Available online at www.sciencedirect.com

ScienceDirect

Procedia Technology 22 (2016) 582 – 591

Procedia Technology

9th International Conference Interdisciplinarity in Engineering, INTER-ENG 2015, 8-9 October 2015, Tirgu-Mures, Romania

Theoretical Study of the Nonlinear Control Algorithms with Continuous and Discrete-Time State Dependent Riccati Equation

Katalin György[a,*], László Dávid[a], András Kelemen[a]

[a]Department of Electrical Engineering, Faculty of Technical and Human Sciences, Sapientia University, Tîrgu Mureş, România

Abstract

The State-Dependent Riccati Equation (SDRE) control strategy is one of the most efficient approaches for the nonlinear feedback control algorithm by allowing nonlinearities in the system states. This method can be considered as a nonlinear LQR based control design, where the system matrices (and the weight matrices) are functions of the states. SDRE–based techniques have important properties which make them applicable for different nonlinear systems. In this paper we study some properties of these algorithms in order to improve the control efficiency and robustness of nonlinear process control. Also we compare the results of applied suboptimal and optimal SDRE control in continuous and discrete time case.

Peer-review under responsibility of the “Petru Maior” University of Tirgu Mures, Faculty of Engineering

Keywords: SDRE control; Nonlinear system; Optimal control; LQR control; cost function.

1. Introduction

The State-Dependent Riccati Equation (SDRE) approach is one of the most successful methods in nowadays nonlinear optimal control. It has been used in efficient way to many applied nonlinear control problems as missile control, drone control, airplane control, magnetic levitation, etc. [1] [2] [3]. The nonlinear optimal control methods usually involve Jacobian linearization of the system model around each operation point. This linear approximation of the model may introduce errors, especially if the states are far from the equilibrium point, so in these cases the model

* Corresponding author. Tel.: +40-758-662404.
E-mail address: kgyorgy@ms.sapientia.ro, ldavid@ms.sapientia.ro, kandras@ms.sapientia.ro.

Peer-review under responsibility of the “Petru Maior” University of Tirgu Mures, Faculty of Engineering
doi:10.1016/j.protcy.2016.01.123

based control algorithms can introduce some **incompleteness**. The SDRE method is a relatively new technique in nonlinear optimal control, and it offers flexible design using existing linear control algorithms (for example LQR and RHC). The SDRE method can be applied to optimal control of continuous time systems - important for study and analysis of the control algorithm - and also for discrete time systems, important for practical implementation of the control algorithm. Section 2 presents the continuous time SDRE method. After a short overview of continuous state feedback control for LTI and LPV (Linear Parameter Variable) systems [4], this section gives a brief presentation also of sub-optimal and optimal forms of the SDRE method. Section 3 presents the SDRE problem, in case of discrete time systems. Section 4 demonstrates the utility and applicability of the presented methods. In this section we present the implementation of both the sub-optimal and optimal forms of the SDRE method, using two simple theoretical systems.

2. Continuous time SDRE method

The optimal control theory was formulated for general continuous time systems. The LQR approach is the application of this theory for LTI and LPV systems. Thereinafter we present the pure state feedback control theory [5][6]. The investigation of a tracking control problem is similar, but in the tracking equation there appear some extra nonlinear correction expressions, above the modified state dependent Riccati equation.

2.1. The state feedback control for LTI continuous systems

The LQR method was developed first for the time continuous linear systems. The general form of the linear mathematical model is formulated as:

$$\dot{x}(t) = A(t) \cdot x(t) + B(t) \cdot u(t), \tag{1}$$

where $x(t) \in \Re^{nx1}$ is the n dimensional state vector, $u(t) \in \Re^{mx1}$ is the m dimensional input vector, $A(t) \in \Re^{nxn}$ and $B(t) \in \Re^{nxm}$ are time variant matrices. The optimal control can be obtained as the minimization of the cost function given by the following relationship:

$$J(u) = \tfrac{1}{2} x(tf)^T \cdot F \cdot x(tf) + \tfrac{1}{2} \int_{t_0}^{tf} \left[x(t)^T \cdot Q(t) \cdot x(t) + u(t)^T \cdot R(t) \cdot u(t) \right] \cdot dt, \tag{2}$$

where $F, Q(t) \in \Re^{nxn}$ are symmetric positive semi-definite weight matrices, and $R(t) \in \Re^{mxm}$ is a symmetric positive definite matrix. If the pair (A(t), B(t)) is controllable, the linear control action is computed from the following equation [5][6]:

$$u(t) = -R(t)^{-1} \cdot B(t)^T \cdot P(t) \cdot x(t) = -K(t) \cdot x(t) \tag{3}$$

where *P(t)* is the Riccati matrix and it can be calculated from the differential Riccati equation:

$$\dot{P}(t) = -P(t) \cdot A(t) - A(t)^T \cdot P(t) + P(t) \cdot B(t) \cdot R(t)^{-1} \cdot B(t)^T \cdot P(t) - Q(t). \tag{4}$$

