Module 3
Integration :
Review And Some New Techniques

Sub-Module 3.1

Gamma functions with properties

Syllabus

3	**Integration : Review And Some New Techniques**		7	CO 3
	3.1	Beta and Gamma functions with properties		
	3.2	Differentiation under integral sign with constant limits of integration.(without proof)		
		# Self-learning topic: Differentiation under integral sign with variable limits of integration.		

- At the end of the module students can successfully

CO3. Apply concept of Beta – Gamma function and DUIS to solve improper integrals

Gamma Functions

- **Gamma Functions:** The function of $n\ \ (Re(n) > 0)$defined by the integral $\int_0^\infty e^{-x}\, x^{n-1}\ dx$ is called Gamma function and is denoted by $\overline{|n}$
- Thus,

$$\overline{|n} = \int_0^\infty e^{-x}\, x^{n-1}\, dx \qquad\qquad (Re(n) > 0)$$

- $\overline{|n}$ is defined for all n except negative integers (i.e. except $n = -1, -2, -3 \ldots .)$

Properties of Gamma Function

1. $\boxed{1} = 1$
2. $\overline{|n+1} = n\overline{|n}$

Proof : Consider $\overline{|n+1} = \int_0^\infty e^{-x}\, x^n\, dx$

❖ Integrating by Parts,

❖ $\overline{|n+1} = [-x^n e^{-x}] \Big|_0^\infty - \int_0^\infty -e^{-x} n\, x^{n-1}\, dx$

$$= 0 + \int_0^\infty e^{-x} n\, x^{n-1}\, dx$$

$\overline{|n+1} = n\overline{|n} \ldots\ldots\ldots\ldots\ldots\ldots (1)$

Similarly $\overline{|n} = (n-1)\overline{|n-1}$

3. If n is a positive integer by applying (1) repeatedly. We get,

$$\therefore \overline{|n+1} = n\overline{|n} = n(n-1)\overline{|n-1} = n!$$

4. $\boldsymbol{\left|\overline{0}\right.} = \infty$

5. If n is –ve integer, $\left|\overline{n}\right.$ is not defined.

6. Second Form of Gamma Function: (prove!!)

$$\left|\overline{n}\right. = 2\int_0^\infty e^{-x^2}\, x^{2n-1}\, dx$$

Can be used as a definition of Gamma function.

7. $\boldsymbol{\left|\overline{p}\right.\left|\overline{1-p}\right. = \frac{\pi}{\sin p\pi}}$

8. $\boldsymbol{\left|\overline{\tfrac{1}{2}}\right. = \sqrt{\pi}}$

Example:1

- **Evaluate** $\int_0^\infty x^{\frac{1}{4}} e^{-\sqrt{x}} dx$
- **Solution:** Put $\sqrt{x} = t \Rightarrow x = t^2$, ➔ $dx = 2t\,dt$

$$x^{\frac{1}{2}} = t \Rightarrow x^{\frac{1}{4}} = t^{\frac{1}{2}}$$

x	0	∞
t	0	∞

- $I = \int_0^\infty x^{\frac{1}{4}} e^{-\sqrt{x}} dx = \int_0^\infty t^{\frac{1}{2}} e^{-t}\, 2t\,dt$
- $= 2\int_0^\infty t^{\frac{3}{2}} e^{-t}\ dt$
- $= 2\int_0^\infty e^{-t}\, t^{\frac{5}{2}-1}\ dt$
- $= 2\left|\overline{\frac{5}{2}}\right. = 2\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\left|\overline{\frac{1}{2}}\right. = \frac{3}{2}\left|\overline{\frac{1}{2}}\right.$

Example:2

- **Evaluate** $\int_0^\infty e^{-\frac{x^2}{4}}\,dx$

- **Solution:** $\frac{x^2}{4} = t \Rightarrow x^2 = 4t \rightarrow x = (4t)^{\frac{1}{2}} = 2\,t^{\frac{1}{2}} \rightarrow$

  $dx = 2 \cdot \frac{1}{2}\; t^{-\frac{1}{2}}\,dt$

x	0	∞
t	0	∞

- $I = \int_0^\infty e^{-\frac{x^2}{4}}\,dx = \int_0^\infty e^{-t}\,t^{-\frac{1}{2}}\,dt$

- $= \int_0^\infty e^{-t}\,t^{-\frac{1}{2}-1}\,dt$

- $= \left|\overline{\tfrac{1}{2}}\right. = \sqrt{\pi}$

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Example:3

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- **Prove that** $\int_0^\infty \frac{e^{-x^3}}{\sqrt{x}}\ dx \cdot \int_0^\infty y^4\, e^{-y^6} dy = \frac{\pi}{9}$
- **Solution:** Let $I_1 = \int_0^\infty \frac{e^{-x^3}}{\sqrt{x}}\ dx$ and $I_2 = \int_0^\infty y^4\, e^{-y^6}\, dy$
- $\therefore I_1 = \int_0^\infty\ e^{-t}\ t^{-\frac{1}{6}} \frac{1}{3}\ t^{-\frac{2}{3}}\, dt$

