Assignment 6

Q.1 : Given : X_1 and X_2 are independent and σ_1 and σ_2 subgaussian respectively.

To prove : $X_1 + X_2$ is $\sqrt{\sigma_1^2 + \sigma_2^2}$

Proof :

From the properties of subgaussian variables we know :-

$$\forall\, t \in R, \quad \mathbb{E}\left[\exp(tX)\right] \leq \exp\left(\frac{\sigma^2 t^2}{2}\right)$$

if X is σ - subgaussian

Thus,

$$\mathbb{E}\left[\exp\left(t\,(X_1 + X_2)\right)\right]$$

$$= \mathbb{E}\left[\exp(X_1 t) \cdot \exp(X_2 t)\right] \qquad \left[\begin{array}{l}\text{Given : } X_1 \,\&\, X_2 \\ \text{are independent}\end{array}\right.$$

$$\leq \exp\left(\frac{\sigma_1^2 t^2}{2}\right) \cdot \exp\left(\frac{\sigma_2^2 t^2}{2}\right)$$

$$\leq \exp\left(\frac{t^2}{2}\left(\sigma_1^2 + \sigma_2^2\right)\right)$$

$$\leq \exp\left(\frac{t^2}{2}\left(\sqrt{\sigma_1^2+\sigma_2^2}\right)^2\right)$$

Thus :

$X_1 + X_2$ is $\sqrt{\sigma_1^2+\sigma_2^2}$ subgaussian.

Q.2 : To prove :

$$P\left(\sum_{t=1}^{n}(X_t - \mathbb{E}[X_t]) \geq \epsilon\right) \leq \exp\left(-\frac{2\epsilon^2}{\sum_{t=1}^{n}(b_t-a_t)^2}\right)$$

Proof :

For all $1 \leq i \leq n$, define a new variable Z_i as the difference between X_i and its expectation

$\therefore \; Z_i = X_i - \mathbb{E}[X_i]$

This implies : $\mathbb{E}[Z_i] = 0$

The domain of Z_i may be bounded inside $[a_i - \mathbb{E}(X_i), \, b_i - \mathbb{E}(X_i)]$. The length of the interval must still have length $b_i - a_i$.

Let s be some positive value.

Then,

$$P\left(\sum_{i=1}^{n} z_i \geq t\right) = P\left(\exp\left(s\sum_{i=1}^{n} z_i\right) \geq e^{st}\right)$$

$$\leq \frac{\mathbb{E}\left[\prod_{i=1}^{n} e^{sz_i}\right]}{e^{st}} \quad \text{[Chernoff inequality]} \qquad (1)$$

Also

$$\frac{\mathbb{E}\left[\prod_{i=1}^{n} e^{sz_i}\right]}{e^{st}} = \frac{\prod_{i=1}^{n} \mathbb{E}\left[e^{sz_i}\right]}{e^{st}} \quad \text{[By independence of } z_i\text{]}$$

$$\leq e^{-st} \prod_{i=1}^{n} \exp\left(\frac{s^2 (b_i - a_i)^2}{8}\right) \quad \text{[Hoeffding Lemma]}$$

$$= \exp\left(-st + \frac{s^2}{8}\sum_{i=1}^{n}(b_i - a_i)^2\right) \qquad (2)$$

Let, $s = \dfrac{4t}{\sum_{i=1}^{n}(b_i - a_i)^2}$ and $t = \epsilon$

Substituting in eq (1) we get:

$$P\left(\sum_{i=1}^{n} z_i \geq \epsilon\right) = P\left(\left(\frac{1}{n}\sum_{i=1}^{n} x_i - \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[x_i]\right) \geq \epsilon\right)$$

$$\leq \exp\left[-\frac{4\epsilon}{\sum_{i=1}^{n}(b_i - a_i)^2}\epsilon + \frac{1}{8}\left(\frac{4\epsilon}{\sum_{i=1}^{n}(b_i - a_i)^2}\right)^2 \sum_{i=1}^{n}(b_i - a_i)^2\right]$$

