Assignment 8

Q.1 - Let $p, q \in [0,1]$

To prove: Pinsker's Inequality

$$d(p,q) \geq 2(p-q)^2$$

Proof:

Let $g(x) = d(p, p+x) - 2x^2$

$$= p \log\left(\frac{p}{p+x}\right) + (1-p)\log\left(\frac{1-p}{1-p-x}\right) - 2x^2$$

Differentiating w.r.t. x

$$g'(x) = \frac{d}{dx}\left[p\log\left(\frac{p}{p+x}\right)\right] + \frac{d}{dx}\left[(1-p)\log\left(\frac{1-p}{1-p-x}\right)\right] + \frac{d}{dx}(-2x^2)$$

$$= \left[\log\left(\frac{p}{p+x}\right)\cancel{\frac{d}{dx}(p)}^{0} + (p)\frac{d}{dx}\left[\log\left(\frac{p}{p+x}\right)\right]\right]$$

$$+\left[\log\left(\frac{1-p}{1-p-x}\right)\cancel{\frac{d}{dx}(1-p)}^{0} + (1-p)\frac{d}{dx}\log\left(\frac{1-p}{1-p-x}\right)\right]$$

$$-4x$$

$$=\left[p\frac{d}{dx}\left(\log\frac{p}{p+x}\right)\right]+\left[(1-p)\frac{d}{dx}\log\left(\frac{1-p}{1-p-x}\right)\right]-4x$$

Let $y=\dfrac{p}{p+x}$

then $\dfrac{d}{dx}\log\left(\dfrac{p}{p+x}\right) = \dfrac{d}{dy}\log(y)\,\dfrac{d}{dx}\left(\dfrac{p}{p+x}\right)$

$$=\left(\frac{p+x}{p}\right)\left[\frac{(p+x)\cancel{\frac{d}{dx}(p)}^{0}-p\left(\frac{d}{dx}(p+x)\right)}{(p+x)^2}\right]$$

$$=\left(\frac{p+x}{p}\right)\left(\frac{-p}{(p+x)^2}\right)$$

$$=\frac{-1}{p+x}$$

Similarly:

Let $y=\left(\dfrac{1-p}{1-p-x}\right)$

$$\frac{d}{dy}\log(y)\,\frac{d}{dx}\left(\frac{1-p}{1-p-x}\right)$$

$$= \left(\frac{1-p-x}{1-p}\right)\left[\frac{(1-p-x)(0) - (-1)(1-p)}{(1-p-x)^2}\right]$$

$$= \left(\frac{1-p-x}{1-p}\right)\left[\frac{1-p}{(1-p-x)^2}\right]$$

$$= \frac{1}{(1-p-x)}$$

Thus,

$$g'(x) = \frac{-p}{p+x} + \frac{(1-p)}{(1-p-x)} - 4x$$

$$= \frac{-p(1-p-x) + (1-p)(p+x)}{(p+x)(1-p-x)} - 4x$$

$$= \frac{-\cancel{p} + \cancel{p^2} + \cancel{px} + \cancel{p} + x - \cancel{p^2} - \cancel{px}}{(p+x)(1-p-x)} - 4x$$

$$= \frac{x}{(p+x)(1-(p+x))} - 4x$$

$$g'(x) = x\left[\frac{1}{(p+x)(1-(p+x))} - 4\right]$$

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$$g'(x) = x\left[\frac{1}{q(1-q)} - 4\right]$$

We can see that : $g'(0) = 0$.

Also since $q \in [0, 1]$ the maximum value $q(1-q)$ can take is $1/4$, $\therefore q(1-q) \leq 1/4$.

Thus the minimum value of $\frac{1}{q(1-q)}$ is 4 $\therefore$ the value of $\left[\frac{1}{q(1-q)} - 4\right] \geq 0$.

Thus $g'(x) \geq 0$ for $x > 0$ and
$g'(x) \leq 0$ for $x < 0$.

Hence, g is increasing for positive x and decreasing for negative x.

Thus, $x = 0$ is a minimiser of g.

Here, $g(0) = 0$, and so $\underline{g(x) \geq 0 \text{ over } [-p, 1-p]}$.

Hence Proved.