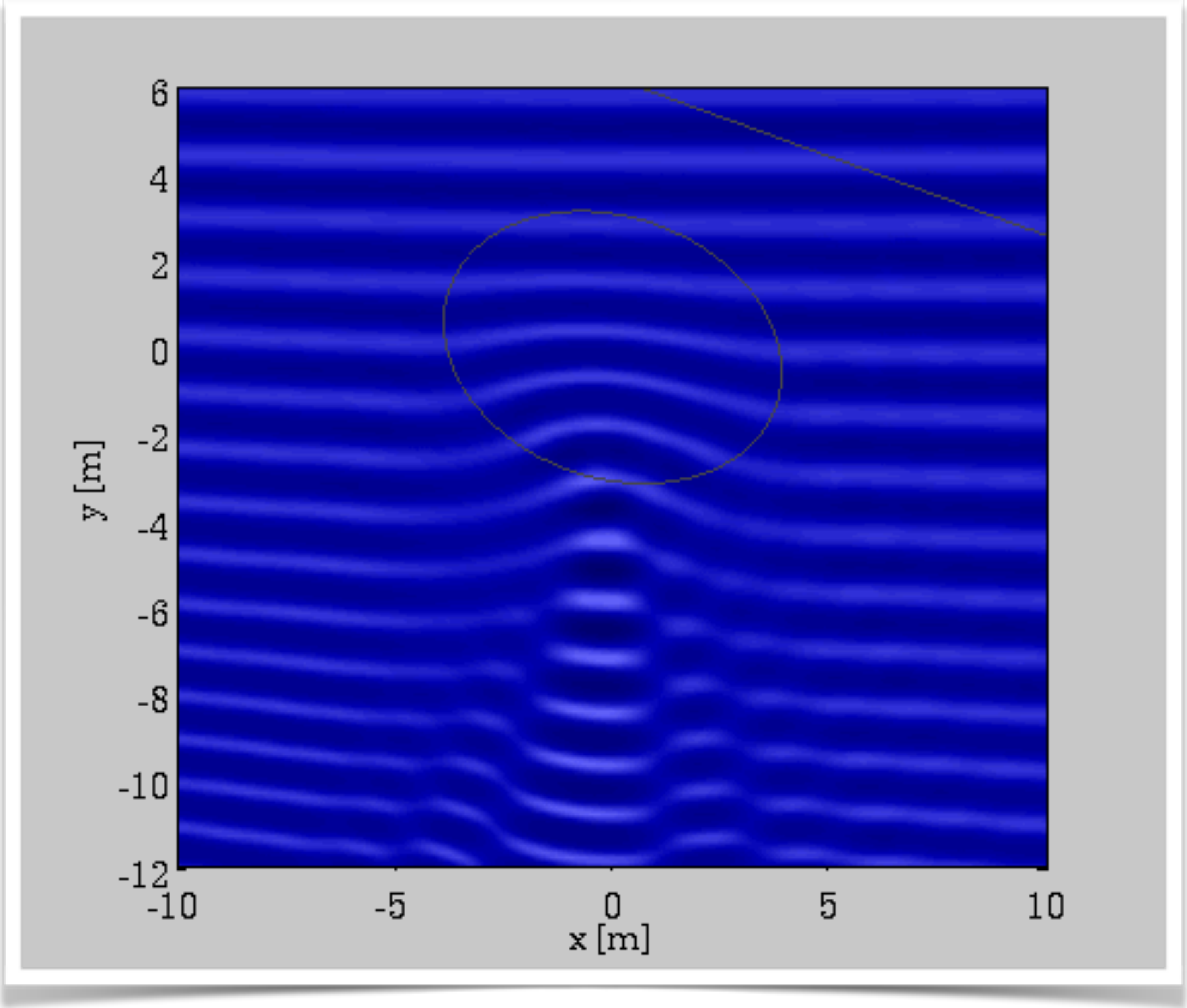


LINEARISED BOUSSINESQ EQUATION

BOUSSINESQ APPROXIMATION AND APPLICATIONS

- Elimination of vertical coordinates from the equations.
- Taylor expansion of horizontal and vertical flow velocity at a certain elevation.
- Extensively used in the non linear wave equations, flows having an unsteady nature and those with large domain sizes
- Advantageous than Finite Volume in these circumstances owing to a much lesser computational time required.

GOVERNING EQUATIONS

- The following governing equations for the wave has been established using Taylor series expansion upto second order, considering even bottom.

$$\frac{\partial \eta}{\partial t} + \left(\frac{\partial (d+\eta)}{\partial x}\right)u + \left(\frac{\partial (d+\eta)}{\partial y}\right)v + (d+\eta)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + g\frac{\partial \eta}{\partial x} - \frac{d}{2}\left(d\frac{\partial^3 u}{\partial t \partial x^2} + d\frac{\partial^3 v}{\partial x \partial y \partial t}\right) = 0$$

- $$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + g\frac{\partial \eta}{\partial y} - \frac{d}{2}\left(d\frac{\partial^3 v}{\partial t \partial^2 y} + d\frac{\partial^3 u}{\partial x \partial y \partial t}\right) = 0$$

SIMPLIFICATION OF EQUATIONS TO SHALLOW WATER EQUATION

- For the present case, the depth is assumed to be constant over time and space. Higher order mixed derivatives are also ignored. The new non linear equations are:

$$\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x}u + \frac{\partial \eta}{\partial y}v + (d+\eta)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + g\frac{\partial \eta}{\partial x} = 0$$

- $$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + g\frac{\partial \eta}{\partial y} = 0$$

SIMPLIFIED SHALLOW WATER EQUATION

- Further simplification can be achieved by considering only the linear terms, and simulations are run both for linear as well as non linear equations for different schemes to check their stability. The linearised form of these equations are :

$$\frac{\partial \eta}{\partial t} + (d + \eta)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

$$\frac{\partial u}{\partial t} + g\frac{\partial \eta}{\partial x} = 0$$

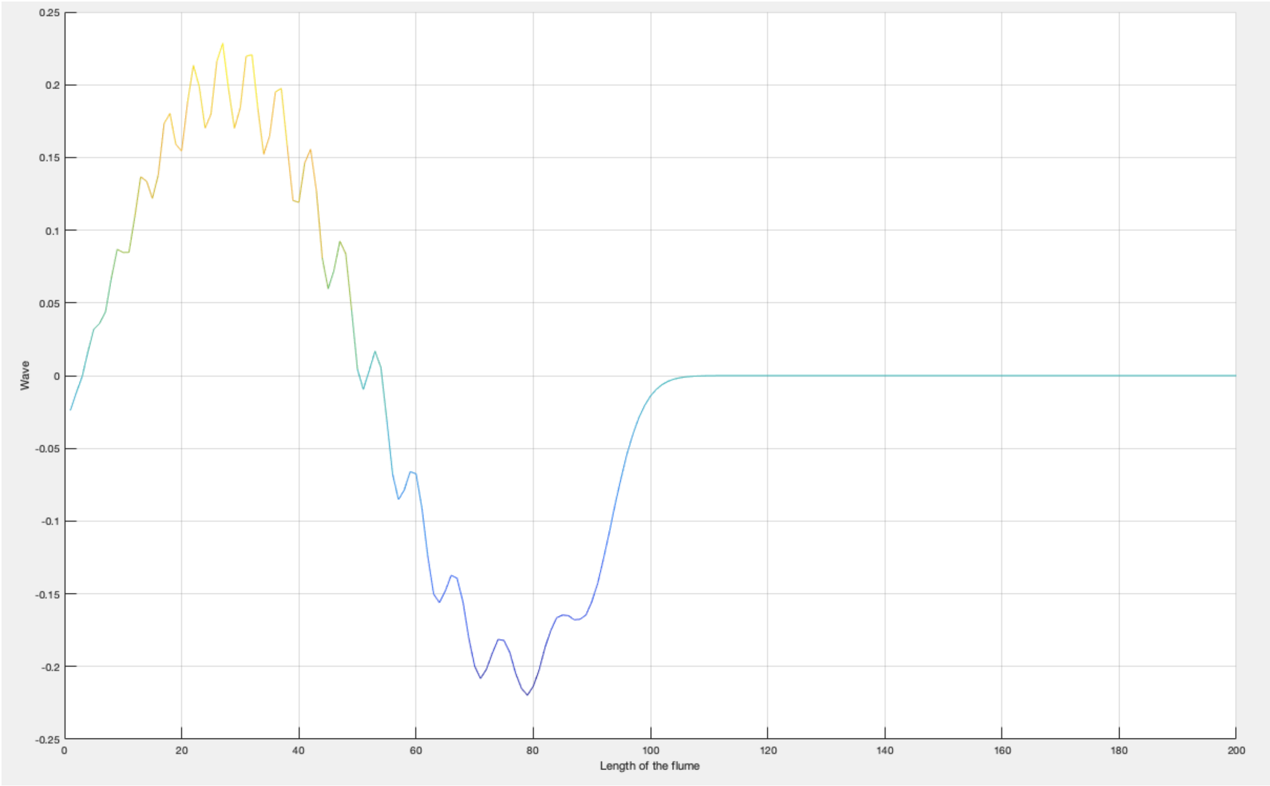
$$\frac{\partial v}{\partial t} + g\frac{\partial \eta}{\partial y} = 0$$

