Module 1
Differential Equations
of
First Order and First Degree

Sub-module 1.2

Linear Differential Equations

Syllabus

Module No.	Unit No.	Details	Hrs.	CO
1	**Differential Equation of First Order and First Degree**		13	CO 1
	1.1	Differential Equation of first order and first degree- Exact differential equations, Equations reducible to exact equations by integrating factors.		
	1.2	Linear differential equations (Review), Equation reducible to linear form. Applications of Differential Equation of first order and first degree		
	1.3	Linear Differential Equation with constant coefficients: Complimentary function, particular integrals of differential equation of the type f(D)y=X, where X is e^{ax}, $\sin(ax+b)$, $\cos(ax+b)$, x^n, $e^{ax}V$		
	1.4	Cauchy's homogeneous linear differential equation		
	1.5	Method of variation of parameters		
		# **Self-learning topic:** Bernoulli's equation. Equation reducible to Bernoulli's equation.		

Linear Differential Equation

- A differential equation is said to be linear if the dependent variable ‘y’ and its derivatives occur only in the first degree.
- In other words, a D. E. is said to be linear if

- Every term of its dependent variable and its derivatives occur with no degree higher than first.

- In no term any two derivatives or a dependent variable and a derivative are multiplied together.
- The dependent variable and its derivatives does not appear either in radical sign or in the denominator.

ie. The terms like $\sqrt{1+\left(\frac{dy}{dx}\right)^2}$ and $\frac{1}{1+2\frac{dy}{dx}}$ should not be present in D.E.

Linear DE of First order First Degree

❖ If y is the dependent variable and x is the independent variable, then the D. E. of the form

$$\frac{dy}{dx} + Py = Q$$

is called a linear D.E. of the first order where P, Q are functions of x only or constants.

Working Rule:

- Check for more than two terms
- Observe Dependent variable appearing once with degree 1 and co-officient of derivative term should be 1.
- $I.F. = e^{\int P dx}$
- Solution is :

(Dependent variable)$(\boldsymbol{I.F.}) = \int \boldsymbol{RHS(I.F.)dx + c}$

Example 1

- **Solve $\frac{dy}{dx}\cos hx = 2\cos h^2 x \sin hx - y \sin hx$**
- **Solution:** Given $\frac{dy}{dx}\cos hx = 2\cos h^2 x \sin hx - y \sin hx$
- Dividing throughout by $\cos hx$, we get
- $\frac{dy}{dx} + \frac{\sin hx}{\cos hx} y = 2 \sin hx \cos hx$
- The equation is in the linear form $\frac{dy}{dx} + Py = Q$
- Here $P = \frac{\sin hx}{\cos hx}, \qquad Q = 2 \sin hx \cos x$
- $I.F. = e^{\int \frac{\sin hx}{\cos hx} dx} = e^{\log \cos hx} = \cos hx$
- G.S. is given by $y \cos hx = \int 2 \sin hx \cos h^2 x \;\, dx + c$
- Put $\cos hx = t, \quad \sin hx\, dx = dt$
- $\therefore y \cos hx = \int 2\, t^2\, dt = \frac{2}{3} t^3 + c$
- $\therefore y \cos hx = \frac{2}{3} \cos h^3 x + c$ is the required G.S.

Example 2

- **Solve** $\frac{dy}{dx} + 2y\tan x = \sin x \ \ at\ y = 0, x = \frac{\pi}{3}$ (HW)
- **Solution:** The given equation is in the linear form $\frac{dy}{dx} + Py = Q$
- Here $P = 2\tan x\,,\ \ Q = \sin x$
- $I.F. = e^{\int Pdx} = e^{2\int \tan x\, dx} = e^{2\log \sec x}$
- $= e^{\log sec^2 x} = sec^2 x$
- G. S. is given by $y(I.F.) = \int Q(I.F.)dx + c$
- $y\, sec^2 x = \int \sin x\,(sec^2 x)dx + c$

- $= \int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} dx + c$
- $= \int \tan x \sec x \ dx + c \quad \left[\because \int \sec x \tan x \, dx = \sec x\right]$
- $= \sec x + c$
- $\therefore y\, sec^2 x = \sec x + c \ldots\ldots\ldots\ldots\ldots..(i)$
- Put $x = \frac{\pi}{3}, y = 0$ in (i), we get
- $0 = \sec\frac{\pi}{3} + c, \ \therefore c = -2$
- Substituting $c = -2$ in (i), we get
- $y\, sec^2 x = \sec x - 2$
- $\therefore y = \cos x - 2cos^2 x$ is the required G.S.

