Module 3
Integration :
Review And Some New Techniques

Sub-Module 3.1

Beta functions with properties

Example 15

- **Evaluate $\int_3^7 \sqrt[4]{(x-3)(7-x)}\,dx$**

Solution: Put $x-3=(7-3)t=4t, \ \ x=4t+3$

$dx=4dt$

When $x=3, \ t=0$

When $x=7, \ \ t=1$

$$\mathrm{I}=\int_{t=0}^{1}\sqrt[4]{(4t)(7-4t-3)}\,4dt$$

$$=\int_{t=0}^{1}\sqrt[4]{(4t)(4-4t)}\,4dt$$

$$=\int_{t=0}^{1}16^{\frac{1}{4}}\cdot 4\cdot t^{\frac{1}{4}}\,(1-t)^{\frac{1}{4}}\,dt$$

$$=8\int_{t=0}^{1}t^{\frac{5}{4}-1}\,(1-t)^{\frac{5}{4}-1}\,dt$$

$$=8\beta\left(\frac{5}{4},\frac{5}{4}\right)=8\times\frac{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{5}{2}\right)}=8\times\frac{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)\frac{1}{4}\Gamma\left(\frac{1}{4}\right)}{\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}}$$

$$=\frac{2}{3}\left(\Gamma\left(\frac{1}{4}\right)\right)^2\frac{1}{\sqrt{\pi}}$$

Example 16

- **Evaluate** $\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}\,dx$
- **Solution:**

$$I = \int_0^\infty \frac{x^4}{(1+x)^{15}}dx + \int_0^\infty \frac{x^9}{(1+x)^{15}}dx$$

$$= \int_0^\infty \frac{x^{5-1}}{(1+x)^{5+10}}dx + \int_0^\infty \frac{x^{10-1}}{(1+x)^{10+5}}dx$$

$$= \beta(5,10) + \beta(10,5) \quad \because \ \beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$$

$$= 2\beta(5,10)$$

$$= 2\,\frac{\lvert\overline{5}\,\lvert\overline{10}}{\lvert\overline{15}}$$

$$= 2 \times \frac{4!\times 9!}{14!}$$

Example 17

- **Evaluate $\int_0^\infty \frac{\sqrt{x}}{1+2x+x^2}\,dx$**
- **Solution:**

$$I = \int_0^\infty \frac{x^{\frac{1}{2}}}{(x+1)^2}\,dx$$

$$= \int_0^\infty \frac{x^{\frac{3}{2}-1}}{(x+1)^{\frac{3}{2}+\frac{1}{2}}}\,dx$$

$$= \beta\left(\frac{3}{2},\frac{1}{2}\right)$$

$$= \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)}$$

$$= \frac{1}{2}\sqrt{\pi}\sqrt{\pi} = \frac{\pi}{2}$$

Example 18

- **Prove that $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}} dt = \frac{\beta(m,n)}{(a+b)^m a^n}$**
- Proof : substitution $x = \frac{at}{a+b-bt}$

$$dx = \frac{(a+b-bt)a-at(-b)}{(a+b-bt)^2} = \frac{a(a+b)}{(a+b-bt)^2}\ dt$$

$$1-x = 1-\frac{at}{a+b-bt} = \frac{a+b-bt-at}{a+b-bt} = \frac{(a+b)-t(a+b)}{a+b-bt} = \frac{(a+b)(1-t)}{a+b-bt}$$

$$a+bx = a+\frac{b\ at}{a+b-bt} = \frac{a^2+ab-abt+abt}{a+b-b\ t} = \frac{a(a+b)}{a+b-bt}$$

$$I = \int_0^1 \frac{a^{m-1}t^{m-1}}{(a+b-bt)^{m-1}} \cdot \frac{(a+b)^{n-1}(1-t)^{n-1}}{(a+b-bt)^{n-1}} \cdot \frac{(a+b-bt)^{m+n}}{a^{m+n}(a+b)^{m+n}} \cdot \frac{a(a+b)}{(a+b-bt)^2}\ dt$$

$$I = \frac{1}{(a+b)^m a^n}\int_0^1 t^{m-1}(1-t)^{n-1} dt = \frac{\beta(m,n)}{(a+b)^m a^n}$$

$$HW.\int_0^1 \frac{x^{m-1}(1+x)^{n-1}}{(1+x)^{m+n}} dx = \frac{\beta(m,n)}{2^m}$$

