Program : **B.Tech**

Subject Name: **Mathematics-I**

Subject Code: **BT-102**

Semester: **1st**

Subject: Mathematics-I Subject Code: BT-102

Module 3: Sequences and series

Convergence of sequence and series, tests for convergence; Power series, Taylor's series, series for exponential, trigonometric and logarithm functions; Fourier series: Half range sine and cosine series, Parseval's theorem

Part-I : Sequence

Definition: A sequence can be written as a list of numbers in a definite order like $a_1, a_2, a_3, \cdots, a_n, \cdots$ The number a_1 is called the first term, a_2 is the second term, and in general a_n is the nth term . In this section we will consider infinite sequence having infinitely many terms. We represent an infinite sequence by $\{a_1, a_2, a_3, \cdots, a_n, \cdots\}$ or $\{a_n\}_{n=1}^{\infty}$ or simply by $\{a_n\}$.

Convergence and Divergence of a sequence:

A sequence $\{a_n\}$ is convergent if $\lim_{n\to\infty} a_n = L$ exists, otherwise the sequence is divergent.

Theorem 1. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$

Increasing and Decreasing sequence:

A sequence $\{a_n\}$ is called increasing if $a_n < a_{n+1}$ for all $n \geq 1$. It called decreasing if $a_n > a_{n+1}$ for all $n \geq 1$. It is called **monotonic** if it is either increasing or decreasing.

Theorem: Every bounded, monotonic sequence is convergent.

Arithmetic and geometric sequences

A sequence of the form $a, a+d, a+2d, a+3d, \cdots, a+(n-1)d, \cdots$ is an arithmetic sequence, where ***a*** is the first term and ***d*** is the common difference.

A sequence of the form $a, ar, ar^2, ar^3, \cdots, ar^{n-1}, \cdots$ is a geometric sequence, where ***a*** is the first term and ***r*** is the common ratio.

Examples Determine whether the sequence converges or diverges, if converges find the limit

1. $a_n = \dfrac{n+1}{3n-1}$. The given sequence is convergent because $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{n+1}{3n-1} = \dfrac{1}{3}$, which is finite.

2. $a_n = \dfrac{(-1)^n n^3}{n^3 + 2n^2 + 1}$. The given sequence is divergent because $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left| \dfrac{(-1)^n n^3}{n^3 + 2n^2 + 1} \right| = 1$.

The sequence converges to 1 when *n* is even, on the other hand it converges to -1 when *n* is odd. The sequence does not converge to single finite number.

3. $a_n = \cos(n/2)$. The given sequence is divergent because $\lim_{n\to\infty} a_n$ does not exist.

4. $a_n = \cos(2/n)$. The given sequence is convergent because $\lim_{n\to\infty} a_n = 1$.

5. $\{a_n\} = \left\{\dfrac{\ln n}{\ln 2n}\right\}$. The given sequence is convergent because $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{\ln n}{\ln 2n} = 1$, by L'H$\hat{o}$ pital rule .

6. Determine whether the sequence is increasing, decreasing, or not monotonic. Is the sequence bounded?

a) $a_n = \dfrac{1}{5^n}$. The given sequence is decreasing $a_n > a_{n+1}$, $\lim_{n\to\infty} a_n = 0$. the sequence is also bounded because $0 < a_n \le \dfrac{1}{5}$

b) $a_n = \dfrac{2n-3}{3n+4}$. The given sequence is increasing because $a_n < a_{n+1}$, $\lim_{n\to\infty} a_n = \dfrac{2}{3}$, the sequence is also bounded because $-\dfrac{1}{7} = a_1 < a_n < \dfrac{2}{3}$ for all $n \ge 1$.

c) $a_n = \dfrac{n}{n^2+1}$. The given sequence is decreasing because $a_n > a_{n+1}$, $\lim_{n\to\infty} a_n = \dfrac{n}{n^2} = 0$, the sequence is also bounded because $\dfrac{1}{2} = a_1 \ge a_n > 0$ for all $n \ge 1$.

d) $a_n = n + \dfrac{1}{n}$. The given sequence is increasing because $a_n < a_{n+1}$, $\lim_{n\to\infty} a_n = \infty$, is not convergent,the sequence is bounded because $2 = a_1 < a_n$ for all $n \ge 1$.

Part-II : Series

An infinite series can be written as $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$

Arithmetic and geometric series

A series of the form $a + (a+d) + (a+2d) + (a+3d) + \cdots + (a+(n-1)d) \cdots$ is an arithmetic series, where ***a*** is the first term and ***d*** is the common difference. The partial sum of n terms of an arithmetic series is given by $\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n = \dfrac{n}{2}(a_1 + l),\ where\ l = a_1 + (n-1)d$, the nth term.

A series of the form $a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots$ is a geometric series, where ***a*** is the first term and ***r*** is the common ratio. The partial sum of n terms of a geometric series is given by $\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n = a_1 \frac{r^n - 1}{r - 1}$.

Convergent series Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$. Suppose $s_n = a_1 + a_2 + a_3 + \cdots + a_n$ be the partial sum of n terms of the infinite series then if $\{s_n\}$ is a convergent sequence and $\lim_{n\to\infty} s_n = s$ exists as a real number, the series written as $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots = s$ is also a convergent series. The number ***s*** is called the sum of the series. Otherwise the series is divergent.

Convergent Geometric series

The geometric series $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots$ is convergent if $|r| < 1$ and its sum is given by $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$. The geometric series is divergent if $|r| \geq 1$

The test of divergence

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$ then the series is divergent.

Examples

1. Find at least 10 partial sums of the given series. Is it convergent or divergent? Expalin.

a) $\sum_{n=1}^{\infty} (0.6)^{n-1}$

$\sum_{n=1}^{\infty} (0.6)^{n-1} = 1 + (0.6) + (0.6)^2 + \cdots$

$s_1 = 1.0,\ s_2 = 1.6,\ s_3 = 1.96,\ s_4 = 2.176,\ s_5 = 2.301,\ s_6 = 2.383,\ s_7 = 2.43,\ s_8 = 2.458,$
$s_9 = 2.475, s_{10} = 2.485$ The given series is a geometric series with $|r| = 0.6 < 1$, and it is convergent, its sum is given by

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} = \frac{1}{1-0.6} = 2.5$$

b) $\sum_{n=1}^{\infty}\frac{2n^2-1}{n^2+1}$

$$\sum_{n=1}^{\infty}\frac{2n^2-1}{n^2+1}=\frac{1}{2}+\frac{7}{5}+\frac{17}{10}+\cdots$$

$s_1=0.5,\ s_2=1.9,\ s_3=3.6,\ s_4=5.42,\ s_5=7.31,\ s_6=9.23,\ s_7=11.17,\ s_8=13.12,$
$s_9=15.08, s_{10}=17.05$

This series is a harmonic series with $\lim_{n\to\infty}\frac{2n^2-1}{n^2+1}=2\neq 0$. By the divergence test it is divergent series.

2. Determine whether the series is convergent or divergent. Find sum if convergent.

a) $\sum_{n=1}^{\infty}\frac{3}{n}$. The given series is not a geometric series. Also $\lim_{n\to\infty}\frac{3}{n}=0\Rightarrow$ divergent test fails. From example 7, page # 717 we know that $\sum_{n=1}^{\infty}\frac{1}{n}$ is a divergent series, since the sequence of partial sums $\{s_n\}$ is divergent. So $\sum_{n=1}^{\infty}\frac{3}{n}$ is also divergent.

b) $\sum_{n=1}^{\infty}\frac{3^n+2^n}{6^n}$. We have $\sum_{n=1}^{\infty}\frac{3^n+2^n}{6^n}=\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n+\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$ are geometric series with $|r|<1$, thus convergent and $\sum_{n=1}^{\infty}\frac{3^n+2^n}{6^n}=\frac{1/2}{1-1/2}+\frac{1/3}{1-1/3}=\frac{3}{2}$.

c) $\sum_{n=1}^{\infty}\left(\frac{3}{n(n+3)}+\frac{5}{4^n}\right)$. We have $\sum_{n=1}^{\infty}\left(\frac{3}{n(n+3)}+\frac{5}{4^n}\right)=\sum_{n=1}^{\infty}\frac{3}{n(n+3)}+\sum_{n=1}^{\infty}\frac{5}{4^n}$. Observe thatthe second series $\sum_{n=1}^{\infty}\left(\frac{5}{4^n}\right)$ is a convergent geometric series and $\sum_{n=1}^{\infty}\left(\frac{5}{4^n}\right)=\frac{5/4}{1-1/4}=\frac{5}{3}$. We need to test the harmonic series $\sum_{n=1}^{\infty}\left(\frac{3}{n(n+3)}\right)$. We use partial fraction to get $\frac{3}{n(n+3)}=\frac{1}{n}-\frac{1}{n+3}$.
Remember the telescoping process to find $\sum_{n=1}^{\infty}\left(\frac{3}{n(n+3)}\right).=1+\frac{1}{2}+\frac{1}{3}-\frac{1}{n+3}=\frac{11}{6}$, when $n\to\infty$. Thus the harmonic series is also convergent. The sum of the series is

$$\sum_{n=1}^{\infty}\left(\frac{3}{n(n+3)}+\frac{5}{4^n}\right)=\frac{11}{6}+\frac{5}{3}=\frac{7}{2}$$

3. Find the values of *x* for which the series $\sum_{n=1}^{\infty}(x-4)^n$ converges. Find the sum of the series for those values of *x*.

