WEEK 1 : TUTORIAL 1

Probability Basics (1)

Sample Space (Ω)

The set of all possible outcomes of an experiment is called the sample space.

Individual elements are denoted by ω and are termed elementary outcomes.

Examples

- (Finite) A single roll of an ordinary die.

$$\Omega = \{1,2,3,4,5,6\}$$

- (Countable) Infinite no. of coin tosses.

$$\Omega = \{H,T\}^{\infty}$$

- (Uncountable) Speed of vehicle measured with infinite precision. $\Omega = R$

Event

Any collection of possible outcomes of an experiment i.e. any subset of Ω

Example

On rolling a die, outcome even (Event $E = \{2,4,6\}$) or odd (Event $O = \{1,3,5\}$)

Set Theory Notations

$$A \subset B \Leftrightarrow x \in A \Rightarrow x \in B$$

$$A = B \Leftrightarrow A \subset B \text{ and } B \subset A$$

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

$$A^c = \{x : x \notin A\}$$

Properties of Set operations

1. Commutativity

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

2. Associativity

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

3. Distributivity

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

4. DeMorgan's Laws

$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$

Disjoint Events

Two events A and B are disjoint (or mutually exclusive) if $A \cap B = \phi$

A sequence of events $A_1, A_2, A_3, \dots$ are pair-wise disjoint if $A_i \cap A_j = \phi$ for all $i \neq j$

Partition

If $A_1, A_2, \dots$ are pair-wise disjoint and $\bigcup_{i=1}^{\infty} A_i = \Omega$, then the collection $A_1, A_2, \dots$ forms a partition of Ω

Sigma Algebra

Given a sample space Ω, a σ-algebra is a collection $\mathcal{F}$ of subsets of Ω, with the following properties

(a) $\phi \in \mathcal{F}$

(b) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$

(c) If $A_i \in \mathcal{F}$ for every $i \in \mathbb{N}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

A set A that belongs to $\mathcal{F}$ is called $\mathcal{F}$-measurable set (event)

Example: Consider $\Omega = \{1,2,3\}$

$\mathcal{F}_1 = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

$\mathcal{F}_2 = \{\phi, \{1,2,3\}\}$

For any Ω (countable or uncountable) 2^{Ω} is always a σ-algebra

For eg., for $\Omega = \{H,T\}$, a feasible σ-algebra is the power set, i.e. $\mathcal{F} = \{\phi, \{H\}, \{T\}, \{H,T\}\}$

However, if Ω is uncountable, then probabilities cannot be assigned to every subset of 2^{Ω}

Probability Measure and Probability Space

A probability measure P on $(\Omega, \mathcal{F})$ is a function $P: \mathcal{F} \rightarrow [0,1]$ satisfying

(a) $P(\phi) = 0 \quad P(\Omega) = 1$

(b) If $A_1, A_2 \dots$ is a collection of pair-wise disjoint members of $\mathcal{F}$, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

The triple $(\Omega, \mathcal{F}, P)$ comprising a set Ω, a σ-algebra $\mathcal{F}$ of subsets of Ω and a probability measure P on $(\Omega, \mathcal{F})$ is called a probability space.

Q Consider a simple experiment of rolling an ordinary die in which we want to identify whether the outcome results in a prime no or not

$\Omega = \{1, 2, 3, 4, 5, 6\}$

$\mathcal{F} = \{\phi, \{1, 4, 6\}, \{2, 3, 5\}, \{1, 2, 3, 4, 5, 6\}\}$

$P : \mathcal{F} \to [0, 1]$

$P(\phi) = 0$

$P(\{1, 4, 6\}) = 0.5$

$P(\{2, 3, 5\}) = 0.5$

$P(\Omega) = 1$

Bonferroni's Inequality

$$P(A \cap B) \geq P(A) + P(B) - 1$$

- It gives a lower bound on the intersection probability which is useful when probability is hard to calculate
- Only useful if the probabilities of individual events are sufficiently large

General Form : $P(\cap_{i=1}^{n} A_i) \geq \sum_{i=1}^{n} P(A_i) - (n-1)$

Boole's Inequality

$P(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i)$, for any sets $A_1, A_2 \ldots$

Gives a useful upper bound for the probability of the union of events.

Conditional Probability

Given two events A and B, if $P(B) > 0$, then the conditional probability that A occurs given that B occurs is defined to be

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Essentially, since event B has occured, it becomes the new sample space.
Conditional probabilities are useful when reasoning in the sense that once we have observed some event, our beleifs or predictions of related events can be updated/improved.

Q A fair coin is tossed twice. What is the probability that both tosses result in heads given that at least one of the tosses resulted in a heads?

$$\Omega = \{HH, HT, TH, TT\}$$

$$P(HH) = P(HT) = P(TH) = P(TT) = \frac{1}{4}$$

~~P(HH)~~ $P(HH \mid \text{at least one toss heads})$

$$= P(HH \mid HT \cup TH \cup HH)$$

$$= \frac{P(HH \cap (HT \cup TH \cup HH))}{P(HT \cup TH \cup HH)}$$

$$= \frac{P(HH)}{P(HT \cup TH \cup HH)} = \boxed{\frac{1}{3}}$$

Bayes' Rule

We have

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = P(A|B)\,P(B)$$
$$P(A \cap B) = P(B|A)\,P(A)$$
$$\therefore \quad P(A|B)\,P(B) = P(B|A)\,P(A)$$

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)} \quad \text{(Bayes' Rule)}$$

It is important as it allows us to compute the conditional probability $P(A|B)$ from the 'inverse' conditional probability $P(B|A)$

Independent Events

Two events A and B, are said to be independent if

$$P(A \cap B) = P(A)\,P(B)$$

More generally, a family $A_i : i \in I$ is called independent if,

$$P(\cap_{i \in J} A_i) = \Pi_{i \in J} P(A_i)$$

for all finite subsets J of I

Conditional Independence

Let A, B and C be three events with $P(C) > 0$. The events A and B are called conditionally independent given C if

$$P(A \cap B | C) = P(A|C)\,P(B|C)$$

or equivalently,

$$P(A | B \cap C) = P(A|C)$$