APMA 1655: Homework #4

We are not collecting this homework due to the midterm exam. The solution will be posted very soon.

1. An expensive item is being insured against early failure. The lifetime of the item is normally distributed with an expected value of seven years and a standard deviation of two years. The insurance will pay $2x$ dollars if the item fails during the first or second year, and x dollars if the item fails during the third or fourth year. If a failure occurs after the fourth year, then the insurance pays nothing. How to choose x such that the expected value of the payment per insurance is 50 dollars?

   Remark: It is common in practice to model nonnegative quantities by normal distributions (lifetimes, SAT scores, height/weight, and so on). Even though normal distributions can take NEGATIVE values, in such modeling the probability of having a negative value is usually negligible. In this problem, you can either pretend life can be negative, or only work with the positive part of this normal distribution. The difference in the final result will be negligible.

2. The joint density function of the continuous random variables X and Y is given by

$$f(x, y) = \begin{cases} cxy & \text{for } 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases}.$$

   Identify the constant c and compute $P(X \leq 0.5)$ and $E[Y]$.

3. Emily has a fair coin and Pete has a coin with $P(H) = p$. Emily tosses her coin twice. Now Pete tosses his coin as many times as the number of heads Emily gets. Let X and Y be the numbers of heads Emily and John get, respectively.

   (a) Find the joint probability mass function of (X, Y). You might want to use a table to describe this pmf.

   (b) What is the (marginal) distribution of X?

   (c) What is the (marginal) distribution of Y? Your answer should be the same as that from Problem 5 of Homework #2.

   (d) Without any calculation, do you think $\text{Cov}(X, Y)$ is positive or negative? Why?

   (e) Compute $\text{Cov}(X, Y)$ to verify your answer in (d).

4. Let X and Y be two random variables with $\text{Var}[Y] > 0$. For each constant β, define

$$L(\beta) \doteq \text{Var}[X + \beta Y].$$

   Show that $L(\beta)$ is minimized at

$$\beta^* = \frac{-\text{Cov}(X, Y)}{\text{Var}[Y]}$$

APMA 1655: Solution to Homework #4

1. Let X be the life time of the item. By assumption, X is $N(7, 2^2)$. The payoff of the insurance is

$$h(X) = \begin{cases} 2x & \text{if } 0 \le X \le 2 \\ x & \text{if } 2 < X \le 4 \\ 0 & \text{if } X > 4 \end{cases} = 2x \cdot 1_{\{X \le 2\}} + x \cdot 1_{\{2 < X \le 4\}}.$$

Thus

$$\begin{aligned} E[h(X)] &= 2xP(0 \le X \le 2) + xP(2 < X \le 4) \\ &= 2xP\left(\frac{0-7}{2} \le \frac{X-7}{2} \le \frac{2-7}{2}\right) + xP\left(\frac{2-7}{2} < \frac{X-7}{2} \le \frac{4-7}{2}\right) \\ &= 2xP(-3.5 \le N(0,1) \le -2.5) + xP(-2.5 < N(0,1) \le -1.5) \\ &= 2x[\Phi(-2.5) - \Phi(-3.5)] + x[\Phi(-1.5) - \Phi(-2.5)] \;=\; x \cdot 0.0726 \;=\; 50. \end{aligned}$$

Thus $x = 688.7$.

Note: Here the assumption the life time is $N(7, 2^2)$. It is just an approximation since normal can take negative values but life time cannot. For this problem, you can see that if we replace $P(0 \le X \le 2)$ by just $P(X \le 2)$, the difference is $\Phi(-3.5) = 2.3263 \times 10^{-4}$ which is really small. It is **also correct** if you use $P(X \le 2)$ in place of $P(0 \le X \le 2)$, in which case, we have $x \cdot 0.0730 = 50$ and $x = 684.9$.

2. The joint density function must integrate to 1. That is,

$$1 = \iint f(x,y)dxdy = \iint_{0<y<x<1} cxydxdy = c \iint_{0<y<x<1} xydxdy.$$

But

$$\iint_{0<y<x<1} xydxdy = \int_0^1 \left[\int_0^x xydy\right] dx = \int_0^1 x \cdot \frac{1}{2}x^2 \, dx = \frac{1}{8}.$$

Thus $c = 8$. The rest is just plain calculation.

$$P(X \le 0.5) = \iint_{\{0<y<x<1, x \le 0.5\}} 8xydxdy = 8 \int_0^{0.5} \left[\int_0^x xydy\right] dx = 8 \int_0^{0.5} x \cdot \frac{1}{2}x^2 \, dx = \frac{1}{16},$$

and

$$E[Y] = \iint_{0<y<x<1} y \cdot 8xydxdy = 8 \int_0^1 \left[\int_0^x xy^2 dy\right] dx = 8 \int_0^1 x \cdot \frac{1}{3}x^3 \, dx = \frac{8}{15}.$$

3. (a) X and Y both take values in $\{0, 1, 2\}$. We have the following table, which represents the joint probability mass function. Please note that by the assumption, $P(X \ge Y) = 1$. We also denote $q \doteq 1 - p$.

