Extensions on Simulating Call Option Delta Hedges Using Monte-Carlo Methods

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Abstract

Delta Hedging is the process of minimizing exposure to price changes in the underlier for a derivative financial instrument. In this paper, we will start by exploring vanilla delta hedging, and tweaking various parameters to see their effect on the result of the delta hedging. In particular, we will use Monte-Carlo simulation to analyze the effects of delta hedging a short call option. We will see that the cost of delta hedging will offset the cost of selling the option in a risk neutral world, i.e. the Black Scholes premium. Additionally, we find that we can achieve decent accuracy even in a non-risk-neutral world and transaction costs.

1. Background

1.1 Black Scholes

The Black Scholes formula is a model to price a European call option. This number can also be used with some other math to price a put option. It can only be used for a European call as the formula only works when an option is excercised on its expiration. Although the Black Scholes model is very important it has to follow many assumptions. The assumptions go as follows there can be no dividends throughout the option (there exists a modified formula for continuous dividends), there are no transaction costs for the options, the underlying asset grows at the risk free rate and has a constant volatility, and that the markets must be random. There is extensive literature covering the derivation of the Black Scholes model and why it works; however, this will not be convered here.

1.2 Delta

Delta Hedging is the process of minimizing delta in order to reduce the risk of price changes from the underlying asset. For a given option its delta is the partial derivative of the option with respect to its underlying asset [2].

$$C_s = \mathcal{N}(d_1)S_t - \mathcal{N}(d_2)K * e^{-rt} \tag{Black Scholes}$$

$$d_1 = \frac{\ln \frac{S_t}{K} + (r + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}; d_2 = d1 - \sigma\sqrt{t} \tag{1}$$

$$\frac{\partial C}{\partial S} = \Phi(d_1) \tag{Δ}$$

Delta will also end up at 1 or 0 at the option expiry date based on if the option is in or out of the money. This makes sense because if the option is ITM then there is intrinsic value to the option and therefore if the underlying asset moves up $1 in price so will the option. If delta is 0 that means the option is out of the money and there is no use to exercise it making any changes in the underlying asset useless. The end result of delta will determine if we own the underlying asset at the option expiry.

1.3 Monte Carlo Simulation

1.3.1 Price Formula

Geometric Brownian Motion is used to simulate the price paths. The formula comprises two key elements the **drift** and the **shock** [1]. Drift is the first part of the equation creating the consistent price growth, and the shock multiplies the growth in either a positive or negative direction. As the shock uses a random variable n price paths will have log-normal prices. This can be more evident as t increases as prices compound upwards and can never fall below 0.

1.3.2 Formulas

Geometric Brownian Motion has a standard form that the expanded form is derived from

$$\frac{ds}{s} = \sigma dt + \sigma dW \tag{Standard Form}$$

$$S_{t+1} = S_t \exp\left[r - \frac{\sigma^2}{2}\Delta t + \sigma dW\right] \tag{2}$$

$$dW \sim \mathcal{N}(0, \Delta t) \tag{3}$$

The dW is the stochastic component of the equation which means it is the ”random” part of the formula

1.3.3 Price Path Results

These graphs depict the results of simulating the price paths using Monte Carlo simulations with the geometric Brownian motion formulas listed in section 1.3.2. As the graphs depict the returns are log-normally distributed and the log returns are normally distributed.

