RGPVNOTES.IN

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Module 4: Vector Spaces

Vector Space, Vector Sub Space, Linear Combination of Vectors, Linearly Dependent, Linearly Independent, Basis of a Vector Space, Linear Transformations

Notation:

1-Space ($\Re$) = {x | x is a real number};

2-Space($\Re^2$) = {(x, y) | x, $y$are real numbers};

3-Space ($\Re^3$) = {(x, y, z) | x, y, z are real numbers};

4-Space ($\Re^4$)={(x_1, x_2, x_3, x_4) | x_1, x_2, x_3, x_4 are real numbers};

…

***n*-Space ($\Re^n$)** ={$(x_1, x_2,..., x_n)$ | $x_1, x_2,..., x_n$ are real numbers}

The elements of $\Re^n$ are called **points** or **vectors**. They are usually denoted by boldface letters as

$$\mathbf{x} = (x_1, x_2,..., x_n) = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}_{nx1} \leftarrow n - \textbf{tuple} \text{ of real numbers}$$

The ith entry of the vector $\mathbf{x} = (x_1, x_2,...x_i,..., x_n)$ is called its ***i*th coordinate** or its ***i*th component**.

The **zero vector** in $\Re^n$ is

0 = (0, 0,…,0).

If $\mathbf{x} = (x_1, x_2,..., x_n)$ and $\mathbf{y} = (y_1, y_2,..., y_n)$ are vectors in $\Re^n$**,** then their **sum** is defined as the vector

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2,..., x_n + y_n).$$

If c is a **scalar** (a real number), then the **scalar multiple** of the vector **x** by the scalar c, denoted by c**x,** isthe vector

$$c\mathbf{x} = (cx_1, cx_2,..., cx_n)$$

Note:

$$(-1)\mathbf{x} = -\mathbf{x} = (-x_1, -x_2,..., -x_n)$$

Vector Space: Let V be a set vectors in which the operations of sum of vectors and of scalar multiplication are defined (that is, given vectors **x** and **y** in V and a scalar c, the vectors **x + y** and *c***x** are also in *V* - in this case *V* is said to be **closed** under vector addition and multiplication by scalars). Then with these operations *V* is called a **vector space** provided that - given any vectors **x, y,** and **z** in *V* and any scalars *a* and *b* - the following properties are true:

a. **x + y = y + x** **(commutativity)**
b. **x + (y + z) = (x + y) + z** **(associativity)**
c. **x + 0 = 0 + x = x** **(zero element)**
d. **x + (-x) = (-x) + x = 0** **(additive inverse)**
e. *a*(**x + y**) = *a***x** + *a***y** **(distributivity)**
f. (*a* + *b*)**x** = *a***x** + *b***y**
g. *a*(*b***x**) = (*ab*)**x**
h. (1)**x** = **x**

Theorem: The *n*-space $\Re^n$ is a **vector space.**

Let *W* be a nonempty subset of the vector space *V*. If *W* is a vector space with the operations of addition and scalar multiplication as defined in *V*, then *W* is a **subspace** of *V*.

Examples:

1. *W* = {**0**} is a subspace of $\Re^n$ (called the **zero subspace**).

2. *W* = $\Re^n$ is a subspace of $\Re^n$ (also called the **improper subspace**).
(all other subspaces of $\Re^n$ are called **proper subspaces**)

Theorem: (Conditions for a subspace)
The nonempty subset *W* of the vector space *V* is a subspace of *V* if and only if it satisfies the following conditions:

a. **0** is in *W;*
a. If **x** and **y** are vectors in *W*, then **x + y** is also in *W*;
b. If **x** is in *W* and *c* is a scalar, then the vector *c***x** is also in *W.*

Theorem: (Solution subspaces)
If *A* is an *m* x *n* matrix of constants, then the solution set of the homogeneous linear system

*A***x** = **0**

is a subspace of $\Re^n$.