The final condition is: $P(t_f) = F$. If the system is LTI, assuming that $tf \to \infty$ and Q and R are constant matrices, the Riccati matrix is a solution of the following algebraic Riccati equation:

$$-P \cdot A - A^T \cdot P + P \cdot B \cdot R^{-1} \cdot B^T \cdot P - Q = 0. \tag{5}$$

The presented LQR algorithm can be applied only if the system model is LTI. If the system is nonlinear the LQR theory is applicable only for the linearized form of the model. The linearization of the system model around an operating point may introduce errors which corrupt the dynamic behavior of controlled system [6]. The SDRE approach is applicable only for a special form of the nonlinear system model, namely in case of state equations with state dependent matrices [1]. Usually this formulation is not unique; the designer has the role to choose the best form.

2.2. The SDRE approach with sub-optimal form of the SDRE method

One possible approach for solving nonlinear optimal control problems is to use the classical LQR methods with substitution of LTI matrices with the state dependent matrices. So for this reason the general continuous SDRE method needs the following pseudo-linear form of the nonlinear model:

$$\dot{x}(t) = f\left(x(t)\right) + g\left(x(t)\right)\cdot u(t) = A(x)\cdot x(t) + B(x)\cdot u(t)\,, \tag{6}$$

where $f(x(t)) = A(x)\cdot x(t)$ and $g(x(t)) = B(x)$. The cost function is with infinite horizon:

$$J(u) = \tfrac{1}{2}\int_0^\infty \left[x(t)^T \cdot Q\cdot x(t) + u(t)^T \cdot R\cdot u(t)\right]\cdot dt \tag{7}$$

where it is considered that the weight matrices (*R* and *Q*) are time invariant only for simplicity of the following calculations. As in the case of an LQR design, if the pair (*A(x), B(x)*) is controllable, the state-feedback control law is calculated as:

$$u(x) = -R^{-1}\cdot B(x)^T\cdot P(x)\cdot x(t) = -K(x)\cdot x(t) \tag{8}$$

Usually the *P(x)* is the unique solution of the following state dependent algebraic Riccati equation (SDRE):

$$-P(x)\cdot A(x) - A(x)^T\cdot P(x) + P(x)\cdot B(x)\cdot R^{-1}\cdot B(x)^T\cdot P(x) - Q = 0 \tag{9}$$

The exact solution of the optimal control problem contains also the solution of the above SDRE equations (8,9) as part of the general problem, but equation (9) must be completed with extra corrective terms. So the solution of SDRE for the system (6) is just suboptimal because there were neglected the derivative terms of the state dependent system matrices (A(x) and B(x)) versus state vector. But the SDRE method (equations 8 and 9) remains very attractive because it can be solved in real time by many well-known methods like: Potter, Schur or Newton [6]. In the following section we present the exact optimal solution of this problem in order to deduce the above mentioned corrective terms.

2.3. Optimal form of the continuous time SDRE control

The exact optimal form of the SDRE control problem can be obtained based on Hamilton-Jacobi theory [5]. The corresponding Hamiltonian expression for the cost function (7) with the state equality constraints (6) is given by:

$$H(x,u,p,t) = \tfrac{1}{2}x(t)^T\cdot Q\cdot x(t) + \tfrac{1}{2}u(t)^T\cdot R\cdot u(t) + p(t)^T\left(A(x)\cdot x(t) + B(x)\cdot u(t)\right), \tag{10}$$

where *p(t)* is the co-state vector: $p(t) = P(x)\cdot x(t)$. The optimal conditions are given by the following Hamilton-Jacobi equations:

$$\frac{\partial H}{\partial u}=0 \Rightarrow R\cdot u(t)+B^T(x)\cdot p(t)=0 \Rightarrow u(t)=-R^{-1}\cdot B^T(x)\cdot p(t)=-R^{-1}\cdot B^T(x)\cdot P(x)\cdot x(t), \tag{11}$$

$$\frac{\partial H}{\partial p}=\dot{x}(t) \Rightarrow A(x)\cdot x(t)+B(x)\cdot u(t)=\dot{x}(t) \Rightarrow \dot{x}(t)=A(x)\cdot x(t)-B(x)\cdot R^{-1}\cdot B^T(x)\cdot P(x)\cdot x(t), \tag{12}$$

$$\frac{\partial H}{\partial x}=-\dot{p}(t) \Rightarrow Q\cdot x(t)+\frac{\partial[A(x)\cdot x(t)]}{\partial x}\cdot p(t)+\frac{\partial[B(x)\cdot u(t)]}{\partial x}\cdot p(t)=-\dot{p}(t), \tag{13}$$

