Module 1
Differential Equations
of
First Order and First Degree

Sub-module 1.3

Linear ODE with constant coefficient

Method of finding P.I.

- We know that $f(D)y = X$
- $y = \frac{1}{f(D)} X$
- Here $\frac{1}{f(D)}$ is known as inverse operator.
- $\therefore$ Particular integral $(P.I.) = \frac{1}{f(D)} X$
- **Case (i)** : If $y = \frac{1}{D} X$

$\frac{1}{D} X = \int X dx$

- **Case (ii) :** If $y = \frac{1}{D-m} X$
- $\frac{1}{D-m} X = e^{mx} \int X\, e^{-mx} dx$
- $\frac{1}{D+m} X = e^{-mx} \int X\, e^{mx} dx$

Standard Cases to Find PI

❖ **Case- I:** When $X = e^{ax}$,

$$P.I. = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax} \text{ [put } D = a] \quad \because f(a) \neq 0$$

❖ If $f(a) = 0$, $P.I. = \frac{1}{(D-a)^r \emptyset(D)} e^{ax} = \frac{x^r}{r!} \frac{e^{ax}}{\emptyset(a)}$

i.e. $P.I. = \frac{1}{(D-1)^r} e^{ax} = \frac{x^r}{r!} e^{ax}$

Note: $\frac{1}{f(D)} e^{ax} = x \cdot \frac{1}{f'(a)} e^{ax}$ if $f'(a) \neq 0$

$$\frac{1}{f(D)} e^{ax} = x^2 \cdot \frac{1}{f''(a)} e^{ax} \quad \text{if } f''(a) \neq 0$$

$$\frac{1}{f(D)} e^{ax} = x^3 \cdot \frac{1}{f'''(a)} e^{ax} \quad \text{if } f'''(a) \neq 0 \text{ and so on.}$$

When X=sin(ax+b)\ cos(ax+b)

- **Case- II:** When $X = \sin(ax+b)\ \ or \cos(ax+b)$

- i. $P.I. = \frac{1}{f(D^2)}\sin(ax+b) = \frac{1}{f(-a^2)}\sin(ax+b)$

$[\text{Put } D^2 = -a^2 \quad \because f(-a^2) \neq 0]$

If $f(-a^2) = 0,\ P.I. = \frac{1}{(D^2+a^2)^r}\sin(ax+b)$

$$= \frac{(-1)^r x^r}{(2a)^r\ r!}\sin\left(ax+b+\frac{r\pi}{2}\right)$$

- ii. $P.I. = \frac{1}{f(D^2)}\cos(ax+b) = \frac{1}{f(-a^2)}\cos(ax+b)$

$[\because f(-a^2) \neq 0]$

If $f(-a^2) = 0,\ P.I. = \frac{1}{(D^2+a^2)^r}\cos(ax+b)$

$$= \frac{(-1)^r x^r}{(2a)^r\ r!}\cos\left(ax+b+\frac{r\pi}{2}\right)$$

When $X = x^n$

❖ **Case- III:** When $X = x^n$,

$P.I. = \frac{1}{f(D)} x^n = [1 \pm f(D)]^{-1} x^n$ Here n is a positive integer.

❖ Method of Solving:

1. Take out the lowest degree term from $f(D)$ and reduce it in the form $[1 \pm f(D)]^n$

2. Take it to the numerator i.e., $[1 \pm f(D)]^{-n}$

3. Expand it with binomial theorem,

Use $(1 + D)^{-1} = 1 - D + D^2 - D^3 + \cdots$.

$(1 - D)^{-1} = 1 + D + D^2 + D^3 + \cdots$.

$(D - 1)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \cdots$

When $\boldsymbol{X = e^{ax} v}$

- When $X = e^{ax} v$,

$$P.I. = \frac{1}{f(D)}\, e^{ax} v$$

$$= e^{ax} \cdot \frac{1}{f(D+a)} v$$

Example 11

- **Solve $(D^3 - 1)y = (e^x + 1)^2$**
- **Solution:** The A.E. is given by $D^3 - 1 = 0$
- $(D-1)(D^2 + D + 1) = 0$
- $\therefore D = 1,\ \frac{-1 \pm \sqrt{1 - 4 \cdot 1 \cdot 1}}{2 \cdot 1}$
- $= 1,\ \frac{-1 \pm \sqrt{-3}}{2}$
- $= 1,\ \frac{-1 \pm i\sqrt{3}}{2}$
- $C.F. = y_c = c_1 e^x + e^{-\frac{x}{2}}\left[c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right]$

