Module 1
Differential Equations
of
First Order and First Degree

1.1 Reducible to Exact DE

Integrating Factor

- Sometimes the given equation is not exact but it becomes exact by multiplication of a suitable factor called the integrating factor.

- **Rule :1**

If the given equation $Mdx + Ndy = 0$ is such that $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ is a function of x only, Then $e^{\int f(x)dx}$ is an integrating factor.

- **Rule :2**

If $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ is a function of y only say $g(y)$. The $e^{\int g(y)dy}$ is an integrating factor.

Or

If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M}$ is a function of y only say $\mathrm{h}(y)$. The $e^{-\int(h(y))dy}$ is an integrating factor.

❖ Rule:3

If the given equation is of the form

$$y\, f_1(xy)dx + x\, f_2(xy)dy = 0$$

where f_1 and f_2 are functions of the product function xy.

Then $\frac{1}{Mx-Ny}$ is an integrating factor where $Mx - Ny \neq 0$.

❖ Rule:4

If the equation $Mdx + Ndy = 0$ is homogeneous and $Mx + Ny \neq 0$ then $\frac{1}{Mx+Ny}$ is an integrating factor.

Example:1

- **Solve $(x^4 + y^4)dx - xy^3dy = 0$**
- **Solution:** This is the form $Mdx + Ndy = 0$
- First check for Exact DE

Here $M = x^4 + y^4$, $N = -xy^3$

$\frac{\partial M}{\partial y} = 4y^3$, $\frac{\partial N}{\partial x} = -y^3$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ Hence, given equation is not exact.

- Integrating factor is required to make it exact.
- Identify which rule is applicable.

- Rule 4
- Rule 3
- Rule 2 or 1

- Here Rule 4 is applicable.

❖ Find Integrating Factor:

The equation is homogenous. ($\left(x^4+y^4\right)dx-xy^3dy=0$)

Also $Mx+Ny=x^5\neq 0$

By rule 4, integrating factor is $\frac{1}{Mx+Ny}$

$$=\frac{1}{x(x^4+y^4)-xy^4}$$

$$=\frac{1}{x^5}$$

❖ Multiply the given equation with Integrating Factor

$$\frac{1}{x^5}\left[\left(x^4+y^4\right)dx-xy^3dy\right]=\frac{1}{x^5}*0$$

$$\therefore\left[\frac{1}{x}+\frac{y^4}{x^5}\right]dx-\frac{y^3}{x^4}=0\ ;\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

This is an exact equation of the form $M'dx+N'dy=0$.

❖ Solve Eq. 2 using method of solving exact DE.

Solution is, $\int M'\,dx + \int N'\,dy = c$

$$\therefore \int \left(\frac{1}{x} + \frac{y^4}{x^5}\right) dx + \int 0\,dy = c$$

$$\therefore \ \log x + y^4 \frac{x^{-4}}{-4} = c$$

$$\therefore \log x - \frac{1}{4x^4} y^4 = c$$

$$\therefore 4x^4 \log x - y^4 = cx^4$$

Working Rule: Reducible to exact

- If given equation is not exact but is of the form $Mdx + Ndy = 0$ then find IF to make it exact & solve.

Steps:

1. First check whether given equation is Exact DE or Non-Exact
2. If not Exact then Identify rule to find IF.
3. Find Integrating Factor using Rule.
4. Multiply the given equation with Integrating Factor
5. New equation obtained ($M'dx + N'dy = 0$) will be exact.
6. Solve using rules of exact DE.

TRY

Example:2

❖ **Solve $(3xy^2 - y^3)dx - (2x^2y - xy^2)dy = 0$**

Solution: This is of the form $Mdx + Ndy = 0$

Here $M = 3xy^2 - y^3$, $N = -2x^2y + xy^2$

$$\frac{\partial M}{\partial y} = 6xy - 3y^2 \qquad \frac{\partial N}{\partial x} = y^2 - 4xy$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, given equation is not exact.

But since the given equation is homogeneous,

$$\text{Integrating Factor} = \frac{1}{Mx + Ny}$$

$$= \frac{1}{x[3xy^2 - y^3] + y[xy^2 - 2x^2y]}$$

$$= \frac{1}{x^2y^2 - xy^3 + xy^3} = \frac{1}{x^2y^2}$$

Now, Multiplying given equation with IF

$\frac{1}{x^2y^2}[3xy^2 - y^3]dx - \frac{1}{x^2y^2}[2x^2y - xy^2]dy = 0$ is an exact equation.