Put $x^3 = t \Rightarrow x = t^{\frac{1}{3}}$

- $x^{\frac{1}{2}} = t^{\frac{1}{6}},\quad dx = \frac{1}{3}\ t^{-\frac{2}{3}}\, dt$
- $= \frac{1}{3}\int_0^\infty\ e^{-t} t^{-\frac{5}{6}}\, dt$
- $= \frac{1}{3}\int_0^\infty\ e^{-t}\ t^{\frac{1}{6}-1}\, dt$
- $= \frac{1}{3}\left|\overline{\frac{1}{6}}\right. \ldots\ldots\ldots\ldots\ldots\ldots (1)$

x	0	∞
t	0	∞

- For I_2
- $I_2 = \int_0^\infty t^{\frac{4}{6}} e^{-t} \frac{1}{6} t^{-\frac{5}{6}} \, dt$
- put $y^2 = t, \;\; y = t^{\frac{1}{6}}, \; dy = \frac{1}{6} t^{-\frac{5}{6}} dt$
- $= \frac{1}{6} \int_0^\infty \; e^{-t} \, t^{-\frac{1}{6}} \, dt$
- $= \frac{1}{6} \Gamma\left(\frac{5}{6}\right) \dots\dots\dots\dots\dots\dots\dots\dots (2)$
- $I_1 = I_1 \times I_2$
- $= \frac{1}{3} \Gamma\left(\frac{1}{6}\right) \cdot \frac{1}{6} \Gamma\left(\frac{5}{6}\right) \quad = \frac{1}{18} \Gamma\left(\frac{1}{6}\right) \Gamma\left(\frac{5}{6}\right)$
- $= \frac{1}{18} 2\pi \qquad \left[\because \Gamma\left(\frac{1}{6}\right) \Gamma\left(\frac{5}{6}\right) = 2\pi\right]$
- $= \frac{\pi}{9}$

x	0	∞
t	0	∞

Example:4

- **Evaluate** $\int_0^1 (\log x)^5 dx$
- **Solution:** $\therefore I = \int_0^1 (\log x)^5 dx$

  Put $\log x = -t \Rightarrow x = e^{-t}$

  $dx = -e^{-t}\, dt$

x	0	1
t	∞	0

- $I = \int_\infty^0 (-t)^5 (-e^{-t}) dt$
- $= \int_\infty^0 -t^5 (-e^{-t}) dt$
- $= \int_\infty^0 t^5 e^{-t} dt$
- $= -\int_0^\infty e^{-t}\, t^5 dt$
- $= -\overline{|6}$
- $= -120$

Example:5

- **Evaluate** $\int_0^1 \frac{dx}{\sqrt{x \log\frac{1}{x}}}$

- **Solution :** $I = \int_0^1 \frac{dx}{\sqrt{x \log\frac{1}{x}}}$

x	0	1
t	∞	0

- Put $\log\frac{1}{x} = t, \log 1 - \log x = t$
- $\Rightarrow \log x = -t$ Or $x = e^{-t}$
- $dx = -e^{-t}dt$
- $I = \int_\infty^0 \frac{-e^{-t}dt}{\sqrt{e^{-t}\,t}}\ dt$

t	0	∞
u	0	∞

- $I = \int_0^\infty \frac{e^{-t}}{e^{-\frac{t}{2}} t^{\frac{1}{2}}}\, dt$

  Put $\frac{t}{2} = u \Rightarrow t = 2u,\; dt = 2du$

- $= \int_0^\infty t^{-\frac{1}{2}} e^{-\frac{t}{2}} dt = \int_0^\infty (2u)^{-\frac{1}{2}} e^{-u}\; 2du$

- $= \frac{2}{\sqrt{2}} \int_0^\infty u^{-\frac{1}{2}} e^{-u}\, du$

- $= \sqrt{2} \int_0^\infty u^{\frac{1}{2}-1} e^{-u} du$

- $= \sqrt{2}\, \Gamma\!\left(\frac{1}{2}\right) = \sqrt{2}\sqrt{\pi} = \sqrt{2\pi}$