Example 19

- **Evaluate** $\int_0^1 \frac{(1-x^4)^{\frac{3}{4}}}{(1+x^4)^2} dx$

Solution: Put $x^4 = t \rightarrow x = t^{\frac{1}{4}}$

$$dx = \frac{1}{4} t^{-\frac{3}{4}} dt$$

$$I = \int_0^1 \frac{(1-t)^{\frac{3}{4}}}{(1+t)^2} \frac{1}{4} t^{-\frac{3}{4}} dt$$

$$= \frac{1}{4} \int_0^1 \frac{t^{\frac{1}{4}-1}(1-t)^{\frac{7}{4}-1}}{(1+t)^{\frac{1}{4}+\frac{7}{4}}}$$

$$= \frac{1}{4} \frac{1}{2^{\frac{1}{4}}} \beta\left(\frac{1}{4}, \frac{7}{4}\right)$$

$$= \frac{1}{4} \frac{1}{2^{\frac{1}{4}}} \frac{\left|\frac{1}{4}\right.\left|\frac{7}{4}\right.}{\left|2\right.}$$

$$= \frac{1}{4\left(2^{\frac{1}{4}}\right)} \left|\frac{1}{4}\right. \frac{3}{4} \left|\frac{3}{4}\right.$$

$$= \frac{3}{16\left(2^{\frac{1}{4}}\right)} \frac{\pi}{\sin\frac{\pi}{4}} = \frac{3\pi\sqrt{2}}{16\sqrt[4]{2}}$$

Example:20

❖ **Prove that $\boldsymbol{\beta(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx}$**

❖ **Proof:** Consider $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$$

$$= I_1 + I_2 \ldots\ldots\ldots\ldots\ldots(1)$$

Consider $I_2 = \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$

Put $x = \frac{1}{y} \Rightarrow dx = -\frac{1}{y^2}dy$

$$\therefore I_2 = \int_{y=1}^0 \left(\frac{1}{y}\right)^{m-1} \frac{1}{\left(1+\frac{1}{y}\right)^{m+n}} \left(-\frac{1}{y^2}\right)dy$$

$$I_2 = \int_1^0 \frac{1}{y^{m-1}} \frac{1}{\left(\frac{y+1}{y}\right)^{m+n}} \left(-\frac{1}{y^2}\right) dy$$

$$= \int_0^1 \frac{1}{y^{m+1}} \frac{y^{m+n}}{(1+y)^{m+n}}\ dy$$

$$= \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}}\ dy$$

$$= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\ dx$$

$$\therefore by\ (1), \beta(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\ dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\ dx$$

$$= \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\ dx$$

Example:21

❖ **Prove that** $\boldsymbol{\beta(m,m)\cdot\beta\left(m+\frac{1}{2},m+\frac{1}{2}\right)=\frac{\pi}{m}2^{1-4m}}$

❖ **Solution:** $\beta(m,m)\cdot\beta\left(m+\frac{1}{2},m+\frac{1}{2}\right)=\dfrac{\left|\overline{m}\right.\left|\overline{m}\right.}{\left|\overline{2m}\right.}\,\dfrac{\left|\overline{m+\frac{1}{2}}\right.\left|\overline{m+\frac{1}{2}}\right.}{\left|\overline{2m+1}\right.}$

$$=\left[\frac{\left|\overline{m}\right.\left|\overline{m+\frac{1}{2}}\right.}{\left|\overline{2m}\right.}\right]^2\frac{1}{2m}$$

$$=\frac{\pi}{2^{4m-2}}\frac{1}{2m}$$

$$=\frac{\pi}{2^{4m-1}}\cdot\frac{1}{m}$$

$$=\frac{\pi}{2}\,2^{1-4m}$$

x	0	∞
t	0	∞

Example:22

❖ **Prove That** $\int_0^\infty \frac{dx}{(e^x+e^{-x})^n} = \frac{1}{4}\beta\left(\frac{n}{2},\frac{n}{2}\right)$ **and hence evaluate** $\int_0^\infty \operatorname{sec}h^8 x\ dx$

Solution: $I = \int_0^\infty \frac{dx}{(e^x+e^{-x})^n} = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{(e^x+e^{-x})^n}$

Put $e^x = \tan\theta$, $e^x dx = \sec^2\theta\, d\theta$

$$dx = \frac{\sec^2\theta\, d\theta}{\tan\theta}$$

When $x = -\infty, \tan\theta = 0 \to \theta = 0$

$x = \infty, \tan\theta = \infty \to \theta = \frac{\pi}{2}$

$$I = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sec^2\theta\, d\theta}{\tan\theta\,(\tan\theta+\cot\theta)^n}$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\frac{1}{\cos^2\theta}\, d\theta}{\frac{\sin\theta}{\cos\theta}\left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right)^n}$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1\, d\theta}{\cos\theta \sin\theta \left(\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}\right)^n}$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sin^{\mathrm{n}}\theta\ \cos^{\mathrm{n}}\theta}{\sin\theta\cos\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^{n-1}\theta \cos^{n-1}\theta\ d\theta$$

$$= \frac{1}{2} \times \frac{1}{2}\beta\left(\frac{n-1+1}{2}, \frac{n-1+1}{2}\right)$$

$$= \frac{1}{4}\beta\left(\frac{n}{2}, \frac{n}{2}\right)$$

$$\frac{e^x + e^{-x}}{2} = \cos hx$$

$$e^x + e^{-x} = 2\cos hx$$

$$(e^x + e^{-x})^8 = 2^8 \cos h^8\, x$$

$$\therefore \int_0^\infty \frac{dx}{(e^x+e^{-x})^8} = \int_0^\infty \frac{dx}{2^8 \cos h^8 x}$$

$$= \frac{1}{4}\beta\left(\frac{8}{2},\frac{8}{2}\right)$$

$$\therefore \int_0^\infty \sec h^8 \ dx \ = \frac{2^8}{4}\beta(4,4)$$

$$= 2^6 \frac{\lceil 4 \lceil 4}{\lceil 8}$$

$$= \frac{64\times 3!\times 3!}{7!}$$

$$= \frac{64\times 6\times 6}{7\times 6\times 5\times 4\times 3\times 2}$$

$$= \frac{16}{35}$$

Example:23

❖ **Show that** $\int_0^a \frac{dx}{(a^n - x^n)^{\frac{1}{n}}} = \frac{\pi}{n} cosec \frac{\pi}{n}$

