RGPVNOTES.IN

Program : **B.Tech**

Subject Name: **Mathematics-I**

Subject Code: **BT-102**

Semester: **1st**

UNIT-II

Syllabus : Definite Integral as a limit of sum, Application in summation of series, Beta and gamma Functions, Multiple Integral(Double and triple), Evaluate surface area and volume of revolution, Change of Variables and change of Order of Integration, Applications of Multiple Integral in Area and Volume .

Definite & Indefinite Integrals

There are 2 types of integral

(i) *Indefinite*, in which we *aren't* given the limits of integration, *i.e.* $x=a$ to $x=b$, so we just calculate a generic, all purpose solution, and

(ii) *Definite*, in which we are told a and b and so we can calculate an explicit value for an area.

Indefinite integrals

If the differential of x^3 is $3x^2$, then

$$\int 3x^2.dx = x^3$$

But $3x^2$ is also the differential of $x^3 - 1$ and $x^3 + 8$, *etc*. so that this reversal is not unique - we've 'lost' the *constant*! So in general, $3x^2$ is the differential of $(x^3 + k)$ where k is any constant – this is known as the *'constant of integration'*.

We write this as: $\int 3x^2.dx = x^3 + k$

(Later on, you'll see that if we're given more information, we can work out the value for k, but for now, we just leave it as it is).

Integration 'magic' Formula

Since integration is the reverse of differentiation, for any polynomial $y(x) = x^n$, we can simply reverse the differentiation procedure, so that the integral is given by

$$\boxed{\int x^n.dx = \frac{x^{n+1}}{(n+1)} + k}$$ (except for $n = -1$)

In words: "Add one to the power, then divide by the new power. Then add k."

Examples

1. $\int x^2.dx = \frac{x^3}{3} + k$

2. $\int 20x^4.dx = 4x^5 + k$

Variations on Nomenclature

Because constants don't affect the integration, it is common to bring them in front of the integration sign to make things clearer.

For example: $\int abx^3.dx = ab\int x^3.dx = ab\frac{x^4}{4} + k$

or: $\int 5q^2.dq = 5\int q^2.dq = \frac{5q^3}{3} + k$

Also, the position of the .dx is usually last in the line, but it can, in principle, be anywhere inside the integral. You may sometimes see the .dx written first (usually in Physics textbooks).

For example: Area = $\int dr\,(r^3 - r^5)$ This is identical to: $\int (r^3 - r^5).dr$

Roots follow the same rule

6. $\int \sqrt{x}\,.dx = \int x^{½}.dx = \frac{x^{3/2}}{(3/2)} = \frac{2\sqrt{x^3}}{3} + k$

Inverse powers also follow the same rule

7. $\int \frac{1}{x^3}.dx = \int x^{-3}.dx = \frac{x^{-2}}{-2} = -\frac{1}{2x^2} + k$

This is true so long as the exponent is *not* -1. $\int \frac{1}{x}$ cannot be calculated using this formula because we get a divide-by-zero error.

Other variables

8. $\int 2m^2.dm = \frac{2m^3}{3} + k$

9. $\int 5\sqrt{\lambda}.d\lambda \quad = \quad \frac{10\sqrt{\lambda^3}}{3} + k$

10. $\int \frac{1}{\sqrt{\theta}}.d\theta \quad = \quad \int \theta^{-½}.d\theta = \quad \frac{\theta^{1/2}}{1/2} \quad = \quad 2\sqrt{\theta} + k$

Sums of terms

Just as in differentiation, a function can by integrated term-by-term, and we only need one constant of integration.

11. $\int 3x^2 + 7x.dx \quad = \quad \int 3x^2 + \int 7x \quad =$

12. $\int \sqrt{x} + \frac{1}{x^2} + \frac{5}{3}x^2 + 4x^3.dx \quad =$

Definite Integrals

We now know how to integrate simple polynomials, but if we want to use this technique to calculate *areas*, we need to know the *limits* of integration. If we specify the limits $x = a \rightarrow x = b$, we call the integral a *definite integral.*

To solve a definite integral, we first integrate the function as before (*i.e*. find its indefinite integral), then feed in the 2 values of the limits. Subtracting one from the other gives the *area*.

Example

1. What is the area under the curve $y(x) = 2x^2$ between x=1 and x=3? (Note: this is the same problem we did graphically earlier).

Area $= \int_{x=1}^{x=3} 2x^2.dx$ we write the limits at the top and bottom of the integration sign

$= \left[\frac{2x^3}{3} + k\right]_{x=1}^{x=3}$ we use square brackets to indicate we've calculated the indefinite integral

$= (18 + k) - ({}^2/3 + k)$ feed in the larger value, then the smaller, and subtract the two.

$= 18 - {}^{2}/3$

$= \underline{17\,{}^{1}/3 \text{ sq. units}}$ (compare the approximate value we got graphically of 17¾).

Note: the *k*'s cancel. So when we evaluate a *definite integral* we can ignore the constant of integration.

2. What is the area under the curve $y(x) = 2x^3 - 6x$ between $x = -1$ and $x = 0$?

$$A = \int_{x=-1}^{x=0} 2x^3 - 6x \, .\mathrm{d}x$$

$$= \left[\frac{x^4}{2} - 3x^2 \right]_{-1}^{0}$$

$= (0 - 0) - (½ - 3)$

$= \underline{2½ \text{ sq.units}}$

3.What is the enthalpy of a gas at 20 K given that its heat capacity as a function of temperature is given by $C = 2T^2$, over the range $T = 0$ K to 20 K?

You'll learn in chemistry lectures that the enthalpy of a gas, *H*, is given by the area under the curve of heat capacity *vs* temperature. In most cases, we approximate it by saying that the heat capacity doesn't change much with *T*, so is in fact a constant. If we take an average value between 0 and 20 K of 10 K, then $C \sim 2\times10^2 = 200$ J K^{-1} mol^{-1}. In this case the enthalpy is just given by

$H = \int_{T_1}^{T_2} C \, .\mathrm{d}T$ (with C = constant = 200) $= \int_{T_1}^{T_2} 200 .\mathrm{d}T$

$= [200T]_{T_1}^{T_2}$ $= 200(T_2 - T_1)$ $= 200(30 - 0)$ $= \underline{6.0 \text{ kJ mol}^{-1}}$

However in this question, we are asked for a more accurate answer, and are told *C* is not constant, it's a function of *T*.

So $H = \int_{T_1}^{T_2} C\,.\mathrm{d}T = \int_{T_1=0}^{T_2=20} 2T^2.\mathrm{d}T = \left[\frac{2T^3}{3}\right]_0^{20} = (16000 / 3) - 0$

= <u>5.3 kJ mol</u>$^{-1}$ (compare this with the approximate answer we obtained when we assumed *C* was constant).

4.What is the area under the curve $y(x) = \frac{2}{x^2}$ between $x = 1$ and $x = \infty$?

[This may seem odd...how can you calculate an area up to *x* = infinity? But if you draw the graph, you'll see that although *x* goes to infinity, the curve is getting closer and closer to the *y* axis and so the area is getting smaller. So in this case, it is possible to calculate a finite area, even though we are integrating to infinity].

$$A = \int_1^\infty \frac{2}{x^2}.\mathrm{d}x$$

$$= \int_1^\infty 2x^{-2}.\mathrm{d}x$$

$$= [-2x^{-1}]_1^\infty = \left[-\frac{2}{x}\right]_1^\infty$$

$$= (0) - (-2)$$

= <u>2 sq.units</u>.

Negative Integrals

Consider the function $y(x) = 2x$ within the limits $x = -2$ to $+1$. Let's calculate the area 'under' this curve using the standard procedure:

$A = \int_{-2}^{+1} 2x.dx = [x^2]_{-2}^{1} = (1^2) - (-2^2) =$ **-3 sq. units**

What does *negative area* mean?

The area A_1 = ½ × 4 × 2 is *below* the *x* axis and is counted as *-ve.*The area A_2 = ½ × 1 × 2 is *above* the *x* axis and is counted as *+ve.*

Therefore it is always a good idea to sketch a curve before you integrate, to see if it goes -ve anywhere between the limits.

Integrals of Common Functions

For any function for which the differential has been established, reversal of the process gives the integral. Learn these!

Exponential Functions

$$\boxed{\int e^x.dx = e^x + k}$$

$$\boxed{\int e^{ax}.dx = \frac{1}{a}e^{ax} + k}$$

Example

1.What is the area under the curve $y(x) = 3e^{-5x}$ from $x = 1$ to $x = \infty$?

$A = \int_1^{\infty} 3\exp(-5x).dx = \left[-\frac{3}{5}e^{-5x}\right]_1^{\infty} = (0) - (-0.004) =$ <u>0.004 sq.units</u>

Logarithmic Functions

$$\boxed{\int \frac{1}{x}.dx = \ln x + k}$$

(this is the one we cannot do using the 'magic formula' and is very important in Physical chemistry).

$$\boxed{\int \ln x.dx = x.(\ln x - 1) + k}$$

Example: (From 2nd year thermodynamics) Calculate the work done when an ideal gas is expanded infinitely slowly from a starting volume V_1 to a final volume V_2.