TOY PROBLEM AND FINITE DIFFERENCE METHODS USED

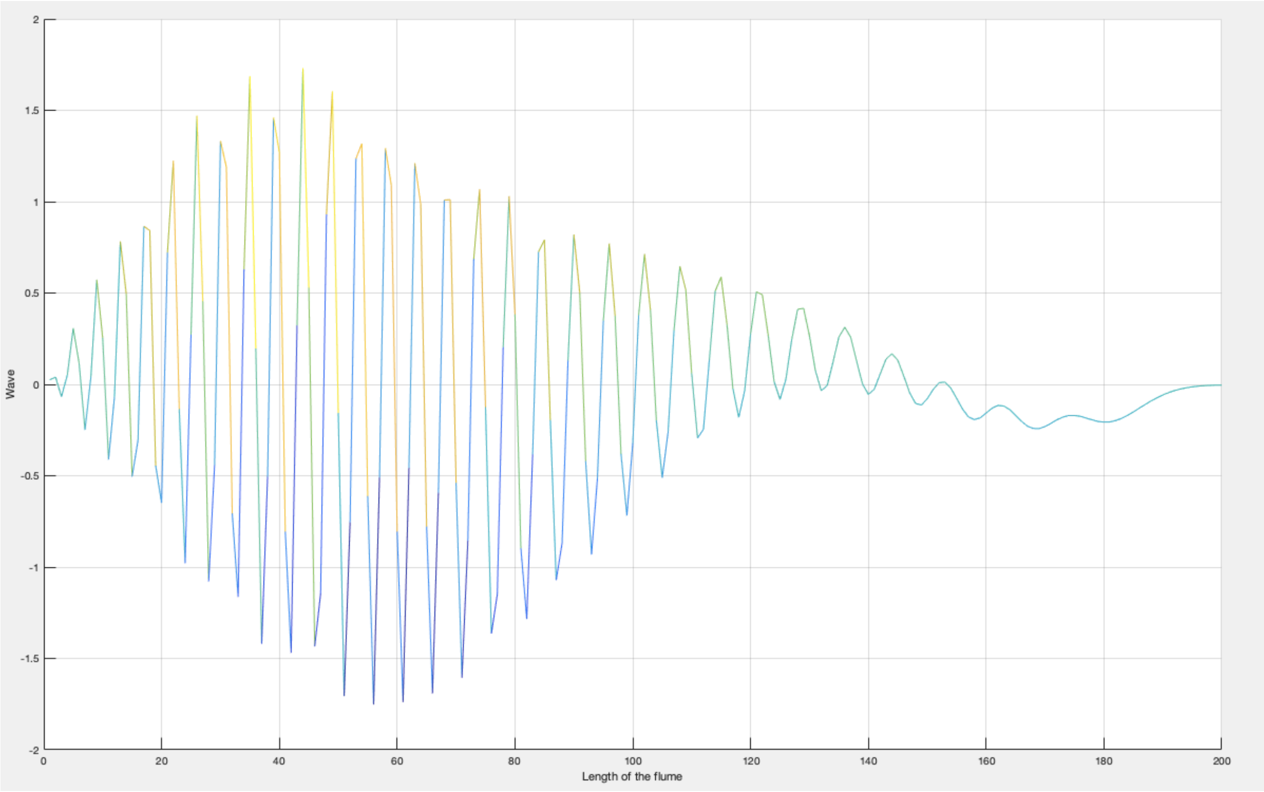
- A shallow wave basin of depth = 2.5 m is used for simulation. A wave paddle generates waves of amplitude 0.2 m at x = 0, which is transmitted through the wave basin. To model this problem, three different time differencing schemes are used:
  - Euler Explicit
  - Leap Frog
  - Predictor corrector
- For each time differencing schemes, both linear as well as non linear shallow water equation is modelled.

EULER EXPLICIT SCHEME - LINEARISED SWE

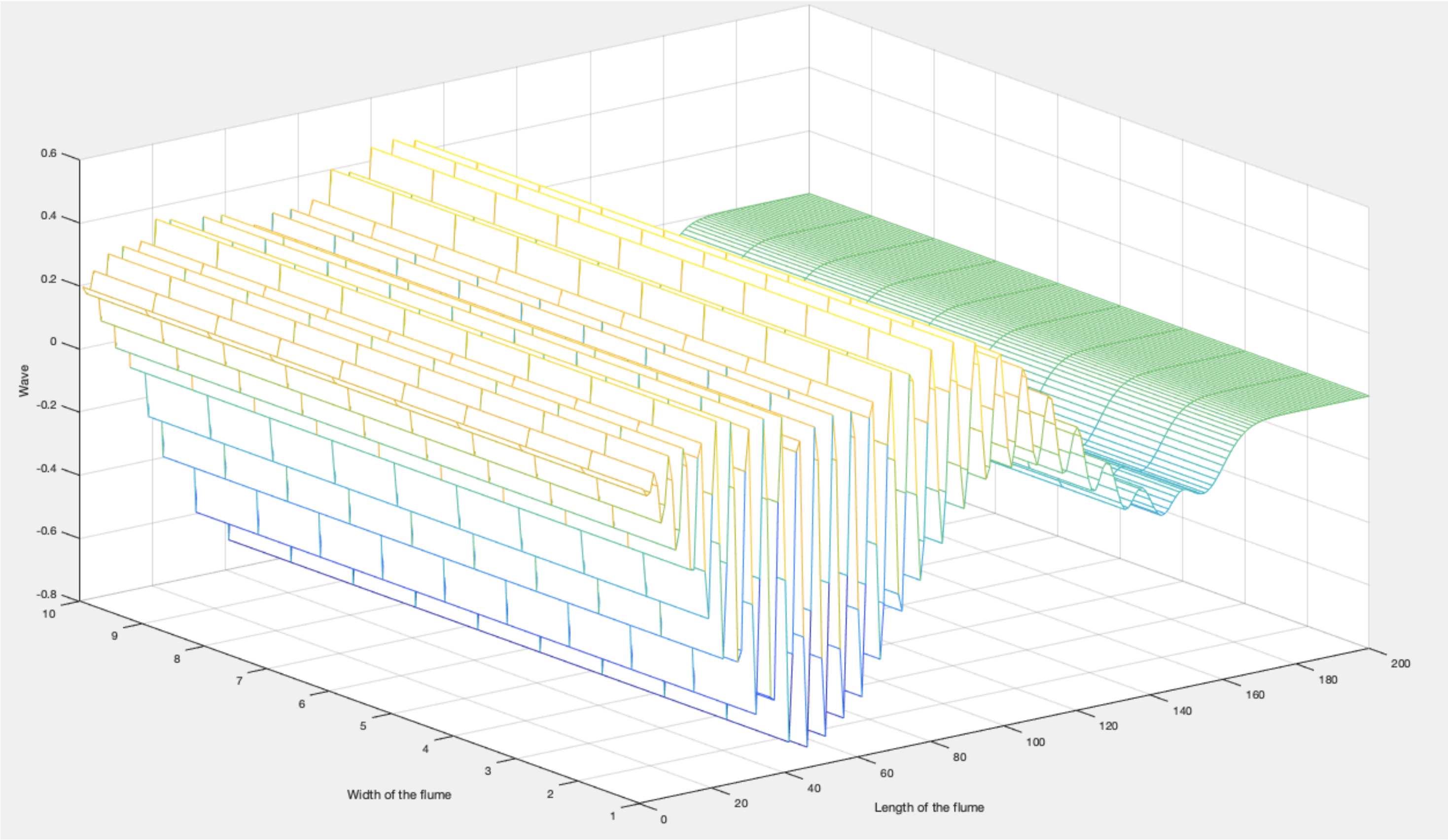
- Linearised Equation is first attempted using this method. The equation is unconditionally unstable, as can be seen from the graph in the next slide, which clearly highlights breaking of the simulation.



