Module 2
Successive Differentiation, Expansion Of Functions, Indeterminate Forms

2.1 Successive differentiation

Module 2

2	**Successive Differentiation, Expansion Of Functions, Indeterminate Forms**	
	2.1	Successive differentiation: nth derivative of standard functions. Leibnitz's Theorem (without proof) and problems.
	2.2	Taylor's Theorem (only statement) and Taylor's series, Maclaurin's series(only Statement) Expansion of e^x, $sinx, cosx, tanx$
		#Self-learning topic: Expansion of sinh(x), cosh(x), tanh(x), log(1 + x) , Indeterminate forms, L'Hospital Rule, problemsinvolving series

Successive Differentiation

- If $y = f(x)$then
  - first derivative of $f(x)$ is denoted by $f'(x) or\ y_1$
  - Second derivative of $f(x)$ is denoted by $f''(x) or\ y_2$
  - Similarly nth derivative of $f(x)$ is denoted by $f^{(n)}(x) or\ y_n$
- nth Derivative of $f(x)$ at $x = a$ is denoted by

$$f^{(n)}(a) = f^n(a) = \left[\frac{d^n y}{dx^n}\right] \text{at } x = a$$

- If $y = x^m$ *then*

$y_1 = mx^{m-1}$

$y_2 = m(m-1)x^{m-2}$

$y_3 = m(n-1)(m-2)x^{m-3}$

$y_n = m(m-1)(m-2)\dots(m-n)x^{m-n} \quad where\; n < m$

$= m(m-1)(m-2)\dots 3.2.1\, x^{m-m} = m! \;\; if\; n = m$

$= 0 \quad where\; n > m$

- If $y = (ax+b)^m$ then

$y_n = m(m-1)(m-2)\dots(m-n+1)a^n(ax+b)^{m-n}\;; if\;\; n < m$

$= m!\, a^n$; if $m = n$

$= 0$; if $n > m$

Remember

Sr. No	$y = f(x)$	y_n
1	$(ax+b)^m$	$= m(m-1)(m-2)\cdots(m-n+1)a^n (ax+b)^{m-n}$ if $n<m$
2	$(ax+b)^{-m}$	$= (-1)^n m\,(m+1)(m+2)\cdots(m+n-1)a^n (ax+b)^{-m-n}$ $= (-1)^n \cdot \dfrac{(m+n-1)!}{(m-1)!} \cdot \dfrac{a^n}{(ax+b)^{m+n}}$
3	x^m	$= m(m-1)(m-2)\cdots(m-n+1)x^{m-n}$ if $n<m$
4	1/ x^m	$= (-1)^n \cdot \dfrac{(m+n-1)!}{(m-1)!} \cdot \dfrac{1}{(x)^{m+n}}$

Remember

Sr. No	$y = f(x)$	y_n
4	$\dfrac{1}{(ax+b)}$	$= \dfrac{(-1)^n \cdot n! a^n}{(ax+b)^{n+1}}$
5	$\log(ax+b)$	$= \dfrac{(-1)^{n-1} \cdot (n-1)! a^n}{(ax+b)^n}$
6	a^{mx}	$= m^n a^{mx} (\log a)^n$
7	e^{mx}	$= m^n e^{mx}$

Remember

Sr. No	$y = f(x)$	y_n
8	$\sin(ax+b)$	$= a^n \sin\left(ax+b+\frac{n\pi}{2}\right)$
9	$\cos(ax+b)$	$= a^n \cos\left(ax+b+\frac{n\pi}{2}\right)$
10	$e^{ax}\sin(bx+c)$	$= r^n e^{ax}\sin(bx+c+n\phi)$ where $r=\sqrt{a^2+b^2}$ & $\phi=\tan^{-1}\left(\frac{b}{a}\right)$
11	$e^{ax}\cos(bx+c)$	$= r^n e^{ax}\cos(bx+c+n\phi)$ where $r=\sqrt{a^2+b^2}$ & $\phi=\tan^{-1}\left(\frac{b}{a}\right)$

