\sec x}[0 - e^{-ax}(-x)]dx$
- $= \int_0^\infty e^{-ax}\cdot\cos x \ \ dx$
- $= \frac{e^{-ax}}{a^2+1}[-a\cos x + \sin x]\Big|_0^\infty$
- $= 0 - \frac{e^0}{a^2+1}[-a\cos 0 + \sin 0]$
- $\frac{dI}{da} = \frac{a}{a^2+1}$

- Integrating both sides w.r.to a we get,
- $I(a)=\frac{1}{2}\log(a^2+1)+c\ldots\ldots\ldots(2)$
- To find c put $a=1$
- $I(1)=\frac{1}{2}\log 2+c$
- But, from (1) $I(1)=0$
- $\therefore c=-\frac{1}{2}\log 2$
- $\therefore$ from (2),
- $I(a)=\int_0^\infty \frac{e^{-x}-e^{-ax}}{x\sec x}dx=\frac{1}{2}\log(a^2+1)-\frac{1}{2}\log 2$
- $=\frac{1}{2}\log\left(\frac{a^2+1}{2}\right)$

Example 5

- **Prove that,** $\int_0^\infty \frac{e^{-x}}{x}\left(a-\frac{1}{x}+\frac{1}{x}\,e^{-ax}\right)dx$
- **Solution:** Let, $I(a)=\int_0^\infty \frac{e^{-x}}{x}\left(a-\frac{1}{x}+\frac{1}{x}\,e^{-ax}\right)dx \ldots\ldots\ldots\ldots(1)$
- Using differentiation under integral sign w. r. t. 'a'

$$\frac{dI}{da}=\int_0^\infty \frac{\partial}{\partial a}\left[\frac{e^{-x}}{x}\left(a-\frac{1}{x}+\frac{1}{x}\,e^{-ax}\right)\right]dx$$

$$=\int_0^\infty \frac{e^{-x}}{x}\left[1+\frac{1}{x}e^{-ax}(-x)\right]dx$$

$$\frac{dI}{da}=\int_0^\infty \frac{e^{-x}}{x}\left[1-e^{-ax}\right]dx \ldots\ldots\ldots.(2)$$

- Again using differentiation under integral sign w. r. t. a,

$$\frac{d^2I}{da^2}=\int_0^\infty \frac{e^{-x}}{x}\left[0-e^{-ax}(-x)\right]dx$$

$$=\int_0^\infty e^{-(1+a)x}\;dx=\frac{e^{-(1+a)x}}{-(1+a)}\Bigg|_0^\infty=-\frac{1}{1+a}\left[e^{-\infty}-e^{0}\right]=-\frac{1}{1+a}[0-1]$$

$$\frac{d^2I}{da^2}=\frac{1}{1+a}$$

- Integrating both sides w.r.to a we get,

$$\frac{dI}{da} = \log(1+a) + c \ldots\ldots\ldots\ldots (3)$$

- To find c put $a = 0$

$$\frac{dI}{da}(0) = c$$

But, from (2) , $\frac{dI(0)}{da} = 0$

- $\therefore c = 0$

$\therefore$ From (3), $\quad \frac{dI}{da} = \log(1+a)$

- Again integrating both sides w.r.to a we get,

$$I(a) = \int \log(1+a)\, da + c_1$$

$$= \log(1+a) \cdot a - \int \frac{1}{1+a} \cdot a \; da + c_1$$

$$I(a) = a\log(1+a) - \int\left(1 - \frac{1}{1+a}\right)da + c_1$$

$$I(a) = a\log(1+a) - a + \log(1+a) + c_1 \dots\dots\dots\dots.(4)$$

- To find c_1 put $a = 0$,

$$I(0) = c_1$$

- But from (1)

$I(0) = 0 \therefore c_1 = 0$

- $\therefore$ From (4)

$$I(a) = \int_0^\infty \frac{e^{-x}}{x}\left(a - \frac{1}{x} + \frac{1}{x}e^{-ax}\right)dx = (1+a)\log(1+a) - a$$