Pulse variation at half time



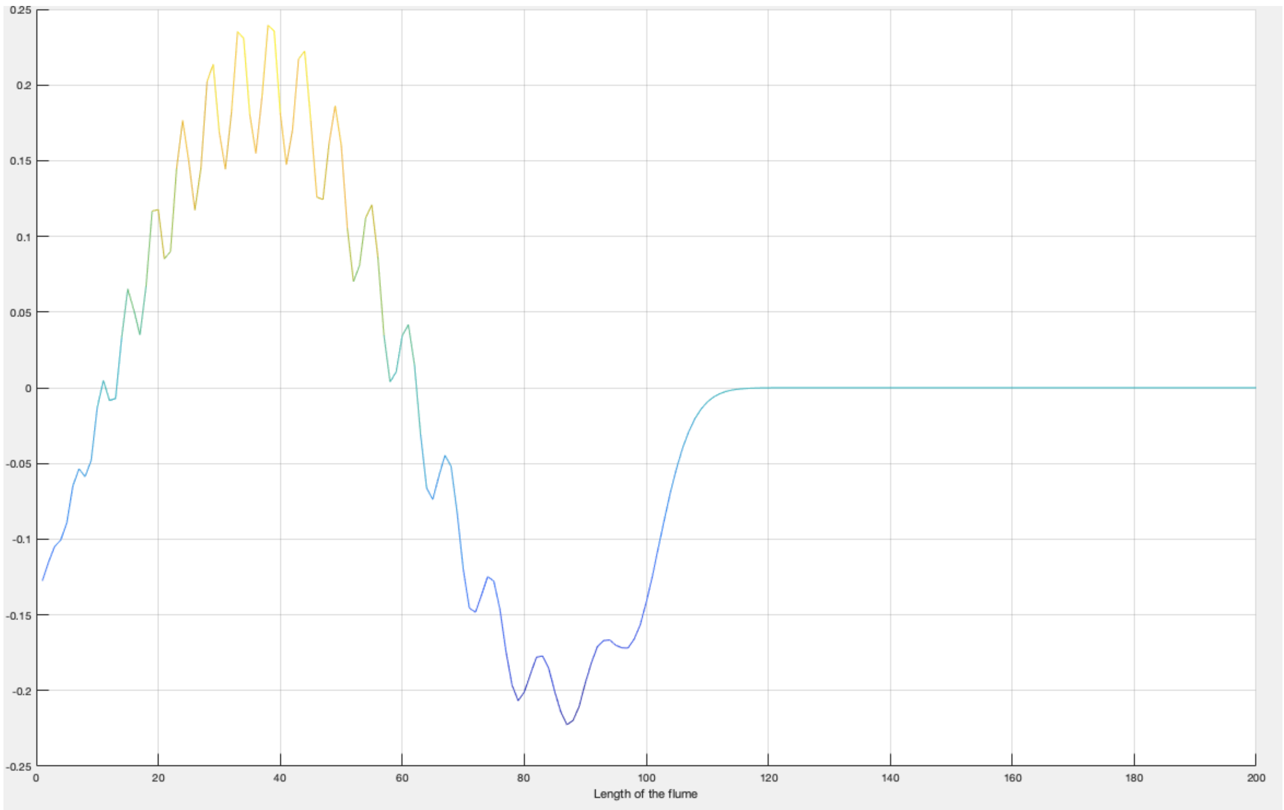
Wave
Length of the flume



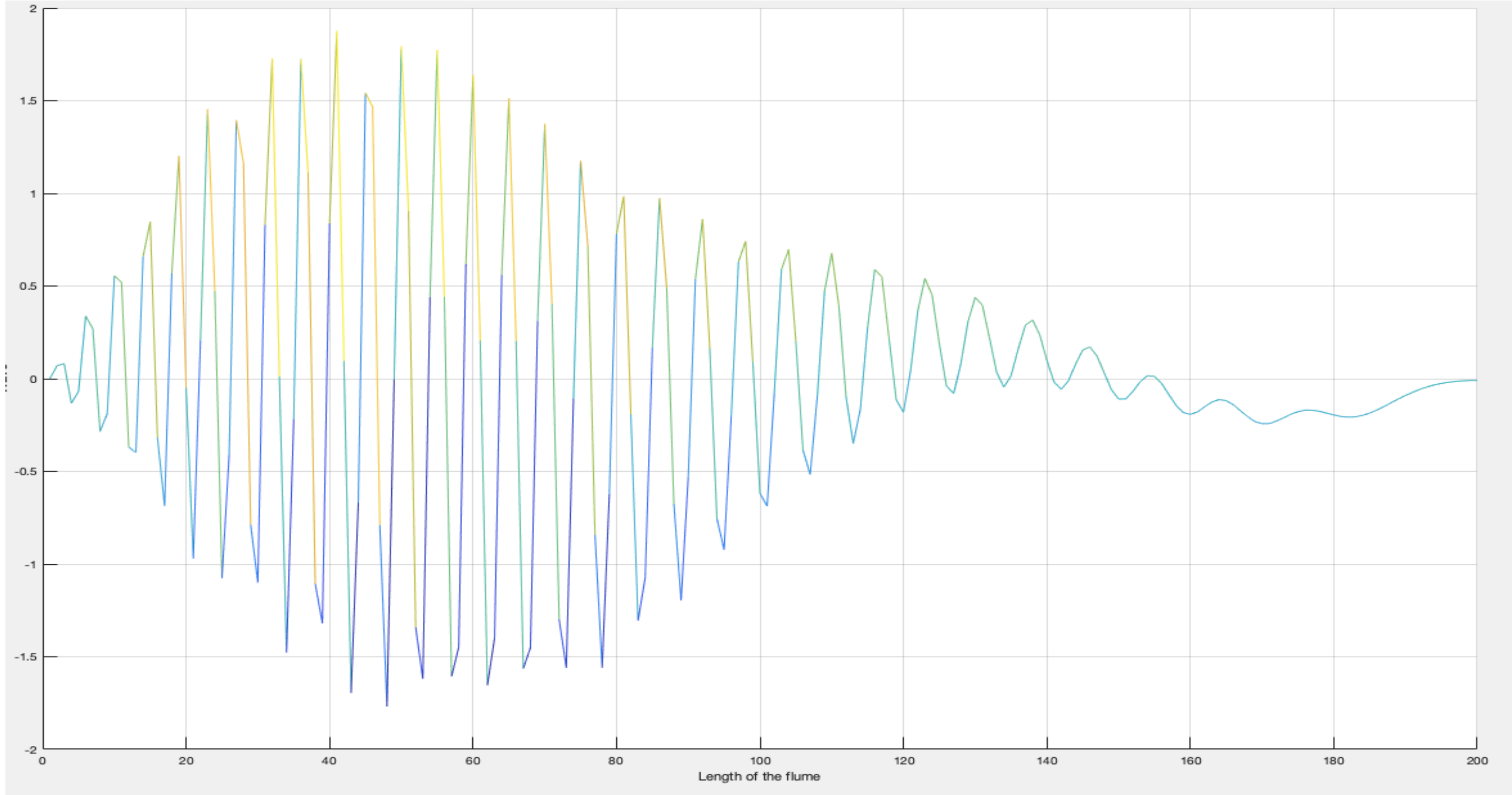
Wave
Width of the flume
Length of the flume

EULER EXPLICIT SCHEME - NON LINEAR SWE

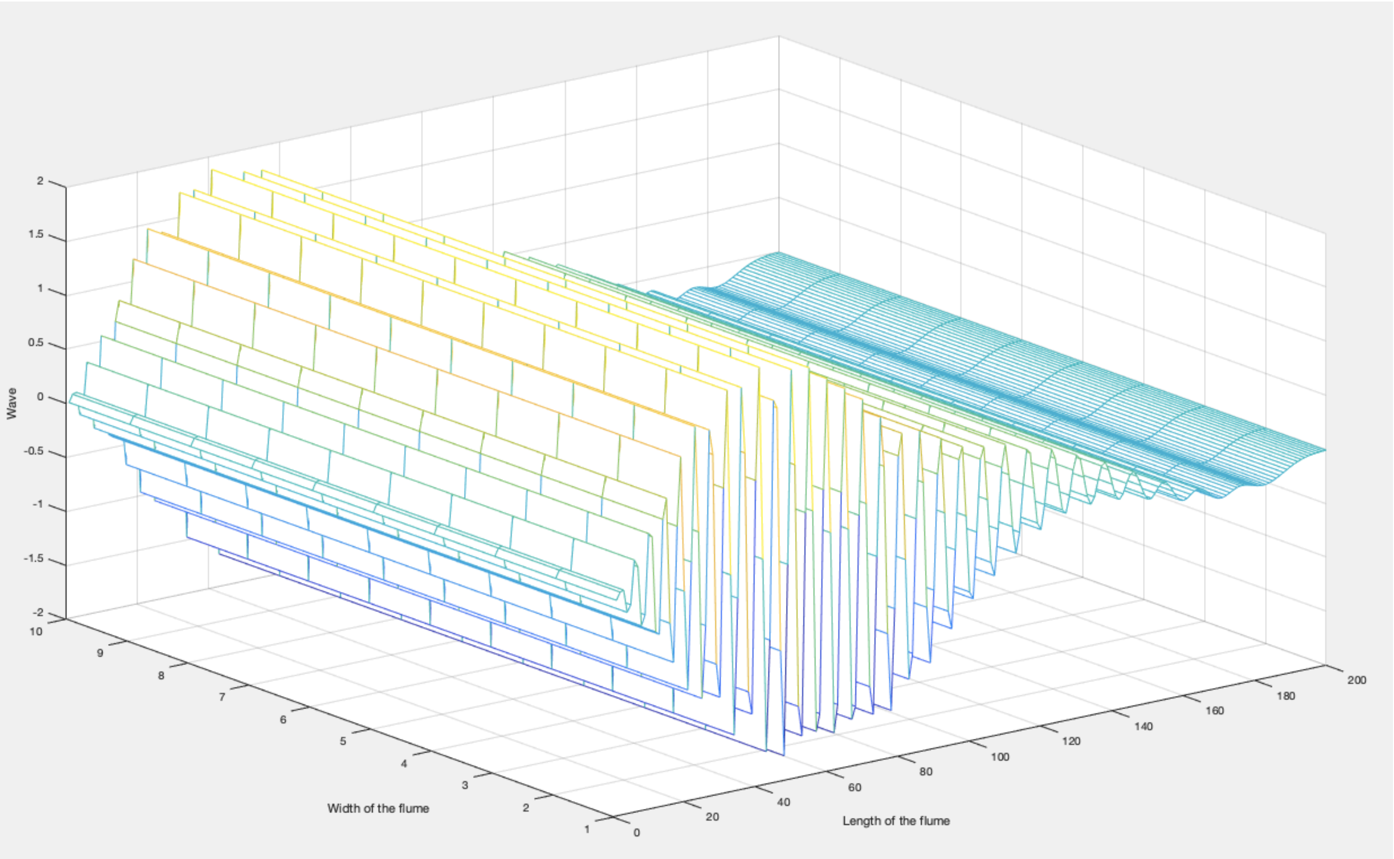
- Non Linearised Equation is next studied using the Euler Explicit method. Since Euler explicit scheme can't handle linearised wave equation, it is inevitable that it can't handle the non linear equation too, which is visible from the graph.