The given series is a geometric series will converge if $|r|=|x-4|<1\Rightarrow -1<x-4<1$.

Solving the inequality we find $3 < x < 5$ and the sum is $\sum_{n=1}^{\infty}(x-4)^n = \frac{x-4}{1-(x-4)} = \frac{x-4}{5-x}$.

The integral test and estimates of sums:

The integral test

Suppose f is continuous, positive decreasing function on [1, ∞] and let $a_n = f(n)$ then the series $\sum_{n=1}^{\infty} a_n$ is convergent iff the improper integral $\int_1^{\infty} f(x)dx$ is convergent. Otherwise it will be divergent.

The *p* – series

The *p* – series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ is convergent if *p*> 1. Otherwise it is divergent.

Reminder estimates for integral test

Suppose $f(k) = a_k$, f is a continuous positive decreasing function for $x \geq n$ and $\sum_{n=1}^{\infty} a_n$ is convergent. If $R_n = s - s_n$, where $\lim_{n\to\infty} s_n = s$, then $\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_n^{\infty} f(x)dx$.

Examples

1. Determine using integral test whether the series convergent or divergent

a) $\sum_{n=1}^{\infty}\frac{1}{n^4}$ is a p – series with p = 4 > 1, which is convergent. Now we will verify using integral test. $\int_1^{\infty}\frac{1}{x^4}dx = \lim_{b\to\infty}\left[-\frac{1}{3x^3}\right]_1^b = \lim_{b\to\infty}\left[-\frac{1}{3b^3}+\frac{1}{3}\right] = \frac{1}{3}$ converges, thus the series converges.

b) $\sum_{n=1}^{\infty}\frac{2}{n^{0.85}}$ is a p – series with p = 0.85 < 1, which is divergent. Now we will verify using integral test. $\int_1^{\infty}\frac{2}{x^{0.85}}dx = \lim_{b\to\infty}\left[\frac{2x^{.015}}{0.15}\right]_1^b = \lim_{b\to\infty}\left[\frac{2b^{0.15}}{0.15} - \frac{2}{0.15}\right] = \infty$ does not exist.

c) $\sum_{n=1}^{\infty}\frac{n+2}{n+1}$ is not a p – series. We will test convergence using integral test.

$\int_1^{\infty}\frac{n+2}{n+1}dx = \lim_{b\to\infty}\left[x+\ln|x+1|\right]_1^b = \infty$ does not exist. One may observe that the given series is not a decreasing series as well. The series is divergent. (One can use divergent test: $\lim_{n\to\infty}\frac{n+2}{n+1}=1\neq 0$)

d) $\sum_{n=1}^{\infty}\frac{\ln n}{n^2}$ is not a p – series. We will test the convergence using integral test.

$\int_2^{\infty}\frac{\ln x}{x^2}dx = \lim_{b\to\infty}\left[-\frac{\ln x}{x}-\frac{1}{x}\right]_2^b = \lim_{b\to\infty}\left[\frac{2b^{0.15}}{0.15}-\frac{2}{0.15}\right] \approx 0.85$ is finite and exists. It is convergent.

e) $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ is not a p – series. We will test the convergence using integral test.

$\int_2^{\infty}\frac{1}{x\ln x}dx = \lim_{b\to\infty}\left[\ln|\ln x|\right]_2^b = \infty$ does not exist. It is divergent.

2. Find *p* so that $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^p}$. For convergence $\int_2^{\infty}\frac{1}{x(\ln x)^p}dx = \lim_{b\to\infty}\int_2^{b}\frac{1}{x(\ln x)^p}dx = \lim_{b\to\infty}\left[\frac{(\ln x)^{1-p}}{1-p}\right]_2^{\infty}$ must exist. The integral will exist if $1-p<0 \Rightarrow p>1$

3. Approximate the sum of $\sum_{n=1}^{\infty}\frac{1}{n^3}$ by using the sum of first 10 terms. Estimate the error involved in this approximation. How many terms are required to ensure that the sum is accurate to within 0.0005.

$$\sum_{n=1}^{\infty}\frac{1}{n^3} = 1+\frac{1}{2^3}+\frac{1}{3^3}+\cdots\frac{1}{10^3}\approx 1.1975$$

Now $R_{10} = \int_{10}^{\infty}\frac{1}{x^3}dx = \frac{1}{200} = 0.005$, which is the at most size of the error.

For the required accuracy we need to have $R_n = \int_n^{\infty}\frac{1}{x^3}dx = \frac{1}{2n^2} < 0.005 \Rightarrow n > 31.6$. We need 37 terms.

The comparison test and the limit convergence test:

The comparison test

Suppose that $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ are series with positive terms.

i) If $\sum_{n=1}^{\infty} b_n$ is convergent and $a_n \le b_n$, then is $\sum_{n=1}^{\infty} a_n$ also convergent.

ii) If $\sum_{n=1}^{\infty} b_n$ is divergent and $a_n \ge b_n$, then is $\sum_{n=1}^{\infty} a_n$ also divergent.

The limit comparison test

Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c, \quad c > 0$, finite and positive, then either both series converges or both diverges.

Examples

1. Test the convergence of the series

a) $\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$. We use comparison test, consider $b_n = \frac{5}{2n^2}$, which is a convergent *p* – series with *p* = 2. And also observe that $a_n \le b_n$. So the given series is convergent.

b) $\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$. We use comparison test, consider $b_n = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}}$, which is a divergent *p* – series with *p* = 1/2. And also observe that $a_n \ge b_n$. So the given series is divergent.
To check the inequality one can verify the result

$$\frac{2n^2 + 3n}{\sqrt{5 + n^5}} \ge \frac{2}{\sqrt{n}} \Rightarrow \left(\frac{2n^2 + 3n}{\sqrt{5 + n^5}}\right)^2 \ge \left(\frac{2}{\sqrt{n}}\right)^2$$

$$\Rightarrow 12n^4 + 9n^3 \ge 20$$

is true.

Or, we may use limit test: $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, as $b_n = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}}$ is a divergent series, and the given series is also divergent.

