Quantitative Asset Management: Final Project

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Abstract: Our project is an analysis of various trading strategies. We analyze expected excess returns, standard deviations, Sharpe ratios, certainty equivalent returns, and turnover rates for the following asset weighting systems: equal, value, mean-variance efficient in-sample, mean-variance efficient out-of-sample, and naïve risk-parity. We do not mean to advocate a particular strategy but instead to show that results tend not to perform significantly better—and, in fact, sometimes perform worse—than the rudimentary equal weighting of returns ($1/N$). Our analysis is inspired by the paper "Optimal Versus Naive Diversification: How Inefficient is the $1/N$ Portfolio Strategy?" by DeMiguel, Victor *et al.* However, this is not a replication. The time-frames are different, we did not use exactly the same data, our analysis considers naïve risk-parity and our methodology is, at times, different.

Table 1: Strategies

#	Model	Abbreviation
1.	$1/N$ with rebalancing	eq or $1/N$
2.	Value-weighting	vw
3.	Mean-Variance efficient in-sample	mv_{in}
4.	Mean-Variance efficient out-of-sample	mv_{out}
5.	Naïve risk-parity	rp

We consider an estimation window of $M = 60$ months when we calculate estimated expected returns, variance, and covariance, except for mve_{in} where we use $M = T$.

1 Description of Asset-Allocation Models

As mentioned previously, we considered five asset allocation models within our analysis. We will follow the notation in DeMiguel *et al.* Let R_t denote the vector of N risky-assets at time t, the N-dimensional vector μ_t denote expected returns at time t, and Σ_t denote the covariance matrix at time t. The symbol $\mathbf{1}$ denotes either a column or row vector of N-values of 1 within our work. To denote sample expected values and covariance matrix, we write $\hat{\mu}_t$ and $\hat{\Sigma}_t$, respectively. Let T denote the entire span of time. The value M is the sample window when we consider rolling estimations; we used $M = 0$, $M = 60$ and sometimes $M = T$

within our work.

1.1 Equal-Weighting

The equal-weighted portfolio strategy (ew or $1/N$) supposes all assets within our portfolio are equally weighted. Therefore, the returns in month t are simply the average returns of our N risky assets during that month, *i.e.* the returns at time t are

$$\frac{1}{N} R_t \cdot \mathbf{1}.$$

1.2 Value-Weighting

Value-weighted portfolio strategy (vw) supposes that the weight on each asset is simply the market capitalization of that asset divided by the total market capitalization. Since market capitalization grows at the rate of total returns, by definition, this strategy requires no rebalancing.

Within all of our strategies, we include the Fama and French excess market returns. The equal-weighted excess returns of our universe are simply the market excess returns. This is the convention of DeMiguel *et al.* as well.

In some formulations of the efficient market hypothesis, the market portfolio contains the maximum Sharpe ratio. The hypothesis asserts that, if this were not the case, investors would purchase the higher Sharpe ratio asset until the ratio of excess returns to standard deviation is equivalent to that of the market. Even in our stylistic analysis, we find this hypothesis to be false.

1.3 Mean-Variance Efficient

We consider two cases of the mean variance efficient portfolio: in-sample (mve_{in}) and out-of-sample (mve_{out}). In both cases, we find $\boldsymbol{w}_t$ which solves

$$\begin{aligned} \min_{\boldsymbol{w}_t} \quad & \boldsymbol{w}_t' \Sigma_t \boldsymbol{w}_t \\ \text{subject to} \quad & \mathbf{1} \cdot \boldsymbol{w}_t = 1. \end{aligned}$$

As has been proven in Investments and Empirical Methods, the solution is

$$\boldsymbol{w}_t = \frac{\Sigma_t^{-1}\mu_t}{\mathbf{1}\Sigma_t^{-1}\mu_t}.$$

Clearly, however, we do not know μ_t and Σ_t. Hence, these parameters must be estimated with past data.

Our approach for the estimation of μ_t and Σ_t varies depending on whether we consider an in-sample or out-of-sample estimation. For the in-sample estimation, we used the entire time-period, *i.e.* we supposed $M = T$, while out-of-sample we only use values from time $t - M$ to time $t - 1$ where $M = 60$. In all cases, these intervals are inclusive. The in-sample strategy is not implementable. However, it serves as a benchmark. Both calculations follow the example of DeMiguel *et al.*.