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- $P.I. = y_p = \frac{1}{f(D)} X = \frac{1}{D^3-1}[e^x + 1]^2$
- $= \frac{1}{D^3-1}[e^x + 2e^x + 1]$
- $= \frac{1}{D^3-1} e^{2x} + 2 \cdot \frac{1}{D^3-1} e^x + \frac{1}{D^3-1} e^{ox}$
- $= \frac{1}{8-1} e^{2x} + 2 \cdot x \cdot \frac{1}{3D^2} e^x + \frac{1}{0-1} e^{ox}$

  [Put $D = a = 2$ in 1st part and $D = a = 0$ in 3rd part]
- $= \frac{1}{7} e^{2x} + \frac{2}{3} x\, e^x - 1$
- Complete Solution is $y = y_c + y_p$
- $\therefore y = c_1 e^x + e^{-\frac{x}{2}}\left[c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right]$

  $+ \frac{1}{7} e^{2x} + \frac{2}{3} x\, e^x - 1$

Example 12

- **Solve** $6\frac{d^2y}{dx^2} + 17\frac{dy}{dx} + 12y = e^{-\frac{3}{2}x} + 2^x$
- **Solution**: The A.E. is $6D^2 + 17D + 12 = 0$
- $6D^2 + 9D + 8D + 12 = 0$
- $[3D(2D+3) + 4(2D+3)] = 0$
- $(3D+4)(2D+3) = 0$
- $D = -\frac{4}{3}, -\frac{3}{2}$
- $C.F. = y_c = c_1 e^{-\frac{4}{3}x} + c_2 e^{-\frac{3}{2}x}$

- $P.I. = y_p = \frac{1}{6D^2+17D+12}\left(e^{-\frac{3}{2}x} + 2^x\right)$
- $= \frac{1}{6D^2+17D+12} e^{-\frac{3}{2}x} + \frac{1}{6D^2+17D+12} 2^x$
- $= x\frac{1}{12D+17} e^{-\frac{3}{2}x} + \frac{1}{6D^2+17D+12} e^{x\log 2}$ $\left[\because 2^x = e^{x\log 2}\right]$
- $= x\frac{1}{[12(-\frac{3}{2})+17]} e^{-\frac{3}{2}x} + \frac{1}{6(\log 2)^2+17\log 2+12} e^{x\log 2}$
- $= -x\, e^{-\frac{3}{2}x} + \frac{e^{x\log 2}}{6(\log 2)^2+17\log 2+12}$
- Complete solution is $y = y_c + y_p$

$$= c_1 e^{-\frac{4}{3}x} + c_2 e^{-\frac{3}{2}x} - x\, e^{-\frac{3}{2}x} + \frac{e^{x\log 2}}{6(\log 2)^2+17\log 2 + 12}$$

Example 13

- **Solve $\left(D^2 + 4D + 4\right)y = \cos h\, 2x$**
- **Solution:** The A.E. is $D^2 + 4D + 4 = 0$
- $(D+2)^2 = 0$
- $D = -2, \ \ -2$
- $C.F. = y_c = (c_1 + c_2\, x)e^{-2x}$

- $P.I. = y_p = \frac{1}{(D+2)^2}\left[\frac{e^{2x}+e^{-2x}}{2}\right]$
- $= \frac{1}{2}\,\frac{1}{(D+2)^2}e^{2x} + \frac{1}{2}\cdot\frac{1}{D^2+4D+4}e^{-2x}$
- $= \frac{1}{2}\cdot\frac{1}{16}e^{2x} + \frac{1}{2}x\cdot\frac{1}{2D+4}e^{-2x}$
- $= \frac{1}{32}e^{2x} + \frac{1}{2}x^2\cdot\frac{1}{2}e^{-2x}$
- $= \frac{1}{32}e^{2x} + \frac{1}{4}x^2e^{-2x}$
- Complete Solution is given by $y = y_c + y_p$
- $\therefore\ y = (c_1 + c_2\,x)e^{-2x} + \frac{1}{32}e^{2x} + \frac{1}{4}x^2e^{-2x}$

When X=sin(ax+b)\ cos(ax+b)