Example 6

- **Show that,** $\int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)} dx = \frac{\pi}{2}\log(1+a)$
- **Solution:** Let, $I(a) = \int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)} dx \ldots\ldots\ldots\ldots (1)$
- Using Differentiation under integral sign w. r. t. 'a', we get
- $\frac{dI}{da} = \int_0^\infty \frac{\partial}{\partial a} \frac{\tan^{-1} ax}{x(1+x^2)} dx$

$= \int_0^\infty \frac{1}{x(1+x^2)} \cdot \frac{1}{1+a^2x^2} \cdot x \quad dx$

$= \int_0^\infty \frac{1}{(1+x^2)(1+a^2x^2)} \quad dx$

Let $x^2 = t$ we have,

$$\frac{1}{(1+t)(1+a^2t)} = \frac{A}{1+t} + \frac{B}{1+a^2t} \qquad \text{(Partial Fraction)}$$

$$1 = A(1+a^2t) + B(1+t)$$

For $t = -1$ $\qquad 1 = A(1-a^2) \qquad \therefore A = \frac{1}{1-a^2}$

For $t = -\frac{1}{a^2}$ $\qquad 1 = B\left(1 - \frac{1}{a^2}\right) \qquad \therefore B = \frac{a^2}{a^2-1} = \frac{-a^2}{1-a^2}$

❖ $\frac{dI}{da} = \frac{1}{1-a^2}\int_0^\infty \left(\frac{1}{(1+x^2)} - \frac{a^2}{1+a^2x^2}\right) dx$

$$= \frac{1}{1-a^2}\left[tan^{-1}x - a^2 \cdot \frac{tan^{-1}ax}{a}\right]\Bigg|_0^\infty$$

$$= \frac{1}{1-a^2}[tan^{-1}x - a\, tan^{-1}ax]\Bigg|_0^\infty$$

$$= \frac{1}{1-a^2}\{[tan^{-1}\infty - a\, tan^{-1}\infty] - [0]\}$$

$$= \frac{1}{1-a^2}\left\{\frac{\pi}{2} - a\frac{\pi}{2}\right\}$$

$$= \frac{\pi}{2}\ \frac{1}{(1-a)(1+a)}(1-a)$$

❖ $\frac{dI}{da} = \frac{\pi}{2(1+a)}$

❖ Integrating both sides w.r.to a ,

$$I(a) = \frac{\pi}{2}\log(1+a) + c \ldots\ldots\ldots\ldots (2)$$

❖ To find c put $a = 0,\ I(0) = c$

But from (1), $I(0) = 0 \qquad \therefore c = 0$

$\therefore$ From (2)

$$I(a) = \int_0^\infty \frac{\tan^{-1}\text{ax}}{x(1+x^2)}dx = \frac{\pi}{2}\log(1+a)$$

Example 7

- **Prove That** $\int_0^{\frac{\pi}{2}} \frac{\log(1+a\sin^2 x)}{\sin^2 x} dx = \pi\left[\sqrt{a+1}-1\right]$
- **Solution:** Let $I(a) = \int_0^{\frac{\pi}{2}} \frac{\log(1+a\sin^2 x)}{\sin^2 x} dx \ldots\ldots\ldots\ldots\ldots.(1)$
- Using differentiation under integral sign w. r. t. 'a' we get,
- $\frac{dI}{da} = \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial a} \frac{\log(1+a\sin^2 x)}{\sin^2 x} dx$
- $= \int_0^{\frac{\pi}{2}} \frac{1}{\sin^2 x} \frac{1}{1+a\sin^2 x} \sin^2 x \; dx$
- $= \int_0^{\frac{\pi}{2}} \frac{1}{1+a\sin^2 x} \; dx$
- Divide N and D by $cos^2 x$
- $= \int_0^{\frac{\pi}{2}} \frac{sec^2 x}{sec^2 x + a\tan^2 x} \; dx = \int_0^{\frac{\pi}{2}} \frac{sec^2 x}{1+tan^2 x + a\tan^2 x} \; dx$
- $= \int_0^{\frac{\pi}{2}} \frac{sec^2 x}{1+(1+a)\tan^2 x} \; dx \; = \int_0^{\infty} \frac{dt}{1+(1+a)\,t^2} \; dx$

