Finite Element analysis of laminar flames

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Abstract

1 Governing Equations

1.1 Problem definition

The flow of a compressible fluid is described in terms of the velocity (u), pressure (p), density (ρ), and temperature(T) fields. These fields are solutions of the compressible Navier Stokes equations that describe the dynamics of the system and that are statements of conservation of mass, momentum and energy. The classical compressible navier stokes equation is written as

$$\begin{aligned} \frac{D\rho^*}{Dt^*} \ + \ \rho^*\nabla \cdot u^* &= 0 \qquad (1)\\ \rho^*\frac{Du^*}{Dt^*} &= -\nabla p^* + \ \nabla \cdot \tau^* \ + \ \rho^* g^* \qquad (2)\\ \rho^* c_p^* \frac{DT^*}{Dt^*} \ - \ \beta^* T^* \frac{Dp^*}{Dt^*} &= \tau^* \cdot \nabla u^* + \ \nabla \cdot q^* \qquad (3)\\ & \qquad (4) \end{aligned}$$

Where τ^* is the viscous stresses. g^* is the gravity vector. c_p^* is the specific heat at constant pressure, β^* is the thermal expansion coefficient , q^* is the heat flux vector. For a newtonian fluid
$\tau^* = \mu^*(\nabla u^* + (\nabla u^*)^t)) - \frac{2}{3}\mu^* \nabla . u^* I$
Assuming Fourier's law $q = -k\nabla T$.
μ^* is the viscocity and k is the thermal conductivity.

1.2 Non dimensionalisation

The goal is to perform asymptotic analysis based on low mach number. we first nondimensionalize the governing equation by introduction of following scaling

$$x = \frac{x^*}{L}, \rho = \frac{\rho^*}{\rho_\infty}, \ p = \frac{p^*}{p_\infty}, \ u = \frac{u^*}{u_\infty}, \ T = \frac{T^*}{T_\infty}, \ \mu = \frac{\mu^*}{\mu_\infty}, \ k = \frac{k^*}{k_\infty}, c_p = \frac{c_p^*}{c_{p\infty}}, \ t = \frac{t^*}{L^*/u_\infty}, \ \beta = \frac{\beta^*}{\beta_\infty}$$

Now the derivatives are non dimensionalized as follows
$\frac{\partial}{\partial x^*} = \frac{1}{L}\frac{\partial}{\partial x}, \frac{\partial}{\partial t^*} = \frac{u_0}{L}\frac{\partial}{\partial t}, \frac{D}{Dt^*} = \frac{u_0}{L}\frac{D}{Dt}$
Substituting these values in mass, momentum and energy conservation equations respectively we develop following equations
$\frac{\rho_\infty u_\infty}{L}\frac{D\rho}{Dt} \ + \ + \frac{\rho_\infty u_\infty}{L}\rho \nabla \cdot u = 0$
$\frac{\rho_\infty u_\infty^2}{L}\rho\frac{Du}{Dt} \ = \ -\frac{p_\infty}{L}\nabla p + \ \frac{\mu_\infty u_\infty}{L^2}\nabla \cdot \tau \ + \ \rho_\infty g_\infty \rho g$
$\frac{\rho_\infty c_{p\infty} u_\infty T_\infty}{L}\rho c_p \frac{DT}{Dt} \ - \ \frac{\beta_\infty p_\infty u_\infty T_\infty}{L}\beta T \frac{Dp}{Dt} \ = \ \frac{\mu_\infty u_\infty^2}{L^2}\tau \cdot \nabla u + \frac{k_\infty T_\infty}{L^2} \ \nabla \cdot q$
The mass, momentum and energy equations are now rewritten as follows
$\frac{D\rho}{Dt} \ + \ + \rho \nabla \cdot u = 0$
$\rho \frac{Du}{Dt} \ = \ -\frac{p_\infty}{\rho_\infty u_\infty^2}\nabla p + \ \frac{\mu_\infty}{\rho_\infty u_\infty L}\nabla \cdot \tau \ + \ \frac{g_\infty L}{u_\infty^2}\rho g$

$\rho c_p \frac{DT}{Dt} \; - \; \frac{\beta_\infty p_\infty}{\rho_\infty c_{p_\infty}} \beta T \frac{Dp}{Dt} \; = \; \frac{\mu_\infty u_\infty}{\rho_\infty c_{p_\infty} T_\infty L} \tau \cdot \nabla u + \frac{k_\infty}{\rho_\infty c_{p_\infty} u_\infty L} \; \nabla \cdot q$

We write these equations in terms of various non dimensional number defined as follows