Pulse variation at half time



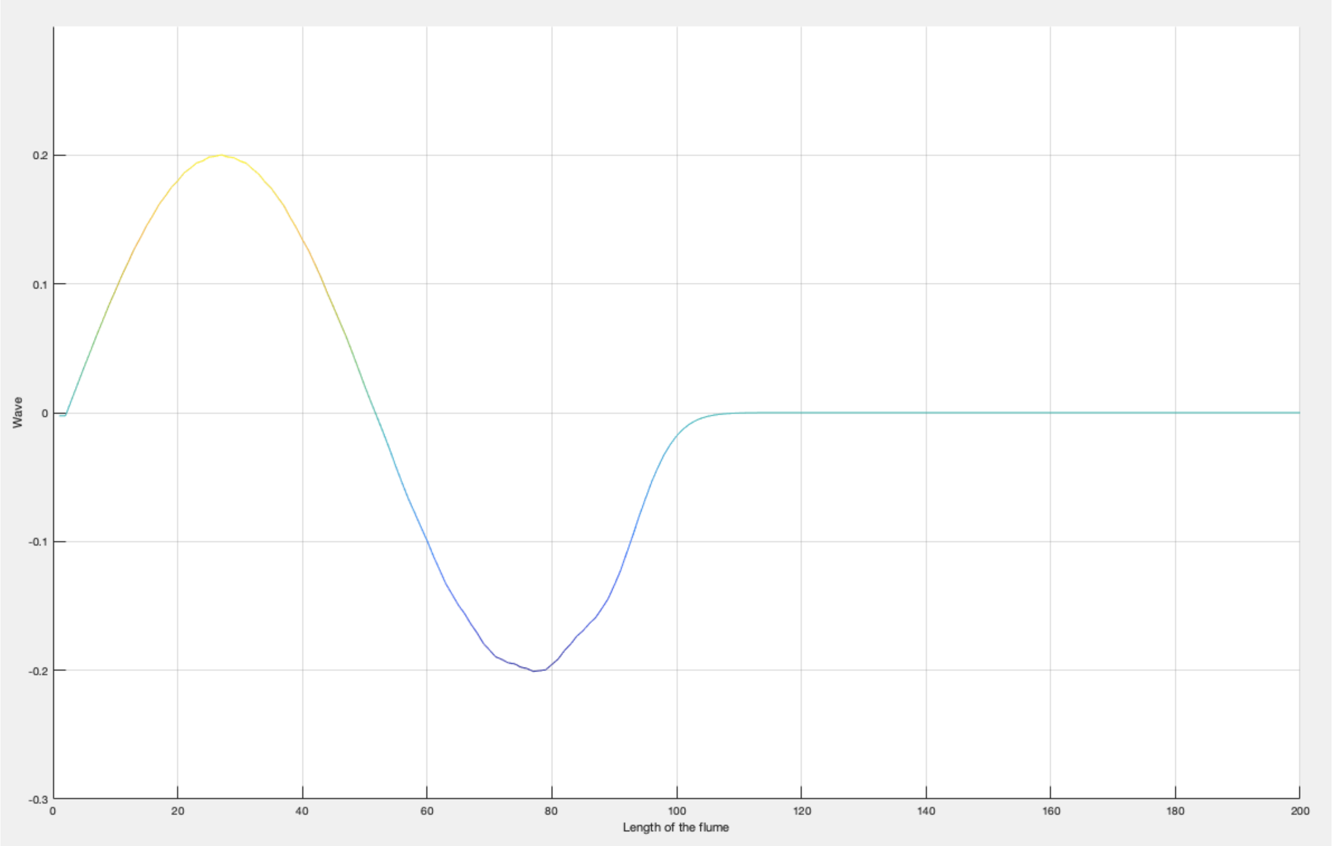
Length of the flume



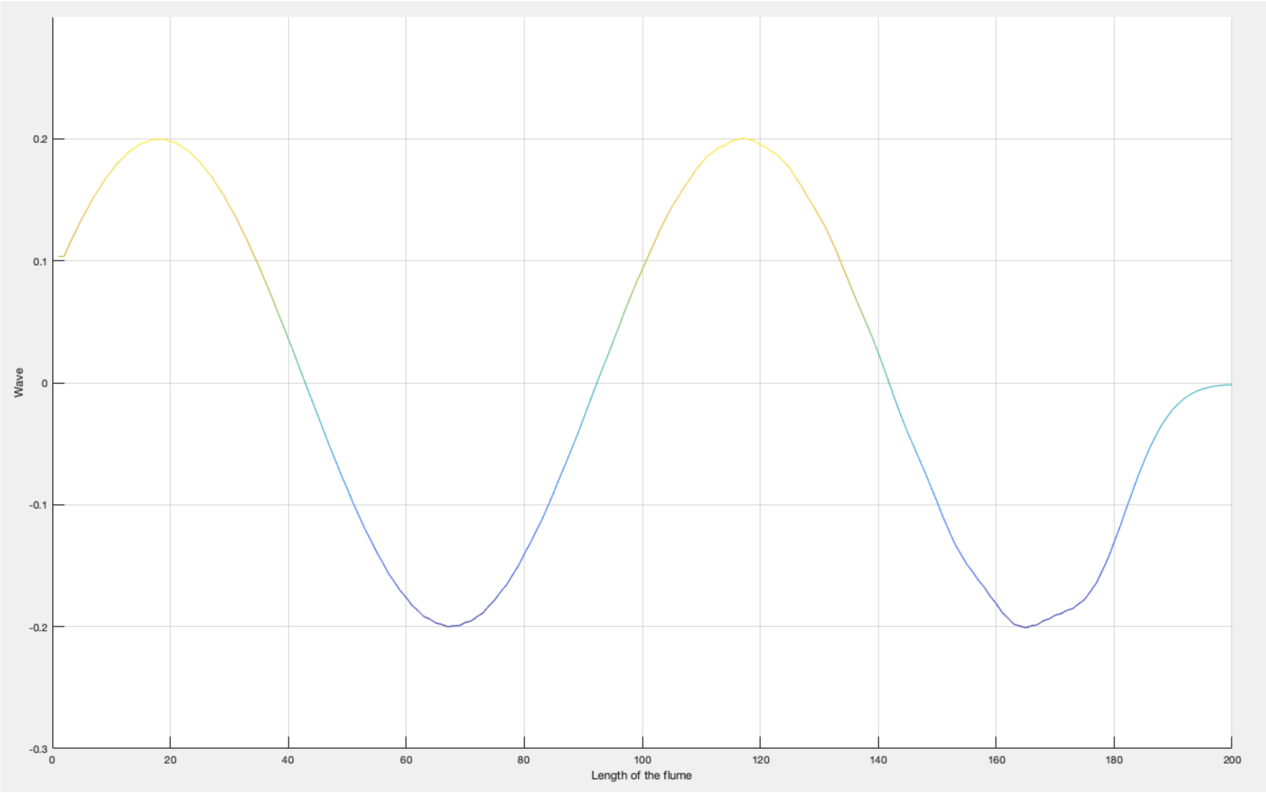
Wave
Width of the flume
Length of the flume