1.4 Naïve Risk-Parity

The naïve risk-parity portfolio strategy (rp) weights risky assets inversely proportional to the assets' risks. If we assume assets are independent and the standard deviations are known, this strategy would equally distribute the risk among the assets. If assets 1, 2, 3, ..., and N have standard deviations σ_t^1, σ_t^2, ..., and σ_t^N, respectively, then the weight on asset i would be

$$w_t^i = \frac{1/\sigma_t^i}{1/\sigma_t^1 + 1/\sigma_t^2 + \ldots + 1/\sigma_t^N}.$$

Under the assumption of independence, this formula can be written as

$$w_t^i = \frac{\Sigma_t^{-1}\boldsymbol{e}_i}{\mathbf{1}\Sigma^{-1}\mathbf{1}},$$

where $\boldsymbol{e}_i$ is a vector with a 1 in the i-th position and 0 otherwise. We estimate each σ_t^i at time t by considering returns of our N risky assets from time $t - M$ to time $t - 1$, inclusive.

In real-life, assets tend to have correlations. Therefore, this strategy does not, in fact, equally distribute risk. However, we opted to consider naïve risk-parity because we learned about risk-parity between bonds and stocks in class and we did not want to go far outside of the scope of the course for our project.

2 Evaluating Performance

We consider five criteria to evaluate performance:

- excess returns,
- standard deviation of excess returns,
- the Sharpe ratio,
- certainty equivalent returns (CEQ), and
- turn-over rate.

We assume that the reader is familiar excess returns, standard deviations, with the Sharpe ratios. The certainty equivalent return for asset k is defined to be

$$CEQ_k = \mu_k - \frac{\gamma}{2}(\sigma_k)^2.$$

We estimate μ_k and σ_k via consideration of the entire sample from September 1989 to December 2019. Because we wish to report a numerical value, we let $\gamma = 1$. The turn-over rate is defined as

$$\frac{1}{T-M}\sum_{t=1}^{T-M}\sum_{i=1}^{N}\left|w_t^i - w_{t^-}^i\right|$$

where $w_{t^-}^i$ denotes the weighting in asset i immediately before rebalancing. Therefore,

$$w_{t^-}^i = \begin{cases} 0, & t = 1 \\ \frac{w_{t-1}^i(1+R_{t-1}^i)}{1+\boldsymbol{w}_{t-1}\boldsymbol{\cdot}R_{t-1}}, & t > 1. \end{cases}$$

3 Data

The Fama French data occasionally has -99.99 values, which we removed. Other missing values were dealt with using our Python and R functions; typically they were ignored.

Table 2: List of Data

#	Dataset and source	N	Abbreviation
1.	S & P Sector data Source: S & P Dow Jones Indices	10 + 1	SPX
2.	Ten industry portfolios and the US equity market portfolio Source: Ken French's Web site	10 + 1	Industry
3.	SMB and HML portfolios and the US equity market portfolio Source: Ken French's Web site	2 + 1	MKT/SMB/HML
4.	Twenty size- and book-to-market portfolios and the US equity MKT Source: Ken French's Web site	20 + 1	FF-1-factor
5.	Twenty size- and book-to-market portfolios and the MKT, SMB, HML, and MOM portfolios Source: Ken French's Web site	20 + 4	FF-4-factor

All data within our analysis is from September 1989 to December 2019, inclusive. The value of N denotes the number of risky-assets. The risk-free rate is the 90-day T-bill rate taken from Ken French's website. As in DeMiguel et al., we remove the five largest market capitalization from the 25 Fama-French portfolios because the 25 Fama-French MKT, HML, and MOM can almost be written as a linear combination of the 25 Fama-French portfolios. All portfolios contain the Fama French excess market returns.

4 Results

Below we list the monthly excess returns. Clearly, the mean-variance efficient portfolio produces the largest results. However, this strategy cannot be replicated because it uses future data to construct the weights. The second best results varied from universe to universe.

Table 3: Excess Returns

Strategy	SPX $N = 11$	Industry Portfolios $N = 11$	Mkt/SMB/HML $N = 3$	FF 1-factor $N = 21$	FF 4-factor $N = 24$	Mean
$1/N$	0.006584	0.006820879	0.002909	0.007769	0.007105	0.006237576
mve_{in}	0.046232	0.011940283	0.004402	0.025654	0.006833	0.019012257
mve_{out}	0.015140	0.021819592	0.003321	0.023020	0.105233	0.033706718
vw	0.006841	0.006557692	0.006558	0.006558	0.006558	0.006614538
rp	0.006694	0.007302720	0.002368	0.008547	0.007191	0.006420544

Monthly excess returns as a decimal. We consider an estimation window of $M = 60$ months for calculations that require previous return information, except for mve_{in} where we use $M = T$.

