Module 1
Differential Equations
of
First Order and First Degree

Sub-module 1.3

Linear ODE with constant coefficient

Syllabus

Module No.	Unit No.	Details	Hrs.	CO
1	**Differential Equation of First Order and First Degree**		13	CO 1
	1.1	Differential Equation of first order and first degree- Exact differential equations, Equations reducible to exact equations by integrating factors.		
	1.2	Linear differential equations (Review), Equation reducible to linear form. Applications of Differential Equation of first order and first degree		
	1.3	Linear Differential Equation with constant coefficients: Complimentary function, particular integrals of differential equation of the type f(D)y=X, where X is e^{ax}, $\sin(ax+b)$, $\cos(ax+b)$, x^n, $e^{ax}V$		
	1.4	Cauchy's homogeneous linear differential equation		
	1.5	Method of variation of parameters		
		# **Self-learning topic:** Bernoulli's equation. Equation reducible to Bernoulli's equation.		

Higher order Diff. equations

- **Linear Differential Equation with constant coefficient:**
- A linear differential equation with constant coefficient of the n^{th} order is given by:
- $a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots \ldots \ldots \ldots \ldots \ldots .. + a_n y = X$
- Where $a_0, a_1, a_2, \ldots \ldots \ldots . a_n$ are constant and X is a function of x alone.
- For solution of differential equation, the operator $\frac{d}{dx}$ is denoted by D.
- $\therefore \frac{d}{dx} = D$
- $\therefore$ The linear differential equation becomes
- $a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \cdots \ldots \ldots + a_n y = X$
- $(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots \ldots \ldots + a_n) y = X$
- $\therefore f(D) y = X$
- Where $f(D) = a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots \ldots \ldots + a_n$

General Solution of Diff Eq:

- The relation between dependent variables and independent variables which satisfy the given differential equation is known as general solution of given differential equation.
- The general solution consists of two parts namely, Complementary function (C. F.) and Particular Integral (P. I.)
- **Complementary Function:**
- A Part of General Solution which contains arbitrary constants (i.e. $c_1, c_2, c_3, \dots \dots \dots$) is known as Complementary function.
- **Particular Integral:**
- A Part of General Solution which does not contain any arbitrary constants is known as Particular Integral.
- The general solution (G. S.) is given by
  $y = C.F. + P.I. \;\; or \;\; y = y_c + y_p$

Methods of Finding CF:

- If DE is $f(D)y = X$ then Complementary function (C. F.) can be obtained by solving Auxiliary equation $f(D) = 0$. Let $m_1, m_2, m_3, \dots \dots \dots m_n$ be the roots of Auxiliary equation. Depending upon the nature of roots, the Complementary function (C. F.) is determined as follows:

- **Case (i)**
- **Roots are real and non-repeated:**
- If $m_1, m_2, m_3, \dots \dots \dots m_n$ are the real roots, then
- $C.F. = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \cdots \dots \dots + c_n e^{m_n x}$

- **Case (ii)**
- **Roots are real and repeated:**
- If m_1 is repeated 4 times, m_2 is repeated twice and m_3 is not repeated then
- $C.F. = (c_1 x^3 + c_2 x^2 + c_3 x + c_4) e^{m_1 x} + (c_5 x + c_6) e^{m_2 x} + c_7 e^{m_3 x}$
- Or
- $C.F. = (c_1 + c_2 x + c_3 x^2 + c_4 x^3) e^{m_1 x} + (c_5 + c_6 x) e^{m_2 x} + c_7 e^{m_3 x}$

- **Case (iii)**
- **Roots are complex and non-repeated:**
- If $m_1 = \alpha + i\beta$, then for every complex root, there will exist its conjugate pair $m_2 = \alpha - i\beta$
- $C.F. = e^{\alpha x}[c_1 \cos(\beta x) + c_2 \sin(\beta x)]$

- **Case (iv)**
- **Roots are complex and repeated:**
- If $m_1 = \alpha \pm i\beta$ is repeated twice and $m_2 = \alpha_1 \pm i\beta_1$ is not repeated then,
- $C.F. = e^{\alpha x}[(c_1 x + c_2)\cos(\beta x) + (c_3 x + c_4)\sin(\beta x)] + e^{\alpha_1 x}[c_5 \cos(\beta_1 x) + c_6 \sin(\beta_1 x)]$

Example 1

- $9\frac{d^2y}{dx^2} + 18\frac{dy}{dx} - 16y = 0$
- **Solution:** The auxiliary equation is given by
- $9D^2 + 18D - 16 = 0$
- $9D^2 + 24D - 6D - 16 = 0$
- $3D(3D + 8) - 2(3D + 8) = 0$
- $(3D - 2)(3D + 8) = 0$
- $D = \frac{2}{3}, \quad \frac{-8}{3}$
- $\therefore y_c = c_1 e^{\frac{2}{3}x} + c_2 e^{\frac{-8}{3}x}$

