We can write the M^{th} order term for a polynomial in D dimensions

$$\sum_{i_1=1}^{D} \sum_{i_2=1}^{D} \cdots \sum_{i_M=1}^{D} w_{i_1 i_2 \ldots i_M} x_{i_1} x_{i_2} \ldots x_{i_M} \quad \text{--- (1)}$$

Now,
coefficients $w_{i_1 i_2 \ldots i_M}$ comprise D^M elements but due to many interchange symmetries of the factor $x_{i_1}, x_{i_2} \ldots x_{i_M}$ no. of independent parameters is small.

We can rewrite this M^{th} order term in the form (to enforce ordering of indic

$$\sum_{i_1=1}^{D} \sum_{i_2=1}^{i_1} \cdots \sum_{i_M=1}^{i_{M-1}} \tilde{w}_{i_1 i_2 \ldots i_M} x_{i_1} x_{i_2} \ldots x_{i_M} \quad \text{--- (2)}$$

In this way no. of independent parameters $n(D, M)$ which appear.

at order m can be written as

$$n(D,m) = \sum_{i_1=1}^{D} \sum_{i_2=1}^{i_1} \cdots \sum_{i_m=1}^{i_{m-1}} 1 \quad -(3)$$

which has m terms.

(3) can be written as

$$n(D,m) = \sum_{i_1=1}^{D} \left\{ \sum_{i_2=1}^{i_1} \cdots \sum_{i_m=1}^{i_{m-1}} 1 \right\} \quad -(4)$$

(4) has $m-1$ terms in braces

which from (3) must be

equal to $n(i_1, m-1)$

Thus $n(D,m) = \sum_{i_1=1}^{D} n(i_1, m-1)$ -(5)

Now we are going to prove by

induction

$$\sum_{i=1}^{D} \frac{(i+m-2)!}{(i-1)!\,(m-1)!} = \frac{(D+m-1)!}{(D-1)!\,m!} \quad (6)$$

Base for $D=1$

$$LHS = \frac{(M-1)!}{0!\,(M-1)!} = 1$$

$$RHS = \frac{(1+M-1)!}{0!\,M!} = 1$$

$\Rightarrow$ LHS = RHS

Now assume for $D = D$ & we have to prove for $D = D+1$

$$\sum_{i=1}^{D+1} \frac{(i+M-2)!}{(i-1)!\,(M-1)!} = \frac{(D+M-1)!}{(D-1)!\,M!} + \frac{(D+M-}{D!\,(M-}$$

$$= \frac{(D+M-1)!\,D! + (D+M-1)!\,M}{D!\,M!}$$

$$= \frac{(D+M)!}{D!\,M!}$$ = RHS for dimensionality $D+1$

Thus by induction (5) must hold true for all values of D

Now we have

$$n(D,M) = \frac{(D+M-1)!}{(D-1)!\,M!} \quad \text{to prove} \qquad -(7)$$

For $M=2$ we obtain the standard result

$$n(D,2) = \frac{1}{2} D(D+1)$$

Now assume (7) is true for $M-1$

so that

$$n(D, M-1) = \frac{(D+M-2)!}{(D-1)!\,(M-1)!} \qquad -(8)$$

From (5) & (8) we obtain

$$n(D,M) = \sum_{i=1}^{D} \frac{(i+M-2)!}{(i-1)!\,(M-1)!}$$

Now using (6) in above we get

$$n(D,M) = \frac{(D+M-1)!}{(D-1)!\,M!}$$

Hence true for all values of M.

We can also Prove in following manner:

(5)

from the multinomial theorem,
we can distribute D degree
among m variable in ${}^{D+M-1}C_{M-1}$
ways.

$$\left[(x_1 + x_2 + \cdots + x_M)^D \Rightarrow \#(\text{terms}) = {}^{D+M-1}C_{M-1} \right]$$

Since D^{th} degree polynomial may have
terms having all possible degrees
less than or equal to D

So
possible terms $= \sum_{d=1}^{D} {}^{m+d-1}C_{M-1}$

$$= \sum_{d=1}^{D} \frac{(M+d-1)!}{(M-1)!\, d!}$$