Module 1
Differential Equations
of
First Order and First Degree

Sub-module 1.5

Method of variation of parameters

Syllabus

Module No.	Unit No.	Details	Hrs.	CO
1	**Differential Equation of First Order and First Degree**		13	CO 1
	1.1	Differential Equation of first order and first degree- Exact differential equations, Equations reducible to exact equations by integrating factors.		
	1.2	Linear differential equations (Review), Equation reducible to linear form. Applications of Differential Equation of first order and first degree		
	1.3	Linear Differential Equation with constant coefficients: Complimentary function, particular integrals of differential equation of the type f(D)y=X, where X is e^{ax}, $\sin(ax+b)$, $\cos(ax+b)$, x^n, $e^{ax}V$		
	1.4	Cauchy's homogeneous linear differential equation		
	1.5	Method of variation of parameters		
		# **Self-learning topic:** Bernoulli's equation. Equation reducible to Bernoulli's equation.		

Method of variation of parameters to solve second order linear DE with constant coefficient: f(D)y=X

- Let $C.F. = y_c = c_1y_1 + c_2y_2$ where c_1 and c_2 are arbitrary constants.

$P.I. = y_p = uy_1 + vy_2$

- Where $u = -\int \frac{y_2X}{W}dx \quad v = \int \frac{y_1X}{W}dx$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

- The complete solution is $y = y_c + y_p$

For Third order

- $C.F. = y_c = c_1y_1 + c_2y_2 + c_3y_3$ where $c_1, c_2 \, and \; c_3$ are arbitrary constants.

$$P.I. = y_p = uy_1 + vy_2 + ty_3$$

- Where $u = \int \frac{(y_2y_3{}' - y_3y_2{}')X}{W} dx$

$$v = \int \frac{(y_3y_1{}' - y_1y_3{}')X}{W} dx \;\&$$

$$t = \int \frac{(y_1y_2{}' - y_2y_1{}')X}{W} dx$$

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1{}' & y_2{}' & y_3{}' \\ y_1{}'' & y_2{}'' & y_3{}'' \end{vmatrix}$$

- The complete solution is $y = y_c + y_p$

Example 1

- **Solve $(D^2 + D)y = \frac{1}{1+e^x}$**
- **Solution:** The A.E. is given by $D^2 + D = 0$

$$D(D+1) = 0$$
$$D = 0, -1$$
$$y_c = c_1 + c_2 e^{-x}$$

- $y_c = c_1 y_1 + c_2 y_2$ (let)

Here $y_1 = 1, y_2 = e^{-x},\ X = \frac{1}{1+e^x}$

- $y_p = u y_1 + v y_2$

$$where \quad u = -\int \frac{y_2 X}{W} dx$$

$$V = \int \frac{y_1 X}{W} dx \qquad \& \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix}$$

- $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 1 & e^{-x} \\ 0 & -e^{-x} \end{vmatrix} = -e^{-x}$
- $u = -\int \frac{e^x}{1+e^x} \cdot \frac{1}{-e^{-x}} dx = \int \frac{1}{1+e^x} dx = \int \frac{e^{-x}}{1+e^{-x}} dx$

  $= -\log(1 + e^{-x})$
- $v = \int 1 \frac{1}{1+e^x} \cdot (\frac{1}{-e^{-x}}) dx = -\int \frac{e^x}{1+e^x} dx = -\log(1 + e^x)$
- $y_p = -\log(1 + e^{-x}) - \log(1 + e^x) e^{-x}$
- $y = y_c + y_p$

  $= c_1 + c_2 e^{-x} - \log(1 + e^{-x}) - e^{-x} \log(1 + e^x)$

is the complete solution.

Example 2

- **Use the method of variation of parameters to solve**

$$y'' + 3y' + 2y = e^{e^x}$$

- **Solution:** Given $y'' + 3y' + 2y = e^{e^x}$
- $(D^2 + 3D + 2)y = e^{e^x}$
- The A.E. is $D^2 + 3D + 2 = 0$
- $D^2 + 2D + D + 2 = 0$
- $D(D+2) + 1(D+2) = 0$
- $(D+1)(D+2) = 0$
- $D = -1, -2$
- $y_c = c_1e^{-x} + c_2e^{-2x}$
- $C.F. = c_1y_1 + c_2y_2$
- Where $y_1 = e^{-x}$, $y_2 = e^{-2x}$, $X = e^{e^x}$
- $W = \begin{vmatrix} y_1, & y_2 \\ y_1', & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -2e^{-3x} + e^{-3x} = -e^{-3x}$