Example 3

- **4. Solve $x \log x \frac{dy}{dx} + y = 2 \log x$**
- **Solution:** Dividing throughout by $x \log x$, we get
- $\frac{dy}{dx} + \frac{1}{x \log x} y = \frac{2}{x}$
- The equation is in the form of $\frac{dy}{dx} + Py = Q$
- Here $P = \frac{1}{x \log x}, \ Q = \frac{2}{x}$
- $I.F. = e^{\int Pdx} = e^{\int \frac{1}{x \log x} dx} = e^{\log \log x} = \log x$
- G.S. is given by $y(I.F.) = \int Q(I.F.)dx + c$
- $y \log x = \int \frac{2}{x} \log x \ dx + c$
- $= (\log x)^2 + c$
- $\therefore y \log x = (\log x)^2 + c$ is the required G.S.

Example 4

- **Solve $\left(1+y^2\right)dx = \left(tan^{-1}y - x\right)dy$**

  Solution: Dividing throughout by $1+y^2$, we get

- $dx = \frac{tan^{-1}y - x}{1+y^2}dy$
- $\frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{tan^{-1}y}{1+y^2}$
- The above equation is in the form of $\frac{dx}{dy} + Px = Q$
- $I.F. = e^{\int P dy}$
- Here $P = \frac{1}{1+y^2}, \quad Q = \frac{tan^{-1}y}{1+y^2}$
- $I.F. = e^{\int \frac{1}{1+y^2}dy} = e^{tan^{-1}y}$

- G. S. is given by $x(I.F.) = \int Q(I.F.)dy + c$
- $x\, e^{tan^{-1}y} = \int \frac{tan^{-1}y}{1+y^2}\, e^{tan^{-1}y} dy + c$
- Put $tan^{-1}y = t$
- $\frac{1}{1+y^2} dy = dt$
- $x\, e^{tan^{-1}y} = \int t\, e^t dt + c$
- $= t\, e^t - e^t + c$
- $\therefore x\, e^{tan^{-1}y} = e^t(t-1) + c$
- $= e^{tan^{-1}y}(tan^{-1}y - 1) + c$
- $\therefore x = tan^{-1}y - 1 + c\, e^{-tan^{-1}y}$ is the required G. S.

Example 5

- Solve $\frac{dy}{dx} + \frac{y \log y}{x - \log y} = 0$
- **Solution:**
  $\frac{dy}{dx} = -\frac{y \log y}{x - \log y}$
- $\frac{dx}{dy} = \frac{x - \log y}{-y \log y}$
- $\frac{dx}{dy} + \frac{1}{y \log y} x = \frac{1}{y}$
- The above equation is linear in x
- Here $P = \frac{1}{y \log y}, \quad Q = \frac{1}{y}$
- $I.F. = e^{\int P dy}$
- $I.F. = e^{\int \frac{1}{y \log y} dy} = e^{\log \log y} = \log y$
- G. S. is given by $x(I.F.) = \int Q(I.F.) dy + c$
- $x \log y = \int \frac{1}{y} \log y \ dy + c$
- $\therefore x \log y = \frac{(\log y)^2}{2} + c$ is the required G. S.

Example 6

- **Solve $dr + (2r\cot\theta + \sin 2\theta)\,d\theta = 0$**
- **Solution:** Given $\frac{dr}{d\theta} + 2\cot\theta\, r = -\sin 2\,\theta$
- Here $P = 2\cot\theta$, $Q = -\sin 2\,\theta$
- $I.F. = e^{\int P d\theta} = e^{\int 2\cot\theta\, d\theta} = e^{2\log\sin\theta} = sin^2\theta$
- G.S. is given by r$(I.F.) = \int Q(I.F.)d\theta + c$
- $r\sin^2\theta = -\int \sin 2\theta\ \sin^2\theta\, d\theta + c$
- $= -\int 2\, sin^3\theta\cos\theta\, d\theta + c$
- Put $\sin\theta = t$
- $\cos\theta\, d\theta = dt$
- $= -\int 2\, t^3 dt + c$
- $= -\frac{2}{4}t^4 + c$
- $= -\frac{1}{2}\sin^4\theta + c$
- $\therefore r\, sin^2\theta = -\frac{1}{2}\sin^4\theta + c$ is the required G. S.

Example 7

- **Solve $x(x-1)\frac{dy}{dx}-(x-2)y=x^3(2x-1)$**
- **Solution:** Given $\frac{dy}{dx}-\frac{x-2}{x(x-1)}y=\frac{x^2(2x-1)}{x-1}$ which is linear in y.
- $I.F.=e^{\int P dx}$
- Here $P=\frac{x-2}{x(x-1)}=\frac{2}{x}-\frac{1}{x-1}$, $\quad Q=\frac{x^2(2x-1)}{x-1}$
- $e^{-\int P dx}=e^{-\int\left(\frac{2}{x}-\frac{1}{x-1}\right)dx}=e^{-\log\left(\frac{x^2}{x-1}\right)}=e^{\log\left(\frac{x-1}{x^2}\right)}=\frac{x-1}{x^2}$
- G.S. is given by $y(I.F.)=\int Q(I.F.)dx+c$
- $y\left(\frac{x-1}{x^2}\right)=\int\frac{x^2(2x-1)}{x-1}\cdot\frac{x-1}{x^2}dx+c$
- $y\left(\frac{x-1}{x^2}\right)=\int(2x-1)dx+c$
- $=x^2-x+c$
- $\therefore y=x^3+\frac{c\,x^2}{x-1}$ is the required G.S.

