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Module 5: Matrices

Contents

Rank of a Matrix, Solution of Simultaneous Linear Equations by Elementary Transformation, Consistency of Equation, Eigen Values and Eigen Vectors, Diagonalization of Matrices, Cayley-Hamilton theorem and its applications to find inverse

Matrix : It is a rectangular arrangement of elements.

It is denoted by capital letter, while their elements by small letter.

i.e. $A = [a_{ij}]_{m \times n}$, Where$m = No.of\ rowes$,$n = No.of\ columns$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

i.e. A matrix is simply a rectangular (or square) array of numbers arranged in rows and columns. In fact, the term "array" is used pretty much interchangeably with matrix. Programmers tend to use "array", while mathematicians and biologists tend to use "matrix". The entries in a matrix are termed *elements* of the matrix.

Useful types of Matrix:-

[1] **Rectangular** Matrix: If $m \neq n$ then the Matrix $A = [a_{ij}]_{m \times n}$ is called a **rectangular** Matrix.

For example $\begin{bmatrix} 6 & -3 & 1 \\ 2 & 0 & 9 \end{bmatrix}$

[2]**Square Matrix**: If $m = n$ then the Matrix $A = [a_{ij}]_{m \times n}$ is called a **Square**Matrix.

For example $\begin{bmatrix} 1 & -3 \\ 2 & 7.2 \end{bmatrix}$

[3] Sub-matrix of a Matrix :- A matrix ,which is obtained from a given matrix by deleting any number of rows and number of columns is called a Sub-matrix of a Matrix.

Each matrix has two sub matrix like matrix itself and zero matrix.

For example: For the matrix $A=\begin{bmatrix}6 & -3 & 1\\ 2 & 0 & 9\end{bmatrix}$

The sub matrices are

$A, 0,\ A_1=\begin{bmatrix}6\\2\end{bmatrix}, A_2=\begin{bmatrix}-3\\0\end{bmatrix}, A_3=\begin{bmatrix}1\\9\end{bmatrix}, A_4=\begin{bmatrix}6 & -3\\2 & 0\end{bmatrix}, A_5=\begin{bmatrix}-3 & 1\\0 & 9\end{bmatrix}, A_6=\begin{bmatrix}6 & 1\\2 & 9\end{bmatrix}$

$A_7=[6\ \ -3], A_8=[-3\ \ 1], A_9=[2\ \ 0], A_{10}=[0\ \ 9], A_{11}=[6\ \ 1], A_{12}=[2\ \ 9]'$

$A_{13}=[6], A_{14}=[-3], A_{15}=[1],\ A_{16}=[2], A_{17}=[0], A_{18}=[9]$

Rank of a Matrix: The order of max.ordernon-singular sub-matrix is called the rank of the matrix and it is denoted by $\rho(A)$ *for example If* $A=\begin{bmatrix}6 & -3 & 1\\ 2 & 0 & 9\end{bmatrix}$ *then it's rank i.e.* $\rho(A)=2$

If A is a square matrix of order n *then it's nullity exists and it is* $N(A)=n-\rho(A)$.

Some important conclusions:-

i. Since the non-singular matrix is a square matrix, therefore the rank of a non-singular matrix of order n *is* n and it's *nullity* is zero.
ii. Rank of a Unit matrix of order n *is* n and it's *nullity* is zero.
iii. If A is matrix of order $m\times n$,then it's rank is $\le m$ *or* $\le n$.
iv. The rank of a zero matrix is taken as **zero.**
v. The rank is always a non-negative whole number.

Practice problems: Find the rank of following matrices

$$A=\begin{bmatrix}2 & 1 & -1\\ 0 & 3 & -2\\ 2 & 4 & -3\end{bmatrix},\quad B=\begin{bmatrix}1 & 3 & 4 & 3\\ 3 & 9 & 12 & 3\\ 1 & 3 & 4 & 1\end{bmatrix}$$

Answers : $\rho(A)=2$ because

$$|A| = \begin{vmatrix}2 & 1 & -1\\ 0 & 3 & -2\\ 2 & 4 & -3\end{vmatrix} = 0$$

Therefore $\rho(A)\neq 3$,Now we take a non-singular sub-matrix $A_1=\begin{bmatrix}2 & 1\\ 0 & 3\end{bmatrix}$ of A such that $|A_1| = \begin{vmatrix}2 & 1\\ 0 & 3\end{vmatrix} = 6 \neq 0$,hence $\rho(A)=2$ and $N(A)=n-\rho(A)=3-2=1$

Similarly $\rho(B)=2$.

Elementary transformations or operations on a matrix :- Any one of the following transformation on a matrix is called an elementary transformation :

[1] Interchange of any two rows or columns

i.e.we symbolized as say $R_i \leftrightarrow R_j \;\; or \;\; C_i \leftrightarrow C_j$

[2]Multiplication of a row or a column by a non-zero scalar.

i.e. We symbolized as say $\rightarrow kR_i \;\; or \;\; \rightarrow k\,C_i \;\; where \;\; k \neq 0$

[3] Additionof k times the elements of a row (ora column) to the corresponding elements of another row (or a column).

i.e. We symbolized as say

$$\rightarrow R_i + kR_j \;\; or \;\; \rightarrow C_i + kC_j \;\; where \;\; k \neq 0$$

Normal Form of a Matrix:

By means of elementary transformations every matrix A of order $m \times n$ can be reduced to one of the following forms: $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$, $[I_r \quad 0]$, $[I_r]$

and these are called it's normal forms and r is the rank of A.

Example-1: Reduce the following matrix to normal form $A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 2 \end{bmatrix}$

And hence find it's rank.