The work done is given by the area under the pressure-volume graph, or:

Work = $\int -p(V).dV$

Since we're told it's an ideal gas, we can replace $p(V)$ with nRT/V

Work = $\int -\frac{nRT}{V}.dV$, and since n, R and T are constants, this becomes

Work = $-nRT \times \int_{V_1}^{V_2} \frac{1}{V}.dV$. Applying the rule, above:

Work = $-nRT \times [\ln V]_{V_1}^{V_2} = -nRT \times (\ln V_2 - \ln V_1)$

Work = $-nRT \ln (V_2 / V_1)$

Trigonometrical Functions

$$\boxed{\int \cos x.dx = \sin x + k}$$

$$\boxed{\int \sin x.dx = - \cos x + k}$$

$$\boxed{\int \tan x.dx \quad = \quad -\ln(\cos x) + k}$$

Example: What is the area under the curve $y(\theta) = 3\sin\theta$ between $\theta = 0$ and $\frac{\pi}{4}$?

$$A = \int_0^{\pi/4} 3\sin\theta\, d\theta$$

$= [-3\cos\theta]_0^{\pi/4}$

$= (-3 \times 0.707) - (-3)$

$= 0.879$ sq.units

Integration 'magic' Formula

Since integration is the reverse of differentiation, for any polynomial $y(x) = x^n$, we can simply reverse the differentiation procedure, so that the integral is given by

$$\boxed{\int x^n.dx = \frac{x^{n+1}}{(n+1)} + k}$$

(except for $n = -1$)

In words: "Add one to the power, then divide by the new power. Then add k."

Examples

1. $\int x^2.dx \quad = \quad \frac{x^3}{3} + k$
2. $\int 20x^4.dx \quad = \quad 4x^5 + k$

Variations on Nomenclature

Because constants don't affect the integration, it is common to bring them in front of the integration sign to make things clearer.

For example: $\int abx^3.dx \quad = \quad ab\int x^3.dx \quad = \quad ab\frac{x^4}{4} + k$

or: $\int 5q^2.dq \quad = \quad 5\int q^2.dq \quad = \quad \frac{5q^3}{3} + k$

Also, the position of the .dx is usually last in the line, but it can, in principle, be anywhere inside the integral. You may sometimes see the .dx written first (usually in Physics textbooks).

For example: Area = $\int dr\,(r^3 - r^5)$ This is identical to: $\int (r^3 - r^5).dr$

Roots follow the same rule

6. $\int \sqrt{x}\,.dx \quad = \quad \int x^{½}.dx = \frac{x^{3/2}}{(3/2)} = \frac{2\sqrt{x^3}}{3} + k$

Inverse powers also follow the same rule

7. $\int \frac{1}{x^3}.dx \quad = \quad \int x^{-3}.dx = \frac{x^{-2}}{-2} \quad = \quad -\frac{1}{2x^2} + k$

This is true so long as the exponent is *not* -1. $\int \frac{1}{x}$ cannot be calculated using this formula because we get a divide-by-zero error.

Other variables

8. $\int 2m^2.dm \quad = \quad \frac{2m^3}{3} + k$

9. $\int 5\sqrt{\lambda}\,.d\lambda \quad = \quad \frac{10\sqrt{\lambda^3}}{3} + k$

10. $\int \frac{1}{\sqrt{\theta}}.d\theta \quad = \quad \int \theta^{-½}.d\theta = \frac{\Theta^{1/2}}{1/2} \quad = \quad 2\sqrt{\Theta} + k$

Sums of terms

Just as in differentiation, a function can by integrated term-by-term, and we only need one constant of integration.

11. $\int 3x^2 + 7x.dx \quad = \quad \int 3x^2 + \int 7x \quad =$

12. $\int \sqrt{x} + \frac{1}{x^2} + \frac{5}{3}x^2 + 4x^3.dx \quad =$

Example: Find the limit when $n \to \infty$ of the series $\sum_{r=1}^{n-1} \frac{1}{\sqrt{n^2-r^2}}$

Solution: since $\sum_{r=1}^{n-1} \frac{1}{\sqrt{n^2-r^2}} = \lim_{n\to\infty} \sum_{r=1}^{n-1} \frac{1}{\sqrt{n^2-r^2}} = \lim_{n\to\infty} \sum_{r=1}^{n-1} \frac{1}{\sqrt{1-(\frac{r}{n})^2}}$

$$= \int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

$$= [sin^{-1}x]^2 = \pi/2$$

Beta and Gamma Function:

B(m,n) = $\boldsymbol{\int_0^1 x^{m-1}(1-x)^{n-1}dx}$ **, where m, n are positive numbers**

$\boldsymbol{\Gamma n = \int_0^\infty x^{n-1} e^{-x} dx}$ **, where n are positive numbers**

Example: prove that $\Gamma 1 = 1$

Sol: since we know that

$$\Gamma n = \int_0^\infty x^{n-1} e^{-x} dx = \Gamma 1 = \int_0^\infty x^{1-1} e^{-x} dx = 1$$

Example: Evaluate $\int_0^\infty e^{-4x} . x^{5/2}$ dx

Sol: since we know that

$$\Gamma n = \int_0^\infty x^{n-1} e^{-x} dx \quad = \int_0^\infty (\frac{z}{4})^{5/2} e^{-z} dz = 1/128 . \Gamma 7/2$$

$$= 1/128.\ 5/2..\ 3/2.\ ½.\sqrt{\pi} = \frac{15\sqrt{\pi}}{1024}$$

Example: Evaluate $\int_0^1 x^{n-1}\,(log(\frac{1}{x}))^{m-1}dx$

Sol: since $\int_0^1 x^{n-1}\,(log(\frac{1}{x}))^{m-1}dx$

Put $log(\frac{1}{x})$ =y or x = $-e^{-y}$

= $\int_0^\infty y^{m-1}\,e^{-ny}$ .dy = $\frac{1}{n^m}\Gamma m$

Example: Evaluate $\int_0^\infty e^{-k^2x^2}.x^n$ dx = $\frac{1}{2k^{n+1}}\,\Gamma\frac{n+1}{2}$, n>-1

Sol: since $I = \int_0^\infty e^{-k^2x^2}.x^n$ dx

Put $k^2x^2 = y$ implies that $2k^2x\,dx = dy$

I= $\frac{1}{2k^{n+1}}\int_0^\infty y^{\frac{n-1}{2}}\,e^{-y}\,dy$

= $\frac{1}{2k^{n+1}}\int_0^\infty y^{\frac{n+1}{2}}-1\,e^{-y}\,dy$

$\int_0^\infty e^{-k^2x^2}.x^n$ dx = $\frac{1}{2k^{n+1}}\,\Gamma\frac{n+1}{2}$, n>-1

Miscellaneous Examples

Example: Evaluate $lim_{n\to\infty}$ [(1 + $\frac{1}{n^2}$) (1 + $\frac{2^2}{n^2}$) (1 + $\frac{3^2}{n^2}$)……….(1 + $\frac{n^2}{n^2}$)]$^{1/n}$

Sol: Let p = $\lim_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\dots\dots\dots\dots(1+\frac{n^2}{n^2})\right]^{\frac{1}{n}}$

Logp = $\lim_{n\to\infty}\frac{1}{n}\left[\log\left(1+\frac{1}{n^2}\right)+\cdots\dots\dots\log(1+\frac{n^2}{n^2})\right]$

= $\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\log(1+\frac{r^2}{n^2})$

= $\int_0^1\log(1+x^2)\,1\,dx$

$\Rightarrow$ Logp - log2 = $\frac{\pi}{2}$ - 2 $\Rightarrow$ $\frac{p}{2}$ = $e^{\frac{\pi}{2}-2}$ $\Rightarrow$ p = $2e^{\frac{\pi}{2}-2}$

Example: Prove that B (m, n) = $\frac{\Gamma m\,\Gamma n}{\Gamma m+n}$, m>0, n>0.

Sol: Now $\Gamma m = \int_0^\infty e^{-t}t^{m-1}dt$, put t=$x^2$ $\Rightarrow$ dt = 2xdx we get

Γm = $2\int_0^\infty e^{-x^2}x^{2m-1}dx$ ....(1)

IIIly Γn = $2\int_0^\infty e^{-y^2}y^{2n-1}dy$ ....(2)

Changing into polar coordinates we get

i.e. x=rcosθ, y = rsinθ , dxdy = rdrdθ , r→ $0\ to\infty$, $\theta \rightarrow 0 to\frac{\pi}{2}$

ΓmΓn = $4\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{\infty} e^{-r^2}r^{2(m+n)-1}cos^{2m-1}\theta sin^{2n-1}\theta dr d\theta$

= $2\int_{\theta=0}^{\frac{\pi}{2}} cos^{2m-1}\theta sin^{2n-1}\theta d\theta \times 2\int_{r=0}^{\infty} e^{-r^2}r^{2(m+n)-1}dr$

$\Rightarrow$ $\Gamma m\Gamma n$ = Γm+n $\beta\,(m,n)$

Example 2: Prove that

(*i*) $\Gamma(n+1) = n\Gamma(n)$

(*ii*) $\Gamma(n+1) = n!$

Solution: (*i*) $\Gamma(n) = \int_0^\infty x^{n-1}\, e^{-x}\, dx$

$$= \left[x^{n-1}\frac{e^{-x}}{-1}\right]_0^\infty - (n-1)\int_0^\infty x^{n-2}\frac{e^{-x}}{-1}dx$$

$$= \left[\lim_{x\to 0}\frac{x^{n-1}}{e^x} = \lim_{x\to 0} 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots + x^{n-1}\right] = 0$$

$$= (n-1)\int_0^\infty x^{n-2}e^{-x}dx$$

$$\Gamma(n) = (n-1)\Gamma(n-1) \qquad \cdots\cdots\cdots\cdots(*)$$

$\Gamma(n+1) = n\Gamma(n)$ Replacing n by $(n+1)$

(*ii*) Replace n by $(n-1)$ in $(*)$ we get

$$\Gamma(n-1) = (n-2)\Gamma(n-2)$$

Putting the value $\Gamma(n-1)$ in $(*)$ we get

$$\Gamma(n) = (n-1)(n-2)\Gamma(n-2)$$

Similarly $\Gamma(n) = (n-1)(n-2)\ldots 3\cdot 2\cdot 1\Gamma(1) \qquad \cdots\cdots\cdots\cdots(**$