- $P.I. = uy_1 + vy_2$ where $u = -\int \frac{y_2 X}{W} dx$
- $u = +\int \frac{e^{-2x} e^{e^x}}{e^{-3x}} dx$
- Put $e^x = t$
- $e^x dx = dt$
- $u = \int e^x \, e^{e^x} dx = \int e^t dt$
- $= e^t$
- $= e^{e^x}$
- $V = \int \frac{y_1 X}{W} dx = -\int \frac{e^{-x} e^{e^x}}{e^{-3x}} = -\int e^{2x} e^{e^x} dx$
- Put $e^x = t, e^x dx = dt$
- $= -\int t \, e^t dt$
- $= -[t \, e^t - e^t] = e^t - t \, e^t$
- $= e^{e^x} - e^x e^{e^x}$

- $y_p = P.I. = e^{e^x}e^{-x} + \left(e^{e^x} - e^x e^{e^x}\right)e^{-2x}$
- $= e^{e^x}e^{-x} + e^{e^x}e^{-2x} - e^{-x}e^{e^x}$
- $= e^{e^x}\, e^{-2x}$
- $y = y_c + y_p$
- $y = c_1 e^{-x} + c_2 e^{-2x} + e^{e^x}e^{-2x}$ is the required G. S.

Example 3

- **Solve $\left(D^2+1\right)y = cosec\, x \cot x$ using variation of parameters.**
- **Solution:** The A.E. is given by $D^2+1=0$
- $D^2=-1$
- $D=\pm i$
- $y_c = c_1 \cos x + c_2 \sin x$
- $y_c = c_1 y_1 + c_2 y_2$
- where $y_1 = \cos x$, $y_2 = \sin x$, $X = cosec\, x \cot x$
- $W = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = cos^2 x + sin^2 x = 1$

- $P.I. = uy_1 + vy_2$ where $u = -\int \frac{y_2 X}{W} dx$
- $u = -\int \frac{\sin x \, cosec\, x \cot x}{1} dx = -\int \cot x \;\, dx = -\log \sin x$
- $v = \int \frac{y_1 X}{W} dx$
- $= \int \frac{\cos x \, cosec\, x \cot x}{1} dx = \int cot^2 x \, dx$
- $= \int \frac{cos^2 x}{sin^2 x} dx = \int \frac{1 - sin^2 x}{sin^2 x} dx$
- $= \int (cosec^2 x - 1) dx$
- $= -\cot x - x$
- $y_p = uy_1 + vy_2$
- $= -\log \sin x \cos x - \sin x \,(\cot x + x)$
- The G. S. is given by $y = y_c + y_p$
- $\therefore y = c_1 \cos x + c_2 \sin x - \log \sin x \cos x - \sin x \,(\cot x + x)$

Example 4

- **By using method of variation of parameters solve $\left(D^2 - 4D + 4\right)y = e^{2x} sec^2 x$**
- **Solution:** The A. E. is given by $(D-2)^2 = 0$
- $D = 2, 2$
- $\therefore C.F. = y_c = (c_1 + c_2 x)e^{2x}$
- $= c_1 e^{2x} + c_2 x e^{2x}$
- $y_c = c_1 y_1 + c_2 y_2$
- Here $y_1 = e^{2x},\ y_2 = xe^{2x},\ \ X = e^{2x} sec^2 x$
- $P.I. = y_p = u y_1 + v y_2$
- Where $u = -\int \frac{y_2 X}{W} dx, \quad v = \int \frac{y_1 X}{W} dx, \quad w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

- $W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = e^{4x} + 2xe^{4x} - 2xe^{4x}$
- $W = e^{4x}$
- $u = -\int \frac{xe^{2x}e^{2x}sec^2x}{e^{4x}} dx = -\int x\, sec^2x\, dx$
- $= -\left[x \int sec^2x\, dx - \int \tan x \cdot 1\, dx\right]$
- $= -x \tan x + \log \sec x$
- $v = \int \frac{y_1 X}{W} dx = \int \frac{e^{2x}e^{2x}sec^2x}{e^{4x}} dx = \int sec^2x\, dx = \tan x$
- $P.I. = y_p = (-x \tan x + \log \sec x)e^{2x} + \tan x \;\; x\, e^{2x}$
- $= e^{2x} \log \sec x$
- G. S. is given by $y = y_c + y_p$
- $\therefore y = c_1 e^{2x} + c_2 x\, e^{2x} + e^{2x} \log \sec x$ is the requires G.S.

