Module 1
Differential Equations
of
First Order and First Degree

Sub-module 1.2

Reducible Linear Differential Equations

Syllabus

Module No.	Unit No.	Details	Hrs.	CO
1	**Differential Equation of First Order and First Degree**		13	CO 1
	1.1	Differential Equation of first order and first degree- Exact differential equations, Equations reducible to exact equations by integrating factors.		
	1.2	Linear differential equations (Review), Equation reducible to linear form. Applications of Differential Equation of first order and first degree		
	1.3	Linear Differential Equation with constant coefficients: Complimentary function, particular integrals of differential equation of the type f(D)y=X, where X is e^{ax}, $\sin(ax+b)$, $\cos(ax+b)$, x^n, $e^{ax}V$		
	1.4	Cauchy's homogeneous linear differential equation		
	1.5	Method of variation of parameters		
		# **Self-learning topic:** Bernoulli's equation. Equation reducible to Bernoulli's equation.		

Reducible to Linear DE

- The equation of the type $f'(y)\frac{dy}{dx}+Pf(y)=Q$ where $P\ and\ Q$ are the functions of x can be reduced to linear form by substituting

$$f(y)=u$$
$$f'(y)\ \frac{dy}{dx}=\frac{du}{dx}$$

$\therefore \frac{du}{dx}+Pu=Q$ is linear

- The equation of the type $f'(x)\frac{dx}{dy}+Pf(x)=Q$ where P and Q are the functions of y and can be reduced to linear form by substituting

$$f(x)=u$$
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$\therefore \frac{du}{dy}+Pu=Q$ is linear.

Example - 1

- **Solve $3y^2\frac{dy}{dx}+2y^3x=4x^3e^{x^2}$**
- **Solution:** Put $y^3=u$
- $3y^2\frac{dy}{dx}=\frac{du}{dx}$
- $\therefore \frac{du}{dx}+2xu=4x^3e^{x^2}$
- Here $P=2x,\quad Q=4x^3e^{x^2}$
- $I.F.=e^{\int Pdx}=e^{\int 2x\,dx}=e^{x^2}$
- G.S. is given by $u(I.F.)=\int Q(I.F.)dx+c$
- $u\left(e^{x^2}\right)=\int 4x^3e^{x^2}\cdot e^{x^2}dx+c$
- Put $x^2=t,\ 2x\,dx=dt$
- $ue^{x^2}=\int 2t\,e^{2t}dt+c$
- $=2\left[t\,\frac{e^{2t}}{2}-\int\frac{e^{2t}}{2}dt\right]+c$
- $=2\left[\frac{t\,e^{2t}}{2}-\frac{e^{2t}}{4}\right]+c$
- $y^3e^{x^2}=2\left[\frac{x^2e^{2x^2}}{2}-\frac{e^{2x^2}}{4}\right]+c$
- $=2\left[\frac{2x^2e^{2x^2}-e^{2x^2}}{4}\right]+c$
- $\therefore y^3e^{x^2}=\frac{e^{2x^2}(2x^2-1)}{2}+c$ is the required G.S.

Example - 2

- **Solve $y\frac{dx}{dy}=x+yx^2\log y$**
- **Solution:** Dividing throughout by y, we get
- $\frac{dx}{dy}=\frac{x}{y}+x^2\log y$
- $\frac{dx}{dy}-\frac{x}{y}=x^2\log y$
- Dividing throughout x^2, we get
- $\frac{1}{x^2}\frac{dx}{dy}-\frac{1}{x}\cdot\frac{1}{y}=\log y$
- Put $-\frac{1}{x}=u$
- $\frac{1}{x^2}\frac{dx}{dy}=\frac{du}{dy}$
- $\therefore \frac{du}{dy}+\frac{1}{y}u=\log y$ which is in linear form.
- Here $P=\frac{1}{y}$, $\quad Q=\log y$
- $I.F.=e^{\int P dy}=e^{\int\frac{1}{y}dy}=e^{\log y}=y$
- G.S. is given by $u(I.F.)=\int Q(I.F.)dy+c$
- $uy=\int y\log y\,dy+c$
- $uy=\log y\frac{y^2}{2}-\int\frac{y^2}{2}\cdot\frac{1}{y}+c$
- $=\log y\frac{y^2}{2}-\frac{y^2}{4}+c$
- $-\frac{y}{x}=\log y\frac{y^2}{2}-\frac{y^2}{4}+c$
- $\therefore \frac{y}{x}+\frac{y^2}{2}\log y-\frac{y^2}{4}=c$ is the required G.S.