$M = \frac{u_\infty}{a_\infty}$, $Re = \frac{\rho_\infty u_\infty L}{\mu_\infty}$, $Pr = \frac{\mu_\infty c_{p_\infty}}{k_\infty}$,$F = \frac{u_\infty}{\sqrt{g_\infty L}}$

Note that by definition

$\beta^* = -\frac{1}{\rho^*} \frac{\partial \rho^*}{\partial T^*} = \frac{1}{T_\infty} \beta$ thus $\beta_\infty = \frac{1}{T_\infty}$

By definition $a^{*2} = \frac{\partial p^*}{\partial \rho^*} = \frac{p_\infty}{\rho_\infty} a^2$ where $a_\infty = \sqrt{p_\infty / \rho_\infty}$

We will also define $\lambda = \frac{c_{p_\infty} T_\infty}{a_\infty^2}$

Now the mass momentum and energy equations are written as follows

$$\frac{D\rho}{Dt} \; + \; + \rho \nabla \cdot u \;\; = \;\; 0 \tag{5}$$

$$\rho \frac{Du}{Dt} \;\; = \;\; -\frac{1}{M^2} \nabla p + \; \frac{1}{Re} \nabla \cdot \tau \; + \; \frac{1}{F^2} \rho g \tag{6}$$

$$\rho c_p \frac{DT}{Dt} \; - \; \frac{1}{\lambda} \beta T \frac{Dp}{Dt} \;\; = \;\; \frac{M^2}{Re\lambda} \tau \cdot \nabla u + \frac{1}{RePr} \; \nabla \cdot q \tag{7}$$

1.3 Asymptotic Analysis

The limit when the Mach number tends to zero can be found using standard procedures of asymptotic analysis described. The first step is to expand all flow variables in power series of the small parameter considered

$\xi(x,t,M) = \xi^{(0)}(x,t) + M\xi^{(1)}(x,t) + M^2 \xi^{(2)}(x,t) + O(M^3)$

Where ξ can be pressure p, density ρ, velocity u and temperature T.

Some asymptotic expansion of equation is shown below.

$\frac{D\rho}{Dt} \; + \; + \rho \nabla \cdot u = 0$ is expanded asymptotically as

$\frac{D\rho}{Dt} = \frac{\partial}{\partial t} (\rho^{(0)} + M\rho^{(1)} + M^2\rho^{(2)} + O(M^3)) + \left(u^{(0)} + Mu^{(1)} + M^2 u^{(2)} + O(M^3) \right) \nabla \left(\rho^{(0)} + M\rho^{(1)} + M^2\rho^{(2)} + O(M$

Combining zero order terms, first order terms we get the following expression

$\frac{D\rho}{Dt} = \frac{\partial \rho^{(0)}}{\partial t} + u^0 . \nabla \rho^0 + M \left(\frac{\partial \rho^{(1)}}{\partial t} + u^0 . \nabla \rho^1 + u^1 . \nabla \rho^o \right) + O(M^2)$

$\rho \nabla \cdot u = (\rho^{(0)} + M\rho^{(1)} + M^2 \rho^{(2)} + O(M^3) \nabla \cdot (u^{(0)} + Mu^{(1)} + M^2 u^{(2)} + O(M^3)$

Again combining zero order terms together

$\rho \nabla \cdot u = \rho^{(0)} \nabla \cdot u^{(0)} + M(\rho^{(1)} \nabla \cdot u^{(0)} + \rho^{(0)} \nabla \cdot u^{(1)}) + O(M^2)$

Thus we have the following zero order $O(M^0)$ for mass equation

$$\frac{D\rho^0}{Dt} + \rho^{(0)} \nabla \cdot u^{(0)} \;\; = \;\; 0 \tag{8}$$

Next we will consider momentum equation

$\rho \frac{Du}{Dt} = \rho^{(0)} + M\rho^{(1)} + M^2\rho^{(2)} + O(M^3) \left(\frac{\partial}{\partial t} (u^{(0)} + Mu^{(1)} + M^2 u^{(2)} + O(M^3)) \; + \; \left(u^{(0)} + Mu^{(1)} + M^2 u^{(2)} + O(M^3) \right) \right.$

Combining zero order terms, first order terms we get the following expression

$\rho \frac{Du}{Dt} = \rho^0 \frac{Du^0}{Dt} + M \left(\rho^1 \frac{Du^0}{Dt} + \rho^0 \frac{\partial u^1}{\partial t} + \rho^0 u^0 . \nabla u^1 + \rho^0 u^1 . \nabla u^0 \right) + O(M^2)$

$\frac{1}{M^2} \nabla p = \frac{1}{M^2} \nabla p^{(0)} + Mp^{(1)} + M^2 p^{(2)} + O(M^3)$

$\frac{1}{M^2} \nabla p = \frac{1}{M^2} \nabla p^0 + \frac{1}{M} \nabla p^1 + \nabla p^2$

If we want pressure to behave in $\lim_{M \to 0}$ we get $p^0 = p^0(t)$ and $p^1 = p^1(t)$.