		Y		
		0	1	2
X	0	$1/4$	0	0
	1	$q/2$	$p/2$	0
	2	$q^2/4$	$pq/2$	$p^2/4$

For example,

$$P(X=2,Y=1)=P(Y=1|X=2)P(X=2).$$

Since

$$P(X=2)=P(\text{Emily tosses two heads})=\frac{1}{4}$$

and

$$\begin{aligned} P(Y=1|X=2) &= P(\text{John tosses twice and gets one heads}) \\ &= P(HT)+P(TH) \ = \ pq+qp \\ &= 2pq. \end{aligned}$$

Thus

$$P(X=2,Y=1)=(2pq)/4=pq/2.$$

(b) The (marginal) distribution of X is $B(2,0.5)$. One way to see this that X is the total number of heads in two tosses of a fair coin. Thus, by definition it is $B(2,0.5)$.

Another way is to compute the (marginal) probability mass function of X by summing up the rows

		Y			
		0	1	2	
X	0	$1/4$	0	0	$1/4$
	1	$q/2$	$p/2$	0	$1/2$
	2	$q^2/4$	$pq/2$	$p^2/4$	$(p+q)^2/4=1/4$

In other words,

$$P(X=0)=\frac{1}{4},\quad P(X=1)=\frac{1}{2},\quad P(X=2)=\frac{1}{4},$$

which is exactly $B(2,0.5)$.

(c) By Problem 5 of Homework 2, the distribution of Y is $B(2,p/2)$.

Another way is to compute the (marginal) probability mass function of Y by summing up the columns

		Y		
		0	1	2
X	0	$1/4$	0	0
	1	$q/2$	$p/2$	0
	2	$q^2/4$	$pq/2$	$p^2/4$
		$(1+q)^2/4$	$p(1+q)/2$	$p^2/4$

That is, (note that $1+q=2-p$)

$$P(Y=0)=\frac{(1+q)^2}{4}=\binom{2}{0}\left(\frac{p}{2}\right)^0\left(1-\frac{p}{2}\right)^2$$

$$P(Y=1) = \frac{p(1+q)}{2} = \binom{2}{1}\left(\frac{p}{2}\right)^1\left(1-\frac{p}{2}\right)^{2-1}$$

$$P(Y=2) = \frac{p^2}{4} = \binom{2}{2}\left(\frac{p}{2}\right)^2\left(1-\frac{p}{2}\right)^{2-2}.$$

In other words, Y is $B(2, p/2)$.

Remark: This exercise justifies that the marginal distribution of (say) X is simply the distribution of X. The terminology "marginal" can be safely ignored.

(d) The covariance $\mathrm{Cov}(X, Y)$ should be positive. The reason is that X and Y change in more-or-less the same direction. That is, when X gets larger (resp. smaller), Y tends to get larger (resp. smaller) as well.

(e) By definition, $\mathrm{Cov}(X, Y) = E[XY] - E[X]E[Y]$. To calculate $E[XY]$, we just follow definition and

$$E[XY] = 1\cdot 1\cdot \frac{p}{2} + 2\cdot 1\cdot \frac{pq}{2} + 2\cdot 2\cdot \frac{p^2}{4} = \frac{p}{2} + pq + p^2 = \frac{p}{2} + p(p+q) = \frac{p}{2} + p = \frac{3p}{2}.$$

Since X is $B(2, 0.5)$, we have $E[X] = 2 \times 0.5 = 1$. Similarly, since Y is $B(2, p/2)$, we have $E[Y] = 2 \times p/2 = p$. It follows that

$$\mathrm{Cov}(X, Y) = E[XY] - E[X]E[Y] = \frac{3p}{2} - 1\cdot p = \frac{p}{2} > 0$$

4. By the general variance formula,

$$\begin{aligned} L(\beta) &= \mathrm{Var}[X + \beta Y] = \mathrm{Var}[X] + \mathrm{Var}[\beta Y] + 2\mathrm{Cov}(X, \beta Y) \\ &= \mathrm{Var}[X] + \beta^2 \mathrm{Var}[Y] + 2\beta \mathrm{Cov}(X, Y). \end{aligned}$$

That is, $L(\beta)$ is a quadratic function of β. Taking derivative with respect to β and setting it to 0, we have

$$2\beta \mathrm{Var}[Y] + 2\mathrm{Cov}(X, Y) = 0.$$

Therefore, the function $L(\beta)$ is minimized at

$$\beta^* = -\frac{\mathrm{Cov}(X, Y)}{\mathrm{Var}[Y]}$$