Numerical PDE II HW 3: Differential Geometry and Transfinite Interpolation

Dhruv Balwada

March 20, 2015

1 Statement of Problem

Generate grids for two domains using transfinite interpolation. The domains are :

1. The quarter annulus.
2. The nozzle shape.

Details on the attached question sheet.

2 Description of the Mathematics

Transfinite interpolation is an algebraic grid generation technique. If the bounding curves are $\Gamma_1(\zeta)$,$\Gamma_2(\eta)$, $\Gamma_3(\zeta)$ and $\Gamma_4(\eta)$ counterclockwise then the grid can be generated simply as

$$\vec{X}(\zeta,\eta) = (1-\zeta)\Gamma_4(\eta)+\zeta\Gamma_2(\eta)+(1-\eta)\Gamma_1(\zeta)+\eta\Gamma_3(\zeta)-(1-\zeta)((1-\eta)\Gamma_1(0)+\eta\Gamma_3(0))-\zeta((1-\eta)\Gamma_1(1)+\eta\Gamma_3(1)) \quad (1)$$

The derivatives of the grid are

$$\vec{X}(\zeta,\eta)_\zeta = -\Gamma_4(\eta)+\Gamma_2(\eta)+(1-\eta)\Gamma_1'(\zeta)+\eta\Gamma_3'(\zeta)+((1-\eta)\Gamma_1(0)+\eta\Gamma_3(0))-((1-\eta)\Gamma_1(1)+\eta\Gamma_3(1)) \quad (2)$$

$$\vec{X}(\zeta,\eta)_{\zeta\zeta} = (1-\eta)\Gamma_1''(\zeta)+\eta\Gamma_3''(\zeta) \quad (3)$$

etc.

Thus the derivatives anywhere in the domain basically depend on the derivatives of the boundary curves. So the properties of the derivatives of the boundary curves will help us learn about the properties of the grid in the domain.

We also know that the mapping introduces an error into the local truncation error of the form

$$\tau_{grid} = \left(-\frac{1}{6}\frac{X_{\zeta\zeta\zeta}}{X_\zeta^3}f_x - \frac{1}{2}\frac{X_{\zeta\zeta}}{X_\zeta^2}f_{xx}\right)\triangle x^2 \quad (4)$$

So to reduce errors introduced by the grid one should aim for larger first derivatives and smaller higher derivatives. Perfect grids are linear like a square grid.

To get a more uniform grid spacing along a curved boundary the parameterization is usually done using arc length. One way to do this is to solve

$$\frac{dt}{d\zeta} = \frac{L}{\sqrt{X_t^2 + Y_t^2}} \tag{5}$$

Where L is the length of the curve and the curve is in initially given in parametric form $(X(t), Y(t))$. Solving this gives us t as a function of ζ and hence $(X(\zeta), Y(\zeta))$ along the boundary.

3 Description of the Algorithm

The algorithm is very simple and essentially just a calculation. First we generate the boundary curves, both the X and Y positions of the curves as functions of ζ or η. This is done simply using the analytical formulae for the quarter annulus boundaries. For the nozzle this is simple the side and bottom walls for three sides. The fourth side requires a slightly more involved process. This is done by first evaluating the length of the curve using Simpson's rule for integration. Following this equation 5 is solved numerically using the RK4 solver. This gives us X as a function of ζ and then the formula for area given in the problem is used to find Y at the same points. Once we have all the curves the interpolation is simply calculating the value of equation 1 at every grid point. The derivatives were calculated using centered finite differences for the plotting algorithm.

4 Results

4.1 The Quarter Annulus

I used transfinite interpolation to generate a grid on a quarter annulus. The boundary curves are specified as

$$X(\zeta, \eta) = (\zeta + 1)cos(\eta\pi/2)$$
$$Y(\zeta, \eta) = (\zeta + 1)sin(\eta\pi/2)$$

Such that $\Gamma_1 = \vec{X}(\zeta, 0)$, $\Gamma_2 = \vec{X}(1, \eta)$, $\Gamma_3 = \vec{X}(\zeta, 1)$ and $\Gamma_4 = \vec{X}(0, \eta)$. All the derivatives exist as sin and cos are analytic functions with all derivatives existing everywhere. The first derivatives are shown in figure 1 and 2. The higher derivatives were checked and they were smooth(not shown here).