- **Case- II:** When $X = \sin(ax + b)$ $or \cos(ax + b)$
- i. $P.I. = \frac{1}{f(D^2)} \sin(ax + b) = \frac{1}{f(-a^2)} \sin(ax + b)$

$[\text{Put } D^2 = -a^2 \quad \because f(-a^2) \neq 0]$

If $f(-a^2) = 0$, $P.I. = \frac{1}{(D^2+a^2)^r} \sin(ax + b)$

$$= \frac{(-1)^r x^r}{(2a)^r\, r!} \sin\left(ax + b + \frac{r\pi}{2}\right)$$

- ii. $P.I. = \frac{1}{f(D^2)} \cos(ax + b) = \frac{1}{f(-a^2)} \cos(ax + b)$

$[\because f(-a^2) \neq 0]$

If $f(-a^2) = 0$, $P.I. = \frac{1}{(D^2+a^2)^r} \cos(ax + b)$

$$= \frac{(-1)^r x^r}{(2a)^r\, r!} \cos\left(ax + b + \frac{r\pi}{2}\right)$$

Example 14

- **Solve :** $\left(\mathbf{cosec}\ \boldsymbol{x}\right)\dfrac{d^2y}{dx^2}+\left(cosec\ x\right)y=\sin 2x$
- **Solution:** Given $(D^2+1)y=\sin x\sin 2x$
- The A.E. is $D^2+1=0$
- $D^2=-1$
- $D=\pm i$
- $y_c=C.F.=c_1\cos x+c_2\sin x$

- $y_p = P.I. = \frac{1}{D^2+1}\left[\frac{1}{2}(\cos x - \cos 3x)\right]$
- $= \frac{1}{2}\left[\frac{1}{D^2+1}\cos x - \frac{1}{D^2+1}\cos 3x\right]$
- $= \frac{1}{2}\left[x \cdot \frac{1}{2D}\cos x - \frac{1}{-9+1}\cos 3x\right]$
- $= \frac{x}{4}\sin x + \frac{1}{16}\cos 3x$
- G.S. is given by $y = y_c + y_p$
- $\therefore y = c_1 \cos x + c_2 \sin x + \frac{x}{4}\sin x + \frac{1}{16}\cos 3x$

Example 15

- **Solve $\left(D^2+4\right)y=\cos(2x+3)$**
- **Solution:** The A.E. is given by $f(D)=0$
- $D^2+4=0$
- $D=\pm 2i$
- $y_c=e^{ox}[c_1\cos 2x+c_2\sin 2x]$
- $y_p=\frac{1}{f(D)}X=\frac{1}{D^2+4}\cos(2x+3)$
- $=x\cdot\frac{1}{2D}\cos(2x+3)$
- $=\frac{x}{2}\frac{\sin(2x+3)}{2}$
- $=\frac{x}{4}\sin(2x+3)$

Example 16

- **Solve $(D-1)^2\left(D^2+1\right)y = e^x + \sin^2\frac{x}{2}$**
- **Solution:** The A.E. is $(D-1)^2(D^2+1)=0$
- $D = 1,\ \ 1,\ \ \pm i$
- $C.F. = y_c = (c_1 + c_2\,x)e^x + (c_3\cos x + c_4\sin x)$
- $P.I. = y_p = \frac{1}{(D-1)^2(D^2+1)}e^x + \frac{1}{(D^2-2D+1)(D^2+1)}\left[\frac{1-\cos x}{2}\right]$
- $= \frac{1}{(D-1)^2(2)}e^x + \frac{1}{2}\left[\frac{1}{(D^2-2D+1)(D^2+1)}e^{ox} - \frac{1}{(D^2-2D+1)(D^2+1)}\cos x\right]$
- $= \frac{1}{2}\cdot x\cdot\frac{1}{2(D-1)}e^x + \frac{1}{2}\left[\frac{1}{(0-0+1)}e^{ox} - \frac{1}{(-1-2D+1)(D^2+1)}\cos x\right]$
- $= \frac{x^2}{4}e^x + \frac{1}{2}\left[1 - \frac{1}{(-2D)(D^2+1)}\cos x\right]$