Remember

Sr. No	$y = f(x)$	y_n
12	$k^x \sin(bx+c)$	$= r^n k^x \sin(bx+c+n\phi)$ where $r = \sqrt{(\log k)^2 + b^2}$ & $\phi = \tan^{-1}\left(\dfrac{b}{\log k}\right)$
13	$k^x \cos(bx+c)$	$= r^n k^x \cos(bx+c+n\phi)$ where $r = \sqrt{(\log k)^2 + b^2}$ & $\phi = \tan^{-1}\left(\dfrac{b}{\log k}\right)$

Example:1

- If $y = \frac{1}{1+x+x^2+x^3}$ then find y_n
- **Solution**

Example:2

- If $y = \frac{8x}{x^3-2x^2-4x+8}$ then find y_n
- **Solution**

If $v = \dfrac{1}{(x-2)^2}$, $\quad v_n = \dfrac{(-1)^n (n+1)!(1)^n}{(x-2)^{n+2}}$

If $w = \dfrac{1}{x+2}$, $\quad w_n = \dfrac{(-1)^n n!(1)^n}{(x+2)^n}$

$$\therefore \quad y_n = \frac{(-1)^n n!}{(x-2)^{n+1}} + 4 \cdot \frac{(-1)^n (n+1)!}{(x-2)^{n+2}} - \frac{(-1)^n}{(x+2)^n}.$$

Example:3

- If $y = \dfrac{x}{(x+1)^4}$, find y_n.
- Solution: $y = \dfrac{x}{(x+1)^4} = \dfrac{(x+1)-1}{(x+1)^4}$

$$\therefore \quad y_n = \frac{(-1)^n (n+2)!}{2!(x+1)^{n+3}} - \frac{(-1)^n (n+3)!}{3!(x+1)^{n+4}}$$

$$= \frac{(-1)^n (n+2)!}{6(x+1)^{n+3}} \left[3 - \frac{(n+3)}{(x+1)} \right]$$

$$= \frac{(-1)^n (n+2)!}{6(x+1)^{n+4}} (3x-n).$$

Example:4

- If $y = x\log\left(\dfrac{x-1}{x+1}\right)$, prove that

$$y_n = (-1)^{n-2}(n-2)!\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right].$$

- **Solution:** We have $y = x\log(x-1) - x\log(x+1)$

$$\therefore \quad y_1 = \frac{x}{x-1} + \log(x-1) - \frac{x}{x+1} - \log(x+1)$$

$$= \log(x-1) - \log(x+1) + \frac{x}{x-1} - \frac{x}{x+1}$$

$$\therefore \quad y_1 = \log(x-1) - \log(x+1) + \frac{(x-1)+1}{x-1} - \frac{(x+1)-1}{x+1}$$

$$\therefore \quad y_1 = \log(x-1) - \log(x+1) + 1 + \frac{1}{x-1} - 1 + \frac{1}{x+1}$$

Changing n to $n-1$, for the first two terms, we get

$$y_n = \frac{(-1)^{n-2}(n-2)!}{(x-1)^{n-1}} - \frac{(-1)^{n-2}(n-2)!}{(x+1)^{n-1}} + \frac{(-1)^{n-1}(n-1)!}{(x-1)^n} + \frac{(-1)^n(n-1)!}{(x+1)^n}$$

$$= (-1)^{n-2}(n-2)!\left[\frac{1}{(x-1)^{n-1}} - \frac{1}{(x+1)^{n-1}} + \frac{(-1)(n-1)}{(x-1)^n} + \frac{(-1)(n-1)}{(x+1)^n}\right]$$

$$= (-1)^{n-2}(n-2)!\left[\frac{x-1}{(x-1)^n} - \frac{x+1}{(x+1)^n} + \frac{-n+1}{(x-1)^n} + \frac{-n+1}{(x+1)^n}\right]$$

$$\therefore \quad y_n = (-1)^{n-2}(n-2)!\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right]$$

Example:5

- Prove that the value of n^{th} differential coefficient of $x^3/(x^2-1)$ for $x=0$ is 0 if n is even and $-n!$ if n is odd and greater than 1.