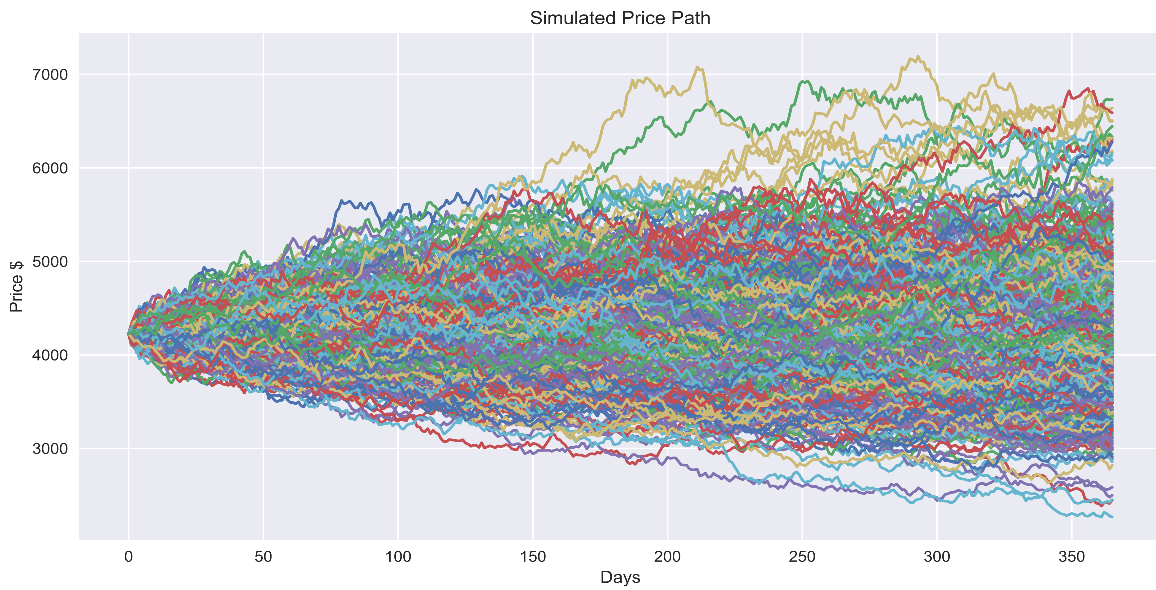

Figure 1: Simulated price paths for 365 days

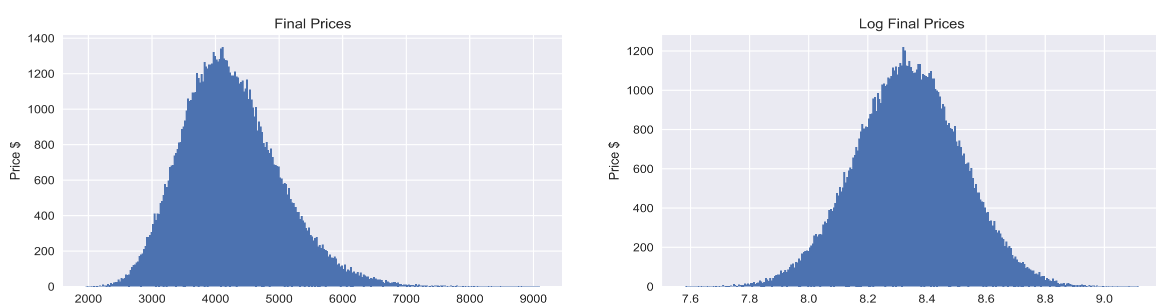

Figure 2: Final prices compression normal vs log

2. Parameters and Variables

Most of the simulations are run with the same set of variables and parameters and had 100,000 trials. μ and the transaction costs vary for some simulations.

$$
\begin{aligned}
S_0 &= K = 4225 \\
\mu &= 0 \\
r &= 0.005 \\
\sigma &= 0.18 \\
t &= 1 \text{ (year)}
\end{aligned}
$$

3. Simulating Delta Hedging

3.1 Discrete Delta Hedging

In this section we will be simulating discrete delta hedging. Discrete delta hedging is hedging the call option a discrete number of times rather than continuously. Theoretically continuous delta hedging would result in no money lost; however, in practice we cannot perform continuous delta hedging. In these samples we will vary how often we hedge starting from not hedging at all to hedging every day. Hedging 0 times is just putting the option premium in a bank account and having it adjust with interest. This price is not the exact same as the option premium as the result for no hedging also depends on if the simulated price path is ITM or OTM.