- PI$=\frac{x^2}{4}e^x+\frac{1}{2}+\frac{1}{4}\cdot\frac{1}{D(D^2+1)}\cos x$
- $=\frac{x^2}{4}e^x+\frac{1}{2}+\frac{1}{4}\cdot x\frac{1}{(2D)D}\cos x$
- $=\frac{x^2}{4}e^x+\frac{1}{2}+\frac{x}{8D^2}\cos x$
- Put $D^2=-1$
- $=\frac{x^2}{4}e^x+\frac{1}{2}+\frac{x}{8(-1)}\cos x$
- $=\frac{x^2}{4}e^x+\frac{1}{2}-\frac{x}{8}\cos x$
- G. S. is given by $y=y_c+y_p$
- $\therefore y=(c_1+c_2\,x)e^x+(c_3\cos x+c_4\sin x)+\frac{x^2}{4}e^x+\frac{1}{2}-\frac{x}{8}\cos x$

Example 17

- **HW) Solve $(D^3 + 1)y = \cos(2x - 1)$**
- **Solution:** A.E. is given by $f(D) = 0$
- $\therefore D^3 + 1 = 0$
- $(D + 1)(D^2 - D + 1) = 0$
- $(D + 1)\left[\frac{1 \pm \sqrt{1-4}}{2}\right] = 0$
- $(D + 1)\left[\frac{1 \pm i\sqrt{3}}{2}\right] = 0$
- $D = -1, \frac{1}{2} + \frac{i\sqrt{3}}{2}, \frac{1}{2} - \frac{i\sqrt{3}}{2}$
- $y_c = c_1 e^{-x} + e^{\frac{x}{2}}\left[c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right]$

- $y_p = \frac{1}{f(D)}X = \frac{1}{D^3+1}\cos(2x-1)$
- $= \frac{1}{-4D+1}\cos(2x-1)$
- $[\text{Put } D^2 = -4]$
- $= \frac{1+4D}{1-16D^2}\cos(2x-1)$
- $= \frac{1+4D}{1+64}\cos(2x-1)$
- $= \frac{1}{65}(1+4D)\cos(2x-1)$
- $= \frac{1}{65}\cos(2x-1) + \frac{4}{65}D\cos(2x-1)$
- $= \frac{1}{65}\cos(2x-1) + \frac{8}{65}(-\sin(2x-1))$
- $= \frac{1}{65}[\cos(2x-1) - 8\sin(2x-1)]$
- G.S. is given by $y = y_c + y_p$
- $\therefore y = c_1e^{-x} + e^{\frac{x}{2}}\left[c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right] + \frac{1}{65}[\cos(2x-1) - 8\sin(2x-1)]$

- **Solve $\left(D^3 + D^2 + D + 1\right)y = \sin^2 x$**
- **Solution:** Given $(D^3 + D^2 + D + 1)y = \sin^2 x$
- The A.E. is given by $f(D) = 0$
- $D^3 + D^2 + D + 1 = 0$
- $(D + 1)(D^2 + 1) = 0$
- $D = -1, \ \ \pm i$
- $C.F. = c_1 e^x + c_2 \cos x + c_3 \sin x$

- $P.I. = \frac{1}{D^3+D^2+D+1}\left(\frac{1-\cos 2x}{2}\right)$
- $= \frac{1}{2}(1) - \frac{1}{2} \cdot \frac{1}{D^3+D^2+D+1} \cos 2x$ Put $D^2 = -4$
- $= \frac{1}{2} - \frac{1}{2}\frac{1}{(-3D-3)} \cos 2x$
- $= \frac{1}{2} + \frac{1}{(2)(3)} \left(\frac{D-1}{D^2-1}\right) \cos 2x$
- $= \frac{1}{2} + \frac{1}{6(-5)} [-2\sin 2x - \cos 2x]$
- $= \frac{1}{2} + \frac{1}{30} [2\sin 2x + \cos 2x]$
- G.S. is given by $y = y_c + y_p$
- $y = c_1 e^x + c_2 \cos x + c_3 \sin x + \frac{1}{2} + \frac{1}{30}[2\sin 2x + \cos 2x]$

