Algebra - Full Course

Exponent Rules:

x^n

$2^5 \to \underbrace{2 \times 2 \times 2 \times 2 \times 2}_{5 \text{ times}}$

$x^n = \underbrace{x \cdot x \cdot x \cdot \ldots \cdot x}_{n \text{ times}}$

5 ← Exponent or Power
2 ← base

1> **The Product Rule:**

$$x^n \cdot x^m = x^{n+m}$$

$$2^3 \cdot 2^4 = 2^7$$

2> **The Quotient Rule:**

$$\frac{x^n}{x^m} = x^{n-m}$$

3> **The Power Rule**

$$(x^n)^m = x^{n \cdot m}$$

$$(5^4)^3 = 5^{4 \cdot 3} = 5^{12}$$

4> **Power to Zero**

$$x^0 = 1$$ (with Exception : 0^0 is undefined)

5> **Negative Exponent**

$$x^{-n} = \frac{1}{x^n}$$

$$5^7 \cdot 5^{-7} = 5^{7+(-7)} = 5^0 = 1$$

6) Fractional Exponents

$$x^{1/n} = \sqrt[n]{x}$$

$$64^{1/3} = \sqrt[3]{64} = 4$$

$$9^{1/2} = \sqrt{9} = 3$$

$$(5^{1/3})^3 = 5$$

7) Distribute an Exponent over a Product

$$(x \cdot y)^n = x^n \cdot y^n$$

$$(5 \cdot 7)^3 = 5^3 \cdot 7^3$$

8) Distribute an Exponent over a Quotient

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

$$\left(\frac{2}{7}\right)^5 = \frac{2^5}{7^5}$$

$$(a+b)^n \neq a^n + b^n$$

$$(a-b)^n \neq a^n - b^n$$

Simplifying with Exponent Rules

a) $\dfrac{3x^{-2}}{x^4} = 3x^{-6} = 3/x^6$

b) $\dfrac{4y^3}{y^{-5}} = 4y^8$

c) $\dfrac{y^3 z^5}{7z^{-2}y^7} = \dfrac{z^7}{7y^4} = \dfrac{z^7 y^{-4}}{7}$

2

> $\left(\frac{25x^4y^{-5}}{x^{-6}y^3}\right)^{3/2} = \left[\frac{25x^{10}}{y^8}\right]^{3/2}$

First method:

$$=\left(\frac{25x^{10}}{y^8}\right)^{3\times 1/2}$$

$$=\left(\frac{(25x^{10})^3}{(y^8)^3}\right)^{1/2} \qquad = \cancel{\left(\frac{(25x^{10})^3}{(y^8)^3}\right)^{1/2}}$$

$$=\sqrt{\frac{125x^{30}}{y^{24}}} \qquad = \cancel{\sqrt{\left(\frac{25x^{10}}{y^8}\right)^3}}$$

$$=\frac{(25^3)^{1/2}x^{30/2}}{y^{24/2}} \qquad = \cancel{\sqrt{\frac{125x^{30}}{y^{24}}}}$$

$$=\frac{(5^3)x^{15}}{y^{12}}$$

Second method:

$$=\frac{(25)^{3/2}(x^{10})^{3/2}}{(y^8)^{3/2}}$$

$$=\frac{(25^{1/2})^3(x^5)^3}{(y^4)^3}$$

$$=\frac{(5)^3x^{15}}{y^{12}}$$

Simplifying Radicals

$\sqrt[n]{x}$

$\sqrt[3]{8}=2$

$(2)^3=8$

$\sqrt{25}=5 \qquad 5^2=25$

Rules for Radicals:

$$\sqrt[n]{a \cdot b} = \sqrt[n]{a}\sqrt[n]{b} \qquad = \sqrt{9 \cdot 16} = \sqrt{9} \cdot \sqrt{16} = 3$$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \qquad \sqrt[3]{\frac{64}{8}} = \frac{\sqrt[3]{64}}{\sqrt[3]{8}} = \frac{4}{2}$$

$$\sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b}$$

$$\sqrt[n]{a-b} \neq \sqrt[n]{a} - \sqrt[n]{b}$$

$$(a/b)^{1/n} = \frac{a^{1/n}}{b^{1/n}}$$

$$a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$

$\rightarrow (a^m)^{1/n}$

$\rightarrow (a^{1/n})^m$

Ex) Compute $(25)^{-3/2}$

$$= 5^{-3}$$

$$= 1/5^3 = 1/125$$

Ex) $\sqrt{60x^2y^6z^{-11}}$

$$\sqrt{\frac{60x^2y^6}{z^{11}}} \qquad \frac{(60)^{1/2}xy^3}{z^{11/2}} = xy^3\sqrt{\frac{60}{z^{11}}}$$

(4) $2xy^3\sqrt{\frac{15}{z^{10} \cdot z}}$ $\qquad \frac{xy^3\sqrt{4 \times 3 \times 5}}{z^{11}}$

$\left(= \frac{2xy^3}{z^5}\sqrt{\frac{15}{z}}\right)$ $\qquad 2xy^3\sqrt{15/z^{11}}$

Rationali

$\frac{3x}{\sqrt{x}}$

=

=

rst m

c

Rationalize the dinominator $\boxed{\frac{3x}{\sqrt{x}}}$

$$\frac{3x}{\sqrt{x}} \times \frac{\sqrt{x}}{\sqrt{x}}$$

$$\frac{3x^{1-1/2}}{x} \qquad \frac{3x^{-1/2}}{x}$$

1st method : $\longrightarrow$

$$3x^{-1/2-1}$$
$$= 3x^{-3/2}$$

Second method:

$$\frac{3x}{\sqrt{x}} \times \frac{\sqrt{x}}{\sqrt{x}} \qquad \frac{3x\sqrt{x}}{x} = \boxed{3\sqrt{x}}$$

Factoring

$30 = 6.5 = 2.3.5$

$x^2 + 5x + 6 = x^2 + \underbrace{2x + 3x} + 6$

$= x(x+2) + 3(x+2)$

$= (x+3)(x+2)$

note:

multiplication should be $\rightarrow$ +ve 6

Addition should be $\rightarrow$ +ve 5

Ex. Pull out the Greatest common factor

1) Factor $15 + 25x$

$$5(3 + 5x)$$

2) $x^2y + y^2x^3$

$$x^2y(1 + yx)$$
$$x^2y(1 + xy)$$

factoring by grouping

Ex. factor $x^3 + 3x^2 + 4x + 12$

$= x^2(x+3) + 4(x+3)$

$= (x^2 + 4)(x+3)$

(5)

3. factor Quadratics

Ex. Factor x^2-6x+8

$$x^2-4x-2x+8$$
$$=x(x-4)-2(x-4)$$
$$=(x-2)(x-4)$$

factor $10x^2+11x-6$

-60

11

$$10x^2+15x-4x-6$$
$$5x(2x+3)-2(2x+3)$$
$$(5x-2)(2x+3)$$

60
↑
30×2
2×30
12×5
5×12
20×3
3×20
6×10
10×6
15×4
4×15

4. Difference of Squares

a^2-b^2

Ex. factor $x^2-16 = x^2-4^2$

$$a^2-b^2 = (a+b)(a-b)$$
$$x^2-4^2 = (x+4)(x-4)$$

Ex. factor $9p^2-1$

$$(3p)^2-(1)^2$$
$$=(3p+1)(3p-1)$$

5) Difference or Sum of Cubes

$$a^3-b^3 = (a-b)(a^2+ab+b^2)$$
$$a^3+b^3 = (a+b)(a^2-ab+b^2)$$

Ex. factor y^3+27

$$y^3+3^2 = (y+3)(y^2-3y+3^2)$$
$$=(y+3)(y^2-3y+9)$$

1) x^2+x

$x(x+1)$

2) $x^2-25 \to x^2-5^2$

$\to (x+5)(x-5)$

3) $x^2+4 \to x^2+2^2$

4) $x^3+2x^2+3x+6 \to x^2(x+2)+3(x+2)$

$\to (x^2+3)(x+2)$

5) $5x^2-14x+8 \to 5x^2-10x-4x+8$

$5x(x-2)-4(x-2)$

$(5x-4)(x-2)$

Extra Ex. factor $2z^2+3z-14$

$2z^2+7z-4z-14$

$z(2z+7)-2(2z+7)$

$(z-2)(2z+7)$

50×5

-250

$\to 25 \times 10$

5×50

Ex. $-5v^2-45v+50$

$-5v^2-50v+5v+50$

$-5v(v+10)+5(v+10)$

$(-5v+5)(v+10)$

Rational Expressions

$\frac{x+2}{x^2-3}$

$+$ $-$

$\times$ $\div$

simplify

Ex: simplify $\frac{21}{45}$ by reducing to lower terms

$\frac{21}{45} = \frac{7 \times 3}{15 \times 3} = \frac{7}{15}$

Ex. simplify $\frac{3x+6}{x^2+4x+4}$

$\frac{3x+6}{(3x+6)+x+x^2+2}$

$\frac{3x+6}{(3x+6)+x^2+x+2)}$

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$$\frac{3x+6}{x^2+4x+4} = \frac{3(x+2)}{x^2+4(x+1)}$$

$$\frac{3x+6}{(3x+6)+x^2+x-2}$$

$$\frac{3x+6}{(3x+6)+(x^2+x-2)} = \frac{3x+6}{(3x+6)+(x^2-2x+x-2)}$$

$$\frac{3x+6}{(3x+6)+(x(x-2)+1(x-2))}$$

$$= \frac{3x+6}{(3x+6)+(x+1)(x-2)}$$

$$= 1 + \frac{3x+6}{(x+1)(x-2)}$$

$$1 + \frac{3(x+2)}{(x+1)(x-2)}$$

$$\frac{3x+6}{x^2+4x+4} = \frac{3x+6}{x^2+2x+2x+4}$$

$$\frac{3x+6}{x(x+2)+2(x+2)}$$

$$\frac{3x+6}{(x+2)(x+2)}$$

$$= \frac{3\cancel{(x+2)}}{(x+2)\cancel{(x+2)}}$$

$$= 3/x+2$$

Multiplying & Dividing

Ex. Compute

a) $\frac{4}{3} \cdot \frac{2}{5} \rightarrow \frac{8}{15}$

b) $\frac{4/5}{2/3} \rightarrow \frac{4 \times 3}{5 \times 2} = \frac{12}{10} = \frac{6}{5}$

Ex. $\frac{(x^2+x)(x^2-16)}{(x+4)(x+1)}$ $\quad \frac{(x^2+x)\cancel{(x+4)}(x-4)}{\cancel{(x+4)}(x+1)}$

$$\frac{(x^2+x)(x-4)}{(x+1)}$$

$$\frac{x\cancel{(x+1)}(x-4)}{\cancel{(x+1)}}$$

$$= x(x-4)$$

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subtract $\frac{7}{6} - \frac{4}{15}$

$$\frac{7\times5}{6\times5} - \frac{4\times2}{15\times2}$$

3	6	15
2	2	5
5	1	5
	1	1

$6\times5 = 30$

$$\frac{35-8}{30} = \frac{27}{30} = \frac{9}{10}$$

Ex. $\frac{3}{2x+2} + \frac{5}{x^2-1}$

$2x+2 = 2(x+1)$

$x^2-1 = (x+1)(x-1)$

$$\frac{3(x-1)}{2(x+1)(x-1)} + \frac{5\times2}{2(x+1)(x-1)}$$

$$= \frac{3(x-1)+10}{2(x+1)(x-1)} = \frac{3x+7}{2(x^2-1)}$$

Solving Quadratic Eqⁿs.

Contains → x^2 in eqⁿ.

$Ax^2 + bx + C = 0$

where a, b, c represent real numbers & $a \neq 0$

$3x^2 + 7x - 2 = 0$

$a = 3,\ b = 7,\ c = -2$

Ex. find all real solⁿ. for eqⁿ. $y^2 = 18 - 7y$

$y^2 + 7y - 18 = 0$

$y^2 + 9y - 2y - 18 = 0$

$y(y+9) - 2(y+9) = 0$

$(y-2)(y+9) = 0$

$y = -9$ or $y = 2$

-18
9 -2
7

9

Ex. $w^2 = 121$

$w^2 - 121 = 0$

$w^2 - 11^2 = 0$

$(w + 11)(w - 11) = 0$

$w = 11$ or $w = -11$

Ex. all real solⁿ. for the eqⁿ. $x(x+2) = 7$

$x^2 + 2x - 7 = 0$

$a = 1 \quad b = 2 \quad c = -7$

Note: $\boxed{x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$

$= \dfrac{-2 \pm \sqrt{4 - 4(1)(-7)}}{2(1)} = \dfrac{-2 \pm \sqrt{4 + 28}}{2}$

$= \dfrac{-2 \pm \sqrt{32}}{2}$

$= \dfrac{-2 \pm 4\sqrt{2}}{2}$

$= \dfrac{-2}{2} \pm \dfrac{4\sqrt{2}}{2}$

$= -1 \pm 2\sqrt{2}$

$= -1 + 2\sqrt{2}$ or $-1 - 2\sqrt{2}$

Ex. find all real solⁿ. for the eqⁿ. $\frac{1}{2}y^2 = \frac{1}{3}y - 2$

$= \frac{1}{2}y^2 - \frac{1}{3}y + 2 = 0$

$\dfrac{-(-1/3) \pm \sqrt{(1/3)^2 - 4(1/2)(2)}}{2(1/2)}$

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no. real sol? $\Longrightarrow$ $= \dfrac{1/3 \pm \sqrt{1/9 - 4}}{1} = 1/3 \pm \sqrt{\dfrac{-35}{9}}$
$= 1/3 \pm \dfrac{\sqrt{-35}}{3}$

Solve $\frac{x}{x+3} = 1 + 1/x$

$$\frac{x}{x+3} = \frac{x+1}{x}$$

$$x^2 = (x+3)(x+1)$$

$$x^2 = x^2 + x + 3x + 3$$

$$0 = 4x + 3$$

$$4x = -3$$

$$\boxed{x = -3/4}$$

Ex. Solve $\frac{4c}{c-5} - \frac{1}{c+1} = \frac{3c^2+3}{c^2-4c-5}$

$$\frac{4c(c+1) - (c-5)}{(c-5)(c+1)} = \frac{3c^2+3}{c^2-4c-5}$$

$$\frac{4c^2+4c-c+5}{c^2+c-5c-5} = \frac{3c^2+3}{c^2-4c-5}$$

$$\frac{4c^2+3c+5}{c^2-4c-5} = \frac{3c^2+3}{c^2-4c-5}$$

$$4c^2 - 3c^2 + 3c + 5 - 3 = 0$$

$$c^2 + 3c + \cancel{8}\,2 = 0$$

2×2 / $8 \times$ ✓

$$\cancel{\frac{-3 \pm \sqrt{9-4(-8)}}{2(1)}}$$

$$\cancel{= \frac{-3 \pm \sqrt{41}}{2}}$$

$$\cancel{= \frac{-3}{2} + \frac{\sqrt{41}}{2}} \text{ or } \cancel{\frac{-3}{2} - \frac{\sqrt{41}}{2}}$$

$$c^2 + 3c + 2 = 0$$

$$c^2 + 2c + c + 2 = 0$$

$$c(c+2) + 1(c+2) = 0$$

$$c(c+1)(c+2) = 0$$

$c = -1$ or $c = -2$

Ans.

$$\frac{-4}{-6} - \frac{1}{0} =$$

$$\frac{-8}{-7} - \frac{1}{-1} = \frac{3(4)+3}{4+8-5}$$

$$\frac{8}{7} + 1 = \frac{12+3}{7}$$

$$\frac{15}{7} = \frac{15}{7}$$

(11)

Solving Radical Eqⁿ.

$x + \sqrt{x} = 12$

Ex. find all real solⁿ. for the eq. $x + \sqrt{x} = 12$

$$x + \sqrt{x} - x = 12 - x$$
$$\sqrt{x} = 12 - x$$
$$x = (12 - x)^2$$
$$x = 144 - 2(12)(x) + x^2$$
$$x = 144 - 24x + x^2$$
$$x^2 - 25x + 144 = 0$$
$$x^2 - 9x - 16x + 144 = 0$$
$$x(x - 9) - 16(x - 9) = 0$$
$$(x - 16)(x - 9) = 0$$
$$x = 16 \text{ or } x = 9$$

144, -9, -16, -25

-1	-144
-2	-72

$16 + 4 =$

$9 + 3 = 12$

Ex. find the all real solⁿ. for the eqⁿ. $2p^{4/5} = 1/8$

(1)
$$2p^{4/5} = 1/8$$
$$p^{4/5} = 1/16$$
$$\left(p^{4/5}\right)^5 = \left(\frac{1}{16}\right)^5$$
$$p^4 = \left(\frac{1}{16}\right)^5$$
$$(p^4)^{1/4} = \left(\left(\frac{1}{16}\right)^5\right)^{1/4}$$
$$p = \pm\left(\frac{1}{16}\right)^{5/4}$$

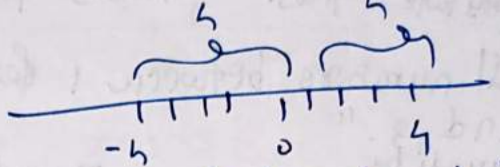
$$p = \pm\left(\sqrt[4]{\frac{1}{16}}\right)^5$$
$$p = \pm\left(\frac{1}{2}\right)^5$$
$$p = \pm\left(\frac{1}{32}\right)$$
$$\frac{1}{32} \text{ or } -\frac{1}{32}$$

(12)

Solving Absolute Value Eq^n.

