Week 4 – Lecture 19

Lemma: Let $w_1, w_2 \ldots$ be the sequence produced by FTL.

Then $\forall u \in S$, we have

$$R(n, FTL, u) = \sum_{t=1}^{n} f_t(w_t) - \sum_{t=1}^{n} f_t(u)$$

$$\leq \sum_{t=1}^{n} f_t(w_t) - \sum_{t=1}^{n} f_t(w_{t+1}) \quad \text{— (1)}$$

← To prove this

Showing this is same as showing

$$\sum_{t=1}^{n} f_t(w_{t+1}) \leq \sum_{t=1}^{n} f_t(u) \quad \text{— (2)}$$

Proving by induction:

For $n=1$ ~~the~~ (2) holds by choice of w_2

Assume the inequality holds for $(n-1)$ rounds

$$\sum_{t=1}^{n-1} f_t(w_{t+1}) \leq \sum_{t=1}^{n-1} f_t(u)$$

add $f_n(w_{n+1})$ on both sides

$$\sum_{t=1}^{n} f_t(w_{t+1}) \leq f_n(w_{n+1}) + \sum_{t=1}^{n-1} f_t(u)$$

Choose $u = w_{n+1}$ then:

$$\leq f_n(w_{n+1}) + \sum_{t=1}^{n-1} f_t(w_{n+1})$$

$$= \sum_{t=1}^{n} f_t(w_{n+1})$$

$$= \min_{u \in S} \sum_{t=1}^{n} f_t(u) \quad \leftarrow \text{from the algo}$$

$$\sum_{t=1}^{n} f_t(w_{t+1}) \leq \sum_{t=1}^{n} f_t(u) \quad \text{for any } u$$

Exploring the algorithm & bound for a convex fun

Taking a quadratic loss function

$$f_t = \frac{1}{2} \| w_t - v_t \|_2^2$$

In each round

$$w_t = \arg\min_{w} \sum_{t=1}^{t-1} \frac{1}{2} \| w - v_i \|_2^2$$

$$w_t = \frac{1}{t-1} \sum_{i=1}^{t} V_i$$

$$w_{t+1} = \frac{1}{t} \sum_{i=1}^{t} V_i = \frac{(t-1)}{(t-1)t} \left(\sum_{i=1}^{t-1} V_i + V_t \right)$$

$$= \left(1 - \frac{1}{t}\right) w_t + \frac{1}{t} V_t$$

$$\Rightarrow w_{t+1} - V_t = \left(1 - \frac{1}{t}\right)(w_t - V_t) \quad \text{———} \quad (3)$$

Now plugging it back in eq (1)

$$f_t(w_t) - f_t(w_{t+1})$$

$$= \frac{1}{2} \| w_t - V_t \|_2^2 - \frac{1}{2} \| w_{t+1} - V_t \|_2^2$$

$$\frac{1}{2} \left(1 - (1 - 1/t)\right)^2 \| w_t - V_t \|_2^2 \quad \text{[Plugging in values for (3)}$$

$= \frac{1}{2}$ [illegible]

$$= \frac{1}{2}\left(-\frac{1}{t^2} + \frac{2}{t}\right) \|w_t - v_t\|_2^2$$

$$\leq \frac{1}{t}\|w_t - v_t\|_2^2 \qquad \left[\text{Ignoring } -1/t^2\right]$$

Assume that $\|v_t\|_2^2 \leq L$

Since w_t are averaging v_t

$\therefore \|w_t\|_2^2 \leq L$ [not for $t=1$, ignore that]

$$\therefore f_t(w_t) - f_t(w_{t+1}) \leq \frac{1}{t}\|w_t - v_t\|_2^2$$

$$= \frac{1}{t}\left(\|w_t\|_2^2 + \|v_t\|_2^2\right) \qquad \left[\text{Applying triangular inequality}\right]$$

$$= \frac{1}{t}(2L)$$

Taking summation on both sides

$$\sum_{t=1}^{n} f_t(w_t) - \sum_{t=1}^{n} f_t(w_{t+1}) \leq \sum_{t=1}^{n} \frac{1}{t}(2L)$$

$$\leq 2L(\log n + 1) \quad \text{——} \quad 4$$

Thus we have a bound that is independent of the u $\qquad \left[\int \frac{1}{x}\,dx = \log x\right]$

Hence:

$$R(n, FTL) \leq 2L(\log n + 1)$$

This bound holds for quadratic function, but does not hold for linear functions.

Hence a regularizer term needs to be added.

Next Lecture