On the other hand:

$$\dot{p}(t)=\dot{P}(x)\cdot x(t)+P(x)\cdot\dot{x}(t)=-Q\cdot x(t)-\frac{\partial[A(x)\cdot x(t)]}{\partial x}p(t)-\frac{\partial[B(x)\cdot u(t)]}{\partial x}p(t) \tag{14}$$

After some mathematical calculation, the modified state dependent Riccati equation results as:

$$\begin{aligned}&\dot{P}(x)=-P(x)A(x)-A^T(x)P(x)+P(x)\cdot B(x)R^{-1}B^T(x)\cdot P(x)-Q-\ldots\\&\ldots-\frac{\partial[A(x)\cdot x(t)]}{\partial x}\cdot P(x)+A^T(x)P(x)-\frac{\partial[B(x)\cdot u(t)]}{\partial x}P(x)=0\end{aligned} \tag{15}$$

The equation SDRE (9) and the modified optimal form of SDRE (15) have a similar structure, but in the second one there appear some extra terms. These extra terms can be considered as correction expression for the SDRE equation:

$$E_c(x)=\dot{P}(x)+\left(\frac{\partial[A(x)\cdot x(t)]}{\partial x}-A^T(x)+\frac{\partial[B(x)\cdot u(t)]}{\partial x}\right)\cdot P(x) \tag{16}$$

If this expression $E_c(x)$ is zero then equations (9) and (15) are identical, namely the results of both control laws coincide. The derivatives from (16) are:

$$\frac{\partial[A(x)\cdot x(t)]}{\partial x}+\frac{\partial[B(x)\cdot u(t)]}{\partial x}=\frac{\partial A(x)}{\partial x}\cdot x(t)+\frac{\partial x^T(t)}{\partial x}A^T(x)+\frac{\partial B(x)}{\partial x}u(t)=\frac{\partial A(x)}{\partial x}\cdot x(t)+A^T(x)+\frac{\partial B(x)}{\partial x} \tag{17}$$

where $\frac{\partial A(x)}{\partial x}$ and $\frac{\partial B(x)}{\partial x}$ are three dimensional tensors, but $\frac{\partial[A(x)\cdot x(t)]}{\partial x}$ and $\frac{\partial[B(x)\cdot u(t)]}{\partial x}$ terms are two dimensional matrices. So the control problem is to solve using efficient numerical methods the nonlinear matrix differential equation (15) which contains the above corrective terms. Section 4 contains comparative examples and results using this method.

3. Discrete time SDRE method

The optimal LQR control theory can be deduced directly for discrete time systems, it can be important for practical implementation of these algorithms. Thereinafter the state feedback control algorithm will be presented in two cases, with finite and infinite horizons.

3.1. The state feedback control for discrete time LTI systems

The DLQR method was developed for the discrete time linear systems [6]. The general form of the linear mathematical model is formulated as:

$$x_{k+1} = \Phi_k \cdot x_k + \Gamma_k \cdot u_k , \tag{18}$$

where $x_k \in \Re^{nx1}$ is the n-dimensional discrete time state vector, $u_k \in \Re^{mx1}$ is the m-dimensional discrete time input vector, $\Phi_k \in \Re^{nxn}$ and $\Gamma_k \in \Re^{nxm}$ are discrete time system matrices.

