SOMAIYA VIDYAVIHAR UNIVERSITY
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Module 3
Integration :
Review And Some New Techniques

Sub-Module 3.1

Beta functions with properties

Syllabus

3	**Integration : Review And Some New Techniques**		**7**	**CO 3**
	3.1	Beta and Gamma functions with properties		
	3.2	Differentiation under integral sign with constant limits of integration.(without proof)		
		# Self-learning topic: Differentiation under integral sign with variable limits of integration.		

Beta Functions

- **Beta Functions:** The definite integral

$\int_0^1 x^{m-1}(1-x)^{n-1}dx,\ \ (Re(m), Re(n) > 0)$ is called Beta function denoted by $\beta(m,n)$. It is a function of parameters m and n.

- Thus,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

- $\beta(m,n)$ is defined for all n except negative integers (i.e. except $n = -1, -2, -3 \dots.)$

Properties of Beta Function

1. $\beta(m,n) = \beta(n,m)$

2. Relation between Beta & Gamma function :

$$\beta(m,n) = \frac{\overline{|m}\,\overline{|n}}{\overline{|m+n}}$$

3. $\beta(m,n) = \int_0^\infty \frac{x^{(m-1)}}{(1+x)^{m+n}}\,dx$ (Definition 2)

❖ **Proof:** Consider the integral $\int_0^\infty \frac{x^{(m-1)}}{(1+x)^{m+n}}\,dx$

Put $x = \frac{t}{1-t} \Rightarrow dx = \frac{(1-t)-t(-1)}{(1-t)^2}\,dt$

$$\Rightarrow dx = \frac{1}{(1-t)^2}\,dt$$

When $x = 0,\ \frac{t}{1-t} = 0 \Rightarrow t = 0$

When $x = \infty,\ \frac{t}{1-t} = \infty \Rightarrow 1 - t = 0 \Rightarrow t = 1$

$$\therefore \int_0^\infty \frac{x^{(m-1)}}{(1+x)^{m+n}}dx = \int_0^1 \left(\frac{t}{1-t}\right)^{m-1} \frac{1}{\left(1+\frac{t}{1-t}\right)^{m+n}} \frac{1}{(1-t)^2}\ dt$$

$$= \int_0^1 \frac{t^{m-1}}{(1-t)^{m-1}} \frac{1}{\left(\frac{1-t+t}{1-t}\right)^{m+n}} \frac{1}{(1-t)^2}\ dt$$

$$= \int_0^1 \frac{t^{m-1}}{(1-t)^{m-1+2}}\ (1-t)^{m+n}\ dt$$

$$= \int_0^1 t^{m-1}\ (1-t)^{n-1}\ dt$$

$$= \beta(m,n)$$

❖ $\boldsymbol{\beta(m,n) = \int_0^\infty \frac{x^{(m-1)}}{(1+x)^{m+n}} dx}$

Definition 3

- $\boldsymbol{\beta(m,n) = 2\int_0^{\pi/2} sin^{2m-1}\theta cos^{2n-1}\,\theta d\theta}$

Proof: Consider $\beta(m,n) = \int_0^1 x^{m-1}\,(1-x)^{n-1}dx$

Put $x = \sin^2\theta,\ \ dx = 2\sin\theta\cos\theta\,d\theta$

When $x = 0,\ \ \sin^2\theta = 0,\ \ \ \Rightarrow \theta = 0$

$x = 1,\ \ \sin^2\theta = 1,\ \ \ \Rightarrow \theta = \frac{\pi}{2}$

$$\therefore \beta(m,n) = \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{m-1}\,(1-\sin^2\theta)^{n-1}\,2\sin\theta\cos\theta\ \ d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2m-2}\theta\,(\cos^2\theta)^{n-1}\,\sin\theta\cos\theta\ \ d\theta$$

$$\therefore \beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \ \ d\theta$$

Put $2m - 1 = p$ and $2n - 1 = q$

Then we have,

$$\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta \ \ d\theta$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) \text{..................}(1)$$

- $\boldsymbol{\beta(m,n) = 2\int_0^{\pi/2} sin^{2m-1}\theta cos^{2n-1}\,\theta d\theta}$
- $\boldsymbol{\beta(m,n) = \int_0^{\infty} \frac{x^{(m-1)}}{(1+x)^{m+n}} dx}$
- $\beta(m,n) = \int_0^1 t^{m-1} \ (1-t)^{n-1} \, dt$