- Solution $y = \dfrac{x^3}{x^2-1} = x + \dfrac{x}{x^2-1}$

$$\therefore \quad y = x + \frac{1}{2}\left[\frac{1}{x-1} + \frac{1}{x+1}\right]$$

$$\therefore \quad y_n = \frac{1}{2}\left[\frac{(-1)^n n!}{(x-1)^{n+1}} + \frac{(-1)^n n!}{(x+1)^{n+1}}\right]$$

Putting $x=0$, $\quad y_n(0) = \dfrac{(-1)^n n!}{2}\left[\dfrac{1}{(-1)^{n+1}} + \dfrac{1}{(1)^{n+1}}\right]$

When n is even, $(n+1)$ is odd,

$$\therefore \quad y_n(0) = \frac{(-1)^n n!}{2}[-1+1] = 0$$

When n is odd, $(n+1)$ is even,

$$\therefore \quad y_n(0) = \frac{(-1)^n n!}{2}[1+1] = -n!$$

Practice Problems

- $\dfrac{x^4}{(x-1)(x-2)}$
- $\dfrac{x^2}{(x-1)(2x+3)}$
- $\dfrac{x}{1-4x^2}$
- $\dfrac{1}{x^4-a^4}$
- If $y = x \cot h^{-1} x$, prove that

$$y_n = \frac{(-1)^n (n-2)!}{2}\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right].$$

$$\frac{x^2+4x+1}{x^2+2x^2-x-2}$$

Example:6

- **If y** = sinrx+cosrx then prove that
  $y_n = r^n[1 + (-1)^n sin2rx]^{1/2}$
- **Solution**

$$y_n = r^n\left[\sin\left(rx + \frac{n\pi}{2}\right) + \cos\left(rx + \frac{n\pi}{2}\right)\right]$$

$$= r^n\left[\left\{\sin\left(rx + \frac{n\pi}{2}\right) + \cos\left(rx + \frac{n\pi}{2}\right)\right\}^2\right]^{1/2}$$

$$= r^n\left[\sin^2\left(rx + \frac{n\pi}{2}\right) + \cos^2\left(rx + \frac{n\pi}{2}\right) + 2\sin\left(rx + \frac{n\pi}{2}\right)\cos\left(rx + \frac{n\pi}{2}\right)\right]^{1/2}$$

$$= r^n\left[1 + 2\sin\left(rx + \frac{n\pi}{2}\right)\cdot\cos\left(rx + \frac{n\pi}{2}\right)\right]^{1/2}$$

$$= r^n\left[1+2\sin\left(rx+\frac{n\pi}{2}\right)\cdot\cos\left(rx+\frac{n\pi}{2}\right)\right]^{1/2}$$

$$= r^n\left[1+\sin 2\left(rx+\frac{n\pi}{2}\right)\right]^{1/2}$$

$$= r^n\left[1+\sin(2rx+n\pi)\right]^{1/2}$$

since $\sin(n\pi+\theta) = (-1)^n \sin\theta.$

$$= r^n\left[1+(-1)^n \sin 2rx\right]^{1/2}$$

Example:7

- If $y = sin^2xcos^3x$ then find y_n
- **Solution** $y = \sin^2 x \cos^2 x \cdot \cos x = \frac{1}{4}(\sin 2x)^2 \cos x$

$$= \frac{1}{8}(1 - \cos 4x)\cos x$$

$$= \frac{1}{8}(\cos x - \cos 4x \cos x)$$

$$= \frac{1}{8}\cos x - \frac{1}{16}[\cos 5x + \cos 3x]$$

$$y_n = \frac{1}{8}\cos\left(x + \frac{n\pi}{2}\right) - \frac{1}{16}\cdot 5^n \cos\left(5x + \frac{n\pi}{2}\right)$$

$$- \frac{1}{16}\cdot 3^n \cos\left(3x + \frac{n\pi}{2}\right)$$

Example:8

- **If $y = 2^x \sin^2 x \cos^3 x$ then find y_n**
- **Solution:** note that $2^x = e^{x \log 2} = e^{ax}$ where $a = \log 2$

$$\therefore \quad \sin^2 x \cos^3 x = \frac{1}{8}\cos x - \frac{1}{16}(\cos 5x + \cos 3x)$$

$$\therefore \quad y = \frac{1}{8}e^{ax}\cos x - \frac{1}{16}e^{ax}\cos 5x - \frac{1}{16}e^{ax}\cos 3x$$