Example 3: Find two solution vectors **u** and **v** for the following homogeneous system such that the solution space is the set of all linear combinations of the form *a***u** +*b***v:**

$$\begin{aligned} 2x+4y \quad -5z+3w &= 0 \\ 3x+6y \quad -7z+4w &= 0 \\ 5x+10y-11z+6w &= 0 \end{aligned}$$

We reduce the coefficient matrix to echelon form by applying the following sequence of EROs: $-3R_1+2R_2,\ -5R_1+2R_3,\ -3R_2+R_3$.

The echelon matrix we obtain is

$$\begin{bmatrix} 2 & 4 & -5 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence *x* and *z* are the **leading variables**, and *y* and *w* are the **free variables**. Back substitution yields the general solution

$$y = a,\ w = b,\ z = b,\ x = -2a + b$$

Thus the general solution vector of the system has the form

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -2a+b \\ a \\ b \\ b \end{bmatrix} = a\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = a\mathbf{u} + b\mathbf{v}$$

where**u** = (-2, 1, 0, 0) and **v** = (1, 0, 1, 1).
The solution space of the system is completely determined by the vectors **u** and **v** by the formula **x =** *a***u** **+***b***v.**

The vector **y** is called a **linear combination** of the vectors $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}$ provided that there exists scalars $c_1, c_2, \ldots, c_n$ such that

$$\mathbf{y} = c_1\mathbf{x_1} + c_2\mathbf{x_2} + \ldots + c_n\mathbf{x_n}$$

Let *S* = $\{\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}\}$be a set of vectors in the vector space *V*. The set of all linear combinations of $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}$ is called the **span** of the set *S*, denoted by span(*S*) or span($\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}$).

Theorem: span(*S*) is a subspace of *V*.

The set *S* = $\{\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}\}$of vectors in the vector space *V* is a **spanning set** for *V* provided that every vector in *V* is a linear combination of the vectors in *S*.

The set of vectors *S* = $\{\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}\}$in a vector space *V* is said to be **linearly independent** provided that the equation

$$c_1\mathbf{x_1} + c_2\mathbf{x_2} + \ldots + c_n\mathbf{x_n} = \mathbf{0}$$

has only the trivial solution $c_1 = c_2 = \ldots = c_n = 0$

Example 4: The **standard unit vectors**in $\Re^n$ **,** viz.**,**

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots \quad e_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

are linearly independent.

Note:
Any **subset of a linearly independent** set is a **linearly independent set**.

The coefficients in a linear combination of the vectors in a linearly independent set are unique.

A set of vectors is called **linearly dependent** if it is not linearly independent.

Example 5: The vectors **u =** (1, -1, 0)**, v =** (1, 3, -1)**,** and **w** = (5, 3, -2) are linearly dependent since 3**u** + 2**v** - **w** = 0.

Exercise: Determine whether the following vectors in $\Re^4$ are linearly dependent or independent.

(1, 3, -1, 4), (3, 8, -5, 7), (2, 9, 4, 23).

The vectors $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}$ are linearly dependent if and only if at least one of them is a linear combination of the others.

Theorem: The *n* vectors $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}$ in $\Re^n$ are linearlyindependentif and only if the *n* x *n* matrix

$$A = [\ \mathbf{x_1}\ \ \mathbf{x_2}\ \ \ldots\ \ \mathbf{x_n}\ ]$$

with the vectors as columns has nonzero determinant.

Theorem: Consider *k* vectors in $\Re^n$**,** with $k < n$. Let

$$A = [\ \mathbf{x_1}\ \ \mathbf{x_2}\ \ \ldots\ \ \mathbf{x}_k\ ]$$

be the *n* x *k* matrix having the *k* vectors as columns. Then the vectors $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x}_k$ are linearly independent if and only if some *k* x *k* submatrix of *A* has nonzero determinant.

A finite set *S* of vectors in a vector space *V* is called a **basis** for *V* provided that
(a) The vectors in *S* are linearly independent;
(b) The vectors in *S* span *V*.