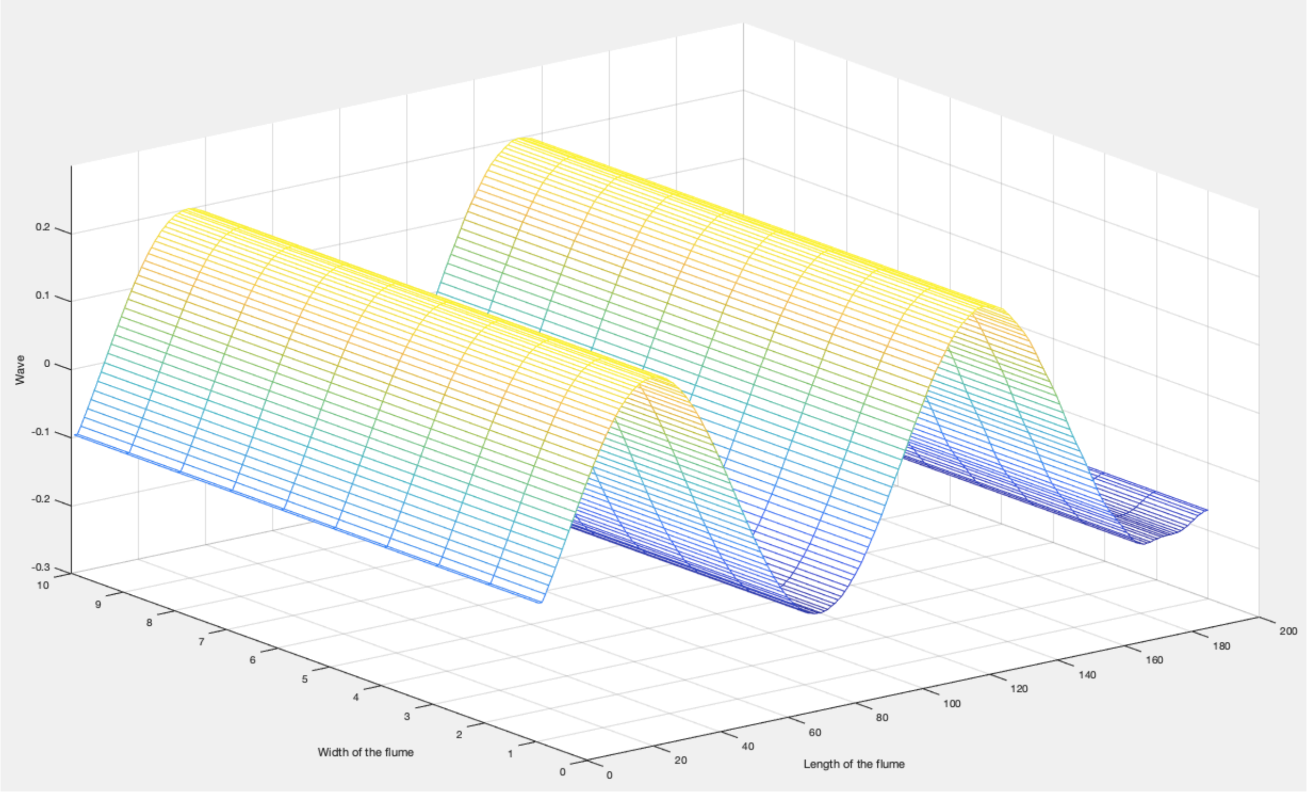
LEAP FROG SCHEME - LINEAR SWE

- Linearised equation is used with the leap frog scheme first. This method gives a stable result. On the right, a screenshot of simulation as the wave reaches L = 100 is taken.