Using formula (11), we get,

$$y_n = \frac{1}{8}r_1^n e^{ax}\cos(x + n\Phi_1) - \frac{1}{16}r_2^n e^{ax}\cos(5x + n\Phi_2) - \frac{1}{16}r_3^n e^{ax}\cos(3x + n\Phi_3)$$

$$= \frac{1}{8}r_1^n 2^x \cos(x + n\Phi_1) - \frac{1}{16}r_2^n 2^x \cos(5x + n\Phi_2) - \frac{1}{16}r_3^n 2^x \cos(3x + n\Phi_3)$$

where, $r_1 = \sqrt{(\log 2)^2 + 1^2}$, $\Phi_1 = \tan^{-1}\left(\frac{1}{\log 2}\right)$,

$r_2 = \sqrt{(\log 2)^2 + 5^2}$, $\Phi_2 = \tan^{-1}\left(\frac{5}{\log 2}\right)$,

$r_3 = \sqrt{(\log 2)^2 + 3^2}$, $\Phi_3 = \tan^{-1}\left(\frac{3}{\log 2}\right)$.

Practice Problems

- Find nth derivative of the following functions.
- $\sin 2x \sin 3x \cos 4x$
- $\sin^3 3x$
- $e^x \cos^2 x \cos x$
- $2^x \sin^2 x \cos x$
- $\cos^4 x$

- If $y = \cos h\, 2x$, prove that
  $y_n = 2^n \sin h\, 2x$ if n is odd and
  $y_n = 2^n \cos h\, 2x$ if n is even.

Example:9

- If $y = \dfrac{x}{x^2 + a^2}$, prove that

  $y_n = (-1)^n \cdot n! a^{-n-1} \sin^{n+1}\theta \cos(n+1)\theta$ where $\theta = \tan^{-1}(a/x)$.

- **Solution :** $y = \dfrac{x}{x^2 + a^2} = \dfrac{x}{(x+ai)(x-ai)}$

$$= \frac{1}{2}\left[\frac{1}{x+ai} + \frac{1}{x-ai}\right]$$ [By partial fractions]

$$y_n = \frac{1}{2}\left[\frac{(-1)^n n!}{(x+ai)^{n+1}} + \frac{(-1)^n \cdot n!}{(x-ai)^{n+1}}\right]$$

Let $x = r\cos\theta$, $a = r\sin\theta$, $r^2 = (x^2 + a^2)$ and $\theta = \tan^{-1}(a/x)$

$$\therefore \quad \frac{1}{(x+ai)^{n+1}} = \frac{1}{r^{n+1}(\cos\theta + i\sin\theta)^{n+1}}$$

By De Moivre's Theorem

$$= \frac{1}{r^{n+1}} \cdot \frac{1}{\cos(n+1)\theta + i\sin(n+1)\theta}$$

$$\therefore \quad \frac{1}{(x+a\,i)^{n+1}} = \frac{1}{r^{n+1}} \cdot [\cos(n+1)\theta - i\sin(n+1)\theta]$$

And,

$$\frac{1}{(x-a\,i)^{n+1}} = \frac{1}{r^{n+1}(\cos\theta - i\sin\theta)^{n+1}}$$

$$= \frac{1}{r^{n+1}} \cdot \frac{1}{\cos(n+1)\theta - i\sin(n+1)\theta}$$

$$= \frac{1}{r^{n+1}} \cdot [\cos(n+1)\theta + i\sin(n+1)\theta]$$

Adding the two results,

$$\frac{1}{(x-a\,i)^{n+1}} + \frac{1}{(x+a\,i)^{n+1}} = \frac{1}{r^{n+1}} \cdot 2\cos(n+1)\theta$$

$$\therefore \quad y_n = (-1)^n \cdot n!\; r^{-n-1}\cos(n+1)\theta$$

Example:10

- **If $\mathbf{y} = \boldsymbol{tan^{-1}x}$ then find $y_n = \frac{(-1)^{n-1}(n-1)!}{(x^2+1)^{\frac{n}{2}}} \sin(n\, tan^{-1}\frac{1}{x})$**

