Week 3 — Lecture 15

Regret bound for Exp3

To show:

$$\bar{R}_n \leq \frac{K}{2} \sum_{t=1}^{n} \eta_t + \frac{\ln K}{\eta_n} \qquad \text{——— (4)}$$

We know: $\hat{l}_{ti} = \frac{l_{ti}}{P_{ti}} \mathbb{1}\{I_t = i\}$

Also $\mathop{\mathbb{E}}_{I_t \sim P_t}[\hat{l}_{ti}] = l_{ti}$

However $\mathop{\mathbb{E}}_{i \sim P_t}[\hat{l}_{ti}] = \sum_{j=1}^{k} \hat{l}_{tj} P_{tj}$

$$= \sum_{j=1}^{k} \frac{l_{tj}}{P_{tj}} \mathbb{1}\{I_t = j\} P_{tj}$$

$$= l_{tI_t}$$

First step:

$$\sum_{t=1}^{n} l_{tI_t} - \sum_{t=1}^{n} l_{tK} = \sum_{t=1}^{t} \mathop{\mathbb{E}}_{i \sim P_t}[\hat{l}_{ti}] - \sum^{t} \mathbb{E}[\hat{l}_{tK}]$$

$\displaystyle t=1 \quad I_t \sim P_t$

Expressing $\mathbb{E}_{i\sim P_t}\left[\hat{l}_{ti}\right]$ in terms of its moment generating function, we have :

$$\mathbb{E}_{i\sim P_t}\left[\hat{l}_{ti}\right] = \frac{1}{\eta_t} \ln \mathbb{E}_{i\sim P} \exp\left(-\eta_t\left(\hat{l}_{ti} - \mathbb{E}_{k\sim P_t} \hat{l}_{tk}\right)\right) \leftarrow (i)$$

$$- \frac{1}{\eta_t} \ln \mathbb{E}_{i\sim P_t} \exp\left(-\eta_t \hat{l}_{ti}\right) \leftarrow (ii)$$

<u>Second Step</u> (bound (i)) :

$$\ln \mathbb{E}_{i\sim P_t} \exp\left(-\eta_t\left(\hat{l}_{ti} - \mathbb{E}_{k\sim P_t} \hat{l}_{tk}\right)\right)$$

$$\boxed{1 - x \leq e^{-x} \leq 1 - x + \frac{x^2}{2} \quad \leftarrow \text{tight bounds on exp} \quad \text{(A)}}$$

$$= \ln \mathbb{E}_{i\sim P_t} \exp\left(-\eta_t \hat{l}_{ti}\right) + \eta_t \mathbb{E}_{k\sim P_t} \hat{l}_{tk}$$

$\uparrow$ is a constant

$$\boxed{\ln x \leq x - 1 \quad \forall\, x \geq 0 \quad \text{(B)}}$$

$$= \mathbb{E}_{i\sim P_t} \exp\left(-\eta_t \hat{l}_{ti}\right) - 1 + \eta_t \mathbb{E}_{k\sim P_t} \hat{l}_{tk} \quad \left[\text{from (B)}\right]$$

$$= \mathop{\mathbb{E}}_{i \sim P_t} \left(\exp(-\eta_t \hat{l}_{ti}) - 1 + \eta_t \hat{l}_{tk} \right)$$

$$\leq \mathop{\mathbb{E}}_{i \sim P_t} \left(\frac{\eta_t^2 \hat{l}_{ti}^2}{2} \right) \quad \text{[from (A)]}$$

$\frac{\eta_t^2}{2}$ is a constant, hence

$$\mathop{\mathbb{E}}_{i \sim P_t} \hat{l}_{ti}^2 = \sum_j \hat{l}_{tj}^2 P_{tj}$$

$$= \sum_j \left(\frac{l_{tj}}{P_{tj}} \mathbb{1}\{I_t = j\} \right)^2 P_{tj}$$

$$= \frac{l_{tI_t}^2}{P_{tI_t}} \leq \frac{1}{P_{tI_t}}$$

[Assuming all losses are between [0, 1]

Hence:

$$\mathop{\mathbb{E}}_{i \sim P_t} \left(\frac{\eta_t^2 \hat{l}_{ti}^2}{2} \right) \leq \frac{\eta_t^2}{2} \frac{1}{P_{tI_t}}$$

<u>Third step</u> (bound (ii)):

$$-\frac{1}{\eta_t} \ln \mathop{\mathbb{E}}_{i \sim P_t} \exp(-\eta_t \hat{l}_{ti})$$

$$= -\frac{1}{\eta_t} \ln \sum_{i=1}^{k} \exp\left(-\eta_t \hat{l}_{ti}\right) P_{ti}$$

def^{n} of P_{ti} from Exp3 Alg

$$= -\frac{1}{\eta_t} \ln \sum_{i=1}^{k} \exp\left(-\eta_t \hat{l}_{ti}\right) \frac{\exp\left(-\eta_t \hat{L}_{(t-1)i}\right)}{\sum_{k} \exp\left(-\eta_t \hat{L}_{(t-1)i}\right)}$$

$$= -\frac{1}{\eta_t} \ln \frac{\sum_{i=1}^{k} \exp\left(-\eta_t \hat{L}_{ti}\right)}{\sum_{i=1}^{k} \exp\left(-\eta_t \hat{L}_{(t-1)i}\right)}$$

$$= \Phi_{t-1}(\eta_t) - \Phi_t(\eta_t)$$

Define :

$$\Phi_t(\eta) = \frac{1}{\eta} \log \frac{1}{K} \sum_{i} \exp\left(-\eta_t \hat{L}_{t\right.$$