Sunday, 11 January 2026　　12:42 PM

@　$G = c = 1$, Equatorial plane $(\theta = \pi/2)$.

Schwarzschild line element is,

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2 d\phi^2$$

$\tau \rightarrow$ Achilles' proper time

$$\boxed{\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \quad \text{where} \quad \dot{x} = \frac{dx}{d\tau}$$

In general relativity, motion of a free falling particle is given by geodesic. We can describe this motion using a Lagrangian $\mathcal{L}$.

From provided Schwarzschild metric, non zero components of the metric tensor $g_{\mu\nu}$ are:

$$g_{tt} = -\left(1 - \frac{2M}{r}\right) \qquad g_{\phi\phi} = r^2$$

$$g_{rr} = \left(1 - \frac{2M}{r}\right)^{-1}$$

Substituting these into the given Lagrangian formula,

$$\boxed{\mathcal{L} = \frac{1}{2}\left[-\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\phi}^2\right]}$$

→ A coordinate x^α is said to be cyclic if the Lagrangian does not explicitly depend on it. $\left(\dfrac{\partial \mathcal{L}}{\partial x^\alpha} = 0\right)$

- Time $(t) \longrightarrow$ only appears as a derivative $\dot{t}$. There is no t term on its own,
  ∴ $\boxed{t \text{ is cyclic}}$

- Azimuthal angle (ϕ): ϕ only appears as $\dot{\phi}$. No ϕ term on its own.
  · $\boxed{\phi \text{ is cyclic}}$

$$\ddot{x} - \ldots$$

→ According to [Noether's theorem], every cyclic coordinate corresponds to a conserved quantity.

- Canonical momentum p_α is defined as:

$$\boxed{p_\alpha = \frac{\partial L}{\partial \dot{x}^\alpha}}$$

For the time coordinate (t):

$$p_t = \frac{\partial L}{\partial \dot{t}} = \frac{1}{2}\left[-2\left(1-\frac{2M}{r}\right)\dot{t}\right] = -\left(1-\frac{2M}{r}\right)\dot{t}$$

This is associated with energy of the particle at infinity.

By convention,

$$E = -p_t = \left(1 - \frac{2M}{r}\right)\dot{t}$$

For angular coordinate (ϕ):

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \frac{1}{2}\left[2r^2\dot{\phi}\right] = r^2\dot{\phi}$$

∴ This is the Angular Momentum (L) of the particle: $L = p_\phi = r^2\dot{\phi}$

→ Why equatorial plane can be taken?
- Spherical Symmetry of the Schwarzschild metric means that the mass does not prefer any specific direction
- In a central potential, motion of particles is confined to a single plane.
- Thus, we are free to ~~rot~~ rotate our plane of motion so that it aligns with the equatorial plane.

⑧ Normalization conditions
→ $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -1$

Using Schwarzschild metric for equatorial plane, $(\theta = \pi/2)$,

we expand $g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = -1$

$\Rightarrow$ $g_{tt}\dot{t}^2 + g_{rr}\dot{r}^2 + g_{\phi\phi}\dot{\phi}^2 = -1$

$\Rightarrow$ $-\left(1-\dfrac{2M}{r}\right)\dot{t}^2 + \left(1-\dfrac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\phi}^2 = -1$

From part(a)

$$\boxed{\longrightarrow \quad \dot{t} = \dfrac{E}{1-\frac{2M}{r}} \qquad \longrightarrow \dot{\phi} = \dfrac{L}{r^2}}$$

$-\left(1-\dfrac{2M}{r}\right)\left[\dfrac{E}{1-\frac{2M}{r}}\right]^2 + \left(1-\dfrac{2M}{r}\right)^1 \dot{r}^2$

$+ \; r^2 \left[\dfrac{L}{r^2}\right]^2 = -1$

$\Rightarrow \quad -\dfrac{E^2}{\left(1-\frac{2M}{r}\right)} + \dfrac{\dot{r}^2}{\left(1-\frac{2M}{r}\right)} + \dfrac{L^2}{r^2} = -1$

$\Rightarrow$ Multiplying entire equation by $\left(1-\frac{2M}{r}\right)$

$\Rightarrow \quad -E^2 + \dot{r}^2 + \dfrac{L^2}{r^2}\left(1-\dfrac{2M}{r}\right) = -\left(1-\dfrac{2M}{r}\right)$

$\Rightarrow \quad \left(1-\dfrac{2M}{r}\right) + \dfrac{L^2}{r^2}\left(1-\dfrac{2M}{r}\right) + \dot{r}^2 = E^2$

$\Rightarrow \quad \left(1-\dfrac{2M}{r}\right)\left(1+\dfrac{L^2}{r^2}\right) + \dot{r}^2 = E^2 \quad —①$

Given radial form $\Rightarrow \dot{r}^2 + V^2(r) = E^2 \quad —②$

Comparing ① and ②,

$$\boxed{V^2(r) = \left(1-\dfrac{2M}{r}\right)\left(1+\dfrac{L^2}{r^2}\right)} \rightarrow \text{Hence, proven.}$$

© Specialize to $L = 0$.