Here $M' = \frac{1}{x^2y^2}[3xy^2 - y^3] = \frac{3}{x} - \frac{y}{x^2}$

$N' = -\frac{1}{x^2y^2}[2x^2y - xy^2] = -\frac{2}{y} + \frac{1}{x}$

Solution is of the form, $\int M'\,dx + \int (terms\ of\ N'\, free\ from\ x)\,dy = c$

$$\therefore \int \left(\frac{3}{x} - \frac{y}{x^2}\right)dx + \int -\frac{2}{y}dy = c$$

$$\therefore\ 3\log x - y\left(\frac{x^{-1}}{-1}\right) - 2\log y = c$$

$$\therefore\ 3\log x + \frac{y}{x} - 2\log y = c$$

$$\therefore\ \log x^3 - \log y^2 + \frac{y}{x} = c$$

$$\therefore\ \log\frac{x^3}{y^2} = c - \frac{y}{x}$$

$$\therefore\ \frac{x^3}{y^2} = e^{c-\frac{y}{x}}$$

Example:3

- **Solve $y(1+xy)dx + x(1-xy)dy = 0$**

- **Solution:** Here $M = y(1+xy)$ $\quad N = x(1-xy)$

$$\frac{\partial M}{\partial y} = 1 + 2xy, \qquad \frac{\partial N}{\partial x} = 1 - 2xy$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ Hence, given equation is not exact.

- This is of the form $y\, f_1(xy)dx + x\, f_2(xy)dy = 0$

$$Mx - Ny = xy(1+xy) - yx(1-xy)$$
$$= xy(1 + xy - 1 + xy)$$
$$= 2x^2y^2 \neq 0$$

$\therefore \frac{1}{Mx - Ny} = \frac{1}{2x^2y^2}$ is the integrating factor.

- Hence $\frac{1}{2x^2y^2}y(1+xy)dx+\frac{1}{2x^2y^2}x(1-xy)dy=0$ is an exact equation.

$$M'=\frac{1}{2x^2y^2}y(1+xy) \qquad N'=\frac{1}{2x^2y^2}x(1-xy)$$

$$=\frac{1}{2x^2y}+\frac{1}{2x} \qquad =\frac{1}{2xy^2}-\frac{1}{2y}$$

- Solution is
- $\int M'\,dx+\int N'\,dy=c$
- $\int\left(\frac{1}{2x^2y^2}+\frac{1}{2x}\right)dx\int\left(-\frac{1}{2y}\right)dy=c$
- $\frac{1}{2y}\left(\frac{x^{-1}}{-1}\right)+\frac{1}{2}\log x-\frac{1}{2}\log y=c$
- $-\frac{1}{2xy}+\frac{1}{2}\log x-\frac{1}{2}\log y=c$
- Solution is $-\frac{1}{xy}+\log\left(\frac{x}{y}\right)=c$

Example:4

- Solve $\frac{dy}{dx} = -\frac{x^2y^3+2y}{2x-2x^3y^2}$
- **Solution:** $(x^2y^3 + 2y)dx + (2x - 2x^3y^2)dy = 0$

$M = y(2 + x^2y^2)$ $\qquad$ $N = x(2 - 2x^2y^2)$

$\frac{\partial M}{\partial y} = 2 + 2x^2y,$ $\qquad$ $\frac{\partial N}{\partial x} = 2 - 4xy^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ Hence, given equation is not exact.

- $y(2 + x^2y^2)dx + x(2 - 2x^2y^2)dy = 0$

This is of the form $y\, f_1(xy)dx + x\, f_2(xy)dy = 0$

$Mx - Ny = xy(2 + x^2y^2) - xy(2 - 2x^2y^2)$

$= xy(2 + x^2y^2 - 2 + 2x^2y^2\ )$

$= 3x^3y^3 \neq 0$

$\therefore \frac{1}{Mx-Ny} = \frac{1}{3x^3y^3}$ is an integrating factor.

