Module 2
Successive Differentiation, Expansion Of Functions, Indeterminate Forms

2.2 Expansion Of Functions

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Module 2

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2	**Successive Differentiation, Expansion Of Functions, Indeterminate Forms**	
	2.1	Successive differentiation: nth derivative of standard functions. Leibnitz's Theorem (without proof) and problems.
	2.2	Taylor's Theorem (only statement) and Taylor's series, Maclaurin's series(only Statement) Expansion of e^x, $sinx, cosx, tanx$
		#Self-learning topic: Expansion of sinh(x), cosh(x), tanh(x), log(1 + x), Indeterminate forms, L'Hospital Rule, problemsinvolving series

Taylor's Theorem

- Let $f(x)$be defined in $[a, a+h]$ and

i)$f(x), f'(x), \ldots f^{(n-1)}(x)$ be continuous in $[a, a+h]$

ii)$f^{(n)}(x)$ exists in $(a, a+h)$ then,

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + Rn \text{ ....(1)}$$

Where Rn is the remainder after n terms which is given by,

$\frac{h^n}{n!}f^{(n)}(a+\theta h)$ where $0 < \theta < 1$is called Lagrange's form of Remainder.

- Note:

1. In Taylor's theorem, if $Rn \to 0$as $n \to \infty$, we get the Taylor's Series. It can be written as

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^n}{n!}f^{(n)}(a) \text{...(2)}$$

Or $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots + \frac{h^n}{n!}f^{(n)}(x) + \cdots$

2. Interchanging x & h, we get,

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!}f''(h) + \cdots + \frac{x^n}{n!}f^{(n)}(h) + \cdots$$

3. In (2) replace h by $x - a$ we get,

$$f(a + x - a) = f(x)$$

$$= f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots \text{.........(3)}$$

The above series is called the Taylor's series about $x = a$.

4. In (3) if we put $a = 0$ then,

$$f(x) = f(0) + (x)f'(0) + \frac{(x)^2}{2!}f''(0) + \cdots \text{..........(4).}$$

This series is called the Maclaurin's series.

Standard Expansions

- **Example 1: Expand $sinx$ in powers of x.**
- Solution: Let $f(x) = sinx$ $\quad f(0) = 0$
- $f'(x) = \cos x$ $\quad f'(0) = 1$
- $f''(x) = -sinx$ $\quad f''(0) = 0$
- $f'''(x) = -\cos x$ $\quad f'''(0) = -1$
- $f^{iv}(x) = sinx$ $\quad f^{iv}(0) = 0$
- $f^{v}(x) = \cos x$ $\quad f^{v}(x) = 1$

& so on

By Maclaurin's Series,

$$f(x) = f(0) + (x)f'(0) + \frac{(x)^2}{2!}f''(0) + \cdots$$

$$\therefore sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

- Similarly we can find expansions of
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$
- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$
- $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$
- $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$
- (Try proving !!!)

Example 2

- **Prove that** $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$
- Proof: $y = f(x) = tanx$ $\qquad f(0) = 0$
- $y_1 = f'(x) = sec^2x = 1 + tan^2x = 1 + y^2$ $\qquad f'(0) = 1$
- $y_2 = f''(x) = 2yy_1$ $\qquad f''(0) = 0$
- $y_3 = f'''(x) = 2yy_2 + 2y_1y_1$ $\qquad f'''(0) = 2$
- $y_4 = f^{iv}(x) = 4y_1y_2 + 2y_1y_2 + 2yy_3$ $\qquad f^{iv}(0) = 0$
- $y_5 = f^{v}(x) = 6y_2y_2 + 8y_1y_3 + 2yy_4$ $\qquad f^{v}(0) = 16$

& so on

By Maclaurin's Series,

$$f(x) = f(0) + (x)f'(0) + \frac{(x)^2}{2!}f''(0) + \cdots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

Self Study

- Prove that $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - + \cdots$
- Expand $tanhx$ in powers of x.