Solution: Given that $A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 2 \end{bmatrix}$

$$\rightarrow R_2 - R_1 \;\; and \;\; \rightarrow R_3 - 3R_1$$

$$\sim\begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 5 \end{bmatrix}$$

$$\to C_2 - C_1, \to C_3 - C_1 \text{ and } \to C_4 + C_1$$

$$\sim\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 5 \end{bmatrix}$$

$$\to R_3 - R_2$$

$$\sim\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\to C_3 - 2C_2 \text{ and } \to C_4 - 5C_2$$

$$\sim\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

Hence the $\rho(A) = 2$.

Computation of two non-singular matrices by the normal form : It is well explained by the following example.

Example-2 : Find the non-singular matrices $P \text{ and } Q$ such that PAQ is in normal form of A

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

Solution: Here the order of given matrix A is 4×3 ,Therefore , the matrices P and Q must be of order 4 and 3 respectively.

Thus to find the non-singular matrices P and Q ,we write

$$A = I_4 A I_3$$

Now , reduce the matrix A on the L.H.S. to it's normal form by the application of elementary transformations .

Each elementary rowtransformation is effected by performing the same on the prefactor unit matrix I_4 of A and each elementary column transformation is effected by performing the same on the post factor unit matrix I_3 of A . i.e.

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\to R_2 - 3R_1 \ , \ \to R_3 - R_1 \ and \to R_4 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & 1 & -1 \\ 0 & -3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\to C_2 - 2C_1 , and \to C_3 - 3C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & -8 \\ 0 & 1 & -1 \\ 0 & -3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -4 & -8 \\ 0 & -3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -3 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\to R_3+4R_2 \ \ and \to R_4+3R_2$$

$$\sim\begin{bmatrix}1&0&0\\0&1&-1\\0&0&-12\\0&0&-6\end{bmatrix}=\begin{bmatrix}1&0&0&0\\-1&0&1&0\\-7&1&4&0\\-5&0&3&1\end{bmatrix}A\begin{bmatrix}1&-2&-3\\0&1&0\\0&0&1\end{bmatrix}$$

$$\to C_3+C_2$$

$$\sim\begin{bmatrix}1&0&0\\0&1&-1\\0&0&-12\\0&0&-6\end{bmatrix}=\begin{bmatrix}1&0&0&0\\-1&0&1&0\\-7&1&4&0\\-5&0&3&1\end{bmatrix}A\begin{bmatrix}1&-2&-5\\0&1&1\\0&0&1\end{bmatrix}$$

$$\to\frac{1}{6}C_3$$

$$\sim\begin{bmatrix}1&0&0\\0&1&-1\\0&0&-2\\0&0&-1\end{bmatrix}=\begin{bmatrix}1&0&0&0\\-1&0&1&0\\-7&1&4&0\\-5&0&3&1\end{bmatrix}A\begin{bmatrix}1&-2&-5/6\\0&1&1/6\\0&0&1/6\end{bmatrix}$$

$$\to -1/2\ R_3$$

$$\sim\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&-1\end{bmatrix}=\begin{bmatrix}1&0&0&0\\-1&0&1&0\\-7/2&-1/2&-2&0\\-5&0&3&1\end{bmatrix}A\begin{bmatrix}1&-2&-5/6\\0&1&1/6\\0&0&1/6\end{bmatrix}$$

$$\to R_4+R_3$$

$$\sim\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}=\begin{bmatrix}1&0&0&0\\-1&0&1&0\\-7/2&-1/2&-2&0\\-3/2&-1/2&1&1\end{bmatrix}A\begin{bmatrix}1&-2&-5/6\\0&1&1/6\\0&0&1/6\end{bmatrix}$$

$$\sim \begin{bmatrix} I_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -7/2 & -1/2 & -2 & 0 \\ -3/2 & -1/2 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -5/6 \\ 0 & 1 & 1/6 \\ 0 & 0 & 1/6 \end{bmatrix} = PAQ$$

$$\therefore P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -7/2 & -1/2 & -2 & 0 \\ -3/2 & -1/2 & 1 & 1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1 & -2 & -5/6 \\ 0 & 1 & 1/6 \\ 0 & 0 & 1/6 \end{bmatrix}$$

Echelon Form of Matrix: The matrix obtained by applying a series of elementary row transformations such that

[1] All the rows with zero elements will be below the non-zero rows .i.e. All zero rows below the non-zero rows.

[2] The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

For example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 5 & 2 \end{bmatrix}$

Note:- 1. Rank = No. of non-zero rows in the Echelon form: $\rho(A)$=3 and $\rho(B)$=3.

2. Condition [1] is not compulsory.

Example -1: Reduce the following matrix to Echelon form $A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

Solution: Given matrix is $A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

$$\rightarrow R_2 + 2R_1 \;, \; \rightarrow R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$\rightarrow 1/3\, R_2$ *and* $\rightarrow 1/2\, R_3$

$$\sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$\rightarrow R_3 + R_2\,,\ \rightarrow R_4 - R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It is in Echelon form

Therefore $\rho(A) = No.of\ non-zero\ rows = 2$

Solution of system of Non-homogeneous linear algebraic equations:-

Let us consider a system of m **equations and** n in unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

The Augmented matrix is

$$[A:B] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

Now by applying a series of elementary row transformations, we reduce the above matrix to **Echelon form and we will find say** $[A_1 : B_1]$ *of* $[A:B]$

$$\rho[A:B] = No. of\ non-zero\ rows\ in\ the\ echelon\ form\ of\ [A:B] = r\ and$$

Now the $\rho[A] = No. of\ non-zero\ rows\ in\ the\ echelon\ form\ of\ [A] = s\ and\ n$

Conclusions:

[1] Consistent:If $\rho[A:B] = \rho[A]$ then system is consistent.

