Module 1
Differential Equations
of
First Order and First Degree

Sub-module 1.4

Cauchy’s Homogeneous Linear Differential Equations

Syllabus

Module No.	Unit No.	Details	Hrs.	CO
1	**Differential Equation of First Order and First Degree**		13	CO 1
	1.1	Differential Equation of first order and first degree- Exact differential equations, Equations reducible to exact equations by integrating factors.		
	1.2	Linear differential equations (Review), Equation reducible to linear form. Applications of Differential Equation of first order and first degree		
	1.3	Linear Differential Equation with constant coefficients: Complimentary function, particular integrals of differential equation of the type f(D)y=X, where X is e^{ax}, $\sin(ax+b)$, $\cos(ax+b)$, x^n, $e^{ax}V$		
	1.4	Cauchy's homogeneous linear differential equation		
	1.5	Method of variation of parameters		
		# **Self-learning topic:** Bernoulli's equation. Equation reducible to Bernoulli's equation.		

Cauchy’s Homogeneous Linear D.E

- An equation of the form
- $x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_1 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots \ldots \ldots . + a_{n-1} x \frac{dy}{dx} + a_n y = X$ ....(1)
- is called Cauchy’s homogeneous linear differential equation of order n or simply known as Cauchy’s equation. Here $a_1, a_2, a_3 \ldots \ldots \ldots \ldots \ldots \ldots \ldots a_n$ are constants and X is the function of x.
- **Method of Substitution:**
- $Put\ z = \log x \qquad \therefore x = e^z$
- $\frac{dz}{dx} = \frac{1}{x}$
- $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$
- $\therefore x\frac{dy}{dx} = \frac{dy}{dz} = Dy \quad where\ D = \frac{d}{dz}$

- Similarly,
- $x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dz^2}-\frac{dy}{dz}=D^2y-Dy=D(D-1)y\ where\ D=\frac{d}{dz}$
- $x^3\frac{d^3y}{dx^3}=D(D-1)(D-2)y\ and\ so\ on.$
- Proof: $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$

$$=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x}\frac{d^2y}{dz^2}\cdot\frac{dz}{dx}$$

$$=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x}\frac{d^2y}{dz^2}\left(\frac{1}{x}\right)$$

$$\frac{d^2y}{dx^2}=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x^2}\frac{d^2y}{dz^2}$$

- $\therefore x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dz^2}-\frac{dy}{dz}=D^2y-Dy=D(D-1)y$
- Similarly, Equation (1) will be reduced to linear differential equation with constant coefficients by substituting $x=e^z$ on the R.H.S. X.

Example 1

- **Solve $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = \cos\log x + x\sin\log x$**
- **Solution:** Put $z = \log x \quad \therefore x = e^z$,
- $x\frac{dy}{dx} = Dy,\ x^2\frac{d^2y}{dx^2} = D(D-1)y$
- Given equation get reduced to
- $[D(D-1) - D + 4]y = \cos z + e^z \sin z$
- $[D^2 - 2D + 4]y = \cos z + e^z \sin z$;...........(1)
- The A.E. is given by $D^2 - 2D + 4 = 0$
- $\therefore D = \frac{2\pm\sqrt{4-4\cdot1\cdot4}}{2\cdot1} = \frac{2\pm\sqrt{12}}{2} = 1 \pm i\sqrt{3}$
- $C.F. = y_c = e^z\left[c_1\cos\sqrt{3}\,z + c_2\sin\sqrt{3}\,z\right]$

- $P.I. = y_p = \frac{1}{f(D)}X = \frac{1}{D^2-2D+4}\cos z + \frac{1}{D^2-2D+4}e^z \sin z$
- $= \frac{1}{-1+4-2D}\cos z + e^z \frac{1}{(D+1)^2-2(D+1)+4}\sin z$
- $= \frac{1}{3-2D}\cos z + e^z \frac{1}{D^2+2D+1-2D-2+4}\sin z$
- $= \frac{3+2D}{9-4D^2}\cos z + e^z \cdot \frac{1}{D^2+3}\sin z$
- $= \frac{3+2D}{9+4}\cos z + e^z \frac{1}{-1+3}\sin z$
- $= \frac{1}{13}[3\cos z + 2(-\sin z)] + \frac{e^z}{2}\sin z$

- Complete Solution is given by $y = y_c + y_p$
- $\therefore y = e^z\left[c_1 \cos\sqrt{3}\,z + c_2 \sin\sqrt{3}\,z\right]$
$$+\frac{1}{13}[3\cos z + 2(-\sin z)] + \frac{e^z}{2}\sin z$$
- $y = x\left[c_1 \cos\sqrt{3}\log x + c_2 \sin\sqrt{3}\log x\right]$
$$+\frac{1}{13}[3\cos\log x - \ 2\sin\log x] + \frac{x}{2}\sin\log x$$

is the required G.S.