Spaceship starts from rest at $r = R$.

radial velocity at starting point, $\boxed{\dot{r} = 0}$

$\therefore\ L = 0,\ \dot{r} = 0,\ r = R$ in our radial

equation, $\dot{r}^2 + V^2(r) = E^2$

$$0^2 + \left(1 - \frac{2M}{R}\right)\left(1 + \frac{0^2}{R^2}\right) = E^2$$

$\Rightarrow$ $\boxed{E^2 = \left(1 - \frac{2M}{R}\right)}$ — ⑪

Now, again,

$$\dot{r}^2 + \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{R^2}\right) = E^2$$

Here, r changes but $L = 0$,

$$\dot{r}^2 + \left(1 - \frac{2M}{r}\right)(1 + 0) = E^2$$

Substituting, ⑪

$\Rightarrow$ $\dot{r}^2 + \left(\cancel{1} - \frac{2M}{r}\right) = \left(\cancel{1} - \frac{2M}{R}\right)$

$\Rightarrow$ $\boxed{\dot{r}^2 = 2M\left(\frac{1}{r} - \frac{1}{R}\right)}$

$$\frac{dr}{d\tau} = -\sqrt{2M\left(\frac{1}{r} - \frac{1}{R}\right)}$$

→ Achilles is starting from a high point
   $r = R$ and its radial distance decreases
   to 0.

∴ We take the negative sign.

$$r = \frac{R}{2}\left(1 + \cos x\right) \quad \text{when } x = 0$$
$$r = R$$
$$\text{when } x = \pi_1$$
$$r = 0.$$

$$\Rightarrow \quad \frac{dr}{dx} = -\frac{R}{2}\sin x$$

$$\boxed{\frac{dr}{dz} = \frac{dr}{dx}\frac{dx}{dz}}$$

$r(x)$ into r^2 equation,

$$\left(\frac{dr}{dx} \cdot \dot{x}\right)^2 = 2M\left(\frac{2}{R(1+\cos x)} - \frac{1}{R}\right)$$

$$\Rightarrow \left(-\frac{R}{2}\sin x \cdot \dot{x}\right)^2 = 2M\left(\frac{2}{R \times 2\cos^2\left(\frac{x}{2}\right)} - \frac{1}{R}\right)$$

$$\Rightarrow \left(\frac{R^2}{4}\sin^2 x \cdot \left(\dot{x}\right)^2\right) = \frac{2M}{R}\left(\frac{1 - \cos^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)}\right)$$

$$\Rightarrow \frac{R^2}{4}\sin^2 x \cdot \left(\frac{dx}{d\tau}\right)^2 = \frac{2M}{R}\left(\frac{\sin^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)}\right)$$

$$\Rightarrow \frac{R^2}{4}\sin^2 x \left(\frac{dx}{d\tau}\right)^2 = \frac{2M}{R}\tan^2\left(\frac{x}{2}\right)$$

$$\Rightarrow \frac{R^3}{8}\left(\frac{dx}{d\tau}\right)^2 \sin^2 x = M\tan^2\left(\frac{x}{2}\right)$$

$$\Rightarrow \frac{R^3}{8M}\left(\frac{dx}{d\tau}\right)^2 4\sin^2\frac{x}{2}\cos^2\frac{x}{2} = \frac{\sin^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)}$$

$$=) \quad \frac{R^3}{8M} \left(\frac{dx}{d\tau} \right)^2 = \frac{1}{4\cos^4(x/2)}$$

$$=) \frac{R^3}{8M} \times 4\cos^4(x/2) = (d\tau)^2$$

$$=) \quad \frac{R^3}{8M} \left(2\cos^2(x/2) \right)^2 (dx)^2 = (d\tau)^2$$

$$=) \quad \frac{R^3}{8M} \left(1 + \cos x \right)^2 (dx)^2 = (d\tau)^2$$

$$=) \quad \sqrt{\frac{R^3}{8M}} \left(1 + \cos x \right) dx = d\tau$$

$$=) \text{ Integrating both sides,}$$

$$\int_0^x \frac{R^{3/2}}{2\sqrt{2M}} \left(1 + \cos x \right) dx = \int_0^\tau d\tau$$