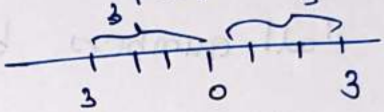
$|4| = 4$

$|-4| = 4$



$|-4| = 4 \quad |4| = 4$

$|x| = 3$



Ex. Solve the eq^n.

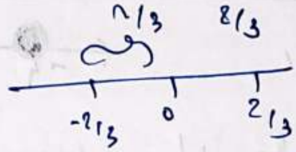
$3|x| + 2 = 4$

$3|x| + 2 = 4$

$3|x| = 4 - 2$

$3|x| = 2$

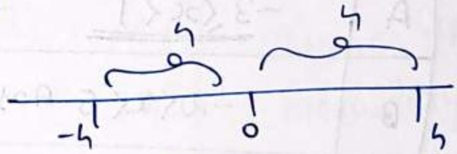
$|x| = 2/3$



$3|-2/3| + 2 = 4$

$3 \times \frac{2}{3} + 2 = 4$, $3|2/3| + 2 = 4$

Ex. Solve eq^n. $|3x+2| = 4$



$3x + 2 = 4$ or $3x + 2 = -4$

$3x = 2$ or $3x = -6$

$x = 2/3$ or $x = -2$

$|3(2/3) + 2| = 4$

$|3(-2) + 2| = 4$

$|-6 + 2| = 4$

Ex. Solve the eq^n. $5|4P - 3| + 16 = 1$

$5|4P - 3| = -15$

$|4P - 3| = -3$

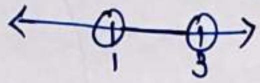
then^n is dis. from 0 to -3

So, sol^n. to this eq^n.

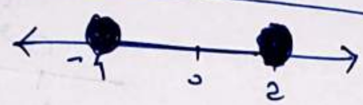
Interval Notation

Ex. Translate these phrases into standard math notation:

a) "all numbers between 1 and 3 but not including 1 and 3."

Inequality	number line	Interval Notation
$1 < x < 3$		$(1, 3)$

b) "all numbers b/w -4 and 2, including -4 and 2."

Inequality	number line	Interval Notation
$-4 \le x \le 2$,		$[-4, 2]$

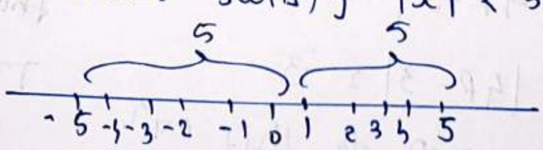
Ex. Translate b/w inequality & interval notation.

	Inequality Notation	Interval Notation
A	$-3 \le x < 1$	$[-3, 1)$ Ans.
B	$-\infty < x < 5$ Ans.	$(-\infty, 5)$
C	$-15 < x$	$(-15, \infty)$ Ans.
D	$4 \ge x > 0$ Ans.	$(0, 4]$ Ans.

always lower value on left

Absolute Value Inequalities

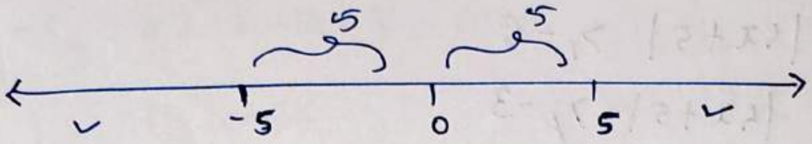
Ex.) What x-values satisfy $|x| < 5$?



$-5 < x < 5$

$(-5, 5)$

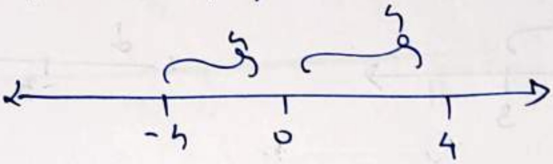
Ex. What x values satisfy $|x| \geq 5$?



$x \geq 5$ or $x \leq -5$

$(-\infty, -5] \cup [5, \infty)$

Ex. Solve $|3-2t| < 4$



$-4 < 3-2t < 4$

$-4-3 < -2t < 4-3$

$-7 < -2t < 1$

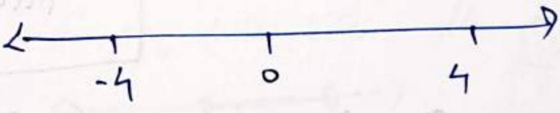
$-7/-2 > t > 1/-2$

$7/2 > t > -\frac{1}{2}$

$(-1/2, \frac{7}{2})$

Note:
Multiply with -ve sign changes the Inequality.

Ex. Solve $|3-2t| > 4$

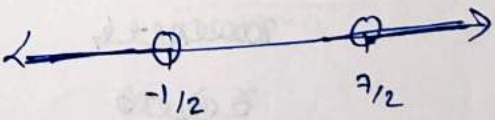


$3-2t < -4$ or $3-2t > 4$

$-2t < -7$ or $-2t > 1$

$t > \frac{7}{2}$ or $t < -\frac{1}{2}$

$(-\infty, -\frac{1}{2}) \cup (\frac{7}{2}, \infty)$



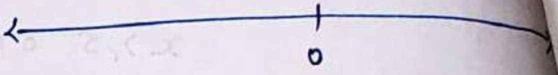
Ex. Solve $2|4x+5|+7 \geq 1$

$2|4x+5| \geq -6$

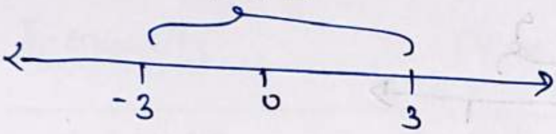
$|4x+5| \geq -3$

always true

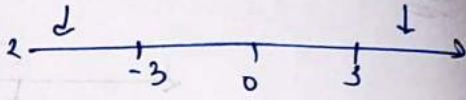
$(-\infty, \infty)$



$|x+2| < 3$



$|x+2| > 3$



Solving linear Inequalities

$3x+2 \leq x-7$

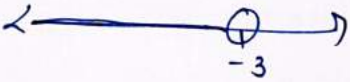
Ex. Soln : $-5(x+2)+3 > 8$

$-5(x+2) > 5$

$x+2 < -1$

$x < -3$

$(-\infty, -3)$



Note:

If you multiply or divide by negative numbers then you have to change the inequality

$-1(-x) < -1(-5)$

$x > 5$

Ex. Solve $3x-4 \leq x-8$ or $6x+1 > 10$

$3x \leq x-4$ or $6x > 9$

$x > 9/6$

$x > 3/2$

$2x \leq -4$

$x \leq -2$

$(-\infty, -2] \cup (3/2, \infty)$

Sol: $-\frac{2}{3}y > -12$ AND $-4y+2>5$

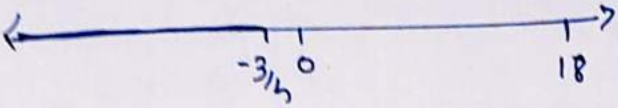
$-\frac{2}{3}y \times 3 > -12 \times 3$ AND $-4y > 3$

$-2y > -36$

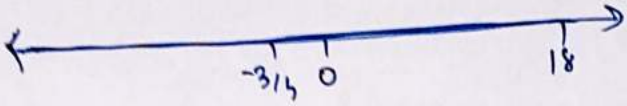
$2y < 36$

$y < 18$

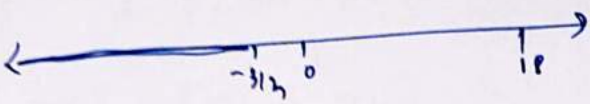
$y < -\frac{3}{4}$



$y < -3/4$



$y < 18$



$y < -3/4$ AND $y < 18$

OR $\rightarrow \cup$
AND $\rightarrow \cap$

Ex. $-3 \leq 6x-2 < 10$

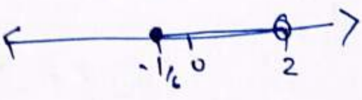
$\cancel{\frac{1}{7} \geq \frac{6x-2}{-3} > \frac{10}{3}}$

$\cancel{-\frac{10}{3} \leq \frac{6x-2}{-3} \leq 1}$

$-3 \leq 6x-2 < 10$

$-1 \leq 6x < 12 \quad +2$

$-1/6 \leq x < 2 \quad /6 \quad [-1/6, 2)$



Polynomial and Rational Inequalities

$x^3 \geq 5x^2+6x$

$\frac{x^2+6x+9}{x-9} \leq 0$

Ex. Solve $x^2 < 4$

$x^2 < 4$

$x^2 - 4 < 0$

$\downarrow$

$x^2 - 4 = 0$

$(x-2)(x+2) = 0$

$x = 2 \quad x = -2$

$\boxed{-2 < x < 2}$

$\oplus \quad \ominus \quad \oplus$

$\longleftarrow -2 \quad 0 \quad 2 \longrightarrow$

$(-3)^2 - 4$

$9 - 4$

$= 5$

$x = 0$

$0^2 - 4$

-4

$x = 10$

$(10)^2 - 4 = 100 - 4 = 96$

Ex. Solve: $x^3 \geq 5x^2 + 6x$

$x^3 - 5x^2 - 6x \geq 0$

$x^3 - 5x^2 - 6x = 0$

$x(x^2 - 5x - 6) = 0$

$x(x^2 - 6x + x - 6) = 0$

$x(x(x-6) + 1(x-6)) = 0$

$x((x+1)(x-6)) = 0$

$x = 0 \quad x = -1 \quad x = 6$

$\ominus \quad \oplus \quad \ominus \quad \oplus$

$\longleftarrow -1 \quad 0 \quad 6 \longrightarrow$

$-2(-8)(-1)$

$2(-8) = -16$

$(-1, 2)$

$[-1, 0] \cup [6, \infty)$

Ex. Solve: $\dfrac{x^2 + 6x + 9}{x - 1} \leq 0$

$\dfrac{x^2 + 3x + 3x + 9}{x - 1} \leq 0$

$\dfrac{x(x+3) + 3(x+3)}{(x-1)}$

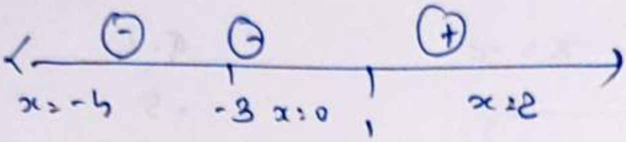
$\dfrac{(x+3)(x+3)}{(x-1)} \leq 0$

$$\frac{(x+3)(x+3)}{(x-1)} \le 0$$

$$\frac{(x+3)(x+3)}{(x-1)} = 0$$

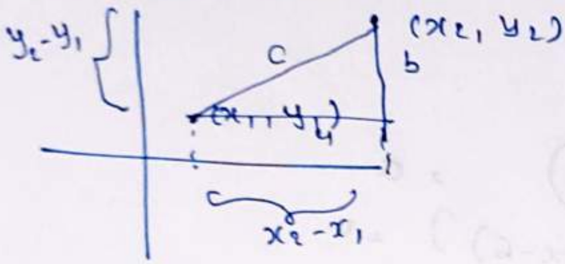
$$(x+3)^2 = 0$$

$$x = -3$$



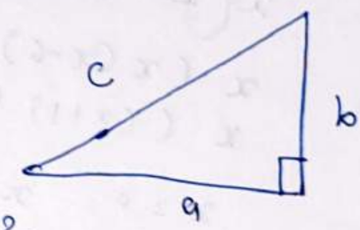
Ans. $\boxed{x < 1 \quad (-\infty, 1)}$

Distance formula



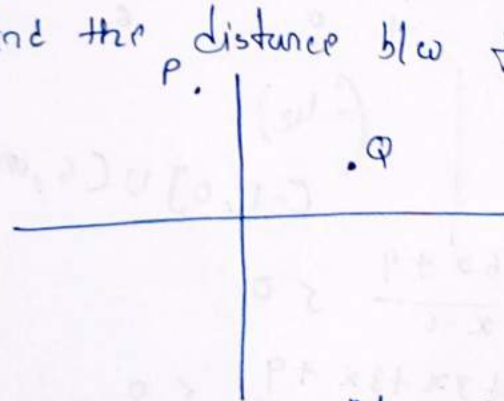
$$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

$$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$



$$(x_2-x_1)^2 + (y_2-y_1)^2 = d^2 \qquad a^2+b^2=c^2$$

Ex. find the distance b/w the points P(−1, 5) & Q(4, 2)



1st method:

$$(4+1)^2 + (2-5)^2$$

$$5^2 + (-3)^2$$

$$25 + 9 = 34$$

$$d = \sqrt{34}$$

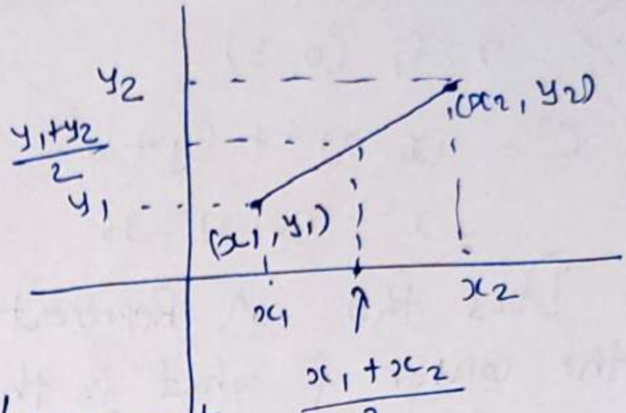
2nd meth

$$= \sqrt{(-1-4)^2 + (5-2)^2}$$

$$= \sqrt{25+9} = \sqrt{34}$$

Midpoint formula

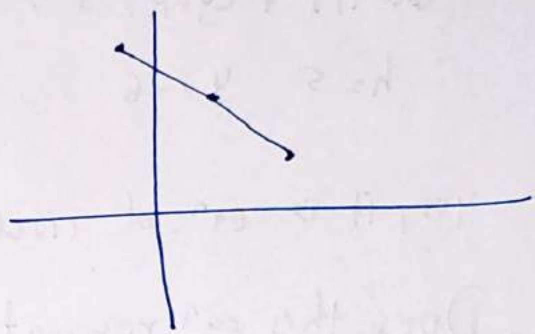
$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$



Ex:> find the midpoint of the segment between the points $P(-1,5)$ & $Q(4,2)$

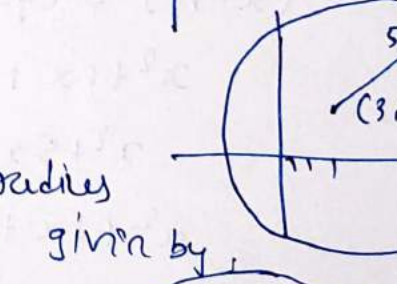
$$\left(\frac{-1+4}{2}, \frac{5+2}{2}\right)$$

$$\left(\frac{3}{2}, \frac{7}{2}\right)$$



Graph & Equations

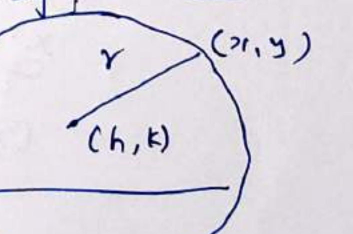
Ex. find the equation of a circle of radius 5 centered at the point $(3,2)$



$$5 = \sqrt{(x-3)^2+(y-2)^2}$$

$$25 = (x-3)^2+(y-2)^2$$

$$\boxed{(x-3)^2+(y-2)^2 = 25}$$



Note: The eqn. of a circle with radius r centered at point (h,k) is given by

$$\text{dist.} = r = \sqrt{(x-h)^2+(y-k)^2}$$

$$r^2 = (x-h)^2+(y-k)^2$$

$$\boxed{(x-h)^2+(y-k)^2 = r^2}$$

Ex.) $r = 6$, $(0,-3)$

$$6^2 = (x-0)^2 + (y+3)^2$$

$$= x^2 + (y+3)^2 = 36$$

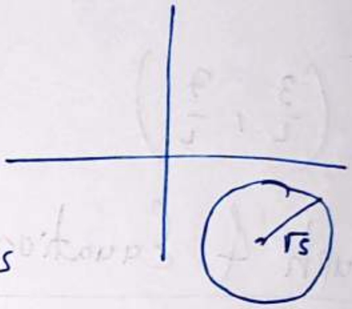
Ex. Does this eqn. Represent a circle? If so, what is the center & what is the radius?

$$(x-5)^2 + (y+6)^2 = 5$$

$$(x-h)^2 + (y-k)^2 = r^2$$

$h = 5$ $\quad y = -6$ $\qquad r^2 = 5$

$r = \sqrt{5}$



• yes, it is eqn. of circle.