- $\frac{1}{3x^3y^3}y(2+x^2y^2)dx + \frac{1}{3x^3y^3}x(2-2x^2y^2)dy = 0$

is an exact equation.

$$\left(\frac{2}{3x^3y^2}+\frac{1}{3x}\right)dx + \left(\frac{2}{3x^2y^3}-\frac{2}{3y}\right)dy = 0$$

- Solution is,
- $\int M'\,dx + \int \text{Term in } N' \text{ free from x} \quad dy = c$
- $\int \left(\frac{2}{3x^3y^2}+\frac{1}{3x}\right)dx + \int -\frac{2}{3y}dy = c$
- $\frac{2}{3y^2}\left(\frac{x^{-2}}{-2}\right)+\frac{1}{3}\log x - \frac{2}{3}\log y = c$
- $\frac{-1}{3x^2y^2}+\frac{1}{3}\log x - \frac{2}{3}\log y = c$
- Solution is $\frac{1}{3}\log\frac{x}{y^2} - \frac{1}{3x^2y^2} = c$

Example:5

- **Solve $(4xy+3y^2-x)dx+x(x+2y)dy=0$**
- **Solution:** Here $M=4xy+3y^2-x$, $N=x(x+2y)$

$\frac{\partial M}{\partial y}=4x+6y$, $\frac{\partial N}{\partial x}=2x+2y$

$\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ Hence, given equation is not exact.

- $\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=2x+4y\,;$

divisible by N so that we get pure function of x.

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{2x+4y}{x(x+2y)}=\frac{2}{x}\left(\frac{x+2y}{x+2y}\right)=\frac{2}{x}=f(x)$$

- Integrating factor $=e^{\int f(x)dx}$

$$=e^{\int\frac{2}{x}dx}=e^{2\log x}=e^{\log x^2}=x^2$$

- $x^2[4xy + 3y^2 - x]dx + x^2[x(x+2y)]dy = 0$ is an exact equation.

$M' = x^2[4xy + 3y^2 - x]$
$N' = x^3(x + 2y)$

- Solution is,

- $\int M'\,dx + \int \text{Term in}'N''\text{free from x}\;\; dy = c$
- $\int x^2[4xy + 3y^2 - x]dx + \int 0\,dy = c$
- $y\dfrac{4x^4}{4} + y^2 \cdot \dfrac{3x^3}{3} - \dfrac{x^4}{4} = c$
- Solution is $x^4y + x^3y^2 - \dfrac{x^4}{4} = c$

Example:6

- **Solve $\left(x^4e^x - 2mxy^2\right)dx + 2mx^2y\,dy = 0$**
- **Solution:** Here $M = x^4e^x - 2mxy^2, \qquad N = 2mx^2y$
- $\dfrac{\partial M}{\partial y} = -4mxy \qquad \dfrac{\partial N}{\partial x} = 4mxy$

$\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$ Hence, given equation is not exact.

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{-8mxy}{2mx^2y} = -\frac{4}{x} = f(x)$$

Integrating factor $= e^{\int -\frac{4}{x}} = e^{-4\log x} = e^{\log x^{-4}} = x^{-4} = \dfrac{1}{x^4}$

Now $\dfrac{1}{x^4}(x^4e^x - 2mxy^2)dx + \dfrac{1}{x^4}(2mx^2y)\,dy = 0$is an exact equation.

$$M' = e^{x} - \frac{2my^2}{x^3}, \qquad N' = \frac{2my}{x^2}$$

Solution is of the form,

$$\int M'\,dx + \int \text{Term in}'N''\text{free from x}\,dy = c$$

$'y'$constant

$$\int \left(e^{x} - \frac{2my^2}{x^3}\right)dx + \int 0\,dy = c$$

$$e^{x} - 2my^2\left(\frac{x^{-2}}{-2}\right) = c$$

Solution is $e^{x} + \frac{my^2}{x^2} = c$

Example:7

- **Solve** $\left(3x^2y^4 + 2xy\right)dx + \left(2x^3y^3 - x^2\right) dy = 0$
- **Solution:** Here $M = 3x^2y^4 + 2xy, \quad N = 2x^3y^3 - x^2$

$$\frac{\partial M}{\partial y} = 3x^2(4y^3) + 2x = 12x^2y^3 + 2x$$

$$\frac{\partial N}{\partial x} = 6x^2y^3 - 2x$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ Hence, given equation is not exact.

$$-\left[\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M}\right] = -\left[\frac{6x^2y^3+4x}{3x^2y^3+2xy}\right] = -\frac{2}{y}\frac{(3x^2y^3+2x)}{(3x^2y^3+2x)} = -\frac{2}{y}$$

$$I.F. = e^{\int -\frac{2}{y}dy} = e^{-2\log y} = e^{\log y^{-2}} = \frac{1}{y^2}$$