Example 6: The set of **standard unit vectors**in $\Re^n$**,** viz.**,**

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots \quad e_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

form the **standard basis** for $\Re^n$.

Note: Any set of *n* linearly independent vectors in $\Re^n$ is a **basis**for $\Re^n$ **.**
Example 7: u = (-2, 1, 0, 0) and **v** = (1, 0, 1, 1) is a basis of the solution space of the homogeneous system given in **Example 3**. The dimension of the solution space is 2.

Theorem: Let *S* = $\{\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_n}\}$be a basis for the vector space *V*. Then any set of more than *n* vectors in *V* is linearly dependent**.**

Theorem: Any two bases of a vector space consist of the same number of vectors.

A vector space *V* is called **finite dimensional** if it has a basis consisting of a finite number of vectors. The unique number of vectors in each basis for *V* is called the **dimension** of *V* and is denoted by **dim(*V*)**.

A vector space that is not finite dimensional is called **infinite-dimensional.**

Let $A = [a_{ij}]_{mxn}$ be a matrix. The row vectors of *A* are the *m* vectors in $\Re^n$ given by

$$\begin{aligned} \mathbf{r}_1 &= (a_{11}, a_{12}, \dots, a_{1n}) \\ \mathbf{r_2} &= (a_{21}, a_{22}, \dots, a_{2n}) \\ &\vdots \\ \mathbf{r_m} &= (a_{m1}, a_{m2}, \dots, a_{mn}) \end{aligned}$$

The subspace of $\Re^n$ spanned by the *m* row vectors $\mathbf{r_1}, \mathbf{r_2}, ..., \mathbf{r_m}$ is called the **row space** of the matrix *A* and is denoted by **Row(*A*)**.

The dimension of the row space Row(*A*) is called the **row rank** of the matrix *A*.

Theorem: The nonzero row vectors of an echelon matrix are linearly independent and therefore form a basis for its row space Row (*A*).

Theorem: If two matrices are equivalent, then they have the same row space.

Note: ***To find a basis for the row space of a matrix, reduce the matrix to echelon form. Then the nonzero row vectors of the echelon matrix form a basis for the row space.***

Let $A=[a_{ij}]_{mxn}$ be a matrix. The column vectors of *A* are the *n* vectors in $\Re^m$ given by

$$\mathbf{c_1}=\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix},\ \mathbf{c_2}=\begin{bmatrix} a_{12}, \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix},\ \cdots\ \mathbf{c_n}=\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

The subspace of $\Re^m$ spanned by the *n* column vectors $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c_n}$ is called the **column space** of the matrix *A* and is denoted by **Col(*A*)**.

The dimension of the row space Col(*A*) is called the **column rank** of the matrix *A*.

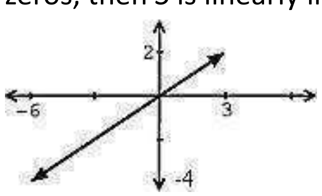
Note: ***To find a basis for the column space of a given matrix, reduce the matrix to echelon form. Then the column vectors of the given matrix that correspond to the pivot columns of the echelon matrix form a basis for the column space.***

Subspace, *W*: A subset *W* of *V* is subspace of $V \Leftrightarrow \begin{cases} (1)\ x+y \in W, \quad where\ x \in W\ and\ y \in W \\ (2)\ ax \in W, \quad where\ a \in F\ and\ x \in W \\ (3)\ 0\ in\ V \Rightarrow 0 \in W \\ (4)\ x+y=0\ \ for\ x \in W \Rightarrow y \in W \end{cases}$

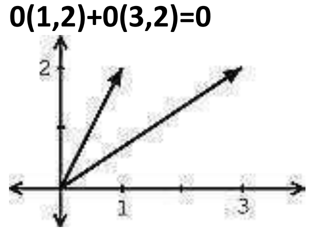
Theorem: **Any intersection of subspaces of a vector space *V* is a subspace of *V*.**
Theorem: W_1 and W_2 are subspaces of *V*, then $W_1 \cup W_2$ is a subspace $\Leftrightarrow$ $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

1-2 Linear Dependence and Linear Independence

Linear dependence & linear independence: For $x_1, x_2, \ldots, x_n \in S$, $\sum_{i=1}^{n} a_i x_i = 0$ if $\exists$ $a_1, a_2, \ldots, a_n$ are all zeros, then *S* is linearly independent; otherwise, *S* is linearly dependent.