Ex.) Does this eqn represent a Circle? If so, what is the center & what is the radius?

$$9x^2 + 9y^2 + 72x - 18y + 36 = 0$$

$$9x^2 + 72y + 18 + 9y^2 - 18y + 36 = 0.$$

$$x^2 + 8x + 2 + y^2 - 2y + 4 = 0.$$

$$(x-h)^2 + (y-k)^2 = r^2$$

$$x^2 + 8x + y^2 - 2y + 4 = 0$$

$$x^2 + 8x + y^2 - 2y = -4$$

$$x^2 + 8x + 16 + y^2 - 2y = -4 + 16$$

$$\underbrace{x^2 + 8x + 16} + \underbrace{y^2 - 2y + 1} = -4 + 16 + 1$$

$$(x+4)^2 + (y-1)^2 = 13$$

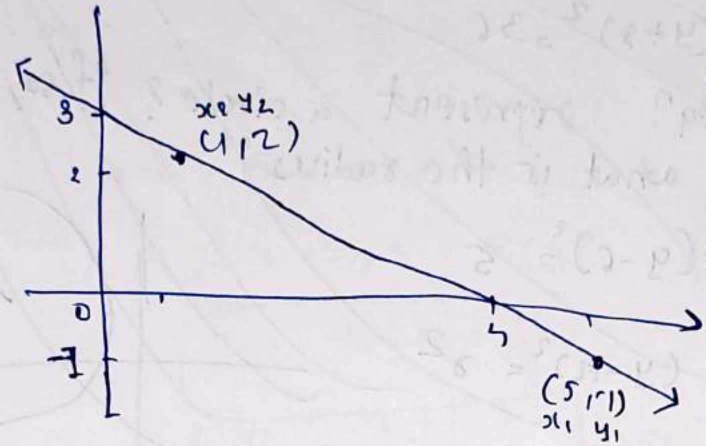
$$(x-h)^2 = (x-h)(x-h)$$

$$(x-h)^2 = (x-h)(x-h)$$
$$= x^2 - hx - hx + h^2$$
$$= x^2 - 2hx + h^2$$

$$\boxed{\begin{array}{l} \text{center} : (-4, 1) \\ \text{Radius} : \sqrt{13} \end{array}}$$

// lines : Graph & Equation

Ex. find the eqn. of this line.



$y = mx + b$

slope ↑ ; y-intercept ↑

slope = $\frac{rise}{run} = \frac{\Delta y}{\Delta x}$

$= \frac{y_2 - y_1}{x_2 - x_1}$

$\frac{2+1}{1-5} = \frac{3}{-4}$

$\frac{-1-2}{5-1} = -\frac{3}{4}$.

$y = mx + b$

$y = -3/4\, x + b$

pt: (1, 2) for (x, y)

$2 = -3/4(1) + b$

$2 = -\frac{3}{4} + b$

$8 = -3 + 4b$

$11 = 4b$

$11/4 = b$

$2 + \frac{3}{4} = b$

$\frac{8+3}{4} = b$

$\frac{11}{4} = b$

$$\boxed{y = -\frac{3}{4}x + \frac{11}{4}}$$

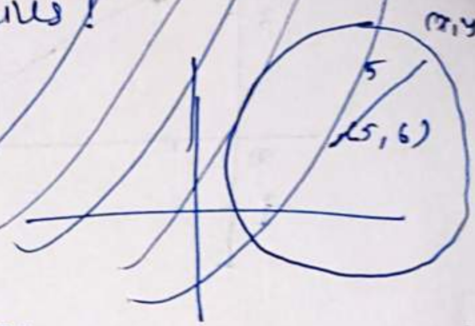
Ex.) $r = 6$, $(0,-3)$

$$6^2 = (x-0)^2 + (y+3)^2$$
$$= x^2 + (y+3)^2 = 36$$

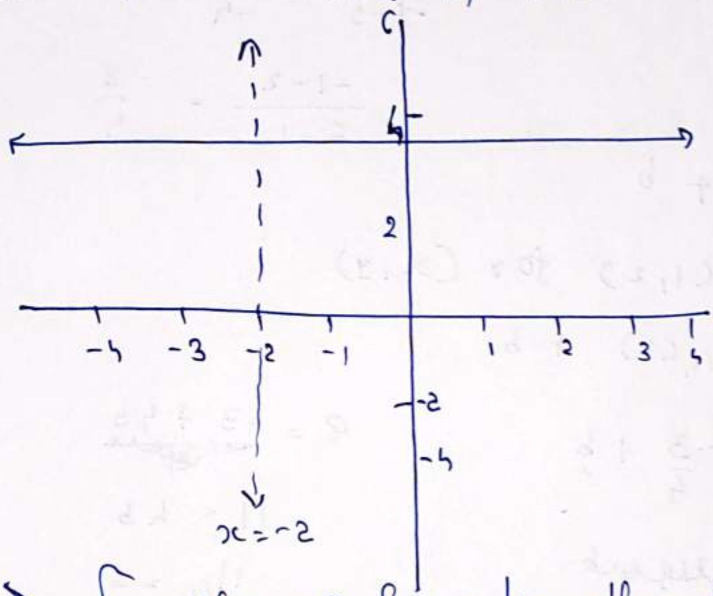
Ex.) Does this eq^n represent a circle? If so, what is the center and what is the radius?

$$(x-5)^2 + (y-6)^2 = 5$$

$$(x-h)^2 + (y-k)^2 = r^2$$



Ex. find the eq^n of these lines.



horizontal line

$y = mx + b$

$y = b$ $\quad m = 0$

$y = 3.5$

Ex.) find the eq^n of the line through the points $\overset{x_2\, y_2}{(1,2)}$ and $(4,-3)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2+3}{1-4} = \frac{5}{-3}$$

Method 1: $y = mx + b$

$y = -5/3\, x + b$ $\quad$ P.f. $(1,2)$

$2 = -5/3 + b$

$2 + 5/3 = b$

$11/3 = b$

$$y = -5/3\, x + \frac{11}{3}$$

Method 2:

$$\boxed{y - y_0 = m(x - x_0)}$$

where (x_0, y_0) is a pt on the line, m is slope.

$m = \frac{-3-2}{4-1} = \frac{-5}{3}$

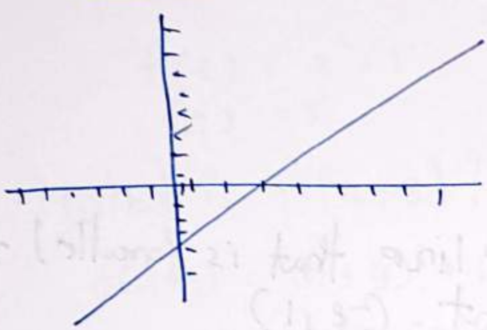
pt. $(1, 2)$

$y - 2 = -\frac{5}{3}(x - 1)$

$y - 2 = -5/3\, x + 5/3$

$$\boxed{y = -5/3\, x + \frac{11}{3}}$$

Parallel & Perpendicular lines



Parallel lines have the same slope.

For perpendicular

m_1	m_2 opposite Reciprocal
2	$-1/2$
$-1/3$	3
$\frac{7}{2}$	$-\frac{2}{7}$

Ex. Find the Eqn. of a line that is parallel to the line $3y - 4x + 6 = 0$ & Goes through the point $(-3, 2)$.

$3y = 4x - 6$

$y = \frac{4}{3}x - 2$

$m_1 = \frac{4}{3}$, $m_2 = \frac{4}{3}$

$m_1 = m_2$ for parallel

New line,

$y = \frac{4}{3}x + b$

$2 = \frac{4}{\cancel{3}}(-\cancel{3}) + b$

$2 = -4 + b$

$b = 6$

$y = \frac{4}{3}x + 6$

$3y = 4x + 18$

$3y - 4x - 18 = 0$

Ex.1) Find the eqn. of a line that is perpendicular line $6x+3y=4$ and Goes through the point

$$6x+3y=4$$

$$3y=-6x+4$$

$$y=-2x+4/3 \qquad m_1=-2$$

$$m_1=\frac{1}{2}$$

$$y=1/2x+b \qquad \text{Pt. } (4,1)$$

$$1=\frac{1}{2}(4)+b$$

$$1=2+b$$

$$b=-1$$

$$\boxed{y=1/2x-1}$$

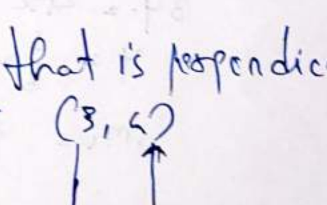
Ex.b) Find the eqn. of the line that is parallel to $y=3$ & Goes through the point $(-2,1)$.

$$\rightarrow \quad y=mx+b$$

$$1=0\times -2+b$$

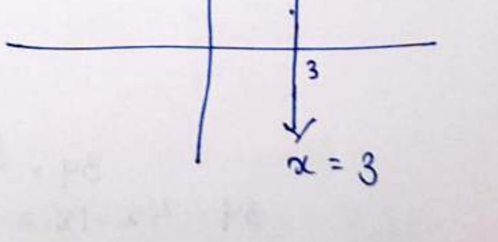
$$1=b$$

$$\boxed{y=1}$$



Ex.c) Find the Eqn. of the line that is perpendicular to $y=4$ & Goes through the point $(3,4)$

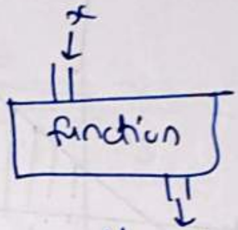
$$x=3$$



Functions

Defination: A function is corrispondance between input numbers (x-values) and output numbers (y-value) that sends each input number (x-value) to exactly one output number (y-value).

Sometimes, a function is described with an eq?



Ex. $y = x^2+1$, which can also be written as $f(x) = x^2+1$, what is $f(2)$? $f(5)$?

$f(2) = 2^2+1$ $\quad$ $f(5) = 25+1 = 26$

$f(2) = 5$

what is $f(a+3)$?

$f(a+3) = (a+3)^2+1$

$= a^2+6a+10$

Note:
$f(x)$ is notation for fun not a multiplication

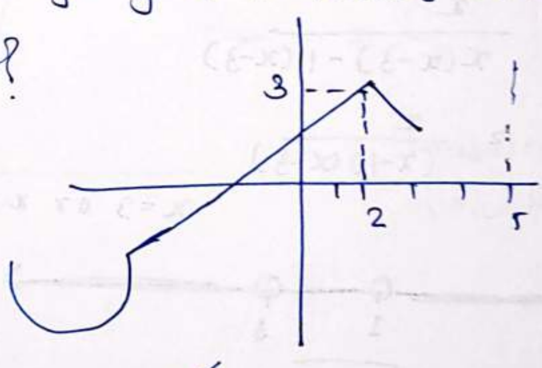
Sometimes, a function is described with a Graph

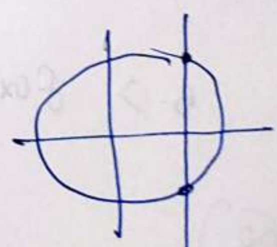
Ex. The Graph of $y = g(x)$ is shown below

$g(2) = ?$ $g(5) = ?$

$g(2) = 3$

$g(5) =$ undefined does not exist

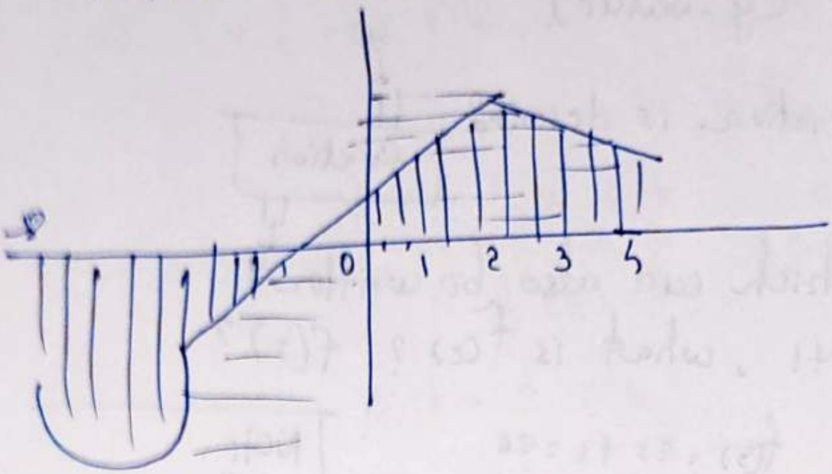




not a function

Defination: The domain of a function is all Possible
x-values. The range is the y-values.

Ex: What is the domain of & Range of the fun^n g(x) graphed below?



domain for g(x)

$-8 \leq x \leq 4$

$[-8, 4]$

Range for g(x)

$-5 \leq y \leq 3$

$[-5, 3]$

Ex: What are the domains of these function?

A. $g(x) = \dfrac{x}{x^2-4x+3}$

$= \dfrac{x}{x^2-3x-x+3}$

$= \dfrac{x}{x(x-3)-1(x-3)}$

$g(x) = \dfrac{x}{(x-1)(x-3)}$

To find domain,
1) Exclude x-values that solve denominator 0
2) Exclude x-value that make an invalid complex

$x=3$ or $x=1$ <- Exclude these values

$(-\infty, 1) \cup (1, 3) \cup (3, \infty)$

B. $f(x) = \sqrt{3-2x}$

Exclude $3-2x < 0$

$3-2x \geq 0$

$3 \geq 2x$

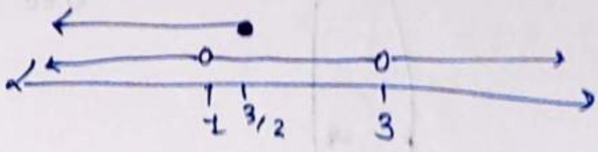
$x \leq 3/2$

$(-\infty, 3/2]$

$$f(x) = \frac{\sqrt{3-2x}}{x^2-4x+3}$$

Need $x^2-4x+3 \neq 0 \iff x \neq 3$ and $x \neq 1$

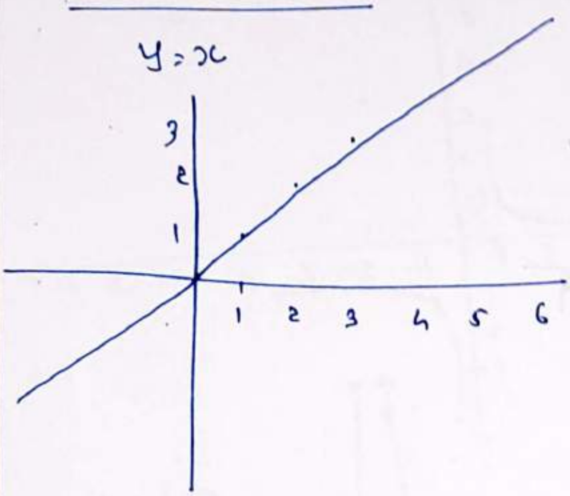
$3-2x \geq 0 \iff x \leq 3/2$



$(-\infty, 1) \cup (1, 3/2]$

Toolkit function.