Now $\frac{1}{y^2}(3x^2y^4 + 2xy)dx + \frac{1}{y^2}(2x^3y^3 - x^2)\, dy = 0$ is an exact equation.

$$M' = 3x^2y^2 + \frac{2x}{y}$$

$$N' = 2x^3y - \frac{x^2}{y^2}$$

Solution is of the form,

$$\int M'\,dx + \int \text{Term in } 'N'' \text{ free from x } dy = c$$

$$\int (3x^2y^2 + \frac{2x}{y})_{\,dx} + \int 0\,dy = c$$

$$3y^2\frac{x^3}{3} + \frac{2}{y}\,\frac{x^2}{2} = c$$

$$x^3y^2 + \frac{x^2}{y} = c$$

Solution is $x^3y^3 + x^2 = cy$

❖ Solve

8) $\left(y^4 + 2y\right)dx + \left(xy^3 + 2y^4 - 4x\right)dy = 0$

9) $\left[y\left(1 + \frac{1}{x}\right) + \cos y\right]dx + (x + \log x - x\sin y)dy = 0$

10) $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$

11) $y(x + y)dx - x(y - x)dy = 0$

12) $y(\sin xy + xy\cos xy)dx + x(xy\cos xy - \sin xy)dy$

13) $(2x\log x - xy)dy + 2y\,dx = 0$

14) $\left(2xy^4e^y + 2xy^3 + y\right)dx + \left(x^2y^4e^y - x^2y^2 - 3x\right)dy = 0$

Example:8

- **Solve $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$**
- **Solution:** Here $M = (y^4 + 2y)$, $N = (xy^3 + 2y^4 - 4x)$
- $\frac{\partial M}{\partial y} = 4y^3 + 2$ $\frac{\partial N}{\partial x} = y^3 - 4$
- $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ Hence, given equation is not exact.
- $-\left[\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M}\right] = \frac{y^3 - 4 - 4y^3 - 2}{y^4 + 2y} = \frac{-3y^3 - 6}{y^4 + 2y}$
- $= \frac{3(y^3+2)}{y(y^3+2)} = -\frac{3}{y}$
- Integrating factor $e^{\int -\frac{3}{y}dy} = e^{-3\log y} = e^{\log y} = \frac{1}{y^3}$
- Now $\frac{1}{y^3}(y^4 + 2y)dx + \frac{1}{y^3}(xy^3 + 2y^4 - 4x)dy = 0$ is an exact function.

- $M' = \frac{1}{y^3}(y^4 + 2y) = y + \frac{2}{y^2}$
- $N' = \frac{1}{y^3}(xy^3 + 2y^4 - 4x) = x + 2y - \frac{4x}{y^3}$
- Solution is of the form,
- $\int M'\,dx + \int N'\,dy = c$
- $'y'$constant Term in $'N''$ free from x
- $\int \left(y + \frac{2}{y^2}\right)_{dx} + \int 2y\,dy = c$
- $x\left(y + \frac{2}{y^2}\right) + \frac{2y^2}{2} = c$
- Solution is $xy + \frac{2x}{y^2} + y^2 = c$

Example:9

- **Solve $\left[y\left(1+\frac{1}{x}\right)+\cos y\right]dx+(x+\log x-x\sin y)dy=0$**
- **Solution:** This is the form, $Mdx+Ndy=0$
- Where $M=y\left(1+\frac{1}{x}\right)+\cos y$, $\quad N=x+\log x-x\sin y$
- $\frac{\partial M}{\partial y}=1+\frac{1}{x}-\sin y \qquad \frac{\partial N}{\partial x}=1+\frac{1}{x}-\sin y$
- Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ the given equation is exact.
- Solution is of the form
- $\int M\,dx+\int N\,dy=c$
- $'y'$ constant $\qquad$ Term in $'N'$ free from x
- $\int y\left(1+\frac{1}{x}\right)+\cos y\ dx+\int 0\,dy=c$
- $y[x+\log x]+x\cos y=c$