Example 8

- **Solve $(x^2-1)\sin x\frac{dy}{dx}+[2x\sin x+(x^2-1)\cos x]y=(x^2-1)\cos x$**
- **Solution**: Dividing throughout by $(x^2-1)\sin x$, we get
- $\frac{dy}{dx}+\left(\frac{2x}{x^2-1}+\frac{\cos x}{\sin x}\right)y=\frac{\cos x}{\sin x}$
- $\therefore\ \frac{dy}{dx}+\left(\frac{2x}{x^2-1}+\cot x\right)y=\cot x$
- The above equation is in the linear form $\frac{dy}{dx}+Py=Q$
- Here $P=\frac{2x}{x^2-1}+\cot x,\quad Q=\cot x$
- $I.F.=e^{\int P dx}=e^{\int\left(\frac{2x}{x^2-1}+\cot x\right)dx}=e^{\log(x^2-1)+\log\sin x}=e^{\log\{(x^2-1)\sin x\}}$
  $=(x^2-1)\sin x$
- G.S. is given by $y(I.F.)=\int Q(I.F.)dx+c$
- $y(x^2-1)\sin x=\int\cot x(x^2-1)\sin x\ dx$
- $=\int\cos x\,(x^2-1)dx+c$
- $=(x^2-1)\sin x-2x(-\cos x)+2(-\sin x)+c$
- $\therefore y(x^2-1)\sin x=(x^2-1)\sin x+2x\cos x-2\sin x+c$ is the required G.S.

Example 9

- **Solve $(x+y+1)\frac{dy}{dx}=1$**
- **Solution:**
- $\frac{dy}{dx}=\frac{1}{x+y+1}$
- $\therefore \frac{dx}{dy}=x+y+1$
- $\frac{dx}{dy}-x=y+1$
- The above equation is in the form of $\frac{dx}{dy}+Px=Q$
- Here $P=-1,\ Q=y+1$
- $I.F.=e^{\int Pdy}=e^{-\int dy}=e^{-y}$
- G. S. is given by $x(I.F.)=\int Q(I.F.)dy+c$
- $xe^{-y}=\int(y+1)e^{-y}dy+c$
- $\quad\quad =-(y+1)^{-y}e^{-y}+c$
- $x=-(y+1)-1+c\,e^{y}$
- $x=-y-2+c\,e^{y}$
- $\therefore x+y+2=ce^{y}$ is the required G. S.

Example 10

- **Solve $(x + 2y^3)dy = y\,dx$**
- **Solution:** Given $y\dfrac{dx}{dy} = x + 2y^3$
- $\dfrac{dx}{dy} = \dfrac{x}{y} + 2y^2$
- $\therefore\ \dfrac{dx}{dy} - \dfrac{1}{y}x = 2y^2$
- The above equation is in the form of $\dfrac{dx}{dy} + Px = Q$
- Here $P = -\dfrac{1}{y},\ \ Q = 2y^2$
- $I.F. = e^{\int P dy} = e^{-\int \frac{1}{y} dy} = e^{-\log y} = \dfrac{1}{y}$
- G.S. is given by $x(I.F.) = \int Q(I.F.)dy + c$
- $x\left(\dfrac{1}{y}\right) = \int 2y^2 \cdot \dfrac{1}{y} dy + c$
- $\dfrac{x}{y} = y^2 + c$
- $\therefore x = y^3 + cy$ is the required G. S.

Example 11

- **Solve $(1+\sin y)\frac{dx}{dy} = [2y\cos y - x(\sec y + \tan y)]$**
- **Solution:** Given Equation can be written as
- $\frac{dx}{dy} = \frac{2y\cos y}{1+\sin y} - \frac{x(\sec y+\tan y)}{1+\sin y}$
- $\frac{dx}{dy} + \left(\frac{\sec y+\tan y}{1+\sin y}\right)x = \frac{2y\cos y}{1+\sin y}$
- $\frac{dx}{dy} + \sec y\, x = \frac{2y\cos y}{1+\sin y}$
- The above equation is in the form of $\frac{dx}{dy} + Px = Q$
- Here $P = \sec y$, $Q = \frac{2y\cos y}{1+\sin y}$
- $I.F. = e^{\int P dy}$
- $= e^{\int \sec y\, dy} = e^{\log(\sec y+\tan y)} = \sec y + \tan y$
- G.S. is given by $x(I.F.) = \int Q(I.F.)dy + c$
- $x(\sec y + \tan y) = \int \frac{2y\cos y}{1+\sin y}(\sec y + \tan y)dy + c$
- $= \int 2y\, dy + c$
- $\therefore x(\sec y + \tan y) = y^2 + c$ is the required G.S.