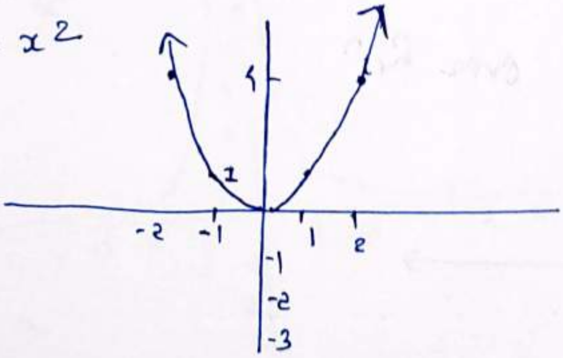
$y = x$



$x = 1,\ y = 1$
$x = 2,\ y = 2$

line

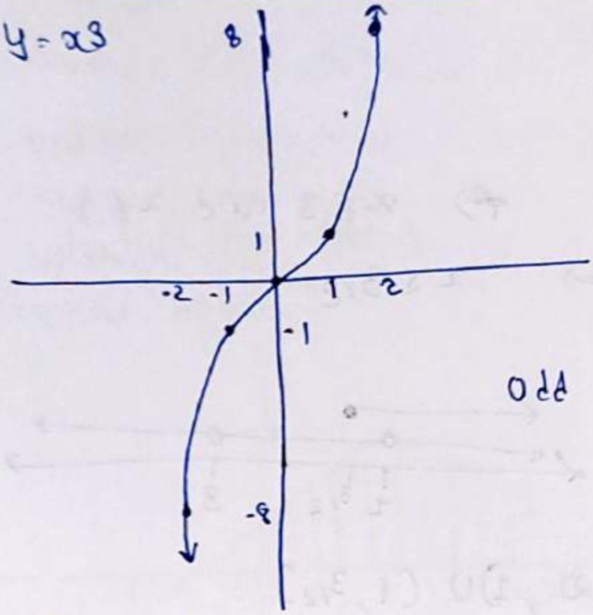
$y = x^2$



$x = 1,\ y = 1$
$x = -1,\ y = 1$

Parabola
Even function

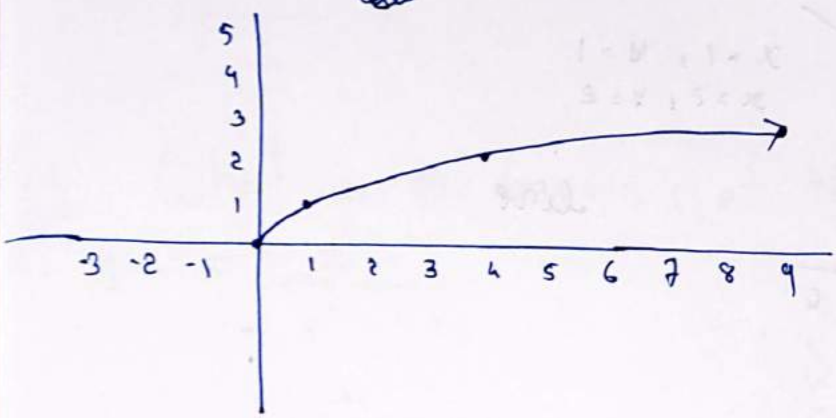
$y = x^3$



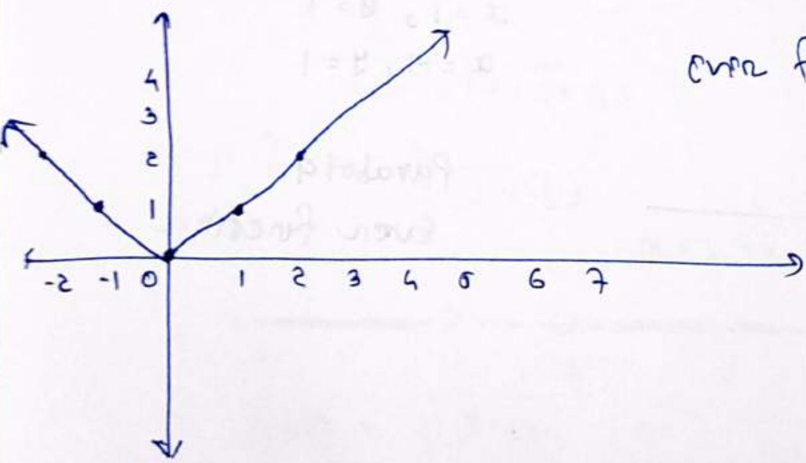
x	y
0	0
1	1
-1	-1
2	8
-2	-8

Odd function

$y = \sqrt{x}$ domain $x \geq 0$

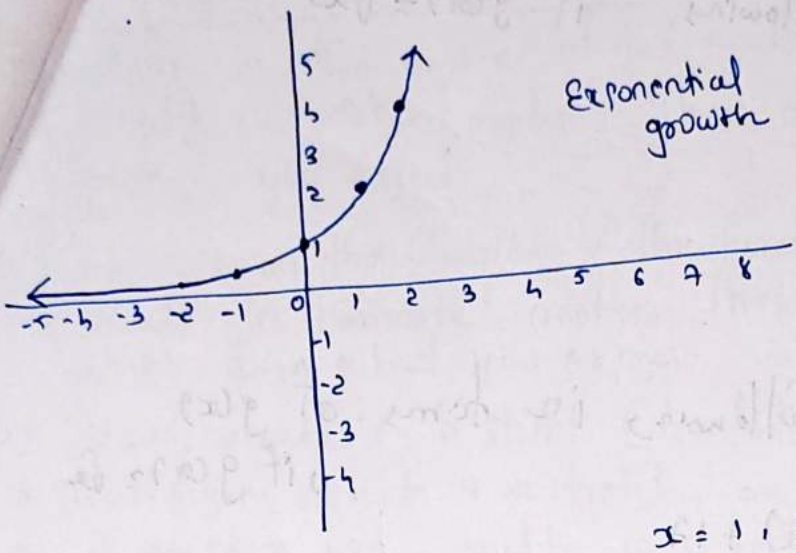


$y = |x|$



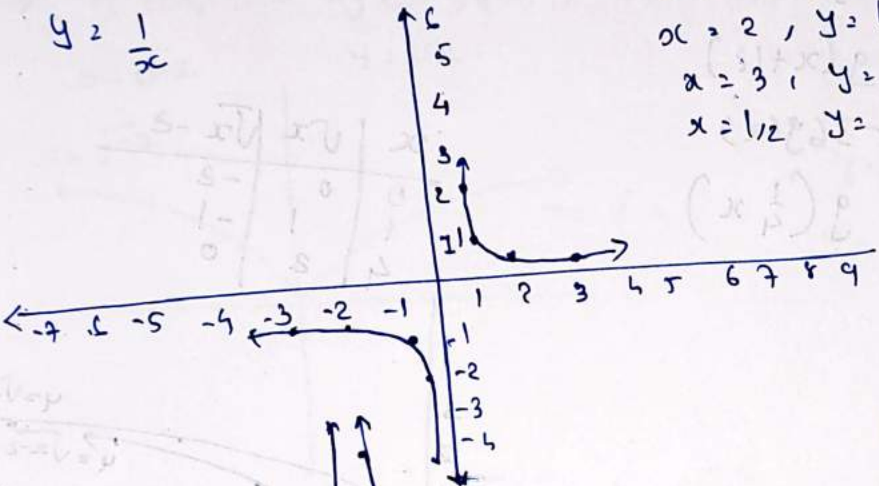
even fn.

2^x :- Exponential function



$x = 0,\ y = 1$
$x = 1,\ y = 2$
$x = 2,\ y = 4$
$x = -1,\ y = 1/2$
$2^{(-1)}$
$= 1/2$
$x = -2,\ \frac{1}{2^2} = \frac{1}{4}$

$y = \frac{1}{x}$



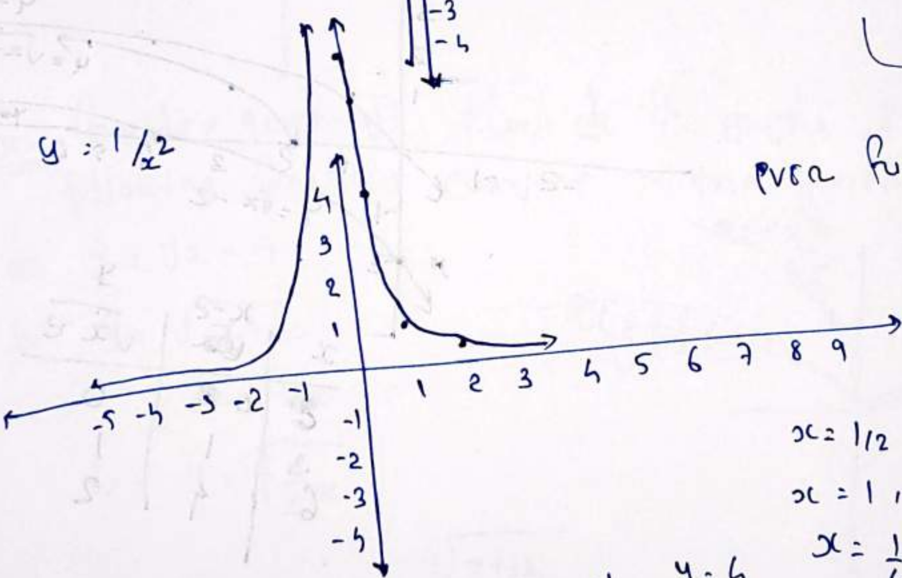
$x = 1,\ y = 1$
$x = 2,\ y = 1/2$
$x = 3,\ y = 1/3$
$x = 1/2\ \ y = 2$

$x = -1,\ y = -1$
$x = -2,\ y = -1/2$
$x = -3,\ y = -1/3$
$x = -1/2\ \ y = -2$

hyporbola
odd fu^n

$y = 1/x^2$

even function



$x = 1/2,\ y = 4$
$x = 1,\ y = 1$
$x = \frac{1}{4},\ y = 16$
$x = -1/2,\ y = 4$
$x = -1,\ y = 1$
$x = 1/3\ \ y = 9$
$x = 2,\ y = 1/4$

Transformation of function

Ex. Rewrite the following. if $g(x) = \sqrt{x}$

a) $g(x) - 2 = \sqrt{x} - 2$
b) $g(x-2) = \sqrt{x-2}$
c) $g(3x) = \sqrt{3x}$
d) $3g(x) = 3\sqrt{x}$
e) $g(-x) = \sqrt{-x}$

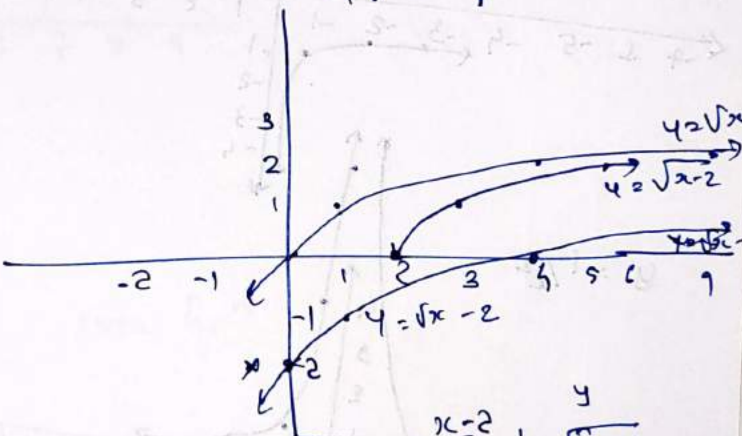
Ex. Rewrite the following in terms of $g(x)$, if $g(x) = \sqrt{x}$

f) $\sqrt{x} + 17 = g(x) + 17$
g) $\sqrt{x+12} = g(x+12)$
h) $-36 \cdot \sqrt{x} = -36g(x)$
i) $\sqrt{\frac{1}{4}x} = g\left(\frac{1}{4}x\right)$

Ex. Graph.

$y = \sqrt{x}$
$y = \sqrt{x} - 2$
$y = \sqrt{x-2}$

x	$\sqrt{x}$	$\sqrt{x} - 2$
0	0	-2
1	1	-1
4	2	0



x	$x-2$	y $\sqrt{x-2}$
2	0	0
3	1	1
6	4	2

Rules for transformation:

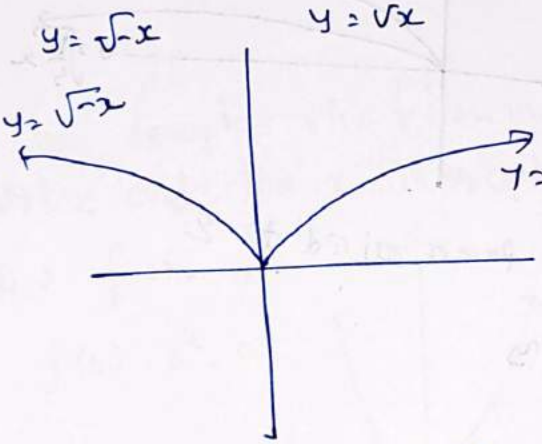
Numbers on the outside of the function affect the y-values & result in vertical motions. These motions are in the direction you expect.

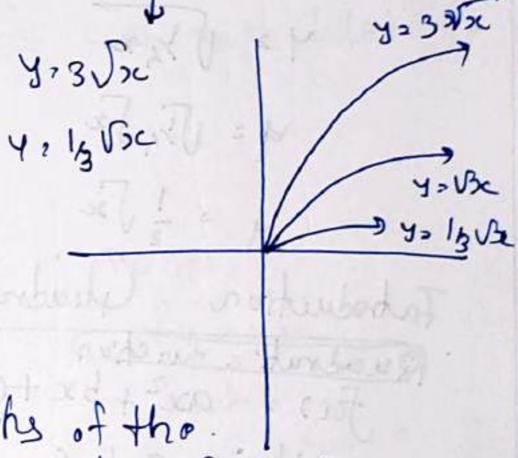
→ Numbers on the Outside of the function affect the x-values & result in horizontal motions. These motions go in the opposite direction from what you expect.

→ Adding results in a shift (translations)

→ Multiplying results in a stretch or shrink

→ A negative sign results in a reflection





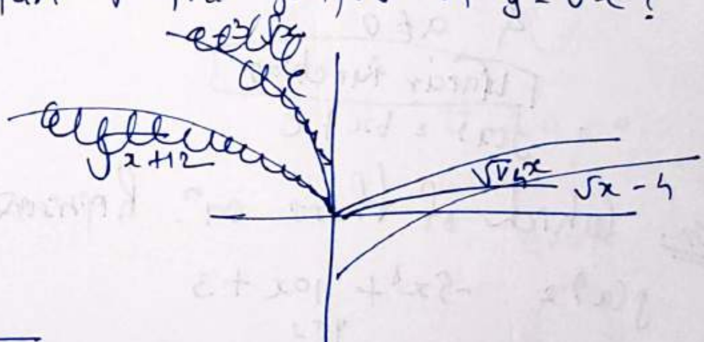
Ex. Consider $g(x) = \sqrt{x}$. How do the graphs of the following functions compare to the graph of $y = \sqrt{x}$?

a) $y = \sqrt{x} - 4$

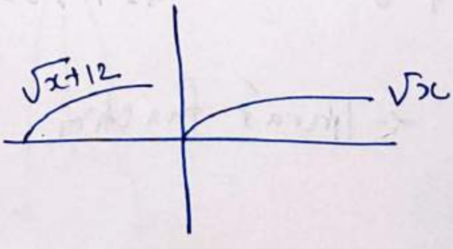
b) $y = \sqrt{x+12}$

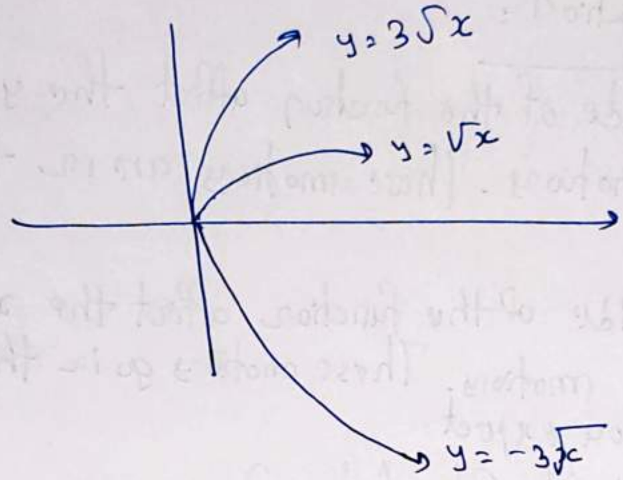
c) $y = -3.\sqrt{x}$

d) $y = \sqrt{\frac{1}{4}x}$



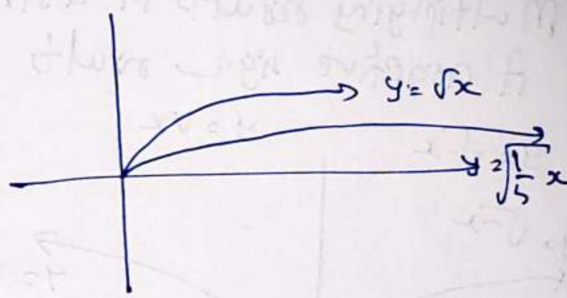
$\sqrt{x+12}$





$\Rightarrow y = \sqrt{\frac{1}{4}x}$

stretching horizontally by factor of 4



$y = \sqrt{\frac{1}{4}x}$

$y = \sqrt{\frac{1}{4}}\sqrt{x}$

$y = \frac{1}{2}\sqrt{x}$ → power raised to 2

Introduction Quadratic functions

Quadratic function

$f(x) = ax^2 + bx + c$

where a, b, c are real values

& $a \neq 0$

Linear function

$f(x) = bx + c$

Ex. Which of these eq^n. Represent Quadratic function?

1) $g(x) = -5x^2 + 10x + 3$
yes

2) $f(x) = x^2 + 0x + 0$ $\quad a = 1, b = 0, c = 0$
yes

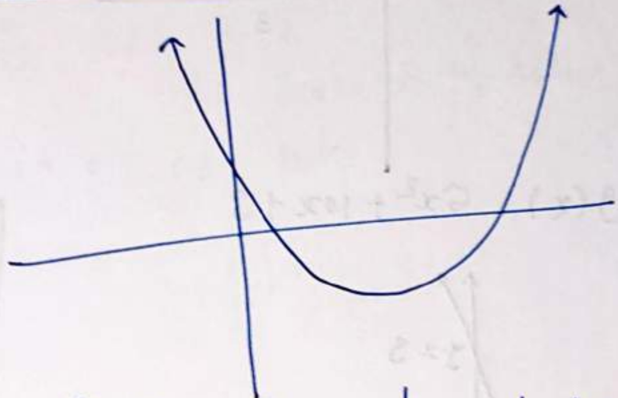
3) $y = 3x - 2$
no. ∝ - linear function

34

$$y = 2(x-3)^2 + 4$$
$$= 2(x^2 - 6x + 9) + 4$$
$$= 2x^2 - 12x + 22$$ yes

Graphing Quadratic functions

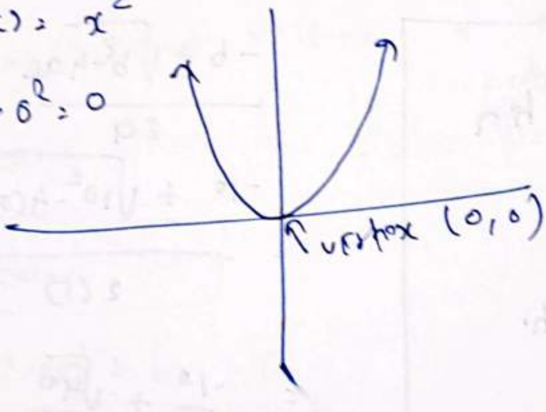
$$y = ax^2 + bx + c$$
$$y = a(x-h)^2 + k$$



Ex. Graph the following functions. For each graph label the vertex and the x-intercepts.