4.2 The Nozzle

Grids was generated for the nozzle as shown in Figures 3-6. The nozzle is defined by three sides being the horizontal or vertical lines. However the top boundary is empirically defined, which can have discontinuous/non-existent derivatives. We see that the higher derivatives blow up near the throat and at the left side where the curved part of the nozzle meets the linear part(around X = 0.875). These are points where the derivatives are non-existent in the continuous sense and so the finite differenced derivatives have high values(positive and negative). These are the potential points where the errors in the local truncation would be huge. These errors linearly propagate into the domain.

5 Conclusions

Grids were generated for irregular domains using transfinite interpolation for a quarter annulus and a nozzle. The quarter annulus is a smooth grid as all derivatives exist. The nozzle is smooth unto first derivative but the higher derivatives blow up and the grid is non smooth. This also leads to high truncation errors as the truncation error is dependent on the higher derivatives of the grid.

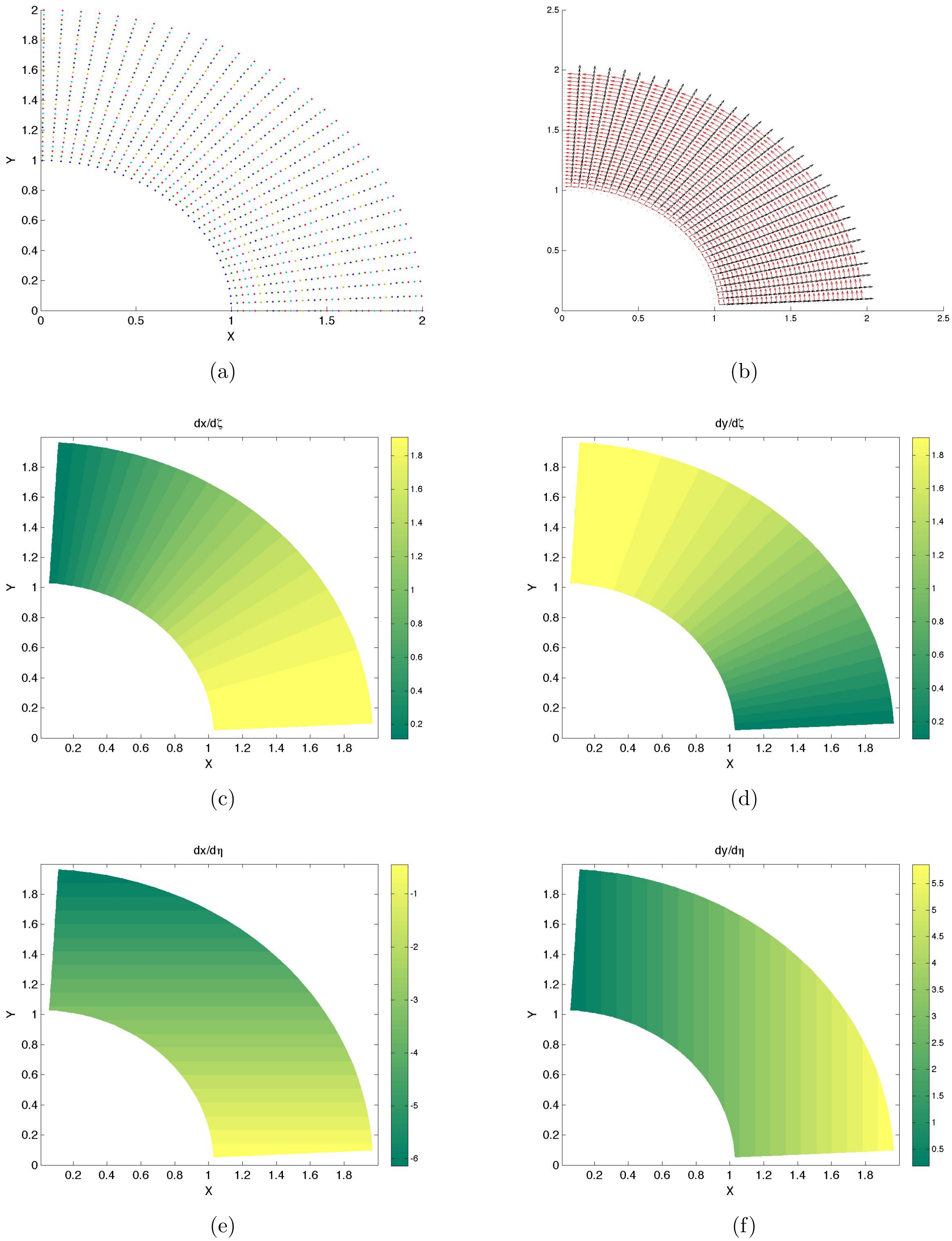
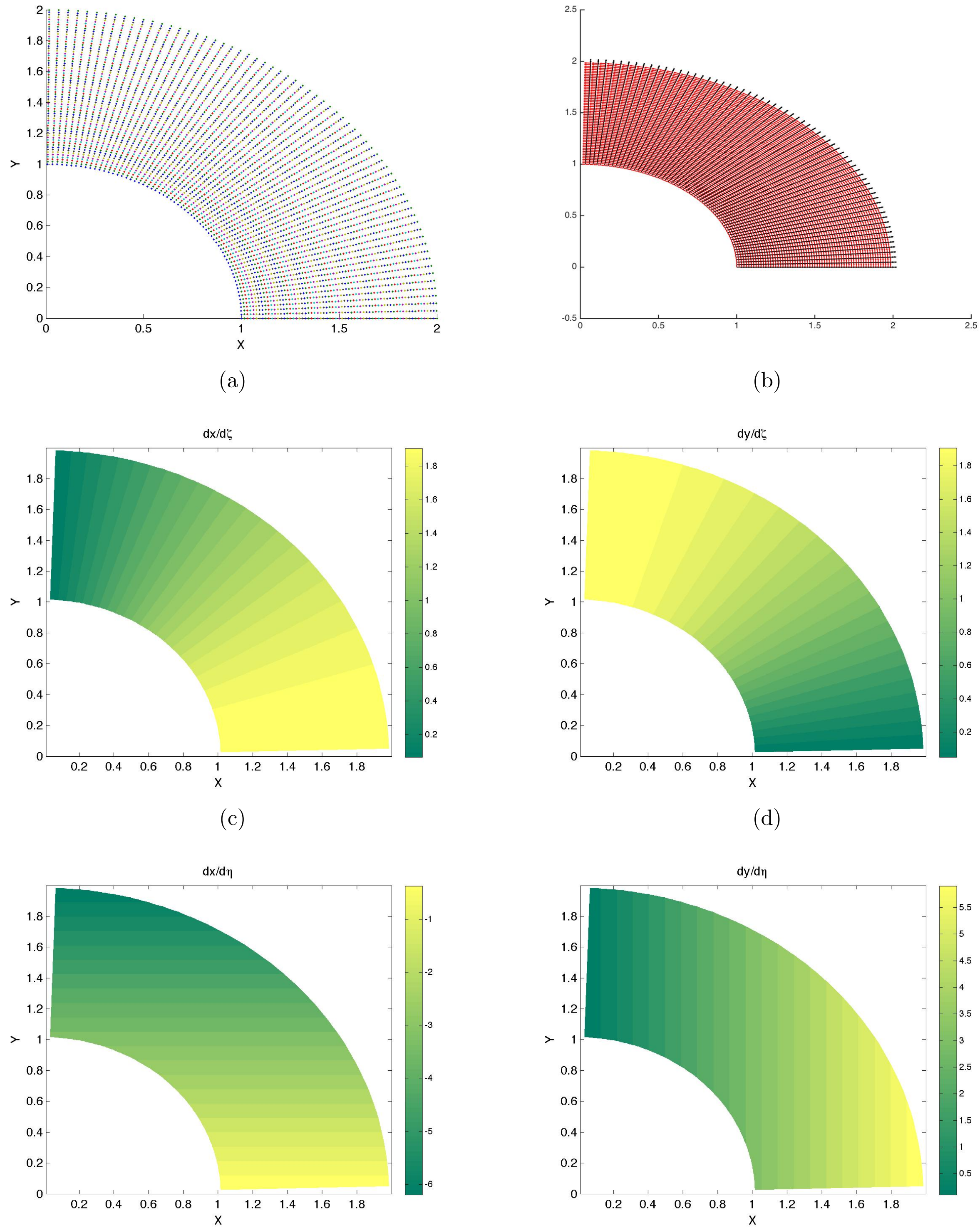