- **Solution:** Differentiating $y = \tan^{-1} x$, w.r.t. x, we get,

$$y_1 = \frac{1}{x^2+1} = \frac{1}{(x+i)(x-i)} = \frac{1}{2i}\left[\frac{1}{(x-i)} - \frac{1}{(x+i)}\right]$$

Differentiating $(n-1)$ times,

$$y_n = \frac{1}{2i}\left[\frac{(-1)^{n-1}(n-1)!}{(x-i)^n} - \frac{(-1)^{n-1}(n-1)!}{(x+i)^n}\right] \quad ....(1)$$

Putting $r = r\cos\theta,\ 1 = r\sin\theta$ $\ r = \sqrt{1+x^2}$ and $\theta = \tan^{-1}(1/x)$,

$$\frac{1}{(x-i)^n} = \frac{1}{r^n(\cos\theta - i\sin\theta)^n} = \frac{1}{r^n}(\cos n\theta + i\sin n\theta)$$

Similarly,

$$\frac{1}{(x+i)^n} = \frac{1}{r^n(\cos\theta + i\sin\theta)^n} = \frac{1}{r^n}(\cos n\theta - i\sin n\theta)$$

$$\therefore \quad \frac{1}{(x-i)^n} - \frac{1}{(x+i)^n} = \frac{2i}{r^n} \sin n\theta$$

from (1), we get,

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2i} \cdot \frac{2i}{r^n} \sin n\theta$$

$$= (-1)^{n-1}(n-1)! \cdot \frac{1}{r^n} \sin n\theta \qquad \text{..........................(2)}$$

Now, we put $r = \sqrt{1+x^2}$ and $\theta = \tan^{-1}\left(\frac{1}{x}\right)$ in (2),

$$\therefore \quad y_n = \frac{(-1)^{n-1}(n-1)!}{(x^2+1)^{n/2}} \sin\left[n \tan^{-1}\left(\frac{1}{x}\right)\right]$$

(HW) **If $\mathbf{y} = \boldsymbol{tan^{-1}x}$** then Prove that

$y_n = (-1)^{n-1}(n-1)!\, sin^n\theta \sin n\theta$; where $\theta = tan^{-1}(\frac{1}{x})$

Hint: In (2) put $r = \frac{1}{sin\theta}$

Example:11

- If $y = \cos^{-1}\left(\dfrac{x - x^{-1}}{x + x^{-1}}\right)$, prove that $y_n = 2\,(-1)^{n-1}\,(n-1)\,!\,\sin^n\theta\,\sin n\theta$ where $\theta = \tan^{-1}(1/x)$.

- **Solution :** Putting $x = \tan\alpha \quad \therefore \quad \alpha = \tan^{-1} x$.

$$y = \cos^{-1}\left(\frac{x^2 - 1}{x^2 + 1}\right) = \cos^{-1}\left(\frac{\tan^2\alpha - 1}{\tan^2\alpha + 1}\right)$$

$$\therefore \quad y = \cos^{-1}\left[-\frac{(\cos^2\alpha - \sin^2\alpha)/\cos^2\alpha}{1/\cos^2\alpha}\right]$$

$$= \cos^{-1}[-(\cos^2\alpha - \sin^2\alpha)]$$

$$= \cos^{-1}[-(\cos 2\alpha)] = \cos^{-1}\cos(\pi + 2\alpha)$$

$$= \pi + 2\alpha = \pi + 2\tan^{-1} x$$

$$\therefore \quad y_1 = 2 \cdot \frac{1}{x^2 + 1}$$

(Proceed as per previous example)

Leibnitz's Theorem

- If u and v are functions of x such that their n^{th} derivatives exist,

  then the n^{th} derivative of their product is given by

$$(u\,v)_n = u_n v + n_{C_1} u_{n-1} v_1 + n_{C_2} u_{n-2} v_2 + \cdots + n_{C_r} u_{n-r} v_r + \cdots + u v_n$$

$$(u\,v)_n = u_n v + n_{C_1} u_{n-1} v_1 + n_{C_2} u_{n-2} v_2 + \cdots + n_{C_r} u_{n-r} v_r + \cdots + u v_n$$

where u_r and v_r represent r^{th} derivatives of u and v respectively.