Test the following series:

c) $\sum_{n=1}^{\infty} \frac{1}{n^2 + n + 1}$ is convergent d) $\sum_{n=1}^{\infty} \frac{2}{n^3 + 4}$ is convergent e) $\sum_{n=2}^{\infty} \frac{1}{n - \sqrt{n}}$ is divergent

f) $\sum_{n=1}^{\infty} \frac{n+1}{n^2}$ is divergent g) $\sum_{n=2}^{\infty} \frac{n^2 + 1}{n^3 - 1}$ is divergent h) $\sum_{n=0}^{\infty} \frac{1 + \sin n}{10^n}$ is convergent (Hint: consider $b_n = \frac{2}{10^n}$ a convergent geometric series with $|r| = 0.1 < 1$, $a_n \le b_n$)

i) $\sum_{n=0}^{\infty}\frac{2+(-1)^n}{n\sqrt{n}}$ is convergent j) $\sum_{n=0}^{\infty}\frac{1+2^n}{1+3^n}$ is convergent

The alternating series test :

An alternating series is a series whose terms are alternately positive and negative. An alternating series $\sum_{n=0}^{\infty}(-1)^n b_n$ is convergent if i) $b_{n+1}\le b_n$ and ii) $\lim_{n\to\infty} b_n =0$, for all n and $b_n >0$

Estimation: If $s=\sum_{n=0}^{\infty}(-1)^{n-1} b_n$ is the sum of an alternating series, with i) $0\le b_{n+1}\le b_n$ and ii) $\lim_{n\to\infty} b_n =0$ then $|R_n|=|s-s_n|\le b_{n+1}$

Examples

1. Test the convergence of the series

a) $\sum_{n=1}^{\infty}\frac{(-1)^n n}{n+2}$. We have $b_n =\frac{n}{n+2}$, and $\lim_{n\to\infty}\frac{n}{n+2}=1\ne 0$. So the given series is divergent by the alternating series.

b) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$. We have $b_n =\frac{1}{\sqrt{n}}$, and $\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0$, $b_{n+1}\le b_n$. So the given series is convergent by the alternating series.

c) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{3^{n-1}}$. We have $b_n =\frac{1}{3^{n-1}}$, and $\lim_{n\to\infty}\frac{1}{3^{n-1}}=0$, $b_{n+1}\le b_n$. So the given series is convergent by the alternating series.

d) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2n}{4n^2+1}$. We have $b_n =\frac{2n}{4n^2+1}$, and $\lim_{n\to\infty}\frac{2n}{4n^2+1}=0$, $b_{n+1}\le b_n$. So the given series is convergent by the alternating series.

e) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}e^{1/n}}{n}$. We have $b_n =\frac{e^{1/n}}{n}$, and $\lim_{n\to\infty} b_n =0$, $b_{n+1}\le b_n$. So the given series is convergent by the alternating series.

f) $\sum_{n=1}^{\infty}\frac{\sin(n\pi/2)}{n!}$. We have $\sum_{n=1}^{\infty}(-1)^n b_n =\sum_{n=1}^{\infty}\frac{\sin(n\pi/2)}{n!}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}$, and $\lim_{n\to\infty}\frac{1}{n!}=0$, $b_{n+1}\le b_n$. So the given series is convergent by the alternating series.

g) $\sum_{n=1}^{\infty}\frac{(-1)^n n^n}{5^n}$. We have $b_n=\left(\frac{n}{5}\right)^n$, and $\lim_{n\to\infty} b_n \neq 0$. So the given series is divergent by the alternating series.

h) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{3/2}}$. We have $b_n=\frac{1}{n^{3/2}}$, and $\lim_{n\to\infty} b_n = 0$, $b_{n+1}\le b_n$. So the given series is convergent by the alternating series.

The Absolute convergence: Ratio and root test :

1. A series $\sum_{n=1}^{\infty} a_n$ is called **absolutely convergent** if the series of absolute values $\sum_{n=1}^{\infty}|a_n|$ is convergent.
2. A series $\sum_{n=1}^{\infty} a_n$ is called **conditionally convergent** if the series is convergent but not absolutely convergent
3. **Theorem** If a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent then the series is convergent.

The ratio test:

i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and therefore convergent.

ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$, or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$ then the series $\sum_{n=1}^{\infty} a_n$ is divergent

iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L = 1$, then the ratio test is inconclusive.

The root test:

i) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and therefore convergent.

ii) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L > 1$, or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$ then the series $\sum_{n=1}^{\infty} a_n$ is divergent

iii) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L = 1$, then the root test is inconclusive.

Examples

1. Test the convergence of the series

a) $\sum_{n=1}^{\infty}\frac{(-1)^n n^3}{3^n}$. We have $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1/3 < 1$. So the given series is absolutely convergent by **the ratio** test.

b) $\sum_{n=1}^{\infty}(-1)^n\frac{(3n+2)^n}{(4n+3)^n}$. We have $\lim_{n\to\infty}\sqrt[n]{|a_n|} = 3/4 < 1$. So the given series is absolutely convergent by **the root** test.

2. Apply ratio test to verify that the given series are absolutely convergent and thereby convergent.

a) $\sum_{n=1}^{\infty}\frac{1}{(3n)!}$ b) $\sum_{n=1}^{\infty}\frac{(-10)^n}{n!}$ c) $\sum_{n=1}^{\infty}\frac{(-1)^n 3^n n}{4^{n-1}}$ d) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1} n^2 2^n}{n!}$

e) $\sum_{n=1}^{\infty}\frac{(-1)^n n^3}{3^n}$

3. Show that following series are conditionally convergent:

a) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt[4]{n}}$. We have $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, by ratio test the it is inconclusive. But by absolute convergence test $\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\frac{1}{n^{1/4}}$ is a p - series with p =1/4 < 1, divergent, on the other hand by alternating series test it is convergent, since $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n^{1/4}} = 0,\ b_n > b_{n+1}$. So the given series is conditionally convergent.

b) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1} n}{n^2+1}$. We have $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, by ratio test the it is inconclusive. But by limit comparision test (Section 11.4) $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n/(n^2+1)}{1/n} = 1 > 0$ is divergent since $b_n = \frac{1}{n}$ is a divergent p - series with p =1. On the other hand by alternating series test it is convergent, since $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{n}{n^2+1} = 0,\ b_n > b_{n+1}$. So the given series is conditionally convergent.

Strategy for Testing Series:

We have learnt the following:

1. $\sum_{n=1}^{\infty}\frac{1}{n^p}\ b_n = \frac{1}{n}$ is a divergent $p-$ series, converges when p > 1 and diverges when $p \le 1$.
2. $\sum_{n=1}^{\infty} ar^{n-1}$ or $\sum_{n=1}^{\infty} ar^n$ is a geometric series, converges when $|r| < 1$, diverges when $|r| \ge 1$.
3. If $\lim_{n\to\infty} b_n \ne 0$ the series diverges (Divergent test)

4. Series with factorials use ratio test

Test the convergence off the following series:

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)}{n^2+1}$, by alternating series test the series is convergent

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n-1}{n^2+1} = 0,\ b_n > b_{n+1}, n \geq 3$

2. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)}{n^2+1}$, by alternating series test the series is convergent

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n-1}{n^2+1} = 0,\ b_n > b_{n+1}, n \geq 3$.

3. $\sum_{n=1}^{\infty} \frac{n-1}{n^2+1}$, by limit comparison test $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n-1}{n^2+1} \cdot \frac{n}{1} = 1 > 0$, the series is divergent

since $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n}$ is a divergent p – series with p =1.

4. $\sum_{n=1}^{\infty} \frac{2^n n!}{(n+3)!}$, by ratio test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2^{n+1}(n+1)!}{(n+4)!} \cdot \frac{(n+3)!}{2^n n!} = 2 > 1 > 0$, the series is divergent.

The Power Series :

A power series is a series of the form $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$ where x is a variable and c_n's are called the coefficients of the series. The sum of a power series is a function $f(x) = c_0 + c_1 x + c_2 x^2 + \cdots$ whose domain is the set of all x for which the series is convergent.

A power series of the form $\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$ is power series in (*x* – *a*) or a power series centered at *a* or a power series about *a*.

Theorem: The power series $\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$ may have three possibilities:

1) the series converges for $x = a$
2) the series converges for all x
3) There exists R > 0 such that the series converges if $|x - a| < R$, diverges for $|x - a| > R$

The number R is called the radius of convergence.

There four possibilities of interval of convergence a) I = ($a - R$, $a + R$), b)) I = ($a - R$, $a + R$], c) I = [$a - R$, $a + R$), d)) I = [$a - R$, $a + R$]

Examples

1. Find the radius of convergence and interval of convergence of the power series

a) $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n}$. The power series is convergent if (by absolute convergent test)

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-5)^{n+1}}{n+1}\cdot\frac{n}{(x-5)^n}\right| = |x-5| < 1$, The radius of convergence $R = 1$ and the possible interval of convergence is (4, 6). We need to check the convergence separately for x = 4 and
x =6.

When x= 4 we have our series $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, by alternating series test it is convergent.

When x= 6 we have our series $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$, Harmonic series which is divergent.Therefore the interval of convergence is I = [4, 6)

b) $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n+1}$. The alternating power series is convergent if (by absolute convergent test)

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+2}\cdot\frac{n+1}{x^n}\right| = |x| < 1$, The radius of convergence $R = 1$ and the possible interval of convergence is (-1, 1). We need to check the convergence separately for x = -1 and
x =1

When x= -1 we have our series $\sum_{n=1}^{\infty} \frac{1}{n+1}$, a harmonic series, divergent.