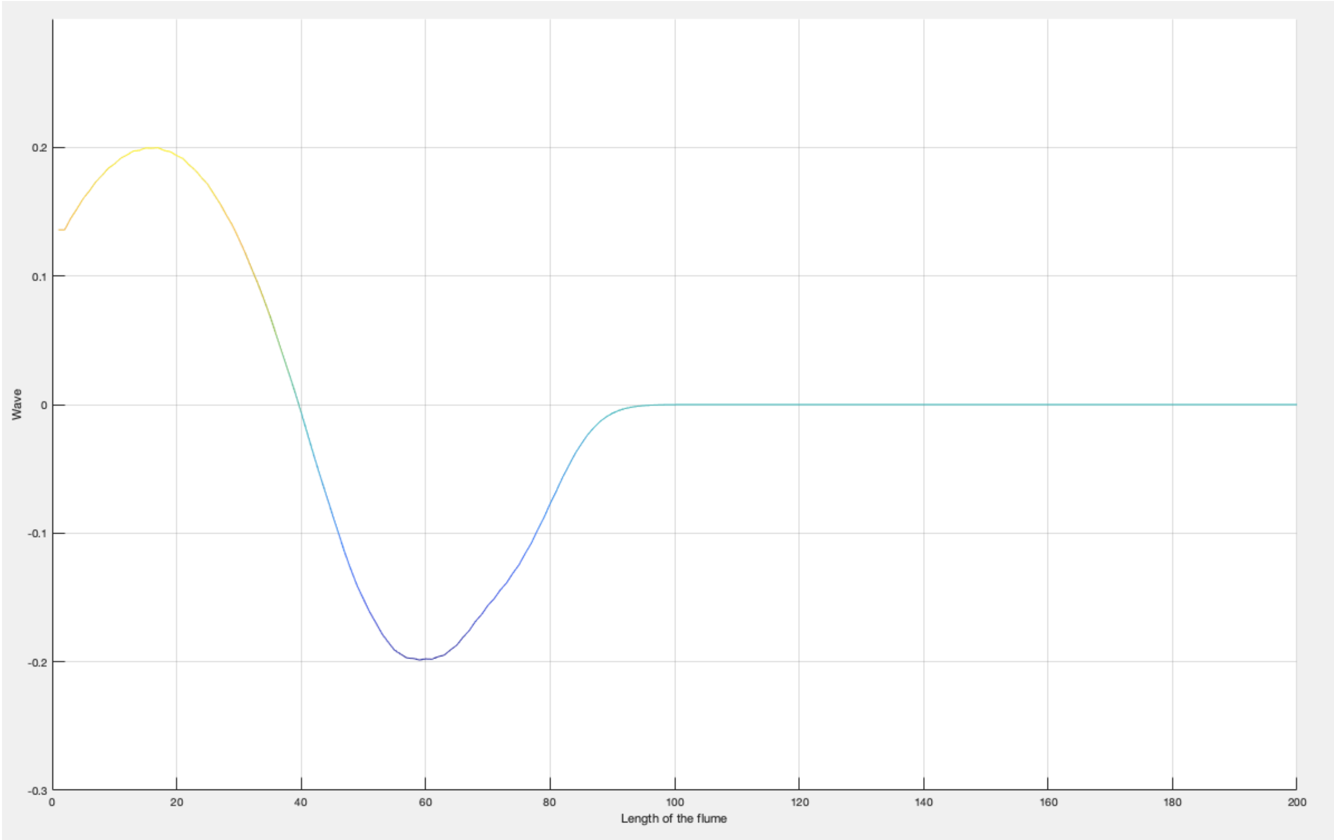
Pulse variation at half time


Wave
Length of the flume

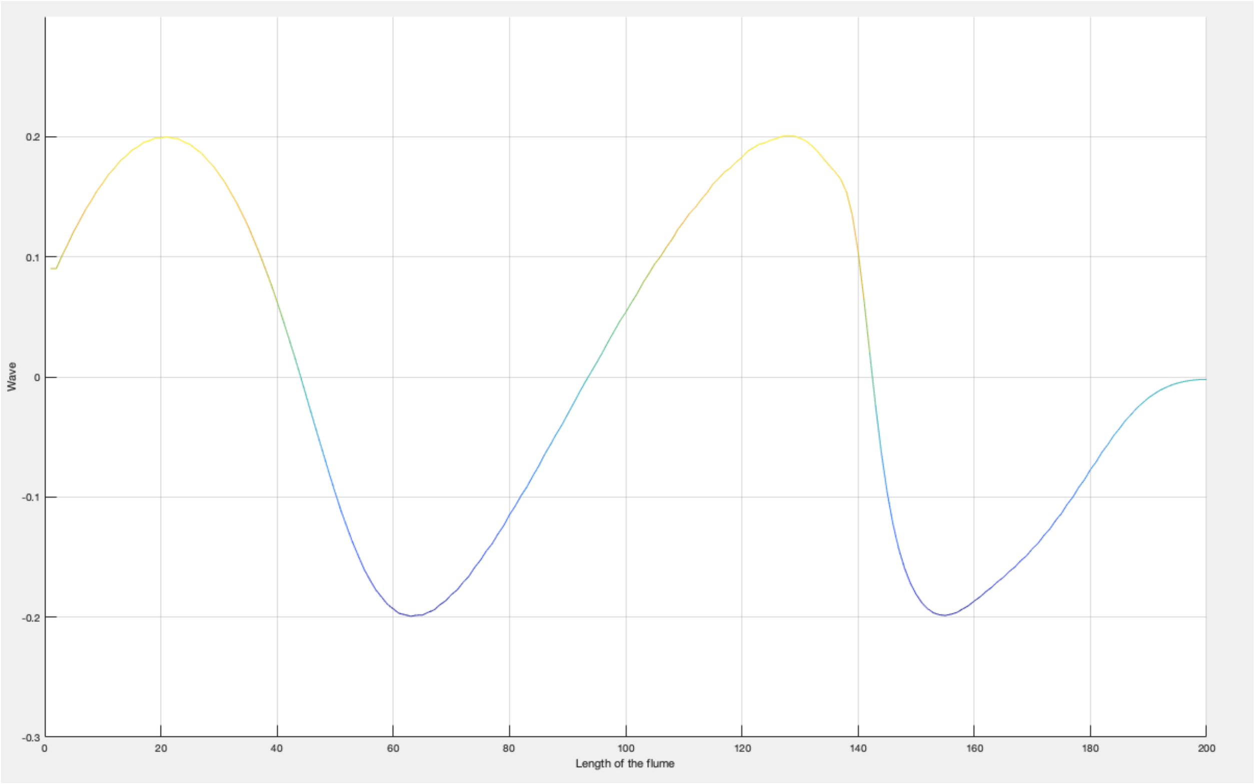

Wave
Width of the flume
Length of the flume

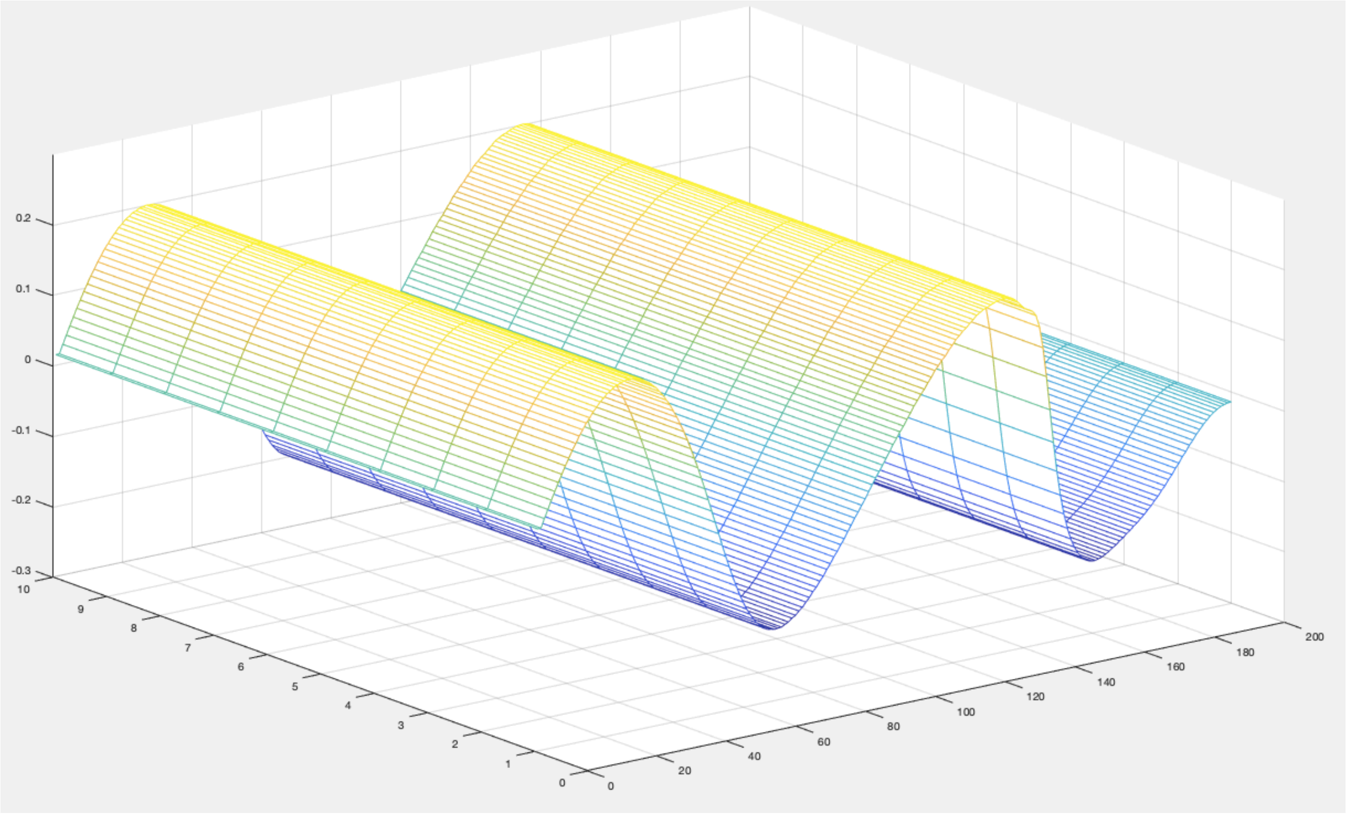
LEAP FROG SCHEME - NON LINEAR SWE

- Non-linear equation is shown. This method gives a stable result, but distorted wave is generated. On the right, a screenshot of simulation as the wave reaches L = 100 is taken.