Example:10

- **Solve** $\frac{dy}{dx}+\frac{y\cos x+\sin y+y}{\sin x+x\cos y+x}=0$
- **Solution:** $(y\cos x+\sin y+y)dx+(\sin x+\ x\cos y+x)dy=0$
- This is the form of $Mdx+Ndy=0$
- Here $M\ = y\cos x+\sin y+y,\qquad N=\sin x+\ x\cos y+x$
- $\frac{\partial M}{\partial y}=\cos x+\cos y+1\ ,\quad \frac{\partial N}{\partial x}=\cos x+\cos y+1$
- Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ the given equation is exact.
- Solution is of the form,
- $\int M\ dx+\int N\ dy=c$
- $'y'$constant $\qquad\qquad$ Term in $'N'$ free from x
- $\int(y\cos x+\sin y+y)dx+\int 0\ \ dy=c$
- $y\sin x+x(\sin y+y)=c$
- $y\sin x+x\sin y+xy=c$

Example:11

- **Solve $y(x+y)dx - x(y-x)dy = 0$**
- **Solution:** This is of the form $Mdx + Ndy = 0$
- Here $M = y(x+y)$, $\quad N = -x(y-x)$
- $\frac{\partial M}{\partial y} = x + 2y$, $\quad \frac{\partial N}{\partial x} = -y + 2x$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ Hence, given equation is not exact.

- Since the equation is homogeneous we use the integrating factor.
- $I.F. = \frac{1}{Mx+Ny} = \frac{1}{xy(x+y)-xy(y-x)}$
- $= \frac{1}{xy(x+y-y+x)}$
- $= \frac{1}{2x^2y}$
- Now $\frac{1}{2x^2y}y(x+y)dx - \frac{1}{2x^2y}x(y-x)dy = 0$ is an exact equation.

- $M' = \frac{1}{2x^2y}y(x+y) = \frac{yx+y^2}{2x^2y} = \frac{1}{2x} + \frac{y}{2x^2}$
- $N' = -\frac{1}{2x^2y}(xy - x^2) = -\frac{1}{2x} + \frac{1}{2y}$
- Solution is of the form,

$$\int M'\,dx + \int N'\,dy = c$$

- $'y'$constant Term in N' free from x
- $\int\left(\frac{1}{2x} + \frac{y}{2x^2}\right)dx + \int\left(-\frac{1}{2} + \frac{1}{2y}\right)dy = c$
- $\frac{1}{2}\log x + \frac{y}{2}\left(\frac{x^{-1}}{-1}\right) + \frac{1}{2}\log y = c$
- $\frac{1}{2}\log x - \frac{y}{2x} + \frac{1}{2}\log y = c$
- $\frac{1}{2}\log xy = c + \frac{y}{2x}$
- Solution is $\log\sqrt{xy} = \frac{y}{2x} + c$

Example:12

- **Solve $y(\sin xy + xy\cos xy)dx + x(xy\cos xy - \sin xy)dy$**
- **Solution:** This is of the form $y\, f_1(xy)dx + x\, f_2(xy)dy = 0$
- Where $M = y\sin(xy) + xy^2\cos(xy)$

$$\frac{\partial M}{\partial y} = xy\cos xy + \sin(xy) + x[y^2(-\sin xy)x + \cos(xy)\,(2y)]$$
$$= xy\cos xy + \sin(xy) - x^2y^2\sin(xy) + 2xy\cos(xy)$$
$$= 3\,xy\cos(xy) + (1 - x^2y^2)\sin(xy)$$
$$N = -x\sin(xy) + x^2y\cos(xy)$$
$$\frac{\partial N}{\partial x} = -yx\cos xy - \sin(xy) + y[x^2(-\sin xy)y + \cos(xy)\,(2x)]$$
$$= -xy\cos(xy) - \sin(xy) - x^2y^2\sin(xy) + 2xy\cos(xy)$$
$$= xy\cos(xy) + (-1 - x^2y^2)\sin(xy)$$

- $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ Hence, given equation is not exact.

$Mx - Ny \; = xy(\sin xy + xy\cos xy) - xy(xy\cos xy - \sin xy)$

$= xy(\sin xy + xy\cos xy - xy\cos xy + \sin xy) = 2xy\sin xy$

$\frac{1}{Mx-Ny} = \frac{1}{2\,xy\sin xy}$ is an integrating factor

$\frac{1}{2\,xy\sin xy}y(\sin xy + xy\cos xy)dx + \frac{1}{2\,xy\sin xy}x(xy\cos xy - \sin xy) = 0$ is an exact equation.