When x= 1 we have our series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$, convergent by alternating series test. The interval of convergence is I = (-1, 1]

c) $\sum_{n=1}^{\infty}\sqrt{n+1}x^{n+1}$. The power series is convergent if (by Ratio test)

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\sqrt{n+2}\,x^{n+2}}{\sqrt{n+1}\,x^{n+1}}\right| = |x| < 1$, The radius of convergence R = 1 and the possible interval of convergence is (-1, 1). We need to check the convergence separately for x = -1 and x =1.

When x= -1 we have our series $\sum_{n=1}^{\infty}(-1)^{n+1}\sqrt{n+1}$, the alternating series is divergent.

When x= 1 we have our series $\sum_{n=1}^{\infty}\sqrt{n+1}$, also diverges, the interval of convergence is I = (-1, 1)

d) Check that $\sum_{n=1}^{\infty}n^3(x-5)^n$ has radius of convergence R= 1 and interval of convergence is I = (4, 6)

e) The series $\sum_{n=2}^{\infty}(-1)^n\frac{(2x+3)^n}{n\ln n}$ has radius of convergence R = 1, and interval of convergence is given by I = (-2, -1] (Hint: check with problem number 21, section 11.3 and problem number 22, section 11.6)

Representation of a function as a Power Series:

Representations of functions as power series

In this section we have the following representations: We use radius of convergence = R, and interval of convergence is I.

1. $f(x) = e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty}\frac{x^n}{n!}$, R = 1, $I = (-\infty, \infty)$
2. $f(x) = \frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n \cdots = \sum_{n=0}^{\infty}x^n$, $|x| < 1$, R = 1
3. $f(x) = \frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots + x^n \cdots = \sum_{n=0}^{\infty}(-1)^n x^n$, $R = 1$,
4. $f(x) = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n}\cdots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$, R = 1,
5. $f(x) = \ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots - \frac{x^n}{n}\cdots = -\sum_{n=1}^{\infty}\frac{x^n}{n}$, R = 1

6. $f(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} \cdots = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

7. $f(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

Differentiation and integration of power series:

The sum of a power series is a function

$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots$ whose domain is the interval of convergence of the series.

Theorem: The power series $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ has radius of convergence R > 0, then the function is differentiable and therefore continuous on the interval (*a – R, a + R*).

Observe that

1) $f'(x) = \sum_{n=1}^{\infty} nc_n (x-a)^{n-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$

2) $\int f(x)\,dx = c + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} = c + c_0 x + c_1 \frac{(x-a)^2}{2} + c_2 \frac{(x-a)^3}{3} + \cdots$

3. $\frac{d}{dx} \sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} \frac{d}{dx} c_n (x-a)^n$

4. $\int \sum_{n=0}^{\infty} c_n (x-a)^n dx = \sum_{n=0}^{\infty} \int c_n (x-a)^n dx$

Examples

1. Find a power series representation of the function and determine interval of convergence.

a) $f(x) = \frac{3}{1-x^4}$

$= 3(1-x^4) = 3\sum_{n=0}^{\infty} (x^4)^n$, R = 1, and I =(-1, 1)

b) $f(x) = \frac{x^2}{a^3 - x^3}$

$= x^2 a^3 \left(1 - \left(\frac{x}{a}\right)^3\right)^{-1} = x^2 a^3 \sum_{n=0}^{\infty} \left(\frac{x}{a}\right)^{3^n}$, R = $|a^3|$ 1, and I =(- $|a^3|$, $|a^3|$)

c) $f(x) = \dfrac{7x-1}{3x^2+2x-1)}$

$$f(x) = \frac{7x-1}{3x^2+2x-1)} = \frac{7x-1}{(3x-1)(x+1)} = \frac{2}{1+x} + \frac{1}{3x-1} = 2(1+x)^{-1} - (1-3x)^{-1}$$

$$= 2\sum_{n=0}^{\infty}(-1)^n x^n - \sum_{n=0}^{\infty}(3x)^n$$

We have that the interval of convergence of $2\sum_{n=0}^{\infty}(-1)^n x^n$ is $I = (-1,1)$ and $\sum_{n=0}^{\infty}(3x)^n$ is $I = (-1/3,\ 1/3)$. Thus the interval of convergence of the given series is $I = (-1/3,\ 1/3)$.

d) $f(x) = \dfrac{x^3}{2+x}$

$f(x) = \dfrac{x^3}{2(1+x/2)} = \dfrac{x^3}{2}(1+x/2)^{-1} = \dfrac{x^3}{2}\sum_{n=0}^{\infty}(-1)^n (x/2)^n$. Thus the interval of convergence of the given series is $I = (-2,\ 2)$.

e) $f(x) = \dfrac{x^2}{(1-2x)^2}$

We consider

$$\frac{1}{(1-2x)} = (1-2x)^{-1} = \sum_{n=0}^{\infty}(2x)^n$$

Differentiating on both sides we find

$$\frac{2}{(1-2x)^2} = \sum_{n=0}^{\infty}2n(2x)^{n-1} \Rightarrow \frac{1}{(1-2x)^2} = \sum_{n=0}^{\infty}n(2x)^{n-1}$$

$f(x) = \dfrac{x^2}{(1-2x)^2} = \sum_{n=0}^{\infty}nx^2(2x)^{n-1} = \dfrac{1}{2}\sum_{n=0}^{\infty}nx(2x)^n$, Thus the interval of convergence of the given series is $I = (-1/2, 1/2)$.

2. a) Find a power series representation of $f(x) = \ln(1+x)$, what is the radius of convergence of ? b) Find the power series representation of $f(x) = \ln(1+x^2)$

c) Find the power series representation of $f(x) = x\ln(1+x)$

We have $F(x) = \dfrac{1}{1+x} = (1+x)^{-1} = \sum_{n=0}^{\infty}(-1)^n x^n$

Integrating on both sides w. r. to x, we find

$$\ln(1+x) = \int\frac{1}{1+x}dx = \sum_{n=0}^{\infty}(-1)^n\int x^n dx = \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1} + C$$

using the initial condition $x = 0$, we find $C = 0$. Then

$$\ln(1+x) = \int \frac{1}{1+x}dx = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n},\ with\ R=1$$

b) Using result from a) we can write $\ln(1+x^2) = \sum_{n=0}^{\infty}(-1)^n \frac{(x^2)^{n+1}}{n+1} = \sum_{n=2}^{\infty}(-1)^{n-1}\frac{x^{2n}}{n},\ with\ R=1$

c) Using result from a) we can write $x\ln(1+x) = x\sum_{n=0}^{\infty}(-1)^n \frac{(x)^{n+1}}{n+1} = \sum_{n=2}^{\infty}(-1)^{n}\frac{x^{n}}{n-1},\ with\ R=1$

3. Evaluate as a power series and find radius of convergence.

   a) $\int \frac{\ln(1-x)}{x}dx$

We have $\ln(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n} \Rightarrow \frac{\ln(1-x)}{x} = \sum_{n=1}^{\infty}\frac{x^{n-1}}{n}$

Now $\int \frac{\ln(1-x)}{x}dx = -\sum_{n=1}^{\infty}\int\frac{x^{n-1}}{n}dx = -\sum_{n=1}^{\infty}\frac{x^n}{n^2},\ where\ R=1$

   b) $\int \frac{x-\tan^{-1}x}{x^3}dx$

We have $\tan^{-1}x = \int\frac{1}{1+x^2}dx = \int(1+x^2)^{-1}dx = \int\sum_{n=0}^{\infty}(-1)^n x^{2n}dx = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1},\ C=0$

Now $\int\frac{x-\tan^{-1}x}{x^3}dx = C+\sum_{n=0}^{\infty}(-1)^{n+1}\int\frac{x^{2n-2}}{2n+1}dx = C+\sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n-1}}{4n^2-1}$

Part-III: Fourier Series

1.1 Introduction

A Taylor series is an infinite series of the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

where a_o, a_1, a_2, … are constants, called the **coefficients** of the series. A Taylor series does not include terms with negative powers. A Fourier series is an infinite series expansion in terms of trigonometric functions

$$f(x) = a_o + \sum_{n=1}^{\infty}(a_n\cos(nx) + b_n\sin(nx)) \quad (1.1\text{-}1)$$

Any piecewise smooth function defined on a finite interval has a Fourier series expansion.