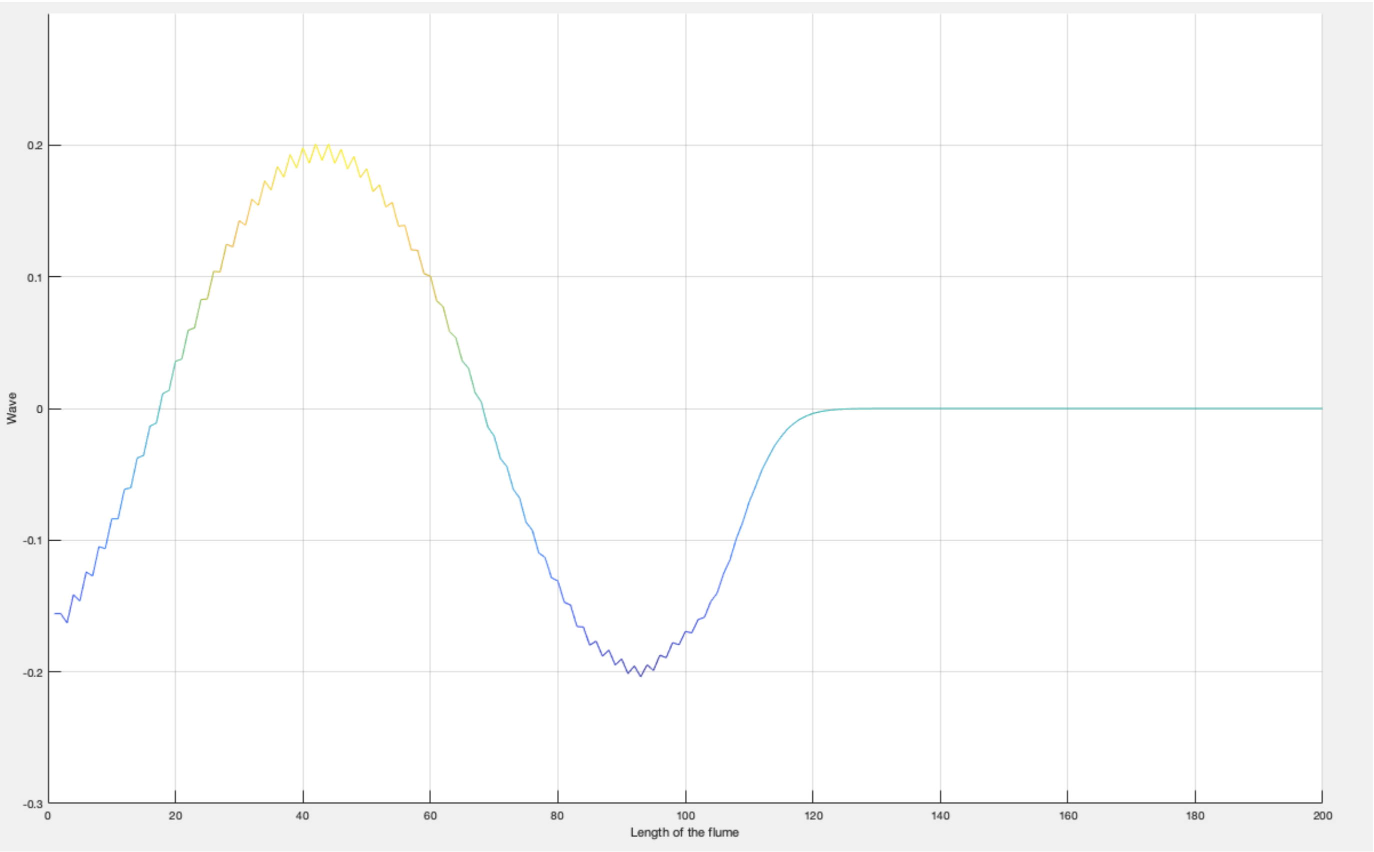
Pulse variation at half time


Wave
Length of the flume

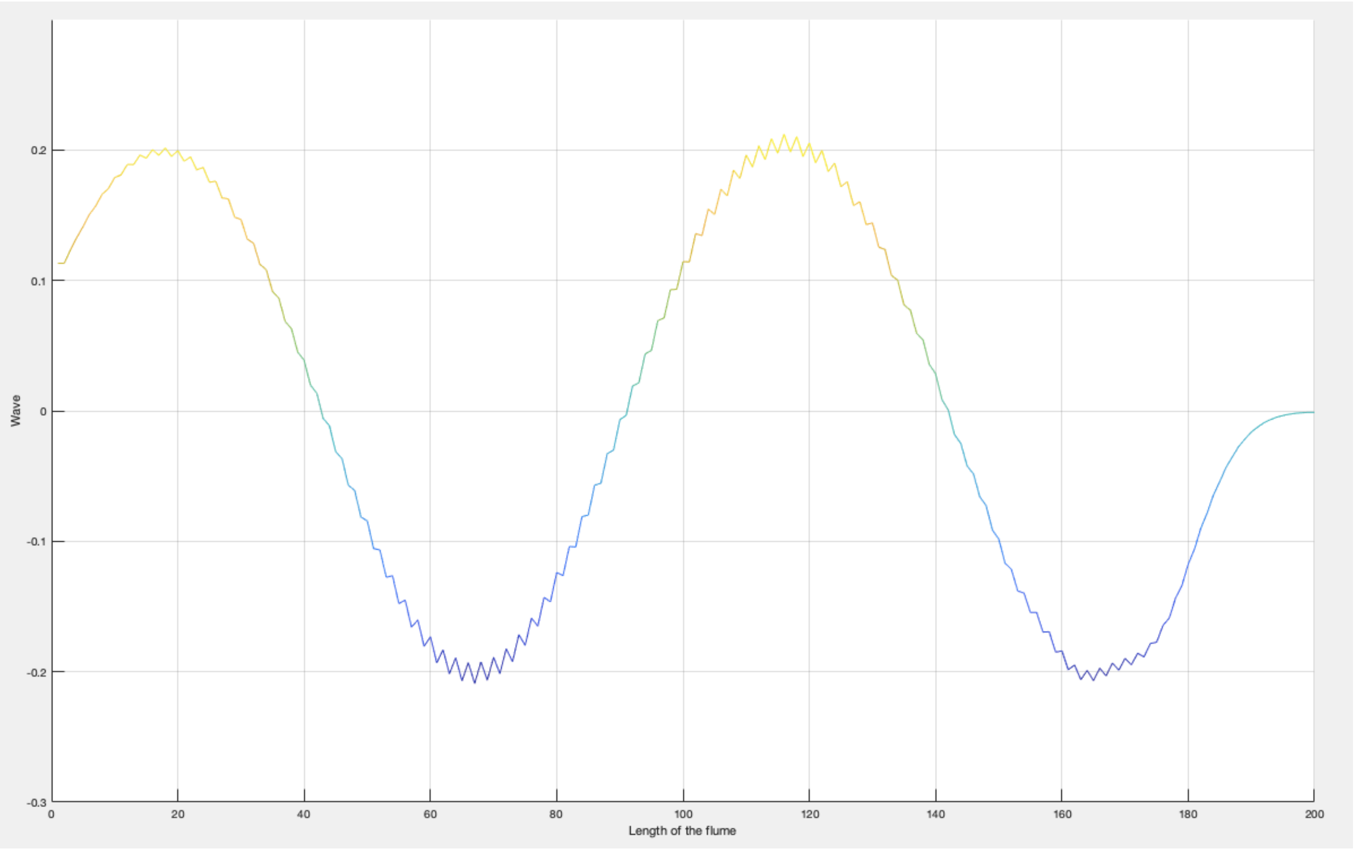


PREDICTOR CORRECTOR SCHEME - LINEAR SWE

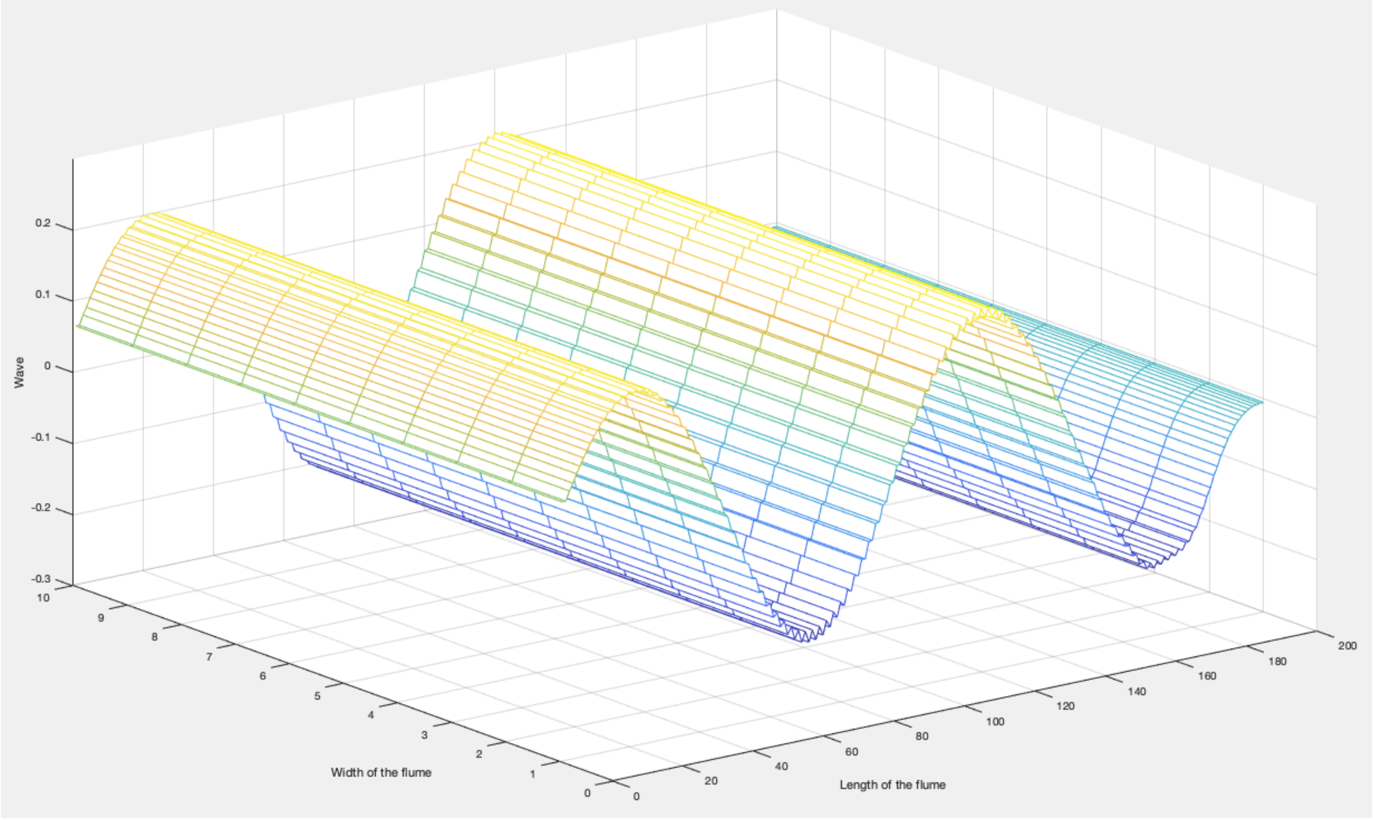
- This scheme gives very accurate results with both linear as well as non linear Boussinesq equation. This and the next slide shows the simulation of the linear wave.



Pulse variation at half time



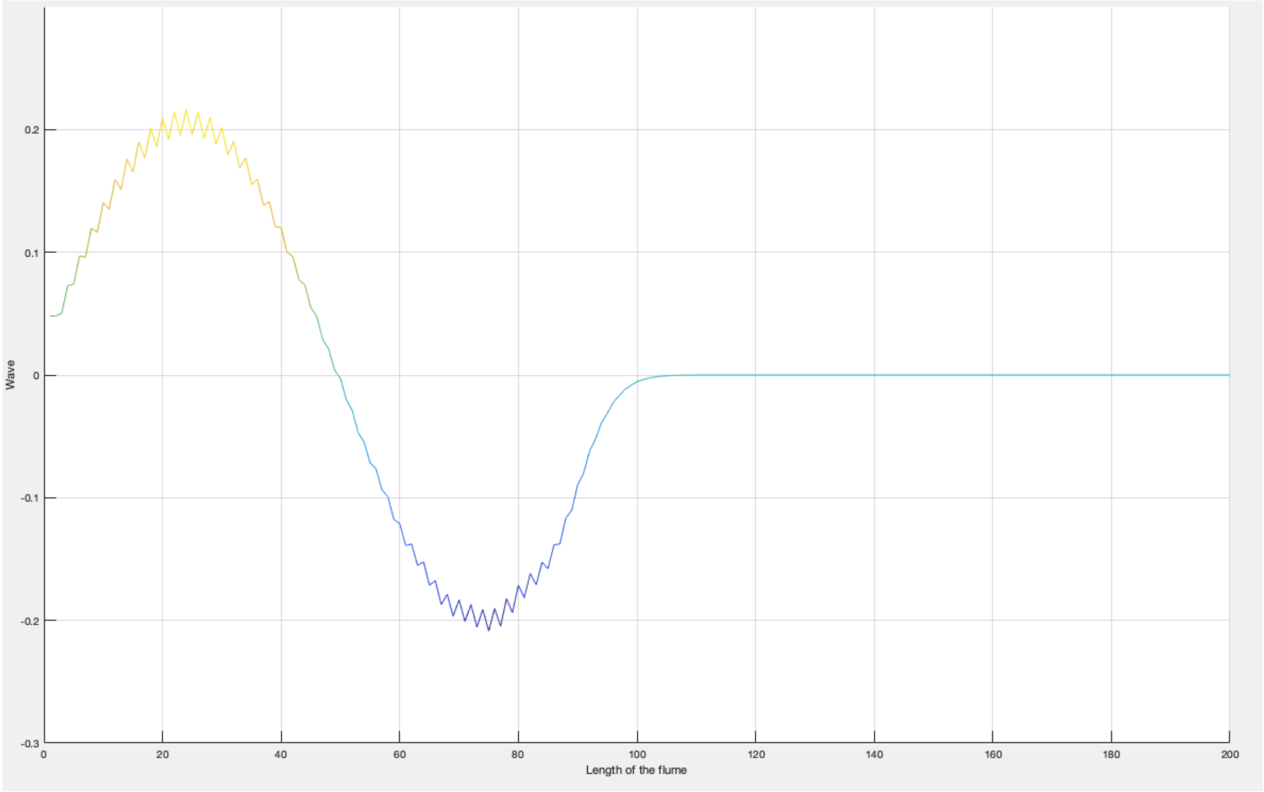
Wave
0.2
0.1
0
-0.1
-0.2
-0.3
0
20
40
60
80
100
120
140
160
180
200
Length of the flume



