Assignment 5

1. Update rule for Weighted Majority :

PART (i)

$$\tilde{\omega}_i^{(t+1)} = \tilde{\omega}_i^{(t)} e^{-\eta z_{ti}} \qquad \text{for } \eta = \sqrt{\frac{2\log d}{n}} \quad \text{— (1)}$$

Convex optimization Problem :

$$f_t(\omega) = \langle \omega, z_t \rangle \quad \text{where } z_t \in [0,1]^d \ \forall t$$

$$S = \{ \omega : \|\omega\|_1 = 1, \ \omega > 0 \}$$

FoReL update rule :

$$\forall t, \quad \omega_t = \underset{\omega \in S}{\operatorname{argmin}} \sum_{i=1}^{t-1} f_i(\omega) + R(\omega) \quad \text{— (2)}$$

Entropy Regularizer

$$R(\omega) = \frac{1}{\eta} \sum_{i=1}^{d} \omega_i \log(\omega_i) \quad \text{— (3)}$$

FoReL update rule with Entropy Regularizer

$$\omega_t = \underset{\omega \in S}{\operatorname{argmin}} \sum_{i=1}^{t-1} f_i(\omega) + \frac{1}{\eta} \sum_{k=1}^{d} \omega_k \log(\omega_k) \quad \text{— (4)}$$

Given $f_t(\omega) = \langle \omega_t, z_t \rangle$

Let $F_t = \sum_{i=1} f_t(\omega) + R(\omega)$

$$= \sum_{i=1}^{t-1} \langle \omega, z_i \rangle + \frac{1}{\eta} \sum_{k=1}^{d} \omega_k \log(\omega_k)$$

Differentiating w.r.t. ω

$$F_t' = \sum_{i=1}^{t} z_i + \frac{1}{\eta} \sum_{k=1}^{d} \left[\log(\omega_k) + \omega_k / \omega_k \right]$$

$$F_t' = \sum_{i=1}^{t} z_i + \frac{1}{\eta} \sum_{k=1}^{d} \left[\log(\omega_k) + 1 \right]$$

$$= \sum_{i=1}^{t} z_i + \frac{d}{\eta} \sum_{k=1}^{d} \log(\omega_k)$$

Setting $F_t' = 0$ [as $\omega_t = \text{argmin } F_t$]

$$0 = \sum_{i=1}^{t} z_i + \frac{d}{\eta} \sum_{k=1}^{d} \log(\omega_k)$$

$$\Rightarrow \sum_{k=1}^{d} \log(\omega_k) = -\frac{\eta}{d} \sum_{i=1}^{t} z_i$$

Considering for one arm k at round t, we have

$$\log(\omega_t) = -\eta z_{tk}$$

$$\Rightarrow \omega_t = \exp(-\eta z_{tk})$$

$$\Rightarrow \omega_t = \underline{\omega_{t-1} \exp(-\eta z_{(t-1)k})} \quad \text{——} \quad 5$$

Hence the update rule from FoReL with entropy regularizer is equivalent to the update rule of weighted majority.

PART (ii)

Regret of Weighted Majority :

$$R(n) = \sum_{t=1}^{n} \langle w_t , z_t \rangle - \min_{i \in [d]} \sum_{t=1}^{n} V_{ti} \leq \sqrt{2 \log(d) n} \quad \text{--- (6)}$$