Two Point Evaluation Expectation

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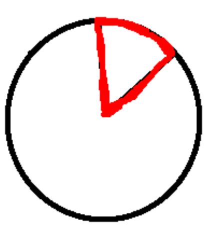
Setup. Let's work with a function $f(x) : \mathbb{R}^2 \mapsto \mathbb{R}$, so I can draw pictures and explain the intuition. I'm pretty sure that we can generalize this to arbitrary dimensions. Let's say we are running gradient descent and are at some point x. We don't have access to the gradient, but can evaluate the cost of a function at x as well as some nearby point. Let's pick a small radius, and consider a sphere (circle in our case) of this radius centered at the origin. Define r to be a vector sampled uniformly from this circle. Define w as follows:

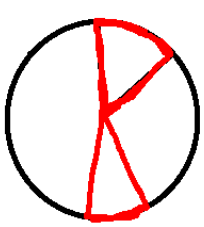
$$w = \begin{cases} r, & \text{if } f(x+r) \geq f(x) \\ -r, & \text{if } f(x+r) < f(x) \end{cases} \tag{1}$$

I will show with a proof sketch and intuitive argument that $\mathbb{E}[w^\top \nabla f(x)] \geq 0$, in the case of the LQR, assuming that the radius was chosen small enough.

Intuition. In the LQR setting, we know that we satisfy the PL Inequality (very close to strong convexity). This implies that all stationary points, including local minima, are actually global minima. We also know that there is a unique optimum controller K^*, which is the solution we want to achieve with gradient descent. Now let's define Θ to be all points r on the boundary of the circle where $f(x+r) \geq f(x)$. Assume that in the pictures below, the red area on the boundary defines Θ. Clearly, the direction which corresponds to the gradient lies in this red region. Crucially, this looks something like the following:



and not like the following:



The reason the second picture is wrong is that this would imply there are two "different" regions of ascent, with different regions of descent between them. This means that we have different local minima, which contradicts the LQR setting as I explained it above.

If this argument is wrong, let me know, I think the proof could still be adapted so that my result still holds. But my intuition for the argument came from this thinking, and the plots I made earlier.

Theorem. *If we select w in the fashion above, then $\mathbb{E}[w^\top \nabla f(x)] \geq 0$.*

Proof. Define θ to be the angle that Θ forms on the boundary of the circle. First consider the case when $\theta \geq \pi$. Clearly, $\mathbb{E}[w^\top \nabla f(x)] \geq 0$. Now consider the second case when $\theta < \pi$. Define H to be the half-sphere which is formed by dividing the circle using the plane defined by using the gradient as the perpendicular to the plane. Define F to be the other half-sphere. Define x to be the angle formed by r with the horizontal axis. Then, after rotation we have that:

$$
\begin{aligned}
\mathbb{E}[w^\top \nabla f(x)] &= \mathbb{E}[w^\top \nabla f(x) | r \in \Theta] P(r \in \Theta) + \mathbb{E}[w^\top \nabla f(x) | r \in H \text{ and } r \notin \Theta] P(r \in H \text{ and } r \notin \Theta) && (2)\\
&\quad + \mathbb{E}[w^\top \nabla f(x) | r \in F] P(r \in F) && (3)\\
&= \mathbb{E}[r^\top \nabla f(x) | r \in \Theta] P(r \in \Theta) + \mathbb{E}[-r^\top \nabla f(x) | r \in H \text{ and } r \notin \Theta] P(r \in H \text{ and } r \notin \Theta) && (4)\\
&\quad + \mathbb{E}[(-r)^\top \nabla f(x) | r \in F] P(r \in F) && (5)\\
&= \int_0^\theta \|r\| \, \|\nabla f(x)\| \cos(x) \frac{1}{\theta} \, \mathrm{d}x (\frac{\theta}{2\pi}) - \int_\theta^\pi \|r\|_2 \, \|\nabla f(x)\|_2 \cos(x) \frac{1}{\pi - \theta} \, \mathrm{d}x (\frac{\pi - \theta}{2\pi}) && (6)\\
&\quad - \int_\pi^0 \|w\| \, \|\nabla f(x)\| \cos(x) \frac{1}{\pi} \, \mathrm{d}x (\frac{\pi}{2\pi}) && (7)\\
&= \frac{\|w\| \, \|\nabla f(x)\|}{2\pi} [\int_0^\theta \cos(x) \, \mathrm{d}x - \int_\theta^\pi \cos(x) \, \mathrm{d}x - \int_\pi^0 \cos(x) \, \mathrm{d}x] && (8)\\
&= \frac{\|\nabla f(x)\|_2 \sin(\theta)}{\pi} && (9)\\
&\geq 0 && (10)
\end{aligned}
$$

So, with the two cases we have shown the result. $\square$