Putting the value of $\Gamma(1)$ in $(**)$ we have:

$$\Gamma(n) = (n-1)(n-2)\ldots 3\cdot 2\cdot 1\cdot 1$$

$$\Gamma(n) = (n-1)!$$

Replacing n by $(n+1)$, we have

Example 4: Evaluate $\int_0^\infty \sqrt[4]{x} e^{-\sqrt{x}} dx$

Solution: Let $I = \int_0^\infty x^{1/4} e^{-\sqrt{x}} dx$ .... (1)

Putting $\sqrt{x} = t$ or $x = t^2$, $dx = 2tdt$ in (1), we get

$$I = \int_0^\infty t^{1/2} e^{-t}\, 2tdt = 2\int_0^\infty t^{3/2} e^{-t} dt$$

$= 2\Gamma(\frac{5}{2})$ By definition

$= 2.\frac{3}{2}\Gamma\left(\frac{3}{2}\right) = 2.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{2}\sqrt{\pi}$

Example 5: Evaluate $\int_0^\infty \sqrt{x}\, e^{-\sqrt[3]{x}} dx.$

Solution: Let $I = \int_0^\infty \sqrt{x} e^{-\sqrt[3]{x}} dx$ .... (1)

Putting $\sqrt[3]{x} = t$ or $x = t^3$, $dx = 3t^2 dt$ in (1) we get

$I = \int_0^\infty t^{3/2} e^{-t}\, 3t^2 dt = 3\int_0^\infty t^{7/2} e^{-t}\, dt = 3\Gamma\left(\frac{9}{2}\right) = 3 \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right.$

Example 6: Evaluate $\int_0^\infty x^{n-1}\, e^{-h^2x^2} dx.$

Solution: Let $I = \int_0^\infty x^{n-1}\, e^{-h^2x^2} dx.$ .... (1)

Putting $t = h^2x^2$ or $x = \frac{\sqrt{t}}{h}$, $dx = \frac{dt}{2h\sqrt{t}}$

Thus (1) becomes $I = \int_0^\infty \left(\frac{\sqrt{t}}{h}\right)^{n-1} e^{-t} \frac{dt}{2h\sqrt{t}}$

$= \frac{1}{2h^2}\int_0^\infty t^{\frac{n-1}{2}} e^{-t} \frac{dt}{\sqrt{t}} = \frac{1}{2h^2}\int_0^\infty t^{\frac{n-2}{2}} e^{-t} dt$

$= \frac{1}{2h^2}\Gamma\left(\frac{n}{2}\right)$

Practice set:

Example 3: Evaluate each of the following integrals.

a) $\int_0^1 x^4(1-x)^3 dx = B(5,4) = \frac{\Gamma(5)\Gamma(4)}{\Gamma(9)} = \frac{4!3!}{8!} = \frac{1}{280}$

b) $\int_0^2 \frac{x^2 dx}{\sqrt{2-x}}$. Letting $x = 2v$, the integral becomes

$$4\sqrt{2}\int_0^1 \frac{v^2}{\sqrt{1-v}} dv = 4\sqrt{2}\int_0^1 v^2(1-v)^{-\frac{1}{2}} dv = 4\sqrt{2}\, B\left(3, \frac{1}{2}\right) = \frac{4\sqrt{2}\Gamma(3)\Gamma(1/2)}{\Gamma\left(\frac{7}{2}\right)} = \frac{64\sqrt{2}}{15}$$

c) $\int_0^a y^4\sqrt{(a^2-y^2)}\, dy$. Letting $y^2 = a^2x$ or $y = a\sqrt{x}$, the integral becomes $a^6\int_0^1 x^{3/2}(1-x)^{1/2} dx = a^6 B(5/2, 3/2) = \frac{a^6\Gamma(5/2)\Gamma(3/2)}{\Gamma(4)} = \frac{\pi a^6}{16}$

Example 4: Evaluate (a) $\int_0^{\pi/2} \sin^6\theta\, d\theta$ (b) $\int_0^{\pi/2} \sin^4\theta \cos^5\theta\, d\theta$ (c) $\int_0^{\pi} \cos^4\theta\, d\theta$

a) Let $2u-1 = 6, 2v-1 = 0$ i.e. $u = 7/2, v = 1/2$,

Then the required integral has the value $\frac{\Gamma(7/2)\Gamma(1/2)}{2\Gamma(4)} = \frac{5\pi}{32}$

b) Letting $2u-1 = 4, 2v-1 = 5$, the required integral has the value $\frac{\Gamma(5/2)\Gamma(3)}{2\Gamma(11/2)} = \frac{8}{315}$

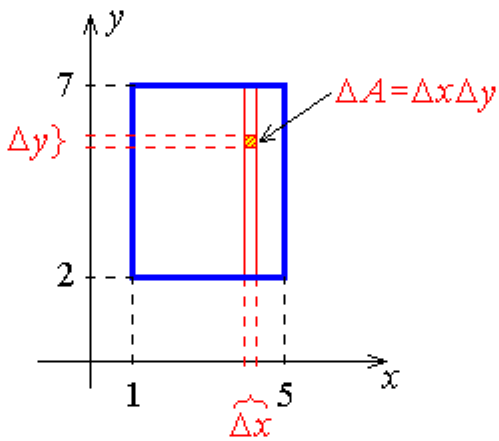
Multiple Integration

This chapter provides only a very brief introduction to the major topic of multiple integration. Uses of multiple integration include the evaluation of areas, volumes, masses, total charge on a surface and the location of a centre-of-mass.

Double Integrals (Cartesian Coordinates)

Example

Find the area shown (assuming SI units).



$$\text{Area of strip} \approx \left(\sum_{y=2}^{7} \Delta y \right) \Delta x$$

$$\text{Total Area} \approx \sum_{x=1}^{5} \left(\left(\sum_{y=2}^{7} \Delta y \right) \Delta x \right)$$

As $\Delta x \to 0$ and $\Delta y \to 0$, the summations become integrals:

$$(\text{Total Area}) \quad \to \quad A = \int_{x=1}^{x=5} \left(\int_{y=2}^{y=7} 1 \, dy \right) dx$$

The inner integral has no dependency at all on *x*, in its limits or in its integrand.

It can therefore be extracted as a “constant” factor from inside the outer integral.

$$\Rightarrow \quad A = \left(\int_{y=2}^{y=7} 1 \, dy \right) \int_{x=1}^{x=5} 1 \, dx$$

$$= [y]_2^7 [x]_1^5 = (7-2) \times (5-1) = 5 \times 4 = \underline{\underline{20 \text{ m}^2}}$$

<u>Example :</u>

Suppose that the surface density on the rectangle is $\sigma = x^2 y$. Find the mass of the rectangle.

The element of mass is

$\Delta m = \sigma \Delta A = \sigma \Delta x \, \Delta y$

$$\rightarrow \quad m = \int_1^5 \int_2^7 \sigma \, dy \, dx = \int_1^5 \int_2^7 x^2 y \, dy \, dx$$

$$= \int_1^5 x^2 \left(\int_2^7 y \, dy \right) dx = \left(\int_2^7 y \, dy \right) \int_1^5 x^2 \, dx$$

$$= \left[\frac{y^2}{2} \right]_2^7 \cdot \left[\frac{x^3}{3} \right]_1^5 = \frac{49-4}{2} \times \frac{125-1}{3} = 15 \times 62$$

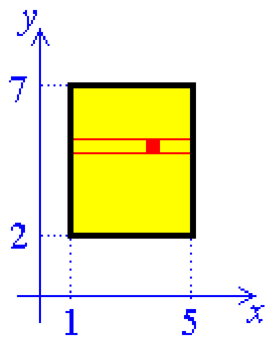
Therefore the mass of the rectangle is m = 930 kg.

OR

We can choose to sum horizontally first:

$$m = \int_2^7 \int_1^5 x^2 y \, dx \, dy$$

$$m = \int_2^7 y \left(\int_1^5 x^2 \, dx \right) dy$$



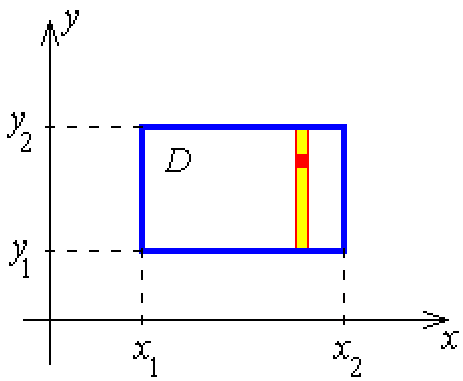
The inner integral has no dependency at all on *y*, in its limits or in its integrand. It can therefore be extracted as a "constant" factor from inside the outer integral.

$$m = \left(\int_1^5 x^2\,dx\right)\left(\int_2^7 y\,dy\right)$$

which is exactly the same form as before, leading to the same value of 930 kg.

A double integral $\iint_D f(x,y)\,dA$ may be separated into a pair of single integrals if

- the region D is a rectangle, with sides parallel to the coordinate axes; and
- the integrand is separable: $f(x, y) = g(x)\,h(y)$.