[i] If $\rho[A:B] = \rho[A] = n$,then system has a unique solution or a trivial solution.

[ii] If $\rho[A:B] = \rho[A] < n$,then system has a many solutions or a non-trivial solution.

[2] In-Consistent: If $\rho[A:B] \neq \rho[A]$ then system is in-consistent.

Example -1: Examine whether the following system of equations are consistent or not and if it is consistent then solve it.

[i] $x + y - 2z = 5$
$x - 2y + z = -2$
$-2x + y + z = 4$

Solution: The Augmented matrix is

$$[A:B] = \begin{bmatrix} 1 & 1 & -2 & 5 \\ 1 & -2 & 1 & -2 \\ -2 & 1 & 1 & 4 \end{bmatrix}$$

$$\rightarrow R_2 - R_1\ ,\ \rightarrow R_3 + 2R_1$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 5 \\ 0 & -3 & 3 & -7 \\ 0 & 3 & -3 & 14 \end{bmatrix}$$

$$\rightarrow R_3 + R_2$$

$$\sim\begin{bmatrix}1 & 1 & -2 & 5\\ 0 & -3 & 3 & -7\\ 0 & 0 & 0 & 7\end{bmatrix}$$

$$\rightarrow 1/-3\ R_2\ and\ \rightarrow 1/7\ R_3$$

$$\sim\begin{bmatrix}1 & 1 & -2 & 5\\ 0 & 1 & -1 & 7/3\\ 0 & 0 & 0 & 1\end{bmatrix}\sim[A_1:B_1]$$

It is in Echelon form

Therefore $\rho[A:B]=3\ and\ \rho(A)=2$,Hence $\rho[A:B]\neq\rho[A]$ i.e. system is in-consistent.

[ii] $x+2y-z=1$
$x+y+2z=9$
$2x+y-z=2$

Solution: The Augmented matrix is

$$[A:B]=\begin{bmatrix}1 & 2 & -1 & 1\\ 1 & 1 & 2 & 9\\ 2 & 1 & -1 & 2\end{bmatrix}$$

$$\rightarrow R_2-R_1\ ,\ \rightarrow R_3-2R_1$$

$$\sim\begin{bmatrix}1 & 2 & -1 & 1\\ 0 & -1 & 3 & 8\\ 0 & -3 & 1 & 0\end{bmatrix}$$

$$\rightarrow(-1)R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -3 & -8 \\ 0 & -3 & 1 & 0 \end{bmatrix}$$

$$\to R_3 + 3R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -3 & -8 \\ 0 & 0 & -8 & -24 \end{bmatrix}$$

$$\to (-1/8)R_3$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -3 & -8 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim [A_1 : B_1]$$

It is in Echelon form

Therefore $\rho[A:B]=3$ and $\rho(A)=3$, Hence

$\rho[A:B]=\rho[A]$ and $n=3$

Hence $\rho[A:B]=\rho[A]=n=3$ *i.e. System is consistent and has a unique or a trivial solution.*

It is find as follows

$A_1X = B_1$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -8 \\ 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} x+2y-z \\ y-3z \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -8 \\ 3 \end{bmatrix}$$

Therefore,

$x+2y-z=1$ ....(i)

$y-3z=-8$.......(ii)

$z=3$, *Hence from above* $y=1$ *and* $x=2$

[iii] $x - y + 2z = 4$
$3x + y + 4z = 6$
$x + y + z = 1$

Solution: The Augmented matrix is

$$[A : B] = \begin{bmatrix} 1 & -1 & 2 & 4 \\ 3 & 1 & 4 & 6 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\rightarrow R_2 - 3R_1 \ , \ \rightarrow R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 4 & -2 & -6 \\ 0 & 2 & -1 & -3 \end{bmatrix}$$

$$\rightarrow (1/2)R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 2 & -1 & -3 \\ 0 & 2 & -1 & -3 \end{bmatrix}$$

$$\rightarrow R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 2 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim [A_1 : B_1]$$

It is in Echelon form

Therefore $\rho[A:B]=2 \ \ and\ \rho(A)=2$, Hence

$\rho[A:B]=\rho[A] \quad and \ \ n=3$

$Hence \ \ \rho[A:B]=\rho[A]=2<n=3$

$i.e. \ \ System \ \ is \ consistent \ and \ has \ many \ solution \ or \ a \ Non-trivial \ solution.$

It is find as follows

$A_1X=B_1$

$$\sim\begin{bmatrix}1 & -1 & 2\\0 & 2 & -1\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}4\\-3\\0\end{bmatrix}$$

$$\sim\begin{bmatrix}x-y+2z\\2y-z\\0\end{bmatrix}=\begin{bmatrix}4\\-3\\0\end{bmatrix}$$

$Therefore,$

$x-y+2z=4 \ \ ....(i)$

$2y-z=-3.......(ii)$

$\therefore 3-2=1 \ \ i.e. we \ have \ one \ independent \ variable.$

$let \ \ z=k, where \ k\in R \ , Hence \ from \ above \ \ y=\dfrac{k-3}{2} and \ by \ equqtion\,(i)\, x=\dfrac{5-3k}{2}$

Example -2: By the method of elementary transformation of matrices investigate the value of λ and μ for which the system of equations

$$x+y+z=6$$

$$x+2y+3z=10$$

$$x+2y+\lambda z=\mu$$

have (i) no solution (ii) a unique solution (iii) an infinite number of solution.

