Q4

Let N be the input values

$\mathbf{X} = (x_1, x_2, \ldots x_N)^T$ and their corresponding target values be

$\mathbf{t} = (t_1, t_2 \ldots t_N)^T$

Assuming Gaussian distribution,

$$p(t \mid x, \omega, \beta) = \mathcal{N}(t \mid \phi(x)^T \omega, \beta^{-1})$$

where

$$\phi = (\phi_0, \phi_1, \phi_2 \ldots \phi_{M-1})^T$$

and in its simplest form $\phi_i(x) = x^i$

$$p(\mathbf{t} \mid \mathbf{X}, \omega) = \prod_{i=1}^{N} p(t_i \mid x_i, \omega)$$

$$= \prod_{i=1}^{N} \sqrt{\frac{\beta}{2\pi}} \exp\left\{-\frac{\beta}{2}(t_i - \phi(x)^T \omega)^2\right\}$$

$$= \frac{1}{(2\pi/\beta)^{n/2}} \exp\left\{\frac{-1}{2/\beta} \left|t - \phi(x)^T \omega\right|^2\right\}$$

$$= \mathcal{N}(t \mid \Phi(x)^T \omega, \beta^{-1} I)$$

where $\phi(x)$ is given by,

$$\Phi = \begin{pmatrix} \phi_0(x_1) & \phi_1(x_1) & \phi_2(x_1) & \cdots & \phi_{M-1}(x_1) \\ \phi_0(x_2) & & & & \vdots \\ \vdots & & & & \vdots \\ \phi_0(x_n) & & & & \phi_{M-1}(x_n) \end{pmatrix}$$

Let us also assume Gaussian as prior distribution ~~of~~ for ω,

$$p(\omega|\alpha) = \mathcal{N}(\omega|0, \alpha^{-1}I)$$

$$= \left(\frac{\alpha}{2\pi}\right)^{M/2} \exp\left\{-\frac{\alpha}{2}\omega^T\omega\right\}$$

using Bayes theorem,

(posterior distribution) $\propto$ (likelihood) (prior)

$$p(\omega|t, X, \alpha) \propto p(t|X, \omega)\, p(\omega|\alpha)$$

$$\propto \exp\left\{\frac{-\beta}{2}(t - \phi(X)^T\omega)^T(t - \phi(X)^T\omega)\right\} \exp\left\{-\frac{\alpha}{2}\omega^T\omega\right\}$$

$$\propto \exp\left\{\frac{-1}{2}(\omega - \bar{\omega})^T C^{-1}(\omega - \bar{\omega})\right\}$$

where $\bar{\omega} = \beta(\beta\,\phi(X)\,\phi(X)^T + \alpha I)^{-1}\phi(X)\,t$

and covariance matrix

$$C = (\beta\,\phi(X)\,\phi(X)^T + \alpha I)^{-1}$$

So $p(\omega|t, X, \alpha) \sim N(\bar{\omega} = \beta C\,\phi(X)t, C)$