In the following table, we considered the standard deviation of excess returns. The mean-variance efficient portfolio produces very large standard deviations out-of-sample. This is not a contradictory result because returns are heteroscedastic; the covariance matrix changes over time and is not equal to the sample covariance matrix from $t - M - 1$ to $t - 1$.

Table 4: Standard Deviation of Excess Returns

Strategy	SPX $N = 11$	Industry Portfolios $N = 11$	Mkt/SMB/HML $N = 3$	FF 1-factor $N = 21$	FF 4-factor $N = 24$	Mean
$1/N$	0.04014436	0.03930094	0.019409	0.050617	0.044485	0.03879126
mve_{in}	0.10504217	0.15207191	0.025830	0.049555	0.012363	0.068972416
mve_{out}	0.16135713	0.36925550	0.210896	0.161758	1.733706	0.527394526
vw	0.04049606	0.04217485	0.042175	0.042175	0.042175	0.041839182
rp	0.03914803	0.03675082	0.018359	0.050516	0.041462	0.03724717

Monthly standard deviation of excess returns as a decimal. We consider an estimation window of $M = 60$ months for calculations that require previous return information, except for mve_{in} where we use $M = T$.

Table 5: Sharpe Ratios

Strategy	SPX $N = 11$	Industry Portfolios $N = 11$	Mkt/SMB/HML $N = 3$	FF 1-factor $N = 21$	FF 4-factor $N = 24$	Mean
$1/N$	0.164018	0.17355510	0.149891	0.153490	0.159727	0.16013622
mve_{in}	0.440128	0.07851735	0.170433	0.517681	0.552708	0.35189347
mve_{out}	0.093668	0.05909077	0.015748	0.142310	0.060698	0.074302954
vw	0.161893	0.15548821	0.155488	0.155488	0.155488	0.156769042
rp	0.170995	0.19870901	0.128997	0.169190	0.173446	0.178085003

Monthly Sharpe ratios. We consider an estimation window of $M = 60$ months for calculations that require previous return information, except for mve_{in} where we use $M = T$.

When we adjust risk by the variance, the results vary substantially. However, the out-of-sample mean variance efficient portfolio consistently performs the worst.

Table 6: Certainty Equivalent Returns

Strategy	SPX $N = 11$	Industry Portfolios $N = 11$	Mkt/SMB/HML $N = 3$	FF 1-factor $N = 21$	FF 4-factor $N = 24$	Mean
$1/N$	0.005778	0.0060485970	0.002721	0.006488	0.006116	0.005980866
mve_{in}	0.024096	0.0003773499	0.004069	0.024426	0.006757	0.013242087
mve_{out}	0.002077	-0.0463552197	-0.018917	0.009937	-1.397635	-0.290178644
vw	0.005948	0.005668333	0.005668	0.005668	0.005668	0.005738083
rp	0.005927	0.0066274083	0.002200	0.007271	0.006332	0.005671482

Monthly Certainty equivalent returns. We chose $\gamma = 1$ within our calculations. We consider an estimation window of $M = 60$ months for calculations that require previous return information, except for mve_{in} where we use $M = T$.

We list absolute asset turn-over rates. The mean-variance efficient portfolio in-sample is omitted because it is not tradable. Clearly, the mean-variance efficient portfolio out-of-sample has the highest asset turn-over. Naïve risk-parity also tends to perform poorly.

Table 7: Asset Turn-Over

Strategy	SPX $N = 11$	Industry Portfolios $N = 11$	Mkt/SMB/HML $N = 3$	FF 1-factor $N = 21$	FF 4-factor $N = 24$	Mean
$1/N$	0.006584	0.02194351	0.023718	0.018814	0.022231	0.018658102
mve_{in}	-	-	-	-	-	-
mve_{out}	26.845508	26.46202826	6.078604	32.830331	46.131542	27.66960265
vw	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
rp	0.071080	0.02829952	1.996711	19.937500	22.927632	8.992244504

Monthly mean asset turn-over. We did not followed the convention of DeMiguel et al. for this table. Mean-variance portfolio in-sample is omitted because this strategy is not tradable. We consider an estimation window of $M = 60$ months for calculations that require previous return information.

5 Conclusion

The $1/N$ was competitive with the other implementable strategies we considered. Though some strategies performed better in some circumstances, there was no clear winner. There seems to be no premium for sophistication.