Periodic Functions:-

A function satisfying the identity $f(x) = f(x + T)$ for all x, where $T > 0$, is called periodic or T-periodic as shown in Figure 1.1-1.

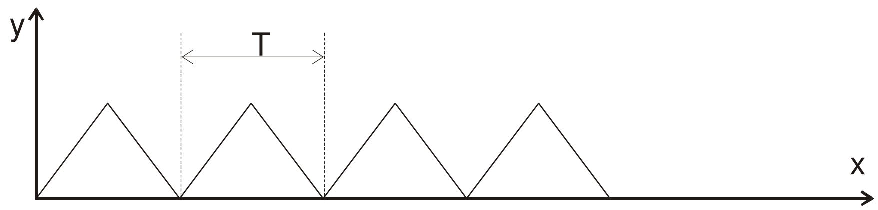


Figure 1.1-1 A T-periodic function.

For a T-periodic function

$$f(x) = f(x + T) = f(x + 2T) = f(x + 3T) = \dots = f(x + nT)$$

If T is a period then nT is also a period for any integer $n > 0$. T is called a fundamental period. The definite integral of a T-periodic function is the same over any interval of length T. Example 1.1-1 will use this property to integrate a 2-periodic function shown in Figure 1.1-2.

Example 1.1-1: Let f be the 2-periodic function and N is a positive integer. Compute $\int_{-N}^{N} f^2(x)dx$ if $f(x) = -x + 1$ on the interval $0 \le x \le 2$

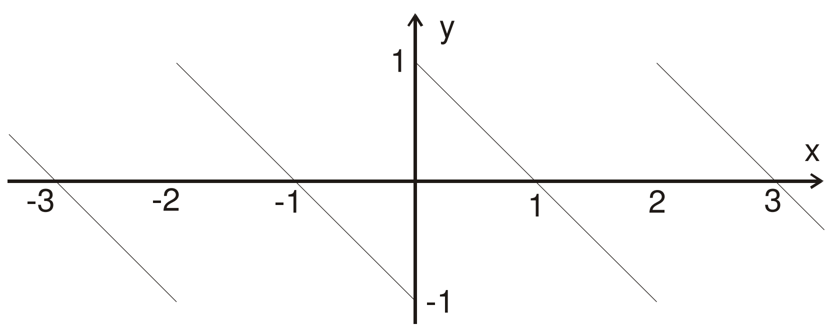


Figure 1.1-1. A 2-periodic function.

Solution:

$$\int_{-N}^{N} f^2(x)dx = \int_{-N}^{-N+2} f^2(x)dx + \int_{-N+2}^{-N+4} f^2(x)dx + \dots + \int_{N-2}^{N} f^2(x)dx$$

$$\int_{-N}^{N} f^2(x)dx = N\int_{N-2}^{N} f^2(x)dx = N\int_{0}^{2}(-x+1)^2 dx = N\left\{-\frac{1}{3}(-x+1)^3\right\}\Bigg|_0^2$$

$$\int_{-N}^{N} f^2(x)dx = -\frac{N}{3}[-1-1] = \frac{2}{3}N$$

The most important periodic functions are those in the (2π-period) trigonometric system

1, cos x, cos $2x$, cos $3x$, …, cos mx, …,

sin$x$, sin $2x$, sin $3x$, …, sin nx, …,

Orthogonal functions:-

If $\int_a^b f(x)g(x)dx = 0$ then f and g are orthogonal over the interval [a, b].

Examples of orthogonal functions:

$$\int_{-\pi}^{\pi} \cos mx \cos nx dx = 0 \text{ for } m \neq n$$

$$= \pi \text{ for } m = n$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx dx = 0 \text{ for } m \neq n$$

$$= \pi \text{ for } m = n$$

$$\int_{-\pi}^{\pi} \cos mx \sin nx dx = 0 \text{ for all } m \text{ and } n$$

Fourier series are special expansions of functions of the form

$$f(x) = a_o + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) \qquad (1.1\text{-}1)$$

where the coefficients a_0, a_1, a_2, …, b_1, b_2, … must be evaluated.

The coefficient a_0 is determined by integrating both sides of Eq. (2.1-1) over the interval $[-\pi, \pi]$.

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} a_0 dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} (a_n \cos(nx) + b_n \sin(nx))dx$$

Since $\int_{-\pi}^{\pi} \cos nx dx = \int_{-\pi}^{\pi} \sin nx dx = 0$ for $n = 1, 2, \ldots$

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} a_0 dx = 2\pi a_0 \Rightarrow a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

The coefficient a_n is determined by multiplying both sides of Eq. (2.1-1) with cos mx and integrating the resulting equation over the interval $[-\pi, \pi]$.

$$\int_{-\pi}^{\pi} f(x)\cos(mx)dx = \int_{-\pi}^{\pi} a_0 \cos(mx)dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} a_n \cos(nx)\cos(mx)dx$$

$$+ \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} b_n \sin(nx)\cos(mx)dx$$

Since $\int_{-\pi}^{\pi} \cos mx dx = 0$, $\int_{-\pi}^{\pi} \cos mx \sin nx dx = 0$ for all m and $\int_{-\pi}^{\pi} \cos mx \cos nx dx = 0$ for $m \neq n$

$$\int_{-\pi}^{\pi} f(x)\cos(mx)dx = a_n \int_{-\pi}^{\pi} (\cos nx)^2 dx = \pi a_n$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)dx = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\cos(mx)dx$$

Similarly the coefficient b_n is determined by multiplying both sides of Eq. (2.1-1) with sin mx and integrating the resulting equation over the interval $[-\pi, \pi]$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)dx = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\sin(mx)dx$$

Example 1.1-2. Solve the one dimensional heat equation with no heat generation, zero boundary conditions (0^oC), and constant initial temperature of 100^oC.

Solution:

The partial differential equation for one-dimensional heat conduction is

$$\rho C_p \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} + Q(x, t)$$

Since there is no heat generation $Q(x, t) = 0$

$$\rho C_p \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} \Rightarrow \frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2} \text{ where } \alpha = \frac{k}{\rho C_p}$$

The boundary and initial conditions are

$$T(0, t) = T(L, t) = 0^oC; \; T(x, 0) = 100^oC.$$

The solution for the temperature is

$$T(x, t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\exp\left[-\left(\frac{n\pi}{L}\right)^2 \alpha t\right]$$

At $t = 0$, $$T(x, 0) = 100 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

This is the Fourier sine series (with $a_0, a_1, a_2, \ldots = 0$) where the coefficients $b_1, b_2, \ldots$ can be determined by by multiplying both sides of the above equation with sin mx and integrating the resulting equation over the interval $[0, L]$.

$$100\int_0^L \sin\frac{m\pi x}{L}dx = \sum_{n=1}^{\infty} b_n\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx$$

Since $\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx = 0$ for $m \neq n$

$$100\int_0^L \sin\frac{n\pi x}{L}dx = b_n\int_0^L \left(\sin\frac{n\pi x}{L}\right)^2 dx$$

Using the identity $\sin^2 x = \frac{1-\cos 2x}{2}$, the above equation becomes

$$100\frac{L}{n\pi}\left[-\cos\frac{n\pi x}{L}\right]_0^L = b_n\int_0^L \frac{1}{2}\left(1-\cos\frac{2n\pi x}{L}\right)dx$$

$$100\frac{L}{n\pi}[-\cos n\pi + 1] = \frac{b_n}{2}\left[x - \frac{L}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right)\right]_0^L = \frac{b_n}{2}L$$

$$b_n = \frac{200}{n\pi}[1 - \cos n\pi]$$

For n = even, $\cos(n\pi) = 1 \Rightarrow b_n = 0$

For n = odd, $\cos(n\pi) = -1 \Rightarrow b_n = \frac{400}{n\pi}$

The Fourier expansion for 100 is then given by

$$f(x) = 100 = \sum_{n=1}^{\infty} b_{2n-1}\sin\frac{(2n-1)\pi x}{L} \text{ where } b_{2n-1} = \frac{400}{(2n-1)\pi}$$

The plot of $f(x) = \sum_{n=1}^{51} b_{2n-1}\sin\frac{(2n-1)\pi x}{L}$ for 51 terms is a good approximation of 100 away from the end points as shown in Figure 1.1-1. There is a 18 % overshoot called Gibbs phenomenon near the end points. Gibbs phenomenon occurs only when a finite series of eigenfunctions approximates a discontinuous function.