Solution: The Augmented matrix is

$$[A:B]=\begin{bmatrix}1 & 1 & 1 & 6\\1 & 2 & 3 & 10\\1 & 2 & \lambda & \mu\end{bmatrix}$$

$$\to R_2-R_1 \ , \ \to R_3-R_1$$

$$\sim\begin{bmatrix}1 & 1 & 1 & 6\\0 & 1 & 2 & 4\\0 & 1 & \lambda-1 & \mu-6\end{bmatrix}$$

$$\rightarrow R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{bmatrix} \sim [A_1 : B_1]$$

It is in Echelon form and their rank depends upon the values of λ *and* μ.

Hence ,we have

(i) **No solution** : We know that for this case $\rho[A:B] \neq \rho[A]$ i.e. system is in-consistent .therefore $\lambda = 3$ *and* $\mu \neq 10$ *i.e* $\mu \in R$ *excluding* $\mu = 10$.

(ii) **A unique solution :** We know that for this case

$\rho[A:B] = \rho[A]$ *and* $n = 3$

i.e. $\rho[A:B] = \rho[A] = n = 3$ *i.e. System is consistent and has a unique or a trivial solution.*

Hence, we have $\lambda \neq 3$ *and* $\mu \in R$.

(iii) An infinite number of solution :We know that for this case

$\rho[A:B] = \rho[A]$ *and* $n = 3$

Hence $\rho[A:B] = \rho[A] = 2 < n = 3$ *i.e.*

System is consistent and has many solution or a Non-trivial solution.

Hence, $\lambda = 3$ *and* $\mu = 10$

Solution of system of Homogeneous linear algebraic equations:-

Let us consider a system of m **equations and** n in unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

The Augmented matrix is

$$[A:B]=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 \end{bmatrix}$$

Now by applying a series of elementary row transformations, we reduce the above matrix to **Echelon form and we will find say** $[A_1:B_1]$ *of* $[A:B]$

Now the
$$\rho[A:B] = No.of\ non-zero\ rows\ \text{in}\ the\ echelon\ form\ of\ [A:B] = r\ and$$
$$\rho[A] = No.of\ non-zero\ rows\ \text{in}\ the\ echelon\ form\ of\ [A] = s$$
$$and\ n\ is\ the\ number\ of\ un-knows.$$

Conclusions:

If $\rho[A:B]=\rho[A]$ then system is consistent.

[i] If $\rho[A:B]=\rho[A]=n$,then system has a unique solution or a trivial solution and it is $x_1 = x_2 = \cdots x_n = 0$

[ii] If $\rho[A:B]=\rho[A]<n$,then system has a many solutions or a non-trivial solutions.

Note :- [1] The **system of Homogeneous linear algebraic equations are always** consistent.

[2] If $|A| \neq 0, then, we\ have\ unique\ solution,\ x_1 = x_2 = \cdots x_n = 0$.

[3] If $|A| = 0, then, we\ have\ many\ solution\ or\ Non-trivial\ solutions.$

Example -1: Examine whether the following system of equations are consistent or not and if it is consistent then solve it.

[i] $x + 3y - 2z = 0$
$2x - y + 4z = 0$
$x - 11y + 14z = 0$

Solution: The Augmented matrix is

$$[A:B]=\begin{bmatrix} 1 & 3 & -2 & 0 \\ 2 & -1 & 4 & 0 \\ 1 & -11 & 14 & 0 \end{bmatrix}$$

$$\to R_2 - 2R_1 \ , \ \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 0 \\ 0 & -7 & 8 & 0 \\ 0 & -14 & 16 & 0 \end{bmatrix}$$

$$\to R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 0 \\ 0 & -7 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim [A_1 : B_1]$$

It is in Echelon form

Therefore $\rho[A:B]=2 \ and \ \rho(A)=2$, Hence

$\rho[A:B]=\rho[A] \quad and \ n=3$

$Hence \ \rho[A:B]=\rho[A]=2<n=3$

i.e. System is consistent and has many solution or a Non – trivial solution.

It is find as follows

$A_1X = B_1$

$$\sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} x+3y-2z \\ -7y+8z \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore,

$x+3y-2z=0 \ ....(i)$

$-7y+8z=0.......(ii)$

$\therefore 3-2=1$ *i.e. we have one independent variable.*

let $z=k$, *where* $k \in R$, *Hence from above* $y=\dfrac{8k}{7}$ *and by equqtion* (i) $x=\dfrac{-10k}{7}$

Example -2 Determine $'b'$ for which the system of equations

$$2x + y + 2z = 0$$

$x + y + 3z = 0$

$4x + 3y + bz = 0$

have (i) a unique solution i.e. trivial solution (iii) an infinite number of solution i.e. a non-trivial solutions. Find the non-trivial solutions.

Solution: The Augmented matrix is

$$[A:B] = \begin{bmatrix} 2 & 1 & 2 & 0 \\ 1 & 1 & 3 & 0 \\ 4 & 3 & b & 0 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 3 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 3 & b & 0 \end{bmatrix}$$

$$\to R_2 - 2R_1 \;, \; \to R_3 - 4R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 4 & -1 & b-12 & 0 \end{bmatrix}$$

$$\to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & b-8 & 0 \end{bmatrix} \sim [A_1 : B_1]$$

It is in Echelon form and their rank depends upon the values of b.

Hence ,we have

(i) **A unique solution or a trivial solution :** We know that for this case

$\rho[A:B] = \rho[A]$ and $n = 3$

i.e. $\rho[A:B] = \rho[A] = n = 3$ i.e. System is consistent and has a unique or a trivial solution.