Example:6

- **Evaluate $\int_0^\infty \frac{x^4}{4^x}\, dx$**
- **Solution:**
- $\therefore I = \int_0^\infty \frac{x^4}{4^x}\, dx$
- Put $4^x = e^t$
- $x \log 4 = t \Rightarrow x = \frac{t}{\log 4}$
- $dt = \log 4 \;\; dx$
- $I = \int_0^\infty \left(\frac{t}{\log 4}\right)^4 \frac{1}{e^t} \frac{1}{\log 4}\, dt$
- $= \frac{1}{(\log 4)^5} \int_0^\infty t^4\, e^{-t}\, dt$
- $= \frac{1}{(\log 4)^5} \overline{|5}$

x	0	∞
t	0	∞

Example:7

- **Evaluate $\int_0^\infty 3^{-4x^2}\,dx$**
- **Solution:**
- $\therefore I = \int_0^\infty 3^{-4x^2}\,dx$
- Put $3^{-4x^2} = e^{-t}$
- $-4x^2 \log 3 = -t$
- $x = \dfrac{\sqrt{t}}{2\sqrt{\log 3}}$
- $\therefore dx = \dfrac{1}{4\sqrt{t}\sqrt{\log 3}}\,dt$

x	0	∞
t	0	∞

- $I = \int_0^\infty e^{-t} \cdot \frac{1}{4\sqrt{\log 3}} \frac{1}{\sqrt{t}}\, dt$
- $= \int_0^\infty \frac{1}{4\sqrt{\log 3}}\, t^{-\frac{1}{2}} e^{-t} dt$
- $= \frac{1}{4\sqrt{\log 3}} \left|\overline{\frac{1}{2}}\right.$
- $= \frac{1}{4\sqrt{\log 3}} \sqrt{\pi}$

- Integrals of the form $\int_0^\infty e^{-ax^n}\,dx$ and $\int_0^\infty x^m\, e^{-ax^n}\,dx$

- we use substitution $ax^n = t$.

- Integrals of the form $\int_0^1 (\log x)^n dx$ and $\int_0^1 x^m\,(\log x)^n dx$

- Put $\log x = -t$ (check limit!!)

- Integrals of the form $\int_0^\infty \frac{x^a}{a^x} dx$ and $\int_0^\infty a^{-kx^2}\,dx$

- Substitution is $a^x = e^t$ and $a^{-kx^2} = e^{-t}$

Practice Problems

1. Evaluate $\int_0^\infty e^{-x^5}\,dx$
2. Evaluate $\int_0^\infty x^n\, e^{-\sqrt{ax}}\;dx$
3. Show that $\int_0^\infty x\, e^{-x^8}\;dx \cdot \int_0^\infty x^2\, e^{-x^4}dx = \frac{\pi}{16\sqrt{2}}$
4. Evaluate $\int_0^\infty \sqrt{x}\, e^{-x^2}\;dx \cdot \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\;dx$
5. Evaluate $\int_0^\infty x^2\, e^{-x^4}\;dx \cdot \int_0^\infty e^{-x^4}\;dx$
6. Evaluate $\int_0^1 \frac{dx}{\sqrt{-\log x}}$
7. Evaluate $\int_0^1 \sqrt{x \log\left(\frac{1}{x}\right)}\;dx$
8. Evaluate $\int_0^\infty 5^{-4x^2}\,dx$
9. Evaluate $\int_0^\infty \frac{x^7}{7^x}\,dx$
10. If $I_n = \dfrac{\frac{\sqrt{\pi}}{2}\left|\overline{\frac{n+1}{2}}\right.}{\left|\overline{\frac{n}{2}+1}\right.}$, show that $I_n + 2 = \frac{n+1}{n+2} I_n$ and hence find I_5

Example:8

- **Show that** $\int_0^{\infty} x^{m-1} \cos ax \; dx = \frac{\lfloor m}{a^m} \cos\left(\frac{m\pi}{2}\right)$
- **Solution :** Since, $e^{-iax} = \cos ax - i\sin ax$,
- $\cos ax = Real\ Part\ of\, e^{-iax}$
- $\therefore I = \int_0^{\infty} x^{m-1} \cos ax \; dx$
- $= \int_0^{\infty} x^{m-1} \left(Real\ Part\ of\, e^{-iax}\right) dx$
- $= Real\ Part\ of\ \int_0^{\infty} x^{m-1} \left(e^{-iax}\right) dx$

Put $iax = \mathrm{t} \therefore dx = \frac{dt}{ia}$

x	0	∞
t	0	∞

- $\therefore I = Real\ Part\ of\ \int_0^{\infty} \left(\frac{t}{ia}\right)^{m-1} e^{-t} \; \frac{dt}{ia}$
- $= Real\ Part\ of\, \frac{1}{(ia)^m} \int_0^{\infty} t^{m-1} e^{-t} dt$