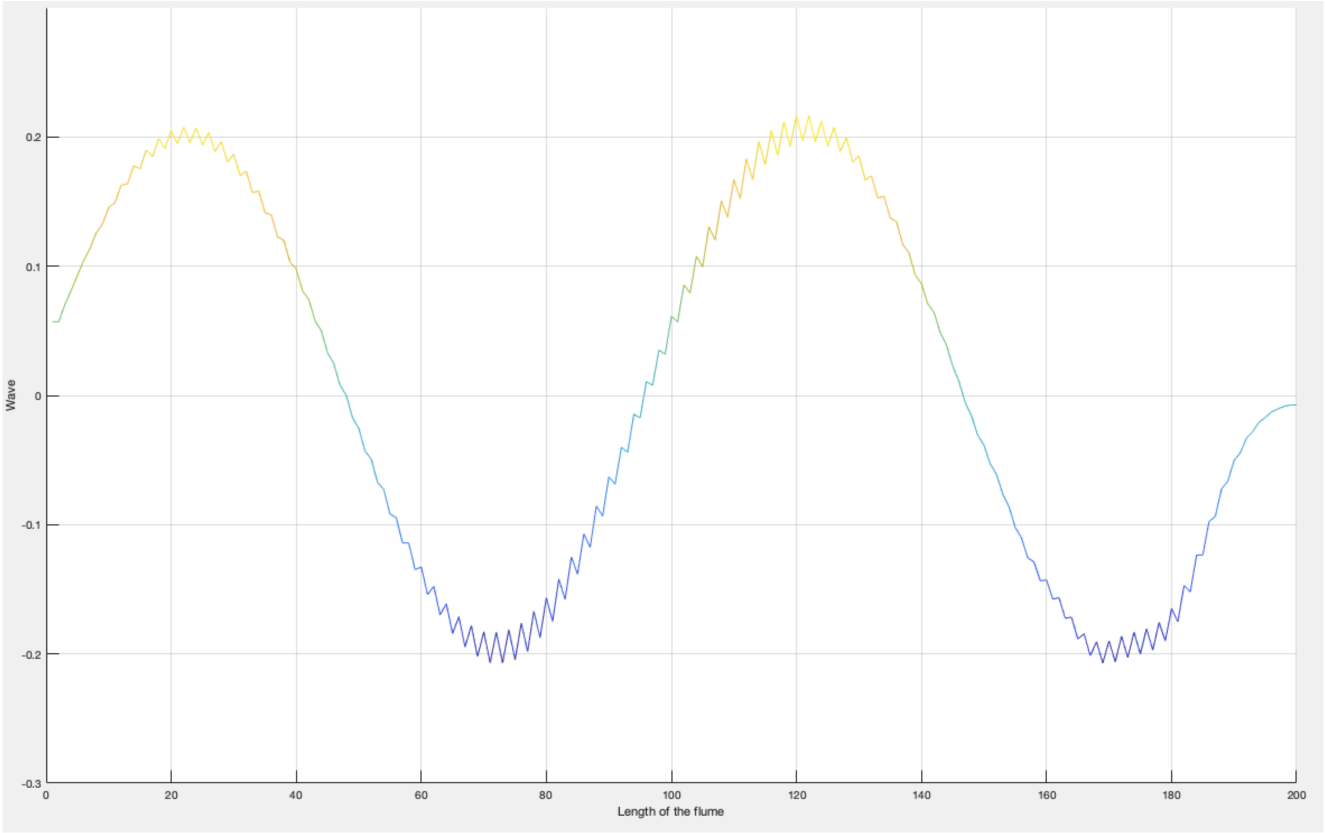
Wave
0.2
0.1
0
-0.1
-0.2
-0.3
10
9
8
7
6
5
4
3
2
1
0
Width of the flume
0
20
40
60
80
100
120
140
160
180
200
Length of the flume

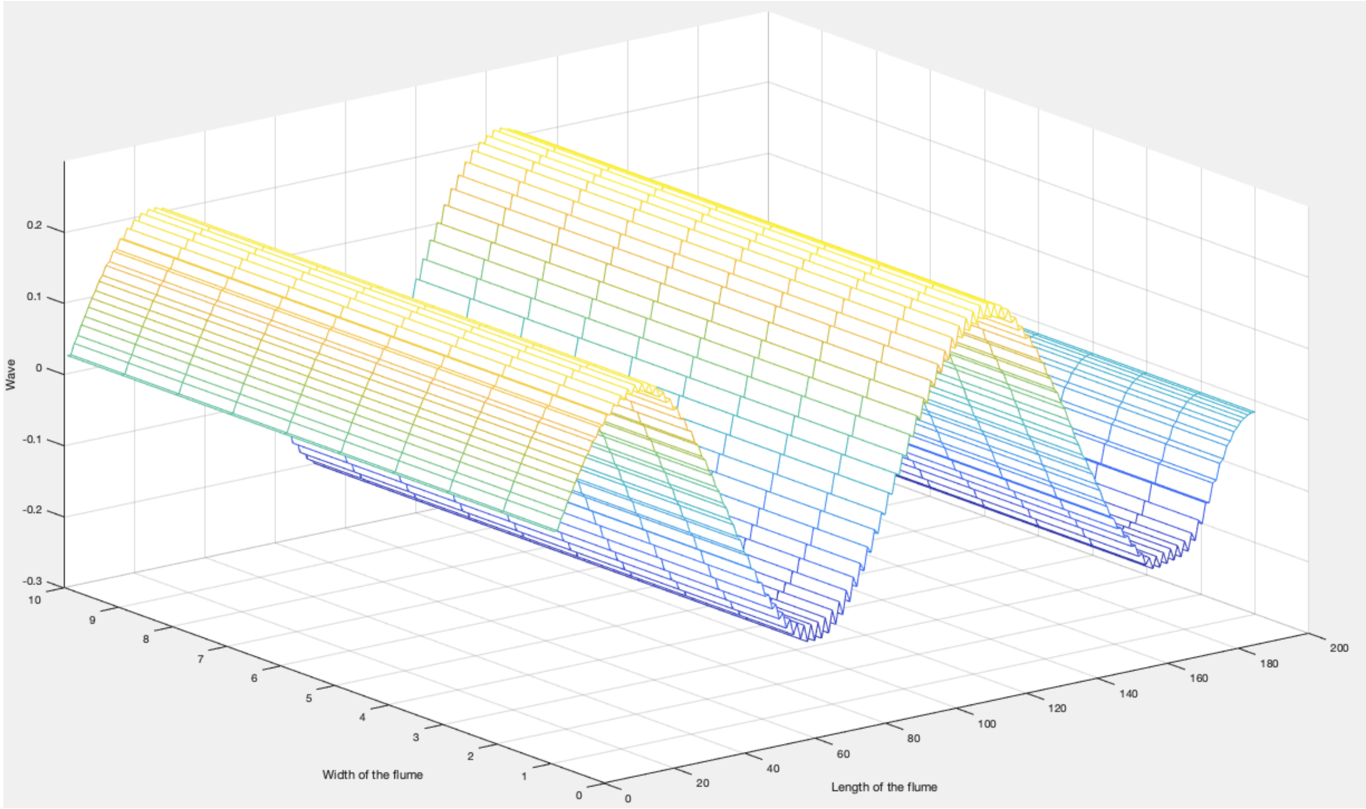
PREDICTOR CORRECTOR SCHEME - NON LINEAR SWE

- Finally, the predictor corrector method is used to estimate the non linear wave equation. It doesn't show any instability.



Pulse variation at half time


0.2
0.1
0
-0.1
-0.2
-0.3
Wave
0
20
40
60
80
100
120
140
160
180
200
Length of the flume


0.2
0.1
0
-0.1
-0.2
-0.3
Wave
10
9
8
7
6
5
4
3
2
1
0
Width of the flume
0
20
40
60
80
100
120
140
160
180
200
Length of the flume

RESULTS AND CONCLUSIONS

- Truncated Boussinesq Equation is equivalent to shallow water equation.
- Euler Explicit Schemes cannot model linear and non linear wave equation, and simulation breaks down as the wave progresses in space.
- Leap Frog Method provides an accurate representation of wave propagation for linear SWE, and a distorted wave, with the wave crest and trough tilted towards the direction of wave propagation is obtained.
- Predictor Corrector methods trumps over both these methods and gives accurate representation of both linear as well as non linear SWE.

REFERENCES

- Jiang, Henn, Sharma, Wash Waves Generated by Ships Moving on Fairways of Varying Topography, 2002.
- Lee, Yates, Wu, Experiments and analyses of upstream-advancing solitary waves generated by moving disturbances, 1988.