$\frac{1}{Re} \nabla \cdot \tau = \frac{1}{Re} \nabla \cdot \tau^0 + \frac{M}{Re} \nabla \cdot \tau^1 + O(M^2)$

Thus we have the following zero order $O(M^0)$ for momentum equation

$$\rho^0 \frac{Du^0}{Dt} \;\; = \;\; -\nabla p^2 + \; \frac{1}{Re} \nabla \cdot \tau^0 \; + \; \frac{1}{F^2} \rho^0 g \tag{9}$$

The energy equation is also asymptotically expanded in the same way as mass and momentum equation. However we have already seen that $p^0 = p^0(t)$ and $p^1 = p^1(t)$. Therefore

$\frac{Dp^0}{Dt} = \frac{dp^0}{dt}$.
The visous dissipation term is already $O(M^2)$.

$q = k\nabla T = (k^{(0)} + Mk^{(1)} + M^2 k^{(2)} + O(M^3))\nabla(T^{(0)} + MT^{(1)} + M^2 T^{(2)} + O(M^3))$

Combining similar order terms together
$\nabla.q = \nabla.(k^{(0)}\nabla T^{(0)}) + O(M)$.
Thus we have following zero order energy equation

$$\rho^0 c_p^0 \frac{DT^0}{Dt} - \frac{1}{\lambda}\beta^0 T^0 \frac{dp^0}{dt} = \frac{1}{RePr}\nabla \cdot (k^{(0)}\nabla T^{(0)}) \tag{10}$$

1.4 State Equation and its nondimensionalisation

The equation of state is

$\frac{D\rho^*}{Dt^*} = \beta^* \rho^* \frac{DT^*}{Dt^*} + \alpha^* \rho^* \frac{Dp^*}{Dt^*}$ where $\beta^* = -\frac{1}{\rho^*}\left.\frac{\partial \rho^*}{\partial T^*}\right|_p$ and $\alpha^* = \frac{1}{\rho^*}\left.\frac{\partial \rho^*}{\partial p^*}\right|_T$

First we nondimensionalise as before

$\frac{\rho^* u^*}{L}\frac{D\rho^*}{Dt^*} = -\frac{\beta_\infty \rho_\infty T_\infty u_\infty}{L}\beta\rho\frac{DT}{Dt} + \frac{\alpha_\infty \rho_\infty p_\infty u_\infty}{L}\alpha\rho\frac{Dp}{Dt}$ as before $\beta_\infty = \frac{1}{T_\infty}\left.\frac{\partial \rho^*}{\partial T^*}\right|_p$ as before $\beta^*_\infty = \frac{1}{T_\infty}$ and $\alpha^* = \frac{1}{\rho_\infty \rho}\frac{\rho_\infty}{p_\infty}\frac{\partial \rho}{\partial p} = \frac{1}{p_0}\alpha$ where $\alpha_\infty = \frac{1}{p_\infty}$.

We get non dimensional state equation as

$\frac{D\rho}{Dt} = -\beta\rho\frac{DT}{Dt} + \alpha\rho\frac{Dp}{Dt}$

The asymptotic analysis of the state equation yields following relationship

$\frac{D\rho^0}{Dt} = -\beta^0\rho^0\frac{DT^0}{Dt} + \alpha^0\rho^0\frac{Dp^0}{Dt}$

From mass conservation we have $\frac{D\rho^0}{Dt} = -\rho^{(0)}\nabla \cdot u^{(0)}$ and we know that $p^0 = p^0(t)$ and $p^1 = p^1(t)$. Therefore we have the following relationship

$-\rho^{(0)}\nabla \cdot u^{(0)} = -\beta^0\rho^0\frac{DT^0}{Dt} + \alpha^0\rho^0\frac{dp^0}{dt}$

Therefore divergence of velocity is obtained as

$$-\nabla \cdot u^{(0)} = -\beta^0\frac{DT^0}{Dt} + \alpha^0\frac{dp^0}{dt} \tag{11}$$

$$-\nabla \cdot u^{(0)} = -\frac{1}{T_0}\frac{DT^0}{Dt} + \frac{1}{p_0}\frac{dp^0}{dt} \tag{12}$$