Practice Problems

- Solve $(D^2 - 2D + 1)y = e^x + 1$
- Solve $\frac{d^2y}{dx^2} + 3\ \frac{dy}{dx} + 2y = e^{-x}$
- Solve $(D^4 - 4D^3 + 8D^2 - 8D + 4)y = e^x + 1$
- Solve $\frac{d^2y}{dx^2} - (a+b)\frac{dy}{dx} + \ ab\ y = e^{ax} + e^{bx}$
- Solve $\frac{d^3y}{dx^3} - 4\ \frac{dy}{dx} = 2\cos h^2\ 2x$
- Solve $(2D+1)^2 y = 4e^{-\frac{x}{2}}$
- Solve $(D^4 + 1)y = \cos h\ \ 4x \sin h\ \ 3x$
- Solve $\frac{d^2y}{dx^2} + y = \sin x \sin 2x$
- Solve $(D^3 + D^2 + D + 1)y = \sin^2 x$
- Solve $(D^4 + 10D^2 + 9)y = 96 \sin 2x \cos x$

When $X = x^n$

- **Case- III:** When $X = x^n$,

$$P.I. = \frac{1}{f(D)} x^n = [1 \pm f(D)]^{-1} x^n$$ Here n is a positive integer.

- Method of Solving:

1. Take out the lowest degree term from $f(D)$ and reduce it in the form $[1 \pm f(D)]^n$
2. Take it to the numerator i.e., $[1 \pm f(D)]^{-n}$
3. Expand it with binomial theorem,

Use $(1 + D)^{-1} = 1 - D + D^2 - D^3 + \cdots$.

$(1 - D)^{-1} = 1 + D + D^2 + D^3 + \cdots$.

$(D - 1)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \cdots$

Example 18

- **Solve $\left(D^3 + 2D^2 + D\right)y = x^2 + x$**
- **Solution:** Given $D(D^2 + 2D + 1)y = x^2 + x$
- The A.E. is given by $D(D^2 + 2D + 1) = 0$
- $D(D + 1)^2 = 0$
- $D = 0, -1, -1$
- $C.F. = y_c = c_1 + (c_2 + c_3 x)e^{-x}$
- $P.I. = y_p = \frac{1}{D^3+2D^2+D}(x^2 + x)$
- $= \frac{1}{D(1+2D+D^2)}(x^2 + x)$
- $= \frac{1}{D}[1 + (2D + D^2)]^{-1}(x^2 + x)$
- $= \frac{1}{D}[1 - (2D + D^2) + (2D + D^2)^{2} - \cdots..](x^2 + x)$

- $= \frac{1}{D}[1 - 2D - D^2 + 4D^2](x^2 + x)$
- $= \frac{1}{D}[1 - 2D + 3D^2](x^2 + x)$
- $= \frac{1}{D}[(x^2 + x) - 2(2x + 1) + 3(2)]$
- $= \frac{1}{D}[x^2 + x - 4x - 2 + 6]$
- $= \frac{1}{D}[x^2 - 3x + 4]$
- $= \int (x^2 - 3x + 4)\, dx$
- $= \frac{x^3}{3} - \frac{3x^2}{2} + 4x$
- $\therefore$ Complete solution is given by $y = y_c + y_p$
- $\therefore y = c_1 + (c_2 + c_3 x)e^{-x} + \frac{x^3}{3} - \frac{3x^2}{2} + 4x$

Example 19

- **Solve $\left(D^2 - 4D + 4\right)y = 8\left(x^2 + \sin 2x + e^{2x}\right)$**
- **Solution:** The A.E. is given by $f(D) = 0$
- $D^2 - 4D + 4 = 0$
- $(D - 2)^2 = 0$
- $\therefore D = 2,2$
- $y_c = (c_1 + c_2 x)e^{2x}$
- $y_p = \frac{1}{D^2 - 4D + 4} 8(x^2 + \sin 2x + e^{2x})$
- $= P.I._1 + P.I._2 + P.I._3 \ (Say)$

- $P.I._1 = \frac{1}{D^2-4D+4} 8x^2 = 8 \cdot \frac{1}{4} \cdot \frac{1}{\left(1-\frac{D}{2}\right)^2} x^2$
- $= 2\left(1-\frac{D}{2}\right)^{-2} x^2$
- $= 2\left[1+2\left(\frac{D}{2}\right)+3\left(\frac{D}{2}\right)^2+\cdots\ldots.\right]x^2$
- $= 2[1+D+\frac{3}{4}D^2+\cdots.]x^2$
- $= 2\left[x^2+2x+\frac{3}{4}(2)\right]$
- $= 2x^2+4x+3$