Example - 3

- **Solve**

$$y\sin x\,\frac{dx}{dy}-\cos x=2y^3\cos^2 x$$

- **Solution:** Dividing Throughout by $y\,cos^2 x$, we get
- $\tan x\sec x\frac{dx}{dy}-\frac{\sec x}{y}=2y^2$
- Put $\sec x=u$
- $\sec x\tan x\frac{dx}{dy}=\frac{du}{dy}$
- $\frac{du}{dy}-\frac{u}{y}=2y^2$ which is linear in u.
- Here $P=-\frac{1}{y}\quad Q=2y^2$
- $I.F.=e^{\int P dy}=e^{-\int\frac{1}{y}dy}=e^{-\log y}=\frac{1}{y}$
- G.S. is given by $u(I.F.)=\int Q(I.F.)dy+c$
- $\frac{u}{y}=\int 2y^2\cdot\frac{1}{y}dy+c$
- $\frac{u}{y}=y^2+c$
- $\therefore\ \frac{\sec x}{y}=y^2+c$
- $\therefore\ \sec x=y^3+cy$ is the required G.S.

Example - 4

- **Solve:** $\frac{dy}{dx} - 1 = xe^{-y}$
- **Solution:** Dividing Throughout by e^{-y}, we get
- $e^{y}\frac{dy}{dx} - e^{y} = x$
- Put $e^{y} = u$
- $e^{y}\frac{dy}{dx} = \frac{du}{dx}$
- $\frac{du}{dx} - u = x$ which is a linear D.E.
- Here $P = -1 \quad Q = x$
- $I.F. = e^{\int P dx} = e^{-\int 1\, dx} = e^{-x}$
- G.S. is given by $u(I.F.) = \int Q(I.F.)dx + c$
- $ue^{-x} = \int xe^{-x}dx + c$
- $ue^{-x} = -xe^{-x} + -e^{-x} + c$
- $e^{y}e^{-x} = -xe^{-x} + -e^{-x} + c$is the required G.S.
- HW. **Solve:** $x\frac{dy}{dx} - 1 = xe^{-y}$

Example - 5

- **Solve**

$$\frac{dy}{dx} + \frac{1}{x}\tan y = \frac{1}{x^2}\tan y \sin y$$

- **Solution:** Dividing throughout by $\tan y \sin y$, we get
- $\cot y \; cosec\, y \frac{dy}{dx} + cosec\, y \cdot \frac{1}{x} = \frac{1}{x^2}$
- Put $cosec\, y = u$
- $-cosec\, y \cot y \frac{dy}{dx} = \frac{du}{dx}$
- $\frac{-du}{dx} + \frac{1}{x} u = \frac{1}{x^2}$
- $\frac{du}{dx} - \frac{1}{x} u = -\frac{1}{x^2}$ is linear D. E.
- Here $P = -\frac{1}{x} \quad Q = -\frac{1}{x^2}$
- $I.F. = e^{\int P dx} = e^{-\int \frac{1}{x} dx} = e^{-\log x} = \frac{1}{x}$
- G.S. is given by $u(I.F.) = \int Q(I.F.)dx + c$
- $u \cdot \frac{1}{x} = -\int \frac{1}{x^2} \cdot \frac{1}{x} dx + c$
- $\frac{u}{x} = -\int \frac{1}{x^3} dx + c$
- $\frac{cosec\, y}{x} = \frac{1}{2x^2} + c$
- $\therefore 2x\, cosec\, y = 1 + c\, 2x^2$ is the required G.S.
- HW: **Solve:**