Fourier series forthe interval (-l, l):-

Let f(x) be a periodic function defined in the interval (-l,l) i.e. 2l , Then the Fourier series is given by

$f(x) = \frac{ao}{2} + \sum_{n=1}^{\infty}(a_n cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l})$

Where a_0, a_n, b_n are Fourier coefficients and their values are given by

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)cos\frac{n\pi x}{l}dx$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx$$

Now Fourier series for period 2π:

Simply l is replace by π in above equations, we have

f(x) = $\frac{ao}{2}$ + $\sum_{n=1}^{\infty}(a_n cos\frac{n\pi x}{\pi} + b_n \sin\frac{n\pi x}{\pi})$

f(x) = $\frac{ao}{2}$ + $\sum_{n=1}^{\infty}(a_n cosnx + b_n \sin nx)$

Where a_0, a_n, b_n are Fourier coefficients and their values are given by

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cosnxdx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

Fourier Series for General Interval: For(k,k+2l)

f(x) = $\frac{ao}{2}$ + $\sum_{n=1}^{\infty}(a_n cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l})$

Where a_0, a_n, b_n are Fourier coefficients and their values are given by

$$a_0 = \frac{1}{l}\int_{k}^{k+2l} f(x)dx$$

$$a_n = \frac{1}{l}\int_{k}^{k+2l} f(x)cos\frac{n\pi x}{l}dx$$

$$b_n = \frac{1}{l}\int_{k}^{k+2l} f(x)\sin\frac{n\pi x}{l}dx$$

Dirichlet'sConditions:-

Any periodic waveform of period *p* = 2*L*, can be expressed in a Fourier series provided that

(a) it has a finite number of discontinuities within the period 2*L*;

(b) it has a finite average value in the period 2*L*;

(c) it has a finite number of positive and negative maxima and minima.

When these conditions, called the Dirichlet's conditions, are satisfied, the Fourier series for the function *f*(*t*) exists.

Each of the examples in this chapter obey the Dirichlet's Conditions and so the Fourier Series exists.

Q1.: Find the Fourier series for e^x in the interval $(-\pi, \pi)$.

Solution: Suppose

f(*x*) = $\frac{ao}{2} + \sum_{n=1}^{\infty}(a_n cosnx + b_n \sin nx)$

Where a_0, a_n, b_n are Fourier coefficients and their values are given by

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x dx$$

$$a_0 = \frac{1}{\pi}(2\, sinh\pi)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x cosnxdx$$

After solving,

$$a_n = \frac{2(-1)^n \sinh \pi}{\pi(1+n^2)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin nx\, dx$$

After Solving,

$$b_n = \frac{-2(-1)^n n \sinh \pi}{\pi(1+n^2)}$$

Now putting values in equation

$$e^x = \frac{1}{\pi}(sinh\pi) + \sum_{n=1}^{\infty}(\frac{2(-1)^n \sinh \pi}{\pi(1+n^2)} cosnx + \frac{-2(-1)^n n \sinh \pi}{\pi(1+n^2)} sinnx)$$

This is the required Fourier series.

Q.2 :Find the Fourier series for the function defined as

f(x) = -1 for $-\pi \le x < 0$

0 for x = 0

1 for $0 < x \le \pi$

Solution: Let f(x) = $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n cosnx + b_n sinnx)$, where

$a_0 = \frac{2}{2\pi}[\int_{-\pi}^{0}(-1)\,dx + \int_{0}^{\pi} 1.dx] = 0,$

$a_n = \frac{1}{\pi}[\int_{-\pi}^{0}(-1)\,cosnxdx + \int_{0}^{\pi} cosnx\,dx = 0$,$a_1 = a_2 = \ldots = 0$

$b_n = \frac{1}{\pi}[\int_{-\pi}^{0}(-1)\,sinnxdx + \int_{0}^{\pi} sinnx\,dx = 2[1-(-1)^n]/n\pi$

Ifn is even , $(-1)^n = 1$, $b_n = 0$, i.e. , $b_2 = b_4 = \ldots = 0$

If n is odd. $(-1)^n = -1$, $b_n = \frac{4}{n\pi}$, $b_1 = \frac{4}{\pi}$ etc. $a_n = \frac{1}{\pi}[\int_{-\pi}^{0}(-1)\,cosnxdx + \int_{0}^{\pi} cosnx\,dx = 0$

Putting $x = \frac{\pi}{2}$ in (2) we get $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$ ....

Q.3: Find the Fourier expansion of the Function:

$$f(x) = -\pi \quad for -\pi \le x < 0$$
$$= x \quad for 0 < x \le \pi$$

Solution: Let $a_0 = \frac{1}{\pi}\int_{-\pi}^{0}(-\pi)\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\,dx = -\frac{\pi}{2}$

$a_n = \frac{1}{\pi}[\int_{-\pi}^{0} -\pi cosnxdx + \int_{0}^{\pi} xcosnxdx] = \frac{cosn\pi - 1}{\pi n^2}$

If n is even, $cosn\pi = 1$, then $a_n = 0$ i.e. $a_2 = a_4 = \ldots = 0$

IF n is odd, cos $n\pi$ = -1 , then $a_n = \frac{-2}{\pi n^2}$

$b_n = \frac{1}{\pi}[\int_{-\pi}^{0} -\pi sinnxdx + \int_{0}^{\pi} xsinnxdx] = \frac{1-2\cos n\pi}{n}$

If n is even, cos $n\pi = 1, then,$ $b_n = \frac{-1}{n}$

If n is odd , cos n$\pi = -1$, then $b_n = \frac{3}{n}$, putting all these values we get

f(x) = $\frac{-\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2}cosx + \frac{1}{3^2}cos3x + \cdots \ldots \ldots.\right] + [\frac{3}{1}sinx - \frac{1}{2}sin2x + \cdots \ldots \ldots.]$.............(3)

putting x = 0 in R.H.S. of (3) we get $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \ldots$ ....(4)

IIIY if we put x = π in (3), we get (4)

Q.4: Develop the Fourier series for the function $f(x) = x + x^2$ in the interval $-\pi < x < \pi$. Hence show that $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots.$

Solution: Let f(x) = x +x^2, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2 \ ) \, dx = \frac{2\pi^2}{3}$

$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 cosnxdx = \frac{4}{n^2}(-1)^n$

$b_n = \frac{2}{\pi}\int_0^{\pi} xsinnxdx = \frac{-2}{n}(-1)^n$, putting all these values in the series we get

x + x^2= $\frac{\pi^2}{3} - 4\left(cosx - \frac{1}{2^2}cos2x \ldots \ldots \ldots \ldots\right) + 2\left(sinx - \frac{1}{2}sin2x \ldots \ldots \ldots ..\right)$..................(1)

Putting x= $\pi \& x = -\pi$, successively, in (1) we get

$\pi + \pi^2 = \frac{\pi^2}{3} + 4\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots..\right)$......(2)

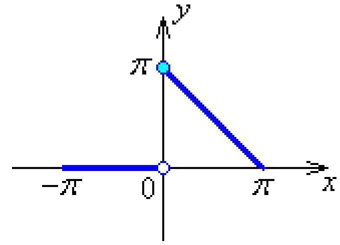
& $-\pi + \pi^2 = \frac{\pi^2}{3} + 4\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots.\right)$..........(3)

Adding (2) & (3)

we get $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots\ldots..$

Q5. Expand $f(x) = \begin{cases} 0 & (-\pi < x < 0) \\ \pi - x & (0 \le x < +\pi) \end{cases}$ in a Fourier series.

Solution: *Here L* $=\pi$.