Figure 1: 32X32 grid generated on a quarter annulus using transfinite interpolation (a) grid points, (b) Covariant vectors, (c)$dX/d\zeta$, (d)$dY/d\zeta$ (e) $dX/d\eta$ (f)$dY/d\eta$



(a) (b)

(c) (d)

(e) (f)

Figure 2: 64X64 grid generated on a quarter annulus using transfinite interpolation. Same plot types as figure 1.

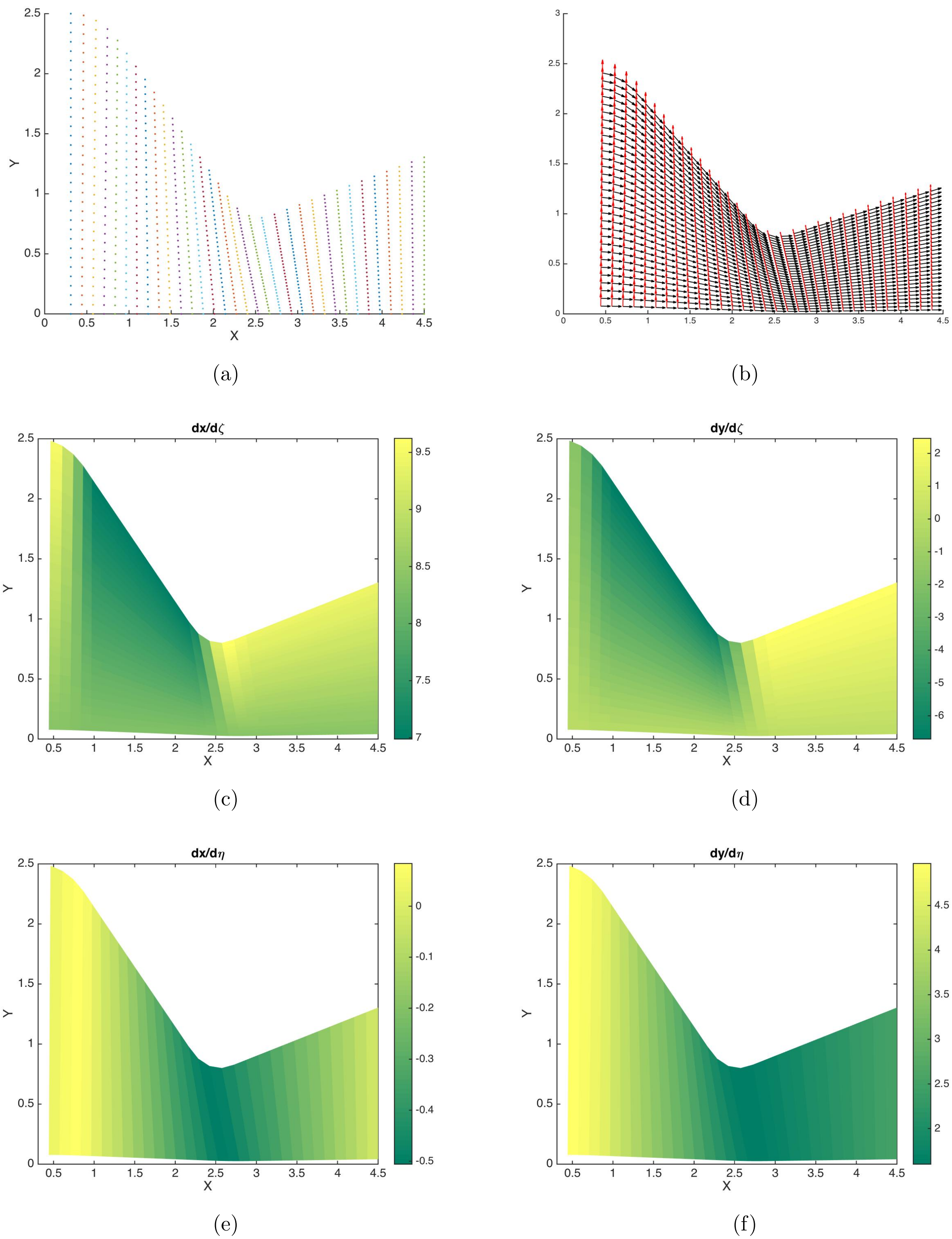


Figure 3: 32X32 grid generated for a nozzle using transfinite interpolation. Same plot types as figure 1.

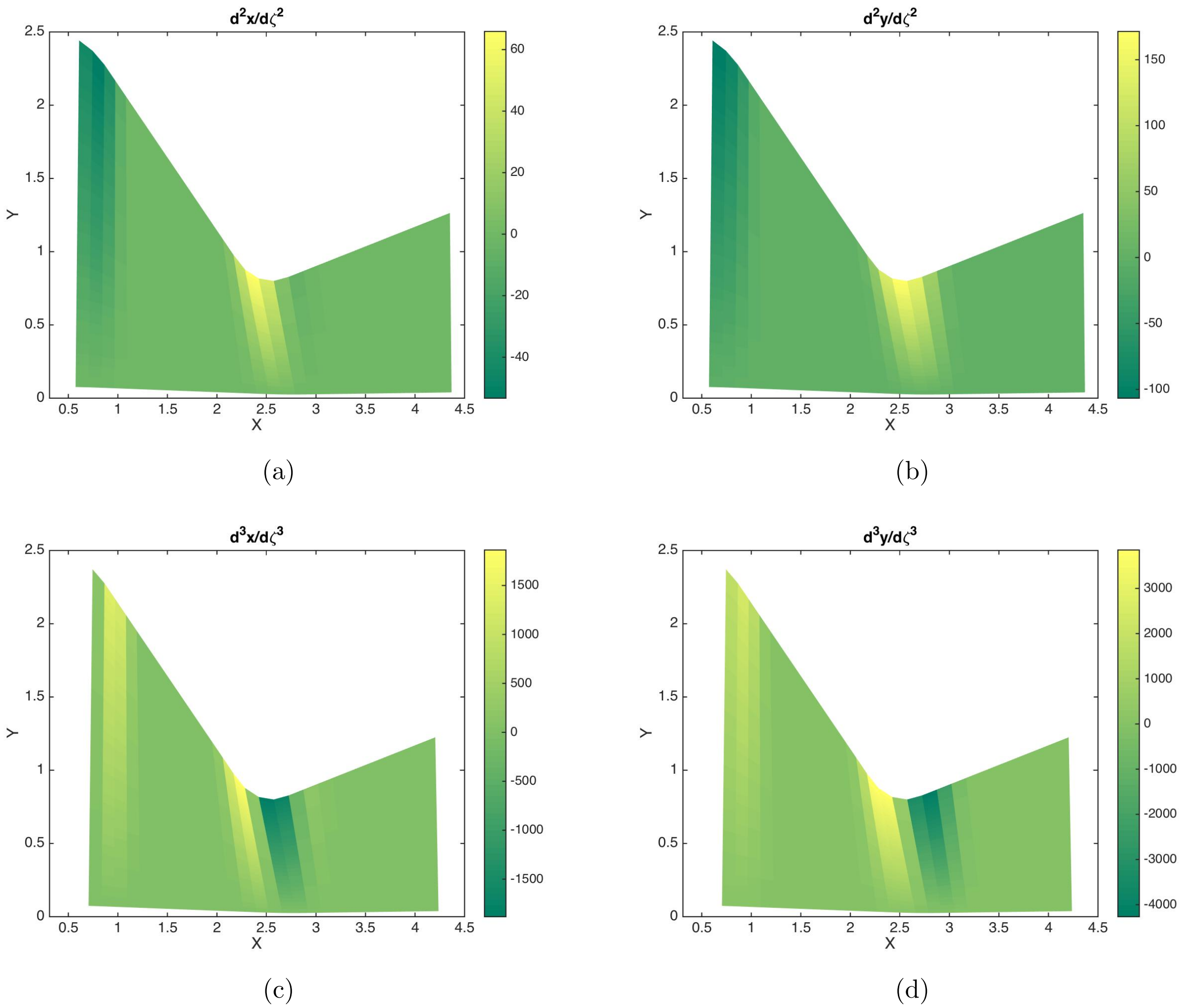


Figure 4: Higher derivatives on the 32X32 grid generated for a nozzle using transfinite interpolation.