Example 2

- **Solve** $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=\frac{x^3}{x^2+1}$
- **Solution**: Put $z=\log x \quad and \quad x=e^z$
- Given $[D(D-1)+D-1]y=\frac{e^{3z}}{e^{2z}+1}$
- The A.E. is given by $f(D)=0$
- $D^2-1=0$
- $D=\pm 1$
- $C.F.=y_c=c_1e^z+c_2e^{-z}$

- $P.I. = y_p = \frac{1}{D^2-1} e^{3z}(e^{2z}+1)^{-1}$
- $= \frac{1}{D+1} \cdot \frac{1}{D-1} \frac{e^{3z}}{e^{2z}+1}$
- $= \frac{1}{D+1} e^{z} \int \frac{e^{-z}e^{3z}}{e^{2z}+1} dz$
- $= \frac{1}{D+1} e^{z} \int \frac{e^{2z}}{e^{2z}+1} dz$ $\quad [Put\ e^{z} = t]$
- $= \frac{1}{D+1} e^{z} \int \frac{t}{t^2+1} dt$
- $= \frac{1}{D+1} e^{z} \cdot \frac{1}{2} \log(t^2+1)$
- $= \frac{1}{D+1} e^{z} \frac{1}{2} \log(e^{2z}+1)$
- $= \frac{1}{2} e^{-z} \int e^{z} e^{z} \log(e^{2z}+1) dz$
- $= \frac{1}{2} e^{-z} \int e^{2z} \log(e^{2z}+1) dz$

- $y = \frac{1}{2}e^{-z}\left[\log(e^{2z}+1)\frac{e^{2z}}{2} - \int \frac{e^{2z}}{2}\ \frac{e^{2z}\,2}{(e^{2z}+1)}dz\right]$
- To find the integral put $e^{z} = t$
- $\therefore \int \frac{t^3}{t^2+1}dt = \int \left[t - \frac{t}{t^2+1}\right]dt$
- $= \frac{t^2}{2} - \frac{1}{2}\log(t^2+1) = \frac{e^{2z}}{2} - \frac{1}{2}\log(e^{2z}+1)$
- $P.I. = y_p = \frac{1}{2}e^{-z}\left[\frac{e^{2z}}{2}\log(e^{2z}+1) - \frac{e^{2z}}{2} + \frac{1}{2}\log(e^{2z}+1)\right]$
- $= \frac{e^{z}}{4}\log(e^{2z}+1) - \frac{e^{z}}{4} + \frac{e^{-z}}{4}\log(e^{2z}+1)$
- $\therefore$The Complete Solution is $y = y_c + y_p$
- $y = c_1e^{z} + c_2e^{-z} + \frac{e^{z}}{4}\log(e^{2z}+1) - \frac{e^{z}}{4} + \frac{e^{-z}}{4}\log(e^{2z}+1)$
- $\therefore y = c_1x + \frac{c_2}{x} + \frac{x}{4}\log(x^2+1) - \frac{x}{4} + \frac{1}{4x}\log(x^2+1)$

Example 3

- **Solve $x^2\frac{d^3y}{dx^3}+3x\frac{d^2y}{dx^2}+\frac{dy}{dx}=x^2\log x$**
- **Solution:** Put $x=e^z, \quad and \;\; z=\log x$
- Multiplying given equation throughout by x, we get
- $x^3\frac{d^3y}{dx^3}+3x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=x^3\log x$
- Given $[D(D-1)(D-2)+3D(D-1)+D]y= \; e^{3z}\,z$
- $=[D^3-3D^2+2D+3D^2-3D+D]y=z\,e^{3z}$
- $\therefore D^3y=z\,e^{3z}$
- The A.E. is given by $D^3=0$
- $D=0, \;\; 0, \;\; 0$
- $C.F.=y_c=c_1+c_2z+c_3z^2$
- $=c_1+c_2\log x+c_3(\log x)^2$

- $P.I. = y_p = \frac{1}{f(D)} X = \frac{1}{D^3} z\, e^{3z}$
- $= e^{3z} \cdot \frac{1}{(D+3)^3} z$
- $= e^{3z} \cdot \frac{1}{D^3+9D^2+27D+27} z$
- $= \frac{e^{3z}}{27} \left[1 + \frac{D^3+9D^2+27D}{27}\right]^{-1} z$
- $= \frac{e^{3z}}{27} [1 - D] z$
- $= \frac{e^{3z}}{27} [z - 1]$
- $= \frac{1}{27} x^3 (\log x - 1)$
- Complete Solution is given by $y = y_c + y_p$
- $\therefore y = c_1 + c_2 \log x + c_3 (\log x)^2 + \frac{1}{27} x^3 (\log x - 1)$

Example 4

- **Solve $x^3 \frac{d^2y}{dx^2} + 3x^2 \frac{dy}{dx} + xy = \sin \log x$**
- **Solution:** Put $x = e^z,\ z = \log x$
- Dividing throughout by x, we get
- $x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + y = \frac{1}{x} \sin \log x$
- Given $[D(D-1) + 3D + 1]y = \frac{1}{e^z} \sin z$
- The A.E. is given by $D^2 + 2D + 1 = 0$
- $(D+1)^2 = 0$
- $D = -1, -1$
- $C.F. = y_c = (c_1 + c_2 z)e^{-z} = \frac{1}{x}[c_1 + c_2 \log x]$