Example 3

❖ **Expand $f(x) = e^x$ in powers of $(x-3)$.**

❖ Solution: From the Taylor's series expansion about $x = a$,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots \quad ....(1)$$

Here, $a = 3$

$$f(x) = e^x = f'(x) = \cdots = f^{(n)}(x)$$

By (1), $e^x = f(3) + (x-3)f'(3) + \frac{(x-3)^2}{2!}f''(3) + \cdots$

$$e^x = e^3 + (x-3)e^3 + \frac{(x-3)^2}{2!}e^3 + \cdots$$

HW. **Expand $f(x) = tan^{-1}x$ in powers of $\left(x - \frac{\pi}{4}\right)$.**

Expand $f(x) = \frac{1}{1-x}$ in powers of $(x+2)$

Example 4

- **Expand $f(x) = 2x^3 + 7x^2 + x - 6$ in powers of $(x-2)$. Hence find approximate value of $f(21/10)$.**
- Solution: From the Taylor's series expansion about $x = a$,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots \text{.......(1)}$$

Here, $a = 2$

$f(x) = 2x^3 + 7x^2 + x - 6$ $\qquad$ $f(2) = 40$

$f'(x) = 6x^2 + 14x + 1$ $\qquad$ $f'(2) = 53$

$f''(x) = 12x + 14$ $\qquad$ $f''(2) = 38$

$f'''(x) = 12$ $\qquad$ $f'''(2) = 12$

$f^{iv}(x) = 0 = f^{v}(x) = \cdots = f^{(n)}(x) = 0$

By (1),

$$2x^3 + 7x^2 + x - 6 = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \cdots$$

$$2x^3 + 7x^2 + x - 6$$

$$= 40 + 53(x-2) + 38\,\frac{(x-2)^2}{2!} + 12\,\frac{(x-2)^3}{3!}$$

$$= 40 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3 \ldots\text{.(2)}$$

f(21/10)

In (2) put $x = 2.1$ & $x - 2 = 0.1$

$$\therefore f\left(\frac{21}{10}\right) = 40 + 53(0.1) + 19(0.01) + 2(0.001) = 45.492$$

HW. Expand $f(x) = x^5 - x^4 + x^3 - x^2 + x - 1$ in powers of $(x-1)$Hence find f(11/10) & also f(0.99).

Example 5

- **Expand**

$\boldsymbol{f(x) = (x+2)^4 + 5(x+2)^3 + 6(x+2)^2 + 7(x+2) + 8}$

in ascending powers of $(x+1)$.

- Solution: From the Taylor's series expansion about $x = a$,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots \text{.......(1)}$$

Here, $a = -1$

$$f(x) = (x+2)^4 + 5(x+2)^3 + 6(x+2)^2 + 7(x+2) + 8$$

$$f(-1) = 27$$

$$f'(x) = 4(x+2)^3 + 15(x+2)^2 + 12(x+2) + 7 \qquad f'(-1) = 38$$

$$f''(x) = 12(x+2)^2 + 30(x+2) + 12 \qquad f''(-1) = 54$$

$$f'''(x) = 24(x+2) + 30 \qquad f'''(-1) = 54$$

$$f^{iv}(x) = 24 \qquad f^{iv}(-1) = 24$$

$$f^{v}(x) = \cdots = f^{(n)}(x) = 0$$

By (1),

$$f(x) = f(-1) + (x+1)f'(-1) + \frac{(x+1)^2}{2!}f''(-1) + \frac{(x+1)^3}{3!}f'''(-1) + \frac{(x+1)^4}{4!}f'^{v}(-1) + \cdots$$

$$= 27 + 38(x+1) + 27(x+1)^2 + 9(x+1)^3 + (x+1)^4$$

Example 6

- **Using Taylor's theorem , arrange in power's of x.**

$-(x+2)^5+(x+2)^4+3(x+2)^3+(x+2)+7$

- **Solution:** Given function can be written as $f(x+2)$ and can be compared with $f(x+h)$.

Here, $f(x)=-x^5+x^4+3x^3+x+7$ & $h=2$ and as expansion required in powers of x, we can use

$$f(x+h)=f(h)+xf'(h)+\frac{x^2}{2!}f''(h)+\cdots \text{....(1)}$$

$$f(x)=-x^5+x^4+3x^3+x+7 \qquad f(2)=17$$
$$f'(x)=-5x^4+4x^3+9x^2+1 \qquad f'(2)=-11$$
$$f''(x)=-20x^3+12x^2+18x \qquad f''(2)=-76$$
$$f'''(x)=-60x^2+24x+18 \qquad f'''(2)=-174$$
$$f^{iv}(x)=-120x+24 \qquad f^{iv}(2)=-216$$
$$f^{v}(x)=-120 \ \& \ f^{vi}(x)=0 \qquad f^{v}(2)=-120$$