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi}(\pi - x)\,dx$$

$$= 0 + \frac{1}{\pi}\left[\frac{(\pi - x)^2}{-2}\right]_0^{\pi} = \frac{\pi}{2}$$

D	I
$\pi - x$	$\cos nx$
-1	$\frac{1}{n}\sin nx$
0	$-\frac{1}{n^2}\cos nx$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0 + \frac{1}{\pi}\int_0^{\pi}(\pi - x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{n(\pi - x)\sin nx - \cos nx}{n^2}\right]_0^{\pi} = \frac{1 - (-1)^n}{n^2\pi}$$

D	I
$\pi - x$	nx
-1	$-\frac{1}{n}\cos nx$
0	$-\frac{1}{n^2}\sin nx$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0 + \frac{1}{\pi}\int_0^{\pi}(\pi - x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{n(\pi - x)\cos nx + \sin nx}{-n^2}\right]_0^{\pi} = \frac{1}{n}$$

Therefore the Fourier series for $f(x)$ is

$$\boxed{f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2\pi}\cos nx + \frac{1}{n}\sin nx\right) \qquad (-\pi < x < +\pi)}$$

Q6. Find the Fourier series expansion for the standard square wave,

$$f(x) = \begin{cases} -1 & (-1 < x < 0) \\ +1 & (0 \le x < +1) \end{cases}$$

Solution: Here $L = 1$.

The function is odd ($f(-x) = -f(x)$ for all x).

Therefore an = 0 for all n. We will have a Fourier sine series only.

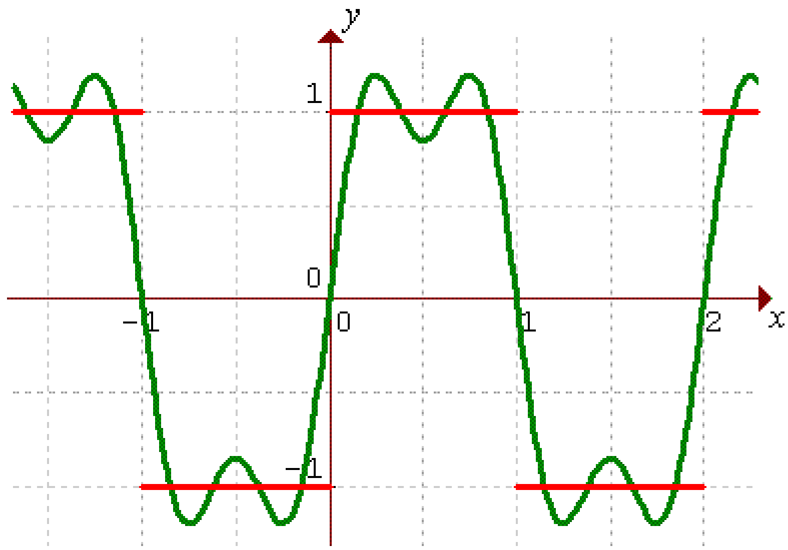
$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin n\pi x\,dx = \int_{-1}^{0} -\sin n\pi x\,dx + \int_{0}^{1} \sin n\pi x\,dx$$

(can use symmetry)

$$= \left[\frac{\cos n\pi x}{n\pi}\right]_{-1}^{0} + \left[\frac{-\cos n\pi x}{n\pi}\right]_{0}^{1} = \frac{2\left(1-(-1)^n\right)}{n\pi}$$

$$\Rightarrow \quad f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n}\sin n\pi x\right) = \frac{4}{\pi}\sum_{k=1}^{\infty}\left(\frac{1}{2k-1}\sin(2k-1)\pi x\right)$$

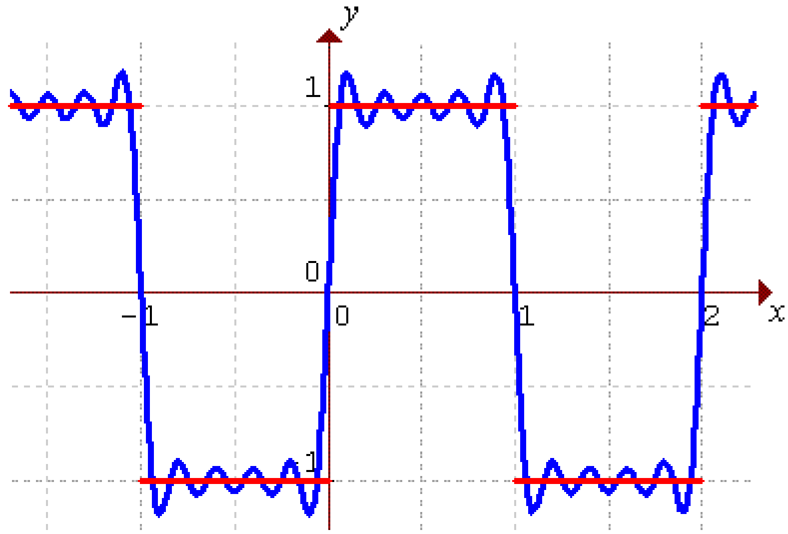
The graphs of the third and ninth partial sums (containing two and five non-zero terms respectively) are displayed here, together with the exact form for $f(x)$, with a **periodic extension** beyond the interval (–1, +1) that is appropriate for the square wave.



$y = S_3(x)$

<u>Q.6</u> (continued)

$y = S_9(x)$

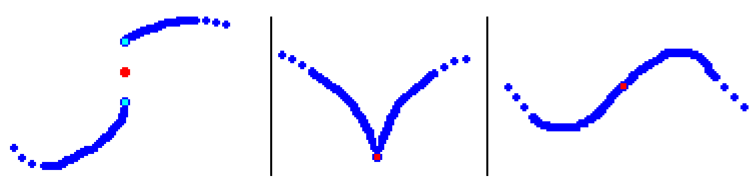


Convergence :-

At all points $x = x_0$ in $(-L, L)$ where $f(x)$ is continuous and is either differentiable or the limits $\lim_{x \to x_0^-} f'(x)$ and $\lim_{x \to x_0^+} f'(x)$ both exist, the Fourier series converges to $f(x)$.

At finite discontinuities, (where the limits $\lim_{x \to x_0^-} f'(x)$ and $\lim_{x \to x_0^+} f'(x)$ both exist), the Fourier series converges to $\dfrac{f(x_0-) + f(x_0+)}{2}$,

(using the abbreviations $f(x_0-) = \lim_{x \to x_0^-} f(x)$ and $f(x_0+) = \lim_{x \to x_0^+} f(x)$).



$f(x)$ not continuous at $x = x_0$ continuous but not differentiable continuous and differentiable

In all cases, the Fourier series at $x = x_0$ converges to $\dfrac{f(x_0-) + f(x_0+)}{2}$ (the red dot).

Note :- It is note that the cosine functions (and the function 1) are even, while the sine functions are odd.

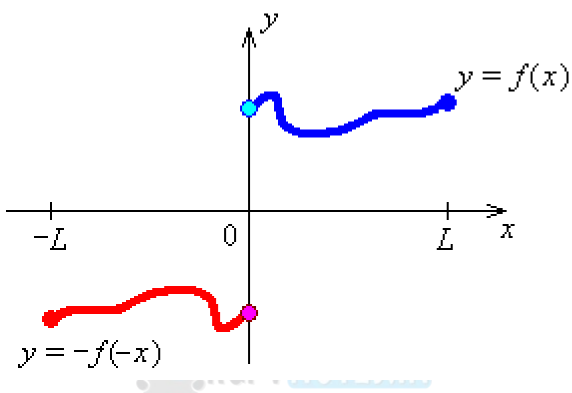
If $f(x)$ is even ($f(-x) = +f(x)$ for all x), then $b_n = 0$ for all n, leaving a Fourier cosine series (and perhaps a constant term) only for $f(x)$.

If $f(x)$ is odd ($f(-x) = -f(x)$ for all x), then $a_n = 0$ for all n, leaving a Fourier sine series only for $f(x)$.

Half-Range Fourier Series :-

A Fourier series for $f(x)$, valid on $[0, L]$, may be constructed by extension of the domain to $[-L, L]$.

An odd extension leads to a **Fourier sine series**:

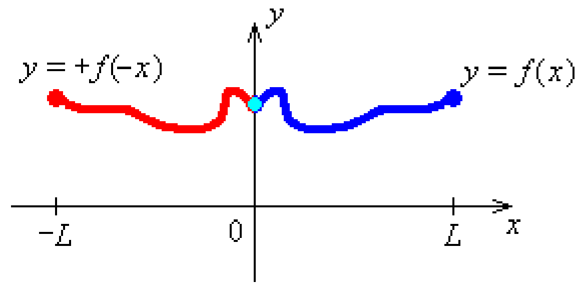


$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad (n = 1, 2, 3, \ldots)$$

An even extension leads to a **Fourier cosine series**:



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

Where $a_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad (n = 0, 1, 2, 3, \ldots)$

and there is automatic continuity of the Fourier cosine series at $x = 0$ and at $x = \pm L$.

