Assignment 5

1. Update rule for Weighted Majority :

PART (i)

$$\tilde{\omega}_i^{(t+1)} = \tilde{\omega}_i^{(t)} e^{-\eta z_{ti}} \quad \text{--- (1)} \qquad \text{for } \eta = \sqrt{\frac{2 \log d}{n}}$$

Convex optimization Problem :

$$f_t(\omega) = \langle \omega, z_t \rangle \quad \text{where } z_t \in [0,1]^d \ \forall t$$

$$S = \{ w : \|\omega\|_1 = 1, \ \omega > 0 \}$$

FoReL update rule :

$$\forall t, \quad \omega_t = \underset{\omega \in S}{\operatorname{argmin}} \sum_{i=1}^{t-1} f_i(\omega) + R(\omega) \quad \text{--- (2)}$$

Entropy Regularizer

$$R(\omega) = \frac{1}{\eta} \sum_{i=1}^{d} \omega_i \log(\omega_i) \quad \text{--- (3)}$$

FoReL update rule with Entropy Regularizer

$$\omega_t = \underset{\omega \in S}{\operatorname{argmin}} \sum_{i=1}^{t-1} f_i(\omega) + \frac{1}{\eta} \sum_{k=1}^{d} \omega_k \log(\omega_k) \quad \text{--- (4)}$$

Given $f_t(\omega) = \langle \omega_t, z_t \rangle$

$$\sum^{t-1} f_i(\omega) + R(\omega)$$

Let $F_t = \sum_{i=1} f_t(\omega) + R(\omega)$

$$= \sum_{i=1}^{t-1} \langle \omega, z_i \rangle + \frac{1}{\eta} \sum_{k=1}^{d} \omega_k \log(\omega_k)$$

Differentiating w.r.t. ω

$$F'_t = \sum_{i=1}^{t} z_i + \frac{1}{\eta} \sum_{k=1}^{d} \left[\log(\omega_k) + \omega_k / \omega_k \right]$$

$$F'_t = \sum_{i=1}^{t} z_i + \frac{1}{\eta} \sum_{k=1}^{d} \left[\log(\omega_k) + 1 \right]$$

$$= \sum_{i=1}^{t} z_i + \frac{d}{\eta} \sum_{k=1}^{d} \log(\omega_k)$$

Setting $F'_t = 0$ [as $\omega_t = \operatorname{argmin} F_t$]

$$0 = \sum_{i=1}^{t} z_i + \frac{d}{\eta} \sum_{k=1}^{d} \log(\omega_k)$$

$$\Rightarrow \sum_{k=1}^{d} \log(\omega_k) = -\frac{\eta}{d} \sum_{i=1}^{t} z_i$$

Considering for one arm k at round t, we have

$$\log(\omega_t) = -\eta z_{tk}$$

$$\Rightarrow \omega_t = \exp(-\eta z_{tk})$$

$$\Rightarrow \omega_t = \underline{\omega_{t-1} \exp(-\eta z_{(t-1)k})} \quad \text{—— } 5$$

Hence the update rule from FoReL with entropy regularizer is equivalent to the update rule of weighted majority.

PART (ii)

Regret of Weighted Majority :

$$R(n) = \sum_{t=1}^{n} \langle \omega_t , z_t \rangle - \min_{i \in [d]} \sum_{t=1}^{n} V_{ti} \leq \sqrt{2 \log(d)\, n} \qquad (6)$$

Theorem :

$$R(n) \leq R(u) - \min_{u \in S} R(u) + nL^2/\sigma \qquad (7)$$

For FoReL with entropy regularizer

$$\text{Regret}(n) \leq R(u) + \max_{u \in S} \left(-R(u)\right) + \frac{L^2 n}{\sigma}$$

$$\leq R(u) + \frac{1}{\eta} \log d + L^2 n \eta$$

$$\leq \frac{1}{\eta} \sum_{i=1}^{d} u_i \log u_i + \frac{1}{\eta} \log d + L^2 n \eta$$

$$-R(u) = \frac{1}{\eta} \sum_{i=1}^{d} u_i \log \frac{1}{u_i}$$

$$\leq \frac{1}{\eta} \log d \sum_{i=1}^{d} 1/d$$

$$\leq \frac{1}{\eta} \log d$$

Since $\ell(\omega) = \langle \omega_t , z_t \rangle$

given : $f_t(w) = \ldots$

where $S = \{w : \|w\|_1 = 1, w > 0\}$

Entropy regularizer is $1/\eta$ - strongly conv

Also $U \subset S$ and is set of unit vectors

Thus $U \in \{e_1, e_2, e_3 \ldots e_d\}$ where

$e_1 = \{1, 0, \ldots 0\}$

$e_2 = \{0, 1, \ldots 0\}$

$\vdots$

$e_d = \{0, 0, \ldots 1\}$

Now considering u to be any e_i, we have

$$\frac{1}{\eta} \sum u_i \log u_i = 0$$

Thus simplifies to :

$$\text{Regret}(n, u) \leq \frac{1}{\eta} \log d + L^2 n \eta$$

Setting $\eta = \frac{\sqrt{\log d}}{L\sqrt{2T}}$ we get

$$\text{Regret}(n, u) \leq L\sqrt{2n \log d} \quad \text{—— } 8$$

For the prediction with expert advice $L = 1$

$$\therefore R(n, \text{FoReL}, u) \leq \sqrt{2n \log d} \quad \text{—— } 9$$

is similar as the regret bound obtained for weighted majority algorithm.