❖ Putting these values in (1) we get,

$$-(x+2)^5+(x+2)^4+3(x+2)^3+(x+2)+7$$

$$=f(2)+xf'(2)+\frac{x^2}{2!}f''(2)+\frac{x^3}{3!}f'''(2)+\frac{x^4}{4!}f^{iv}(2)+\frac{x^5}{5!}f^{v}(2)$$

$$=17-11x-38x^2-29x^3-9x^4-x^5$$

Example 7

- **Expand $tan^{-1}(x+h)$ in powers of h up to h^3 and find the value of $tan^{-1}(1.0003)$ upto five places of decimals.**

Solution: $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots$ …….(1)

$$f(x) = tan^{-1}x$$

$$f'(x) = \frac{1}{1+x^2}$$

$$f''(x) = -\frac{2x}{(1+x^2)^2}$$

$$f'''(x) = -2\left[\frac{\left(1+x^2\right)^2.1 - x.2\left(1+x^2\right)2x}{(1+x^2)^4}\right]$$

$$= -2\left[\frac{1+x^2-4x^2}{(1+x^2)^3}\right] = \frac{2(3x^2-1)}{(1+x^2)^3}$$

- Put these values in (1)
- $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots$ ....(1)

$$tan^{-1}(x+h) = tan^{-1}x + h\frac{1}{1+x^2} + \frac{h^2}{2!}\left(-\frac{2x}{(1+x^2)^2}\right)$$

$$+\frac{h^3}{3!}\left(\frac{2(3x^2-1)}{(1+x^2)^3}\right) + \cdots .....(2)$$

Now for finding $\boldsymbol{tan^{-1}(1.0003)}$ **, comparing with** $\boldsymbol{f(x+h)}$

$x = 1, h = 0.003$

By (2)

$$tan^{-1}(1+0.003) = tan^{-1}1 + \frac{0.0003}{2} + \frac{(0.0003)^2}{2}\left[-\frac{2}{4}\right]$$

$$+\frac{(0.0003)^3}{3!}\left[\frac{4}{8}\right] + \cdots$$

$$= 0.78541$$

Example 8

- **Show that** $\boldsymbol{sin\left(\frac{\pi}{4}+\theta\right)=\frac{1}{\sqrt{2}}\left(1+\theta-\frac{\theta^2}{2!}-\frac{\theta^3}{3!}+\cdots\right)}$
- Solution: $f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\cdots$ …….(1)

In (1) put $x=\frac{\pi}{4}$ & $h=\theta$ Here $\mathrm{f(x)}=sinx$,

$f'(x)=\cos x$ $\qquad f'(\frac{\pi}{4})=\frac{1}{\sqrt{2}}$

$f''(x)=-sinx$ $\qquad f''\left(\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$

$f'''(x)=-\cos x$ $\qquad f'''\left(\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$

$f^{iv}(x)=sinx$ $\qquad f^{iv}(\frac{\pi}{4})=\frac{1}{\sqrt{2}}$

$f^{v}(x)=\cos x$ $\qquad f^{v}(\frac{\pi}{4})=\frac{1}{\sqrt{2}}$

By (1), $sin\left(\frac{\pi}{4}+\theta\right)=f\left(\frac{\pi}{4}\right)+\theta f'\left(\frac{\pi}{4}\right)+\frac{\theta^2}{2!}f''\left(\frac{\pi}{4}\right)+\cdots$

$$=\frac{1}{\sqrt{2}}\left(1+\theta-\frac{\theta^2}{2!}-\frac{\theta^3}{3!}+\cdots\right)$$

- **HW. Expand logtan$\left(\frac{\pi}{4}+x\right)$ in powers of x**

Example 9

- **HW. Using Taylor's series, Find approximate value of $f(21/10)$ for $f(x) = 2x^3 + 7x^2 + x - 6$.**
- $f\left(\frac{21}{10}\right) = f(2.1) = f(2 + 0.1) = f(x + h)$
- By Taylor's Series we know that
- $f(x + h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots$ …….(1)
- Here $x = 2$ & $h = 0.1$

$f(x) = 2x^3 + 7x^2 + x - 6$	$f(2) = 40$
$f'(x) = 6x^2 + 14x + 1$	$f'(2) = 53$
$f''(x) = 12x + 14$	$f''(2) = 38$
$f'''(x) = 12$	$f'''(2) = 12$