Eg. (3,2) and (-6,-4) are linearly dependent because of 2(3,2)+1(-6,-4)=0, but (1,2) and (3,2) are linearly independent because of only 0(1,2)+0(3,2)=0



Theorem *V* is a vector space, $S_1 \subseteq S_2 \subseteq V$.

(a) **If S_1 is linearly dependent, then S_2 is also linearly dependent.**
(b) **If S_2 is linearly independent, then S_1 is also linearly independent.**

Basis: A basis β for a vector space *V* is a linearly independent subset of *V* that generates *V*.
Dimension, *dim* (*V*): The unique number of elements in each basis for *V*.

Theorem If $V=W_1\oplus W_2$, then $\dim(V)=\dim(W_1)+\dim(W_2)$.

Example. $\forall \begin{bmatrix} a \\ b \end{bmatrix} \in R^2$, **we have** $\begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. **Thus** $\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$ **is the basis of R^2 and $\dim(R^2)=2$.**

Example. $\forall \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in R^3$, **we have** $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. **Thus** $\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\}$ **is the basis of R^3 and $\dim(R^3)=3$.**

Example: For $W=\{(a_1,a_2,a_3,a_4,a_5)\in R^5: a_1+a_3+a_5=0,\ a_2=a_4\}$, find a basis of *W* and $\dim(W)$.
Sol. : $a_1+a_3+a_5=0$. Set $a_1=r$, $a_3=s$, $a_5=-r-s$, and set $a_2=a_4=t$.

$(a_1, a_2, a_3, a_4, a_5)=(r,t,s,t,-r-s)=r(1,0,0,0,-1)+t(0,1,0,1,0)+s(0,0,1,0,-1)$

Basis of *W*: {(1,0,0,0,-1),(0,1,0,1,0),(0,0,1,0,-1)} and $\dim(W)=3$.

Example. Show that in case $\beta=\{x_1, x_2, x_3\}$ be a basis in R^3, then $\beta'=\{x_1, x_1+x_2, x_1+x_2+x_3\}$ is also a basis in R^3.

Proof : Set $a_1x_1+a_2(x_1+x_2)+a_3(x_1+x_2+x_3)=0$ ...(1). If $a_1=a_2=a_3=0$, then $x_1, x_1+x_2, x_1+x_2+x_3$ are linearly independent

$(1)\Rightarrow(a_1+a_2+a_3)x_1+(a_2+a_3)x_2+a_3x_3=0$ ...(2)

$\Rightarrow\begin{cases} a_1+a_2+a_3=0 \\ a_2+a_3=0 \\ a_3=0 \end{cases} \Rightarrow\begin{cases} a_1=0 \\ a_2=0 \\ a_3=0 \end{cases}$, $\therefore x_1, x_1+x_2, x_1+x_2+x_3$ are linearly independent.

$\because \dim(R^3)=3$, $\therefore \beta'$ is a basis of R^3.

Another method: $\begin{bmatrix} x_1 \\ x_1+x_2 \\ x_1+x_2+x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\det(\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix})=1\neq 0$,

$\therefore \beta'$ is also a basis of R^3.

Example . Determine whether the given set of vectors is linearly independent?
(a) {(1,0,0),(1,1,0),(1,1,1)} in R^3.
(b) {(1,-2,1),(3,-5,2),(2,-3,6),(1,2,1)} in R^3.
(c) {(1,-3,2),(2,-5,3),(4,0,1)} in R^3.

(Sol.) (a) $\det(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix})=1\neq 0$, $\therefore$ linearly independent. (b) 4 vectors in R^3, $\therefore$ linearly dependent.