Example 5

- **Use method of variation of parameters to solve the equation.**

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-2x}sec^2x(1 + 2\tan x)$$

- **Solution:** Given $(D^2 + 5D + 6)y = e^{-2x}sec^2x(1 + 2\tan x)$
- The A.E. is given by $D^2 + 5D + 6 = 0$
- $(D + 2)(D + 3) = 0$
- $\therefore D = -2, -3$
- $y_c = c_1y_1 + c_2y_2 = c_1e^{-2x} + c_2\, e^{-3x}$
- Here

$$y_1 = e^{-2x},\ y_2 = e^{-3x}, W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2x} & e^{-3x} \\ -2e^{-2x} & -3e^{-3x} \end{vmatrix}$$

- $= -3e^{-5x} + 2e^{-5x} = -e^{-5x}$
- $X = e^{-2x}sec^2x(1 + 2\tan x)$

- $P.I. = y_p = uy_1 + vy_2$
- Where $u = -\int \frac{y_2 X}{W} dx = -\int \frac{e^{-3x} e^{-2x} sec^2 x(1+2\tan x)}{-e^{-5x}} dx$
- $u = \int sec^2 x(1 + 2\tan x)\, dx$
- Put $1 + 2\tan x = t,\ 2\, sec^2 x\, dx = dt,\ sec^2 x\, dx = \frac{1}{2} dt$
- $u = \frac{1}{2}\int t\, dt = \frac{t^2}{4} = \frac{1}{4}[1 + 2\tan x]^2$

- $v = \int \frac{y_1 X}{W} dx = \int \frac{e^{-2x} e^{-2x} sec^2 x(1+2\tan x)}{-e^{-5x}} dx$

$$= -\int e^x [1 + 2\tan x] sec^2 x \, dx$$

- $\therefore v = -e^x \frac{(1+2\tan x)}{2}$
- $\therefore$ The complete solution is $y = y_c + y_p$
- $y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{1}{4}(1 + 2\tan x)^2 - \frac{e^x}{2}(1 + 2\tan x)$

- **Use method of variation of parameters to solve the equation** $(D^3+D)y = cosecx$
- $AE = D^3 + D = 0, D = 0, i, -i$
- $C.F. = y_c = c_1 + c_2 cosx + c_3 sinx$
- $C.F. = y_c = c_1 y_1 + c_2 y_2 + c_3 y_3$

where $c_1, c_2 and\ c_3$ are arbitrary constants &

$$y_1 = 1$$
$$y_2 = cosx$$
$$y_3 = sinx$$

Let $P.I. = y_p = uy_1 + vy_2 + ty_3$

Let $P.I. = y_p = uy_1 + vy_2 + ty_3$

- Where $u = \int \frac{(y_2 y_3{}' - y_3 y_2{}')X}{W} dx$

$$v = \int \frac{(y_3 y_1{}' - y_1 y_3{}')X}{W} dx \ \&$$

$$t = \int \frac{(y_1 y_2{}' - y_2 y_1{}')X}{W} dx$$

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1{}' & y_2{}' & y_3{}' \\ y_1{}'' & y_2{}'' & y_3{}'' \end{vmatrix} = \begin{vmatrix} 1 & cosx & sinx \\ 0 & -sinx & cosx \\ 0 & -cosx & -sinx \end{vmatrix} = 1$$

- $u = \int \frac{(y_2 y_3{}' - y_3 y_2{}')X}{W} dx = \int cosec\, x\, dx$
  $= \log(cosecx - cotx)$
- $v = \int \frac{(y_3 y_1{}' - y_1 y_3{}')X}{W} dx$
  $= \int (sinx * 0 - cosx * 1) cosecx\, dx$
  $= \int -cotx\ \ dx = -\log sinx$

$t = \int \frac{(y_1 y_2{}' - y_2 y_1{}')X}{W} dx = \int (-sinx * 1 - cosx * 0) cosecx\, dx$

$= \int -1\ \ dx = -\mathrm{x}$

- PI=$\log(cosecx - cotx) - (\log sinx)cosx + xsinx$
- The complete solution is $y = y_c + y_p$

Practice Problems

1. Solve $\frac{d^2y}{dx^2} + y = x \sin x$

❖ Answer: $\left[y = c_1 \cos x + c_2 \sin x + \frac{x}{4} \sin x - \frac{x^2}{4} \cos x + \frac{1}{8} \cos x\right]$

2. Solve $(D^2 - 4D + y) = \frac{e^{2x}}{x}$

❖ Answer: $[y = c_1 e^{2x} + c_2 x\, e^{2x} - x\, e^{2x} + x \log x\, e^{2x}]$

3. Solve $(D^2 + 3D + 2)y = 10e^{3x} + 4x^2$

❖ Answer: $\left[y = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{2} e^{3x} + 2x^2 - 6x + 7\right]$

4. Solve $\frac{d^2y}{dx^2} - y = e^{-x} \sin(e^{-x}) + \cos(e^{-x})$

❖ Answer: $[y = c_1 e^{x} + c_2 e^{-x} - e^{x} \sin(e^{-x})]$

5. Solve $(D^2 + 1)y = \sec x$

❖ Answer: $y = c_1 \cos x + c_2 \sin x + \cos x \log(\cos x) + x \sin x$