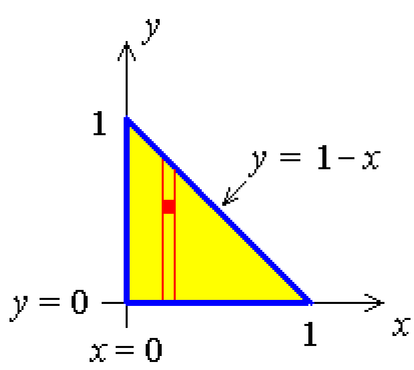
$$\iint_D f(x,y)\,dA = \int_{x_1}^{x_2}\int_{y_1}^{y_2} g(x)\,h(y)\,dy\,dx$$

$$= \left(\int_{x_1}^{x_2} g(x)\,dx\right)\left(\int_{y_1}^{y_2} h(y)\,dy\right)$$

Example :

The triangular region shown here has surface density $\sigma = x + y$.

Find the mass of the triangular plate.



Element of mass:

$\Delta m = \sigma\,\Delta A = \sigma\,\Delta x\,\Delta y$

$$\text{Mass of strip} \approx \left(\sum_{y=0}^{1-x} \sigma\,\Delta y\right)\Delta x$$

$$\text{Total Mass} \approx \sum_{x=0}^{1}\left(\left(\sum_{y=0}^{1-x}\sigma\,\Delta y\right)\Delta x\right)$$

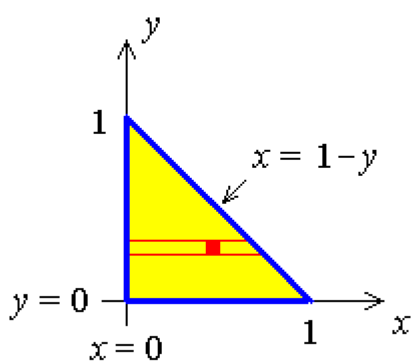
$$\to \quad m = \int_0^1\int_0^{1-x}(x+y)\,dy\,dx$$

$$= \int_0^1\left[xy + \frac{y^2}{2}\right]_0^{1-x} dx = \int_0^1\left(x(1-x) + \frac{(1-x)^2}{2} - 0 - 0\right)dx$$

$$= \int_0^1 \frac{1-x^2}{2}\,dx = \left[\frac{x}{2} - \frac{x^3}{6}\right]_0^1 = \frac{1}{2} - \frac{1}{6} - 0 + 0 = \underline{\underline{\frac{1}{3}\text{kg}}}$$

OR

We can choose to sum horizontally first (re-iterate):



$$m = \int_0^1\int_0^{1-y}(x+y)\,dx\,dy$$

$$= \int_0^1\left[\frac{x^2}{2} + xy\right]_0^{1-y} dy = \cdots = \underline{\underline{\frac{1}{3}\text{kg}}}$$

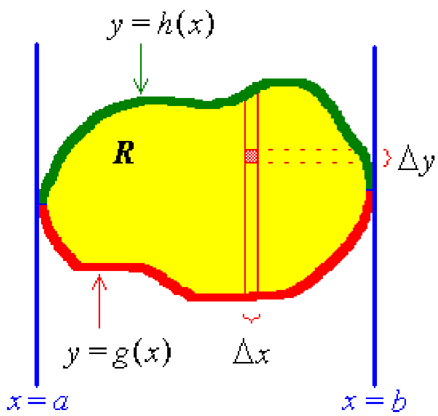
Generally:

In Cartesian coordinates on the *xy*-plane, the rectangular element of area is

$$\Delta A = \Delta x \Delta y.$$

Summing all such elements of area along a vertical strip, the area of the elementary strip is

$$\left(\sum_{y=g(x)}^{h(x)} \Delta y \right) \Delta x$$



Summing all the strips across the region *R*, the total area of the region is:

$$A \approx \sum_{x=a}^{b} \left(\left(\sum_{y=g(x)}^{h(x)} \Delta y \right) \Delta x \right)$$

In the limit as the elements Δx and Δy shrink to zero, this sum becomes

$$A = \int_{x=a}^{b} \int_{y=g(x)}^{h(x)} 1 \, dy \, dx$$

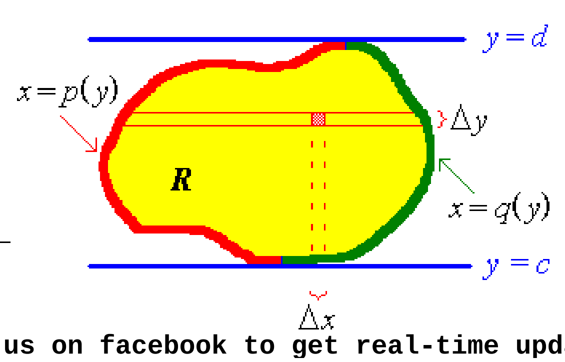
If the surface density σ within the region is a function of location, $\sigma = f(x, y)$, then the mass of the region is

$$m = \int_{x=a}^{b} \left(\int_{y=g(x)}^{h(x)} f(x, y) \, dy \right) dx$$

The inner integral must be evaluated first.

Re-iteration:

One may reverse the order of integration by summing the elements of



area ΔA horizontally first, then adding the strips across the region from bottom to top. This generates the double integral for the total area of the region

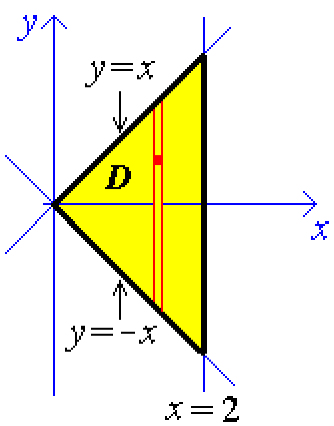
$$A = \int_{y=c}^{d} \left(\int_{x=p(y)}^{q(y)} 1\,dx \right) dy$$

The mass becomes

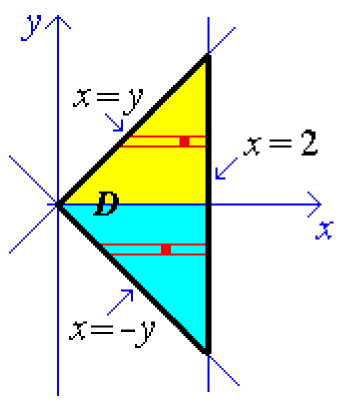
$$m = \int_{y=c}^{d} \left(\int_{x=p(y)}^{q(y)} f(x,y)\,dx \right) dy$$

Choose the orientation of elementary strips that generates the simpler double integration.

For example,



is preferable



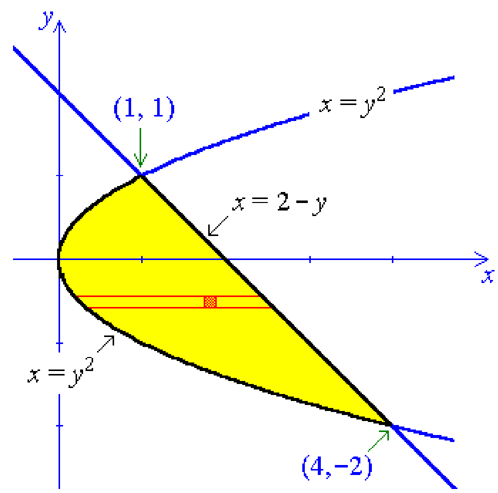
to .

$$\int_0^2 \int_{-x}^{x} f(x,y)\,dy\,dx \quad = \quad \int_{-2}^{0} \int_{-y}^{2} f(x,y)\,dx\,dy + \int_0^2 \int_{+y}^{2} f(x,y)\,dx\,dy$$

Example :

Evaluate $I = \iint_R \left(6x + 2y^2\right) dA$

where R is the region enclosed by the parabola $x = y^2$ and the line $x + y = 2$.



The upper boundary changes form at $x = 1$.
The left boundary is the same throughout R.
The right boundary is the same throughout R.
Therefore choose horizontal strips.

$$I = \int_{-2}^{1}\int_{y^2}^{2-y} \left(6x + 2y^2\right) dx\, dy$$

$$I = \int_{-2}^{1} \left[3x^2 + 2xy^2\right]_{x=y^2}^{x=2-y} dy$$

$$= \int_{-2}^{1} \left(\left(3(2-y)^2 + 2(2-y)y^2\right) - \left(3y^4 + 2y^4\right)\right) dy$$

$$= \int_{-2}^{1} \left(\left(12 - 12y + 3y^2\right) + \left(4y^2 - 2y^3\right) - 5y^4\right) dy$$

$$= \int_{-2}^{1} \left(12 - 12y + 7y^2 - 2y^3 - 5y^4\right) dy$$

$$= \left[12y - 6y^2 + \frac{7}{3}y^3 - \frac{1}{2}y^4 - y^5\right]_{-2}^{+1}$$

$$= \left(12 - 6 + \frac{7}{3} - \frac{1}{2} - 1\right) - \left(-24 - 24 - \frac{56}{3} - 8 + 32\right)$$

Therefore

$$I = \frac{99}{2}$$

Polar Double Integrals

The Jacobian of the transformation from Cartesian to plane polar coordinates (Example 2.4.1 on page 2-17) is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \left\| \begin{matrix} x_r & y_r \\ x_\theta & y_\theta \end{matrix} \right\| = r$$

The element of area is therefore

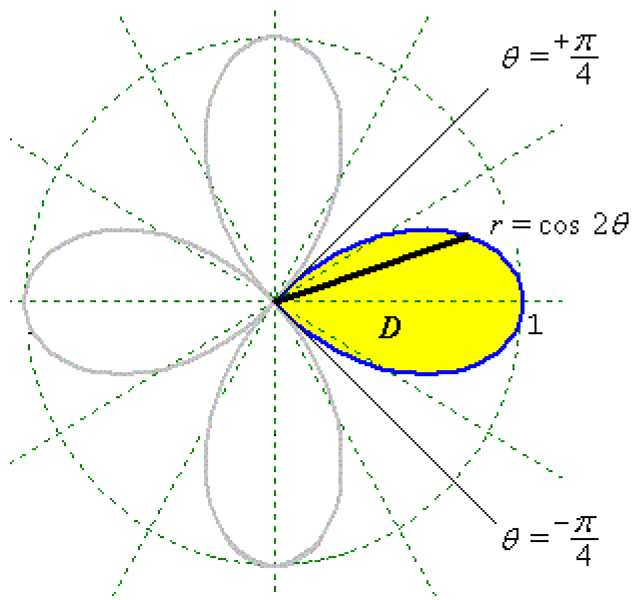
$$dA = dx\,dy = r\,dr\,d\vartheta$$

Example :

Find the area enclosed by one loop of the curve $r = \cos 2\vartheta$.