Remark: The term which vanishes after differentiating finitely should be taken as v.

Example:12

- Find the n^{th} derivative of $x \log x$
- **Solution:** Let $u = \log x$ and $v = x$

$\because n^{th}$ derivative of $\log(ax+b)$ is given by $(-1)^{n-1}\frac{(n-1)!\,a^n}{(ax+b)^n}$

Then $u_n = (-1)^{n-1}\frac{(n-1)!}{x^n}$ and $u_{n-1} = (-1)^{n-2}\frac{(n-2)!}{x^{n-1}}$

Now, by Leibnitz's theorem, we have

$$(u\,v)_n = u_n v + n_{C_1} u_{n-1} v_1 + n_{C_2} u_{n-2} v_2 + \cdots + n_{C_r} u_{n-r} v_r + \cdots + u v_n$$

$$\Rightarrow (x \log x)_n = (-1)^{n-1}\frac{(n-1)!}{x^n}x + n(-1)^{n-2}\frac{(n-2)!}{x^{n-1}} \cdot + 0$$

$$= (-1)^{n-1}\frac{(n-1)!}{x^{n-1}} + n(-1)^{n-2}\frac{(n-2)!}{x^{n-1}}$$

$$= (-1)^{n-2}\frac{(n-2)!}{x^{n-1}}[-(n-1)+n] = (-1)^{n-2}\frac{(n-2)!}{x^{n-1}}$$

Example:13

- If $y = a\cos(\log x) + b\,sin(\log x)$, show that

  $x^2 y_{n+2} + \ (2n+1)xy_{n+1} + n(n+1)y_n = 0$

- Solution: Here $y = a\cos(\log x) + b\sin(\log x)$

$$\Rightarrow y_1 = \ -\frac{a}{x}\sin(\log x) + \frac{b}{x}\cos(\log x)$$

$$\Rightarrow xy_1 = \ -a\sin(logx) + \ b\,cos(logx)$$

Differentiating both sides w.r.t. x , we get

$$xy_2 + y_1 = \ -\frac{a}{x}\cos(logx) + \frac{-b}{x}\ sin(logx)$$

$$\Rightarrow x^2 y_2 + xy_1 = \ -\{acos(logx) + b\ sin(logx)\} = -y$$

$$\Rightarrow x^2 y_2 + xy_1 + y = 0$$

Using Leibnitz's theorem, we get

$$(y_{n+2}x^2 + n_{C_1}y_{n+1}2x + n_{C_2}y_n.2) + (y_{n+1}x + n_{C_1}y_n.1) + y_n = 0$$

$$\Rightarrow y_{n+2}x^2 + y_{n+1}2nx + n(n-1)y_n + y_{n+1}x + ny_n + y_n = 0$$

$$\Rightarrow x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$$

Example:14

- If $y = \log\left(x + \sqrt{1+x^2}\right)$, prove that

$$(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2 y_n = 0$$

- Solution: $y = \log\left(x + \sqrt{1+x^2}\right)$

$$\Rightarrow y_1 = \frac{1}{x+\sqrt{1+x^2}}\left(1 + \frac{1}{2\sqrt{1+x^2}}2x\right) = \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow (1+x^2){y_1}^2 = 1$$

Differentiating both sides w.r.t. x, we get

$$(1+x^2)2y_1y_2 + 2x{y_1}^2 = 0$$

$$\Rightarrow (1+x^2)y_2 + xy_1 = 0$$

Using Leibnitz's theorem

$$[y_{n+2}(1+x^2) + n_{C_1} y_{n+1} 2x + n_{C_2} y_n . 2] + \left(y_{n+1} x + n_{C_1} y_n . 1\right) = 0$$

$$\Rightarrow y_{n+2}(1+x^2) + y_{n+1} 2nx + n(n-1) y_n + y_{n+1} x + n y_n = 0$$

$$\Rightarrow (1+x^2) y_{n+2} + (2n+1) x y_{n+1} + n^2 y_n = 0$$

Example:15

- If $y = \sin(m\, sin^{-1}x)$, show that
  $(1-x^2)y_{n+2} = (2n+1)xy_{n+1} + (n^2 - m^2)y_n$. Also find $y_n(0)$