A.) $f(x) = x^2$

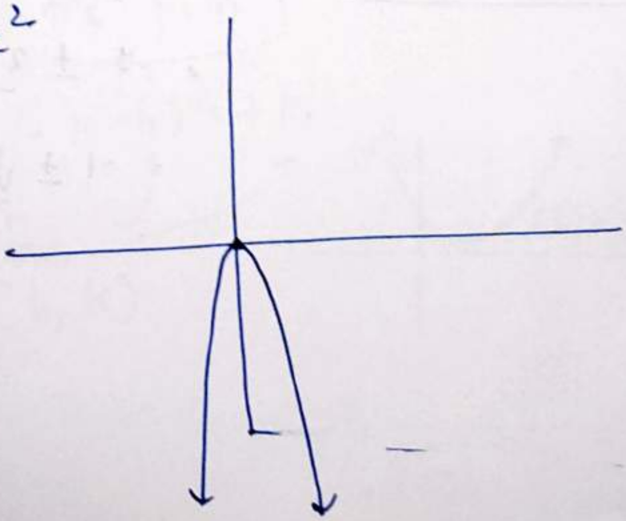
$f(0) = 0^2 = 0$



note:
If $a > 0$
Parabola
upward

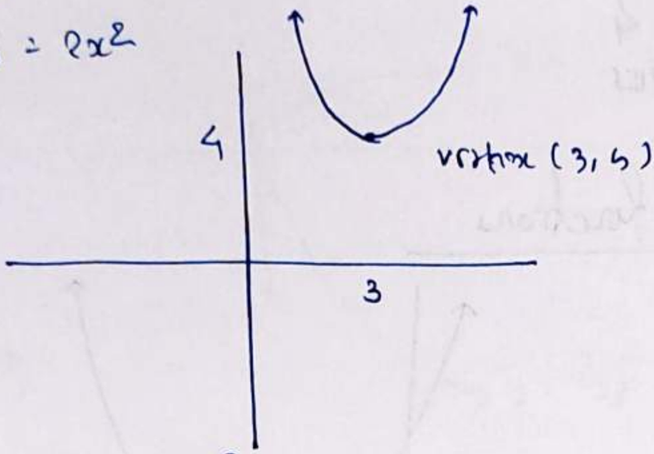
If $a < 0$
Parabola
down

B.) $y = -3x^2$

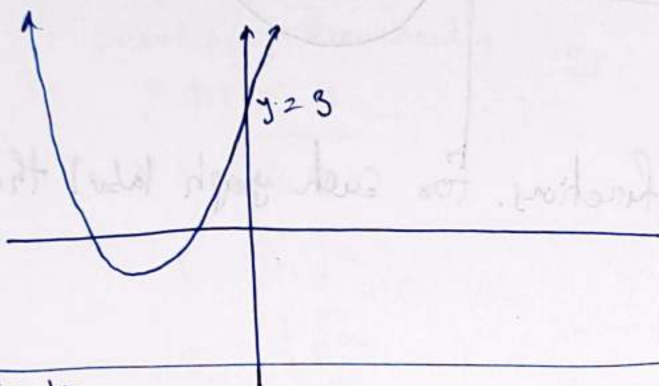


C) $y = 2(x-3)^2 + 4$

$y = 2x^2$



D) $g(x) = 5x^2 + 10x + 3$



Note:
when Quadratic eqn. written in standard form.

$y = ax^2 + bx + c$

vertex has an x coordinate has $-b/2a$

any function written in form of

$y = a(x-h)^2 + k$

has vertex at (h, k)

vertex form

---

x intercepts at $y = 0$

$0 = 5x^2 + 10x + 3$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-10 \pm \sqrt{10^2 - 4(5)(3)}}{2(5)}$$

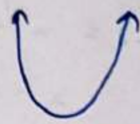
$$= \frac{-10}{10} \pm \frac{\sqrt{40}}{10}$$

$$= -1 \pm \frac{2\sqrt{10}}{10}$$

$$= -1 \pm \frac{\sqrt{10}}{5}$$

Summary:

- Graph a quadratic function $f(x) = Ax^2 + bx + c$
1) The graph has the shape of parabola



2) The parabola opens up if $a > 0$ & down if $a < 0$
3) To find the x-intercepts set $y = 0$ i.e. $f(x) = 0$ & solve for x.
4) To find vertex read it off as h, k

$$y = a(x-h)^2 + k$$

or

use vertex formula

x coordinate $= -b/2a$ if function in standard form.

$$y = ax^2 + bx + c$$

y = coordinate vertex :- plug in values from x coordinate

Standard form & Vertex form for Quadratic Functions

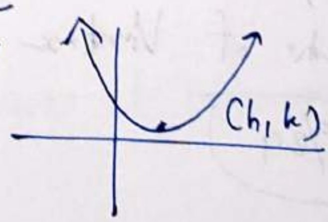
Standard form

$$y = ax^2 + bx + c$$

Vertex form

$$y = a(x-h)^2 + k$$

vertex (h, k)



Ex. Convert this quadratic function to standard form: $f(x) = -4(x-3)^2+1$

$$f(x) = -4(x-3)(x-3)+1$$
$$= -4(x^2-6x+9)+1$$
$$f(x) = -4x^2+24x-35$$

Ex. Convert this Quadratic fn. to Vertex form.

$g(x) = 2x^2+8x+6$ $\qquad g(x) = a(x-h)^2+k$

$= 2x^2+8x+6$

$= 2(x^2+4x+3)+1$

$= 2(x^2+2x+1)+1$

$x = \frac{-b}{2a}$

$= \frac{-8}{2(2)}$

$x = \frac{-8}{4} = -2$

$g(-2) = 2(-2)^2+8(-2)+6$

$= 8-16+6$

$= 14-16$

$g(-2) = -2$

vertex $(-2, 2)$

$g(x) = a(x-(-2))^2+(-2)$

$g(x) = a(x+2)^2-2$

$$\boxed{g(x) = 2(x+2)^2-2}$$

Justification of the vertex formula

$2(x^2+4x+4)-2$

Vertex formula

x coordinate of Vertex

$$\boxed{-b/2a}$$

(36)

find the x - intercepts and the vertex for $y = 3x^2+7x-5$

x intercepts

$0 = 3x^2 + 7x - 5$

$x = \dfrac{-7 \pm \sqrt{7^2 - 4(3)(-5)}}{2(3)}$

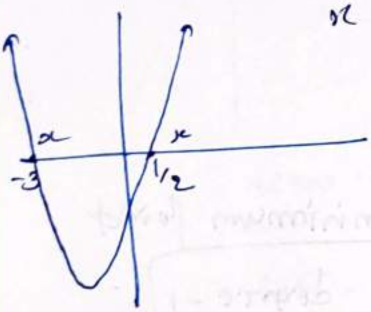
$x = \dfrac{-7 \pm \sqrt{49+60}}{6}$

$x = \dfrac{-7}{6} \pm \dfrac{\sqrt{109}}{6}$

$= -7/6 + \dfrac{\sqrt{109}}{6}$, $\dfrac{-7}{6} - \dfrac{\sqrt{109}}{6}$

$\simeq -\dfrac{7}{6} + \dfrac{10}{6}$ $\simeq -\dfrac{7}{6} - \dfrac{10}{6}$

$x \simeq \frac{1}{2}$ $x \simeq -3$



$x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

x coordinate of vertex formula.

Polynomials

$f(x) = 3x^4 - 2x^3 - 7x^2 + 6x + 7$

Definition:

Degree: The Degree of polynomials is the largest exponent. x^4

The leading term:

(39) the term with the largest Exponent.

$\rightarrow 3x^4$

Leading Coefficient → The leading coefficient is the number in the leading term.

The Constant term → 18

Example: for $p(x) = 5x^3 - 3x^2 - 7x^4 + 2x + 18$, what is the

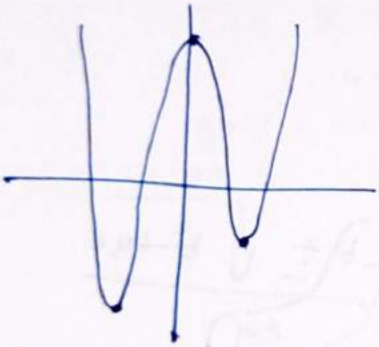
→ degree? → 4

leading term? → $-7x^4$

leading coefficient? → -7

Constant term? → 18

Defination: In the Graph of $f(x) = x^4 + 2x^3 - 15x^2 - 12x + 36$ below, the marked points are called ... turning point



turning point

or

local extreme point

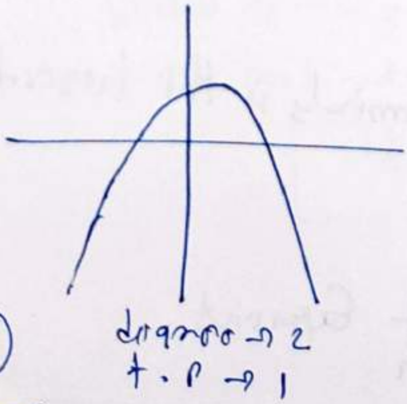
or

local maximum & minimum point

fact. turning pts ≤ degree − 1

Compare the degrees of the polynomials to the number of turning point.

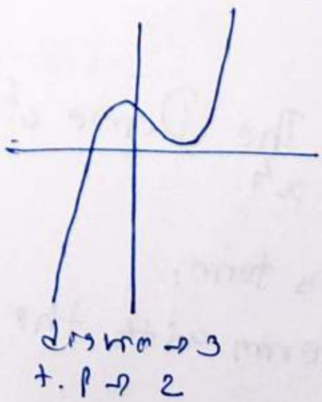
$f(x) = -2x^2 + 2x + 0$



degree → 2

t.p → 1

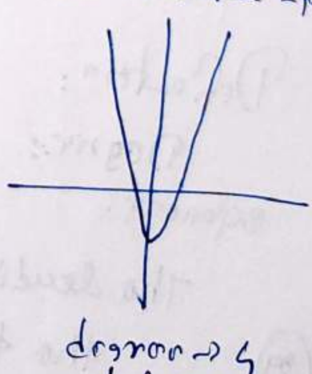
$f(x) = 3x^3 - 5x^2 - 7x + 13$



degree → 3

t.p → 2

$f(x) = x^4 + 6x^4 - 17$



degree → 4

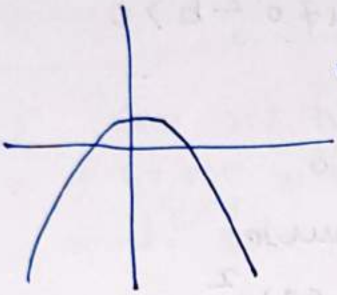
t.p → 1

Definition: The end behaviour of a function is how the "ends" of the function look as $x \to \infty$ & $x \to -\infty$.

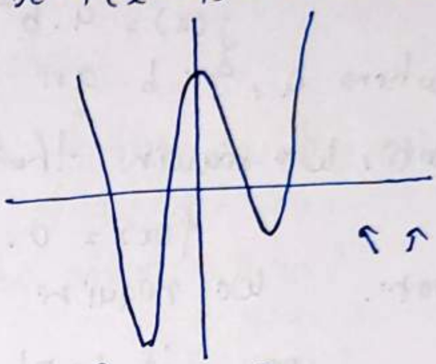
Consider the end behavior for these polynomials:

$f(x) = -2x^2 + 2x + 8$

$f(x) = x^4 + 2x^3 - 15x^2 - 12x + 36$



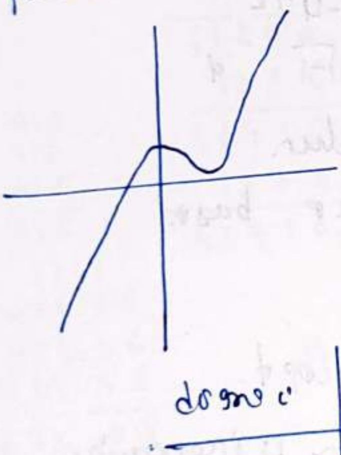
Ans: ↙ ↘



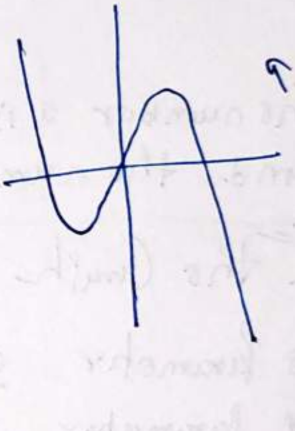
↑ ↗

$f(x) = 3x^3 - 5x^2 - 7x + 13$

$f(x) = -x^3 + 12x$



↙ ↗



↖ ↘

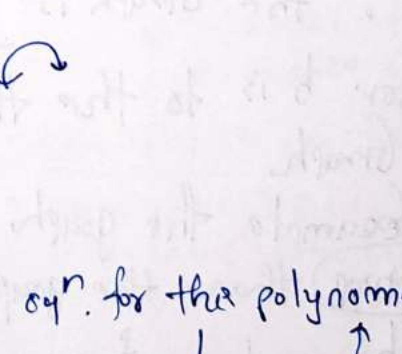
leading coeff.

degree	positive	negative
even	↑ ↗	↙ ↘
odd	↙ ↗	↖ ↘

$y = x^2$ $\quad$ $y = -x^2$

$y = x^3$ $\quad$ $y = -x^3$

Ex. What can you tell about the eq^n. for the polynomial graphed below?



degree n is odd

leading coeff. is positive

degree ≥ 5

Exponential functions

Defination: An exponential function is a function that can be written in the form

$$f(x) = a \cdot b^{x}.$$

where a, & b are real numbers, $a \neq 0$, & $b > 0$

note: We require that $a \neq 0$ because

$$f(x) = 0 \cdot b^{x} \rightarrow f(x) = 0$$

note. We require that $b > 0$ because

ex. if $b = -1$ $\quad f(x) = a \cdot (-1)^{x}$

$$f(1/2) = a \cdot (-1)^{1/2}$$

$$= a \cdot \sqrt{-1}$$

The number a is called the initial value and the number b is called the base.

Facts:

In the Graph of $y = a \cdot b^{x}$

The parameter a Gives the y-intercept.

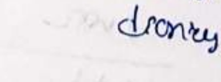
The parameter b tells how the graph is increasing or decreasing.

If $b > 1$, the Graph is increasing

If $b < 1$, the Graph is decreasing

The closer b is to the the number 1, the flatter the Graph

So, for example the graph of $y = 0.25^{x}$ is (circle one) more steep / flatter than the graph of $y = 0.5^{x}$.



Fact: The graph of an exponential function $y = a \cdot b^{x}$

- has a horizontal asymptote at line $y = 0$
- has domain $(-\infty, \infty)$
- has range $(0, \infty)$ if $a > 0$
  $(-\infty, 0)$ if $a < 0$

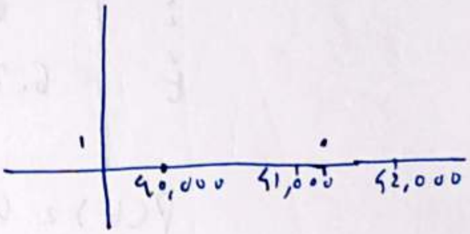
$f(x) = e^x$ <- most famous
$f(x) = \exp(x)$.
Euler's number
$e^x = 2.718281....$

Applications of Exponential functions

Ex.) You are hired for a job & the starting salary is \$40,000 with annual raise of 3% per year.
How much will ~~salary~~ your salary be after 1 year?
2 year?
5 year?
t year?

```
          40,000
           1200
         ------
          41,200   <- 1 year
           1236
         ------
412       42436.00 <- 2 year
412        1274.08
412      ---------
----      43710.08 <- 3 year
1236

424.36
424.36
424.36
------
1273.08
```



# year since hired	Salary
0	40,000
1	$1 \times 40,000 + 0.03 \times 40,000$ $= 40000(1+0.03)$ $= 40,000(1.03)$
2	$1 \times 40,000(1.03) + 0.03 \times 40,000(1.03)$ $= 40,000(1.03)(1+0.03)$ $= 40,000(1.03)(1.03)$ $= 40,000(1.03)^2$
3	$= 40,000(1.03)^2 \cdot (1.03)$ $= 40,000(1.03)^3$
t	$= 40,000(1.03)^t$

Ex. The united Nations estimated the world population in 2010 was 6.79 billion, growing at a rate of 1.1% per year. Assume that the growth rate stays the same. Write an equation for the population at t years after the year 2010.