3.2 Discrete Delta Hedging Results

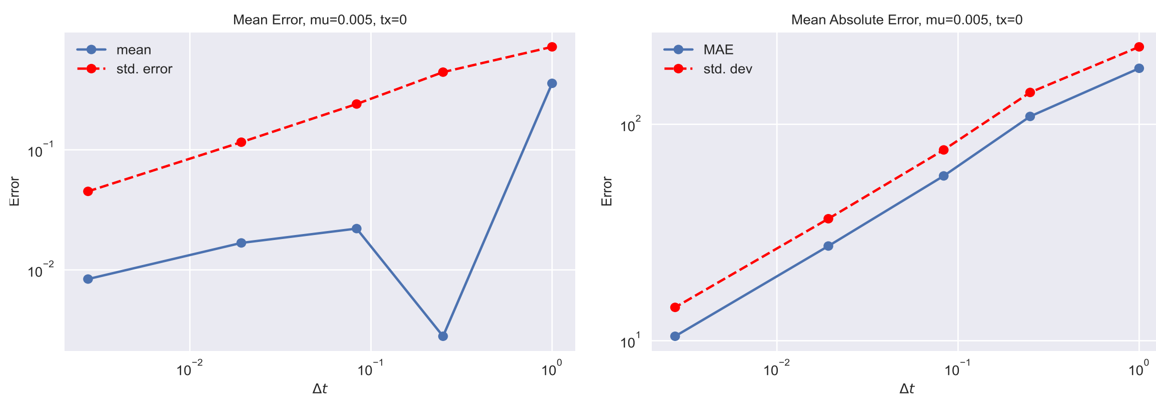

Figure 3: Graphs of error with respect to the different Δt

rehedge	Mean $\mathcal{E}$	Mean Absolute $\mathcal{E}$	Standard Dev.	Standard Error
0	2.928141	380.139610	504.951023	1.596795
1	0.358113	181.479375	227.994974	0.720983
4	0.002800	108.770324	140.382802	0.443929
12	0.022084	57.756108	76.153806	0.240819
52	0.016773	27.394315	36.622535	0.115811
365	0.008388	10.487060	14.261835	0.045100

Table 1: Data for discrete time hedging

Analysis

As Δt approaches 0 (continuous hedging) the strategy converges to the price of the Black Scholes. This can be seen in Figure 3 as the Δt starts at 1/365 (daily hedging) the mean error is rather low with a small standard error. The MAE is also linearly decreasing showing that if there was continuous delta hedging that most if not all of the hedging simulations would converge to the Black Scholes option price.

3.3 Varying μ

So far all of the prior simulation have been done in the Q measure as the price has been simulated using a constant risk free rate. In the next set of simulations μ will change so that it is no longer matching the risk free rate. Changing μ means that the prices will be simulated using a real world return; however, the Black Scholes formula and Delta will both be calculated using the risk free rate. This change in μ is effectively moving the simulations from the Q measure to the P measure.