$$= \exp\left[\frac{-4\epsilon^2}{\sum_{i=1}^{n}(b_i - a_i)^2} + \frac{2\epsilon^2}{\left[\sum_{i=1}^{n}(b_i - a_i)^2\right]^2} \sum_{i=1}^{n}(b_i - a_i)^2 \right]$$

$$= \exp\left[\frac{-4\epsilon^2 + 2\epsilon^2}{\sum_{i=1}^{n}(b_i - a_i)^2} \right]$$

$$= \exp\left[\frac{-2\epsilon^2}{\sum_{i=1}^{n}(b_i - a_i)^2} \right]$$

$$\Rightarrow P\left[\sum_{i=1}^{n} (X_i - E[X_i]) \geq \epsilon \right) \leq \exp\left[\frac{-2\epsilon^2}{\sum_{i=1}^{n}(b_i - a_i)^2} \right]$$

Q.3: Given: Regret for an optimally tuned ETC algorithm for subgaussian 2-armed bandi with means μ_1 & $\mu_2 \in \mathbb{R}$ and $\Delta = |\mu_1 - \mu_2|$ is

$$\{T\Delta, \Delta + 4(1 + \max\{0, \log(\frac{T\Delta^2}{4})\}$$

$R_T \leq \min \{ T\Delta, \ldots \frac{\quad}{\Delta} (\quad (\quad \ldots)) \}$

<u>Show that</u>: $R_T \leq \Delta + C\sqrt{T}$ where $C > 0$ is a universal constant

<u>Proof</u>:

$$R_T \leq \min \Big\{ \underbrace{T\Delta}_{\text{first term}}, \underbrace{\Delta + \frac{4}{\Delta}\Big(1 + \max\Big\{0, \log\Big(\frac{T\Delta^2}{4}\Big)\Big\}\Big)}_{\text{2nd term}} \Big\}$$

<u>Scenario 1</u>:

If $\Delta \leq \frac{1}{\sqrt{T}}$, the 2nd term will become too large.

Hence taking the minimum term, we have:

$$R_T \leq T\Delta$$
$$\leq \sqrt{T}$$

<u>Scenario 2</u>:

If $\Delta > \frac{1}{\sqrt{T}}$, first term would be large,

Hence taking the minimum term, we have:

$$R_T \leq \Delta + \frac{4}{\Delta}\Big(1 + \max\Big\{0, \log\Big(\frac{T\Delta^2}{4}\Big)\Big\}\Big)$$

$$\leq \Delta + 4\sqrt{T} + \max_{x>0} \frac{1}{x} \log_+\Big(\frac{Tx^2}{4}\Big) \quad \text{--- (1)}$$

[with $\Delta = \frac{1}{}$

$\sqrt{T}$

Let $F = \frac{1}{x} \log\left(\frac{T x^2}{4}\right)$ — ② L

Differentiating and setting $F' = 0$, we have:

$$\frac{dF}{dx} = \frac{1}{x}\frac{d}{dx}\left(\log\left(\frac{Tx^2}{4}\right)\right) + \log\left(\frac{Tx^2}{4}\right)\frac{d}{dx}\left(\frac{1}{x}\right)$$

$$\Rightarrow 0 = \frac{2}{x^2} - \frac{1}{x^2}\log\left(\frac{Tx^2}{4}\right)$$

$$\Rightarrow 0 = \frac{1}{x^2}\left[2 - \log\left(\frac{Tx^2}{4}\right)\right]$$

$$\Rightarrow 2 - \log\left(\frac{Tx^2}{4}\right) = 0$$

$$\Rightarrow x^2 = \frac{4}{T}e^2$$

$$\Rightarrow x = \frac{2e}{\sqrt{T}}$$

Substituting value of x in ②

$$\frac{1}{x}\log\left(\frac{Tx^2}{4}\right) = \frac{\sqrt{T}}{2e}\log\left(e^2\right)$$

$$= \sqrt{T}\, e^{-1}$$

Substituting this in eq ①, we have

$$R_T \leq \Delta + 4\sqrt{T} + e^{-1}\sqrt{T}$$
$$\leq \Delta + (4 + e^{-1})\sqrt{T}$$
$$R_T \leq \Delta + C\sqrt{T}$$ where $C = 4 + e^{-1}$ is always positive