Hence, we have $b - 8 \neq 0 \Rightarrow b \neq 8$.

(ii) An infinite number of solution or a non-trivial solution :We know that for this case

$$\rho[A:B]=\rho[A] \quad and \ \ n=3$$

$$Hence \ \ \rho[A:B]=\rho[A]=2<n=3 \ \ i.e.$$

$$System \ \ is \ consistent \ and \ has \ many \ solution \ or \ a \ Non-trivial \ solution.$$

$$Hence, b-8=0 \Rightarrow b=8.$$

It is find as follows

$$A_1X=B_1$$

$$\sim\begin{bmatrix}1 & 1 & 3\\ 0 & -1 & -4\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\sim\begin{bmatrix}x+y+3z\\ -y-4z\\ 0\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Therefore,

$$x+y+3z=0 \ ....(i)$$

$$-y-4z=0.......(ii)$$

$$\therefore 3-2=1 \ \ i.e. we \ have \ one \ independent \ variable.$$

$$let \ \ z=k, where \ \ k\in R \ , Hence \ from \ above \ \ y=-4k \ and \ by \ equqtion \ (i) \ x=k$$

Characteristic Equation: If A *is a square matrix of order n*, Then the equation $|A-\lambda I|=0$ *is called a* **Characteristic Equation** of A. Since it is a polynomial of degree n , therefore , it has n roots say $\lambda_1, \lambda_2, \lambda_3 \ldots\ldots\ldots. \lambda_n$ are called **Characteristic roots or latent roots or eigen values.**

Characteristic Value Problem: If $A is a square matrix of order n,, determine the scalars \lambda and the non-zero vectors X, which simultaneously satisfy the equation$

$$AX=\lambda X or [A-\lambda I]X=0.$$

This is known as **Characteristic value problem . X are called Characteristic vectors or latent vectors or eigen vectors.**

Note: If $\lambda_1, \lambda_2, \lambda_3 \ldots\ldots\ldots. \lambda_n$ be the **eigen values of A, then A^n has the eigen values** $\lambda_1^n, \lambda_2^n, \ldots\ldots\ldots \lambda_n^n$.

Example -1: Find the Eigen values of A^4 , where $A=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$

Solution: We know that for Eigen value of **A**, $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix}1-\lambda & 1\\ 1 & 1-\lambda\end{vmatrix}=0 => (1-\lambda)^2-1=0 => \lambda=0,2$$

Hence the **Eigen values of A^4 are $0^4, 2^4$ i.e. 0, 16**

Example -2: Find the Eigen values of $4A^{-1} + 3A + 2I$, where $A = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}$

Solution: We know that for Eigen value of $\mathbf{A}$, $|A - \lambda I| = 0$

⇨ $\begin{vmatrix} 1-\lambda & 0 \\ 2 & 4-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(4-\lambda) - 0 = 0 \Rightarrow \lambda = 1,4$

Now , the **Eigen values of A^{-1} are $1^{-1}, 4^{-1}$ i.e. $1, \frac{1}{4}$ and those of $4A^{-1}$ are $4.1, 4.\frac{1}{4}$** $=>$ 4,1 .The **Eigen values of $3A$, are 3.1 , 3.4 ,i.e. 3 , 12 .**

Hence , the eigen values of

$\mathbf{4A^{-1} + 3A are 4 + 3, 1 +}$
$\mathbf{12 i.e. 7, 13 .Since the eigen values if unit matrix I of 2x2 order are 1, 1 , therefore the}$
$\mathbf{eigen values}$ of $\mathbf{2I}$ are 2,2.

Thus ,the Eigen values of $\mathbf{4A^{-1} + 3A + 2I are 7 + 2 = 9 and 13 + 2 = 15}$.

Example -3: Find the Eigen values and Eigen vectors of

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Solution: We know that for Eigen value of $\mathbf{A}$, $|A - \lambda I| = 0$

$$=> \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$=> (8-\lambda)[(7-\lambda)(3-\lambda) - 16] - (-6)[-6(3-\lambda) + 8] + 2[24 - 2(7-\lambda)] = 0$
$=> -\lambda^3 + 18\lambda^2 - 45\lambda = 0$
$=> (-\lambda)(\lambda - 3)(\lambda - 15) = 0 => \lambda = 0,3,15$

Now, we will find the corresponding Eigen vectors.

For this , we have the characteristic value problem

$[A - \lambda I\,]X = 0$

$$\sim \begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} --------[1]$$

First put $\lambda = 0$,in equation[1]

$$\sim \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow \frac{1}{2} R_1$

$$\sim \begin{bmatrix} 4 & -3 & 1 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 2 & -4 & 3 \\ -6 & 7 & -4 \\ 4 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow R_3 + 3R_1 \;, \rightarrow R_3 - 2R_1$

$$\sim \begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 5 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow R_3 - R_2$

$$\sim \begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow \frac{1}{5} R_2$

$$\sim \begin{bmatrix} 2 & -4 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

It is in Echelon form ,

Therefore $\rho[A-\lambda I]=2 \;\; for \lambda = 0 \; and \;\; n=3$, Hence

$Hence \;, No. \;\; of \;\; Linearly \; Independant \;\; Eigen \; vector \; = 3-2=1$

$i.e. \quad 2x-4y+3z=0........[i]$

$-y+z=0$

$\Rightarrow y=z=k_1 \;\; where \;\; k_1 \in R, Hence \; by \, (i), x = \frac{k_1}{2}$

$$Thus \;\; the \;\; Eigen \;\; vector \;\; X = X_1 = X_{\lambda=0} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{k_1}{2} \\ k_1 \\ k_1 \end{bmatrix}$$