3.3.1 $\mu = 0.1$

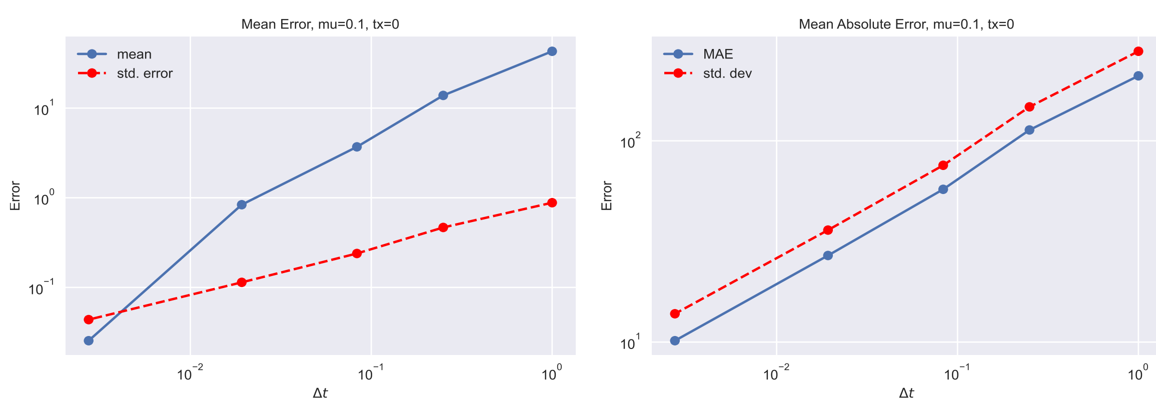

Figure 4: Comparing error with a changing $\mu = 0.1$

	Mean $\mathcal{E}$	Mean Absolute $\mathcal{E}$	Standard Dev.	Standard Error
0	274.628099	521.495655	680.534508	2.152039
1	43.031233	210.338377	278.533898	0.880802
4	13.852162	113.220787	147.491224	0.466408
12	3.696978	57.413832	75.578675	0.239001
52	0.836475	26.929354	36.012850	0.113883
365	0.025446	10.158392	13.825974	0.043722

Table 2: Data for discrete hedging with a $\mu = 0.1$

3.3.2 $\mu = 0.2$

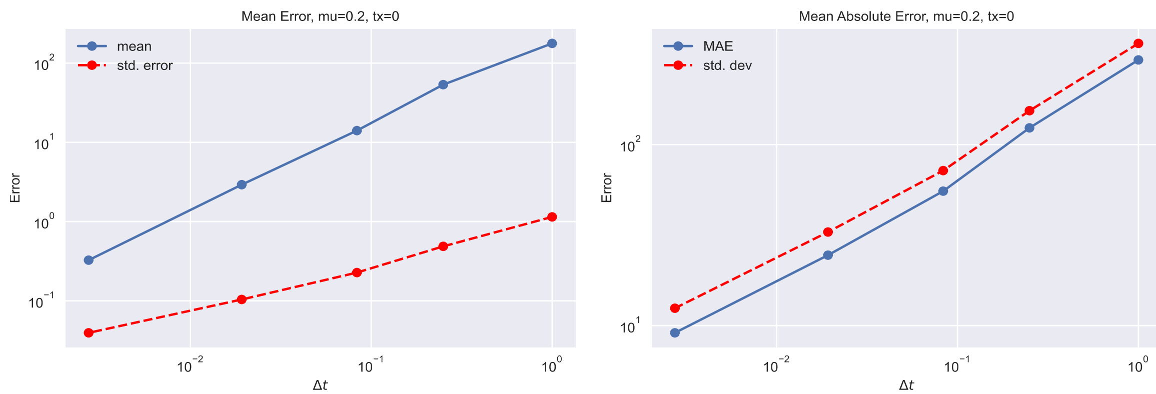

Figure 5: Comparing error with a changing $\mu = 0.2$

	Mean $\mathcal{E}$	Mean Absolute $\mathcal{E}$	Standard Dev.	Standard Error
0	673.422586	804.201667	854.189258	2.701184
1	177.154022	293.630024	362.273632	1.145610
4	53.463836	123.729162	153.960544	0.486866
12	14.064544	55.328133	71.858137	0.227235
52	2.922065	24.503504	32.886721	0.103997
365	0.326016	9.123852	12.477976	0.039459