Example 2

- $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 6y = 0$
- **Solution:** The Auxiliary equation is given by

$D^3 + 2D^2 - 5D - 6 = 0$

- By Synthetic Division:

$$\begin{array}{c|cccc} 2 & 1 & 2 & -5 & -6 \\ & 0 & 2 & 8 & 6 \\ \hline & 1 & 4 & 3 & 0 \end{array}$$

- $(D-2)(D^2+4D+3)=0$
- $(D-2)(D^2+3D+D+3)=0$
- $(D-2)[D(D+3)+1(D+3)]=0$
- $(D+1)(D+3)(D-2)=0$
- $\therefore D=-1,-3,2$
- $C.F.$ is given by $y_c = c_1e^{-x} + c_2e^{-3x} + c_3e^{-2x}$

Example 3

- $\left(\boldsymbol{D^4 - 5D^3 + 5D^2 + 5D - 6}\right)\boldsymbol{y = 0}$
- **Solution:** The Auxiliary Equation is given by
- $D^4 - 5D^3 + 5D^2 + 5D - 6 = 0$
- $(D-1)(D^3 - 4D^2 + D + 6) = 0$
- $(D-1)(D+1)(D^2 - 5D + 6) = 0$
- $(D-1)(D+1)(D-2)(D-3) = 0$
- $D = -1, 1, 2, 3$
- $\therefore y_c = c_1 e^{-x} + c_2 e^{x} + c_3 e^{2x} + c_4 e^{3x}$

Example 4

- $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4y = 0$
- **Solution:** The Auxiliary equation is $D^3 - 3D^2 + 4 = 0$
- $(D+1)(D^2 - 4D + 4) = 0$
- $(D+1)(D-2)^2 = 0$
- $D = -1, 2, 2$
- $\therefore y_c = c_1 e^{-x} + (c_2 + c_3 x)e^{2x}$

Example 5

- $\frac{d^4y}{dx^4} - 18\frac{d^2y}{dx^2} + 81y = 0$
- **Solution:** The Auxiliary equation is
- $D^4 - 18D^2 + 81 = 0$
- $(D^2 - 9)^2 = 0$
- $(D-3)^2(D+3)^2 = 0$
- $D = 3, 3, -3, -3$
- $\therefore y_c = (c_1x + c_2)e^{3x} + (c_3x + c_4)e^{-3x}$

Example 6

- **Solve $\frac{d^3y}{dx^3} + 8y = 0$**
- **Solution:** The auxiliary equation is $D^3 + 8 = 0$
- $(D+2)(D^2 - 2D + 4) = 0$
- $D = -2,\ 1 \pm i\sqrt{3}$
- $\therefore y_c = c_1 e^{-2x} + e^{x}\left[c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x\right]$

Example 7

- $(\boldsymbol{D^2 - 2D + 2)^2 y = 0}$
- **Solution:** The auxiliary equation is given by
- $(D^2 - 2D + 2)^2 = 0$
- $\therefore D = 1 \pm i, \;\; 1 \pm i \qquad [\alpha = 1, \beta = 1]$
- $\therefore y_c = e^x[(c_1 x + c_2) cos\, x + (c_3 x + c_4) \sin x]$

Example 8

- $(D^4 + 4)y = 0$
- **Solution:** The auxiliary equation is $D^4 + 4 = 0$
- $D^4 + 4D^2 + 4 - 4D^2 = 0$
- $(D^2 + 2)^2 - (2D)^2 = 0$
- $(0^2 + 2D + 2)(D^2 - 2D + 2) = 0$
- $D = \frac{-2 \pm \sqrt{-4}}{2}, \quad \frac{2 \pm \sqrt{-4}}{2}$
- $D = -1 \pm i, \quad 1 \pm i$
- $\therefore y_c = e^{-x}[c_1 \cos x + c_2 \sin x] + e^{x}[c_3 \cos x + c_4 \sin x]$

Example 9

- $\boldsymbol{(D^4 + 8D^2 + 16)y = 0}$
- **Solution** The auxiliary equation is given by
- $D^4 + 8D^2 + 16 = 0$
- $(D^2 + 4)^2 = 0$
- $D = \pm 2i, \quad \pm 2i$
- $y_c = (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x$

Example 10

- $\{(D^2+1)^3(D^2+D+1)^2\}y=0$
- **Solution:** The auxiliary equation is given by $(D^2+1)^3(D^2+D+1)^2=0$
- $D=\pm i,\pm i,\pm i,-\frac{1}{2}\pm i\frac{\sqrt{3}}{2},-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$
- $\therefore y_c=$ $(c_1+c_2x+c_3x^2)\cos x+(c_4+c_5x+c_6x^2)\sin x+$ $e^{-x/2}\left[(c_7+c_8x)\cos\frac{\sqrt{3}}{2}x+(c_9+c_{10}x)\sin\frac{\sqrt{3}}{2}x\right]$