$$\sec y \frac{dy}{dx} + 2\,x \sin y = 2\,x \cos y$$

Example - 6

- **HW: Solve $e^x(x+1)dx+(y^2e^{2y}-xe^x)dy=0$**
- **Solution:** Given $e^x(x+1)\frac{dx}{dy}-xe^x=-y^2e^{2y}$
- Put $xe^x=u$
- $(xe^x+e^x)\frac{dx}{dy}=\frac{du}{dy}$
- $e^x(x+1)\frac{dx}{dy}=\frac{du}{dy}$
- $\frac{du}{dy}-u=-y^2e^{2y}$ which is linear.
- Here $P=-1,\ \ Q=-y^2e^{2y}$
- $I.F.=e^{\int Pdy}=e^{-\int dy}=e^{-y}$
- G.S. is given by $u(I.F.)=\int Q(I.F.)dy+c$
- $u\,e^{-y}=-\int y^2e^{2y}e^{-y}dy+c$
- $=-\int y^2e^ydy+c$
- $=-\left[y^2e^y-\int e^y\cdot 2y\,dy\right]+c$
- $=-y^2e^y-2\left[e^yy-\int e^ydy\right]+c$
- $=-[y^2e^y-2ye^y-2e^y]+c$
- $x\,e^x\,e^{-y}=-y^2e^y-2ye^y-2e^y+c$ is the required G.S.

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Problem - 7

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- **Solve:** $\boldsymbol{e^x(x+1)dx + (y^2e^{2y} - xe^x)dy = 0}$
- **Solution:** Given

$$e^x(x+1)\frac{dx}{dy} - xe^x = -y^2e^{2y}$$

- Put $xe^x = u$
- $(xe^x + e^x)\frac{dx}{dy} = \frac{du}{dy}$
- $e^x(x+1)\frac{dx}{dy} = \frac{du}{dy}$
- $\frac{du}{dy} - u = -y^2e^{2y}$ which is linear.
- Here $P = -1, \quad Q = -y^2e^{2y}$
- $I.F. = e^{\int P dy} = e^{-\int dy} = e^{-y}$
- G.S. is given by

$$u(I.F.) = \int Q(I.F.)dy + c$$

- $u\, e^{-y} = -\int y^2e^{2y}e^{-y}dy + c$
- $= -\int y^2e^ydy + c$
- $= -\left[y^2e^y - \int e^y \cdot 2y\, dy\right] + c$
- $= -y^2e^y - 2\left[e^yy - \int e^ydy\right] + c$
- $= -[y^2e^y - 2ye^y - 2e^y] + c$
- $x\, e^x\, e^{-y} = -y^2e^y - 2ye^y - 2e^y + c$ is the required G.S.

Example - 8

- **Solve** $\frac{dy}{dx} = e^{x-y}(e^x - e^y)$
- **Solution:** Given $\frac{dy}{dx} = \frac{e^x}{e^y}(e^x - e^y)$
- $e^y \frac{dy}{dx} = e^{2x} - e^x e^y$
- $e^y \frac{dy}{dx} + e^x e^y = e^{2x}$
- Put $e^y = u$
- $e^y \frac{dy}{dx} = \frac{du}{dx}$
- $\therefore \frac{du}{dx} + e^x u = e^{2x}$ is linear in u.
- Here $P = e^x \quad Q = e^{2x}$
- $I.F. = e^{\int P dx} = e^{\int e^x dx} = e^{e^x}$
- G.S. is given by $u(I.F.) = \int Q(I.F.)dx + c$
- $ue^{e^x} = \int e^{2x} e^{e^x} dx + c$
- Put $e^x = t$
- $e^x dx = dt$
- $ue^{e^x} = \int te^t dt + c$
- $= t\,e^t - e^t + c$
- $= e^t(t-1) + c$
- $\therefore e^y e^{e^x} = e^{e^x}(e^x - 1) + c$
- $e^y = e^x - 1 + c\,e^{-e^x}$ is the G.S.
- Hw. **Solve** $\frac{dx}{dy} = e^{y-x}(e^y - e^x)$