→

2010	6.79 billion
2011	$6.79 + 0.011 \times 6.79$
	$6.79(1+0.011)$
	$6.79(1.011)$
2012	$6.79(1.011)^2$
⋮	
t	$6.79(1.011)^t$

$P(t) = 6.79(1.011)^t$

$P(40) = 6.79(1.011)^{40}$
$= 10.5$ billion

Ex Seroquel is metabolized & eliminated from the body at a rate of 11% per hour. If 400 mg are given, how much remains in the body after 24 hr?

time in hr	# mg of seroquel
0	400
1	$400 - 0.11 \times 400 = 400(1-0.11)$ $= 400 \times 0.89$ $= 400(0.89)$
2	$400 . 0.89 . 0.89$ $= 400(0.89)^2$
⋮	
t	$400(0.89)^t$

$f(t) = 400(0.89)^t$

$f(t)$ = No. mg of seroquel

t = # hr since dose

Applications of Exponential functions

$$f(t) = A \cdot b^t$$

A ↑ Initial amount; b ↑ growth factor

Example: An antique car is worth \$ 50,000 now & its value increases by 7% each year. Write an eqn. to model value x years from now.

$$\boxed{V(x) = A \cdot b^x = A(1+r)^x}$$

$$V(x) = 50,000 \, (1.07)^x$$

Ex. My Toyota Prius is worth \$ ~~30000~~ 3000 now & its value decreases by 5% each year. Write an eqn. to model its value x years in the future.

$\Rightarrow$ after 1 year $= 3000 - 0.05 \times 3000$

$= 3000\,(1-0.05)$

$3000\,(0.95)$

after 2 year value $= 3000\,(0.95)^2$

after x year, value $3000\,(0.95)^x$

$$V(x) = 3000\,(0.95)^x$$

$$\boxed{V(x) = Ab^x = A(1-r)^x}$$

Ex. The number of bacteria in a petri dish is modelled by the eqn. $f(x) = 12.(1.45)^x$

where $f(x)$ represents the number of bacteria in thousands, and x represents the number of hours since noon.

→ what was the number of bacteria at noon?

→ By what percent is the number of bacteria increasing every year?

$$f(x) = 12\,(1.45)^x$$

$$f(x) = 12\,(1+0.45)^x$$

$\boxed{\text{Ans. } 45\% \text{ increase}}$

Ex. The population of salamanders $S(x)$ on an island is modeled by $S(x) = 3000 \cdot 0.78^x$

where x is the number of years since 2015.

→ what was the population in 2015?

By what percent is the population decreasing every years?

$3000(1-0.22)^x$

22 % ← decreasing.

$= 3000(0.78)$

$= 2,340.$

Compound Interest

Ex. Suppose you invest \$200 in a bank account that earns 3% interest every year. If you make no deposits or withdrawals, how much money will you have after 10 years?

$200 + 200 \times 0.03$

$200(1+0.03)$

$200(1.03)^{10}$

$$\boxed{200(1.03)^{10} = 268.78 \$}$$

$P(t) = 200(1.03)^t$

Interest A allows at an annual interest rate of r for t years

$P(t) = A(1+r)^t$

r 0.03 for 3%.

Ex. You deposit \$300 in an account that earns 4.5% annual interest compounded semi-annually. How much money will you have 7 years?

→ 4.5 % annual interest rate compounded 2 time a year

$\frac{4.5\%}{2}$ interest every 6 months

$= 2.25\%$ interest each 1/2 year

0.0225

every time we earn interest, money get multiplied by (1.0225)

# years	# half years	money
0	0	300
1/2	1	300 (1.0225)
1	2	$300 (1.0225)^2$
1.5	3	$300 (1.0225)^3$
2	4	$300 (1.0225)^4$
t	2t	$300 (1.0225)^{2t}$

$P(t) = 300 (1.0225)^{2t}$

t = No. of years

$P(7) = 300 (1.0225)^{2 \cdot 7}$

$= 409.65$

Ex. You take out a loan for $1,200 at annual interest rate of 6%, compounded monthly. If you pay back the loan with interest as a lump sum, how will you owe after 3 years?

→ 6% annual interest rate compounded monthly

compounding 12 times per year

6/12 % interest each time interest is added

0.5% interest decimal 0.005

time in year	# months	money
0	0	1200
1	12	$1200 (1.005)^{12}$
2	24	$1200 (1.005)^{24}$
3	36	$1200 (1.005)^{36}$
t	12t	$1200 (1.005)^{12t}$

$P(t) = 1200 (1.005)^{12t}$

t = no. of year

$= 1200 (1.005)^{36}$

$= 1436.02$

Ex You invest \$4000 in an account that gives 2.5% interest compounded continuosly. How much money will you have after 5 years?

A = initial amount

r = annual interest rate

compounded n times per year

$$\boxed{P(t) = A\left(1+\frac{r}{n}\right)^{nt}}$$

→ formula for <u>Continous compounding</u> $P(t) = A \cdot e^{rt}$

$P(t)$ = amount of money

t = time in year

A = initial amount of mon

r = annual interest rate written as decimal.

$$P(t) = 4000 \cdot e^{0.025t}$$

$$P(5) = 4000 \cdot e^{0.025(5)} = 4532.59$$

Summary:

→ Let r represent → annual interest rate, written as decimal.

→ let t represent → number of year

→ let A represent the initial amount of money

1) Annual interest.

$$P(t) = A(1+r)^t$$

2) Compound interest, compounded n times per year,

$$P(t) = A(1+r/n)^{nt}$$

3) Compound interest, compounded Continously:

$$P(t) = A \cdot e^{rt}$$

Logarithms

Logarithms are way of writing Exponents.

Definition: $\log_a b = c$ means $a^c = b$.

You can think of logarithms as exponents. $\log_a b$ is the exponent (or "power") that you have to raise a to, in order to get b. The number a is called the base of the logarithm. The base is required to be a positive number.

$$\boxed{\log_a b = c \quad (a \text{ base},\ c \text{ power})}$$

Ex. $\log_2 8 = 3$ because $2^3 = 8$

$\log_2 4 = \square$ means $2^{\square} = 4$

$\log_a b = c$

Ex.

1) $\log_2 16 = \underline{4}$
2) $\log_2 2 = \underline{1}$
3) $\log_2 1/2 = \underline{-1}$
4) $\log_2 1/8 = \underline{-3}$
5) $\log_2 1 = \underline{0}$

$\boxed{\log_a b \text{ asks what power do you raise to, to get } b}$

$2^{-1} = 1/2$

$2^{-3} = 1/8$

$2^{0} = 1$

Ex

1) $\log_{10} 1000{,}000 = \underline{6}$ $\qquad 10^6 = 1000000$
2) $\log_{10} 0.001 = \underline{-3}$ $\qquad 10^{-3} = 0.001$
3) $\log_{10} 0 =$ can't be determined
4) $\log_{10} -100 =$ can't be determined $\qquad 10^{0} = 0$

$\frac{1}{10^3} = \frac{1}{1000} = 0.001$

$10^0 = 1 \qquad \boxed{x^0 = 1}$

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Note: It is possible to take the logs of numbers that are >0 but not of numbers that are ≤ 0. In other words, the domain of the function $f(x) = \log_a(x)$ is:

$(0, \infty)$

Note: $\ln x$ means $\log_e x$, and is called the natural log. ($e = 2.718...$)

$\log x$, with no base, means $\log_{10} x$ and is called the common log.

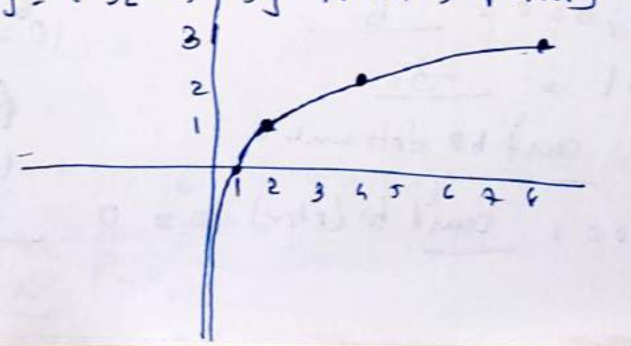
You can find $\ln x$ and $\log x$ for various values of x using the buttons on your calculator.

Ex Rewrite using Exponents

	Ans.
a) $\log_3 1/9 = -2$	$3^{-2} = 1/9$
b) $\log 13 = 1.11393$	$10^{1.11393} = 13$
c) $\ln \frac{1}{e} = -1$	$e^{-1} = 1/e$
d) $3^4 = 9.78$	$\log_3 9.78 = 4$
b) $e^{3t+7} = 4-y$	$\log_e(4-y) = 3t+7$

log functions & their Graphs

Ex Graph $y = \log_2(x)$, by plotting points



$2^y = x$

$y = 1, x = 2$

$y = 2, x = 4$

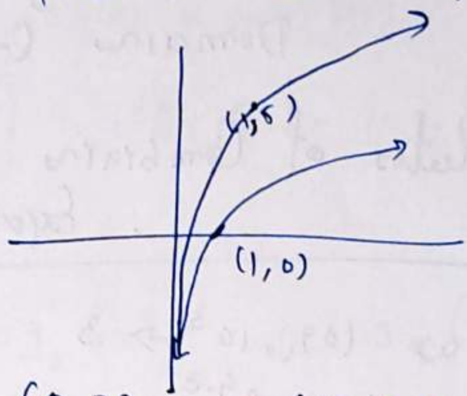
$y = 3, x = 8$

$e^0 = 0; x = 1$

Domain $x > 0$ $(0, \infty)$
Range : all real numbers $(-\infty, \infty)$
vertical assymptote at y axis i.e. at the $x = 0$

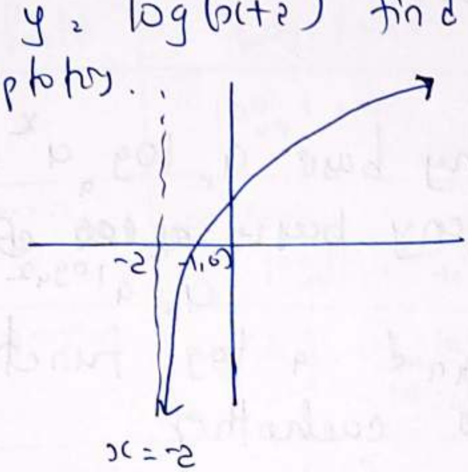
$\boxed{\text{Note: } y = \log_a(x) \quad a > 1, \text{ looks pretty much the same.}}$

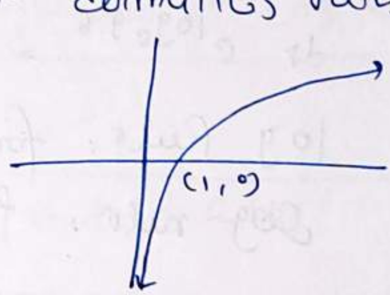
Example: Graph $y = \ln(x) + 5$. Find the domain, range, and symptotes.



VA $x = 0$, Rans $(-\infty, \infty)$, Domain $(0, \infty)$ ← $y = \ln(x)$
VA. $x = 0$, Ran: $(-\infty, \infty)$, Domain $(0, \infty)$ ← $y = \ln(x) + 5$

Ex. Graph $y = \log(x+2)$ find the domain, range and asymptotes.





Domain	$y = \log(x)$	$y = \log(x+2)$
	$(0, \infty)$	$(-2, \infty)$
Range	$(-\infty, \infty)$	$(-\infty, \infty)$
VA.	$x = 0$	$x = -2$

(51)

* find the domain of $f(x) = \ln(2-3x)$

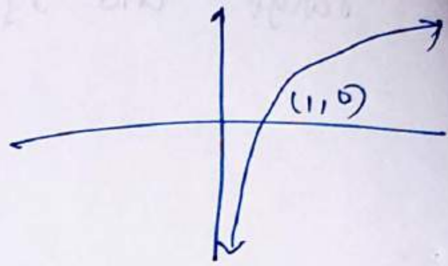
need $2 - 3x > 0$

$2 > 3x$

$2/3 > x$

$x < 2/3$

Domain $(-\infty, 2/3)$



Rules of Combining logs & Exponents

Ex.

a) $\log_{10} 10^{3} \Rightarrow 3$ $\quad 10^{\boxed{3}} \Rightarrow 10^{3}$

b) $\log_{e} e^{4.2} \Rightarrow 4.2$ $\quad 4.2$

$10 \times 10^{2} = 1000$

c) $10^{\log_{10} 1000} = 1000$

d) $e^{\log_{e} 9.6} = 9.6$

log Rule: for any base a, $\log_{a} a^{x} = x$

log rule: for any base a, $a^{\log_a x} = x$

An exponential fun and a log function with the same base undo eachother.

Ex. * Find $3^{\log_3 1.4} = 1.4$ $\quad 3^{\square} = 1.4$

* find $\ln(e^{x}) = x$

$\log_e e^{x} = x$

* find $10^{\log(32)} = 32$

* True or false: $\ln 10^{x} = x$

False $\quad \log_e 10^{x}$

$e^{\square} = 10^{x}$

Log Rules

1. $2^0 = 1$

$\log_2 1 = 0$

2. Product rule: $2^m \cdot 2^n = 2^{m+n}$

$\log_2(xy) = \log_2(x) + \log_2(y)$

3. Quotient rule: $\log_2\left(\frac{x}{y}\right) = \log_2(x) - \log_2(y)$

$\frac{2^m}{2^n} = 2^{m-n}$

4. Power rule:

$(2^m)^n = 2^{m \cdot n}$ $\quad \log_2(x^n) = n \cdot \log_2(x)$

Note: The logarithm rules hold for any base, not just base 2

Exponent Rule	Log Rule	Name of Log Rule
$a^0 = 1$	$\log_a 1 = 0$	-
$a^m a^n = a^{m+n}$	$\log_a(xy) = \log_a(x) + \log_a(y)$	Product Rule
$\frac{a^m}{a^n} = a^{m-n}$	$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$	Quotient Rule
$(a^m)^n = a^{mn}$	$\log_a(x^n) = n \log_a(x)$	Power Rule.

Ex.

a) $\log\left(\frac{x}{yz}\right) = \log(x) - \log(yz)$

$= \log(x) - (\log y + \log z)$

$\log x - \log y - \log z$

b) $\log(5 \cdot 2^t) = \log 5 + t\log 2$

Ex. a) $\log_5 a - \log_5 b + \log_5 c$

$\log_5\left(\frac{ac}{b}\right)$

b) $\ln(x+1) + \ln(x-1) - 2\ln(x^2-1)$

$= \log_e(x+1) + \log_e(x-1) - 2\log_e(x^2-1)$

$= \log_e((x+1)(x-1)) - 2\log_e(x^2-1)$

$= \log_e\left(\frac{(x+1)(x-1)}{(x^2-1)^2}\right)$

$= \ln\left(\frac{(x^2-1)}{(x^2-1)^2}\right)$

$= \ln\left(\frac{(x^2-1)}{(x^2-1)^2}\right)$

$= \ln\left(\frac{1}{(x^2-1)}\right)$

Solving Eq? using log]

Ex. Solve $5 \cdot 2^{x+1} = 17$

$2^{x+1} = 17/5$

$(x+1)\log_{10} 2 = \log_{10} 17 - \log_{10} 5$

$x\log_{10} 2 + \log_{10} 2 = \log_{10} 17 - \log_{10} 5$

$x = \frac{\log_{10} 17/5 - \log(2)}{\log_{10}(2)} = 0.765$

Solve: $2^{2x-3} = 5^{x-2}$

$$\log 2^{2x-3} = \log 5^{x-2}$$

$$(2x-3)\log_{10}2 = (x-2)\log_{10}5$$

$$2x\log_{10}2 - 3\log_{10}2 = x\log_{10}5 - 2\log_{10}5$$

$$2x\log_{10}2 - x\log_{10}5 = 3\log_{10}2 - 2\log_{10}5$$

$$x(2\log_{10}2 - \log_{10}5) = 3\log 2 - 2\log 5$$

$$x = \frac{(3\log 2 - 2\log 5)}{(2\log 2 - \log 5)}$$

$$x = 5.106$$

Sol. Solve: $5.e^{-0.05t} = 3.e^{0.2t}$

$$e^{-0.05t} = 3/5\, e^{0.2t}$$

$$\ln e^{-0.05t} = \ln\left(\tfrac{3}{5}\right)e^{0.2t}$$

$$-0.05t\ln e = \ln\left(\tfrac{3}{5}\right) + \ln e^{0.2t}$$

$$-0.05t\ln e = \ln 3/5 + 0.2t\ln e$$

$$-0.05t = \ln 3/5 + 0.2t$$

$$-0.05t - 0.2t = \ln 3/5$$

$$t(-0.05 - 0.2) = \ln 3/5$$

$$t = \frac{\ln(3/5)}{(-0.05-0.2)}$$

$$t = 2.0433$$

Solving log.. Eqn

Ex.. Solve.. $2\ln(2x+5)-3=1$

$$2\ln(2x+5)=1+3$$

$$\ln(2x+5)=\frac{4}{2}$$

$$\ln(2x+5)=2$$

$$\log_e(2x+5)=2$$

$$e^{\ln(2x+5)}=e^2$$

$$(2x+5)=e^2$$

$$2x=e^2-5$$

$$x=\frac{e^2-5}{2}$$

solns.