- $f(x + h) = f(2 + 0.1) = f(2.1) = f(\frac{21}{10})$

$$= f(2) + 0.1f'(2) + \frac{(0.1)^2}{2!}f''(2) + \frac{(0.1)^3}{3!}f'''(2) + 0$$

$$= 40 + 53(0.1) + 19(0.01) + 2(0.001) = 45.492$$

Example 10

- **Prove that $\log(secx) = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \cdots$**
- Proof: $y = f(x) = logsecx$ $\quad$ $y(0) = f(0) = 0$
- $y_1 = f'(x) = tanx$ $\quad$ $y_1(0) = f'(0) = 0$
- $y_2 = f''(x) = sec^2x = 1 + tan^2x = 1 + y_1{}^2$ $\quad$ $y_2(0) = f''(0) = 1$
- $y_3 = f'''(x) = 2y_1y_2$ $\quad$ $y_3(0) = f'''(0) = 0$
- $y_4 = f^{iv}(x) = 2y_2y_2 + 2y_1y_3$ $\quad$ $y_4(0) = f^{iv}(0) = 2$
- $y_5 = f^{v}(x) = 4y_3y_2 + 2y_3y_2 + 2y_1y_4$ $\quad$ $y_5(0) = f^{v}(0) = 0$
- $y_6 = f^{vi}(x) = 6y_3y_3 + 8y_2y_4 + 2y_1y_5$ $\quad$ $y_6(0) = f^{v}(0) = 16$

By Maclaurin's Series,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \frac{x^5}{5!}f^{v}(0) + \cdots$$

$$\log(secx) = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \cdots$$

Example 11

Expand $e^x \cos x$ by Maclaurin series.

Example 12

❖ **Expand $\log(1+e^x)$ *in powers of x up to* x^4.**

Solution:By Maclaurin's Series,

$$f(x) = f(0) + (x)f'(0) + \frac{(x)^2}{2!}f''(0) + \cdots......(1)$$

$$f(x) = \log(1+e^x) \qquad f(0) = \log(1+1) = log2$$

$$f'(x) = \frac{1}{(1+e^x)}e^x \qquad f'(0) = \frac{1}{2}$$

$$f''(x) = \frac{(1+e^x)e^x - e^x e^x}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2} \qquad f''(0) = \frac{1}{4}$$

$$f'''(x) = \frac{(1+e^x)^2 e^x - e^x 2(1+e^x)e^x}{(1+e^x)^4} = \frac{e^x - e^{2x}}{(1+e^x)^3} \qquad f'''(0) = 0$$

$$f^{iv}(x) = \frac{(1+e^x)^3(e^x - 2e^{2x}) - (e^x - e^{2x})3(1+e^x)^2 e^x}{(1+e^x)^6} \qquad f^{iv}(0) = -\frac{1}{8}$$

- By (1),

$$\log(1+e^x) = log2 + \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{192}x^4 + \cdots$$

- **HW**
- **Expand in powers of x, $e^x secx$**
- **Show that $e^x \log(1+x) = x + \frac{x^2}{2!} + 2\frac{x^3}{3!} + \cdots$**

Example 13

- Expand e^{xsinx} in powers of x up to x^6
- Solution: We know that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
- $\therefore\ e^{xsinx} = 1 + xsinx + \frac{(xsinx)^2}{2!} + \frac{(xsinx)^3}{3!} + \cdots$ ....(1)
- Since, sinx $= x - \frac{x^3}{3!} + \cdots$, by (1)

$$e^{xsinx} = 1 + x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots\right) + \frac{x^2\left(x-\frac{x^3}{3!}+\cdots\right)^2}{2!} + \frac{x^3(x-\frac{x^3}{3!}+\cdots)^3}{3!} + \cdots$$

$$= 1 + (x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} + \cdots) + \frac{x^2}{2!}\left(x^2 - \frac{2x^4}{3!} + \frac{x^6}{(3!^2)} + \cdots\right)$$

$$+ \frac{x^3}{3!}(x^3 - \cdots) \quad + \cdots$$

- $e^{xsinx} = 1 + x^2 - \frac{x^4}{6} + \frac{x^6}{120} + \frac{x^4}{2} - \frac{x^6}{3!} + \frac{x^6}{3!} + \cdots$
- $= 1 + x^2 + \frac{x^4}{3} + \frac{x^6}{120} + \cdots$