The cost function to be minimized is given by the following relationship:

$$J(u) = \tfrac{1}{2} x_N^T \cdot F \cdot x_N + \tfrac{1}{2} \sum_{k=0}^{N-1} (x_k^T \cdot Q_k \cdot x_k + u_k^T \cdot R_k \cdot u_k) , \tag{19}$$

where N is the horizon. The linear control sequence can be computed as [6]:

$$u_k = -\left(R_k + \Gamma_k{}^T \cdot P_{k+1} \cdot \Gamma_k\right)^{-1} \Gamma_k{}^T \cdot P_{k+1} \cdot \Phi_k \cdot x_k = -K_k \cdot x_k , \tag{20}$$

where P_{k+1} is the Riccati matrix which can be calculated solving the following discrete time Riccati equation:

$$P_k = \Phi_k{}^T P_{k+1} \Phi_k + Q_k - \Phi_k{}^T P_{k+1} \Gamma_k \left(R_k + \Gamma_k{}^T P_{k+1} \Gamma_k\right)^{-1} \Gamma_k{}^T P_{k+1} \Phi_k , \tag{21}$$

with the final condition $P_N = F$. If the system is linear and time invariant, assuming that $N \to \infty$ and Q and R are constant weight matrices, then the Riccati matrix is the unique solution of the following discrete time algebraic Riccati equation:

$$P = \Phi^T \cdot P \cdot \Phi + Q - \Phi^T \cdot P \cdot \Gamma \cdot \left(R + \Gamma^T \cdot P \cdot \Gamma\right)^{-1} \Gamma^T \cdot P \cdot \Phi \tag{22}$$

The presented discrete time LQR algorithm can be applied only if the system model is LTI. This equation can be solved using several well known numerical methods, for example by discrete time Potter , Schur, [6] etc. methods.

3.2. The discrete time sub-optimal SDRE approach

The general discrete time SDRE method can be applied for the following pseudo-linear form of the nonlinear system model:

$$x_{k+1} = f(x_k) + g(x_k) \cdot u_k = \Phi(x_k) \cdot x_k + \Gamma(x_k) \cdot u_k , \tag{23}$$

The cost function is

$$J(u) = \tfrac{1}{2} \sum_{k=0}^{\infty} (x_k^T \cdot Q \cdot x_k + u_k^T \cdot R \cdot u_k) , \tag{24}$$

where it is considered that the weight matrices R and Q are constants (only for simplicity of the following calculations). Similar to the classical LQR design the state-feedback control sequence is calculated as:

$$u(x_k) = -\left(R + \Gamma(x_k)^T \cdot P(x_k) \cdot \Gamma(x_k)\right)^{-1} \Gamma(x_k)^T \cdot P(x_k) \cdot \Phi(x_k) \cdot x_k = -K(x_k) \cdot x_k \tag{25}$$

Usually the P(x) matrix is the unique solution of the following discrete time state dependent Riccati equation (SDRE):

$$P(x_k) = Q + \Phi(x_k)^T \cdot (P(x_k) - P(x_k) \cdot \Gamma(x_k) \cdot \left(R + \Gamma(x_k)^T \cdot P(x_k) \cdot \Gamma(x_k)\right)^{-1} \Gamma(x_k)^T \cdot P(x_k)) \cdot \Phi(x_k) \tag{26}$$

The solution of SDRE (26) is only a sub-optimal solution, because there were neglected derivatives of the system matrices ($\Phi(x_k)$ and $\Gamma(x_k)$) similar to the continuous case [7]. The following section will present the exact solution of the discrete time optimal control problem, as a generalization of the above SDRE approach.

3.3. Optimal form of the discrete time SDRE control

Optimal solution of the SDRE problem can be obtained using the general form of Hamilton-Jacobi theory [5]. The Hamiltonian corresponding to the cost function (24) with the equality constraints (23) is given by [7][8]:

$$H(x_k, u_k, p_{k+1}) = \tfrac{1}{2} x_k^T \cdot Q \cdot x_k + \tfrac{1}{2} u_k^T \cdot R \cdot u_k + p_{k+1}^T (\Phi(x_k) \cdot x_k + \Gamma(x_k) \cdot u_k). \tag{27}$$

The optimality conditions for the minimization problem are given by the following equations:

$$\frac{\partial H}{\partial u_k} = 0 \quad \Rightarrow R \quad \cdot u_k + (p_{k+1}^T \cdot \Gamma(x_k))^T = 0 \Rightarrow u_k = -R^{-1} \cdot \Gamma^T(x_k) \cdot p_{k+1} \,, \tag{28}$$

$$\frac{\partial H}{\partial p_{k+1}} = x_{k+1} \quad \Rightarrow \quad x_{k+1} = f(x_k) + g(x_k) \cdot u_k = \Phi(x_k) \cdot x_k + \Gamma(x_k) \cdot u_k = f(x_k) - \Gamma(x_k) \cdot R^{-1} \cdot \Gamma^T(x_k) \cdot p_{k+1} \,, \tag{29}$$

$$\frac{\partial H}{\partial x_k} = p_k \quad \Rightarrow \quad Q \cdot x_k + \frac{\partial}{\partial x}\left[f(x_k) + \Gamma(x_k) \cdot u_k\right]^T \cdot p_{k+1} = p_k \,. \tag{30}$$