Ex. $\log(x+3)+\log(x)=1$

$$10^{\log(x+3)+\log(x)}=10^1$$

$$10^{\log(x+3)}\cdot 10^{\log(x)}=10$$

$$(x+3)\cdot(x)=10$$

$$x^2+3x-10=0$$

$$x^2+5x-2x-10=0$$

$$x(x+5)-2(x+5)=0$$

$$(x+5)(x-2)=0$$

$$x=-5 \text{ or } x=2 \checkmark$$

check $\log(-2)+\log(-5)=1$ ✗

check $\log(5)+\log(2)=1$

$$\log(5\cdot 2)=1$$

$$(\log)_{10}10=1$$

Solving Exponential Eqn. - Applications

Suppose you invest \$1600 in bank account that earns 6.5% annual interest, compounded annually. How many years will it take until the account has \$2000 in it. Assuming you make no further deposits or withdrawals?

$\Rightarrow$ $f(t) = 1600 \cdot (1.065)^t$

$2000 = 1600(1.065)^t$

$\frac{2000}{1600} = (1.065)^t$

$5/4 = (1.065)^t$

$\ln 5/4 = t \ln(1.065)$

$t = \frac{\ln 5/4}{\ln 1.065}$

Ex. A population of bacteria contains 1.5 million bacteria & is growing by 12% per day. What is the doubling time?

$P(t) = 1.5(1.12)^t$

$3 = 1.5(1.12)^t$

$2 = (1.12)^t$

$\ln 2 = t \ln 1.12$

$\frac{\ln 2}{\ln 1.12} = t$

$\boxed{t = 6.12 \text{ days}}$

method 2:

$P(t) = A(1.12)^t$

Initial

$2\cancel{A} = \cancel{A}(1.12)^t$

$2 = (1.12)^t$

$\ln 2 = t \ln(1.12)$

$t = \frac{\ln 2}{\ln(1.12)}$

$\boxed{t = 6.12 \text{ days}}$

Ex.

Ex.) Suppose a bacteria population double every 15 minute when, Write an equation for its growth using the ex eqn. $y = ab^t$. where t represents time in minutes.

$\rightarrow$ $y = a \cdot b^{t}$ $\quad 700 = 350\, b^{15}$

$$2 = b^{15}$$

$$2^{1/15} = (b^{15})^{1/15}$$

$$2^{1/15} = b^{1}$$

$$b = 2^{1/15}$$

$$\simeq 1.047294$$

Ex.) The half life of radioactive Carbon-14 is 5750 years. A sample of bone that originally contained 200 grams of C-14 now contains only 40 grams. How old is the sample?

$\rightarrow$ half life = the amount of time it take for a quantity to decrease to half of much

$f(t) = a \cdot e^{rt}$

$f(t)$: amount of radioactive C-14

a: initial amount

$f(t) = a \cdot e^{rt}$ $\qquad H_L = 5750$

what $t = 5750$

$f(t) = 1/2 \cdot a$

$$\frac{1}{2} \cdot a = a \cdot e^{r5750}$$

$$1/2 = e^{r5750}$$

$\ln(\frac{1}{2}) = \ln(e^{r5750})$

$\ln(1/2) = r \cdot 5750$

$$r = \frac{\ln(1/2)}{57}$$

$$40 = 200 \cdot e^{(1-(1/2)/5750\ t)}$$

$$\frac{40}{200} = e^{\ln(1/2)/5750\ t}$$

$$\ln \frac{1}{5} = \ln e^{\ln(1/2)/5750\ t}$$

$$\ln 1/5 = \frac{\ln 1/2}{5750} t$$

$$t = \frac{\ln 1/5 \times 5750}{\ln 1/2}$$

$$t = 13{,}351$$

$f(t) = a \cdot e^{rt}$ $\qquad$ $f(t) = a \cdot b^t$

$= a \cdot (e^r)^t$

System of linear Equations.

Ex. Solve the system of eqn.

$$\begin{cases} 3x - 2y = 4 \\ 5x + 6y = 2 \end{cases}$$

Method 1: Substitution

Method 2. Elimination

$$18x - 12y = 24$$
$$10x + 12y = 4$$

$$28x = 28$$

Ans. $\boxed{x = 1}$ $\quad 18 - 12y = 24$

$\boxed{y = -1/2}$ $\quad -6 = 12y$

(04)

Method 1: Substitution

Isolate one variable using one eqn. Substitute it into other eqn.

$3x - 2y = 4$

$3x = 4 + 2y$

$x = \frac{4+2y}{3} \Rightarrow x = \frac{4}{3} + \frac{2}{3}y$

$5x + 6y = 2$

$5\left(\frac{4}{3} + \frac{2}{3}y\right) + 6y = 2$

$\frac{20}{3} + \frac{10}{3}y + 6y = 2$

$\frac{10}{3}y + 6y = 2 - \frac{20}{3}$

$3\left(\frac{10}{3}y + 6y\right) = 3\left(2 - \frac{20}{3}\right)$

$10y + 18y = 6 - 20$

$28y = -14$

$\boxed{y = -1/2}$

$3x - 2(-1/2) = 4$

$3x + 1 = 4$

$3x = 3$

$\boxed{x = 1}$

Method 2: Elimination

Multiply each eqn. by a coefficient to make coefficient of one variable match

$5 \times \quad 3x - 2y = 4$

$3 \times \quad 5x + 6y = 2$

$15x - 10y = 20$

$15x + 18y = 6$

$- \quad - \quad -$

$-28y = 14$

$y = -\frac{14}{28}$

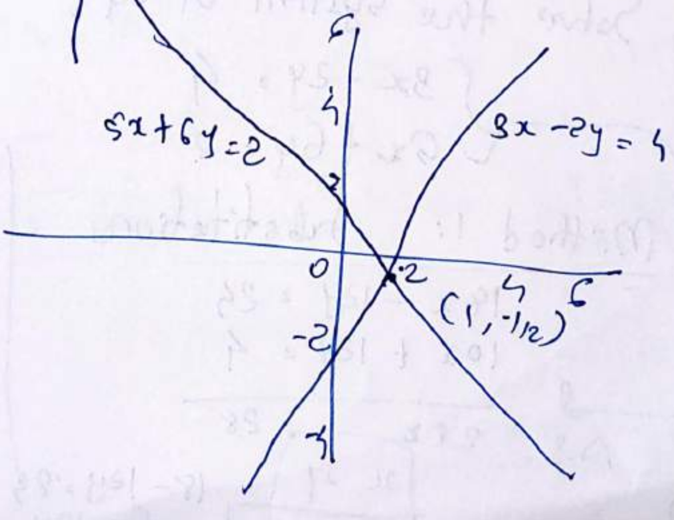
$\boxed{y = -1/2}$

$3x - 2(-1/2) = 4$

$3x + 1 = 4$

$3x = 3$

$\boxed{x = 1}$



Ex. Solve the system of eq^n,

$$8y = 1 + 4x$$
$$3x - 6y = 2$$

incorsisent system

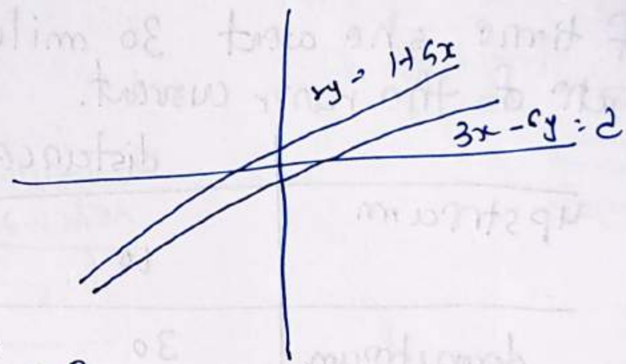
$$4x - 8y = -1 \quad \times 3$$
$$3x - 6y = 2 \quad \times 4$$

$$12x - 24y = -3$$
$$12x - 24y = 2$$

$$24x - 48y = -6$$
$$24x - 48y = 16$$
$$-\quad +$$
$$0 \quad 0 = -24$$
$$0 \neq -24$$

2 | 6 | 8
3 | 3 | 2
2 | 1 | 2
 | 1 | 1

Parallel lines



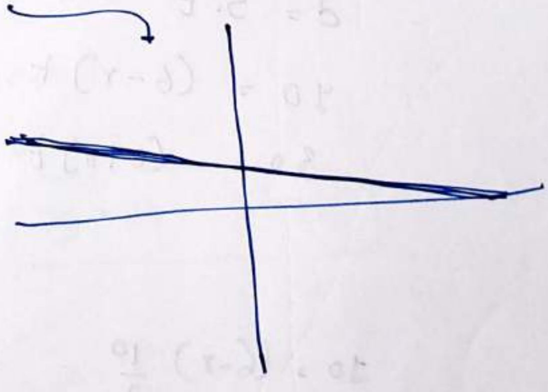
Ex. Solve the system of eq^n:

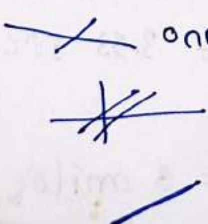
$$x + 5y = 6$$
$$3x + 15y = 18$$

$$x + 5y = 6$$
$$\leftarrow \quad 3(x + 5y) = 3(6)$$

dependent system

⇒ infinitely many Solutions





(61)

Distance, rate, and time problems

$$\text{Speed} = \frac{\text{Distance}}{\text{time}} \qquad s = \frac{d}{t}$$

$$d = s \cdot t$$

$$t = \frac{d}{s}$$

Ex. Elsa's boat has a top speed of 6 miles per hour in still water. While traveling on a river at top speed, she went 10 miles upstream in the same amount of time she went 30 miles downstream. Find the rate of the river current.

	distance	speed ~~rate~~	time
upstream	10	$6-r$	t
downstream	30	$6+r$	t

t = time
s = speed of current

$$d = s \cdot t$$

$$10 = (6-r)\,t \qquad 10 = 6t - rt$$

$$30 = (6+r)\,t \qquad 30 = 6t + rt$$

$$40 = 12t$$

$$t = \frac{40}{12} = \frac{10}{3}$$

$$10 = (6-r)\,\frac{10}{3} \qquad t = 3.33 \text{ sec.}$$

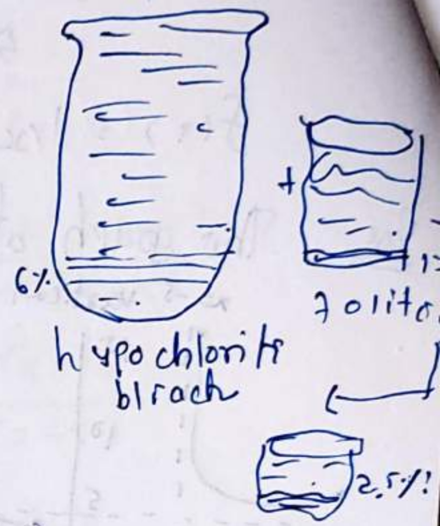
$$3 = 6 - r$$

$$\boxed{r = 3}$$

speed of current ~~speed~~ = 3 miles per hour

Mixture Problems

Ex. Household bleach contains 6% sodium hypochlorite. How much household bleach should be combined with 70 liters of a weaker 1% sodium hypochlorite solution to form a solution that is 2.5% sodium hypochlorite?



NaClO before mixing = NaClO after mixing

water before = water after

solution before = solution after

	Volume of NaClO	volume of water	volume of solution
6% solution	$0.06 \cdot x$	$x - 0.06x$ $0.94x$	x
1% solution	0.01×70 0.7	$0.99 \cdot 70$ 69.3	70
2.5% solution	$0.025 \cdot (70+x)$	$0.975 \cdot (70+x)$	$70+x$

x = volume of household bleach

$$0.06x + 0.7 = 0.025(70+x)$$

$$1000(0.06 + 0.7) = 1000(0.025(70+x))$$

$$60x + 700 = 25(70+x)$$

$$60x + 700 = 1750 + 25x$$

$$35x = 1050 \Rightarrow \boxed{x = 30 \text{ liters}}$$

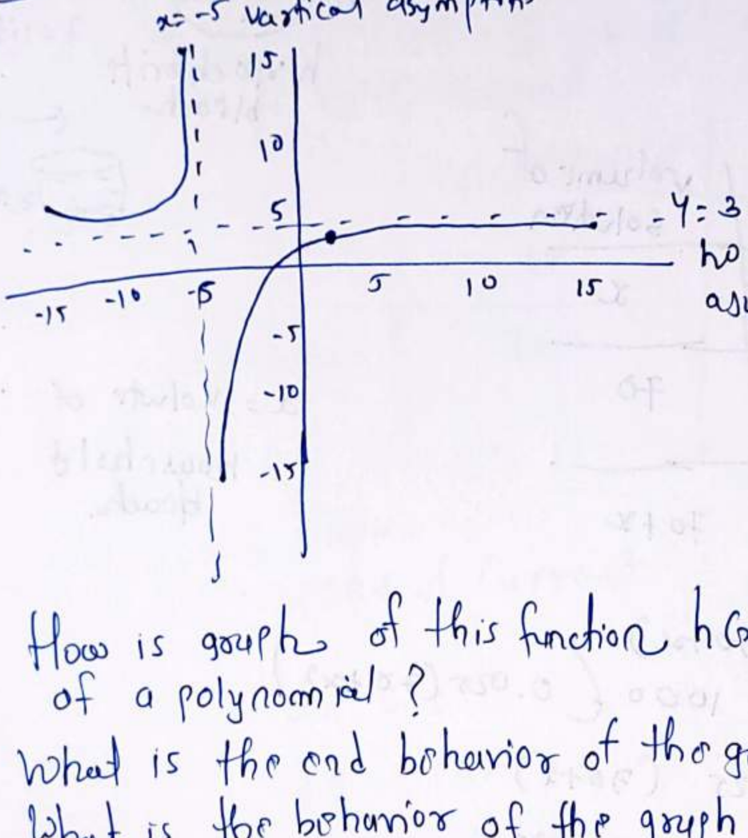
Rational functions

$$f(x) = \frac{P(x)}{Q(x)}$$ ← Polynomials

$$f(x) = \frac{3x^4 - 7x^3 + 6x + 1}{5x^2 - 6x + 7}$$

$f(x) = 1/x$

Ex. The graph of the function $h(x) = \dfrac{3x^2 - 12}{x^2 + 3x - 10}$ is shown below.



How is graph of this function $h(x)$ different from the graph of a polynomial?

What is the end behavior of the graph? → $\boxed{\text{horizontal asymptote at } y = 3}$

What is the behavior of the graph of this function $h(x)$ near $x = -5$?

vertical asymptote at $x = -5$

What is going on at $x = 2$? hole at $x = 2$

[which means function doesn't exist at $x = 2$

$$\frac{3x^2-12}{x^2+3x-10} \rightarrow \frac{3(x^2-4)}{x^2+5x-2x-10}$$

$$= \frac{3(x^2-4)}{x(x+5)-2(x+5)}$$

$$= \frac{3(x^2-4)}{(x-2)(x+5)}$$

$$= \frac{3(x+2)\cancel{(x-2)}}{\cancel{(x-2)}(x+5)}$$

$$= \frac{3(x+2)}{(x+5)}$$

$h(x) = \frac{3(x+2)}{(x+5)}$ as long as $x \neq 2$

$y = \frac{3(2+2)}{2+7} = \frac{12}{9} = \frac{4}{3}$ hole at $(2, 4/3)$

for a rational function:

→ find the vertical asymptotes by → setting by looking when denominator is zero.

→ find the holes by : hole → if denominator and numerator are both zero at factor cancel outs.

→ find the horizontal asymptotes by →
look at highest power term in numerator and denominator.

Ex.

1) $f(x) = \frac{5x+4}{3x^2+5x-4}$ → $\frac{5x}{3x^2} = \frac{5}{3x} \rightarrow \boxed{0}$

degree of Numerator < degree of denominator

H. A. $y = 0$

2) $g(x) = \frac{2x^3+4}{3x^3-7x}$ → $\frac{2x^3}{3x^3} = \frac{2}{3} \rightarrow \frac{2}{3}$ → H.A $\boxed{y = 2/3}$

Degree of numerator = degree of denominator

65

$$h(x) = \frac{x^2+4x-5}{2x-1}$$

$\frac{x^2}{2x} = \frac{x}{2} \to \infty$ when $x \to \infty$

$\to -\infty$ when $x = -\infty$

$\boxed{\text{degree of numerator > degree of denominator}}$ $\to$ no. H.A.