Boundaries:

$$0 \le r \le \cos 2\theta \; ; \quad -\frac{\pi}{4} \le \theta \le +\frac{\pi}{4}$$



Area:

$$A = \iint_D 1\,dA = \int_{-\pi/4}^{+\pi/4} \int_0^{\cos 2\theta} 1\,r\,dr\,d\theta$$

$$= \int_{-\pi/4}^{+\pi/4} \left[\frac{r^2}{2}\right]_0^{\cos 2\theta} d\theta$$

$$= \int_{-\pi/4}^{+\pi/4} \left(\frac{\cos^2 2\theta}{2} - 0 \right) d\theta$$

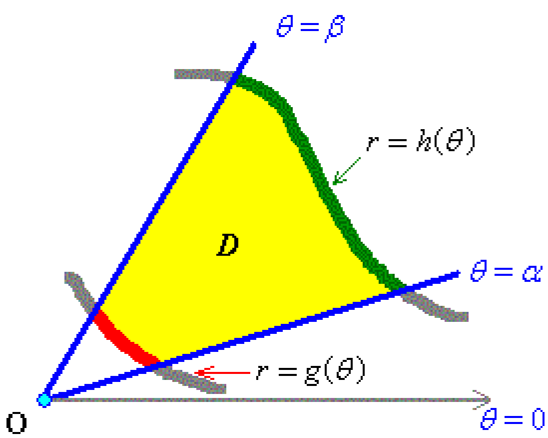
$$= \int_{-\pi/4}^{+\pi/4} \frac{\cos 4\theta + 1}{4} d\theta$$

$$= \left[\frac{\sin 4\theta}{16} + \frac{\theta}{4} \right]_{-\pi/4}^{+\pi/4} = \left(0 + \frac{\pi}{16} \right) - \left(0 - \frac{\pi}{16} \right)$$

Therefore

$$A = \underline{\underline{\frac{\pi}{8}}}$$

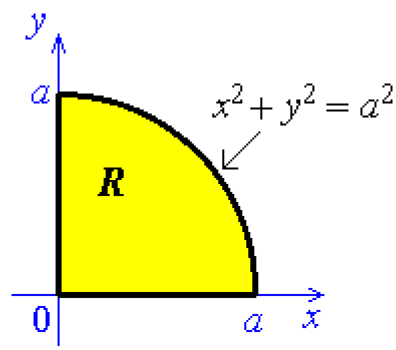
In general, in plane polar coordinates,



$$\iint_D f(x,y)\,dA = \int_{\alpha}^{\beta}\int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Example :

Find the centre of mass for a plate of surface density $\sigma = \frac{k}{\sqrt{x^2+y^2}}$, whose boundary is the portion of the circle $x^2 + y^2 = a^2$ that is inside the first quadrant. k and a are positive constants.



Use plane polar coordinates.

Boundaries:

The positive x-axis is the line $\vartheta = 0$.

The positive y-axis is the line $\vartheta = \pi/2$.

The circle is $r^2 = a^2$, which is $r = a$.

Mass:

Surface density $\sigma = \frac{k}{\sqrt{x^2+y^2}}$.

$$m = \iint_R \sigma\, dA = \int_0^{\pi/2}\int_0^a \frac{k}{r} r\, dr\, d\theta$$

$$= k\int_0^{\pi/2}\int_0^a 1\, dr\, d\theta = k\left(\int_0^a 1\, dr\right)\left(\int_0^{\pi/2} 1\, d\theta\right) = k\,[r]_0^a\,[\theta]_0^{\pi/2}$$

$$m = \frac{k\pi a}{2}$$

Example :

First Moments about the *x*-axis:

$$\Delta M_x = y\,\Delta m \quad \Rightarrow \quad M_x = \iint_R y\,\sigma\,dA$$

$$= \int_0^{\pi/2}\left(\int_0^a (r\,\sin\theta)\frac{k}{r}\,r\,dr\right)d\theta$$

$$= k\int_0^a r\,dr\int_0^{\pi/2}\sin\theta\,d\theta = k\left[\frac{r^2}{2}\right]_0^a\left[-\cos\theta\right]_0^{\pi/2}$$

$$= k\left(\frac{a^2}{2}-0\right)(-0+1)$$

$$\therefore \quad M_x = \frac{k\,a^2}{2}$$

$$\text{But } M_x = m\,\bar{y} \quad \Rightarrow \quad \bar{y} = \frac{M_x}{m} = \frac{k\,a^2}{2}\cdot\frac{2}{k\,\pi\,a} = \frac{a}{\pi}$$

$$\text{By sym.,} \quad \bar{x} = \bar{y}$$

Therefore the centre of mass is at $(\bar{x}, \bar{y}) = \underline{\underline{\left(\frac{a}{\pi}, \frac{a}{\pi}\right)}}$

Triple Integrals

The concepts for double integrals (surfaces) extend naturally to triple integrals (volumes). The element of volume, in terms of the Cartesian coordinate system (*x*, *y*, *z*) and another orthogonal coordinate system (*u*, *v*, *w*), is

$$dV = dx\,dy\,dz = \frac{\partial(x,y,z)}{\partial(u,v,w)}\,du\,dv\,dw$$

and

$$\iiint_V f(x,y,z)\,dV = \int_{w_1}^{w_2}\int_{v_1(w)}^{v_2(w)}\int_{u_1(v,w)}^{u_2(v,w)} f\big(x(u,v,w),y(u,v,w),z(u,v,w)\big)\frac{\partial(x,y,z)}{\partial(u,v,w)}\,du\,dv\,dw$$

The most common choices for non-Cartesian coordinate systems in $\mathbb{R}^3$ are:

Cylindrical Polar Coordinates:

$$x = r\cos\phi$$
$$y = r\sin\phi$$
$$z = z$$

for which the differential volume is

$$dV = \frac{\partial(x,y,z)}{\partial(r,\phi,z)}\,dr\,d\phi\,dz = r\,dr\,d\phi\,dz$$

Spherical Polar Coordinates:

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

for which the differential volume is

$$dV = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}\,dr\,d\theta\,d\phi = r^2\sin\theta\,dr\,d\theta\,d\phi$$

Example :

Verify the formula $V = \frac{4}{3}\pi a^3$ for the volume of a sphere of radius *a*.

$$V = \iiint_V 1\, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= \left(\int_0^a r^2\, dr\right)\left(\int_0^{\pi} \sin\theta\, d\theta\right)\left(\int_0^{2\pi} 1\, d\phi\right)$$

$$= \left[\frac{r^3}{3}\right]_0^a \left[-\cos\theta\right]_0^{\pi} \left[\phi\right]_0^{2\pi} = \left(\frac{a^3}{3} - 0\right)(+1+1)(2\pi - 0)$$

Therefore

$$V = \frac{4}{3}\pi a^3$$

Example:

The density of an object is equal to the reciprocal of the distance from the origin. Find the mass and the average density inside the sphere $r = a$.

Use spherical polar coordinates.

Density:

$$\rho = \frac{1}{r}$$

Mass:

$$m = \iiint_V \rho \, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \frac{1}{r} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \left(\int_0^a r \, dr\right)\left(\int_0^{\pi} \sin\theta \, d\theta\right)\left(\int_0^{2\pi} 1 \, d\phi\right)$$

$$= \left[\frac{r^2}{2}\right]_0^a \left[-\cos\theta\right]_0^{\pi} \left[\phi\right]_0^{2\pi} = \left(\frac{a^2}{2} - 0\right)(+1+1)(2\pi - 0)$$

Therefore

$$m = \underline{\underline{2\pi a^2}}$$

Average density =

$$\bar{\rho} = \frac{\text{mass}}{\text{volume}} = \frac{m}{V} = \frac{2\pi a^2}{\frac{4}{3}\pi a^3} = \frac{3}{2a}$$

Therefore

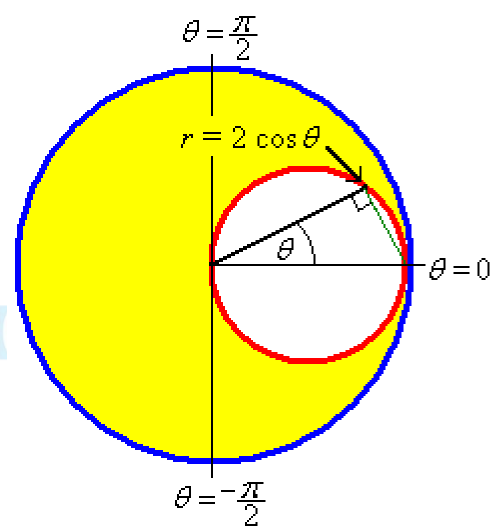
$$\bar{\rho} = \underline{\underline{\frac{3}{2a}}}$$

[Note that the mass is finite even though the density is infinite at the origin!]