The co-state vector is:

$$p_k = P(x_k) \cdot x_k \,. \tag{31}$$

For simplicity we introduce the notation: $P(x_k) = P_k$. From (27) and (29) it results:

$$x_{k+1} = (I + \Gamma(x_k) \cdot R^{-1} \cdot \Gamma^T(x_k) \cdot P_{k+1})^{-1} \cdot f(x_k) \tag{32}$$

From the (30) and (31) co-state equations and using (32) it results:

$$P_k \cdot x_k = Q \cdot x_k + \frac{\partial}{\partial x_k}\left[f(x_k) + \Gamma(x_k) \cdot u_k\right]^T \cdot P_{k+1} \cdot \left[I + \Gamma(x_k) \cdot R^{-1} \cdot \Gamma^T(x_k) \cdot P_{k+1}\right]^{-1} \cdot f(x_k) \tag{33}$$

This regression is a well known equivalent form of discrete optimal control. The comparison with Bellman theory based Riccati equation (26), followed by the determination of the extra corrective terms is possible using the following mathematical expression $(I + X \cdot Y)^{-1} = I - X \cdot (I + Y \cdot X)^{-1} \cdot Y$. In our case $X \to \Gamma(x_k)$ and $Y \to R^{-1} \cdot \Gamma(x_k) \cdot P_{k+1}$. After some relatively simple mathematical calculations, the modified discrete time state dependent Riccati equation becomes :

$$P_k = Q + \frac{\partial}{\partial x_k}\left[f(x_k) + \Gamma(x_k) \cdot u_k\right]^T \cdot P_{k+1} \cdot \left[I + \Gamma(x_k) \cdot \left(R + \Gamma^T(x_k) \cdot P_{k+1} \cdot \Gamma(x_k)\right)^{-1} \cdot \Gamma^T(x_k) \cdot P_{k+1}\right] \cdot \Phi(x_k) \tag{34}$$

Introducing the notation:

$$\Phi^*(x_k) = \frac{\partial}{\partial x_k}\left[f(x_k) + \Gamma(x_k)\cdot u_k\right]^T = \frac{\partial}{\partial x_k}\left[\Phi(x_k)\cdot x_k + \Gamma(x_k)\cdot u_k\right]^T \tag{35}$$

the SDRE equation (26) and the modified SDRE (35) have similar structure. As in the continuous case the discrete time corrective term becomes:

$$E_d(x) = \Phi^*(x_k) - \Phi(x_k) = \frac{\partial}{\partial x_k}\left[\Phi(x_k)\cdot x_k + \Gamma(x_k)\cdot u_k\right]^T - \Phi(x_k) \tag{36}$$

If this $E_d(x)$ corrective expression is null then the results of both control sequences coincide.

4. Numerical simulation. Example.

In this section we present an example - with the corresponding simulation results [9] - to compare the optimal and suboptimal form SDRE solutions for both continuous and discrete time systems.

Consider the following nonlinear continuous system model:

$$\begin{cases} \dot{x}_1 = \sin x_1 + x_2 \\ \dot{x}_2 = x_1 x_2 + u \end{cases} \Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{\sin x_1}{x_1} & 1 \\ 0 & x_1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{37}$$

and the following cost function:

$$J(u) = \tfrac{1}{2}\int_0^\infty (q_1 \cdot x_1^2(t) + q_2 \cdot x_2^2(t) + r \cdot u(t)) \cdot dt \tag{38}$$

For suboptimal form of the SDRE, when the Riccati matrix is symmetric, we have to solve the following system of equations, where all elements of the Riccati matrix are state dependent:

$$\begin{aligned} \dot{p}_{11} &= -2p_{11}\tfrac{\sin x_1}{x_1} + \tfrac{1}{r}p_{12}{}^2 - q_1 \\ \dot{p}_{12} &= -p_{11} - p_{12}x_1 - p_{21}\tfrac{\sin x_1}{x_1} + \tfrac{1}{r}p_{12}p_{22} \\ \dot{p}_{21} &= -p_{11} - p_{12}x_1 - p_{21}\tfrac{\sin x_1}{x_1} + \tfrac{1}{r}p_{12}p_{22} \\ \dot{p}_{22} &= -2p_{21} - 2p_{22}x_1 + \tfrac{1}{r}p_{22}{}^2 - q_2 \end{aligned} \tag{39}$$