Practice Problems

- **Solve the following.**
- $x\cos y\dfrac{dy}{dx}-\sin y=x\sin^2 y$

Answer: $[cosec\ y+x(\log x+c)=0]$

- $\sec^2 y\dfrac{dy}{dx}+2\tan x\tan y=\sin x$

Answer: $[\sec^2 x\tan y=\sec x+c]$

- $y\ \dfrac{dy}{dx}+\dfrac{4x}{3}-\dfrac{y^2}{3x}=0$ Answer: $\left[y^2x^{-\frac{2}{3}}+2x^{\frac{4}{3}}=c\right]$
- $\dfrac{dy}{dx}+x\sin 2y=x^3\cos^2 y$ Answer: $\left[\tan y=\frac{1}{2}(x^2-1)+ce^{-x^2}\right]$
- $\dfrac{dy}{dx}+(2x\tan^{-1}y-x^3)(1+y^2)=0$

  Answer: $\left[\tan^{-1}y=\dfrac{x^2-1}{2}+ce^{-x^2}\right]$
- $\dfrac{dy}{dx}+x^3\sin^2 y+x\sin 2y=x^3$

  Answer: $\left[\tan y\cdot e^{x^2}=\frac{1}{2}e^{x^2}(x^2-1)+c\right]$

Bernoulli's Equation:

- An equation of the form $\frac{dy}{dx} + Py = Qy^n$ is called Bernoulli's equation. Here P and Q are the function of x alone or constants an n is a real number. The above equation can be made linear by dividing throughout by y^n.

Example - 9

- **Solve:** $\frac{dy}{dx} = x^3y^3 - xy$
- **Solution:** Dividing throughout by y^3, we get
- $\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{y^2}x = x^3$
  $(Bernoulii's\ Equation)$
- Put $\frac{1}{y^2} = v$
- $\frac{-2}{y^3}\ \frac{dy}{dx} = \frac{dv}{dx}$
- $\frac{1}{y^3}\ \frac{dy}{dx} = -\frac{1}{2}\ \frac{dv}{dx}$
- $-\frac{1}{2}\ \frac{dv}{dx} + xv = x^3$
- $\therefore \frac{dv}{dx} - 2xv = -2x^3$
  $[Linear\ in\ v]$
- Here $P = -2x$, $Q = -2x^3$
- $I.F. = e^{\int P dx} = e^{-2\int x\, dx} = e^{-x^2}$
- G.S. is given by $v(I.F.) = \int Q(I.F.)dx + c$
- $ve^{-x^2} = -2\int x^3e^{-x^2}dx + c$
- Put $-x^2 = t$
- $-2x\, dx = dt$
- $= -\int t\, e^t dt + c$
- $ve^{-x^2} = -e^t(t-1) + c$
- $\therefore \frac{1}{y^2}e^{-x^2} = -e^{-x^2}(-x^2 - 1) + c$
- $\therefore \frac{1}{y^2}e^{-x^2} = e^{-x^2}(x^2 + 1) + c$
  is the required G.S.

Example - 10

- **Solve** $\frac{dz}{dx}+\frac{z}{x}\log z=\frac{z}{x^2}(\log z)^3$
- **Solution:** Dividing throughout by z, we get
- $\frac{1}{z}\frac{dz}{dx}+\log z\cdot\frac{1}{x}=\frac{(\log z)^3}{x^2}$
- Put $\log z=v$
- $\frac{1}{z}\frac{dz}{dx}=\frac{dv}{dx}$
- $\therefore \frac{dv}{dx}+\frac{1}{x}v=\frac{v^3}{x^2}$
- Dividing throughout by v^3, we get
- $\frac{1}{v^3}\frac{dv}{dx}+\frac{1}{x}\frac{1}{v^2}=\frac{1}{x^2}$
- Put $\frac{1}{v^2}=u$
- $-\frac{2}{v^3}\frac{dv}{dx}=\frac{du}{dx}$
- $\frac{1}{v^3}\frac{dv}{dx}=-\frac{1}{2}\frac{du}{dx}$
- $-\frac{1}{2}\frac{du}{dx}+\frac{1}{x}u=\frac{1}{x^2}$
- $\frac{du}{dx}-\frac{2}{x}u=-\frac{2}{x^2}$ $[Linear\ D.E.]$
- Here $P=-\frac{2}{x}$ $Q=-\frac{2}{x^2}$
- $I.F.=e^{\int Pdx}$