Example:

Find the proportion of the mass removed, when a hole of radius 1, tangent to a diameter, is bored through a uniform sphere of radius 2.

Cross-section at right angles to the axis of the hole:



Use cylindrical polar coordinates, with the *z*-axis aligned parallel to the axis of the cylindrical hole.

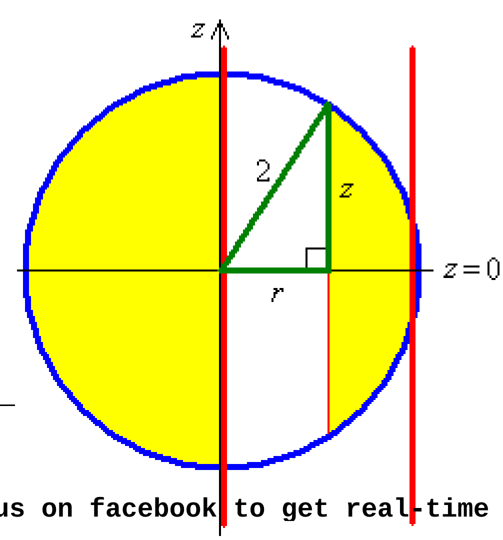
The plane polar equation of the boundary of the hole is then

$$r = 2\cos\vartheta$$

The entire circular boundary is traversed once for

$$-\frac{\pi}{2} \le \theta \le +\frac{\pi}{2}$$

Cross-section parallel to the axis of the hole:



At each value of r, the distance from the equatorial plane to the point where the hole emerges from the sphere is

$$z = \sqrt{2^2 - r^2}$$

The element of volume for the hole is therefore

$$dV = 2z\, dA = 2\sqrt{4-r^2}\left(r\, dr\, d\theta\right)$$

$$V = \int_{-\pi/2}^{+\pi/2} \int_{0}^{2\cos\theta} 2\sqrt{4-r^2}\, r\, dr\, d\theta$$

We cannot separate the two integrals, because the upper limit of the inner integral, ($r = 2\cos\vartheta$),

is a function of the variable of integration in the outer integral.

The geometry is entirely symmetric about $\vartheta = 0$

$$\Rightarrow \quad V = 4\int_{0}^{+\pi/2} \int_{0}^{2\cos\theta} \sqrt{4-r^2}\, r\, dr\, d\theta$$

Example :

$$= 4\int_0^{\pi/2}\left[\frac{\left(4-r^2\right)^{3/2}}{\frac{3}{2}\times(-2)}\right]_0^{2\cos\theta} d\theta$$

$$= -\frac{4}{3}\int_0^{\pi/2}\left(\left(4-4\cos^2\theta\right)^{3/2} - (4-0)^{3/2}\right)d\theta = +\frac{4}{3}\int_0^{\pi/2}\left(\left(-4\sin^2\theta\right)^{3/2} + 4^{3/2}\right)d\theta$$

$$= \frac{4}{3}\int_0^{\pi/2}\left(8 - 8\sin^3\theta\right)d\theta = \frac{32}{3}\int_0^{\pi/2}\left(1 - \sin^3\theta\right)d\theta$$

$$= \frac{32}{3}\left(\int_0^{\pi/2} 1\, d\theta - \int_0^{\pi/2}\sin^2\theta\sin\theta\, d\theta\right)$$

$$= \frac{32}{3}\left(\int_0^{\pi/2} 1\, d\theta - \int_0^{\pi/2}\left(1 - \cos^2\theta\right)\sin\theta\, d\theta\right)$$

Let $u = \cos\vartheta$, then $du = -\sin\vartheta\, d\vartheta$.

$\theta = 0 \Rightarrow u = 1$ and $\theta = \frac{\pi}{2} \Rightarrow u = 0$

$$V = \frac{32}{3}\left(\int_0^{\pi/2} 1\, d\theta - \int_{u=1}^{u=0}\left(1-u^2\right)(-du)\right) = \frac{32}{3}\left(\int_0^{\pi/2} 1\, d\theta - \int_0^1\left(1-u^2\right)du\right)$$

$$= \frac{32}{3}\left([\theta]_0^{\pi/2} - \left[u - \frac{u^3}{3}\right]_0^1\right) = \frac{32}{3}\left(\left(\frac{\pi}{2}-0\right) - \left(\frac{2}{3}-0\right)\right)$$

$$= \frac{16\pi}{3} - \frac{64}{9}$$

The density is constant throughout the sphere. Therefore

$$\frac{m_{\text{hole}}}{m_{\text{sphere}}} = \frac{V_{\text{hole}}}{V_{\text{sphere}}} = \left(\frac{16\pi}{3} - \frac{64}{9}\right)\cdot\frac{3}{4\pi 2^3} = \frac{1}{2} - \frac{2}{3\pi}$$

Therefore the proportion of the sphere that is removed is

$$\frac{1}{2} - \frac{2}{3\pi} \approx 29\%$$

Integration on solids

The triple integral over a solid *T* of a scalar function $\delta(x,y,z)$ is written $\iiint_T \delta(x,y,z)\,dV$. Here *dV* is a symbol for the volume of a small piece of the solid *T*. The triple integral represents the **mass** of the solid if $\delta(x,y,z)$ is the **density** of the solid at point (*x,y,z*) on *T*. To compute the integral, you need to

- describe the solid algebraically, with inequalities involving *x,y,z*;
- choose variables of integration: rectangular coordinates (*x,y,z*), or cylindrical coordinates(r,θ,z), or spherical coordinates (ϕ,θ,ρ);
- express *dV* in terms of these variables;
- set up bounds of integration; and finally
- evaluate the triple integral, beginning with the innermost integral.

1. Descriptions of solids: language and algebra

In these notes, we to use the letters ***T*** to denote a solid, ***S*** a surface,

R a flat surface (usually a region in the *x,y*-plane) , and ***C*** a curve.

There are many ways to use language to describe a solid. Here are some examples.

Try to sketch each solid before reading further.

*T*1 is the solid below $z = 4-(x^2+y^2)$ and above $z = x^2+y^2-4$.

*T*2 is the solid contained between z = 4 and $z = x^2 + y^2 + 2$.

*T*3 is the solid ball bounded by the sphere $x^2 + y^2 + z^2 = 4$.

*T*3*a* is the solid below $z = \sqrt{4-(x^2+y^2)}$ and above $z = -\sqrt{4-(x^2+y^2)}$.

*T*4 is the solid inside $x^2 + y^2 + z^2 = 4$ and above *z* = 1.

*T*5 is the solid inside the cylinder $x^2 + y^2 = 4$, above $z = 0$, and below $z = 10$.

*T*6 is the solid inside $x^2 + y^2 = 4$, above $z = 0$, and below $x + y + z = 10$.

*T*7 is the solid inside the cone $x^2 + y^2 = z^2$, between *z* = 1 and *z* = 2.

*T*8 is the solid contained in the first octant below $3x + 2y + z = 6$.

*T*9 is the solid inside $x^2 + z^2 = 4$ and between $y = 1$ and *y* = 8.

To do problems, you must use inequalities to describe the solid algebraically. These inequalities will translate easily into bounds of integration.

Definition: A solid T is called *z*-simple if the following condition holds:

There are functions $f_B(x,y)$ and $f_T(x,y)$,and a region *R* in the *x,y*-plane such that point (*x,y,z*) is on the solid if and only if (*x,y*) is in *R and* $f_B(x,y) \le z \le f_T(x,y)$.

The surface $z = f_B(x,y)$ for (*x,y*) in *R* is the bottom surface of *T.*

The surface $z = f_T(x,y)$ for (*x,y*) in *R* is the top surface of *T.*

The top and bottom surfaces are *z*-simple surfaces, as described earlier.

To recognize a z-simple solid, use the following crucial result.

Z-simple solid Theorem : Let *S*1: *z* = *f*(*x,y*) and *S*2: *z* = *g*(*x,y*) be two simple surfaces. Let *T* be the solid contained between *S*1 and *S*2, and assume that the solutions (*x,y*) to *f*(*x,y*) = *g*(*x,y*) form a simple closed curve *C*. Let *R* be the region inside *C*.

Pick any point (x_0, y_0) inside C. If $f(x_0, y_0) < g(x_0, y_0)$ then

- T is a z-simple solid with top surface S2 and bottom surface S1.
- *P(x,y,z)* is on the solid if and only if (*x,y*) is in *R and* $f(x,y) \le z \le g(x,y)$.

In other words, to decide which of the surfaces *S*1: *z* = *f*(*x,y*) and *S*2: *z* = *g*(*x,y*) is the top surface, you just need to compare *f* and *g* at one point inside R.