If we assume that $tf \to \infty$ we have to solve the following algebraic equations, where the unknown elements are also state dependent:

$$\begin{aligned} &2p_{11}\tfrac{\sin x_1}{x_1} - \tfrac{1}{r}p_{12}{}^2 + q_1 = 0 \\ &p_{11} + p_{12}x_1 + p_{21}\tfrac{\sin x_1}{x_1} - \tfrac{1}{r}p_{12}p_{22} = 0 \\ &p_{11} + p_{12}x_1 + p_{21}\tfrac{\sin x_1}{x_1} - \tfrac{1}{r}p_{12}p_{22} = 0 \\ &2p_{21} + 2p_{22}x_1 - \tfrac{1}{r}p_{22}{}^2 + q_2 = 0 \end{aligned} \tag{40}$$

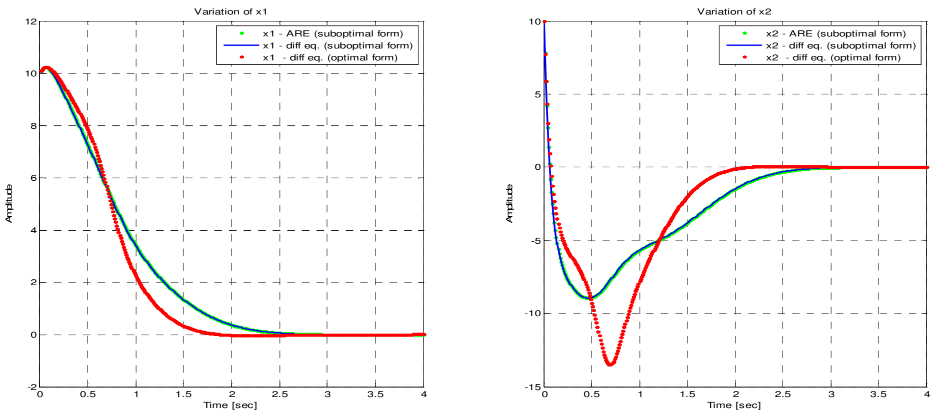
For modified (optimal form) of the SDRE, when the P matrix is not necessarily symmetric, we have to solve the following nonlinear differential equations:

$$\begin{aligned}
\dot{p}_{11} &= -2p_{11}\tfrac{\sin x_1}{x_1} + \tfrac{1}{r}p_{12}{}^2 - q_1 - \left[p_{11}\cdot(\cos x_1 - \tfrac{\sin x_1}{x_1}) + p_{12}\cdot x_2 + \tfrac{1}{r}\cdot p_{12}\cdot(p_{12}-p_{21})\right] \\
\dot{p}_{12} &= -p_{11} - p_{12}x_1 - p_{21}\tfrac{\sin x_1}{x_1} + \tfrac{1}{r}p_{12}p_{22} - \left[p_{12}\cdot\cos x_1 - p_{21}\cdot\tfrac{\sin x_1}{x_1} + p_{22}\cdot x_2 + (p_{21}-p_{12})x_1\right] \\
\dot{p}_{21} &= -p_{11} - p_{12}x_1 - p_{21}\tfrac{\sin x_1}{x_1} + \tfrac{1}{r}p_{12}p_{22} - \left[\tfrac{1}{r}\cdot p_{22}\cdot(p_{12}-p_{21}) + x_1\cdot(p_{21}-p_{12})\right] \\
\dot{p}_{22} &= -2p_{21} - 2p_{22}x_1 + \tfrac{1}{r}p_{22}{}^2 - q_2 - \left[p_{12}-p_{22}\right]
\end{aligned} \tag{41}$$

For the control simulation we tested the following cases:

- ARE - Suboptimal form of SDRE, equations (40),
- Differential Riccati equation - suboptimal form of SDRE equations (39),
- Differential Riccati equation - optimal form of SDRE equations (41).

We use $x(t_0) = [10\,10]^T$, and the weight values are $r=3$, $q_1=100$ and $q_2=10$, namely the behavior of the first state is more important. For the differential equations (39) and (41) we have to specify the parameters $t_f=4$ and $P(t_f)=0$. The state trajectories are shown in Fig. 1, where it can be observed the optimal behavior of x_1 (Fig.1.a) in case if we use the optimal form of SDRE.



a) b)

Fig.1 Variation of the states $x_1(t)$ (a) and $x_2(t)$ (b) if $r=3$, $q_1=100$ and $q_2=10$ (continuous time SDRE method)

If the weight values are $q_1=1$ and $q_2=100$, namely the behavior of the second state is more important, the results of simulation are presented in Fig. 2, where also can be observed the optimal behavior of x_2 (Fig.2.b) in case if we use the optimal form of SDRE.

