# Week 3 - Lecture 16

Revisiting Exp3 Algo:

Input: $\{\eta_t\}$, $K$

Initialization: $\hat{L}_{0i} = 0 \ \forall i$

for $t = 1, 2, \ldots$

$$P_{ti} = \frac{\exp(-\eta_t \hat{L}_{(t-1)i})}{\sum_{j=1}^{K} \exp(-\eta_t \hat{L}_{(t-1)j})}$$

← note $P_{0i} = \frac{1}{K}$ hence uniform distr.

Pick $I_t \sim P_t$ observe $l_{tI_t}$

$\forall i \in [K]$ update $\hat{l}_{ti} = \frac{l_{ti}}{P_{ti}} 1\{I_t = i\}$

and $\hat{L}_{ti} = \hat{L}_{(t-1)i} + \hat{l}_{ti}$

---

Regret Analysis

$$\bar{R}(n, \text{Exp3}) \leq \frac{K}{2} \sum_{t=1}^{n} \eta_t + \frac{\ln K}{\eta_n} \quad \text{——} \ (1)$$

$= \sqrt{2nK\log K}$ with $\eta_t = \sqrt{\frac{2\log K}{\ldots}}$

$$= 2\sqrt{nK\log K} \quad \text{with } \eta_t = \sqrt{\frac{\log K}{tK}}$$

The whole idea is to show eq 1

$$\sum_{t=1}^{n} l_{tI_t} - \sum_{t=1}^{n} l_{tK} = \sum_{t=1}^{n} \mathbb{E}_{i\sim P_t} \hat{l}_{ti} - \sum_{t=1}^{n} \mathbb{E}\, \hat{l}_{kt} \quad \text{— } 2$$

$$\mathbb{E}_{i\sim P_t} \hat{l}_{it} \leq \frac{\eta_t^2}{2P_{I_t}} + \phi_{t-1}(\eta_t) - \phi(\eta_t) \quad \text{— } 3$$

where $\phi_t(\eta) = \frac{1}{\eta} \log \frac{1}{K} \sum_i \exp(-\eta_t \hat{L}_{ti})$

Substituting eq 3 in eq 2, we have

$$\sum l_{tI_t} - \sum_{t=1}^{n} l_{tK} \leq \frac{\eta_t^2}{2P_{I_t}} + \phi_{t-1}(\eta_t) - \phi_t(\eta_t)$$

$$- \sum_{t=1}^{n} \mathbb{E}_{I_t\sim P_t} \hat{l}_{tK} \quad \text{— } 4$$

We are interested in the expectation of $\sum_{t=1}^{n} l_{tI_t} - \sum_{t=1}^{n} l_{tK}$, hence:

$$\mathbb{E}\left[\sum_{t=1}^{n} l_{tI_t} - \sum_{t=1}^{n} l_{tK}\right] \leq \sum_{t=1}^{n} \left\{ \mathbb{E}\left[\frac{\eta_t^2}{2P_{tI_t}}\right] \right.$$

$$+ \mathbb{E}\left[\phi_{t-1}(\eta_t) - \phi_t(\eta_t)\right]$$

$$- \mathbb{E}\Big[ \sum_{t=1}^{n} \mathbb{E}_{I_t \sim P_t} l_{tk} \Big] \Big] \qquad \text{—— (5)}$$

Now we will bound the Expectation of each term in R.H.S.

$$\mathbb{E}\left[ \sum_{t=1}^{n} \frac{\eta_t^2}{2 P_{tI_t}} \right] = \mathbb{E}\left[ \sum_{t=1}^{n} \mathbb{E}_t \left[ \frac{\eta_t}{2 P_{tI_t}} \right] \right]$$

w.r.t the adversary

w.r.t the probability dist of the learner, $P_t$

$$= \mathbb{E}\left[ \sum_{t=1}^{n} \sum_{j=1}^{K} \frac{\eta_t}{2 P_{tj}} \times P_{tj} \right]$$

$$= \mathbb{E}\left[ \sum_{t=1}^{n} K \frac{\eta_t}{2} \right]$$

$$= \frac{K}{2} \sum_{t=1}^{n} \eta_t \qquad \text{—— (6)}$$

$$\sum_{t=1}^{n} \left[ \phi_{t-1}(\eta_t) - \phi_t(\eta_t) \right]$$

$$= \phi_0(\eta_1) - \phi_1(\eta_1) + \phi_1(\eta_2) - \phi_2(\eta_2) + \phi_2(\eta_3) - \phi_3(\eta_3) + \phi_3(\eta_4) - \phi_4(\eta_4) \ldots \phi_{n-1}(\eta_n) - \phi_n(\eta_n)$$

