Module 3
Integration :
Review And Some New Techniques

Sub-Module 3.1

Beta functions with properties

Example:1

- **Evaluate** $\int_0^1 \sqrt{1-\sqrt{x}}\,dx \cdot \int_0^{\frac{1}{2}} \sqrt{2y-4y^2}\;dy$

- **Solution:** Consider $I_1 = \int_0^1 \sqrt{1-\sqrt{x}}\,dx$

Put $\sqrt{x} = t \quad \therefore x = t^2 \rightarrow dx = 2t\,dt$

x	0	1
t	0	1

$$I_1 = \int_0^1 \sqrt{1-t}\,2t\,dt$$

$$= 2\int_0^1 t^1 \,(1-t)^{\frac{1}{2}}\,dt$$

$$= 2\int_0^1 t^{2-1}\,(1-t)^{\frac{3}{2}-1}\,dt$$

$$= 2\beta\left(2, \frac{3}{2}\right)$$

Consider $I_2 = \int_0^{\frac{1}{2}} \sqrt{2y - 4y^2}\ dy$

$$= \int_0^{\frac{1}{2}} \sqrt{2y(1-2y)}\ dy$$

Put $2y = t \therefore 2\,dy = dt \rightarrow dy = \frac{dt}{2}$

y	0	1/2
t	0	1

$$I_2 = \int_0^1 \sqrt{t}\,(1-t)^{\frac{1}{2}} \frac{dt}{2}$$

$$= \frac{1}{2}\int_0^1 t^{\frac{3}{2}-1}\,(1-t)^{\frac{3}{2}-1}\,dt$$

$$= \frac{1}{2}\,\beta\left(\frac{3}{2},\frac{3}{2}\right)$$

$$I = I_1 \cdot I_2 = 2\beta\left(2, \frac{3}{2}\right) \frac{1}{2}\, \beta\left(\frac{3}{2}, \frac{3}{2}\right)$$

$$= \frac{\Gamma(2)\,\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{7}{2}\right)}\;\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(3)}$$

$$= \frac{1 \times \frac{1}{2}\sqrt{\pi}\,\frac{1}{2}\sqrt{\pi}\,\frac{1}{2}\sqrt{\pi}}{\frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi} \times 2}$$

$$= \frac{\frac{\pi}{8}}{\frac{15}{4}} \quad = \frac{\pi}{8} \times \frac{4}{15} = \frac{\pi}{30}$$

Example:2

- **Evaluate** $\int_0^{2a} x^2\sqrt{2ax - x^2}\,dx$
- **Solution:** $I = \int_0^{2a} x^2\sqrt{x(2a-x)}\,dx$

$$= \int_0^{2a} x^{2+\frac{1}{2}}\,(2a-x)^{\frac{1}{2}}\,dx$$

Put $x = 2at,\ \ dx = 2a\,dt$

x	0	$2a$
t	0	1

$$I = \int_0^{2a} (2at)^{\frac{5}{2}}\,(2a-2at)^{\frac{1}{2}}\,2a\,dt$$

$$= \int_0^{1} (2at)^{\frac{5}{2}}\,(2a)^{\frac{1}{2}}\,(1-t)^{\frac{1}{2}}\,2a\,dt = \int_0^{1} (2a)^{\frac{5}{2}+\frac{1}{2}+1}\,t^{\frac{5}{2}}\,(1-t)^{\frac{1}{2}}\,dt$$

$$= 16a^4 \int_0^{1} t^{\frac{7}{2}-1}\,(1-t)^{\frac{3}{2}-1}\,dt$$

$$= 16a^4\,\beta\left(\frac{7}{2},\ \frac{3}{2}\right)$$

$$I = 16a^4\,\beta\left(\frac{7}{2},\ \frac{3}{2}\right)$$

$$= 16a^4\,\frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(5)}$$

$$= \frac{16a^4 \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi} \times \frac{1}{2} \times \sqrt{\pi}}{4 \times 3 \times 2}$$