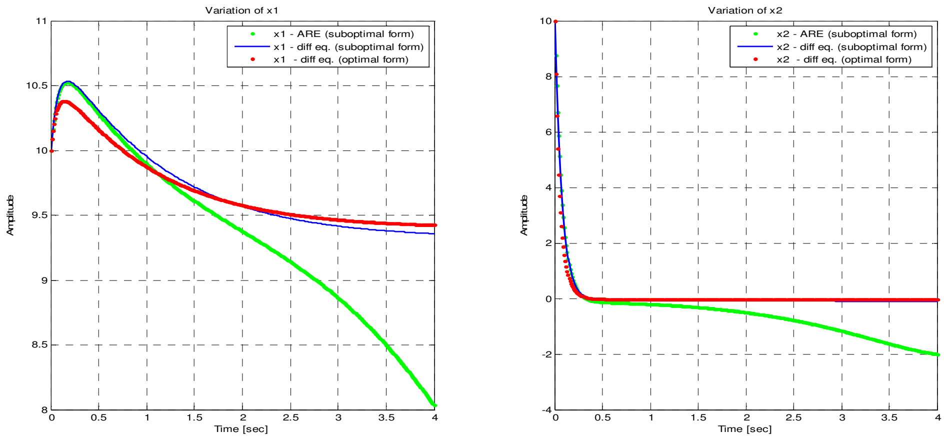
Consider now the discrete time form of the same nonlinear system model:

$$\begin{cases} x_{1,k+1} = x_{1,k} + T_s(\sin x_{1,k} + x_{2,k}) \\ x_{2,k+1} = x_{2,k} + T_s\cdot(x_{1,k}x_{2,k} + u_k) \end{cases} \Rightarrow \begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \end{bmatrix} = \begin{bmatrix} 1 + T_s\dfrac{\sin x_{1,k}}{x_{1,k}} & T_s \\ T_s\cdot x_{2,k} & 1 \end{bmatrix}\cdot\begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} + \begin{bmatrix} 0 \\ T_s \end{bmatrix}\cdot u_k \tag{42}$$

The cost function is similar with (38) but in this case it has a discrete time form. Similar to the continuous control simulation, we tested the following cases:

- DARE - Suboptimal form of the discrete time SDRE,

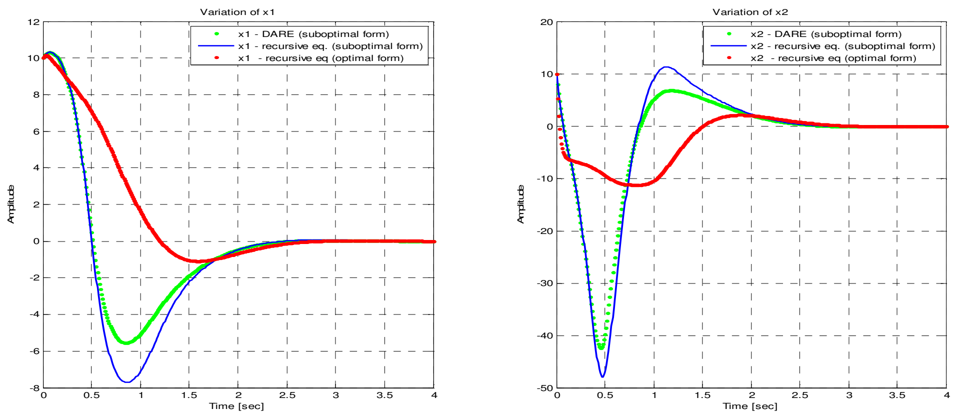
- Recursive form of the Riccati equation - suboptimal form of the discrete time SDRE,
- Recursive form of the Riccati equation - optimal form of discrete time SDRE.



a) b)

Fig.2. Variation of the states $x_1(t)$ (a) and $x_2(t)$ (b) if r=3, q_1 =1 and q_2 =100 (continuous time SDRE method)

The sampling period is fixed at T_s=*0.01* sec and the number of iterations it is 400. We use $x_0 = [10\,10]^T$, the weight values are *r=1*, q_1 *=100* and q_2 *=1,* namely the behavior of the first state is more important. The recursive form of the Riccati equations needs the horizon value (N=80) and the final value of the Riccati matrix (P_N=*0*). The state trajectories are shown in Fig. 3, where we can observe also the optimal behavior of the x_1 (Fig.3.a) in case if we use the optimal form of discrete time SDRE.