- $I \; = Real\ Part\ of\ \frac{1}{(a)^m}\,\overline{|m}\,\frac{1}{(i)^m}$
- But $i = cos\frac{\pi}{2} + isin\frac{\pi}{2}$
- Therefore $i^m = cos\frac{m\pi}{2} + isin\frac{m\pi}{2}$
- Hence,
- $I \; = Real\ Part\ of\ \frac{1}{(a)^m}\,\overline{|m}\,\{cos\frac{m\pi}{2} - isin\frac{m\pi}{2}\}$
- $= \frac{1}{(a)^m}\,\overline{|m}\,\{cos\frac{m\pi}{2}\}$

Example:9

- **Evaluate $\int_0^\infty \cos a x^{\frac{1}{n}}\, dx$**
- **Hint:** put $a x^{\frac{1}{n}} = t$
- $I = \int_0^\infty \cos t \frac{n}{a^n} t^{n-1}\, dt$
- Write $\cos t = Real\ Part\ of\, e^{-it}$ & proceed as per previous example

Example: 10

- **Show that** $\int_0^\infty x\, e^{-ax} \sin bx \; dx = \frac{2ab}{(a^2+b^2)^2}$

& $\int_0^\infty x\, e^{-ax} \cos bx \; dx = \frac{a^2-b^2}{(a^2+b^2)^2}$

Hint: $e^{ibx} = \cos bx + i \sin bx$

Solve $\int_0^\infty x\, e^{-ax}\, e^{ibx} dx = \int_0^\infty x\, e^{-(a-ib)x}\, dx$

Put $(a - ib)x = t$ & solve further

$$\int_0^\infty x\, e^{-ax} \sin bx \; dx = Imaginary\ part\ of \int_0^\infty x\, e^{-(a-ib)x}\, dx$$

$$\int_0^\infty x\, e^{-ax} \cos bx \; dx = Real\ part\ of \int_0^\infty x\, e^{-(a-ib)x}\, dx$$

Example :11

- **Given $\left|\overline{1.8} = 0.9314\right.$ find the value of $\left|\overline{-2.2}\right.$**
- Solution : we know that $\left|\overline{n+1} = n\left|\overline{n}\right.\right. \therefore \left|\overline{n}\right. = \frac{\left|\overline{n+1}\right.}{n}$ ......(1)
- Put $n = -2.2$
- $\left|\overline{-2.2}\right. = \frac{\left|\overline{-2.2+1}\right.}{-2.2} = \frac{\left|\overline{-1.2}\right.}{-2.2}$

$$= \frac{\left|\overline{-1.2+1}\right.}{(-2.2)(-1.2)} = \frac{\left|\overline{-0.2}\right.}{(-2.2)(-1.2)}$$

$$= \frac{\left|\overline{-0.2+1}\right.}{(-2.2)(-1.2)(-0.2)} = \frac{\left|\overline{0.8}\right.}{(-2.2)(-1.2)(-0.2)}$$

$$= \frac{\left|\overline{0.8+1}\right.}{(-2.2)(-1.2)(-0.2)(0.8)} = \frac{\left|\overline{1.8}\right.}{(-2.2)(-1.2)(-0.2)(0.8)}$$

$$= \frac{0.9314}{(-2.2)(-1.2)(-0.2)(0.8)} = -2.21$$

Example:12

- **For any positive integer n, prove that $\Gamma\left(n+\frac{1}{2}\right)=\frac{1\cdot3\cdot5\cdots(2n-1)}{2^n}\sqrt{\pi}$. Hence prove that $\Gamma\left(n+\frac{1}{2}\right)=\frac{(2n)!}{n!\,4^n}\sqrt{\pi}$**

- Solution: $\Gamma\left(n+\frac{1}{2}\right)=\left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right)$

$$=\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\Gamma\left(n-\frac{3}{2}\right)$$

$$=\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)\Gamma\left(n-\frac{5}{2}\right) \text{ \& so on}$$

$$\Gamma\left(n+\frac{1}{2}\right)=\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)\ldots\frac{3}{2}\,\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)$$

$$=\left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\left(\frac{2n-5}{2}\right)\ldots\frac{3}{2}\,\frac{1}{2}\sqrt{\pi}$$

$$=\frac{1\cdot3\cdot5\cdots(2n-1)}{2^n}\sqrt{\pi}$$

- $\left|\overline{n+\frac{1}{2}}\right. = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}\sqrt{\pi}$

$$= \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots(2n-3)\cdot(2n-2)\cdot(2n-1)\cdot 2n}{2\cdot 4\cdots(2n-2)\cdot 2n\cdot 2^n}\sqrt{\pi}$$

$$= \frac{(2n)!}{2^n n!\, 2^n}\sqrt{\pi}$$

$$= \frac{(2n)!}{n!\, 4^n}\sqrt{\pi}$$