$$= \frac{16a^4 \times 15\pi}{16 \times 8 \times 3}$$

$$= \frac{5\pi a^4}{8}$$

Example:3

- **Prove that** $\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{2^{\frac{2}{n}-1}\left(\left|\overline{\frac{1}{n}}\right.\right)^2}{n\left|\overline{\frac{2}{n}}\right.}$
- **Solution:** Put $x^n = t$
- $x = t^{\frac{1}{n}}$
- $dx = \frac{1}{n}\, t^{\frac{1}{n}-1}\, dt$
- $I = \int_0^1 \frac{1}{n} \frac{t^{\frac{1}{n}-1}\, dt}{\sqrt{1-t}}$
- $= \frac{1}{n}\int_{t=0}^1 t^{\frac{1}{n}-1}(1-t)^{\frac{1}{2}-1}\, dt$
- $= \frac{1}{n}\beta\left(\frac{1}{n},\frac{1}{2}\right)$

x	0	1
t	0	1

- $I = \frac{1}{n}\beta\left(\frac{1}{n}, \frac{1}{2}\right)$
- $= \frac{1}{n}\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}$
- $= \frac{1}{n}\frac{\Gamma\left(\frac{1}{n}\right)\sqrt{\pi}}{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}$ ……………………(1)
- By duplication formula, $2^{2m-1}\Gamma(m)\,\Gamma(m+1/2) = \sqrt{\pi}\,\Gamma(2m)$
- Put $m = \frac{1}{n}$
- $2^{\frac{2}{n}-1}\,\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{n}+1/2\right) = \sqrt{\pi}\,\Gamma\left(\frac{2}{n}\right)$

- $$2^{\frac{2}{n}-1}\,\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)=\sqrt{\pi}\,\Gamma\left(\frac{2}{n}\right)$$

- $$\frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}=\frac{2^{\frac{2}{n}-1}\,\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{2}{n}\right)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

- Using (2) in (1)

- $$I=\frac{1}{n}\Gamma\left(\frac{1}{n}\right)\frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}$$

- $$=\frac{1}{n}\Gamma\left(\frac{1}{n}\right)\frac{2^{\frac{2}{n}-1}\,\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{2}{n}\right)}=\frac{2^{\frac{2}{n}-1}\left(\Gamma\left(\frac{1}{n}\right)\right)^{2}}{n\,\Gamma\left(\frac{2}{n}\right)}$$

Example 4

- **Prove That $\int_0^a x^4 (a^2 - x^2)^{\frac{1}{2}} dx = \frac{a^6\pi}{32}$**
- **Solution:** Put $x^2 = a^2 t$

x	0	a
t	0	1

- $x = a\, t^{\frac{1}{2}}, \quad dx = \frac{a}{2}\, t^{\frac{1}{2}-1}\, dt$
- $I = \int_{t=0}^{1} a^4 t^2 (a^2 - a^2 t)^{\frac{1}{2}} \frac{a}{2} t^{\frac{1}{2}-1}\, dt$
- $= a^4 \times a \cdot \frac{a}{2} \int_{t=0}^{1} t^2 (1-t)^{\frac{1}{2}} t^{\frac{1}{2}-1}\, dt$
- $= \frac{a^6}{2} \int_{t=0}^{1} t^{\frac{5}{2}-1} (1-t)^{\frac{3}{2}-1}\, dt$
- $= \frac{a^6}{2} \beta\left(\frac{5}{2}, \frac{3}{2}\right) = \frac{a^6}{2} \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(4)}$
- $= \frac{a^6}{2} \times \frac{\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}\times\frac{1}{2}\times\sqrt{\pi}}{3.2.1} = \frac{a^6\pi}{32}$