Based on the above result, we make the following

Definition: A solid T is a **z-simple solid** provided there is a region *R* in the (*x,y*)-plane with the following property:

Point *P* (*x,y,z*) is on the solid T if and only if (*x,y*) is in *R and* $f_B(x,y) \le z \le f_T(x,y)$. This English sentence is abbreviated as follows

$$T: \begin{cases} (x,y) \text{ in } R \\ f_B(x,y) \le z \le f_T(x,y) \end{cases}$$

Here we have changed the notation slightly:

The Top surface has equation $z = f_T(x,y)$

The Bottom surface has equation $z = f_B(x,y)$

Integration by Pillars

Integration over a z-simple solid $T: \begin{cases} (x,y) \text{ in } R \\ f_B(x,y) \le z \le f_T(x,y) \end{cases}$ is done as follows.

A triple integral over *T* can be computed by first integrating with respect to *x* and *y* on the region *R*, and then integrating with respect to *z*: $\iiint_T \delta(x,y,z)dV = \iint_R \int_{z=f(x,y)}^{g(x,y)} \delta(x,y,z)dz \quad dA$

The outside double integral will evaluate to a (possibly complicated) function of *x* and *y*, and finishing the computation will require integrating that function over the region *R*. As usual, dA = area in *R*, given by *dxdy, dydx*, or or $rdrd\theta$.

This method of integration is sometimes called the pillar method, since the inside integral can be thought of as taking place on the pillar (vertical line segment) running from bottom point $(x,y,f_B(x,y))$ to top point $(x,y,f_T(x,y))$

For each solid listed above, we now discuss how to convert the given language description to an algebraic description of a *z*-simple solid.

Example: **is the solid below** $z = 4-(x^2+y^2)$ **and above** $z = x^2+y^2-4$

This below/above description suggests that

- the top surface's equation is *z* = $f_T(x,y) = 4-(x^2+y^2)$ and
- the bottom surface's equation is *z* = $f_B(x,y) = x^2+y^2-4$.

Solve $f_B(x,y) = f_T(x,y)$ as follows:

$4-(x^2+y^2) = x^2+y^2-4$

$8 = 2(x^2+y^2)$. Thus the solution is the circle *C* : $x^2+y^2=4$, and the region inside C is the disc *R*: $x^2+y^2 \le 4$.

By the z-simple solid Theorem we have the description

T1: $\begin{cases}(x,y)\text{ in }R: x^2+y^2\le 4\\ 4-(x^2+y^2)\le z\le x^2+y^2-4\end{cases}$ and so we calculate $\iiint_{T1}\delta(x,y,z)dV$ =

$\iint_R \int_{z=f_B(x,y)=x^2+y^2-4}^{f_T(x,y)=4-x^2-y^2}\delta(x,y,z)dz\ dA$. Use polar coordinates $x=r\cos\theta, y=r\sin\theta$, and dA = $rdrd\theta$ to rewrite the outside double integral, yielding $\int_{\theta=0}^{2\pi}\int_{r=0}^{2}\int_{z=x^2+y^2-4}^{z=4-(x^2+y^2)}\delta(x,y,z)dzrdrd\theta$.

This triple integral represents the mass of the solid with density $\delta(x,y,z)$. If you use $\delta(x,y,z)$ = 1, you get the volume of the solid, which is

$$\int_{\theta=0}^{2\pi}\int_{r=0}^{2}\int_{z=x^2+y^2-4}^{z=4-(x^2+y^2)}1dzrdrd\theta=\int_{\theta=0}^{2\pi}\int_{r=0}^{2}\left(4-(x^2+y^2)-[x^2+y^2-4]\right)rdrd\theta$$

$$=\int_{\theta=0}^{2\pi}\int_{r=0}^{2}\left(4-(r^2)-[r^2-4]\right)rdrd\theta=\int_{\theta=0}^{2\pi}\int_{r=0}^{2}\left(8r-2r^3\right)drd\theta=\int_{\theta=0}^{2\pi}\left[4r^2-\frac{1}{2}r^4\right]_0^2=$$

$$\int_{\theta=0}^{2\pi}8d\theta=16\pi.$$

Exercise: Work out this integral $\iiint_{T1}\delta(x,y,z)dV$ with $\delta(x,y,z)=y+x^2$. Remember to use polar coordinates to convert (*x*,*y*) to (r,θ).

Example: **is the solid contained between *z* = 4 and $z=x^2+y^2+2$.**

This problem is different. There are no words 'above' and 'below.' Begin by setting the two expressions for *z* equal to each other to get $x^2+y^2+2=4$, which is the circle $x^2+y^2=2$. The region inside that circle is *R*: $x^2+y^2\le 2$.

To figure out which surface is on top, compare *z*-values at any point inside *R*. For example, at the point (*x*,*y*) = (0,0), *z* = 4 on the surface *z* = 4 (of course!), but *z* = 2 on the surface $z=x^2+y^2+2$, which is therefore the bottom surface. Thus we obtain the z-simple solid description T2:

$\begin{cases}(x,y)\text{ in }R: x^2+y^2\le 2\\ x^2+y^2+2\le z\le 4\end{cases}$ and we calculate

$$\iiint_T \delta(x,y,z)dV = \iint_R \int_{z=f_B(x,y)}^{f_T(x,y)} \delta(x,y,z)dz \quad dA = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z=x^2+y^2+2}^{z=4} \delta(x,y,z)dzrdrd\theta$$

Exercise: Use polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, and *dA* = $rdrd\theta$ to work out this integral for the density function $\delta(x,y,z) = x^2 + y^2 + 2z$.

Example: **is the solid ball bounded by the sphere $x^2 + y^2 + z^2 = 4$.**

You know that the equation defines a radius 2 sphere. If you solve for *z,* you get top surface equation $g(x,y) = \sqrt{4-(x^2+y^2)}$ and bottom surface equation $f(x,y) = -\sqrt{4-(x^2+y^2)}$. Set $f(x,y) = g(x,y)$ to get C: $x^2 + y^2 = 4$ and *R*: $x^2 + y^2 \le 4$. Thus T3:

$$\begin{cases} (x,y) \text{ in } R : x^2 + y^2 \le 4 \\ -\sqrt{4-(x^2+y^2)} \le z \le \sqrt{4-(x^2+y^2)} \end{cases}$$

Exercise: Use polar coordinates and density function $\delta(x,y,z) = x^2 + y^2 + z^2$

to evaluate

$$\iiint_T \delta(x,y,z)dV = \iint_R \int_{z=-\sqrt{4-(x^2+y^2)}}^{z=\sqrt{4-(x^2+y^2)}} \delta(x,y,z)dz \quad dA$$

Example: is the solid inside $x^2 + y^2 + z^2 = 4$ and above *z* = 1.

The language here is again a bit different. You have seen this kind of problem before.

$x^2 + y^2 + z^2 = 4$ is a radius 2 sphere. Solve for $z = \pm\sqrt{4-(x^2+y^2)}$. Since $z \ge 1$ on the solid, it follows that *z* is positive and so we use $z = +\sqrt{4-(x^2+y^2)}$ to get the equation of the top surface of solid. Then $z = 1$ on the bottom surface. To find the region *R*, set $z = 1 = \sqrt{4-(x^2+y^2)}$ to obtain C: $x^2 + y^2 = 3$. Therefore *R* is the disc $x^2 + y^2 \le 3$.

Note: **you can't use spherical coordinates on this solid,** because the distance ρ from the origin to point (*x,y,z*) on the bottom surface : $x^2 + y^2 \le 3; z = 1$ is a messy function of ϕ and θ.

Exercise: Find the volume of the solid.

Example: **the solid inside the cylinder** $x^2 + y^2 = 4$**, above** $z = 0$**, and below** $z = 10$**.**

This problem is unlike any previous ones because the planes $z = 0$ and $z = 10$ do not intersect. T5 is a solid cylinder, described algebraically as

T5 $\begin{cases} x^2 + y^2 \le 4 \\ 0 \le z \le 10 \end{cases}$. But this description already fits the definition of *z*-simple surface!

Indeed, use $f_B(x, y) = 0$, $f_T(x, y) = 10$, and region *R*: $x^2 + y^2 \le 4$. Clearly point (*x,y,z*) is on the solid provided (*x,y*) is in *R* and $f_B(x, y) \le z \le f_T(x, y)$. Therefore

$\iiint_{T5} \delta(x, y, z)dV = \iint_R \int_{z=0}^{z=10} \delta(x, y, z)dz \; dA$, which is easy to work out by using polar coordinates on the disc *R*.

Exercise: Find $\iiint_{T5} (x^2 + y^2 + z)dV$

Exercise: Find $\iiint_{T5} (x + y^2 + z^2)dV$

Example: **is the solid inside** $x^2 + y^2 = 4$**, above** $z = 0$**, and below** $x + y + z = 10$

This is similar to the previous problem. It looks like the bottom surface is *z* = 0, the top surface is *z* = 10 – *x* – *y* , and so the solid should be described as a *z*-simple surface

$$\begin{cases} x^2 + y^2 \le 4 \\ 0 \le z \le 10 - x - y \end{cases}.$$

However, it is possible that these inequalities contradict each other.

Certainly the bottom inequality requires $0 \le 10 - x - y$ whenever $x^2 + y^2 \le 4$. In other words, we need to know that $y \le 10 - x$ if $x^2 + y^2 \le 4$. This is a bit annoying to verify algebraically, but much easier to see if you draw a sketch and check that the disc $x^2 + y^2 \le 4$ lies below the line *y* = 10 – *x*.