Table 3: Data for discrete hedging with a $\mu = 0.2$

3.3.3 $\mu = -0.1$

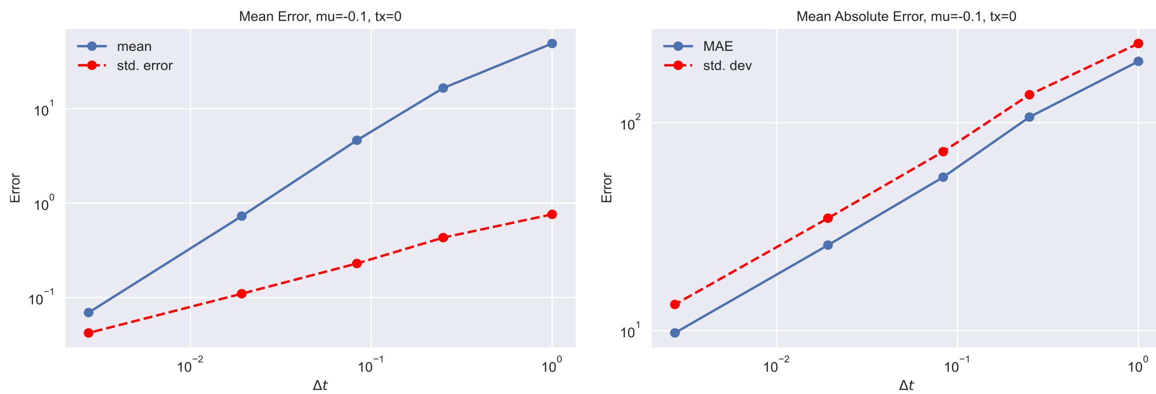

Figure 6: Comparing error with a changing μ = (-0.1)

	Mean $\mathcal{E}$	Mean Absolute $\mathcal{E}$	Standard Dev.	Standard Error
0	181.402244	318.272929	317.593416	1.004319
1	49.082361	197.499569	240.618079	0.760901
4	16.560763	106.520327	136.388315	0.431298
12	4.636949	54.719538	72.594502	0.229564
52	0.728936	25.763985	34.724050	0.109807
365	0.069344	9.737247	13.346946	0.042207

Table 4: Data for discrete hedging with a $\mu = -0.1$

Analysis

Varying μ does not have a large impact on the simulations. All 3 trials with a different μ from 0.005 had similar results from the discrete delta hedging simulations as Δt approaches 0. This can be seen by comparing any of the data from tables 2 to 4 to table 1. In each of the simulations with a different μ the MAE approaches 0 as Δt approaches a continuous delta hedge (0). Although changing the μ results in similar error numbers, while Δt is larger there is much more variance in the data and the errors are considerably larger than discrete delta hedging with a smaller μ Mathematically there is no surprise here. As the prices increase (decrease) depending on a higher (lower) μ, delta is also adjusted accordingly. If the delta scales up along with the prices, the hedge prices will be scaled up and so will the call option price. In the end all the numbers are scaled up making the errors similar as Δt approaches 0.

4. Real World Hedging vs Simulations

In all the tests done it seems that delta hedging works both in the P and the Q measure. In the P measure the errors are larger as the hedge intervals are larger, but in this section we will look at the point where delta hedging will not work in the P measure the same way as it does in the Q measure.

To figure out if delta hedging works in the P measure with the same effectiveness as in the Q measure we will conduct a P-value hypothesis test. In this case Error ($\mathcal{E}$) will be defined as $\epsilon Q - \epsilon P$ or the error from the Black Scholes model in the Q measure subtracted by the P measure error.

In this test the null and alternate hypothesis are listed below.

$$H_0 : \mathcal{E} = 0 \tag{Null}$$
$$H_a : \mathcal{E} \neq 0 \tag{Alternate}$$
$$\alpha = 0.05 \tag{4}$$

In the following table the P-value is calculated by $\Phi(Mean\mathcal{E}) * 2$. It is multiplied by 2 as we need the chance that it is not equal to the mean. The mean is used instead of 0 as errors are never exactly 0. The only results that are not statistically significant is when delta hedging is performed daily and weekly. This means that we cannot reject the null hypothesis and delta hedging could work in the P measure as well. Hedging daily is a reasonable amount and even if it is too frequent weekly delta hedging will also work. Interestingly going below that makes the results very statistically significant. As we can see by delta hedging monthly the errors deviate from the Q measure