Now , put $\lambda = 3$, in equation [1]

$$\sim \begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow R_1 + R_2$

$$\sim \begin{bmatrix} -1 & -2 & -2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow \frac{1}{2} R_2 , \rightarrow \frac{1}{2} R_3$

$$\sim \begin{bmatrix} -1 & -2 & -2 \\ -3 & 2 & -2 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow R_2 - 3R_1 , \rightarrow R_3 + R_1$

$$\sim \begin{bmatrix} -1 & -2 & -2 \\ 0 & 8 & 4 \\ 0 & -4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow \frac{1}{4} R_2 , \rightarrow \frac{1}{2} R_3$

$$\sim \begin{bmatrix} -1 & -2 & -2 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow R_3 + R_2$

$$\sim \begin{bmatrix} -1 & -2 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

It is in Echelon form,

Therefore $\rho[A-\lambda I]=2\ \ for\,\lambda=3 and\ \ n=3$, Hence

$Hence\ \ ,No.\ \ of\ \ Linearly\,Independant\ \ Eigen\ vector\ =3-2=1$

$i.e.\ \ \ -x-2y-2z=0........[i]$

$2y+z=0$

$\Rightarrow 2y=-z\ \ Let\ \ z=k_2\ \ where\ \ k_2\in R,$

$\therefore\ \ y=-\dfrac{k_2}{2}\ \ Hence\ by\,(i),x=-k_2$

$$Thus\ \ the\ \ Eigen\ vector\ \ \ X=X_2=X_{\lambda=3}=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-k_2\\-\dfrac{k_2}{2}\\k_2\end{bmatrix}$$

Now , put $\lambda=15$, in equation [1]

$$\sim\begin{bmatrix}-7&-6&2\\-6&-8&-4\\2&-4&-12\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\rightarrow\frac{1}{2}R_2\ ,\rightarrow\frac{1}{2}R_3$

$$\sim\begin{bmatrix}-7&-6&2\\-3&-4&-2\\1&-2&-6\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$R_1\leftrightarrow R_3$

$$\sim\begin{bmatrix}1&-2&-6\\-3&-4&-2\\-7&-6&2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\rightarrow R_2+3R_1\ ,\rightarrow R_3+7R_1$

$$\sim\begin{bmatrix}1&-2&-6\\0&-10&-20\\-7&-20&-40\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\rightarrow\frac{1}{10}R_2\ ,\rightarrow\frac{1}{20}R_3$

$$\sim\begin{bmatrix}1&-2&-6\\0&-1&-2\\0&-1&-2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\rightarrow R_3-R_2$

$$\sim\begin{bmatrix}1&-2&-6\\0&-1&-2\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

It is in Echelon form,

Therefore $\rho[A-\lambda I]=2 \ for \lambda=3 and \ n=3$, Hence

$Hence \ , No. \ of \ Linearly Independant \ Eigen \ vector \ = 3-2=1$

$i.e. \quad x-2y-6z=0........[i]$

$-y-2z=0$

$\Rightarrow -y=2z \ Let \ z=k_3 \ where \ k_3 \in R,$

$\therefore \quad y=-2k_3 \ Hence \ by (i), x=2k_3$

$$Thus \ the \ Eigen \ vector \ \ X = X_3 = X_{\lambda=15} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2k_3 \\ -2k_3 \\ k_3 \end{bmatrix}$$

Example -4: Find the Eigen values and Eigen vectors of

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Solution: We know that for Eigen value of $\mathbf{A}$, $|A-\lambda I| = 0$

$$=> \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$=> (6-\lambda)[(3-\lambda)^2-1]-(-2)[-2(3-\lambda)+2]+2[2-2(3-\lambda)] = 0$
$=> \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$
Since $\lambda = 2$ satisfies the above equation, $(\lambda-2)$ is one factor of it.
$\lambda^2(\lambda-2) - 10\,\lambda(\lambda-2) + 16(\lambda-2) = 0$
$=> (\lambda-2)(\lambda^2 - 10\,\lambda + 16)$=0
$=> (\lambda-2)(\lambda-2)(\lambda-8) = 0 => \lambda = 2\,,2, 8$
Now, we will find the corresponding Eigen vectors.
For this , we have the characteristic value problem
$[A-\lambda I\,]X = 0$

$$\sim \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} ---------[1]$$

Now , put $\lambda = 2$, in equation [1]

$$\sim \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow \frac{1}{2} R_1$

$$\sim \begin{bmatrix} 2 & -1 & 1 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow R_2 + R_1 , \rightarrow R_3 - R_1$

$$\sim \begin{bmatrix} 2 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

It is in Echelon form,

Therefore $\rho[A-\lambda I]=1 \ for \lambda=2 and \ n=3$, Hence

$Hence \ , No. \ of \ Linearly Independant \ Eigen \ vector =3-1=2$

$i.e. \ \ 2x-y+z=0........[i]$

$Let \ y=k_1 \ \ z=k_2 \ where \ k_1, k_2 \in R,$

$\therefore \ \ Hence \ by (i), \ x=\dfrac{k_1-k_2}{2}$

$$Thus \ the \ Eigen \ vector \ \ X = X_{\lambda=2} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \dfrac{k_1-k_2}{2} \\ k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} \dfrac{k_1}{2} \\ k_1 \\ 0 \end{bmatrix} + \begin{bmatrix} \dfrac{-k_2}{2} \\ 0 \\ k_2 \end{bmatrix} = X_1 + X_2$$