- $P.I._2 = \frac{1}{D^2-4D+4} 8 \sin 2x$
- $= 8 \cdot \frac{1}{-4-4D+4} \sin 2x$
- $= -2 \cdot \frac{1}{D} \sin 2x$
- $= -2 \int \sin 2x \, dx$
- $= \cos 2x$

- $P.I._3 = \frac{1}{D^2-4D+4} 8e^{2x}$
- $= 8 \cdot \frac{1}{(D-2)^2} e^{2x}$
- $= 8x \ \frac{1}{2(D-2)} e^{2x}$
- $= 8x^2 \cdot \frac{1}{2} e^{2x}$
- $= 4x^2 e^{2x}$
- $y = y_c + y_p$
- $\therefore y = (c_1 + c_2 x)e^{2x} + 2x^2 + 4x + 3 + \cos 2x + 4x^2 e^{2x}$

Example 20

- **HW Solve $\left(D^3 - D\right)y = 2e^x + 2x + 1 - 4\cos x$**
- **Solution:** The A.E. is given by $D(D^2 - 1) = 0$
- $D = 0, 1, -1$
- $C.F. = y_c = c_1 + c_2 e^x + c_3 e^{-x}$
- $P.I. = y_p = \frac{1}{D^3 - D}(2e^x + 2x + 1 - 4\cos x)$
- $= 2 \cdot \frac{1}{D^3 - D} e^x + 2 \cdot \frac{1}{D^3 - D} x + \frac{1}{D^3 - D} e^{0x} - 4 \cdot \frac{1}{D^3 - D} \cos x$
- $= 2x \frac{1}{3D^2 - 1} e^x - 2 \cdot \frac{1}{D - D^3} x + x \ \frac{1}{3D^2 - 1} e^{0x} + 4 \cdot \frac{1}{-D - D} \cos x$

- $= 2x\ \frac{1}{3-1} e^{x} - 2 \cdot \frac{1}{D(1-D^2)} x + x \cdot \frac{1}{-1} e^{0x} + \frac{4}{2} \int \cos x\ \ dx$
- $= \frac{2}{2} x e^{x} - \frac{2}{D} (1 - D^2)^{-1} x - x + 2 \sin x$
- $= x e^{x} - \frac{2}{D} (1 + D^2) x - x + 2 \sin x$
- $= x e^{x} - \frac{2}{D} (x) - x + 2 \sin x$
- $= x e^{x} - 2 \int x\ dx - x + 2 \sin x$
- $= x e^{x} - x^2 - x + 2 \sin x$
- Complete Solution is given by $y = y_c + y_p$
- $\therefore\ y = c_1 + c_2 e^{x} + c_3 e^{-x} + x\ e^{x} - x^2 - x + 2 \sin x$

When $X = e^{ax}v$

- When $X = e^{ax}v$,

$$P.I. = \frac{1}{f(D)}\, e^{ax}v$$

$$= e^{ax} \cdot \frac{1}{f(D+a)} v$$

Example 21

- **Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = e^{2x} \sin 3x$**
- **Solution:** Given $(D^2 + D - 6)y = e^{2x} \sin 3x$
- The A.E. is given by $D^2 + D - 6 = 0$
- $D^2 + 3D - 2D - 6 = 0$
- $[D(D + 3) - 2(D + 3)] = 0$
- $(D - 2)(D + 3) = 0$
- $D = 2, -3$
- $\therefore C.F. = y_c = c_1 e^{2x} + c_2 e^{-3x}$

- $P.I. = y_p = \frac{1}{D^2+D-6} e^{2x} \sin 3x$
- $= e^{2x} \cdot \frac{1}{(D+2)^2+(D+2)-6} \sin 3x$
- $= e^{2x} \cdot \frac{1}{D^2+4D+4+D+2-6} \sin 3x$
- $= e^{2x} \cdot \frac{1}{D^2+5D} \sin 3x$
- $= e^{2x} \cdot \frac{1}{5D-9} \sin 3x$
- $= e^{2x} \cdot \frac{5D+9}{25D^2-81} \sin 3x$
- $= e^{2x} \cdot \frac{(5D+9)}{-306} \sin 3x$
- $= \frac{e^{2x}}{-306} [5 \times 3 \cos 3x + 9 \sin 3x]$
- $= \frac{e^{2x}}{-102} [5 \cos 3x + 3 \sin 3x]$
- Complete solution is given by $y = y_c + y_p$
- $\therefore y = c_1 e^{2x} + c_2 e^{-3x} + \frac{e^{2x}}{-102} [5 \cos 3x + 3 \sin 3x]$