- $P.I. = y_p = \frac{1}{f(D)}X = \frac{1}{D^2+2D+1}e^{-z}\sin z$
- $= e^{-z} \cdot \frac{1}{(D-1)^2+2(D-1)+1}\sin z$
- $= e^{-z} \cdot \frac{1}{D^2-2D+1+2D-2+1}\sin z$
- $= e^{-z}\frac{1}{D^2}\sin z$
- $= e^{-z}\frac{1}{D}(-\cos z)$
- $= -e^{-z}\sin z$
- $= \frac{-\sin\log x}{x}$
- Complete solution is given by $y = y_c + y_p$
- $y = \frac{1}{x}[c_1 + c_2\log x - \sin\log x]$

Example 5

- **Find the equation of the curve which satisfies the differential equation $4x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + y = 0$ and crosses the x axis at an angle of $60°$ at $x = 1$.**
- **Solution:** Put $x = e^z,\ z = \log x$
- Given $[4D(D-1) - 4D + 1]y = 0$
- $[4D^2 - 4D - 4D + 1]y = 0$
- $[4D^2 - 8D + 1]y = 0$
- $D = \frac{8\pm\sqrt{64-4(4)(1)}}{2\cdot 4} = \frac{8\pm\sqrt{48}}{8}$
- $= \frac{8\pm 4\sqrt{3}}{8} = \frac{2\pm\sqrt{3}}{2}$
- $y_c = c_1 e^{\left(\frac{2+\sqrt{3}}{2}\right)z} c_2\, e^{\left(\frac{2-\sqrt{3}}{2}\right)z}$
- $\therefore y = c_1 x^{\left(\frac{2+\sqrt{3}}{2}\right)} + c_2\ x^{\left(\frac{2-\sqrt{3}}{2}\right)} \ldots\ldots\ldots\ldots\ldots (1)$

- By Putting $y = 0$ and $x = 1$ in (1), we get
- $0 = c_1 + c_2 \ldots \ldots \ldots \ldots \ldots \ldots (2)$
- $\therefore c_1 = -c_2$
- $\frac{dy}{dx} = slope\ of\ tangent = \tan 60°$
- $\frac{dy}{dx} = c_1(\frac{2+\sqrt{3}}{2})x^{\left(\frac{2+\sqrt{3}}{2}\right)-1} + c_2(\frac{2-\sqrt{3}}{2})x^{\left(\frac{2-\sqrt{3}}{2}\right)-1}$
- At x=1,
- $\Longrightarrow \sqrt{3} = c_1\left(\frac{2+\sqrt{3}}{2}\right) + c_2\left(\frac{2-\sqrt{3}}{2}\right)$
- $\therefore \sqrt{3} = c_1 + \frac{\sqrt{3}}{2}c_1 + c_2 - \frac{\sqrt{3}}{2}c_2$
- $\qquad = (c_1+c_2) + \frac{\sqrt{3}}{2}(c_1 - c_2)$
- $\sqrt{3} = \frac{\sqrt{3}}{2}(c_1 - c_2)\ [\because c_1 + c_2 = 0\ by\ (2)]$
- $c_1 - c_2 = 2 \ldots \ldots \ldots \ldots .. (3)$

- Solving (2) and (3), we get
- $\therefore\ c_1 = 1$
- $\therefore 1 + c_2 = 0$ [substituting c_1 in (2)]
- $c_2 = -1$
- Substituting c_1 and c_2 in (1) , we get the required equation of the curve.
- $\therefore y = x^{\left(\frac{2+\sqrt{3}}{2}\right)} - x^{\left(\frac{2-\sqrt{3}}{2}\right)}$ is the required equation of the curve.

Practice Problems

1. Solve $x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=2\log x$

❖ Answer: $[y=(c_1+c_2\log x)\cdot x+2(\log x+2)]$

2. Solve $x^3\frac{d^3y}{dx^3}-x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-2y=x^3+3x$

❖ Answer: $\left[y=(c_1+c_2\log x)x+c_3x^2+\frac{x^3}{4}-\frac{3x}{2}(\log x)^2\right]$

3. Solve $\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}=\frac{12\log x}{x^2}$

❖ Answer: $[y=c_1+c_2\log x+2(\log x)^3]$

4. Solve $x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=e^x$

❖ Answer: $\left[y=\frac{c_1}{x}+\frac{c_2}{x^2}+\frac{e^x}{x^2}\right]$

5. Solve $(x^2D^2+5xD+3)y=\left(1+\frac{1}{x}\right)^2\log x$

❖ Answer: $\left[y=\frac{c_1}{x}+\frac{c_2}{x^3}+\frac{\log x}{3}-\frac{4}{9}-\frac{1}{x}\left[\frac{(\log x)^2}{2}-\frac{\log x}{2}\right]-\frac{1}{x^2}\log x\right]$