(c) $\det\left(\begin{bmatrix} 1 & 2 & 4 \\ -3 & -5 & 0 \\ 2 & 3 & 1 \end{bmatrix}\right) = -5 - 36 + 40 + 6 = 5 \neq 0$, ∴ Linearly independent

Definition: Let V, W be two vector spaces over F. We call T: V → W a **linear transformation** (linear operator)

if and only if ∀x, y ∈ V and c ∈ F,

a) T(x + y) = T(x) + T(y)

b) T(cx) = cT(x)

Example: T: $\Re^2 \to \Re^2$

$T((a_1, a_2)) = (a_1, -a_2)$ ⇒ Reflection Transformation.

Let $x = (a_1, a_2)$, $y = (b_1, b_2)$.

$T(cx + y) = T(c(a_1, a_2) + (b_1, b_2))$

$= T(ca_1 + b_1, ca_2 + b_2)$

$= (ca_1 + b_1, -(ca_2 + b_2))$

$= c(a_1 - a_2) + (b_1 - b_2)$

$= cT(x) + T(y)$ ⇒This shows that T is a linear transformation.

Example: Let V = C(ℜ) = {f(x) ∈ℜ: f(x) is continuous ∀x ∈ℜ}.

Consider $T(f) = \int^b_a f(t)\,dt$.

Let $f(x)$, g(x) ∈C(ℜ) and c ∈ F.

$T(cf + g) = \int^b_a [cf(t) + g(t)]\,dt$

$= \int^b_a cf(t)\,dt + \int^b_a g(t)\,dt$

$= c\int^b_a f(t)\,dt + \int^b_a g(t)\,dt$

$= cT(f) + T(g)$ ⇒ This shows that T is a linear transformation.

Definition: I_V: V →V defined as $I_V(v) = v$ ∀v∈V is called the identity transformation.

T_0: V → W defined as $T_0(v) = 0_W$ ∀v ∈V is called the zero transformation.

Both of these are linear transformations.

Definition: Let V, W be vector spaces and let T: V → W be linear. The null space (or kernel) is defined as

$N(T) = \{x \in V: T(x) = 0_W\}$.

The range (or image) is R(T) = {y ∈ W: ∃x ∈ V such that T(x) = y} = {T(x): x ∈ V}.

Theorem: Let V, W be vector spaces and T: V → W be linear.

Then N(T) is vector subspace of V and R(T) is a vector subspace of W.

Proof: Let $0_V = 0 \in V$, $0_W = 0 \in W$.

To show G ⊆ V is a vector subspace, need to show (Theorem 1.5)

a) 0 ∈ G

b) x + y ∈ G ∀x, y ∈ G

c) $cx \in G \ \forall x \in G \ \forall c \in F$

a) $T(0_V) = 0_W$
Hence $0_V \in N(T)$ and $0_W \in R(T)$.

b) Let $x, y \in N(T)$.
$T(x) = T(y) = 0_W$
$T(x + y) = T(x) + T(y) = 0_W + 0_W = 0_W$
Hence $x + y \in N(T)$.
Let $x, y \in R(T)$.
Then $\exists x', y' \in V$ such that $T(x') = x$ and $T(y') = y$.
Set $z = x' + y'$. Then $z \in V$.
$T(z) = T(x' + y') = T(x') + T(y') = x + y$
$\exists z$ such that $T(z) = x + y$.
Hence $x + y \in R(T)$.

c) Let $c \in F$ and $x \in N(T)$.
Then $T(x) = 0_W$
$T(cx) = cT(x) = c0_W = 0_W$
Hence $cx \in N(T)$.
Let $c \in F$ and $y \in R(T)$.
$\exists x \in V$ such that $T(x) = y$.
$T(cx) = cT(x) = cy$.
Set $cx = z$.
$\exists z \in V$ such that $T(z) = T(cx) = cy$.

Hence $cy \in R(T)$.