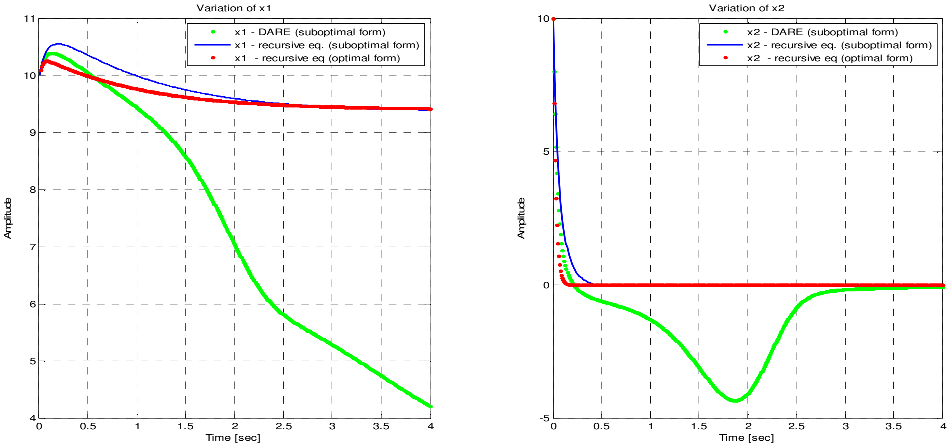
a) b)

Fig.3. The variation of the states $x_1(t)$ (a) and $x_2(t)$ (b) if r=1, q_1 =100 and q_2 =1 (discrete time SDRE method)

If the weight values are q_1 *=1* and q_2 *=100*, the simulation results are presented in Fig. 4.

5. Conclusions

In the paper we presented a theoretical study of the optimal control with continuous time and discrete time SDRE methods. In both cases there were analyzed the errors introduced by the fact that the SDRE method extends the application of the algebraic Riccati equation only due to its tractability, and not as a rigorous derivation from the original optimization problem. We labeled this approach "suboptimal SDRE", while the method that uses the exact solution of the optimization problem, we called "optimal SDRE", because it is based on the introduction of additional, corrective terms in the Riccati equation that help to improve the optimality of the control problem's solution. For comparison of the optimal and suboptimal forms of the SDRE, a theoretical nonlinear optimal control problem has been simulated for continuous and discrete time cases.



a) b)

Fig.4. The variation of the states $x_1(t)$ (a) and $x_2(t)$ (b) if r=1 and q_1 =1 and q_2 =100 (discrete time SDRE method)

References

[1] Erdem E. B, Alleyne A. G., Design of a Class of Nonlinear Controllers via State Dependent Riccati Equations, IEEE Trans. Control Syst. Technol., vol. 12, no. 1, pp. 133–137, January 2004.

[2] Choukroun D, Tekinalp O, Spacecraft attitude and rate estimation and control, using the SDRE method, 5th European Conference for Aeronautics and Space Sciences (EUCASS), Munich, Germany, Jul 2013.

[3] Menon P. K, Sweriduk G. D,. Vaddi S. S, Nonlinear Discrete-time Design Methods for Missile Flight Control Systems, 2004 AIAA Guidance, Navigation, and Control Conference, August 16-19, Providence, RI, 2004.

[4] Bruzelius F, Linear Parameter-Varying Systems, an approach to gain scheduling -Thesis, Department of Signals and Systems, Chalmers University Of Technology,Goteborg, Sweden, 2004.

[5] Kirk D. E., Optimal Control Theory: An Introduction. Englewood Cliffs, NJ: Prentice Hall, 1970, reprinted by Dover, Mineola, NY, 2004

[6] David L., Tehnici de optimizare , Petru Maior University, 2000.

[7] Dutka A. S., Ordys A. W., Grimble M. J., Optimized Discrete-Time State Dependent Riccati Equation Regulator, in Amer. Control Conf., Portland, OR, June 2005, pp. 2293–2298.

[8] Chang I., Bentsman J., Constrained Discrete-Time State-Dependent Riccati Equation Technique: A Model Predictive Control Approach, 52nd IEEE Conference on Decision and Control December 10-13, 2013. Florence, Italy, pp. 5125-5130.

[9] Erdem E. B, Analysis and real-time implementation of state-dependent Riccati equation controlled systems -Thesis, University of Illinois at Urbana-Champaign, 2001.