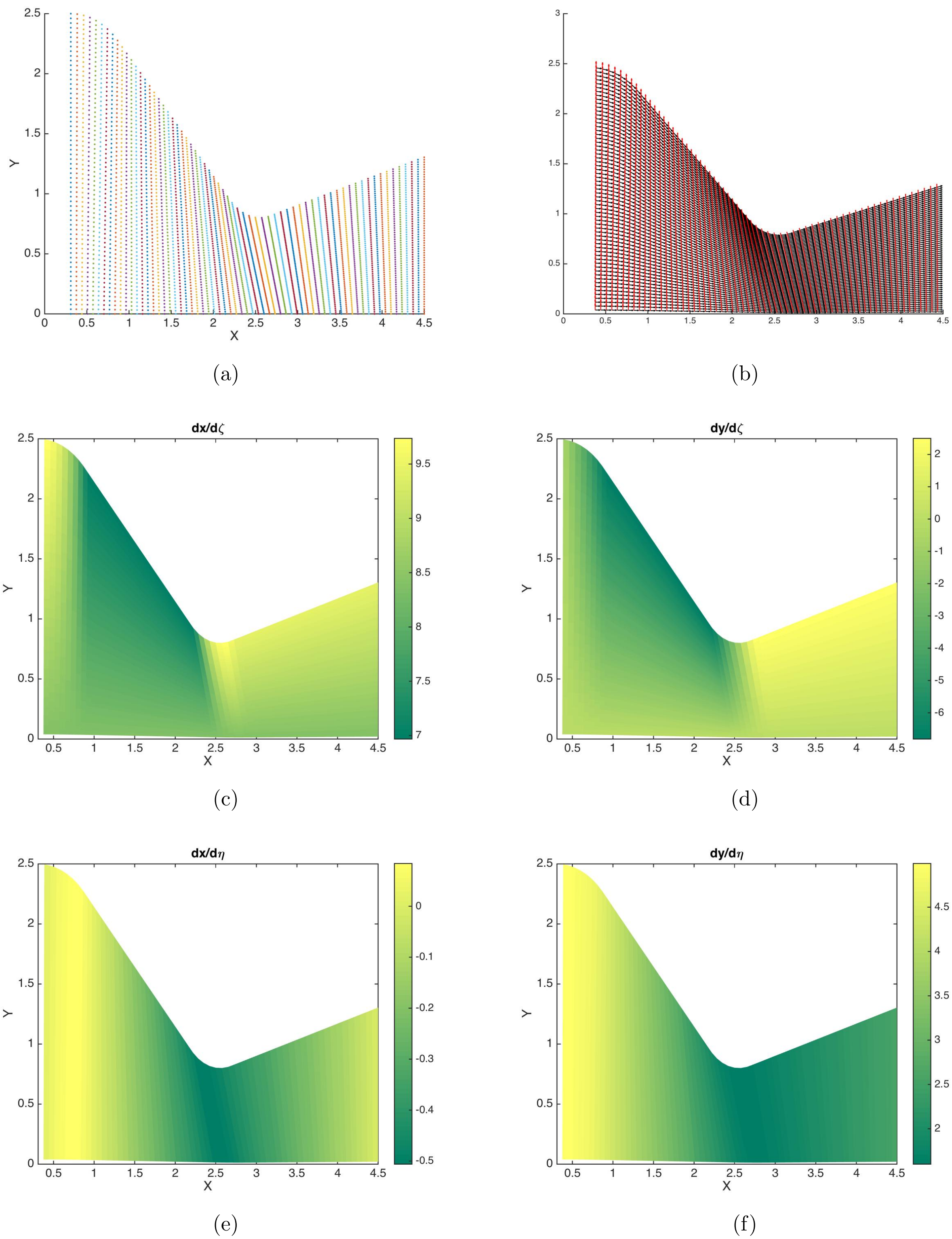


Figure 5: 64X64 grid generated for a nozzle using transfinite interpolation. Same plot types as figure 1.