Example:5

- **Evaluate** $\int_0^1 \frac{x^7}{\sqrt{1-x^2}} dx$
- **Solution:** Put $x = \sin\theta \therefore dx = \cos\theta \ d\theta$

x	0	1
θ	0	$\frac{\pi}{2}$

- $I = \int_0^{\frac{\pi}{2}} \frac{\sin^7\theta}{\sqrt{1-\sin^2\theta}} \cos\theta \ d\theta$
- $= \int_0^{\frac{\pi}{2}} \sin^7\theta \ d\theta$
- $= \int_0^{\frac{\pi}{2}} \sin^7\theta \cos^0\theta \ d\theta = \frac{1}{2}\beta\left(\frac{7+1}{2}, \frac{0+1}{2}\right)$
- $= \frac{1}{2}\beta\left(4, \frac{1}{2}\right)$
- $= \frac{1}{2} \times \frac{\left|\overline{4}\right.\left|\overline{\frac{1}{2}}\right.}{\left|\overline{\frac{9}{2}}\right.}$
- $= \frac{1}{2} \times \frac{3\times2\times\sqrt{\pi}}{\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}} = \frac{16}{35}$

Example 6

- **Evaluate** $\int_0^1 \frac{x^2(4-x^4)}{\sqrt{1-x^2}} dx$
- **Solution:**
- $I = \int_0^1 \frac{4x^2}{\sqrt{1-x^2}} dx - \int_0^1 \frac{x^6}{\sqrt{1-x^2}} dx$
- Put $x = \sin\theta \ \therefore dx = \cos\theta \, d\theta$

x	0	1
θ	0	$\frac{\pi}{2}$

- $I = \int_0^{\frac{\pi}{2}} \frac{4\sin^2\theta}{\cos\theta} \cos\theta \, d\theta - \int_0^{\frac{\pi}{2}} \frac{\sin^6\theta}{\cos\theta} \cos\theta \, d\theta$
- $= 4\int_0^{\frac{\pi}{2}} \sin^2\theta \cos^0\theta \ d\theta - \int_0^{\frac{\pi}{2}} \sin^6\theta \cos^0\theta \ d\theta$
- $= 4 \times \frac{1}{2}\beta\left(\frac{2+1}{2}, \frac{0+1}{2}\right) - \frac{1}{2}\beta\left(\frac{6+1}{2}, \frac{0+1}{2}\right)$

- $I = 2\beta\left(\frac{3}{2},\frac{1}{2}\right) - \frac{1}{2}\beta\left(\frac{7}{2},\frac{1}{2}\right)$
- $= 2\frac{\left|\overline{\frac{3}{2}}\right|\overline{\frac{1}{2}}}{\left|\overline{2}\right.} - \frac{1}{2} \times \frac{\left|\overline{\frac{7}{2}}\right|\overline{\frac{1}{2}}}{\left|\overline{4}\right.}$
- $= 2 \times \frac{1}{2} \times \sqrt{\pi} \times \sqrt{\pi} \; - \frac{\frac{1}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}\times\sqrt{\pi}}{3\times2}$
- $= \pi - \frac{5\pi}{32}$
- $= \frac{27\pi}{32}$

Example:7

- **Evaluate** $\int_0^{\pi} \frac{\sin^4\theta}{(1+\cos\theta)^2}\, d\theta$
- **Solution:**
- $I = \int_0^{\pi} \frac{\sin^4\theta}{\left(2\cos^2\frac{\theta}{2}\right)^2}\, d\theta$
- $= \frac{1}{4}\int_0^{\pi} \sin^4\theta\ \cos^{-4}\frac{\theta}{2}\, d\theta$
- $= \frac{1}{4}\int_0^{\pi} [\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)]^4\ \cos^{-4}\left(\frac{\theta}{2}\right)\, d\theta$
- Put $\frac{\theta}{2} = t \Rightarrow \theta = 2t$
- $d\theta = 2dt$

x	0	1
θ	0	$\frac{\pi}{2}$

- $I = \frac{1}{2}\int_0^{\frac{\pi}{2}} (2\sin t\cos t)^4 \ \cos^{-4} t\, dt$
- $= 8\int_0^{\frac{\pi}{2}} \sin^4 t \ \cos^0 t\, dt$
- $= 8 \times \frac{1}{2}\beta\left(\frac{4+1}{2}, \frac{0+1}{2}\right)$
- $= 4 \times \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(3)}$
- $= 4 \times \frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi} \times \sqrt{\pi} \times \frac{1}{2}$
- $= \frac{3\pi}{2}$