$M' = \frac{1}{2\,xy\sin xy}y(\sin xy + xy\cos xy) = \frac{1}{2x} + \frac{y}{2}\cot xy$

$N' = \frac{1}{2\,xy\sin xy}x(xy\cos xy - \sin xy) = \frac{x}{2}\cot xy - \frac{1}{2y}$

Solution is of the form,

$$\int M'\,dx + \int \text{Term in}'N'\text{free from x}\,dy = c$$

$$\int \frac{1}{2x} + \frac{y}{2}\cot(xy)\,dx + \int -\frac{1}{2y}\,dy = c$$

$$\frac{1}{2}\log x + \frac{1}{2}\log(\sin xy) - \frac{1}{2}\log y = \log c$$

$$\frac{1}{2}\log x + \frac{1}{2}\log(\sin xy) - \frac{1}{2}\log y \ = \log c$$

$$\frac{1}{2}\log\frac{x\sin yx}{y} = \log c$$

$$\rightarrow \frac{x}{y}\sin xy = c$$

Example:13

- **Solve $(2x\log x - xy)dy + 2y\,dx = 0$**
- **Solution:** Here $M = 2y,\qquad N = 2x\log x - xy$

$$\frac{\partial M}{\partial y} = 2 \qquad \frac{\partial N}{\partial x} = 2\left[x \times \frac{1}{x} + \log x\right] - y$$

$$= 2(1 + \log x) - y = 2 + 2\log x - y$$

- $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ Hence, given equation is not exact.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2-2-2\log x+y}{2x\log x - xy} = \frac{y-2\log x}{x(2\log x - y)} = -\frac{1}{x} = f(x)$$

- Integrating factor $= e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = \frac{1}{x}$

Now $\frac{1}{x}(2x\log x - xy)dy + \frac{1}{x}(2y)\,dx = 0$is an exact equation.

$$N' = \frac{1}{x}(2x\log x - xy) = 2\log x - y\,,$$

$$M' = \frac{1}{x}2y = \frac{2y}{x}$$

Solution is of the form,

$$\int M'\,dx + \int \text{Term in } 'N'' \text{ free from x}\; dy = c$$

$$\int \left(\frac{2y}{x}\right)_{dx} + \int -y\,dy = c$$

Solution is $2y\log x - \frac{y^2}{2} = c$

Example:14

- **Solve** $\left(2xy^4e^y + 2xy^3 + y\right)dx + \left(x^2y^4e^y - x^2y^2 - 3x\right)dy = 0$
- **Solution:** Here

$$M = 2xy^4e^y + 2xy^3 + y \qquad N = x^2y^4e^y - x^2y^2 - 3x$$

$$\frac{\partial M}{\partial y} = 2x[y^4e^y + e^y4y^3] + 2x[3y^2] + 1 \qquad \frac{\partial N}{\partial x} = 2xy^4e^y - 2xy^2 - 3$$

$$= 2xy^4e^y - 8xy^3e^y + 6xy^2 + 1$$

- $$\frac{-\frac{\partial M}{\partial y}+\frac{\partial N}{\partial x}}{M} = \frac{-2xy^2-3-8xy^3e^y-6xy^2-1}{2xy^4e^y+2xy^3+y}$$

$$= \frac{-8xy^2-8xy^3e^y-4}{2xy^4e^y+2xy^3+y}$$

$$= \frac{-4\left(2xy^2+2xy^3e^y+1\right)}{y(2xy^3e^y+2xy^2+1\,)}$$

$$= \frac{-4}{y}$$

- Integrating factor $= e^{\int -\frac{4}{y}dy} = e^{-4\log y} = e^{\log y^{-4}} = \frac{1}{y^4}$

$\therefore \frac{1}{y^4}(2xy^4e^y + 2xy^3 + y)dx + \frac{1}{y^4}(x^2y^4e^y - x^2y^2 - 3x)dy = 0$ is an exact equation.

$$\left(2xe^y + \frac{2x}{y} + \frac{1}{y^3}\right)dx + \left(x^2e^y - \frac{x^2}{y^2} - \frac{3x}{y^4}\right)dy = 0$$

$M' = 2xe^y + \frac{2x}{y} + \frac{1}{y^3}$ $\qquad N' = x^2e^y - \frac{x^2}{y^2} - \frac{3x}{y^4}$

Solution is of the form,

$$\int M'\,dx + \int terms\ of\ N' free\ from\ x\ dy = c$$

$$\int \left(2xe^y + \frac{2x}{y} + \frac{1}{y^3}\right)_{dx} + \int 0\,dy = c$$

Solution is $x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = c$