Indeed, the disc lies inside the square $\begin{cases} -2 \le x \le 2 \\ -2 \le y \le 2 \end{cases}$, which clearly lies below *y* = 10 – *x*. Therefore

$\begin{cases} x^2 + y^2 \le 4 \\ 0 \le z \le 10 - x - y \end{cases}$ is a legitimate definition of a *z*-simple surface,

and so $\iiint_{T6} \delta(x, y, z)\,dV = \iint_{R: x^2+y^2 \le 4} \int_{z=0}^{z=10-x-y} \delta(x, y, z)\,dz\,dA$. As usual, evaluate the outside double integral using polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, and *dA* = $r\,dr\,d\theta$.

Exercise: Find $\iiint_{T6} (x + z)\,dV$

Example: **is the solid inside the cone** $x^2 + y^2 = z^2$**, between** *z* = 1 **and** *z* = 2.

The solid can be described algebraically as T7: $\begin{cases} x^2 + y^2 \le z^2 \\ 1 \le z \le 2 \end{cases}$ This problem is completely different from previous ones because **the solid is not *z*-simple**! Unfortunately, the top inequality involves *z* and so the region R that we need also depends on *z.* That stops the surface from being *z*-simple.

Sketch solid T7 to see that it really has two bottom surfaces. We have seen this sort of thing before. Suppose you want to figure out a double integral over the triangle with vertices (1,0), (0,1), and (2,1). The top of the triangle is the horizontal line segment

$y = 1; 0 \le x \le 2$, but the bottom of the triangle is $y = 1 - x$ for $0 \le x \le 1$ and $y = 1 + x$ for $1 \le x \le 2$. Computing the double integral requires splitting the triangle into two regions.

Similarly , since solid T7 has two bottom surfaces, it must be broken into two solids, each with a single top surface. An easy sketch shows that T7 can be broken into two *z*-simple solids,

T7A: $\begin{cases} x^2 + y^2 \le 1 \\ 1 \le z \le 2 \end{cases}$ **and** T7B: $\begin{cases} 1 \le x^2 + y^2 \le 2 \\ \sqrt{x^2 + y^2} \le z \le 2 \end{cases}$ **.** Therefore

$$\iiint_{T7} \delta(x, y, z)dV = \iiint_{T7A} \delta(x, y, z)dV + \iiint_{TB} \delta(x, y, z)dV =$$

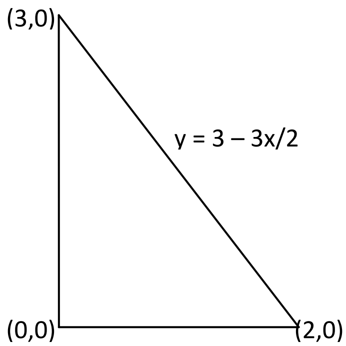
$$\iint_{R_A: x^2+y^2 \le 1} \int_{z=1}^{z=2} \delta(x, y, z)dz \; dA + \iint_{R_B: 1 \le x^2+y^2 \le 2} \int_{z=\sqrt{x^2+y^2}}^{z=2} \delta(x, y, z)dz \; dA$$

Exercise: Find $\iiint_{T7} (x^2 + y^2 + z)dV$

Example: **is the solid contained in the first octant (** $x \ge 0; y \ge 0; z \ge 0$ **) below the plane** $3x + 2y + z = 6$.

The best approach to this problem is to realize that the plane $3x + 2y + z = 6$ meets the coordinate axes at *A*(2,0,0); B(0,3,0) and C(0,0,6). The part of the plane in the first octant is the triangle *ABC*. From the sketch it is apparent that the solid is defined by $0 \le z \le 6 - 3x - 2y$, and that the region *R*, which always represents the top view of T, is the triangle in the (*x,y*) plane with vertices obtained by omitting the z-coordinates of points *A, B, C*. Thus *R* is the triangle with vertices (2,0), (0,3), and (0,0) .

This is a right triangle. Its hypotenuse lies in the plane z = 0 and on the plane $3x + 2y + z = 6$; it follows that equations of that hypotenuse is $3x + 2y = 6$, or (better for our purposes) $y = \frac{6-3x}{2} = 3 - \frac{3}{2}x$ as shown below.



R is then given by $\begin{cases} 0 \le x \le 2 \\ 0 \le y \le 3 - \frac{3}{2}x \end{cases}$

It follows that

$$\iiint_{T8} \delta(x,y,z)dV = \iint_R \int_{z=0}^{6-3x-2y} \delta(x,y,z)dz\, dA = \int_{x=0}^{2} \int_{y=0}^{3-3x/2} \int_{z=0}^{6-3x-2y} \delta(x,y,z)dzdydx$$

In general this would be messy to calculate.

Exercise: Calculate the above triple integral using

a) $\delta(x,y,z) = 1$ and

b) $\delta(x,y,z) = x$

Exercise T3 can be solved instead by using *spherical coordinates* (ϕ, θ, ρ) on T3, which can be described by the inequality $x^2 + y^2 + z^2 \le \alpha^2$ with α = 2.

Spherical Coordinates

At point $P((x,y,z)$ in space, let $\rho = \sqrt{x^2 + y^2 + z^2}$, the distance from the origin (0,0,0) to point P. That point lies on the sphere with equation $x^2 + y^2 + z^2 = \rho^2$. Conversion from rectangular coordinates (x,y,z) to spherical coordinates (ϕ, θ, ρ) is done by substituting ρ for α in the spherical coordinate parametrization of the radius α sphere. This material is covered in Smith-Minton Section 14.7. Here's what you need:

$$\begin{cases} x = \rho\cos\theta\sin\phi \\ y = \rho\sin\theta\sin\phi \\ z = \rho\cos\phi \end{cases} \quad \text{where} \quad \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi \\ 0 \le \rho \le \alpha \end{cases} \quad \text{on the ball } x^2 + y^2 + z^2 \le \alpha^2$$

Furthermore, it's not hard to see that $dV = dS\,d\rho$, where $dS = \rho^2 \sin\phi\, d\phi\, d\theta$ is the element of surface area on the sphere $x^2 + y^2 + z^2 = \rho^2$. Therefore, on a solid ball,

$$dV = \rho^2 \sin\phi\, d\phi\, d\theta\, d\rho$$

For example, let's find the volume of the ball $x^2 + y^2 + z^2 \le \alpha^2$.

$$\iiint_T 1\,dV = \int_{\rho=0}^{\alpha}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} 1\rho^2 \sin\phi\, d\phi\, d\theta\, d\rho = \int_{\rho=0}^{\alpha}\rho^2 \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} \sin\phi\, d\phi\, d\theta\, d\rho = \int_{\rho=0}^{\alpha}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} \rho^2 \sin\phi\, d\phi\, d\theta\, d\rho$$

$$\int_{\rho=0}^{\alpha}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} \rho^2(-\cos\phi)\Big|_0^{\pi} d\phi\, d\theta\, d\rho = \int_{\rho=0}^{\alpha}\int_{\theta=0}^{2\pi} \rho^2(2\pi)\, d\theta\, d\rho = \int_{\rho=0}^{\alpha} 4\pi\rho^2\, d\rho = \frac{4}{3}\pi\rho^3\Big|_0^{\alpha} = \frac{4}{3}\pi\alpha^3$$

Exercise : Use spherical coordinates and density function $\delta(x,y,z) = x^2 + y^2 + z^2$ to evaluate $\iiint_{T3} \delta(x,y,z)\,dV$.

Integration by slices

We can try to avoid breaking T7 into two solids by putting the *dz* integral on the outside and the *dR* integral on the inside. This requires care!!! Note: this technique is not described in Smith-Minton or Stewart, but it should be!

Let's go back to an easier example: T5: **the solid inside the cylinder** $x^2 + y^2 = 4$**, above** $z = 0$**, and below** $z = 10$**.**

This cylinder can be viewed as a vertical stack of disks of radius 2. ......to be continued.

Let's try another method: writing a description of T as a z-sliceable surface. Then

We know that on the slanted face of T, $z = 6 - 3x - 2y$ or $3x + 2y = 6 - z$

Look at the slice of the surface at height z. That slice is a triangle with

vertices (0,0,z), $(0, \frac{6-z}{2}, z)$ (set *x* = 0 in the above equation) and $(\frac{6-z}{3}, 0, z)$

(set *y* = 0). The projection of this triangle to the (*x,y*)-plane is a right triangle with vertices (0,0), $(0, \frac{6-z}{2})$, and $(\frac{6-z}{3}, 0)$. This triangle can be described as

$$R_z : \begin{cases} 0 \le x \le \dfrac{6-z}{2} \\ 0 \le y \le \dfrac{6-z-3x}{2} \end{cases} \quad \text{Then} \quad \iiint_{T8} \delta(x,y,z)\,dV = \int_{z=0}^{6} \iint_{R_z} \delta(x,y,z)\,dA\,dz$$

In general, this is at least as difficult to calculate as it was with the previous method.

However, if the density function is 1, (and so the triple integral gives the volume of T) there is a special simplification. You will get $\iiint_{T8} 1\,dV = \int_{z=0}^{6} \iint_{R_z} 1\,dA\,dz$. Here, the inner double integral is just the area of right triangle R_z , given by Area = 1/2 base*height = $\frac{1}{2}\left(\frac{6-z}{3}\right)\left(\frac{6-z}{2}\right) = \frac{1}{12}(z-6)^2$. Then $\iiint_{T8} 1\,dV =$

$\int_{z=0}^{6} \frac{1}{12}(z-6)^2 = \frac{1}{12}\frac{(z-6)^3}{3}\Big]_0^6 = 6$. You can check that this is correct by using the pyramid volume formula: $V = \frac{1}{3}Bh$, where ***B*** is the base area.

Example: is the solid inside $x^2 + z^2 = 4$ and between $y = 1$ and *y* = 8.

(not yet posted).