Example 22

- **Solve $\left(D^2+4D+4\right)y=\dfrac{e^{-2x}}{x^5}$**
- **Solution:** The A.E. is given by $f(D)=0$
- $D^2+4D+4=0$
- $(D+2)^2=0$
- $D=-2,-2$
- $C.F.=y_c=(c_1+c_2x)e^{-2x}$
- $P.I.=y_p=\frac{1}{D^2+4D+4}e^{-2x}\cdot\frac{1}{x^5}$
- $=e^{-2x}\cdot\frac{1}{(D-2)^2+4(D-2)+4}\cdot\frac{1}{x^5}$
- $=e^{-2x}\cdot\frac{1}{D^2-4D+4+4D-8+4}\;x^{-5}$
- $=e^{-2x}\cdot\frac{1}{D^2}\;x^{-5}$

- $= e^{-2x} \cdot \frac{1}{D} \int x^{-5}\, dx$
- $= e^{-2x} \frac{1}{D} \left[\frac{x^{-4}}{-4}\right]$
- $= \frac{e^{-2x}}{-4} \int x^{-4}\, dx$
- $= \frac{e^{-2x}}{-4} \left[\frac{x^{-3}}{-3}\right]$
- $= \frac{e^{-2x}}{12x^3}$
- Complete Solution is given by $y = y_c + y_p$
- $\therefore y = (c_1 + c_2 x)e^{-2x} + \frac{e^{-2x}}{12x^3}$

Example 23

- Solve $(D^2+2)y = e^x \cos x + x^2 e^{3x}$
- **Solution:** The A.E. is given by $D^2+2=0$
- $D^2=-2, D=\pm\sqrt{2}i$
- $C.F. = y_c = c_1 \cos\sqrt{2}\ x + c_2 \sin\sqrt{2}\ x$
- Let $P.I. = P.I._1 + P.I._2$ where $P.I._1 = \frac{1}{D^2+2} e^x \cos x$
- $P.I._1 = e^x \cdot \frac{1}{(D+1)^2+2} \cos x$
- $= e^x \cdot \frac{1}{D^2+2D+3} \cos x$
- $= e^x \cdot \frac{1}{2+2D} \cos x$
- $= e^x \cdot \frac{2D-2}{4D^2-4} \cos x$
- $= e^x \cdot \frac{(2D-2)}{-8} \cos x$
- $= \frac{e^x}{-8}[-2\sin x - 2\cos x] = \frac{e^x}{4}[\sin x + \cos x]$

- $P.I._2 = \frac{1}{D^2+2} e^{3x} x^2$
- $= e^{3x} \frac{1}{(D+3)^2+2}\ x^2$
- $= e^{3x} \frac{1}{D^2+6D+11} x^2$
- $= \frac{e^{3x}}{11} \left[1 + \frac{6D+D^2}{11}\right]^{-1}\ x^2$
- $= \frac{e^{3x}}{11} \left[1 - \left(\frac{D^2+6D}{11}\right) + \left(\frac{D^2+6D}{11}\right)^2\right] x^2$
- $= \frac{e^{3x}}{11} \left[1 - \left(\frac{D^2+6D}{11}\right) + \frac{36D^2}{121}\right] x^2$
- $= \frac{e^{3x}}{11} \left[x^2 - \frac{2}{11} - \frac{12x}{11} + \frac{72}{121}\right]$
- $= \frac{e^{3x}}{11} \left[x^2 - \frac{12x}{11} + \frac{50}{121}\right]$
- Complete Solution is given by $y = y_c + y_p$

Example 24

- **Solve $(D^2 - 1)y = x^2 \cos x$**
- **Solution:** The A.E. is given by $D^2 - 1 = 0$
- $D^2 = \pm 1$
- $C.F. = y_c = c_1 e^{x} + c_2 e^{-x}$
- $P.I. = y_p = R.P.of\ \frac{1}{D^2-1} x^2 e^{ix}$
- $= R.P.of\ e^{ix} \frac{1}{(D+i)^2-1} x^2$
- $= R.P.of\ e^{ix} \frac{1}{D^2+2Di-2} x^2$
- $= R.P.of\ e^{ix} \frac{1}{(-2)} \left[1 - \left(\frac{D^2+2Di}{2}\right)\right]^{-1} x^2$