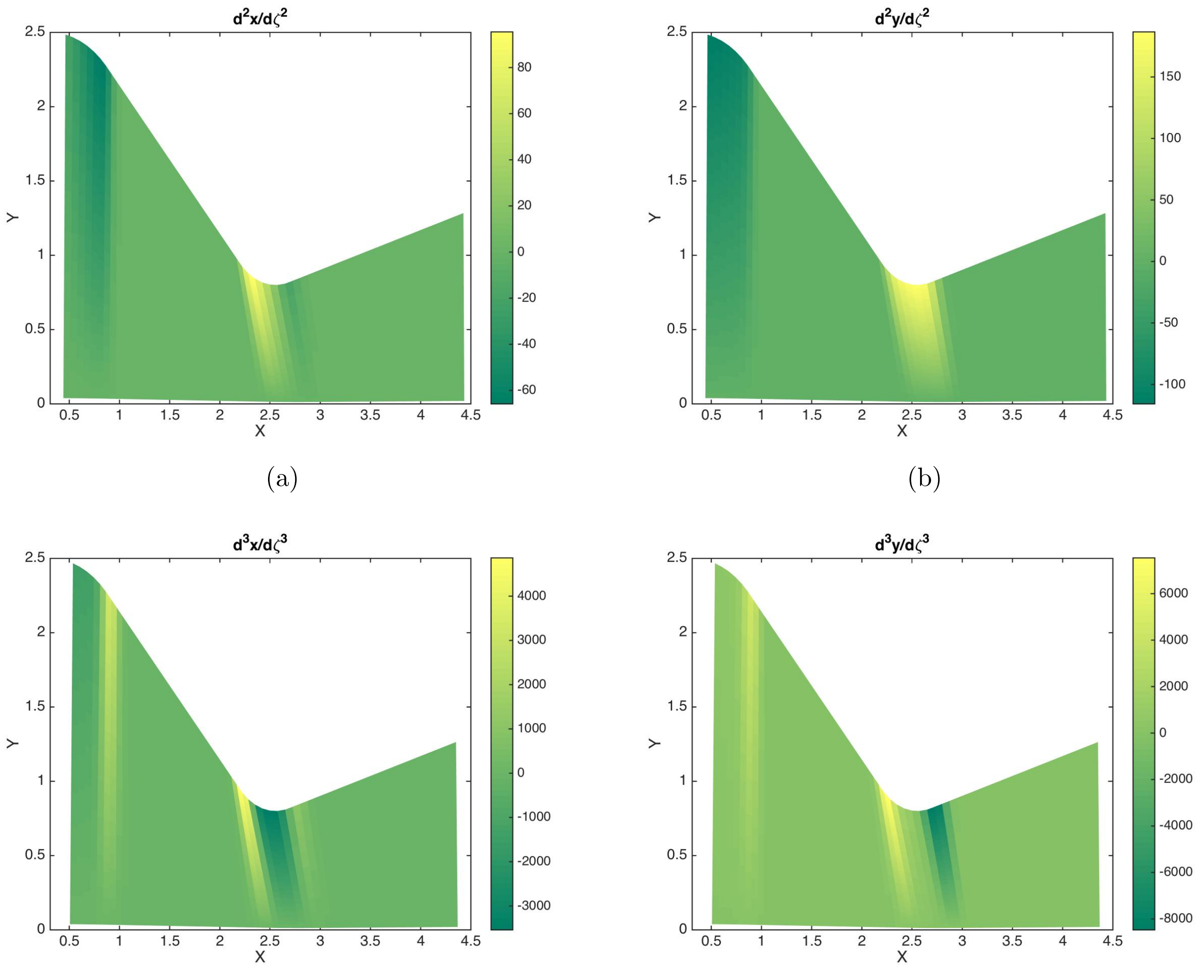


Figure 6: higher derivatives on the 64X64 grid generated for a nozzle using transfinite interpolation.