Product of sines

$$\prod_{k=1}^{n-1} \sin\left(\frac{\pi k}{n}\right) = \frac{n}{2^{n-1}}$$

Proof: Recall that

$$z^n - 1 = \prod_{k=1}^{n} \left(z - \zeta^k\right) \tag{1}$$

where $\zeta = \cos(2\pi/n) + i\sin(2\pi/n)$. Differentiating both sides of Equation (1), we obtain

$$nz^{n-1} = \sum_{j=1}^{n} \prod_{\substack{k=1\\k\neq j}}^{n} \left(z - \zeta^k\right) \tag{2}$$

Setting $z = 1$ in Equation (2), we obtain

$$n = \sum_{j=1}^{n} \prod_{\substack{k=1\\k\neq j}}^{n} \left(1 - \zeta^k\right) \tag{3}$$

Now note that in Equation (3), everyterm except the last term contains $(1 - \zeta^n)$, which is zero. Hence,

$$n = \prod_{k=1}^{n-1} \left(1 - \zeta^k\right) \tag{4}$$

Further,

$$1 - \zeta^k = 1 - \cos\left(\frac{2\pi k}{n}\right) - i\sin\left(\frac{2\pi k}{n}\right) = 2\sin^2\left(\frac{\pi k}{n}\right) - 2i\sin\left(\frac{\pi k}{n}\right)\cos\left(\frac{\pi k}{n}\right) \tag{5}$$

$$= -2i\sin\left(\frac{\pi k}{n}\right)\left(\cos\left(\frac{\pi k}{n}\right) + i\sin\left(\frac{\pi k}{n}\right)\right) = -2i\sin\left(\frac{\pi k}{n}\right)\exp\left(\frac{i\pi k}{n}\right) \tag{6}$$

Therefore, from above and Equation (4), we obtain

$$n = \prod_{k=1}^{n-1} \left(1 - \zeta^k\right) = (-2i)^{n-1} \prod_{k=1}^{n-1} \sin\left(\frac{\pi k}{n}\right) \prod_{k=1}^{n-1} \exp\left(\frac{i\pi k}{n}\right) \tag{7}$$

$$= 2^{n-1}\exp\left(-\frac{i\pi(n-1)}{2}\right)\left(\prod_{k=1}^{n-1} \sin\left(\frac{\pi k}{n}\right)\right)\exp\left(\frac{i\pi(n-1)}{2}\right) \tag{8}$$

Hence, we obtain

$$\boxed{\prod_{k=1}^{n-1} \sin\left(\frac{\pi k}{n}\right) = \frac{n}{2^{n-1}}}$$