[Clubbing all $\phi_1$ s, $\phi_2$, ... $\phi_{n-1}$ ]

$$= \sum^{n-1} \left[ \phi(\eta_{t+1}) - \phi(\eta_t) \right] + \phi\,\overset{0}{\cancel{(\eta_1)}} - \phi_n(\eta_n$$

$$= \sum_{t=1} \left[ \phi_t (\eta_{t+1}) \quad (\eta_t \ldots) \right]$$

[Abel Transformation] $\left[ \phi_0(\eta_1) = 0 \quad (\log 1 = 0) \right]$

Therefore :

$$\sum_{t=1}^{n} \left[ \phi_{t-1}(\eta_t) - \phi_t(\eta_t) \right]$$

$$= \sum_{t=1}^{n} \left[ \phi_t(\eta_{t+1}) - \phi_t(\eta_t) \right] - \phi_n(\eta_n) \qquad \text{--- } (7)$$

Also, continuing this :

$$-\phi_n(\eta_n) = +\frac{1}{\eta_n} \log K - \frac{1}{\eta_n} \log \sum_{i=1}^{K} \exp(-\eta_n \hat{L}_{ni})$$

$$\leq \frac{\log K}{\eta_n} - \cancel{\frac{1}{\eta_n}} \cancel{\log \exp} (-\cancel{\eta_n} \hat{L}_{nK})$$

cummulative loss of $K_{th}$ action

$$= \frac{\log K}{\eta_n} + \sum_{t=1}^{n} \hat{l}_{tK}$$

[From the summation only consider the $K^{th}$ - compo the summation will only becom smaller]

Taking expectation of $-\phi_t(\eta_t)$

$$\mathbb{E}\left[-\phi_t(\eta_t)\right] \leq \frac{\log K}{\eta_n} + \mathbb{E}\left[\sum_{t=1}^{n} \mathbb{E}_{I_t \sim P_t} \hat{l}_{tK}\right] \qquad \text{--- } (7a)$$

[$\mathbb{E}$ is w.r randomne

Substituting (6) & (7a) into (5)

$$\mathbb{E}\left[\sum_{t=1}^{n} l_{tI_t} - \sum_{t=1}^{n} l_{tK}\right] \leq \frac{K}{\ldots} \sum^{n} \eta_t$$

$$- \left[\sum_{t=1} \ell_{t} \quad \sum_{t=1} \right] \quad 2 \sum_{t=1}$$

of learne(inside) & adversar(outside)

$$+ \mathbb{E}\left[\sum_{t=1}^{n} \phi_t(\eta_{t-1}) - \phi_t(\eta_t)\right]$$

$$+ \frac{\log K}{\eta_n} + \cancel{\mathbb{E}\left[\sum_{t=1}^{n} \mathbb{E}\, \hat{\ell}_{tk}\right]}$$

$$- \cancel{\mathbb{E} \sum_{t=1}^{n} \mathbb{E}\left[\hat{\ell}_{tk}\right]}$$

If $\mathbb{E}\left[\sum_{t=1}^{n} \phi_t(\eta_{t-1}) - \phi_t(\eta_t)\right] \leq 0$ —— (8)

Then we have shown (1)

$$= \frac{K}{2} \sum_{t=1}^{n} \eta_t + \frac{\log K}{\eta_n}$$

For proving (8), realize that $\eta$ is chosen s.t., it is either a constant or is decreasing with $t$

$$\eta_t = \sqrt{\frac{\log K}{tK}}$$

Hence $\eta_t < \eta_{t-1}$

To prove eq (8) need to show that

$\phi_t(\eta)$ is increasing in $\eta$

That can be shown by taking the derivative of $\phi_t$, such that $\phi_t'(\eta) \geq 0$

This last step is left for us to do, need to knew KL Divergence or simpley take derivative