Example 14

- **Expand $\log(1+x+x^2+x^3)$ up to x^8.**
- **Solution**: $\log(1+x+x^2+x^3)$

$$= \log\left((1+x)+x^2(1+x)\right)$$

$$= log(1+x)+log(1+x^2)....(1)$$

But we know that, $log\ (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

By (1), $\log(1+x+x^2+x^3)$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8} \ldots$$

$$+x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots$$

$$= x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{3x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} - \frac{3x^8}{8} \ldots$$

Example 15

❖ **Prove that** $xcosecx = 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \cdots$

Solution: $xcosecx = \frac{x}{sinx}$

$$= \frac{x}{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots}$$

$$= \frac{1}{1 - \frac{x^2}{6} + \frac{x^4}{120} - \ldots}$$

$$= \left[1 - \left(\frac{x^2}{6} - \frac{x^4}{120} + \cdots\right)\right]^{-1}$$

$$= 1 + \left(\frac{x^2}{6} - \frac{x^4}{120} + \cdots\right) + \left(\frac{x^2}{6} - \frac{x^4}{120} + \cdots\right)^2 + \cdots$$

$$= 1 + \frac{x^2}{6} + \left(-\frac{x^4}{120} + \left(\frac{x^2}{6}\right)^2\right) + \cdots$$

$$= 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \cdots$$

Example 16

❖ **Prove that $\sqrt{1+sinx} = 1 + \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{48} + \frac{x^4}{384} - \cdots$**

Solution: $\sqrt{1+sinx} = \sqrt{sin^2x + cos^2x + 2sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}$

$$= \sqrt{\left\{sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)\right\}^2}$$

$$= sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)$$

$$= \left(\frac{x}{2}\right) - \frac{1}{3!}\left(\frac{x}{2}\right)^3 + \cdots + 1 - \frac{1}{2!}\left(\frac{x}{2}\right)^2 + \frac{1}{4!}\left(\frac{x}{2}\right)^4 - \cdots$$

$$= \left(\frac{x}{2}\right) - \frac{x^3}{48} + \cdots + 1 - \frac{x^2}{8} + \frac{x^4}{384} - \cdots$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{48} + \frac{x^4}{384} - \cdots$$

Example 17

❖ **Prove that $e^{e^x} = e\left[1 + x + x^2 + \frac{5x^3}{6} + \frac{5x^4}{8} + \cdots\right]$**

We know that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$ ....(1)

Let y $= x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$ ......(2)

By (1), $e^x = 1 + y$

Now, $e^{e^x} = e^{1+y} = e\, e^y$

$$= \mathrm{e}\{1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \cdots\}$$

$$= e\{1 + \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) + \frac{\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)^2}{2!} + \frac{\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+..\right)^3}{3!}$$

$$+ \frac{\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)^4}{4!} + \cdots\}$$

$$= e\{1 + x + \left(\frac{x^2}{2!} + \frac{x^2}{2!}\right) + \left(\frac{x^3}{3!} + \frac{2x.\frac{x^2}{2!}}{2!} + \frac{x^3}{3!}\right) + \left(\frac{x^4}{4!} + \frac{\left(\frac{x^2}{2!}\right)^2}{2!} + \frac{3x^2\frac{x^2}{2!}}{3!} + \frac{x^4}{4!} + ..\right) + \cdots\}$$

$$\boldsymbol{e^{e^x} = e\left[1 + x + x^2 + \frac{5x^3}{6} + \frac{5x^4}{8} + \cdots\right]}$$

Example 18

- **Prove that** $(1+x)^x = 1 + x^2 - \frac{x^3}{2} + \frac{5x^4}{6} - \cdots$
- **Solution**: $y = (1+x)^x$

$\therefore logy = xlog(1+x)$

$$= x\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right)$$

$$= \left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \cdots\right) = \mathrm{z(say)}$$

$$\therefore y = e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$= 1 + \left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \cdots\right) + \frac{1}{2!}\left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \cdots\right)^2 + \cdots$$

$$= 1 + x^2 - \frac{x^3}{2} + \frac{5x^4}{6} - \cdots$$

Practice Problems

- **Expand the following functions in powers of x.**

1. $\log(xcotx)$
2. $(1+x)^{\frac{1}{x}}$
3. $\sin(e^x - 1)$
4. $sinxsinhx$
5. e^{xcosx}
6. $\log(1 - x + x^2 - x^3)$
7. $\left[\frac{1+e^x}{2e^x}\right]^{1/2}$

- **Using Maclaurin's series expand**

1. a^x
2. sec^2x
3. $e^x secx$
4. $\log(1 + tanx)$
5. $\log(secx + tanx)$