Ex.: Find the vertical asymptotes, horizontal asymptotes, and holes.

$$q(x) = \frac{3x^2+3x}{2x^3+5x^2-3x}$$

$$= \frac{3x^2}{2x^3} = \frac{3}{2x} \to \text{H.A.} = 0$$

V.A. & hole

set denominator $= 0$

$2x^3+5x^2-3x = 0$

$x = 0$

$$q(x) = \frac{3x(x+1)}{x(2x^2+5x-3)}$$

$$= \frac{3x(x+1)}{x(2x^2+6x-x-3)}$$

$$= \frac{3x(x+1)}{x(2x(x+3)-(x+3))}$$

$$= \frac{3x(x+1)}{x(2x-1)(x+3)}$$

hole at $x = 0$

hole at $x=0$

y-value of hole

$q(x) = \frac{3(x+1)}{(2x-1)(x+3)}$ when $x \neq 0$

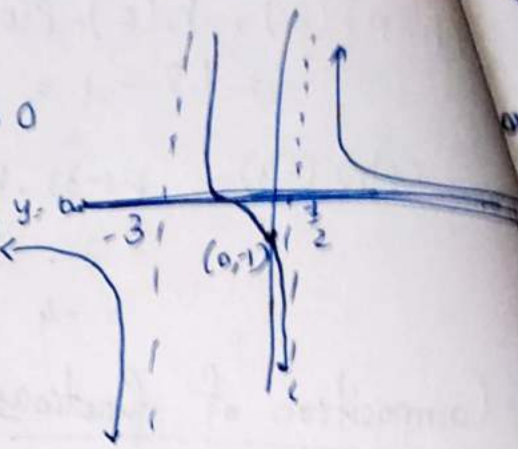
$$\frac{3(0+1)}{(2\cdot0-1)(0+3)} = \frac{3}{-3} = -1$$

hole at $(0,-1)$

V.A. when $(2x-1)(x+3)=0$

when $2x-1=0$ or $x+3=0$

$\boxed{x = \cdot 1/2 \text{ or } x=-3}$



Combining functions

We can Combine functions, such as $f(x)=x+1$ and $g(x)=x^2$ in the following ways:

- Add them together $(f+g)(x)=$

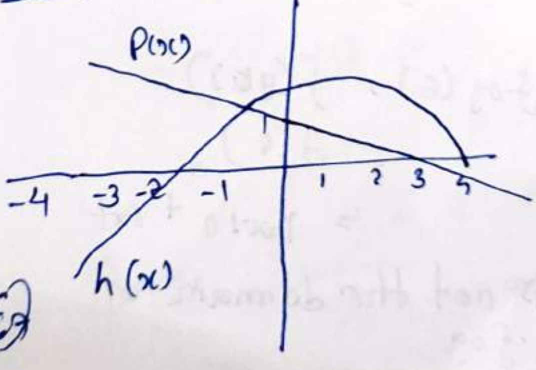
$$f(x)+g(x) = x+1+x^2$$
$$(f+g)(x) = x^2+x+1$$

- Subtract them: $(f-g)(x) = f(x)-g(x)$
$$= x+1-x^2$$
$$(f-g)(1) = 1+1-1^2 = 1$$

+ Multiply them: $(f \cdot g)(x) = f(x) \cdot g(x) = (x+1) \cdot x^2$
$$(fg)(x) = x^3+x^2$$

+ Divide them $\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{x+1}{x^2}$

Ex. The following graphs represent two functions h and p.



find

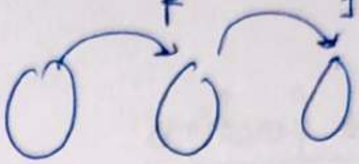
a> $(h-p)(0)$

b> $(ph)(-3)$

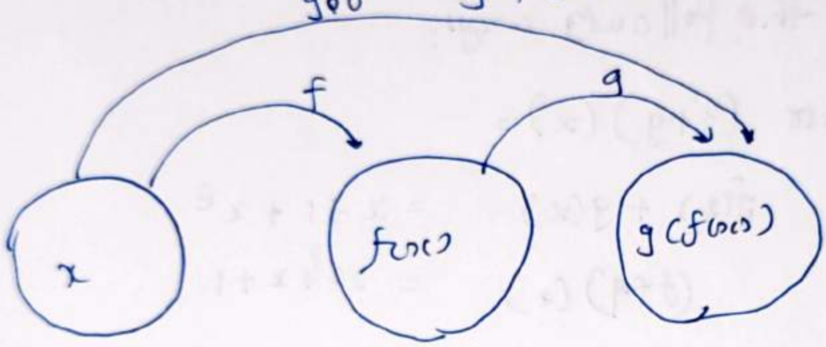
$(h-p)(0) = h(0) - p(0)$
$= 1.8 - 1 = 0.8$

b) $(ph)(-3) = p(-3) \cdot h(-3)$
$= 2 \cdot -2$
$= -4$

Composition of functions



The composition of two functions: $g \circ f(x)$ is defined as:



$g \circ f(x) = g(f(x))$

Example: The tables below define the functions f and g.

x	1	2	3	4	5
$f(x)$	8	3	6	7	4

x	4	5	6	7	8	9
$g(x)$	1	3	8	10	2	2

find

a) $g \circ f(4) = g(f(4))$
$= g(7)$
$= 10$

b) $f \circ g(4) = f(g(4))$
$= f(1)$
$= 8$

c) $f \circ f(2) = f(f(2))$
$= f(3)$
$= 6$

d) $f \circ g(6) = f(g(6))$
$= f(8)$
$=$ Does not exist

6 is not the domain of $f \circ g$.

Let $p(x) = x^2+x$. Let $q(x) = -2x$. Find:

a) $q \circ p(1) \quad q(p(1))$

$= q(2)$

$= -4$

b) $q \circ p(x) = q(p(x))$

$= q(x^2+x)$

$= -2(x^2+x)$

$= -2x^2-2x$

c) $p \circ q(x) = p(q(x))$

$= \cancel{q}\, p(-2x)$

$= (-2x)^2 - 2x$

$= 4x^2-2x$

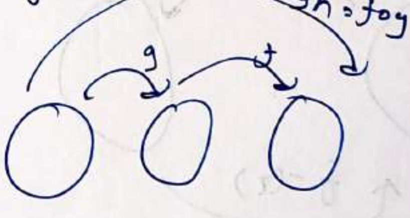
d) $p \circ p(x) = p(x^2+x)$

$= (x^2+x)^2 + x^2+x$

$= x^4+2x^3+x^2+x^2+x$

$= x^4+2x^3+2x^2+x$

Ex. $h(x) = \sqrt{x^2+7}$ find the functions f and g so that $h = f \circ g$



$h(x) = f \circ g(x)$.

$h(x) = f(g(x))$

$g(x) = x^2+7$

$f(x) = \sqrt{x}$

$h = f \circ g(x) = f(g(x))$

$= f(x^2+7)$

$f \circ g(x) = \sqrt{x^2+7}$

$f(x^2+7) = \sqrt{x^2+7}$

Or

$h(x) = \sqrt{x^2+7}$

$g(x) = x^2$

$f(x) = \sqrt{x+7}$

$f \circ g(x) = f(g(x))$
$= f(x^2)$
$= \sqrt{x^2+7}$

Inverse of functions

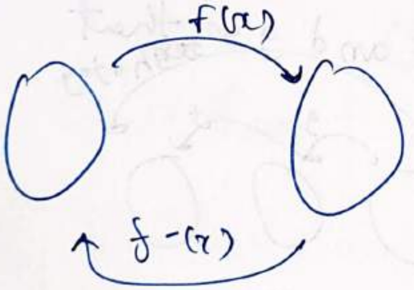
$f(x) = x + 2 \qquad f^{-1}(x) = x - 2$

Ex. Suppose $f(x)$ is the function defined by the chart below:

x	2	3	4	5
$f(x)$	3	5	6	1

In other words
$f(2) = 3$
$f(3) = 5$
$f(4) = 6$
$f(5) = 1$

Defination: The inverse function for f, written $f^{-1}(x)$, undoes what f does



$f^{-1}(3) = 2$

$f^{-1}(5) = 3$

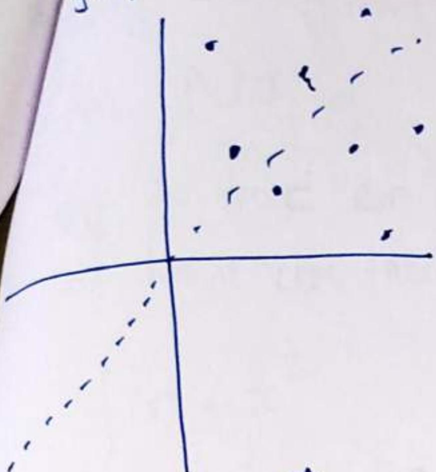
$f^{-1}(6) = 4$

$f^{-1}(1) = 5$

x	3	5	6	1
$y = f^{-1}(x)$	2	3	4	5

Key facts: Inverse functions reverse the roles of y and x.

... $y = f(x)$ and $y = f^{-1}(x)$ on the same axis below.
...at do you notice about the points on the graph of $y = f(x)$ and the points on the graph of $y = f^{-1}$?



x	2	3	4	5
$y = f(x)$	3	5	6	1

x	3	5	6	1
$f^{-1}(x)$	2	3	4	5

Key fact: The graph of $y = f^{-1}(x)$ is obtained from the graph of $y = f(x)$ by reflecting over the line $y = x$

In our same example, Compute,

$f^{-1} \circ f(2) = f^{-1}(3) = 2$

$f^{-1} \circ f(3) = f^{-1}(5) = 3$

$f^{-1} \circ f(4) = f^{-1}(6) = 4$

$f^{-1} \circ f(5) = f^{-1}(1) = 5$

$f \circ f^{-1}(3) = f(f^{-1}(3))$

$= f(2)$

$= 3$

$f \circ f^{-1}(5) = f(3) = 5$

$f \circ f^{-1}(6) = f(4) = 6$

$f \circ f^{-1}(1) = f(5) = 1$

Key fact 3: $f^{-1} \circ f(x) = \underline{x}$ and $f \circ f^{-1}(x) = \underline{x}$
This is the mathematical way of saying that f and f^{-1} undo each other.

Ex. $f(x) = x^3$. Guess what the inverse of f should be. Remember, f^{-1} undoes the work that f does.

$$f(x) = x^3$$

$$f^{-1}(x) = \sqrt[3]{x}$$

$$f \circ f^{-1}(x) = f(\sqrt[3]{x})$$

$$= (\sqrt[3]{x})^3$$

$$= x$$

Ex. find the inverse of the function.

$$f(x) = \frac{5-x}{3x}$$

$$y = \frac{5-x}{3x}$$

$f \circ f^{-1}(x)$

$f(f^{-1}(x))$

$$y = \frac{5-x}{3x}$$

answer the role of x & y

$$x = \frac{5-y}{3y}$$

Solve for y

$$3yx = 5-y$$

$$3yx + y = 5$$

$$y(3x+1) = 5$$

$$y = \frac{5}{3x+1}$$

$$\boxed{f^{-1}(x) = \frac{5}{3x+1}}$$

Note:

$f^{-1}(x)$ means the inverse function for $f(x)$.

Note that

$$f^{-1}(x) \neq \frac{1}{f(x)}.$$

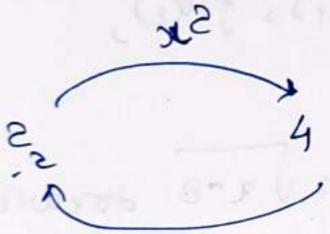
Question: Do all functions have inverse functions? That is, for any function that you might encounter, is there always a function that is its inverse?

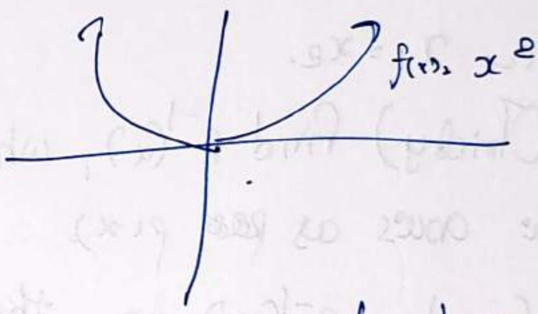
Ans. No

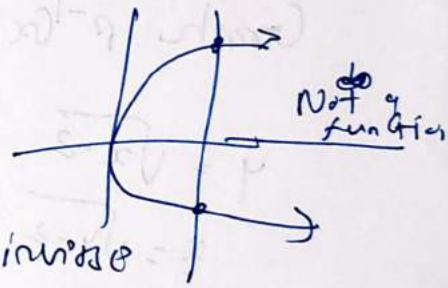
Try to find an example of a function that does not have an inverse function.

function: for each x in domain, there is only one corresponding y-value.

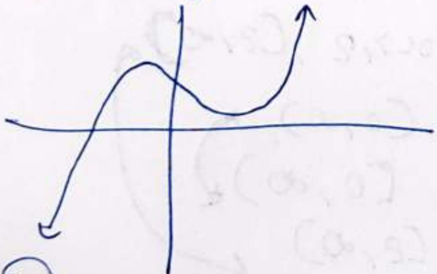
$f(x) = x^2$

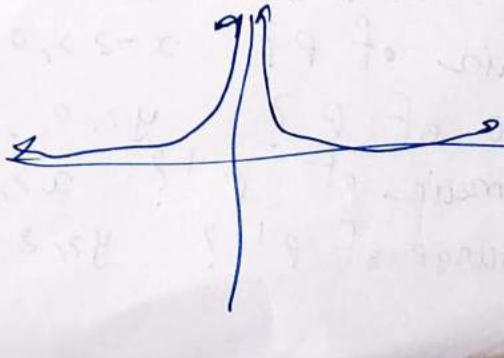


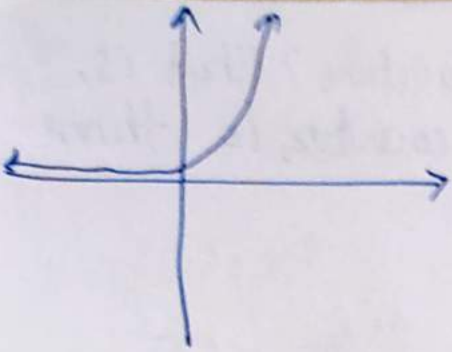


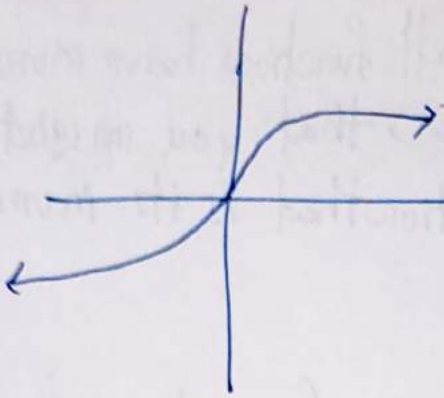


Key fact 4: A function f has an inverse function if and only if the graph of f satisfies the horizontal line test (i.e. every horizontal line intersects the graph of $y = f(x)$ in at most one point)





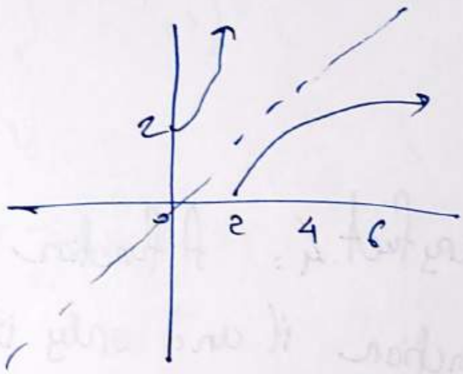




Defination: A function is one to one if it passes the horizontal line test. Equivalently, a function is one to one if for any two different x-values x_1 and x_2, $f(x_1)$ and $f(x_2)$ are different numbers. Sometimes, this is said: f is one to one if, whenever $f(x_1) = f(x_2)$, then $x_1 = x_2$.

Ex. (Tricky) find $f^{-1}(x)$, where $f(x) = \sqrt{x-2}$ draw on same axes as $f(x)$

Graph $f^{-1}(x)$ on the



$$y = \sqrt{x-2}$$

$$x = \sqrt{y-2}$$

$$x^2 = y-2$$

$$y = x^2 + 2, \quad x \geq 0$$

for the function $f(x) = \sqrt{x-2}$, what is:

- the domain of f? $x-2 \geq 0$, $x \geq 2$, $[2,\infty)$
- the range of f? $y \geq 0$, $[0,\infty)$
- the domain of f^{-1}? $x \geq 0$ $[0,\infty)$
- the range of f^{-1}? $y \geq 2$ $[2,\infty)$

my fact 5 : for any invertible function f, the domain of $f^{-1}(x)$, is the range of $f(x)$ and the range of $f^{-1}(x)$ is the domain of $f(x)$