GOP Report

October 29, 2017

1 Problem Formulation

$\min_{\theta,\bar{b}}$
$\sum\limits_{h}\sum\limits_{X_h}\theta_h(X_h)(M{*}\mathrm{b}(X_h)-\bar{b}(X_h))+\sum\limits_{i}\sum\limits_{X_i}\sum\limits_{X_h}\theta_h(X_i,X_h)(M{*}\mathrm{b}(X_i,X_h)-\bar{b}(X_i,X_h))+$
$\sum\limits_{m}\sum\limits_{X_h} b_h^{(m)} log(b_h^{(m)}) - \frac{\lambda}{2}||\theta||^2$

S.t:

1- $\bar{b}(X_h) = \sum\limits_{j=1}^{m} b^{(m)}(X_h)$

$2 - \bar{b}(X_i,X_h) = \sum\limits_{j=1}^{m} b^{(m)}(X_i,X_h)$

$3 - b^{(m)}(X_h) = \sum\limits_{X_i} b^{(m)}(X_i,X_h)$

2 Primal Formulation

$\min_{\theta,\bar{b}}$
$\sum\limits_{h}\sum\limits_{X_h}\theta_h(X_h)(M{*}\mathrm{b}(X_h)-\bar{b}(X_h))+\sum\limits_{i}\sum\limits_{X_i}\sum\limits_{X_h}\theta_h(X_i,X_h)(M{*}\mathrm{b}(X_i,X_h)-\bar{b}(X_i,X_h))+$
$\sum\limits_{m}\sum\limits_{X_h} b_h^{(m)} log(b_h^{(m)}) - \frac{\lambda}{2}||\theta||^2 + \mu_1(\sum\limits_{j=1}^{m} b^{(m)}(X_h) - \bar{b}(X_h)\ ) + \mu_2(\sum\limits_{j=1}^{m} b^{(m)}(X_i,X_h) -$
$\bar{b}(X_i,X_h)\ ) + \sum\limits_{j=1}^{m}\gamma_j(\sum\limits_{X_i} b^{(m)}(X_i,X_h) - b^{(m)}(X_h))$

3 Create upper bound by solving the Primal problem for fixed θ

$\mathrm{P}^k(\theta^k) = \min_{\bar{b}}$
$\sum\limits_{h}\sum\limits_{X_h}\theta_h^k(X_h)(M{*}\mathrm{b}(X_h)-\bar{b}(X_h))+\sum\limits_{i}\sum\limits_{X_i}\sum\limits_{X_h}\theta_h^k(X_i,X_h)(M{*}\mathrm{b}(X_i,X_h)-\bar{b}(X_i,X_h))+$
$\sum\limits_{m}\sum\limits_{X_h} b_h^{(m)} log(b_h^{(m)}) + \mu_1(\sum\limits_{j=1}^{m} b^{(m)}(X_h) - \bar{b}(X_h)\ ) + \mu_2(\sum\limits_{j=1}^{m} b^{(m)}(X_i,X_h) - \bar{b}(X_i,X_h)$

$) + \sum_{j=1}^{m} \gamma_j (\sum_{X_i} b^{(m)}(X_i, X_h) - b^{(m)}(X_h))$

we will take the derivatives to solve for both μ_1, μ_2

$\frac{\partial P^k(\theta^k)}{\partial \bar{b}(X_h)} = -\theta_h^k(X_h) - \mu_1 = 0 \implies -\theta_h^k(X_h) = \mu_1$

$\frac{\partial P^k(\theta^k)}{\partial \bar{b}(X_i, X_h)} = -\theta_h^k(X_i, X_h) - \mu_1 = 0 \implies -\theta_h^k(X_i, X_h) = \mu_2$

Note: I didn't calculate γ because it's irrelevant in the Lower bound construction

4 Relaxed Dual Problem

First let's define L$^k as follow$:

$L^k(\bar{b}^{B_l}, \theta, \mu^k) =$

$\sum_h \sum_{X_h} \theta_h(X_h)(M*\text{b}(X_h)) + \sum_i \sum_{X_i} \sum_{X_h} \theta_h(X_i, X_h)(M*\text{b}(X_i, X_h)) + \frac{\lambda}{2}||\theta||^2 + \sum_h \sum_{X_h} \bar{b}(X_h)$ *(-1 * $(\theta_h(X_h) + \theta_h(X_h)^k)) + \sum_h \sum_{X_h} \bar{b}(X_i, X_h)$ *(-1 * $(\theta_h(X_i, X_h) + \theta_h(X_i, X_h)^k)) =$

$L^k(\theta, \mu^k) + \sum_h \sum_{X_h} \bar{b}(X_h)$ * g(X$_h$) + $\sum_h \sum_{X_h} \bar{b}(X_i, X_h)$ * g(X$_i, X_h$)

Where,

L$^k(\theta, \mu^k) = \sum_h \sum_{X_h} \theta_h(X_h)(M*\text{b}(X_h)) + \sum_i \sum_{X_i} \sum_{X_h} \theta_h(X_i, X_h)(M*\text{b}(X_i, X_h)) + \frac{\lambda}{2}||\theta||^2$,

$g(X_h) = (-1 * (\theta_h(X_h) + \theta_h(X_h)^k))$

$g(X_i, X_h) = (-1 * (\theta_h(X_i, X_h) + \theta_h(X_i, X_h)^k))$

With out loss of generality let's assume that θ is a positive vector (we can force this constraint by shifting all singular values of θ by constant scalar)

Given this assumption we can argue that:
1- $g(X_h)$ is always negative for any selection of $\bar{b}(X_h)$
2- $g(X_i, X_h)$ is always negative for any selection of $\bar{b}(X_i, X_h)$

from 1, 2 only one relaxed dual problem needs to be solved, with a valid under estimator to $\mathrm{L}^k(\bar{b}^{B_l}, \theta, \mu^k), where \bar{b}(X_h)$, and $\bar{b}(X_i, X_h)$ are fixed to their upper bound.