  $=e^{-\int\frac{2}{x}dx}$

  $=e^{-2\log x}=\frac{1}{x^2}$
- G.S. is given by $v(I.F.)=\int Q(I.F.)dx+c$
- $u\cdot\frac{1}{x^2}=\int-\frac{2}{x^2}\cdot\frac{1}{x^2}dx+c$
- $\frac{1}{v^2}\cdot\frac{1}{x^2}=-2\int x^{-4}dx+c$
- $\frac{1}{(\log z)^2}\cdot\frac{1}{x^2}=-2\left[\frac{x^{-3}}{-3}\right]+c$
- $\frac{1}{(\log z)^2}\cdot\frac{1}{x^2}=\frac{2+c}{3x^3}$ is the required G. S.

Example - 11

- **Solve:**

$$\frac{dy}{dx} = 1 - x(y-x) - x^3(y-x)^2$$

- **Solution:** Put $y - x = v$
- $\frac{dy}{dx} - 1 = \frac{dv}{dx}$
- Given equation reduces to $\frac{dv}{dx} + xv = -x^3v^2$
- $\frac{-1}{v^2}\frac{dv}{dx} - \frac{x}{v} = x^3$
- Put $\frac{1}{v} = t$
- $-\frac{1}{v^2}\frac{dv}{dx} = \frac{dt}{dx}$
- $\frac{dt}{dx} - xt = x^3$
- Here $P = -x, \quad Q = x^3$
- $I.F. = e^{\int P dx} = e^{-\int x\, dx} = e^{\frac{-x^2}{2}}$
- G.S. is given by $t(I.F.) = \int Q(I.F.)dx + c$
- Put $-\frac{x^2}{2} = u$, i.e. $x^2 = -2u$

$$x\, dx = -du$$

- G.S. is given by $t\, e^{-\frac{x^2}{2}} = \int x^3 e^{-\frac{x^2}{2}}\, dx + c \;\therefore t\, e^{-\frac{x^2}{2}} = \int 2u\, e^u\, du + c$
- $= 2\, e^{-\frac{x^2}{2}}\left(-\frac{x^2}{2} - 1\right) + c$
- $t = 2\left(-\frac{x^2}{2} - 1\right) c\, e^{\frac{x^2}{2}}$
- Re-substituting the value of t and u, we get

$$\frac{1}{v} = 2\left(-\frac{x^2}{2} - 1\right) + c\, e^{\frac{x^2}{2}}$$

$\therefore \frac{1}{y-x} = -x^2 - 2 + c\, e^{\frac{x^2}{2}}$ is the required G.S.

Practice Problems

- **Solve the following**
- $\frac{dy}{dx}(x^2y^3+xy)=1$
  Answer: $\left[x(2-y^2)+\left(c\,x\,e^{-\frac{y^2}{2}}\right)=1\right]$
- $\frac{dy}{dx}+y\tan x=y^2\sec x$ Answer: $\left[\frac{1}{y}\cos x=-x+c\right]$
- $\frac{dy}{dx}+y\cos x=y^3\sin 2x$ Answer: $\left[\frac{1}{y^2}=(1+2\sin x)+c\,e^{2\sin x}\right]$
- $\frac{dy}{dx}-xy=y^2e^{-\left(\frac{x^2}{2}\right)}\cdot\log x$ Answer: $\left[\frac{1}{y}e^{\frac{x^2}{2}}=x(1-\log x)+c\right]$
- $x\frac{dy}{dx}+y=x^3y^c$ Answer: $\frac{1}{y^5}=\frac{5}{2}x^3+cx^5$
- $(1-x^2)\frac{dy}{dx}+xy=y^3\sin^{-1}x$ Answer: $\left[-2\left[x\sin^{-1}x+\sqrt{1-x^2}\right]+c\right]$