- **Solution:**

Here $y = \sin(m\, sin^{-1}x)$ ...①

$\Rightarrow y_1 = \frac{m}{\sqrt{1-x^2}}\cos(m\, sin^{-1}x)$ ...②

$\Rightarrow (1-x^2){y_1}^2 = m^2\cos^2(m\, sin^{-1}x)$

$\Rightarrow (1-x^2){y_1}^2 = m^2[1-\sin^2(m\, sin^{-1}x)]$

$\Rightarrow (1-x^2){y_1}^2 = m^2(1-y^2)$ ...③

$\Rightarrow (1\text{-}x^2){y_1}^2 + m^2y^2 = m^2$

Differentiating w.r.t. x, we get

$$(1-x^2)2y_1y_2 + {y_1}^2(-2x) + m^2 2yy_1 = 0$$

$$\Rightarrow (1-x^2)y_2 - xy_1 + m^2 y = 0$$

Using Leibnitz's theorem, we get

$$[y_{n+2}(1-x^2) + n_{c_1}y_{n+1}(-2x) + n_{c_2}y_n(-2)] - \left(y_{n+1}x + n_{c_1}y_n 1\right) + m^2 y_n = 0$$

$$\Rightarrow y_{n+2}(1-x^2) - y_{n+1}2nx - n(n-1)y_n - (y_{n+1}x + ny_n) + m^2 y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} = (2n+1)xy_{n+1} + (n^2 - m^2)y_n \qquad \ldots (4)$$

Putting $x = 0$ in (1), (2) and (3)

$$y(0) = 0,\ y_1(0) = m \text{ and } y_2(0) = 0$$

Putting $x = 0$ in (4), we get

$$y_{n+2}(0) = (n^2 - m^2)y_n(0)$$

Putting $n = 1,2,3 \ldots$ in the above equation, we get

$$y_3(0) = (1^2 - m^2)y_1(0) = (1^2 - m^2)m \qquad \because y_1(0) = m$$

$$y_4(0) = (2^2 - m^2)y_2(0) = 0 \qquad \because y_2(0) = 0$$

$$y_5(0) = (3^2 - m^2)y_3(0) = m(1^2 - m^2)(3^2 - m^2)$$

$$\Rightarrow \quad y_n(0) = \begin{cases} 0, & if\ n\ is\ even \\ m(1^2 - m^2)(3^2 - m^2)\ldots[(n-2)^2) - m^2], & if\ n\ is\ odd \end{cases}$$

Practice Problems

1. Find y_n , if $y = x^3 \cos x$

2. Find y_n , if $y = x^2 e^x \cos x$

3. If $y^{\frac{1}{m}} + y^{\frac{-1}{m}} = 2x$, prove that $(x^2 - 1)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$

4. If $y\sqrt{1+x^2} = \log\left(x + \sqrt{1+x^2}\right)$, prove that
$(1+x^2)y_{n+2} + (2n+3)xy_{n+1} + (n+1)^2 y_n = 0$

5. If $y = \left[x + \sqrt{1+x^2}\right]^m$, prove that $(x^2+1)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$

6. If $y = (sinh^{-1}x)^2$, show that
$(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2 y_n = 0$. Also find $y_n(0)$.

7. If $y = \cos(m\, sin^{-1}x)$, show that
$(1-x^2)y_{n+2} = (2n+1)xy_{n+1} + (n^2 - m^2)y_n$. Also find $y_n(0)$.

8. If $f(x) = \tan x$, prove that $f^n(0) - n_{c_2} f^{n-2}(0) + n_{c_4} f^{n-4}(0) - \cdots = sin\,\frac{n\pi}{2}$