1. [12 Pts] Let S defined recursively by (1) 5 ∈ S and (2) if s ∈ S and t ∈ S, then st ∈ S. Let A = {5^i | i ∈ Z+}. Prove that

(a) [6 Pts] A ⊆ S by mathematical induction.
[∀x ∈ A, x ∈ S]
∀n ∈ Z+, P(n): 5^n ∈ S

Basis:
P(1): 5^1 ∈ S. Prove 5^1 = 5 ∈ S.
5 ∈ S by basis of inductive def of S.

Ind Step:
Assume: P(k): 5^k ∈ S
Prove: P(k+1): 5^(k+1) ∈ S
Since x = 5 ∈ S by basis step of inductive def of S and y = 5^k ∈ S by IH, therefore xy = 5(5^k) ∈ S by ind step of ind def of S. So, 5(5^k) = 5^k * 5^1= 5^(k+1) ∈ S
QED

(b) [6 Pts] S ⊆ A by structural induction.
[∀x ∈ S, x ∈ A]
Basis:
By basis step of ind def of S, 5 ∈ S. We prove that 5 ∈ A. Since 5 ^1 = 5 and 1 ∈ Z+, by definition of A, 5 ∈ A.

Ind Step:
Consider s,t ∈ S, assume s,t ∈ A.
By ind step of ind def, st ∈ S. We prove that st ∈ A.
Since, by IH s and t ∈ A, it follows that s = 5^i, i ∈ Z+ and t = 5^j, j ∈ Z+
Therefore, st = 5^i * 5^j = 5^(i+j), where i+j ∈ Z+
So, st ∈ A , as required
QED

2. [5 Pts] Give an inductive definition of the set of palindromes over the alphabet {a, b, c}.You do not need to prove that your construction is correct. Note: a, b, c, aa, cc, aba are all palindromes.

Let Σ be the alphabet.
Σ* = set of palindroms over Σ
Basis Ɛ ∈ Σ*
Ind if w ∈ Σ, r ∈ Σ* then wrw ∈ Σ*

3. [5 Pts] Define the set S = {2^k * 3^m | k, m ∈ Z+} inductively. You do not need to prove that your construction is correct.

Basis 6 ∈ S
Ind if x ∈ S, then x * 3 ∈ S and x * 2 ∈ S

4. [8 Pts] Given the inductive definition of full binary trees (FBTs), define n(T), the number of vertices in tree T , and ℒ(T), the number of leaves in tree T , inductively. Then, use structural induction to prove that for all FBTs T , n(T) = 2ℒ(T) − 1.

The number of vertices in a FBT is:

a) Is one, if T has a single vertex and no children, n(T) = 1
b) If T has children, n(T) = 1 + n(T1) + n(T2)

The number of leaves in a FBT is:

a) Is one of T has no children, ℒ(T) = 1
b) If T has children ℒ(T) = ℒ(T1) + ℒ(T2)

Theorem: For all FBTs n(T) = 2ℒ(T) − 1
Proof:
Basis:

Consider FBT containing a single vertex.
So, n(T) and ℒ(T) = 1 by basis step of ind def of vertices and of leafs.
n(1) = 2ℒ(1) − 1
1 = 2-1 = 1
So, n(T) = 2ℒ(T) − 1

Ind Step
Let T1 and T2 be left and right subtrees of T.
Assume $n(T1) = 2\mathscr{L}(T1) - 1$ and $n(T2) = 2\mathscr{L}(T2) - 1$

Prove that $n(T) = 2\mathscr{L}(T) - 1$
Now, $n(T) = 1 + n(T1) + n(T2)$, by ind def of vertices
$= 1 + 2\mathscr{L}(T1) - 1 + 2\mathscr{L}(T2) - 1$, by IH
$= 2\mathscr{L}(T1) + 2\mathscr{L}(T2) - 1$
$= 2(\mathscr{L}(T1) + \mathscr{L}(T2)) - 1$
$= 2\mathscr{L}(T) - 1$, by ind def of leaves

Thus, $n(T) = 2\mathscr{L}(T) - 1$
QED

5. [15 Pts] Let $L = \{(a, b) \mid a, b \in \mathbb{Z}, (a - b) \bmod 3 = 0\}$. We want to program a robot that can get to each point $(x, y) \in L$ starting at $(0, 0)$.

(a) [5 Pts] Give an inductive definition of L. This will describe the steps you want the robot to take to get to points in L starting at (0, 0). Let L′ be the set obtained by your inductive definition.

Define L'
Basis $(0,0) \in L'$
i) if $(x,y) \in L$, then $(x+3,y) \in L'$,
$(x-3,y) \in L'$,
$(x,y+3) \in L'$,
$(x,y-3) \in L'$

(b) [5 Pts] Prove inductively that $L' \subseteq L$, i.e., every point that the robot can get to is in L.

Basis:
By basis ind step of L', $(0,0) \in L'$
We prove that $(0,0) \in L$. Since (0-0) mod 3 and $0 \in Z$, by def of L, $(0,0) \in L$

Ind Step
Consider $x,y \in L'$, assume $x,y \in L$.
So, $x,y \in Z$, $(x-y) \bmod 3 = 0$
So, $\exists k \in Z$, $x-y \in k$
So, $k \bmod 3 = 0$
By the inductive step of ind def of L', $(x+3,y)$, $(x-3,y)$, $(x,y+3)$,$(x,y-3) \in L'$
We prove that $(x+3,y)$, $(x-3,y)$, $(x,y+3)$,$(x,y-3) \in L$

i) $x+3 -y = k +3$ where $(k+3) \bmod 3 = k \bmod 3 + 3 \bmod 3 = 0 + 0 = 0$
Also, $x+3$ and $y \in Z$, so $(x+3,y) \in L$
ii) $x-3 -y = k - 3$ where $(k-3) \bmod 3 = k \bmod 3 - 3 \bmod 3 = 0 - 0 = 0$
Also, $x-3$ and $y \in Z$, so $(x-3,y) \in L$
iii) $x- (y+3) = x -y -3 = k - 3$ where $(k- 3) \bmod 3 = k \bmod 3 - 3 \bmod 3 = 0 + 0 = 0$
Also, x and $y+3 \in Z$, so $(x,y+3) \in L$
iiii) $x-(y-3) = x -y + 3 = k +3$ where $(k+3) \bmod 3 = k \bmod 3 + 3 \bmod 3 = 0 + 0 = 0$
Also, x and $y-3 \in Z$, so $(x,y-3) \in L$
QED

(c) [5 Pts] Extra Credit Prove that $L \subseteq L'$, i.e., the robot can get to every point in L.
To prove this, you need to give the path the robot would take to get to every point in L
from (0, 0), following the steps defined by your inductive rules.