Now , put $\lambda = 8$, in equation [1]

$$\sim \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow R_2 - R_1, \rightarrow R_3 + R_1$

$$\sim \begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow R_3 - R_2$

$$\sim \begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

It is in Echelon form,

Therefore $\rho[A-\lambda I]=2 \ for \lambda=8 and \ n=3$, Hence

$Hence \ , No. \ of \ Linearly Independant \ Eigen \ vector =3-2=1$

$i.e. \ \ -2x-2y+2z=0........[i]$

$-3y-3z=0$

$\Rightarrow y+z=0 \Rightarrow y=-z \ Let \ z=k_3 \ where \ k_3 \in R,$

$\therefore \ \ y=-k_3 \ Hence \ by (i), x=2k_3$

$$Thus \ the \ Eigen \ vector \ \ X = X_3 = X_{\lambda=8} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2k_3 \\ -k_3 \\ k_3 \end{bmatrix}$$

Diagonalization of Matrices:-

If A $is\ a\ square\ matrix\ of\ order\ n$ has n linearly independent Eigen –vectors, $X_1, X_2, X_3,$ ……………, X_n ,then a matrix B can be found such that $B^{-1}AB$ is a diagonal matrix. Where B is called the Modal Matrix and $B = [X_1 X_2 \ldots \ldots . . X_n]$.

Example: Verify the Diagonalization of Matrix

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Solution: We know that for Eigen value of $\mathbf{A}$, $|A - \lambda I| = 0$

$$=> \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$=> (6-\lambda)[(3-\lambda)^2 - 1] - (-2)[-2(3-\lambda) + 2] + 2[2 - 2(3-\lambda)] = 0$

$=> \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

Since $\lambda = 2$ satisfies the above equation, $(\lambda - 2)$ is one factor of it.

$\lambda^2(\lambda - 2) - 10\,\lambda(\lambda - 2) + 16(\lambda - 2) = 0$

$=> (\lambda - 2)(\lambda^2 - 10\,\lambda + 16)$=0

$=> (\lambda - 2)(\lambda - 2)(\lambda - 8) = 0 => \lambda = 2, 2, 8$

Now, we will find the corresponding Eigen vectors.

For this , we have the characteristic value problem

$[A - \lambda I\,]X = 0$

$$\sim \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} --------[1]$$

Now, put$\lambda = 2$, in equation [1]

$$\sim \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow \frac{1}{2} R_1$

$$\sim \begin{bmatrix} 2 & -1 & 1 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow R_2 + R_1, \rightarrow R_3 - R_1$

$$\sim \begin{bmatrix} 2 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

It is in Echelon form,

Therefore $\rho[A-\lambda I]=1 \ for \lambda=2 \, and \ n=3$, Hence

$Hence \ , No. \ of \ Linearly \, Independant \ Eigen \ vector \ = 3-1=2$

$i.e. \ \ 2x-y+z=0........[i]$

$Let \ y=k_1 \ \ z=k_2 \ \ where \, k_1, k_2 \in R,$

$\therefore \ \ Hence \ by \, (i), \ x=\dfrac{k_1-k_2}{2}$

$$Thus \ the \ Eigen \ vector \ \ X = X_{\lambda=2} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{k_1-k_2}{2} \\ k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} \frac{k_1}{2} \\ k_1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{-k_2}{2} \\ 0 \\ k_2 \end{bmatrix} = X_1 + X_2$$

Now ,put $\lambda = 8$, in equation [1]

$$\sim \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow R_2 - R_1 , \rightarrow R_3 + R_1$

$$\sim \begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow R_3 - R_2$

$$\sim \begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

It is in Echelon form,

Therefore $\rho[A-\lambda I]=2 \ for \lambda=8 \, and \ n=3$, Hence

$Hence \ , No. \ of \ Linearly \, Independant \ Eigen \ vector \ = 3-2=1$

$i.e. \ \ -2x-2y+2z=0........[i]$

$-3y-3z=0$

$\Rightarrow y+z=0 \Rightarrow y=-z \ \ Let \ \ z=k_3 \ \ where \ k_3 \in R,$

$\therefore \ \ y=-k_3 \ \ Hence \ by \, (i), x=2k_3$

$$Thus \ the \ Eigen \ vector \ \ X = X_3 = X_{\lambda=8} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2k_3 \\ -k_3 \\ k_3 \end{bmatrix}$$

Now, the Modal Matrix $B = [X_1 X_2 \dots \dots . . X_n]$

Therefore ,$B = [X_1 X_2 X_3]$
Hence

$$B = \begin{bmatrix} 1/2 & -1/2 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \qquad =$$

$\therefore$

Now forAdj B

$C_{11} = (-1)^{1+1}[1] = 1$,$C_{12} = (-1)^{1+2}[1] = -1$,$C_{13} = (-1)^{1+3}[1] = 1$
$C_{21} = (-1)^{2+1}[-5/2] = 5/2$,$C_{22} = (-1)^{2+2}[1/2] = 1/2$,$C_{23} = (-1)^{2+3}[1/2] = -1/2$
$C_{31} = (-1)^{3+1}[1/2] = 1/2$,$C_{32} = (-1)^{3+2}[-5/2] = 5/2$,$C_{33} = (-1)^{3+3}[1/2] = 1/2$

Hence ,

$$adj\ B = \begin{bmatrix} 1 & 5/2 & 1/2 \\ -1 & 1/2 & 5/2 \\ 1 & -1/2 & 1/2 \end{bmatrix} \quad and\ |B| = 3$$

$$Therefore\ ,\ B^{-1} = \frac{adjB}{|B|} = \begin{bmatrix} 1/3 & 5/6 & 1/6 \\ -1/3 & 1/6 & 5/6 \\ 1/3 & -1/6 & 1/6 \end{bmatrix}$$

$$B^{-1}AB = \begin{bmatrix} 1/3 & 5/6 & 1/6 \\ -1/3 & 1/6 & 5/6 \\ 1/3 & -1/6 & 1/6 \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix} = diag\,(2, 2.8)$$

Hence the given matrix is diagonizable.