	Mean $\mathcal{E}$	P-value	Significant
0	-266.083949	0.000000e+00	True
1	-43.918736	0.000000e+00	True
4	-13.779125	3.403720e-43	True
12	-3.513509	4.422292e-04	True
52	-0.949564	3.423338e-01	False
365	-0.183927	8.540704e-01	False

Table 5: P-values based on hedging intervals

5. Transaction Costs

So far there has also been no transaction costs factored into the equation. Transaction costs tend to mess with the delta hedging because as stated in section 1.1 the Black Scholes model does not take transaction costs into consideration. For this simulation we set a transaction cost of 1% on all the stock purchases made from the hedge besides the first and last. The first and last are larger purchases as in the beginning we have to buy more of the stock then in the middle and at the end we have to buy/sell the entire stock

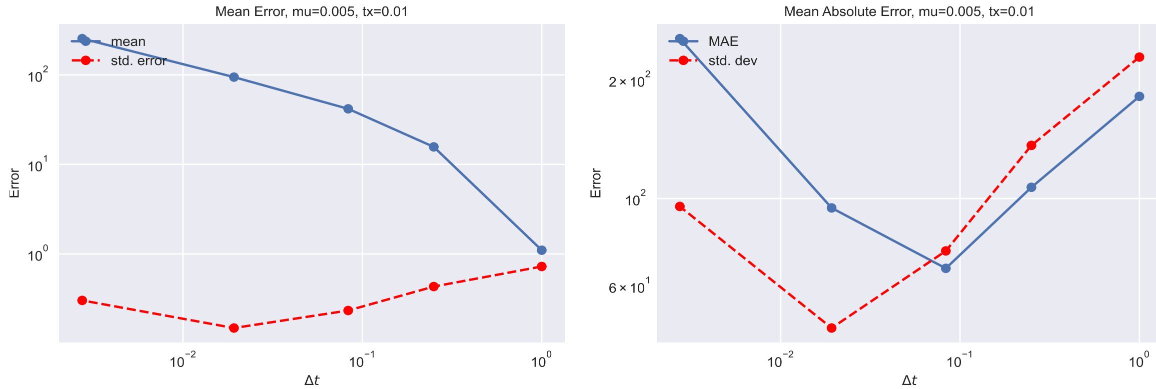

Figure 7: Graphs of error with a transaction cost of 0.01

	Mean $\mathcal{E}$	Mean Absolute $\mathcal{E}$	Standard Dev.	Standard Error
0	1.805585	378.670408	502.019258	1.587524
1	1.098460	181.599253	228.487614	0.722541
4	15.684182	106.750560	136.383026	0.431281
12	41.784505	66.493831	73.619932	0.232807
52	94.509400	94.643961	46.920536	0.148376
365	254.543752	254.543752	95.458617	0.301867

Table 6: Data for delta hedging with a transaction cost of 0.01

Analysis

Adding transaction costs to the equation makes the delta hedging extremely different than without them. In the data with transaction costs there is less error while hedging fewer times. Although the transaction cost seems small it can build up over time. For example if the price stayed constant at $4225 with a transaction cost of 0.01 (1 percent), hedging daily (365 days) would result in about $15,400 dollars lost. Adding transaction costs make delta hedging almost the opposite as if they were not there.

6. Conclusion

In theory delta hedging a call option continuously will result in no money made or lost. Since continuous delta hedging does not exist in the real world the next best thing is discrete delta hedging. This itself is also very accurate as delta hedging a derivative every day results in an almost no profit or loss made regardless if the trials are in the P or the Q measure. The only place where the costs seems to deviate from the Black Scholes model is when transaction costs are applied. In general with proper delta hedging we can increase the premium for an option beyond the Black Scholes valuation to make profit on it. Although the delta hedging seems to work in the simulations there are many more parameters in the real world and the formulas should be altered when using this strategy with real world derivatives.

References

[1] David R. Harper. "How to Use Monte Carlo Simulation With GBM". In: (2022).

[2] John C. Hull. *Options Futures and Other Derivatives*. 9th ed. Pearson, 2014. ISBN: 0-13-345631-5.