Q.1: Obtain the Fourier series for f(x) = $|sinx|$ for – π < x < π.

Solution:

Since $|sinx|$is an even function .Therefore b_n =0.

Now fourierseries is given by

f(x) = $\frac{a_0}{2}$ + $\sum_{n=1}^{\infty} a_n cosnx,$

where , $a_0 = \frac{2}{\pi}\int_0^{\pi} sinx\ \ dx = \frac{4}{\pi}$.

a_n = $\frac{2}{\pi}\int_0^{\pi} sinx\ cosnxdx$

$=\frac{-2}{\pi}\left[\frac{1+(-1)^n}{n^2-1}\right] n \neq 1$

At n=1

$a_1 = \frac{2}{\pi}\int_0^{\pi} sinx\ cosxdx$

=0

Hence the required fourier series is,

$|sinx| = \frac{2}{\pi} - \frac{4}{\pi}[\frac{cos2x}{3} + \frac{cos4x}{15} + \cdots]$

Practice Question:

1) Find Fourier series for f(x)=$x - x^2$ in $(-\pi, \pi)$.
2) Find Fourier series for $f(x) = |x|, (-l, l)$.
3) Find Fourier series for xsinx (0,π).
4) Express f(x)=x,for 0<x<2
   (a) Half Range Fourier Cosine Series
   (b) Half Range Fourier Sine Series

Q2. Find the Fourier sine series and the Fourier cosine series for $f(x) = x$ on [0, 1].

Solution: $f(x) = x$ happens to be an odd function of x for any domain centred on $x = 0$. The odd extension of $f(x)$ to the interval [–1, 1] is $f(x)$ itself.

D		*I*
x	$+$	$\sin n\pi x$
1	$-$	$\frac{-1}{n\pi}\cos n\pi x$
0		$\frac{-1}{(n\pi)^2}\sin n\pi x$

Evaluating the Fourier sine coefficients,

$$b_n = \frac{2}{1}\int_0^1 x \sin\left(\frac{n\pi x}{1}\right) dx, \quad (n = 1, 2, 3, \ldots)$$

$$\Rightarrow \quad b_n = 2\left[-\frac{x}{n\pi}\cos\left(\frac{n\pi x}{1}\right) + \frac{1}{(n\pi)^2}\sin\left(\frac{n\pi x}{1}\right)\right]_0^1$$

$$= \frac{2}{n\pi}\times(-1)^{n+1}$$

Therefore the Fourier sine series for $f(x) = x$ on [0, 1] (which is also the Fourier series for $f(x) = x$ on [–1, 1]) is

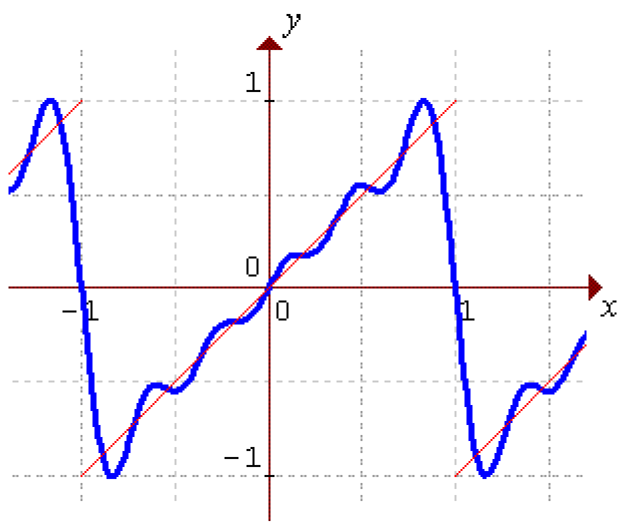
$$f(x) = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(n\pi x)}{n\pi}$$

or

$$f(x) = \frac{2}{\pi}\left(\sin \pi x - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} - \frac{\sin 4\pi x}{4} + \dots\right)$$

This function happens to be continuous and differentiable at $x = 0$, but is clearly discontinuous at the endpoints of the interval ($x = \pm 1$).

Fifth order partial sum of the Fourier sine series for $f(x) = x$ on [0, 1]



Q3. The even extension of $f(x)$ to the interval $[-1, 1]$ is $f(x) = |x|$.

Solution : Evaluating the Fourier cosine coefficients,

D		*I*
x	+	$\cos n\pi x$
1	−	$\frac{1}{n\pi}\sin n\pi x$
0		$\frac{-1}{(n\pi)^2}\cos n\pi x$

$$a_n = \frac{2}{1}\int_0^1 x\cos\left(\frac{n\pi x}{1}\right)dx, \quad (n = 1, 2, 3, \dots)$$

$$\Rightarrow \quad a_n = 2\left[\frac{x}{n\pi}\sin(n\pi x) + \frac{1}{(n\pi)^2}\cos(n\pi x)\right]_0^1$$

$$= \frac{2\left((-1)^n - 1\right)}{(n\pi)^2}$$

and $a_0 = \frac{2}{1}\int_0^1 x\,dx = \left[x^2\right]_0^1 = 1$

Evaluating the first few terms,

$$a_0 = 1,\quad a_1 = \frac{-4}{\pi^2},\quad a_2 = 0,\quad a_3 = \frac{-4}{9\pi^2},\quad a_4 = 0,\quad a_5 = \frac{-4}{25\pi^2},\quad a_6 = 0,\ldots$$

or $$a_n = \begin{cases} 1 & (n=0) \\ \dfrac{-4}{(n\pi)^2} & (n=1,3,5,\ldots) \\ 0 & (n=2,4,6,\ldots) \end{cases}$$

Therefore the Fourier cosine series for $f(x) = x$ on [0, 1] (which is also the Fourier series for $f(x) = |x|$ on [−1, 1]) is

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos\left((2k-1)\pi x\right)}{(2k-1)^2}$$

or

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\left(\cos\pi x + \frac{\cos 3\pi x}{9} + \frac{\cos 5\pi x}{25} + \frac{\cos 7\pi x}{49} + \ldots\right)$$

Lemma 12.1 *(A version of Parseval's Identity)*

Let $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$ $0 < x < L$. *Then* $\boxed{\frac{2}{L}\int_0^L [f(x)]^2\, dx = \sum_{n=1}^{\infty} b_n^2.}$

Proof:

$$\int_0^L [f(x)]^2\, dx = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_m b_n \int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) dx \quad (12.1)$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_m b_n \cdot \delta_{mn} \cdot \frac{L}{2} = \frac{L}{2}\sum_{n=1}^{\infty} b_n^2. \quad (12.2)$$

For a full Fourier Series on $[-L, L]$ Parseval's Theorem assumes the form:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \quad (12.3)$$

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2. \quad (12.4)$$

Example 12.2 Recall for $x \in [0, 2]$ $f(x) = x = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{2}\right)$.

Therefore

$$\begin{aligned}
\frac{2}{L}\int_0^L (f(x))^2\,dx &= \frac{2}{2}\int_0^2 x^2\,dx = \left(\frac{4}{\pi}\right)^2 \sum_{n=1}^{\infty}\frac{1}{n^2} \\
\Rightarrow \quad \frac{x^3}{3}\Big|_0^2 &= \left(\frac{4}{\pi}\right)^2 \sum_{n=1}^{\infty}\frac{1}{n^2} \\
\frac{\pi^2}{6} &= \sum_{n=1}^{\infty}\frac{1}{n^2}
\end{aligned}$$

Note: $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n)^2} = \frac{1}{2^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{4}\left(\frac{\pi^2}{6}\right) = \frac{\pi^2}{24}.$

Also note that

$$\begin{aligned}
\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2} &= \overbrace{\sum_{m=1}^{\infty}\frac{1}{(2m)^2}}^{\text{evens}} + \overbrace{\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}}^{\text{odds}} \\
&= \frac{\pi^2}{24} + \sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}
\end{aligned}$$

Therefore

$$\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2} = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \frac{\pi^2}{8}.$$