❖ $\overline{\left|\frac{1}{2}\right.} = \sqrt{\pi}$

❖ **Proof:** $\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

put p=0 & q=0

$$\int_0^{\frac{\pi}{2}} \sin^0\theta \cos^0\theta d\theta = \frac{1}{2}\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}\frac{\overline{\left|\frac{1}{2}\right.}\,\overline{\left|\frac{1}{2}\right.}}{\overline{\left|\frac{1}{2}+\frac{1}{2}\right.}}$$

$$[\theta]_0^{\frac{\pi}{2}} = \frac{1}{2}\left(\overline{\left|\frac{1}{2}\right.}\right)^2$$

$$\overline{\left|\frac{1}{2}\right.} = \sqrt{\pi}$$

❖ Duplication Formula of Gamma Function:

❖ $2^{2m-1}\overline{|m}\,\overline{\left|m+\frac{1}{2}\right.} = \sqrt{\pi}\,\overline{|2m}$

Proof : $\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

substituting $p = q$ in equation [A]

$$\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^p\theta \; d\theta = \frac{\frac{1}{2}\left(\overline{\left|\frac{p+1}{2}\right.}\right)^2}{\overline{|p+1}}$$

Since $\sin 2\theta = 2\sin\theta\cos\theta$, we have

$$\int_0^{\frac{\pi}{2}} \left(\frac{\sin 2\theta}{2}\right)^p d\theta = \frac{\frac{1}{2}\left(\overline{\left|\frac{p+1}{2}\right.}\right)^2}{\overline{|p+1}}$$

$$\frac{1}{2^p}\int_0^{\frac{\pi}{2}} \sin^p 2\theta \; d\theta = \frac{\frac{1}{2}\left(\overline{\left|\frac{p+1}{2}\right.}\right)^2}{\overline{|p+1}}$$

In the above integral, put $2\theta = t,\ \ 2d\theta = dt,\ \ d\theta = \frac{dt}{2}$

When $\theta = 0,\ \ t = 0$

$\theta = \frac{\pi}{2},\quad t = \pi$

$$\frac{\frac{1}{2}\left(\Gamma\left(\frac{p+1}{2}\right)\right)^2}{\Gamma(p+1)} = \frac{1}{2^p}\int_0^{\frac{\pi}{2}} \sin^p 2\theta\, d\theta = \frac{1}{2^p}\int_0^{\pi} \sin^p t\ \ \frac{1}{2}\, dt = \frac{1}{2^p \cdot 2} \times 2\int_0^{\frac{\pi}{2}} \sin^p t\, dt$$

$$= \frac{1}{2^p}\int_0^{\frac{\pi}{2}} \sin^p t \cos^0 t\, dt$$

$$= \frac{1}{2^p}\cdot\frac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)}$$

$$= \frac{1}{2^{p+1}}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)}$$

Hence, $\dfrac{\frac{1}{2}\left(|\overline{\tfrac{p+1}{2}}\right)^2}{|\overline{p+1}} = \dfrac{\frac{1}{2^{p+1}}|\overline{\tfrac{p+1}{2}}\,|\overline{\tfrac{1}{2}}}{|\overline{\tfrac{p+2}{2}}}$

Put $\frac{p+1}{2} = m$, i.e. $p = 2m - 1$

$$\frac{\frac{1}{2}(|\overline{m})^2}{|\overline{2m}} = \frac{1}{2^{2m}} \frac{|\overline{m}\,|\overline{\tfrac{1}{2}}}{|\overline{\tfrac{2m+1}{2}}}$$

$$\Rightarrow \frac{|\overline{m}}{|\overline{2m}} = \frac{1}{2^{2m-1}} \frac{|\overline{\tfrac{1}{2}}}{|\overline{m+\tfrac{1}{2}}}$$

$$\Rightarrow 2^{2m-1}|\overline{m}\,|\overline{m+\frac{1}{2}} = |\overline{\tfrac{1}{2}}\,|\overline{2m}$$

Using the formula $|\overline{\tfrac{1}{2}} = \sqrt{\pi}$ we get the duplication formula,

$$2^{2m-1}|\overline{m}\,|\overline{m+\frac{1}{2}} = \sqrt{\pi}\,|\overline{2m}$$