❖ **Solution:** $x^n = a^n t, x = at^{\frac{1}{n}}, dx = a\frac{1}{n} t^{\frac{1}{n}-1}\, dt$

When $x = 0, \quad t = 0,$

When $x = a, \quad t = 1$

$$I = \int_0^1 \frac{\frac{a}{n} t^{\frac{1}{n}-1}}{(a^n - a^n t)^{\frac{1}{n}}} dt$$

$$= \frac{a}{n}\int_0^1 \frac{t^{\frac{1}{n}-1}}{(a^n)^{\frac{1}{n}}(1-t)^{\frac{1}{n}}} dt$$

$$= \frac{a}{n} \cdot \frac{1}{a} \int_0^1 \frac{t^{\frac{1}{n}-1}}{(1-t)^{\frac{1}{n}}} dt$$

$$= \frac{1}{n}\int_0^1 t^{\frac{1}{n}-1} (1-t)^{-\frac{1}{n}}\, dt$$

$$= \frac{1}{n}\beta\left(\frac{1}{n}, 1 - \frac{1}{n}\right) = \frac{1}{n} \left|\overline{\frac{1}{n}}\right. \left|\overline{1 - \frac{1}{n}}\right.$$

$$= \frac{1}{n}\frac{\pi}{\sin\frac{\pi}{n}} = \frac{\pi}{n}\, cosec \frac{\pi}{n}$$

Example:24

❖ **Prove that** $\int_0^{\pi} \frac{\sqrt{\sin x}}{(5+3\cos x)^{\frac{3}{2}}}\, dx = \frac{\left(\left|\frac{3}{4}\right)^2}{2\sqrt{2\pi}}$

❖ **Solution:** $I = \int_0^{\pi} \frac{\sqrt{\sin x}}{(5+3\cos x)^{\frac{3}{2}}}\, dx$

Put $t = \tan\frac{x}{2}$; $\sin x = \frac{2t}{1+t^2}$; $\cos x = \frac{1-t^2}{1+t^2}$

$t = \tan\frac{x}{2} \rightarrow dt = \frac{1}{2}\sec^2\frac{x}{2}\, dx$

$\rightarrow dx = \frac{2dt}{\sec^2\frac{x}{2}}$, $dx = \frac{2}{1+t^2} dt$

❖ $I = \int_0^{\infty} \frac{\sqrt{\frac{2t}{1+t^2}}}{\left[5+3\frac{(1-t^2)}{1+t^2}\right]^{\frac{3}{2}}} = \int_0^{\infty} \frac{2\sqrt{2}\,\sqrt{t}}{\frac{(1+t^2)^{\frac{1}{2}}(1+t^2)}{\left[\frac{5(1+t^2)+(1-t^2)}{(1+t^2)}\right]^{\frac{3}{2}}}}\, dt$

$$I = \int_0^\infty \frac{2\sqrt{2}\sqrt{t}}{(8+2t^2)^{\frac{3}{2}}}\,dt$$

$$= \frac{2\sqrt{2}}{2^{\frac{3}{2}}}\int_0^\infty \frac{\sqrt{t}}{(t^2+4)^{\frac{3}{2}}}\,dt$$

Put $t^2 = 4y,\ t = 2\sqrt{y}; dt = 2 \times \frac{1}{2} y^{-\frac{1}{2}}\,dy$

$$I = \int_0^\infty \frac{\sqrt{2}y^{\frac{1}{4}}}{(4y+4)^{\frac{3}{2}}}\, y^{-\frac{1}{2}} dy = \frac{\sqrt{2}}{4^{\frac{3}{2}}}\int_0^\infty \frac{y^{-\frac{1}{4}}}{(y+1)^{\frac{3}{2}}}\,dy$$

$$= \frac{\sqrt{2}}{8}\beta\left(\frac{3}{4},\frac{3}{4}\right)$$

$$= \frac{\sqrt{2}}{8}\frac{\left|\frac{3}{4}\right|\frac{3}{4}}{\left|\frac{3}{2}\right.} = \frac{\sqrt{2}}{8}\left(\sqrt{\frac{3}{\left|\frac{3}{4}\right.4}}\right)^2 \frac{1}{\frac{1}{2}\sqrt{\pi}}$$

$$= \frac{1}{2\sqrt{2\pi}}\left(\left|\frac{3}{4}\right.\right)^2$$