Cayley Hamilton Theorem : Every square matrix satisfies it's own characteristic equation.

Verification : We know that
If $A\ is\ a\ square\ matrix\ of\ order\ n$,Then the characteristic equation
$|A - \lambda I| = 0$. it is a polynomial of degree n.
i.e. $(-1)^n [a_0\, \lambda^n + a_1\, \lambda^{n-1} + a_2\, \lambda^{n-2} + \cdots \dots \dots \dots + a_{n-1}\ \lambda + a_n\ ] = 0$------------[1]
Then to prove the above theorem, ifλ is replace by A, then above equation is satisfied i.e.

$$(-1)^n [a_0\, A^n + a_1\, A^{n-1} + a_2\, A^{n-2} + \cdots \dots \dots \dots + a_{n-1}\ A + a_n\ I] = O$$

For this, let $B = adj(A - \lambda I)$ ....[2]

=> $B = B_0\, \lambda^{n-1} + B_1\, \lambda^{n-2} + B_2\, \lambda^{n-3} + \cdots \dots \dots \dots + B_{n-2}\ \lambda + B_{n-1}$

Now by equation [2], $(A - \lambda I)B = (A - \lambda I) adj(A - \lambda I)$

$=> (A-\lambda I)B = |A-\lambda I|\ I$ Since $A\,adjA = |A|I$

$$=> (A-\lambda I)[B_0\,\lambda^{n-1} + B_1\,\lambda^{n-2} + B_2\,\lambda^{n-3} + \cdots \dots\dots\dots + B_{n-2}\,\lambda + B_{n-1}]$$
$$=.\ (-1)^n[a_0\,\lambda^n + a_1\,\lambda^{n-1} + a_2\,\lambda^{n-2} + \cdots \dots\dots\dots + a_{n-1}\,\lambda + a_n]\,I$$

Now equating the coefficients of like powers of λ on the both sides ,we get

$-B_0\,I = (-1)^n a_0\,I$
$AB_0 - B_1\,I = (-1)^n a_1\,I$
$AB_1 - B_2\,I = (-1)^n a_2\,I$
……………….. ……………..
$AB_{n-2} - B_{n-1}\,I = (-1)^n a_{n-1}\,I$
$AB_{n-1} = (-1)^n a_n\,I$

Now, premultiplying these equations by $A^n, A^{n-1}, A^{n-2}, \dots\dots\dots\dots, A, I$ and then adding, we get

$(-1)^n[a_0\,A^n + a_1\,A^{n-1} + a_2\,A^{n-2} + \cdots \dots\dots\dots + a_{n-1}\,A + a_n\,I] = O \dots\dots\dots [3]$
This shows the Cayley Hamilton theorem.

If A is a non-singular matrix ,then multiply [3] by A^{-1} ,we get

$$A^{-1} = -\frac{1}{a_n}[a_0\,A^{n-1} + a_1\,A^{n-2} + \cdots \dots\dots\dots + a_n\,I]$$

Example-1: Verify Cayley Hamilton theorem to find A^{-1} *and* A^8 if $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

Solution: We know that for Eigen value of **A**, $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0 => -(1-\lambda)^2 - 4 = 0 => \lambda^2 - 5 = 0$$

Now to proveCayley Hamilton theorem ,we will prove that , if λ is replace by A , then above equation is satisfied i.e. $A^2 - 5I = 0 \dots\dots\dots\dots\dots.[1]$

For this,$A^2 = AA = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1+4 & 2-2 \\ 2-2 & 4+1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

$$\therefore L.H.S.of\ (1)is.\ \ A^2 - 5I = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 = R.H.S.of\ (1)$$

Hence the Cayley Hamilton theorem is verified.
Now to find A^{-1}, *Multiply equation* (1) *by* A^{-1} *on the both sides*
We get $A^{-1} = \frac{1}{5}A = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$
Now to find A^8, *for this by Equation* $(1) A^2 = 5I$

$\therefore\ A^4 = A^2 A^2 = 25\,I$

Hence, $A^8 = A^4 A^4 = 625I = 625\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}625 & 0\\0 & 625\end{bmatrix}$

Example -2: Verify Cayley Hamilton theorem and hence find the matrix represented by $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$. $A = \begin{bmatrix}1 & 4\\2 & 3\end{bmatrix}$

Solution: $|A - \lambda I| = 0$ implies $\begin{vmatrix}1-\lambda & 4\\2 & 3-\lambda\end{vmatrix} = 0$

$\Rightarrow \lambda^2 - 4\lambda - 5 = 0.$

For CHT, we have to prove $A^2 - 4A - 5I = 0$.

$$\begin{bmatrix}9 & 16\\8 & 17\end{bmatrix} - \begin{bmatrix}4 & 16\\8 & 12\end{bmatrix} - \begin{bmatrix}5 & 0\\0 & 5\end{bmatrix} = \begin{bmatrix}9-4-5 & 16-16\\8-8 & 17-12-5\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

Now $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = (A^3 - 2A + 3I)(A^2 - 4A - 5I) + A + 5I$

By CHY $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = A + 5I$