- $= R.P.of\ e^{ix}\frac{1}{(-2)}\left[1+\left(\frac{D^2+2Di}{2}\right)+\left(\frac{D^2+2Di}{2}\right)^2+\cdots.\right]x^2$
- $= R.P.of\ e^{ix}\frac{1}{(-2)}\left[1+\frac{D^2+2Di}{2}+\frac{4D^2i^2}{4}\right]x^2$
- $= R.P.of\ e^{ix}\frac{1}{(-2)}\left[x^2+\frac{2}{2}+2xi+2(-1)\right]$
- $= R.P.of\ e^{ix}\frac{1}{(-2)}[x^2+2xi-1]$
- $= R.P.of\ (\cos x+i\sin x)\frac{1}{(-2)}[x^2+2xi-1]$
- $= \frac{1}{(-2)}[x^2\cos x-\cos x-2x\ \sin x\ ]$
- $= x\sin x+\frac{1}{2}(1-x^2)\cos x$
- Complete Solution is given by $y=y_c+y_p$

Example 25

- **Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$**
- **Solution:** Given $(D^2 + 3D + 2)y = e^{e^x}$
- The A.E.is $f(D) = 0$
- $D^2 + 3D + 2 = 0$
- $(D+1)(D+2) = 0$
- $D = -1, -2$
- $C.F. = y_c = c_1e^{-2x} + c_2e^{-x}$
- $P.I. = y_p = \frac{1}{D^2+3D+2}e^{e^x}$
- $= \frac{1}{(D+2)(D+1)}e^{e^x}$
- $= \left(\frac{1}{D+1} - \frac{1}{D+2}\right)e^{e^x}$

- PI $= \frac{1}{D+1} e^{e^x} - \frac{1}{D+2} e^{e^x} \quad \left[\because \frac{1}{D+a} X = e^{-ax} \int e^{ax} X \, dx\right]$
- $= e^{-x} \int e^x \, e^{e^x} dx - e^{-2x} \int e^{2x} \, e^{e^x} dx$
- Put $e^x = t$
- $= e^{-x}(e^t) - e^{-2x}(te^t - e^t)$
- $= e^{-2x} e^{e^x}$
- $\therefore y = c_1 e^{-2x} + c_2 e^{-x} + e^{-2x} e^{e^x}$ is the required complete solution.

Example 26

- **Solve $\left(D^2+5D+6\right)y = e^{-2x}\sec^2 x(1+2\tan x)$**
- **Solution:** The A.E. is given by $D^2+5D+6=0$
- $(D+2)(D+3)=0$
- $D=-2,-3$
- $C.F.=y_c=c_1e^{-2x}+c_2e^{-3x}$
- $P.I.=y_p=\frac{1}{(D+2)(D+3)}\left[e^{-2x}\sec^2 x(1+2\tan x)\right]$
- $=\frac{1}{D+3}\left[e^{-2x}\int e^{2x}e^{-2x}\sec^2 x(1+2\tan x)\,dx\right]$
- $=\frac{1}{D+3}\left[e^{-2x}\int \sec^2 x(1+2\tan x)\,dx\right]$
- Put $\tan x=t, \sec^2 x\,dx=dt$

- PI= $\frac{1}{D+3}\left[e^{-2x}\int(1+2\,\mathrm{t})\,dt\right]$
- $=\frac{1}{D+3}[e^{-2x}(t+t^2)]$
- $=\frac{1}{D+3}[e^{-2x}(\tan x+\tan^2 x)]$
- $=e^{-3x}\int e^{3x}\,e^{-2x}[(\tan x-1)+\sec^2 x]dx$
- $=e^{-3x}\int e^{x}\,[(\tan x-1)+\sec^2 x]dx$
- $=e^{-3x}\int e^{x}(\tan x-1)$
- $\left[\because \int e^{x}\,[f(x)+f'(x)]dx=e^{x}f(x)\right]$
- $=e^{-2x}(\tan x-1)$
- Complete solution is $c_1e^{-2x}+c_2e^{-3x}+e^{-2x}(\tan x-1)$

Practice Problems

- Solve $\frac{d^2y}{dx^2} + y = cosec\, x$
- Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \sin(e^x)$
- Solve $(D^2 - D - 2)y = 2\log x + \frac{1}{x} + \frac{1}{x^2}$