- $\frac{dI}{da} = \frac{tan^{-1}\sqrt{1+a}\,t}{\sqrt{1+a}}\Bigg|_0^\infty$
- $= \frac{1}{\sqrt{1+a}}[tan^{-1}\infty - tan^{-1}0]$
- $\frac{dI}{da} = \frac{1}{\sqrt{1+a}}\ \frac{\pi}{2}$
- Integrating both sides w.r.to a , $I(a) = \frac{\pi}{2}\ 2\sqrt{1+a} + c \ldots\ldots\ldots\ldots (2)$
- To find c put $a = 0$
- $I(0) = \pi + c$
- But from (1) $I(0) = 0 \therefore c = -\pi$
- $\therefore$ From (2)
- $I(a) = \int_0^{\frac{\pi}{2}} \frac{\log(1+a\sin^2 x)}{\sin^2 x} dx = \pi\left[\sqrt{a+1} - 1\right]$

Example 8

Prove $\int_0^\infty e^{-\left(x^2+\frac{a^2}{x^2}\right)} \cdot dx = \frac{\sqrt{\pi}}{2} \quad e^{-2a}$

- Let, $I(a) = \int_0^\infty e^{-\left(x^2+\frac{a^2}{x^2}\right)} \cdot dx \ldots\ldots\ldots\ldots (1)$
- Using Differentiation under integral sign w. r. t. 'a'
- $\frac{dI}{da} = \int_0^\infty \frac{\partial}{\partial a} \quad e^{-\left(x^2+\frac{a^2}{x^2}\right)} \cdot dx$
- $= \int_0^\infty e^{-\left(x^2+\frac{a^2}{x^2}\right)} \cdot \left(\frac{-2a}{x^2}\right) dx$
- $= \int_\infty^0 e^{-\left(y^2+\frac{a^2}{y^2}\right)} \cdot 2dy$
- $= -2 \int_0^\infty e^{-\left(y^2+\frac{a^2}{y^2}\right)} \cdot dy$
- $\frac{dI}{da} = -2I$
- $\therefore \frac{dI}{I} = -2da$

Put $\frac{a}{x} = y$

$-\frac{a}{x^2} dx = dy$

x	0	∞
y	∞	0

- Integrating w.r.to a we get,
- $\log I(a) = -2a + \log c$
- $I(a) = e^{-2a+\log c}$
- $I(a) = e^{-2a} \cdot e^{\log c}$
- $\therefore I(a) = c\, e^{-2a} \ldots\ldots\ldots\ldots (2)$
- To find c Put $a = 0$,
- $I(0) = c$
- But from,(1) $I(0) = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ (given)
- $\therefore c = \frac{\sqrt{\pi}}{2}$
- $\therefore$ From (2)
- $I(a) = \int_0^\infty e^{-\left(x^2+\frac{a^2}{x^2}\right)} \cdot dx = \frac{\sqrt{\pi}}{2} \quad e^{-2a}$

Practice Problems

- **Assuming the validity of differentiation under the integral sign, prove that**
- **1.** $\int_0^\infty \frac{1-\cos ax}{x^2}\,dx = \frac{\pi a}{2}$ $\left(\text{Assume That } \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}\right)$
- **2.** $\int_0^\infty e^{-x^2}\cos 2\,ax\;dx = \frac{\sqrt{\pi}}{2}\,e^{-a^2}$

  $\left(\text{Assume that, } \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}\right)$
- **3.** $\int_0^\pi \frac{dx}{(a-\cos x)} = \frac{\pi}{\sqrt{a^2-1}}$ $(a > 0)$ Deduce that,

i. $\int_0^\pi \frac{dx}{(a-\cos x)^2} = \frac{\pi a}{(a^2-1)^{\frac{3}{2}}}$ **ii.** $\int_0^\pi \frac{dx}{(2-\cos x)^2} = \frac{2\pi}{3\sqrt{3}}$

- **4.** $\int_0^\infty \frac{1-\cos mx}{x} e^{-x}dx = \frac{1}{2}\log(m^2+1)$
- **5.** $\int_0^{\pi/2} \frac{\log(1+\cos\alpha\cos x)}{\cos x}\,dx = \frac{\pi^2}{8} - \frac{\alpha^2}{2}$