$$=) \quad \frac{R^{3/2}}{2\sqrt{2M}} \left(x + \sin x \right) = \tau(x)$$

At $\tau = 0$, $\cos x = -1 \longrightarrow x = \pi$

$$=) \quad \text{at } x = \pi$$

$$\tau_{total} = \frac{R^{3/2}}{2\sqrt{2M}} \left(\pi + \sin \pi \right)$$

$$=) \qquad \boxed{\qquad \qquad 3/2}$$

$$C_{total} = \frac{\pi k}{2\sqrt{2M}}$$

(d) $\dfrac{dr}{dt} = \dfrac{dr/d\tau}{dt/d\tau} = \dfrac{\dot{r}}{\dot{t}}$

we know from previous parts,

$$E = \left(1 - \frac{2M}{r}\right)\dot{t}$$

$\Rightarrow$ $\dot{t} = \dfrac{E}{1 - \frac{2M}{r}}$ $\longrightarrow$ $\dfrac{dr}{dt} = \dfrac{\dot{r}}{\dfrac{E}{\left(1 - \frac{2M}{r}\right)}}$

$\therefore$ $\dfrac{dr}{dt} = \dfrac{\dot{r}}{E}\left(1 - \frac{2M}{r}\right)$

$\Rightarrow$ $\boxed{\dfrac{dr}{dt} = \dfrac{\left(1 - 2M/r\right)\dot{r}}{E}}$

* <u>Tortoise Coordinate</u>

$$r_* = r + 2M \ln\left|\frac{r}{2M} - 1\right|$$

$\dfrac{dr_*}{dr} = 1 + \dfrac{2M}{\left(\frac{r}{2M} - 1\right)} \times \dfrac{1}{2M} = 1 + \dfrac{1}{\frac{r}{2M} - 1}$

$\Rightarrow$ $\dfrac{\frac{r}{2M} - 1 + 1}{\frac{r}{2M} - 1}$ $\Rightarrow$ $\dfrac{\frac{r}{2M}}{\frac{r - 2M}{2M}}$ $\Rightarrow$ $\left(\frac{r - 2M}{r}\right)^{-1}$

$\Rightarrow$ $\left(1 - \frac{2M}{r}\right)^{-1}$

$$\frac{dr_*}{dt} = \frac{dr^*}{dr} \cdot \frac{dr}{dt} =) \quad \left(1 - \frac{2M}{r}\right)^{-1} \cdot \left(1 - \frac{2M}{r}\right) \cdot \frac{\dot{r}}{E}$$

$$\therefore \quad \frac{dr_*}{dt} = \frac{\dot{r}}{E}$$

$$V^2(r) = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{r^2}\right)$$

=)   As Achilles goes to event horizon, $(r \to 2M^+)$

-) $V^2(r) \to 0$

From radial equation, $\dot{r}^2 = E^2 \cdot V^2(r)$

=)   Since $V^2(r) \to 0$ as $r \to 2M^+$

$$\therefore \quad \dot{r}^2 \approx E^2 =) \quad \boxed{\dot{r} \approx -E}$$

→ we choose negative root because Achilles is falling in.

$$\dot{r} \approx -E \longrightarrow \boxed{\frac{dr_*}{dt} \approx \frac{-E}{E} = -1}$$

Integrating,

$$r_* \approx -t + k \qquad \text{where } k \to \text{constant}$$

$$r_* = r + 2M \ln\left|\frac{r}{2M} - 1\right|$$

As $r \to 2M$, logarithmic term dominates,

Approximating, $\Rightarrow$ $r* \approx 2M \ln \left| \dfrac{r}{2M} - 1 \right|$

$\therefore$ $2M \ln \left| \dfrac{r}{2M} - 1 \right| \approx -t + K$

$\Rightarrow$ $\ln \left| \dfrac{r}{2M} - 1 \right| \approx \dfrac{-t}{2M} + \dfrac{K}{2M}$

$\Rightarrow$ $\dfrac{r}{2M} - 1 \approx e^{-t/2M} \cdot e^{K/2M}$

$\Rightarrow$ $r - 2M \approx 2M e^{-t/2M} \cdot e^{K/2M}$

$\Rightarrow$ $\boxed{r \approx C e^{-t/2M} + 2M}$

$\longrightarrow$ To convert $-t/2M$ to seconds, replace $2M$ with Schwarzschild radius, $\boxed{R_S = \dfrac{2GM}{c^2}}$

(c) Does Achilles reach the tortoise in finite Schwarzschild time?

- No. As derived in part (b), $r(t) \approx C e^{-t/2M} + 2M$

$\therefore$ $e^{-t/2M} \to 0$ as $t \to \infty$ which implies,

$r(t) \approx 2M$ as $t \to \infty$.

Resolving this paradox, the two explanations

one both physically valid, yet different because they represent different observers, Achilles as one and the tortoise as the other

- Achilles sees that it reaches the tortoise in finite proper time.

- Tortoise never sees Achilles cross the horizon because the signals take infinite coordinate time to reach it.

→ Another point to be noted is that we are deriving conclusions only on the basis of the mathematical equations, we have solved and the topics at hand are a part of general relativity which is highly variable.